let

- φ, χ range over formulae
- $\varPhi, X, \varPsi$ range over lists of formulae
- $\#X < 2$
- $m \in \{\text{¡}, !, \text{¿}, ?\}$

intuitionistic ordered logic

$$\frac{}{\alpha \Rightarrow \alpha}\ \mathrm{Id}$$

$$\frac{\varPhi, m\varphi, \chi, \varPhi' \Rightarrow X}{\varPhi, \chi, m\varphi, \varPhi' \Rightarrow X}\ \mathrm{EL}_0 \qquad \frac{\varPhi, \varphi, m\chi, \varPhi' \Rightarrow X}{\varPhi, m\chi, \varphi, \varPhi' \Rightarrow X}\ \mathrm{EL}_1$$

$$\frac{\varPhi \Rightarrow X}{\varPhi, !\varphi \Rightarrow X}\ \mathrm{WL}$$

$$\frac{\varPhi, !\varphi, !\varphi \Rightarrow X}{\varPhi, !\varphi \Rightarrow X}\ \mathrm{CL}$$

$$\frac{\varPhi, \varphi, \varPhi' \Rightarrow X}{\varPhi, !\varphi, \varPhi' \Rightarrow X}\ !\mathrm{L} \qquad !\mathrm{R}\ \frac{!\varPhi \Rightarrow \varphi}{!\varPhi \Rightarrow !\varphi}$$

$$\frac{\varPhi, \varphi, \varPhi' \Rightarrow X}{\varPhi, \text{¡}\varphi, \varPhi' \Rightarrow X}\ \text{¡}\mathrm{L} \qquad \text{¡}\mathrm{R}\ \frac{M\varPhi \Rightarrow \varphi}{M\varPhi \Rightarrow \text{¡}\varphi}$$

$$\frac{\varPhi \Rightarrow \varphi \qquad \varPsi, \chi, \varPsi' \Rightarrow X}{\varPsi, \chi \swarrow \varphi, \varPhi, \varPsi' \Rightarrow X}\ {\swarrow}\mathrm{L} \qquad {\swarrow}\mathrm{R}\ \frac{\varPhi, \varphi \Rightarrow \chi}{\varPhi \Rightarrow \chi \swarrow \varphi}$$

$$\frac{\varPhi \Rightarrow \varphi \qquad \varPsi, \chi, \varPsi' \Rightarrow X}{\varPsi, \varPhi, \varphi \searrow \chi, \varPsi' \Rightarrow X}\ {\searrow}\mathrm{L} \qquad {\searrow}\mathrm{R}\ \frac{\varphi, \varPhi \Rightarrow \chi}{\varPhi \Rightarrow \varphi \searrow \chi}$$

$$\frac{\varPhi, \varPhi' \Rightarrow X}{\varPhi, 1, \varPhi' \Rightarrow X}\ 1\mathrm{L} \qquad 1\mathrm{R}\ \frac{}{\Rightarrow 1}$$

$$\frac{\varPhi, \varphi, \chi, \varPhi' \Rightarrow X}{\varPhi, \varphi \times \chi, \varPhi' \Rightarrow X}\ \times\mathrm{L} \qquad \times\mathrm{R}\ \frac{\varPhi \Rightarrow \varphi \qquad \varPhi' \Rightarrow \chi}{\varPhi, \varPhi' \Rightarrow \varphi \times \chi}$$

$$\top\mathrm{R}\ \frac{}{\varPhi \Rightarrow \top}$$

$$\frac{\varPhi, \varphi_i, \varPhi' \Rightarrow X}{\varPhi, \varphi_0 \,\&\, \varphi_1, \varPhi' \Rightarrow X}\ \&\mathrm{L}_i \qquad \&\mathrm{R}\ \frac{\varPhi \Rightarrow \varphi \qquad \varPhi \Rightarrow \chi}{\varPhi \Rightarrow \varphi \,\&\, \chi}$$

$$\frac{}{\varPhi, \bot, \varPhi' \Rightarrow X}\ \bot\mathrm{L}$$

$$\frac{\varPhi, \varphi, \varPhi' \Rightarrow X \qquad \varPhi, \chi, \varPhi' \Rightarrow X}{\varPhi, \varphi + \chi, \varPhi' \Rightarrow X}\ +\mathrm{L} \qquad +\mathrm{R}_i\ \frac{\varPhi \Rightarrow \varphi_i}{\varPhi \Rightarrow \varphi_0 + \varphi_1}$$

co-intuitionistic ordered logic

$$\mathrm{Id}\ \frac{}{\alpha \Rightarrow \alpha}$$

$$\mathrm{ER}_0\ \frac{X \Rightarrow \varPhi', \chi, m\varphi, \varPhi}{X \Rightarrow \varPhi', m\varphi, \chi, \varPhi} \qquad \mathrm{ER}_1\ \frac{X \Rightarrow \varPhi', m\chi, \varphi, \varPhi}{X \Rightarrow \varPhi', \varphi, m\chi, \varPhi}$$

$$\mathrm{WR}\ \frac{X \Rightarrow \varPhi}{X \Rightarrow ?\varphi, \varPhi}$$

$$\mathrm{CR}\ \frac{X \Rightarrow ?\varphi, ?\varphi, \varPhi}{X \Rightarrow ?\varphi, \varPhi}$$

$$\frac{\varphi \Rightarrow ?\varPhi}{?\varphi \Rightarrow ?\varPhi}\ ?\mathrm{L} \qquad ?\mathrm{R}\ \frac{X \Rightarrow \varPhi', \varphi, \varPhi}{X \Rightarrow \varPhi', ?\varphi, \varPhi}$$

$$\frac{\varphi \Rightarrow M\varPhi}{\text{¿}\varphi \Rightarrow M\varPhi}\ \text{¿}\mathrm{L} \qquad \text{¿}\mathrm{R}\ \frac{X \Rightarrow \varPhi', \varphi, \varPhi}{X \Rightarrow \varPhi', \text{¿}\varphi, \varPhi}$$

$$\frac{\chi \Rightarrow \varphi, \varPhi}{\varphi \nwarrow \chi \Rightarrow \varPhi}\ {\nwarrow}\mathrm{L} \qquad {\nwarrow}\mathrm{R}\ \frac{\varphi \Rightarrow \varPhi \qquad X \Rightarrow \varPsi', \chi, \varPsi}{X \Rightarrow \varPsi', \varPhi, \varphi \nwarrow \chi, \varPsi}$$

$$\frac{\varphi \Rightarrow \varPhi \qquad X \Rightarrow \varPsi', \chi, \varPsi}{X \Rightarrow \varPsi', \chi \nearrow \varphi, \varPhi, \varPsi}\ {\nearrow}\mathrm{L} \qquad {\nearrow}\mathrm{R}\ \frac{\chi \Rightarrow \varPhi, \varphi}{\chi \nearrow \varphi \Rightarrow \varPhi}$$

$$\frac{}{0 \Rightarrow}\ 0\mathrm{L} \qquad 0\mathrm{R}\ \frac{X \Rightarrow \varPhi', \varPhi}{X \Rightarrow \varPhi', 0, \varPhi}$$

$$\frac{\chi \Rightarrow \varPhi \qquad \varphi \Rightarrow \varPhi}{\chi \parr \varphi \Rightarrow \varPhi}\ {\parr}\mathrm{L} \qquad {\parr}\mathrm{R}\ \frac{X \Rightarrow \varPhi', \chi, \varphi, \varPhi}{X \Rightarrow \varPhi', \chi \parr \varphi, \varPhi}$$

$$\frac{}{\bot \Rightarrow \varPhi}\ \bot\mathrm{L}$$

$$\frac{X \Rightarrow \varPhi', \varphi_i, \varPhi}{X \Rightarrow \varPhi', \varphi_1 + \varphi_0, \varPhi}\ +\mathrm{L}_i \qquad +\mathrm{R}\ \frac{\chi \Rightarrow \varPhi \qquad \varphi \Rightarrow \varPhi}{\chi + \varphi \Rightarrow \varPhi}$$

$$\top\mathrm{R}\ \frac{}{\varphi \Rightarrow \varPhi', \top, \varPhi}$$

$$\frac{\varphi_i \Rightarrow \varPhi}{\varphi_1 \,\&\, \varphi_0 \Rightarrow \varPhi}\ \&\mathrm{L}_i \qquad \&\mathrm{R}\ \frac{X \Rightarrow \varPhi', \chi, \varPhi \qquad X \Rightarrow \varPhi', \varphi, \varPhi}{X \Rightarrow \varPhi', \chi \,\&\, \varphi, \varPhi}$$