# (co-)intuitionistic logics

#### sumi

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A, B are formulae.  $\Gamma$ ,  $\Delta$ ,  $\Theta$  are lists of formulae. The length of  $\Delta$  is at most 1.

#### **Definition 0 (intuitionistic ordered logic)**

$$\frac{}{A\Rightarrow A} \text{ Refl}$$
 
$$\frac{\Gamma\Rightarrow A \quad \Gamma', A\Rightarrow B}{\Gamma, \Gamma'\Rightarrow B} \text{ Trans}$$

$$\frac{\varGamma, !A, B, \varGamma' \Rightarrow \Delta}{\varGamma, B, !A, \varGamma' \Rightarrow \Delta} \ \mathrm{EL}_0$$

$$\frac{\varGamma,A,!B,\varGamma'\Rightarrow \Delta}{\varGamma,!B,A,\varGamma'\Rightarrow \Delta} \ \mathrm{EL}_1$$

$$\frac{\Gamma, \Gamma' \Rightarrow \Delta}{\Gamma, !A, \Gamma' \Rightarrow \Delta} \text{ WL}$$

$$\frac{\Gamma, !A, !A, \Gamma' \Rightarrow \Delta}{\Gamma, !A, \Gamma' \Rightarrow \Delta} CL$$

$$\frac{\Gamma, A, \Gamma' \Rightarrow \Delta}{\Gamma, !A, \Gamma' \Rightarrow \Delta} !L$$

$$\frac{\Gamma \Rightarrow A \quad \Theta, B, \Theta' \Rightarrow \Delta}{\Theta, B \swarrow A, \Gamma, \Theta' \Rightarrow \Delta} \nearrow L$$

$$\frac{1}{\Theta, B \swarrow A, \Gamma, \Theta' \Rightarrow \Delta} \checkmark I$$

$$\frac{\Gamma \Rightarrow A \quad \Theta, B, \Theta' \Rightarrow \Delta}{\Theta, \Gamma, A \searrow B, \Theta' \Rightarrow \Delta} \searrow L$$

$$\frac{\Gamma, \Gamma' \Rightarrow \Delta}{\Gamma, \mathbf{1}, \Gamma' \Rightarrow \Delta} \text{ 1L}$$

$$\frac{\Gamma, A, B, \Gamma' \Rightarrow \Delta}{\Gamma, A \otimes B, \Gamma' \Rightarrow \Delta} \otimes L \qquad \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma' \Rightarrow B}{\Gamma, \Gamma' \Rightarrow A \otimes B} \otimes R$$

$$\frac{\Gamma, A_i, \Gamma' \Rightarrow \Delta}{\Gamma, A_0 \& A_1, \Gamma' \Rightarrow \Delta} \& L_i \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \& B} \& R$$

$$\frac{}{\Gamma, 0, \Gamma' \Rightarrow \Delta}$$
 0L

$$\frac{\varGamma,A,\varGamma'\Rightarrow \varDelta}{\varGamma,A\oplus B,\varGamma'\Rightarrow \varDelta} \ \oplus \mathbb{L} \ \frac{\varGamma\Rightarrow A_i}{\varGamma\Rightarrow A_0\oplus A_1} \ \oplus \mathbb{R}_i$$

#### Definition 1 (co-intuitionistic ordered logic)

$$\overline{A \Rightarrow A}$$
 Refl

$$\frac{B \Rightarrow A, \Gamma' \qquad A \Rightarrow \Gamma}{B \Rightarrow \Gamma', \Gamma}$$
 Trans

$$\frac{\Delta \Rightarrow \Gamma', B, ?A, \Gamma}{\Delta \Rightarrow \Gamma', ?A, B, \Gamma} ER_0$$

$$\frac{\Delta \Rightarrow \Gamma', ?B, A, \Gamma}{\Delta \Rightarrow \Gamma', A, ?B, \Gamma} ER_1$$

$$\frac{\Delta \Rightarrow \Gamma', \Gamma}{\Delta \Rightarrow \Gamma', ?A, \Gamma}$$
 WR

$$\frac{\Delta \Rightarrow \Gamma', ?A, ?A, \Gamma}{\Delta \Rightarrow \Gamma', ?A, \Gamma} \text{ CR}$$

 $\frac{A \Rightarrow ?\Gamma}{?A \Rightarrow ?\Gamma}$  ?R

$$\frac{\Delta \Rightarrow \Gamma', A, \Gamma}{\Delta \Rightarrow \Gamma', ?A, \Gamma} ?L$$

 $\frac{!\Gamma\Rightarrow A}{!\Gamma\Rightarrow !A}$  !R

 $\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow B \swarrow A} \swarrow R$ 

 $\frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \setminus R} \setminus R$ 

 $\frac{}{\Rightarrow 1}$  1R

 $\frac{}{\Gamma \rightarrow T}$  TR

$$\frac{B \Rightarrow A, \Gamma}{A \searrow B \Rightarrow \Gamma} \searrow L \qquad \frac{A \Rightarrow \Gamma \quad \Delta \Rightarrow \Theta', B, \Theta}{A \Rightarrow \Theta' \quad \Gamma \quad A \searrow B \quad \Theta} \searrow R$$

$$\frac{B \Rightarrow A, \Gamma}{A \nwarrow B \Rightarrow \Gamma} \nwarrow L$$

$$\frac{A \Rightarrow \Gamma \quad \Delta \Rightarrow \Theta', B, \Theta}{\Delta \Rightarrow \Theta', B \nearrow A, \Gamma, \Theta} \nearrow L$$

$$B \Rightarrow \Gamma \quad A \Rightarrow \Gamma$$

$$\frac{B \Rightarrow \Gamma \quad A \Rightarrow \Gamma}{B \ \Re \ A \Rightarrow \Gamma} \ \Re L \qquad \qquad \frac{\Delta \Rightarrow \Gamma', B, A, \Gamma}{\Delta \Rightarrow \Gamma', B \ \Re \ A, \Gamma} \ \Re R$$

$$\frac{B \Rightarrow I \quad A \Rightarrow I}{B \otimes A \Rightarrow \Gamma} \otimes I$$

$$\frac{B \cdot 3 A \Rightarrow I}{0 \Rightarrow \Gamma}$$
 OL

$$\frac{\Delta \Rightarrow \Gamma', A_i, \Gamma}{\Delta \Rightarrow \Gamma', A_1 \oplus A_0, \Gamma} \oplus L_i \qquad \frac{B \Rightarrow \Gamma \quad A \Rightarrow \Gamma}{B \oplus A \Rightarrow \Gamma} \oplus R$$

$$\frac{\Delta \Rightarrow I^{+}, A_{i}, I}{\Delta \Rightarrow \Gamma', A_{1} \oplus A_{0}, \Gamma} \oplus L_{i}$$

$$\bigoplus A_0, \Gamma \oplus L_i$$

$$\frac{}{A \Rightarrow \Gamma', \top, \Gamma} \mathsf{TR}$$

 $\frac{\Delta \Rightarrow \Gamma', \Gamma}{\Delta \Rightarrow \Gamma', \bot, \Gamma} \bot R$ 

$$\frac{A_i \Rightarrow \Gamma}{A_1 \& A_0 \Rightarrow \Gamma} \& L_i \quad \frac{\Delta \Rightarrow \Gamma', B, \Gamma \quad \Delta \Rightarrow \Gamma', A, \Gamma}{\Delta \Rightarrow \Gamma', B \& A, \Gamma} \& R$$

#### **Definition 2 (intuitionistic linear logic)**

$$\frac{\overline{A \Rightarrow A} \text{ Ren}}{\overline{\Gamma \Rightarrow A} \quad \Gamma', A \Rightarrow B}$$

$$\frac{\Gamma \Rightarrow A \quad \Gamma', A \Rightarrow B}{\Gamma, \Gamma' \Rightarrow B} \text{ Trans}$$

$$\frac{\varGamma,A,B,\varGamma'\Rightarrow \Delta}{\varGamma,B,A,\varGamma'\Rightarrow \Delta} \; \mathsf{EL}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta} \text{ WL}$$

$$\frac{\Gamma, !A, !A \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta} \text{ CL}$$

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta} !L$$

$$\frac{\Gamma \Rightarrow A \qquad \Gamma', B \Rightarrow \Delta}{\Gamma, \Gamma', A \rightarrow B \Rightarrow \Delta} \rightarrow L$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, 1 \Rightarrow \Delta} \text{ 1L}$$

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \otimes B \Rightarrow \Delta} \otimes L$$

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \otimes B \Rightarrow \Delta} \otimes L$$

$$\frac{\Gamma, A_i \Rightarrow \Delta}{\Gamma, A_0 \& A_1 \Rightarrow \Delta} \& L_i$$

$$\overline{\Gamma, A_0 \& A_1 \Rightarrow \Delta} \stackrel{\text{def}}{}$$

$$T, 0 \Rightarrow \Delta$$
 0L

$$\frac{\varGamma, A\Rightarrow \Delta \quad \varGamma, B\Rightarrow \Delta}{\varGamma, A\oplus B\Rightarrow \Delta} \ \oplus \mathsf{L}$$

$$\frac{\Gamma \Rightarrow A \qquad \Gamma', A \Rightarrow B}{\Gamma \qquad \Gamma' \Rightarrow A} \text{ Trans}$$

$$\frac{!\Gamma \Rightarrow A}{!\Gamma \Rightarrow !A}$$
 !R

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B} \to \mathbb{R}$$

$$\frac{}{\Rightarrow 1}$$
 1R

$$\frac{\Gamma \Rightarrow A \qquad \Gamma' \Rightarrow B}{\Gamma, \Gamma' \Rightarrow A \otimes B} \otimes \mathbb{R}$$

$$\frac{}{\Gamma \Rightarrow \top} \mathsf{TR}$$

$$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \& B} \& R$$

$$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \& B} \& B$$

$$d \Rightarrow \Delta$$
 OL

$$\frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \oplus A_1} \oplus \mathbf{R}_i$$

### **Definition 3 (co-intuitionistic linear logic)**

$$\frac{}{A \Rightarrow A}$$
 Refl

$$\frac{B \Rightarrow A, \Gamma' \qquad A \Rightarrow \Gamma}{B \Rightarrow \Gamma', \Gamma}$$
 Trans

$$\frac{\Delta \Rightarrow \Gamma', B, A, \Gamma}{\Delta \Rightarrow \Gamma', A, B, \Gamma} ER$$

$$\frac{\Delta \Rightarrow \Gamma}{\Delta \Rightarrow ?A, \Gamma} \text{ WR}$$

$$\frac{\Delta \Rightarrow ?A, ?A, \Gamma}{\Delta \Rightarrow ?A, \Gamma} \text{ CR}$$

$$\frac{A \Rightarrow ?\Gamma}{?A \Rightarrow ?\Gamma} ?L \qquad \qquad \frac{\Delta \Rightarrow A, \Gamma}{\Delta \Rightarrow ?A, \Gamma} ?R$$

$$\frac{A \Rightarrow B, \Gamma}{A \leftarrow B \Rightarrow \Gamma} \leftarrow L \qquad \qquad \frac{\Delta \Rightarrow B, \Gamma' \quad A \Rightarrow \Gamma}{\Delta \Rightarrow B \leftarrow A, \Gamma', \Gamma} \leftarrow R$$

$$\frac{\Delta \Rightarrow \Gamma}{\Delta \Rightarrow \perp, \Gamma} \perp R$$

$$\frac{B \Rightarrow \Gamma \quad A \Rightarrow \Gamma}{B \otimes A \Rightarrow \Gamma} \otimes L$$

$$\frac{\Delta \Rightarrow B, A, \Gamma}{\Delta \Rightarrow B \otimes A, \Gamma} \otimes R$$

$$\frac{A \Rightarrow B, A, \Gamma}{\Delta \Rightarrow B \otimes A, \Gamma} \otimes R$$

$$\frac{\varDelta\Rightarrow A_i, \varGamma}{\varDelta\Rightarrow A_1\oplus A_0, \varGamma} \oplus \mathsf{L}_i \qquad \qquad \frac{B\Rightarrow \varGamma \quad A\Rightarrow \varGamma}{B\oplus A\Rightarrow \varGamma} \oplus \mathsf{R}$$

$$\frac{B \Rightarrow \Gamma \quad A \Rightarrow \Gamma}{B \oplus A \Rightarrow \Gamma} \oplus \mathbb{R}$$

$$\overline{A \Rightarrow \mathsf{T}, \Gamma}$$
 TR

$$\frac{A_i \Rightarrow \Gamma}{A_1 \& A_0 \Rightarrow \Gamma} \& L_i \qquad \qquad \frac{\Delta \Rightarrow B, \Gamma \quad \Delta \Rightarrow A, \Gamma}{\Delta \Rightarrow B \& A, \Gamma} \& R$$

#### **Definition 4 (intuitionistic logic)**

$$\frac{A \Rightarrow A}{A \Rightarrow A} \text{ Keri}$$

$$\frac{\Gamma \Rightarrow A \qquad \Gamma', A \Rightarrow B}{\Gamma, \Gamma' \Rightarrow B} \text{ Trans}$$

$$\frac{\varGamma,A,B,\varGamma'\Rightarrow \Delta}{\varGamma,B,A,\varGamma'\Rightarrow \Delta} \ \mathsf{EL}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \text{ WL}$$

$$\frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \text{ CL}$$

$$\frac{\Gamma \Rightarrow A \qquad \Gamma', B \Rightarrow \Delta}{\Gamma, \Gamma', A \to B \Rightarrow \Delta} \to L \qquad \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B} \to R$$

$$\Gamma, \Gamma', A \to B \Rightarrow \Delta$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, T \Rightarrow \Delta} TL$$

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \land B \Rightarrow \Delta} \land L \qquad \frac{\Gamma \Rightarrow A \quad \Gamma' \Rightarrow B}{\Gamma, \Gamma' \Rightarrow A \land B} \land R$$

$$\frac{\Gamma, A \wedge B \Rightarrow \Delta}{\Gamma, \bot \Rightarrow \Delta} \stackrel{\wedge L}{\to}$$

$$\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta$$

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \lor B \Rightarrow \Delta} \lor L$$

## **Definition 5 (co-intuitionistic logic)**

$$A \Rightarrow A$$
 Refl

$$\frac{B \Rightarrow A, \Gamma' \qquad A \Rightarrow \Gamma}{B \Rightarrow \Gamma', \Gamma} \text{ Trans}$$

$$\frac{\Delta \Rightarrow \Gamma', B, A, \Gamma}{\Delta \Rightarrow \Gamma', A, B, \Gamma} ER$$

$$\frac{\Delta \Rightarrow \Gamma}{\Delta \Rightarrow A, \Gamma} \text{ WR}$$

$$\frac{\Delta \Rightarrow A, A, \Gamma}{\Delta \Rightarrow A, \Gamma} \text{ CR}$$

$$\frac{A \Rightarrow B, \Gamma}{A \leftarrow B \Rightarrow \Gamma} \leftarrow L$$

$$\frac{}{\perp \Rightarrow} \perp L$$

— TR

$$B \Rightarrow \Gamma \quad A \Rightarrow \Gamma$$

$$\frac{B \Rightarrow \Gamma \quad A \Rightarrow \Gamma}{B \lor A \Rightarrow \Gamma} \lor L$$

$$\frac{B \Rightarrow \Gamma \quad A \Rightarrow \Gamma}{B \lor A \Rightarrow \Gamma} \lor L$$

$$\frac{A_i \Rightarrow \Gamma}{A_1 \land A_0 \Rightarrow \Gamma} \land L$$

$$\frac{\Delta \Rightarrow B, \Gamma' \qquad A \Rightarrow \Gamma}{\Delta \Rightarrow B \leftarrow A, \Gamma', \Gamma} \leftarrow \mathbb{R}$$

$$\frac{\Delta \Rightarrow \Gamma}{\Delta \Rightarrow \bot, \Gamma} \bot R$$

$$\frac{\Delta \Rightarrow B, A, \Gamma}{\Delta \Rightarrow B \lor A, \Gamma} \lor R$$

$$\overline{A \Rightarrow \top, \Gamma}$$
 TR

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \vee L \qquad \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \vee A_1} \vee R_i \qquad \qquad \frac{A_i \Rightarrow \Gamma}{A_1 \wedge A_0 \Rightarrow \Gamma} \wedge L_i \qquad \qquad \frac{\Delta \Rightarrow B, \Gamma \quad \Delta \Rightarrow A, \Gamma}{\Delta \Rightarrow B \wedge A, \Gamma} \wedge R$$