(co-)intuitionistic logics

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A, B are formulae. Γ, Δ, Θ are lists of formulae. The length of Δ is at most 1.

Definition 0 (intuitionistic ordered logic)

$$\frac{}{A \Rightarrow A}\ \mathrm{Refl}$$

$$\frac{\Gamma \Rightarrow A \qquad \Gamma', A \Rightarrow B}{\Gamma, \Gamma' \Rightarrow B}\ \mathrm{Trans}$$

$$\frac{\Gamma, !A, B, \Gamma' \Rightarrow \Delta}{\Gamma, B, !A, \Gamma' \Rightarrow \Delta}\ \mathrm{EL}_0$$

$$\frac{\Gamma, A, !B, \Gamma' \Rightarrow \Delta}{\Gamma, !B, A, \Gamma' \Rightarrow \Delta}\ \mathrm{EL}_1$$

$$\frac{\Gamma, \Gamma' \Rightarrow \Delta}{\Gamma, !A, \Gamma' \Rightarrow \Delta}\ \mathrm{WL}$$

$$\frac{\Gamma, !A, !A, \Gamma' \Rightarrow \Delta}{\Gamma, !A, \Gamma' \Rightarrow \Delta}\ \mathrm{CL}$$

$$\frac{\Gamma, A, \Gamma' \Rightarrow \Delta}{\Gamma, !A, \Gamma' \Rightarrow \Delta}\ !L \qquad \frac{!\Gamma \Rightarrow A}{!\Gamma \Rightarrow !A}\ !\mathrm{R}$$

$$\frac{\Gamma \Rightarrow A \qquad \Theta, B, \Theta' \Rightarrow \Delta}{\Theta, B \swarrow A, \Gamma, \Theta' \Rightarrow \Delta}\ {\swarrow}\mathrm{L} \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow B \swarrow A}\ {\swarrow}\mathrm{R}$$

$$\frac{\Gamma \Rightarrow A \qquad \Theta, B, \Theta' \Rightarrow \Delta}{\Theta, \Gamma, A \searrow B, \Theta' \Rightarrow \Delta}\ {\searrow}\mathrm{L} \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \searrow B}\ {\searrow}\mathrm{R}$$

$$\frac{\Gamma, \Gamma' \Rightarrow \Delta}{\Gamma, \mathbf{1}, \Gamma' \Rightarrow \Delta}\ \mathbf{1}\mathrm{L} \qquad \frac{}{\Rightarrow \mathbf{1}}\ \mathbf{1}\mathrm{R}$$

$$\frac{\Gamma, A, B, \Gamma' \Rightarrow \Delta}{\Gamma, A \otimes B, \Gamma' \Rightarrow \Delta}\ \otimes\mathrm{L} \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma' \Rightarrow B}{\Gamma, \Gamma' \Rightarrow A \otimes B}\ \otimes\mathrm{R}$$

$$\frac{}{\Gamma \Rightarrow \top}\ \top\mathrm{R}$$

$$\frac{\Gamma, A_i, \Gamma' \Rightarrow \Delta}{\Gamma, A_0 \,\&\, A_1, \Gamma' \Rightarrow \Delta}\ \&\mathrm{L}_i \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \,\&\, B}\ \&\mathrm{R}$$

$$\frac{}{\Gamma, \mathbf{0}, \Gamma' \Rightarrow \Delta}\ \mathbf{0}\mathrm{L}$$

$$\frac{\Gamma, A, \Gamma' \Rightarrow \Delta \qquad \Gamma, B, \Gamma' \Rightarrow \Delta}{\Gamma, A \oplus B, \Gamma' \Rightarrow \Delta}\ \oplus\mathrm{L} \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \oplus A_1}\ \oplus\mathrm{R}_i$$

□

Definition 1 (co-intuitionistic ordered logic)

$$\frac{}{A \Rightarrow A}\ \mathrm{Refl}$$

$$\frac{B \Rightarrow A, \Gamma' \qquad A \Rightarrow \Gamma}{B \Rightarrow \Gamma', \Gamma}\ \mathrm{Trans}$$

$$\frac{\Delta \Rightarrow \Gamma', B, ?A, \Gamma}{\Delta \Rightarrow \Gamma', ?A, B, \Gamma}\ \mathrm{ER}_0$$

$$\frac{\Delta \Rightarrow \Gamma', ?B, A, \Gamma}{\Delta \Rightarrow \Gamma', A, ?B, \Gamma}\ \mathrm{ER}_1$$

$$\frac{\Delta \Rightarrow \Gamma', \Gamma}{\Delta \Rightarrow \Gamma', ?A, \Gamma}\ \mathrm{WR}$$

$$\frac{\Delta \Rightarrow \Gamma', ?A, ?A, \Gamma}{\Delta \Rightarrow \Gamma', ?A, \Gamma}\ \mathrm{CR}$$

$$\frac{\Delta \Rightarrow \Gamma', A, \Gamma}{\Delta \Rightarrow \Gamma', ?A, \Gamma}\ ?\mathrm{L} \qquad \frac{A \Rightarrow ?\Gamma}{?A \Rightarrow ?\Gamma}\ ?\mathrm{R}$$

$$\frac{B \Rightarrow A, \Gamma}{A \nwarrow B \Rightarrow \Gamma}\ {\nwarrow}\mathrm{L} \qquad \frac{A \Rightarrow \Gamma \qquad \Delta \Rightarrow \Theta', B, \Theta}{\Delta \Rightarrow \Theta', \Gamma, A \nwarrow B, \Theta}\ {\nwarrow}\mathrm{R}$$

$$\frac{A \Rightarrow \Gamma \qquad \Delta \Rightarrow \Theta', B, \Theta}{\Delta \Rightarrow \Theta', B \nearrow A, \Gamma, \Theta}\ {\nearrow}\mathrm{L} \qquad \frac{B \Rightarrow \Gamma, A}{B \nearrow A \Rightarrow \Gamma}\ {\nearrow}\mathrm{R}$$

$$\frac{}{\bot \Rightarrow}\ \bot\mathrm{L} \qquad \frac{\Delta \Rightarrow \Gamma', \Gamma}{\Delta \Rightarrow \Gamma', \bot, \Gamma}\ \bot\mathrm{R}$$

$$\frac{B \Rightarrow \Gamma \qquad A \Rightarrow \Gamma}{B \parr A \Rightarrow \Gamma}\ \parr\mathrm{L} \qquad \frac{\Delta \Rightarrow \Gamma', B, A, \Gamma}{\Delta \Rightarrow \Gamma', B \parr A, \Gamma}\ \parr\mathrm{R}$$

$$\frac{}{\mathbf{0} \Rightarrow \Gamma}\ \mathbf{0}\mathrm{L}$$

$$\frac{\Delta \Rightarrow \Gamma', A_i, \Gamma}{\Delta \Rightarrow \Gamma', A_1 \oplus A_0, \Gamma}\ \oplus\mathrm{L}_i \qquad \frac{B \Rightarrow \Gamma \qquad A \Rightarrow \Gamma}{B \oplus A \Rightarrow \Gamma}\ \oplus\mathrm{R}$$

$$\frac{}{A \Rightarrow \Gamma', \top, \Gamma}\ \top\mathrm{R}$$

$$\frac{A_i \Rightarrow \Gamma}{A_1 \,\&\, A_0 \Rightarrow \Gamma}\ \&\mathrm{L}_i \qquad \frac{\Delta \Rightarrow \Gamma', B, \Gamma \qquad \Delta \Rightarrow \Gamma', A, \Gamma}{\Delta \Rightarrow \Gamma', B \,\&\, A, \Gamma}\ \&\mathrm{R}$$

□

Definition 2 (intuitionistic linear logic)

$$\frac{}{A \Rightarrow A}\ \text{Refl}$$

$$\frac{\Gamma \Rightarrow A \qquad \Gamma', A \Rightarrow B}{\Gamma, \Gamma' \Rightarrow B}\ \text{Trans}$$

$$\frac{\Gamma, A, B, \Gamma' \Rightarrow \Delta}{\Gamma, B, A, \Gamma' \Rightarrow \Delta}\ \text{EL}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta}\ \text{WL}$$

$$\frac{\Gamma, !A, !A \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta}\ \text{CL}$$

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta}\ !L \qquad \frac{!\Gamma \Rightarrow A}{!\Gamma \Rightarrow !A}\ !\text{R}$$

$$\frac{\Gamma \Rightarrow A \qquad \Gamma', B \Rightarrow \Delta}{\Gamma, \Gamma', A \to B \Rightarrow \Delta}\ \to\text{L} \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}\ \to\text{R}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \mathbf{1} \Rightarrow \Delta}\ \mathbf{1}\text{L} \qquad \frac{}{\Rightarrow \mathbf{1}}\ \mathbf{1}\text{R}$$

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \otimes B \Rightarrow \Delta}\ \otimes\text{L} \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma' \Rightarrow B}{\Gamma, \Gamma' \Rightarrow A \otimes B}\ \otimes\text{R}$$

$$\frac{}{\Gamma \Rightarrow \top}\ \top\text{R}$$

$$\frac{\Gamma, A_i \Rightarrow \Delta}{\Gamma, A_0 \,\&\, A_1 \Rightarrow \Delta}\ \&\text{L}_i \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \,\&\, B}\ \&\text{R}$$

$$\frac{}{\Gamma, \mathbf{0} \Rightarrow \Delta}\ \mathbf{0}\text{L}$$

$$\frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \oplus B \Rightarrow \Delta}\ \oplus\text{L} \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \oplus A_1}\ \oplus\text{R}_i$$

□

Definition 3 (co-intuitionistic linear logic)

$$\frac{}{A \Rightarrow A}\ \text{Refl}$$

$$\frac{B \Rightarrow A, \Gamma' \qquad A \Rightarrow \Gamma}{B \Rightarrow \Gamma', \Gamma}\ \text{Trans}$$

$$\frac{\Delta \Rightarrow \Gamma', B, A, \Gamma}{\Delta \Rightarrow \Gamma', A, B, \Gamma}\ \text{ER}$$

$$\frac{\Delta \Rightarrow \Gamma}{\Delta \Rightarrow ?A, \Gamma}\ \text{WR}$$

$$\frac{\Delta \Rightarrow ?A, ?A, \Gamma}{\Delta \Rightarrow ?A, \Gamma}\ \text{CR}$$

$$\frac{A \Rightarrow ?\Gamma}{?A \Rightarrow ?\Gamma}\ ?\text{L} \qquad \frac{\Delta \Rightarrow A, \Gamma}{\Delta \Rightarrow ?A, \Gamma}\ ?\text{R}$$

$$\frac{A \Rightarrow B, \Gamma}{A \leftarrow B \Rightarrow \Gamma}\ \leftarrow\text{L} \qquad \frac{\Delta \Rightarrow B, \Gamma' \qquad A \Rightarrow \Gamma}{\Delta \Rightarrow B \leftarrow A, \Gamma', \Gamma}\ \leftarrow\text{R}$$

$$\frac{}{\bot \Rightarrow}\ \bot\text{L} \qquad \frac{\Delta \Rightarrow \Gamma}{\Delta \Rightarrow \bot, \Gamma}\ \bot\text{R}$$

$$\frac{B \Rightarrow \Gamma \qquad A \Rightarrow \Gamma}{B \parr A \Rightarrow \Gamma}\ \parr\text{L} \qquad \frac{\Delta \Rightarrow B, A, \Gamma}{\Delta \Rightarrow B \parr A, \Gamma}\ \parr\text{R}$$

$$\frac{}{\mathbf{0} \Rightarrow \Gamma}\ \mathbf{0}\text{L}$$

$$\frac{\Delta \Rightarrow A_i, \Gamma}{\Delta \Rightarrow A_1 \oplus A_0, \Gamma}\ \oplus\text{L}_i \qquad \frac{B \Rightarrow \Gamma \qquad A \Rightarrow \Gamma}{B \oplus A \Rightarrow \Gamma}\ \oplus\text{R}$$

$$\frac{}{A \Rightarrow \top, \Gamma}\ \top\text{R}$$

$$\frac{A_i \Rightarrow \Gamma}{A_1 \,\&\, A_0 \Rightarrow \Gamma}\ \&\text{L}_i \qquad \frac{\Delta \Rightarrow B, \Gamma \qquad \Delta \Rightarrow A, \Gamma}{\Delta \Rightarrow B \,\&\, A, \Gamma}\ \&\text{R}$$

□

Definition 4 (intuitionistic logic)

$$\frac{}{A \Rightarrow A}\ \mathrm{Refl}$$

$$\frac{\Gamma \Rightarrow A \qquad \Gamma', A \Rightarrow B}{\Gamma, \Gamma' \Rightarrow B}\ \mathrm{Trans}$$

$$\frac{\Gamma, A, B, \Gamma' \Rightarrow \Delta}{\Gamma, B, A, \Gamma' \Rightarrow \Delta}\ \mathrm{EL}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ \mathrm{WL}$$

$$\frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ \mathrm{CL}$$

$$\frac{\Gamma \Rightarrow A \qquad \Gamma', B \Rightarrow \Delta}{\Gamma, \Gamma', A \to B \Rightarrow \Delta}\ {\to}\mathrm{L} \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}\ {\to}\mathrm{R}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \top \Rightarrow \Delta}\ \top\mathrm{L} \qquad \frac{}{\Rightarrow \top}\ \top\mathrm{R}$$

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\ \wedge\mathrm{L} \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma' \Rightarrow B}{\Gamma, \Gamma' \Rightarrow A \wedge B}\ \wedge\mathrm{R}$$

$$\frac{}{\Gamma, \bot \Rightarrow \Delta}\ \bot\mathrm{L}$$

$$\frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}\ \vee\mathrm{L} \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \vee A_1}\ \vee\mathrm{R}_i$$

□

Definition 5 (co-intuitionistic logic)

$$\frac{}{A \Rightarrow A}\ \mathrm{Refl}$$

$$\frac{B \Rightarrow A, \Gamma' \qquad A \Rightarrow \Gamma}{B \Rightarrow \Gamma', \Gamma}\ \mathrm{Trans}$$

$$\frac{\Delta \Rightarrow \Gamma', B, A, \Gamma}{\Delta \Rightarrow \Gamma', A, B, \Gamma}\ \mathrm{ER}$$

$$\frac{\Delta \Rightarrow \Gamma}{\Delta \Rightarrow A, \Gamma}\ \mathrm{WR}$$

$$\frac{\Delta \Rightarrow A, A, \Gamma}{\Delta \Rightarrow A, \Gamma}\ \mathrm{CR}$$

$$\frac{A \Rightarrow B, \Gamma}{A \leftarrow B \Rightarrow \Gamma}\ {\leftarrow}\mathrm{L} \qquad \frac{\Delta \Rightarrow B, \Gamma' \qquad A \Rightarrow \Gamma}{\Delta \Rightarrow B \leftarrow A, \Gamma', \Gamma}\ {\leftarrow}\mathrm{R}$$

$$\frac{}{\bot \Rightarrow}\ \bot\mathrm{L} \qquad \frac{\Delta \Rightarrow \Gamma}{\Delta \Rightarrow \bot, \Gamma}\ \bot\mathrm{R}$$

$$\frac{B \Rightarrow \Gamma \qquad A \Rightarrow \Gamma}{B \vee A \Rightarrow \Gamma}\ \vee\mathrm{L} \qquad \frac{\Delta \Rightarrow B, A, \Gamma}{\Delta \Rightarrow B \vee A, \Gamma}\ \vee\mathrm{R}$$

$$\frac{}{A \Rightarrow \top, \Gamma}\ \top\mathrm{R}$$

$$\frac{A_i \Rightarrow \Gamma}{A_1 \wedge A_0 \Rightarrow \Gamma}\ \wedge\mathrm{L}_i \qquad \frac{\Delta \Rightarrow B, \Gamma \qquad \Delta \Rightarrow A, \Gamma}{\Delta \Rightarrow B \wedge A, \Gamma}\ \wedge\mathrm{R}$$

□