Contact Handling for Articulated Rigid Bodies Using LCP

Jie Tan, Kristin Siu, and C. Karen Liu

1 Introduction

The following document describes the construction of the matrices responsible for contact handling for articulated rigid bodies in generalized coordinates. We formulate an implicit time-stepping, velocity-based LCP (linear-complementarity problem) to guarantee non-penetration, directional friction, and approximated Coulombs friction cone conditions, similar to Stewart and Trinkle [1]. The document assumes familiarity with physics principles of frictional contact and the mathematical techniques for solving standard LCP. This was originally intended as a guide for the implementation of the LCP solver for the DART dynamics library.

2 The Equations of Motion

We begin our derivation from the following form of the equations of motion for an articulated rigid body system with one contact point:

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q}, \dot{\mathbf{q}}) = \tau + J^T f_n \mathbf{n} + J^T D \mathbf{f}_d \tag{1}$$

The terms of this equation are as follows:

- $\mathbf{q}$: the state vector
- M: the mass matrix
- C: Coriolis, centrifugal, and gravitational forces
- τ: internal generalized forces
- J: the Jacobian matrix evaluated at the contact point
- $\mathbf{n}$: the normal force direction (Figure 1(a))
- f_n: the magnitude of the normal force
- D: discretized friction cone bases (Figure 1(a))
- $\mathbf{f}_d$: the magnitudes of tangent forces along the discretized friction cone bases

We can discretize Equation 1 as follows:

$$M\ddot{\mathbf{q}} = M\frac{(\dot{\mathbf{q}}^{n+1} - \dot{\mathbf{q}}^n)}{\Delta t} \tag{2}$$

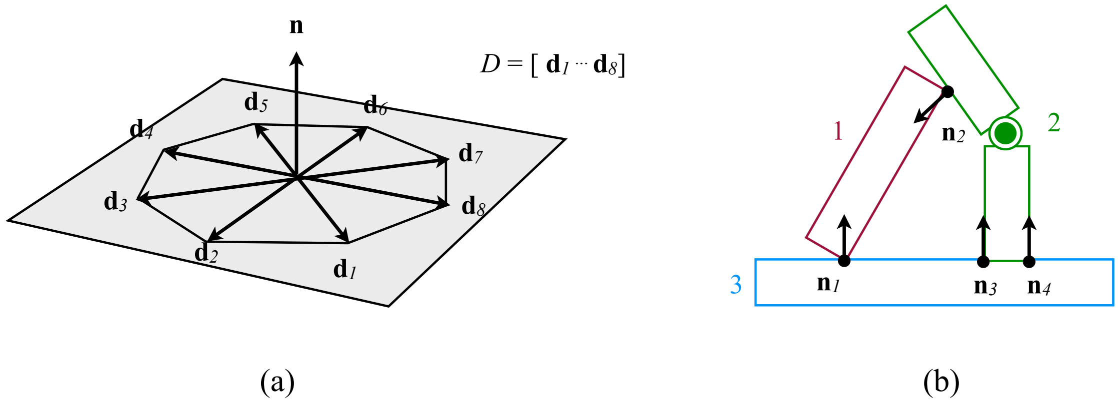

Figure 1: An articulated system.

$$M\frac{(\dot{\mathbf{q}}^{n+1} - \dot{\mathbf{q}}^n)}{\Delta t} = -C(\mathbf{q}^n, \dot{\mathbf{q}}^n) + \tau^n + J^T f_n \mathbf{n} + J^T D\mathbf{f}_d \tag{3}$$

$$M\dot{\mathbf{q}}^{n+1} = M\dot{\mathbf{q}}^n - \Delta t(C(\mathbf{q}^n, \dot{\mathbf{q}}^n) - \tau^n) + \Delta t(J^T f_n \mathbf{n} + J^T D\mathbf{f}_d) \tag{4}$$

where superscripts n and $n+1$ indicate the current and the next time steps. In Equation 4 , the first two terms on the right are known values. We group these into a single term τ^*:

$$\tau^* = M\dot{\mathbf{q}}^n - \Delta t(C - \tau^n) \tag{5}$$

We are then left with:

$$M\dot{\mathbf{q}} = \tau^* + \Delta t(J^T f_n \mathbf{n} + J^T D\mathbf{f}_d) \tag{6}$$

where the unknown variables are the velocity of the state at the next time step ($\dot{\mathbf{q}}$, superscript $n+1$ omitted for simplicity), the magnitude of normal force (f_n), and the magnitudes of tangent forces ($\mathbf{f}_d$).

Now let us consider the equations of motion for a more complex situation shown in Figure 1(b). There are four contact points and three systems involved in the scene. We can stack the equations of motion for each system in the following matrix form:

$$\begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} \tau_1{}^* \\ \tau_2{}^* \\ \tau_3{}^* \end{bmatrix} + \Delta t \begin{bmatrix} J_{11}{}^T\mathbf{n}_1 & J_{12}{}^T\mathbf{n}_2 & 0 & 0 \\ 0 & -J_{22}^T\mathbf{n}_2 & J_{23}^T\mathbf{n}_3 & J_{24}^T\mathbf{n}_4 \\ -J_{31}^T\mathbf{n}_1 & 0 & -J_{33}^T\mathbf{n}_3 & -J_{34}^T\mathbf{n}_4 \end{bmatrix} \begin{bmatrix} f_{n1} \\ f_{n2} \\ f_{n3} \\ f_{n4} \end{bmatrix}$$
$$+\Delta t \begin{bmatrix} J_{11}^T D_1 & J_{12}^T D_2 & 0 & 0 \\ 0 & -J_{22}^T D_2 & J_{23}^T D_3 & J_{24}^T D_4 \\ -J_{31}^T D_1 & 0 & -J_{33}^T D_3 & -J_{34}^T D_4 \end{bmatrix} \begin{bmatrix} \mathbf{f}_{d1} \\ \mathbf{f}_{d2} \\ \mathbf{f}_{d3} \\ \mathbf{f}_{d4} \end{bmatrix} \tag{7}$$

The subscript i for M_i, $\dot{\mathbf{q}}_i$, and τ_i^* indicates the i-th system, while the subscript j for $\mathbf{n}_j$, D_j, f_{nj}, and $\mathbf{f}_{dj}$ indicates the j-th contact. A Jacobian matrix J_{ij} has two subscripts indicating the Jacobian evaluated at contact j for the system i.

For simplicity, we will rewrite some of these terms as matrices N and B as follows:

$$\begin{aligned} N &= \begin{bmatrix} {J_{11}}^T\mathbf{n}_1 & {J_{12}}^T\mathbf{n}_2 & 0 & 0 \\ 0 & -J_{22}^T\mathbf{n}_2 & J_{23}^T\mathbf{n}_3 & J_{24}^T\mathbf{n}_4 \\ -J_{31}^T\mathbf{n}_1 & 0 & -J_{33}^T\mathbf{n}_3 & -J_{34}^T\mathbf{n}_4 \end{bmatrix} \\ B &= \begin{bmatrix} J_{11}^T D_1 & J_{12}^T D_2 & 0 & 0 \\ 0 & -J_{22}^T D_2 & J_{23}^T D_3 & J_{24}^T D_4 \\ -J_{31}^T D_1 & 0 & -J_{33}^T D_3 & -J_{34}^T D_4 \end{bmatrix} \end{aligned} \tag{8}$$

Assuming the total number of degrees of freedom for three systems is m, the number of the contacts is p ($p = 4$ in Figure 1(b)), and the number of friction cone bases is d ($d = 8$ in Figure 1(a)), the dimensions of N and B are $m \times p$ and $m \times pd$, respectively. With these substitutions, Equation 7 reduces to the following:

$$M\dot{\mathbf{q}} = \tau^* + \Delta t(N\mathbf{f}_n + B\mathbf{f}_d) \tag{9}$$

3 LCP Formulation

The problem of contact handling is based on three types of constraints: normal direction constraints, directional friction constraints, and friction cone constraints. Together with equations of motion described earlier, these three sets of constraints constitute a LCP for unknown variables $\dot{\mathbf{q}}$, $\mathbf{f}_n$, $\mathbf{f}_d$, and auxiliary variables λ.

3.1 Normal Direction Constraints

- expression 1: $\mathbf{f}_n \geq 0$
- expression 2: $N\dot{\mathbf{q}} \geq 0$
- expression 3: $(N\dot{\mathbf{q}})^T\mathbf{f}_n = 0$

Expression 1 ensures there is no pulling force. Expression 2 prevents penetration by enforcing a nonnegative normal velocity at contact. Expression 3 constrains the normal force based on the velocity. If $N\dot{\mathbf{q}} > 0$, then $\mathbf{f}_n = 0$ (in takeoff). If $\mathbf{f}_n > 0$ then $N\dot{\mathbf{q}} = 0$ (in contact).

3.2 Directional Friction Constraints

- expression 4: $\mathbf{f}_d \geq 0$
- expression 5: $B^T\dot{\mathbf{q}} + E\lambda \geq 0$
- expression 6: $(B^T\dot{\mathbf{q}} + E\lambda)^T\mathbf{f}_d = 0$

Here $E \in R^{pd\times p}$ is a binary matrix whose structure is defined as follows:

$$E = \begin{bmatrix} \mathbf{e}_1 & & \\ & \ddots & \\ & & \mathbf{e}_p \end{bmatrix} \tag{10}$$

where $\mathbf{e}$ is a vector of ones in R^d. Additionally, λ is a vector in R^p that contains all of the auxiliary variables.

$$\lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_p \end{bmatrix} \tag{11}$$

The goal of directional friction constraints is to ensure that contact slipping on the surface is in the opposite direction of the friction force. Let us first exam the first term of Expression 5, $B^T\dot{\mathbf{q}}$, the velocity at contact projected onto each friction cone basis vector. The projection will end up in one of the three cases:

1. The projected vector is closer to one of the bases than others. The most negative element in $B^T\dot{\mathbf{q}}$ is unique.
2. The projected vector is right in the middle of two basis vectors. The two most negative elements in $B^T\dot{\mathbf{q}}$ are the same.
3. The projected vector is a zero vector.

Because the smallest possible value for any element in $B^T\dot{\mathbf{q}} + E\lambda$ is zero (by Expression 5), the first case has at most one zero element in $B^T\dot{\mathbf{q}} + E\lambda$. Assuming the index of that zero element is i, Expression 5 and Expression 6 together state that $\mathbf{f}_d$ must be a zero vector except for the i-th element. This nonzero element in $\mathbf{f}_d$ determines the direction of the friction force while the corresponding i-th element in $B^T\dot{\mathbf{q}} + E\lambda$ indicates the most negative projection of tangent velocity. Therefore, given a set of basis directions, the friction force is indeed in the most opposite direction of contact slipping. The second case rarely happens, but when it does we can arbitrary pick one of the two bases to break the tie and apply the same reasoning as the first case. The third case indicates either static contact (when $\lambda = 0$) or contact breakage (when $\lambda > 0$). It will be more clear after we introduce friction cone constraints.

Is it valid to choose a large positive λ such that all the elements of $B^T\dot{\mathbf{q}} + E\lambda$ are greater than zero? It does not seem to violate any of the expressions here, but we will see the implication of that in the next subsection.

3.3 Friction Cone Constraints

- expression 7: $\lambda \geq 0$
- expression 8: $\mu\mathbf{f}_n - E^T\mathbf{f}_d \geq 0$
- expression 9: $\lambda^T(\mu\mathbf{f}_n - E^T\mathbf{f}_d) = 0$

Friction cone constraints describe the switch condition between static state and slipping state of a contact. From the previous section, we know that at most one element of $\mathbf{f}_d$ can be nonzero for the first (and the second) case. Therefore, Expression 8 states that the ratio of tangent contact force to normal contact force must be less than or equal to the friction coefficient μ. If the contact force is within the friction cone (inequality case in Expression 8), λ must be zero by Expression 9. When λ is zero, the corresponding Expression 5 becomes

$B^T\dot{\mathbf{q}} \geq 0$. Because the bases of friction cones are arranged in pairs of opposite directions (e.g. $\mathbf{d}_1 = -\mathbf{d}_5$, $\mathbf{d}_2 = -\mathbf{d}_6$), the only way for all elements to be nonnegative is when $B^T\dot{\mathbf{q}}$ is a zero vector (no slipping), hence the friction cone condition.

Now, we can go back to the question about validity of selecting a large positive λ such that all the elements of $B^T\dot{\mathbf{q}} + E\lambda$ are greater than zero. If we do so, $\mathbf{f}_d$ will be all zeros by Expression 6, which leads to $\mu\mathbf{f}_n \geq 0$ by Expression 8. If $\mathbf{f}_n > 0$, λ must be zero by Expression 9, contradicting the assumption that λ is a large positive value. Therefore, $B^T\dot{\mathbf{q}} + E\lambda$ can be greater than zero only when the contact is broken ($\mathbf{f}_n = 0$). As long as the contact exists, λ will always be either zero or a positive value that makes the most negative element of $B^T\dot{\mathbf{q}} + E\lambda$ exactly zero.

3.4 LCP

Putting all the constraints together, we can construct the following linear system of equations:

$$\begin{bmatrix} 0 \\ \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{bmatrix} = \begin{bmatrix} M & -\Delta t N & -\Delta t B & 0 \\ N^T & 0 & 0 & 0 \\ B^T & 0 & 0 & E \\ 0 & \mu & -E^T & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}} \\ \mathbf{f}_n \\ \mathbf{f}_d \\ \lambda \end{bmatrix} + \begin{bmatrix} -\tau^* \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \mathbf{f}_n \\ \mathbf{f}_d \\ \lambda \end{bmatrix} \geq 0, \quad \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{bmatrix} \geq 0, \quad \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{bmatrix}^T \begin{bmatrix} \mathbf{f}_n \\ \mathbf{f}_d \\ \lambda \end{bmatrix} = 0 \tag{12}$$

The first row of the system is based on Equation 9. The remaining three rows, as well as the constraints, encapsulate the nine LCP conditions described above. Unfortunately, the construction described is in MLCP (Mixed LCP) form. To convert it to standard form, we need to make a few adjustments.

4 Standard LCP Form

The standard LCP solves for two vectors $\mathbf{w}$ and $\mathbf{z}$, and is of the following form:

$$\begin{gathered} \mathbf{w} = A\mathbf{z} + \mathbf{q} \\ \mathbf{w} \geq 0 \\ \mathbf{z} \geq 0 \\ \mathbf{w}^T\mathbf{z} = 0 \end{gathered} \tag{13}$$

If we express $\dot{\mathbf{q}}$, as $\dot{\mathbf{q}} = M^{-1}(\Delta t N\mathbf{f}_n + \Delta t B\mathbf{f}_d + \tau^*)$, we can then reorder Equation 12 slightly to get the following:

$$\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{bmatrix} = \begin{bmatrix} \Delta t N^T M^{-1} N & \Delta t N^T M^{-1} B & 0 \\ \Delta t B^T M^{-1} N & \Delta t B^T M^{-1} B & E \\ \mu & -E^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{f}_n \\ \mathbf{f}_d \\ \lambda \end{bmatrix} + \begin{bmatrix} N^T M^{-1} \tau^* \\ B^T M^{-1} \tau^* \\ 0 \end{bmatrix} \tag{14}$$

where:

$$
\begin{aligned}
\mathbf{w} &= \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{bmatrix} \\
A &= \begin{bmatrix} \Delta t N^T M^{-1} N & \Delta t N^T M^{-1} B & 0 \\ \Delta t B^T M^{-1} N & \Delta t B^T M^{-1} B & E \\ \mu & -E^T & 0 \end{bmatrix} \\
\mathbf{z} &= \begin{bmatrix} \mathbf{f}_n \\ \mathbf{f}_d \\ \lambda \end{bmatrix} \\
\mathbf{q} &= \begin{bmatrix} N^T M^{-1} \tau^* \\ B^T M^{-1} \tau^* \\ 0 \end{bmatrix}
\end{aligned} \tag{15}
$$

We can now solve this standard LCP using any LCP solver, such as Lemke's algorithm. Once we have $\mathbf{w}$ and $\mathbf{z}$, we can discard $\mathbf{w}$ and plug values in $\mathbf{z}$ back into the equations of motion to get $\dot{\mathbf{q}}^{n+1}$.

References

[1] David Stewart and J. C. Trinkle. An implicit time-stepping scheme for rigid body dynamics with Coulomb friction. *International Journal of Nnumerical Methods in Engineering*, 39:2673–2691, 1996.