Problem 1: Show that if $n \geq 2$ then among the numbers $(n!+2, n!+3, \ldots n!+n)$ none are prime.

Since we have $n \geq 2$, $n! = n(n-1)\ldots(2\cdot 1)$ if only if $n=2$.

By those information, we have

$2|n!$, $3|n!$, $4|n! \ldots\ldots n|n!$ ← we can see the last number always divides by $n!$

> The def of prime.
> A natural number is called prime number if $p$ only have 1 and $p$ as positive factor.

In our case, we need to show that:

$\gcd(n!, 2) \neq 1$

$\gcd(n!, 3) \neq 1$

$\vdots$

$\gcd(n!, n) \neq 1$

> rewrite $(n!+2, n!+3 \ldots n!+n)$ as
> $[(2)(\frac{n!}{2}+1), (3)(\frac{n!}{3}+1) \ldots\ldots (n)(\frac{n!}{n}+1)]$

First case, $n!+2$, if $n!+2$ is prime, then $n!+2 = a \cdot b = 1 \cdot (n!+2)$

But we can rewrite it as $n!+2 = (2)(\frac{n!}{2}+1)$

Same idea that $n!+3 = (3)(\frac{n!}{3}+1)$ and $n!+n = (n)(\frac{n!}{n}+1)$

This show that

$\gcd(n!, 2) = 2$

$\gcd(n!, 3) = 3$ → which is not equal to 1 and it is not prime

$\vdots$ Because prime number only have 1 and itself.

$\gcd(n!, n) = n$ therefore, we show that all the number are none prime.

Problem 2. Let $a, b, c$ be integers. Assume $\gcd(a,b)=1$ and $\gcd(a,c)=1$, prove that $\gcd(a,bc)=1$.

Pf: Assume by contradiction that $\gcd(a,bc)=d$ and $d\neq 1$,

Since $d \mid a$, $d \mid bc$, so $d \mid a(x)+bc(y)$

I chose a prime $p$ and divides $d$. I get.

$$p \mid bc \quad \text{and} \quad p \mid a$$

Based on Corol: $\gcd(a,b) = ax_0 + by_0 = 1$ for some $x_0, y_0 \in \mathbb{Z}$

$\gcd(a,c) = ax_1 + cy_1 = 1$ for some $x_1, y_1 \in \mathbb{Z}$.

I try multiply second $\gcd(a,c)$ by $b$. so $\Rightarrow ax_1^{\times b} + cy_1^{\times b} = 1^{\times b}$

$\Rightarrow abx_1 + cby_1 = b$

Plugging into the first equation.

$$ax_0 + (abx_1 + cby_1)y_0 = 1$$

$$\Rightarrow ax_0 + abx_1y_0 + cby_1y_0 = 1$$

$$\Rightarrow a(x_0 + bx_1y_0) + bc(y_1y_0) = 1$$

By Bezout Lemma, if $d \mid a$ and $d \mid bc$, then $d \mid 1$.

we have $\gcd(a,bc)=1$ ▨

Problem 3. Let $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ and $m = p_1^{f_1} p_2^{f_2} \cdots p_r^{f_r}$ be a prime factorizations of positive integers n and m respectly, where $e_i, f_i \geq 0$ for all $i$ and $p_1, p_2, p_3 \ldots p_r$. Prove that the GCD of m and n is given by:

$$\gcd(n,m) = p_1^{\min(e_1,f_1)} p_2^{\min(e_2,f_2)} \cdots p_r^{\min(e_r,f_r)}.$$

---

pf: During the class, we will use propety 2. $n \mid m \iff e_i \leq f_i \ \forall i$.

Let $d = p_1^{\min(e_1,f_1)} p_2^{\min(e_2,f_2)} \cdots p^{\min(e_r,f_r)}$

so by the def of gcd, we want to show:

① $d \mid n$ and $d \mid m$ is common divisor

② if c is also a common divisor for $c \mid d$. $\boxed{\text{d is greatest common divisor}}$

pf for ①

$d = p_1^{\min(e_1,f_1)} p_2^{\min(e_2,f_2)} \cdots p_r^{\min(e_r,f_r)}$

$n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ and $m = p_1^{f_1} p_2^{f_2} \cdots p_r^{f_r}$

By propety 2: $n \mid m \iff e_i \leq f_i \ \forall i$, so we rewrite.

$d \mid n \iff \min(e_i,f_i) \leq e_i$ ↘ $e_i$ and $f_i$ is always greater than or equal to $\min(e_i,f_i)$ which is True.

$d \mid m \iff \min(e_i,f_i) \leq f_i$ ↗

pf for ②. suppose $c \mid n$ and $c \mid m$, $C = p_1^{x_1} p_2^{x_2} \cdots p_r^{x_r}$, By property 2

$c \mid n \iff x_i \leq e_i$

$c \mid m \iff x_i \leq f_i$ $\Rightarrow$ but we want to show $c \mid d$.

WTS: $x_i \leq \min(e_i, f_i)$

$x_i \leq \min(e_i, f_i)$, but we know $x_i \leq \boxed{\underset{e_i \text{ or } f_i}{\underset{\shortparallel}{\min(e_i,f_i)}}}$

Because $x_i \leq \min(e_i, f_i)$ is true,

this implies $c \mid d$.

$\Rightarrow d = \gcd(n,m)$ ▨

Problem 4. Let $a, b \in \mathbb{N}$. prove that $a|b$ if and only if $a^n|b^n$ for all $n \geq 1$.

pf: Let $a = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$, $b = p_1^{f_1} p_2^{f_2} \cdots p_r^{f_r}$ $\boxed{\text{This part for} \Longrightarrow}$

we want to show $a^n|b^n \Longleftrightarrow e_i(n) \leq f_i(n)$

$a^n = (p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r})^n = p_1^{e_1 n} p_2^{e_2 n} \cdots p_r^{e_r n}$

$b^n = (p_1^{f_1} p_2^{f_2} \cdots p_r^{f_r})^n = p_1^{f_1 n} p_2^{f_2 n} \cdots p_r^{f_r n}$

prope 2:

$n|m \Rightarrow e_i \leq f_i$ for all $1 \leq i \leq r$, Base on prope 2, we have $e_i \leq f_i$ for $\forall i$, $\Longrightarrow$ since $n \geq 1$, $e_i \cdot n \leq f_i \cdot n$.

($\Longleftarrow$) Conversely if $a^n|b^n$, we know $e_i(n) \leq f_i(n)$

we want to show $a|b \Longleftrightarrow e_i \leq f_i$

we know $n \geq 1$, we can just divide by $n$, then we get $e_i \leq f_i$.

I proof both side, which $a|b \Longleftrightarrow a^n|b^n$ for all $n \geq 1$ $\blacksquare$

Problem 5. Compute $\sigma_0(11!)$ and $\sigma_1(360)$

Part A: $\sigma_0(11!)$

$\sigma_0(n) = (e_1+1)(e_2+1)\dots(e_n+1)$

$n = 11! = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11$

$= 2\cdot 3\cdot (2^2)\cdot 5\cdot (2\cdot 3)\cdot 7\cdot (2^3)\cdot (3^2)\cdot (2\cdot 5)\cdot 11$

$= 2^8\cdot 3^4\cdot 5^2\cdot 7\cdot 11$

I get $p_1 = 2$, $e_1 = 8$

$p_2 = 3$, $e_2 = 4$

$p_3 = 5$, $e_3 = 2$

$p_4 = 7$, $e_4 = 1$

$p_5 = 11$, $e_5 = 1$

$$\boxed{\begin{aligned}\sigma_0(11!) &= (8+1)(4+1)(2+1)(1+1)(1+1)\\ &= 9\cdot 5\cdot 3\cdot 2\cdot 2\\ &= 540\end{aligned}}$$

Part 2: $\sigma_1(360)$

$\sigma_1(n) = \frac{p^{e+1}-1}{p-1}$

$360 = 36\cdot 10 = 6\cdot 6\cdot 2\cdot 5 = 2^2\cdot 3^2\cdot 2\cdot 5 = 2^3\cdot 3^2\cdot 5$

$$\boxed{\begin{aligned}\sigma_1(360) &= \sigma_1(2^3\cdot 3^2\cdot 5)\\ &= \sigma_1(2^3)\cdot \sigma_1(3^2)\cdot \sigma_1(5)\\ &= \left(\frac{2^4-1}{2-1}\right)\cdot\left(\frac{3^3-1}{3-1}\right)\cdot\left(\frac{5^2-1}{5-1}\right)\\ &= 15\cdot 13\cdot \frac{24}{4}\\ &= 15\cdot 13\cdot 6\\ &= 1170\end{aligned}}$$

Problem 6 How many positive integer does 2020 × 2022 and what is the sum of these divisor.

First $\boxed{\sigma_0(2022 \times 2020) = 64 \text{. There are 64 positive integer.}}$

$$2020 = 10 \cdot 202$$
$$= 2 \cdot 5 \cdot 2 \cdot 101$$
$$= 2^2 \cdot 5^1 \cdot 101^1$$

$$2022 = 2 \cdot 1011$$
$$= 2^1 \cdot 3^1 \cdot 337^1$$

$$\sigma_0(2022 \times 2020) \Rightarrow \sigma_0(2^2 \cdot 5^1 \cdot 101^1 \cdot 2^1 \cdot 3^1 \cdot 337^1) \Rightarrow \sigma_0(2^3 \cdot 3^1 \cdot 5^1 \cdot 101^1 \cdot 337^1)$$

$$\sigma_0(2022 \times 2020) = (3+1)(1+1)(1+1)(1+1)(1+1) = (4)(2)(2)(2)(2) = 64$$

---

second (sum of divisor) $\sigma_1(2022 \times 2020) =$

$$\sigma_1(2^3 \cdot 3^1 \cdot 5^1 \cdot 101^1 \cdot 337^1) = 15 \cdot 24 \cdot 102 \cdot 338$$

$$\boxed{\sigma_1(n) = \frac{p^{e+1} - 1}{p-1}}$$

$$\sigma_1(2^3 \cdot 3^1 \cdot 5^1 \cdot 101^1 \cdot 337^1) = \sigma_1(2^3) \cdot \sigma_1(3^1) \cdot \sigma_1(5^1) \cdot \sigma_1(101^1) \cdot \sigma_1(337^1)$$

$$= \left(\frac{2^4 - 1}{2-1}\right) \cdot \left(\frac{3^2 - 1}{3-1}\right) \cdot \left(\frac{5^2 - 1}{5-1}\right) \cdot \left(\frac{101^2 - 1}{101-1}\right) \cdot \left(\frac{337^2 - 1}{337-1}\right)$$

$$= 15 \cdot 4 \cdot 6 \cdot 102 \cdot 338.$$

$$\boxed{= 15 \cdot 24 \cdot 102 \cdot 338}$$

It's a long number, so I keep like that.

$$= 12411360$$

Just in case

Final Answer

Problem 7. (I didn't copy the Question.

pf: $\phi(m,n) = \phi(m)\phi(n)$ for any $m, n \in \mathbb{N}$. $\gcd(m,n)=1$

hint $\phi(p)=p-1$

LHS: Show $\phi(n)$.

we have $n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$, so

$$\phi(n) = \phi(p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}) = \phi(p_1^{e_1})\phi(p_2^{e_2}) \dots \phi(p_r^{e_r})$$

$$\boxed{\phi(p) = p-1, \ \phi(p^r) = p^r - p^{r-1}}$$

we will use $p^r - p^{r-1}$

$\phi(n) = (p_1^{e_1} - p_1^{e_1-1})(p_2^{e_2} - p_2^{e_2-1}) \dots (p_r^{e_r} - p_r^{e_r-1})$ ← LHS is equal to RHS.

RHS: want to show $(n)(1-\frac{1}{p_1}) \dots (1-\frac{1}{p_r})$

we alredy have $n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$

rewrite the $(n)(1-\frac{1}{p_1}) \dots (1-\frac{1}{p_r})$

$\Rightarrow (p_1^{e_1} \dots p_r^{e_r})(1-\frac{1}{p_1}) \dots (1-\frac{1}{p_r})$ ← rewrite the n.

$\Rightarrow (p_1^{e_1} \dots p_r^{e_r})(1-p^{-1}) \dots (1-p_r^{-1})$ ← rewrite $\frac{1}{p_r} = p_r^{-1}$

$\Rightarrow (p_1^{e_1})(p_2^{e_2}) \dots (p_r^{e_r})(1-p^{-1}) \dots (1-p_r^{-1})$ ← rewrite.

$\Rightarrow p_1^{e_1}(1-p_1^{-1}) \cdot p_2^{e_2}(1-p_2^{-1}) \dots \cdot p_r^{e_r}(1-p_r^{-1})$

$\Rightarrow (p_1^{e_1} - p_1^{e_1-1}) \cdot (p_2^{e_2} - p_2^{e_2-1}) \dots \cdot (p_r^{e_r} - p_r^{e_r-1})$ ← This show RHS equal to LHS.

LHS = RHS now! proof done. □