# Calculus Development Approaches by Newton and Leibniz

Math 194: Senior Seminar

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# Contents

# 1 Introduction

## 1.1 The background introduction of Calculus

In the modern world now, Calculus is not just a boring college mathematics. Knowing calculus has become a fundamental math skill or technique for scientists, engineers, and mathematicians across a wide range of disciplines. For example, the study of change and motion is the main emphasis of the mathematical field of calculus. The invention of calculus is an epoch-making feat in the history of mathematics. It can be understood that if there is no calculus, there will be no modern technology, and there will be no brilliant modern civilization of mankind. Human beings are still groping forward in the dark! However, who is the father of calculus? It's a math topic which is always fascinating.

At the end of the seventeenth century, Gottfried Wilhelm Leibniz [28], the greatest mathematician in Germany, and Sir Isaac Newton[27], the greatest mathematician in England. As 17th century mathematicians, both of them created their own development of calculus without any source support. Newton and Leibniz make some math from impossible become possible such as the rates of change and how things change over time while they explore calculus. But there was a secret war that lasted more than 10 years between both of them. Supporters of both Leibniz and Newton accused the other of inappropriate behavior. In this war, both Leibniz and Newton said that they were the real founders of calculus until their respective deaths. On this issue, Newton and Leibniz have been arguing for more than ten years, and their respective fan camps have been arguing for more than half a century, which has become the biggest public case in the mathematics world so far[29]. In my opinion, Calculus is the most important basic mathematical knowledge that has greatly promoted the progress of science; however, this secret calculus war is a major event in the history of science and a tragedy with incalculable losses.

In this research report, I will provide a detailed interpretation, understanding, and analysis of the independent studies of calculus by both Newton and Leibniz. Furthermore, I objectively point out the similarities and differences in their approaches and who's approaches is much useful for mordern calculus.

## 1.2 Notable Precursors and Early Mathematical Methods relate to calculus.

Before calculus's framework was developed by Newton and Leibniz, there are already some famous works and studies for calculus and it was laid by several different countries mathematicians. In ancient times, some ideas that eventually led to integral calculus were introduced, but these concepts were not developed in a rigorous and systematic manner. The calculation of volumes and areas was one of the primary objectives of integral calculus.

Ancient Greek mathematicians, such as Eudoxus and Archimedes, made significant strides in understanding areas and volumes. Archimedes developed the methods for approximating: the area under curves and the value of $\pi$ [7]. He also developed the method of exhaustion: it involved dividing the shape into smaller parts and calculating the sum of their areas or volumes to approximate the total area or volume [7].

Meanwhile, in Asia China and India. The method of exhaustion was independently invented by Chinese mathematician Liu Hui in the 4th century AD. His method and purpose is to find the area of a circle[7]. Later on, another famous mathematician Zu Chongzhi in the 5th century established a method, which later came to be known as Cavalieri's principle, for finding the volume of a sphere[7]. Indian mathematics from Kerala school stated components of calculus such as the Taylor series and infinite series approximations[7]. These contributions from ancient Chinese mathematicians and Indian mathematicians were significant precursors to the development of calculus.

Although these early developments were crucial in shaping the concepts of calculus, it was in the 17th century that Newton and Leibniz independently and systematically formulated the fundamental principles of calculus. Their pioneering work laid the foundation for modern calculus and its immense impact on mathematics and various scientific disciplines.

### 1.3 Who is Sir Isaac Newton?

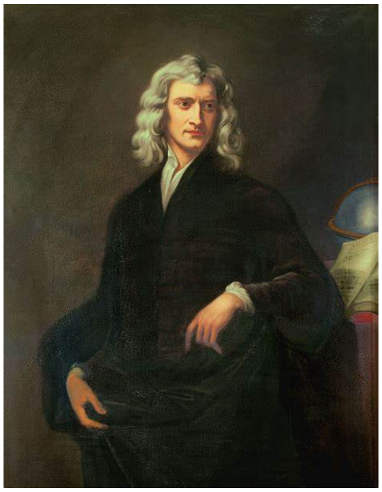

Figure 1: Newton[18]

Let me start by introducing Mathematician Sir Isaac Newton. Sir Isaac Newton, commonly known as Newton, who was a famous and powerful mathematician in the 17th century. Later on he also became a physicist and astronomer. Newton was born in Lincolnshire, England in 1643. Honestly, he wasn't a smart genius during his teenage years; however, he had a keen interest in reading books. In particular, he enjoyed books that introduced simple mechanical models and was inspired to create his own inventions. This early fascination with mechanics would go on to influence his entire career. After completing his early education, Newton pursued his desire of attending to higher education. At the time he attended the University of Cambridge, he was a student at Trinity College. It was here that he began to pursue his studies in mathematics and natural philosophy[8] [27].

During his time at Trinity College, Newton engaged in intensive studies and accomplished many important contributions to mathematics fields. In addition to his work in mathematics, Indeed, Newton is most famous for his physic laws of motion and the law of universal gravitation. Newton also focused on study of optics and light accomplished significant contributions to the study of optics and light. As he discovered and studied that light is made up of different colors and developed the theory of color. He explains how light interacts with different surfaces and objects[27]. Despite the modern world, Newton developing influential theories that would shape the fields of science and mathematics for centuries to come.

To be honest, Newton must absolutely the one of the most famous mathematicians in mathematics history. His groundbreaking discoveries and contributions to the fields of mathematics such as calculus had a profound impact on science and society as a whole. In this research, I will be concentrating on Newton's contributions to the discipline of mathematics; especially his research approaches of calculus

### 1.4 Who is Gottfried Wilhelm Leibniz?

Figure 2: Leibniz[17]

Let me now introducing Mathematician Gottfried Wilhelm Leibniz. He was born in Germany on July 1, 1646. Sometimes people referred to as Leibniz. He was a famous mathematician who specialized in numerous fields in the 17th century. Friedrich Leibnütz, who was the father of Leibniz and a professor at the University of Leipzig, provided as a knowledgeable role model for Leibniz as a child. Unfortunately, Leibnütz died when Leibniz was just six years old, however he left behind a personal library that was fairly well-stocked with amounts written in Latin. Leibniz had unrestricted access to the library starting at age seven and he read a lot of the books. Leibniz was a lot inspired by knowledge and books from the library. He became a extremely bright child and showed an early aptitude for languages and mathematics[28]. Especially mathematics, Leibniz showed an interest in mathematics and early study of geometry and algebra during his young age.

After Leibniz stepped into higher education, he enrolled at the University of Leipzig and focused on the major of law, philosophy, and mathematics. If we take a deep look of his life, we can found out that he completed many of his formulas while riding in a bumpy carriage. Later in 1672, he traveled to France, Paris and began developing his ideas about calculus[28]. After returning to Germany in 1676, Leibniz kept developing his calculus and making it perfect. When he explored calculus, he realized that good mathematical symbols can save a lot of time, which is also a significant skill to be successful in mathematics. Therefore, the calculus notation he created is far superior to Newton's notation, and has had a great influence on the development of calculus until now[5].

When Leibniz explored calculus, he realized that good mathematical symbols can save a lot of time, which is also a significant skill to be successful in mathematics. Therefore, the calculus notation he created is far superior to Newton's notation In this research, I will also focus on Leibniz's calculus and how his calculus notation had a great influence on the development of calculus until now.

## 2 The principle of Newton's calculus

As I introduce Newton in the Introduction part. Newton Sir Isaac Newton is credited with independently developing both the integral and the differential in the history of calculus. In the autumn of 1664, Newton returned to his hometown to avoid the bubonic plague outbreak that was ravaging Cambridge University. It is a golden period for Newton because he made several significant breakthroughs that would lay the foundation for some of his most important work.

Before we go deeper, let me introduce "The Method of Fluxions and Infinite Series". It is a mathematical work completed by Isaac Newton during his golden period and published posthumously in 1736.[3] "Method of Fluxions" states the foundation of Newton's calculus. Newton studied tangents in a coordinate system using velocity components, which not only led to the development of the method of fluxions but also provided the key to its geometric applications. It was one of his most important works and introduced the concept of calculus to the world. In this work, Newton introduces the concept of "moments"[1][3], "infinite series"[1][3] and "fluxions,"[1][3] and how he uses them to calculate rates of change and slopes of curves.

### 2.1 The relationship between Infinitesimals and Moment

Newton introduced the concept of infinitesimals and introduced the term "Moment" to develop the method of fluxions.[1] Moment is used to describing an infinitely small quantity or an infinitely small increment such as time, distance, and velocity. in Newton's calculus or kinematics. Specifically, It's referring to the rate of change of a quantity for time or another variable.

Newton defined the concept of Moment as the precise amount by which a fluent quantity increases during an "infinitely small" time interval. Which we can understand that the increase of $x$ in an infinitesimal time $o$ is the product of the speed of $x \Rightarrow \dot{x}o$. As a consequence, It follows that after this time interval, the value of $x$ will be updated to $x + \dot{x}o$, and similarly, the value of $y$ will be updated to $y + \dot{y}o$[1]. Consequently, an equation which expresses a relationship of fluent quantities without variance at all times will express that relationship equally between $x + \dot{x}o$ and $y + \dot{y}o$ as between $x$ and $y$; Therefore, $x + \dot{x}o$ and $y + \dot{y}o$ may be substituted in place of the latter quantities, $x$ and $y$, in the equation."[1]

## 2.2 Fluent and Fluxion

Let me introduce fluent and fluxions first. Based on the idea of Moment, Newton introduced two new concepts terms, one terms is called "fluent"[1] and another term is "fluxion"[1]. The relationship between both terms is a fluent represents a variable quantity, while its rate of change is referred to as a fluxion. The notation of fluxion is a dot placed above the variable. In Figure 3, given two variables $x$ and $y$ as Fluents. Their respective Fluxion would be represented as $\dot{x}$ and $\dot{y}$. Similarly, the Fluxion of $\dot{x}$ and $\dot{y}$ are $\ddot{x}$ and $\ddot{y}$. Respectively. This pattern continues for higher-order Fluxion. Now we have a better understanding of Newton's calculus

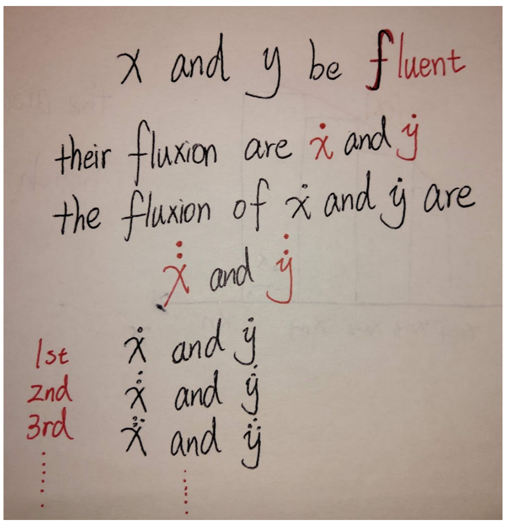


Figure 3: Fluent and Fluxion

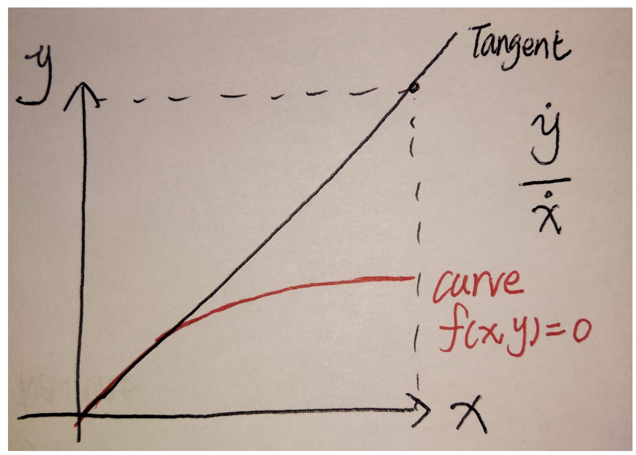


Figure 4: Details of Fluxion

We now have a better idea of what "fluxion" is, but how it refers to the rate of change? What is the formula? In Figure 4, Newton regards curves from function $f(x,y) = 0$, The equation $f(x,y) = 0$ represents a particular relationship between variables. Newton envisioned them as paths traced by a moving point. The point's location on the curve depends on its $x$ and $y$ coordinates, which can change over time. or tracking the object's movement over time. Later on, Newton found a significant observation, which is the slope of the tangent line to the curve $f(x,y) = 0$ at any given point is given by the ratio of $x$ to $y$. Another way to understand is that if we take the change in $\dot{x}$ divided by the change in $\dot{y}$ , it gives us the slope of the tangent line at that particular point on the curve. Because $\dot{x}$ and $\dot{y}$ are flow speeds of fluxion that change while the time is moving. Hence Newton points out the slope of the tangent to the curve $f(x, y) = 0$ is $\frac{\dot{y}}{\dot{x}}$[1], Fluxion represents how a variable is changing over time or with respect to another variable.

## Fundamental Concepts of Newton's Calculus

By applying the knowledge and concepts I just mentioned above, let me get right to it and try to convey Newton's fundamental ideas about calculus: Calculating Fluxions. Here is a fact about fluxion, Newton often thinks of physics when thinking about mathematics namely the motion of particles and in line with that on this coordinates grid[1].

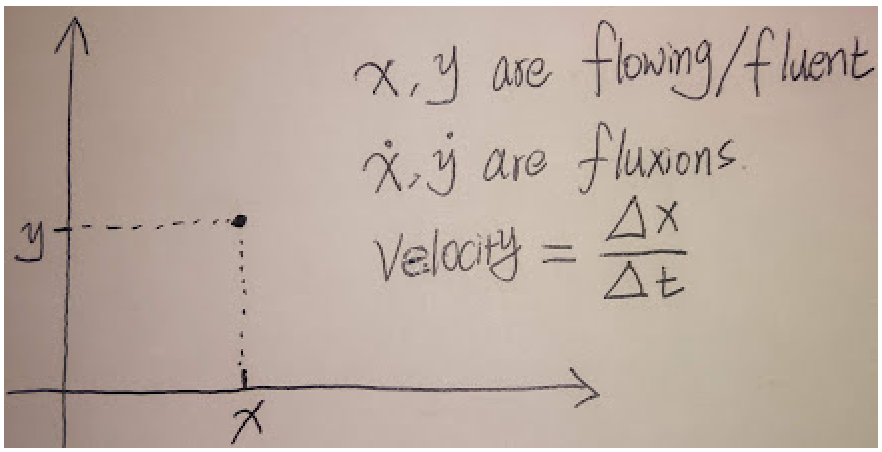


Figure 5: Fundamental ideas

In Figure 5, Newton contemplates the position of a hypothetical particle at a specific moment in time. Newton designs its horizontal coordinate by $X$[16] and vertical coordinate by $Y$[16]. These symbols $X$ and $Y$ often refer to the flowing or the fluent quantities. In the setting of particle motion when we speak about $X$ axis and $Y$ axis, we never intend these symbols to mean a static quantity. A quantity should be permanent over time. Therefore; the $X$ axis and $Y$ axis are going to be called the flowing quantities or the fluent[1][16].

The second important thing is the fundamental idea of velocity. Newton describes velocity as the rate of change of position. when he calls it Delta $X = \triangle X$ divided by Delta $t = \triangle t$.[15] In other words, It's how much we change or displace concerning position divided by how much we change in time. In modern physic, we might notate a velocity in the $X$ direction as $\dfrac{\triangle X}{\triangle t}$[15]; However, in the 17th century, Newton is going to notate $\dot{X}$ stands for the velocity in the $X$ direction and correspondingly $\dot{Y}$ going to stand for the velocity in the $Y$ direction. Both the $\dot{X}$ and $\dot{Y}$ instead of being referred to as fluent quantities are called fluxions.

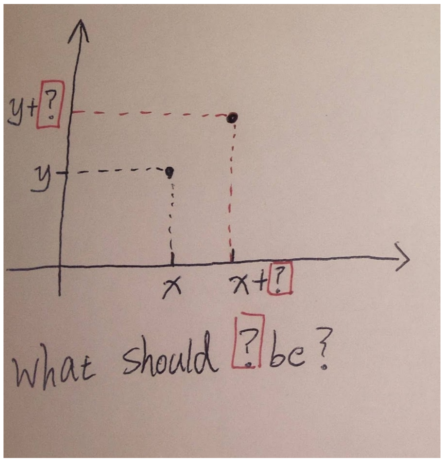


Figure 6: What should ? be?

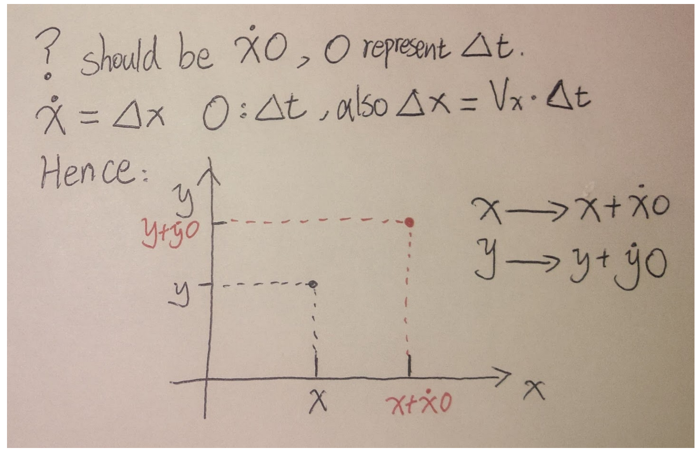


Figure 7: solve for unknown factor

After that, Newton came up with a fundamental question: When the particle move at some later point in time. What is the new $X$ coordinate and new $Y$ coordinate here and what should that be? In Figure 6, Let's mark the position of that particle after having moved over. The fundamental insight here is that the new $X$ coordinate should be $X$ plus some other quantity[16]. Same idea for the Y coordinate. which I called $X+?$ and $Y+?$ for now.

Newton want to find out what should that unknown factor be and in his idea: the new coordinate should be estimated by the velocity $\dot{X}$ times some very small quantity in time which he denotes as $O$. The $O$ is some infinitesimals small change in time. In other words, we could understand that this additional displacement the velocity in the X direction times delta t, which donate as $X + \dot{X}O$. In Figure 7, the horizontal coordinate of the new particle should be $X + \dot{X}O$ and the vertical coordinate should be $Y + \dot{Y}O$. This is the fundamental insight from Newton's calculus that $X \rightarrow X + \dot{X}O$ and $Y \rightarrow Y + \dot{Y}O$[16].

### 2.3 Some Rule From Newton's Calculus

Here are some formula rules for Newton's fluxions[16]. Those rules have been used a lot in modern calculus. Figure 8 is the chain rule, Figure 9 is product rule. The product rule have $o$, It is an infinitely small amount of time. So, the term $o^2$ is second order infinite small term and according to Newton, we can now ignore $o^2$ because of its second order infinite smallness comparing to first order infinite smallness of $o$, Hence it implies $o \to 0$. Figure 10 is Quotient rule and it's based on product rule.

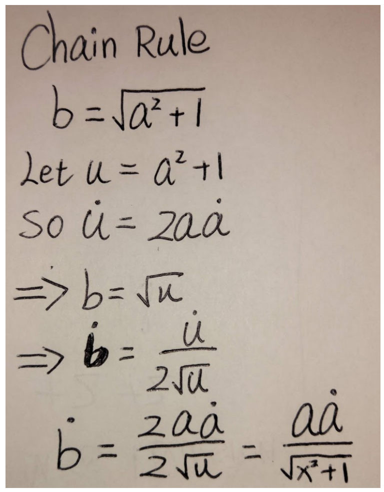


Figure 8: Chain Rule

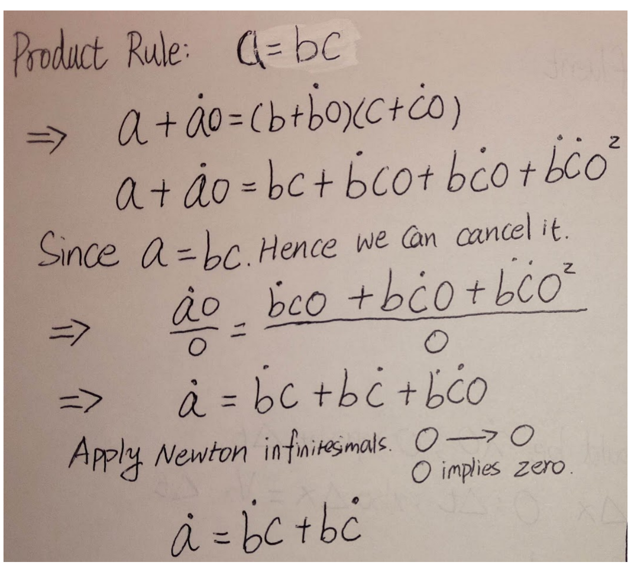


Figure 9: Product Rule

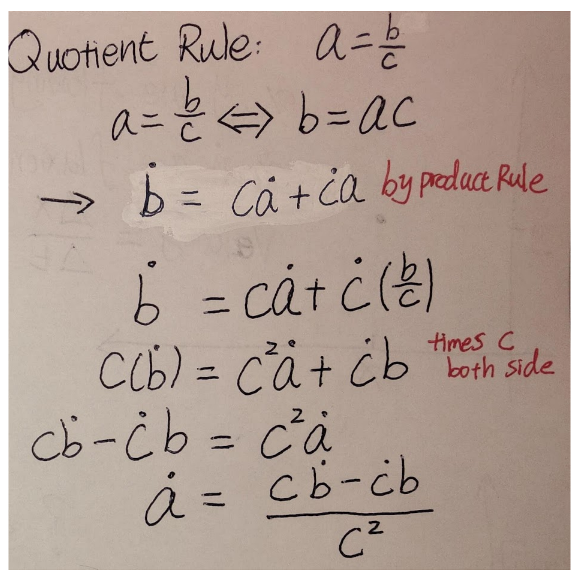


Figure 10: Quotient Rule

### 2.4 Deriving the Newton-Leibniz Formula from Newton's Perspective and Method

Before Newton proved the Newton-Leibniz formula, he first try to proved the following conclusion : $F(x) = \dot{x}\Box f(x)$. This is Newton's calculus notation. A prefixing rectangle $\Box x$ is the Newton's notation for integration. If we rewrite the conclusion in modern calculus notation, it is represented as $F(x) = \frac{d}{dx}\int_0^x f(x)dx$[6]. Now, let's take a look at what $\int_0^x f(x)dx$ is expressing.

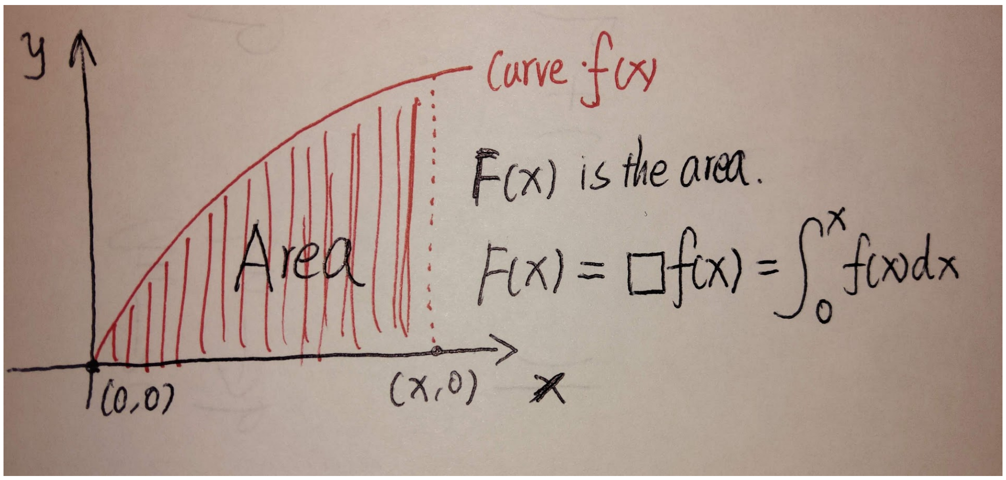


Figure 11: what $\int_0^x f(x)dx$ is expressing?

In calculus, the area under a curve represents a function, which is commonly known as the integral. In Figure 11, Let $f(x)$ be the curve line and let 0 be the start points and $x$ be the end points, but what is the area between $[0, x]$?. Newton use $F(x)$ represent the area under the curve of $x^n$ between 0 and $x$. After Newton defined the formula $F(x) = \int_0^x f(x)dx$[6], his approach can be understood as: Starting from the area function $F(x)$: determine the Fluxion of $F(x)$ is curve $f(x)$.

In Newton's Calculus, the term "derivative" was not used, but there was an equivalent concept known as "fluxion" or "rate of change," and Newton began by seeking the rate of change of $F(x)$. To calculate this rate of change, Newton approached it as follows[6][30]. In Figure 12, Newton considered the height of the rectangle (area function) to be represented by $f(x)$. He introduced a point $P$, located after point $X$. The distance between point $P$ and point $X$ is denoted as a very small increment, which is $O$. Above point

$X$ on the curve, Newton referred to a point as $D$. Similarly, above point $P$ on the curve, he referred to a point as $E$. Take a look at Figure 12, We can see that the area enclosed by connecting points $B, D, E$, and $P$. It forms a region that represents the area under the curve in the interval $[X, P]$.

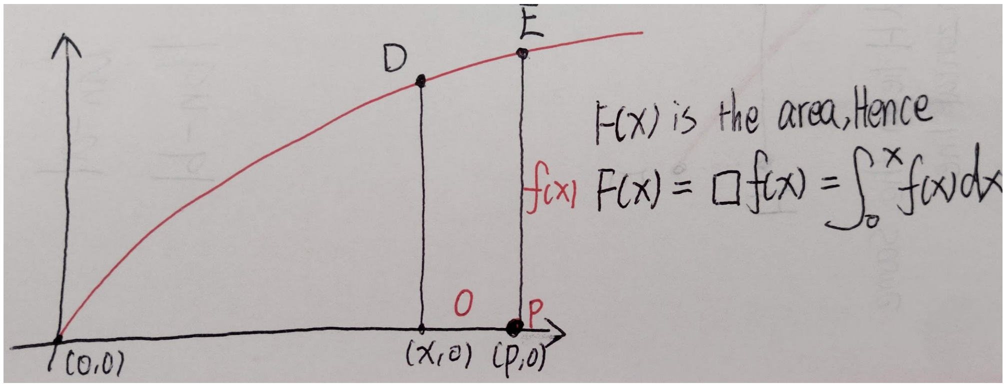


Figure 12: What $F(x)$ represent?

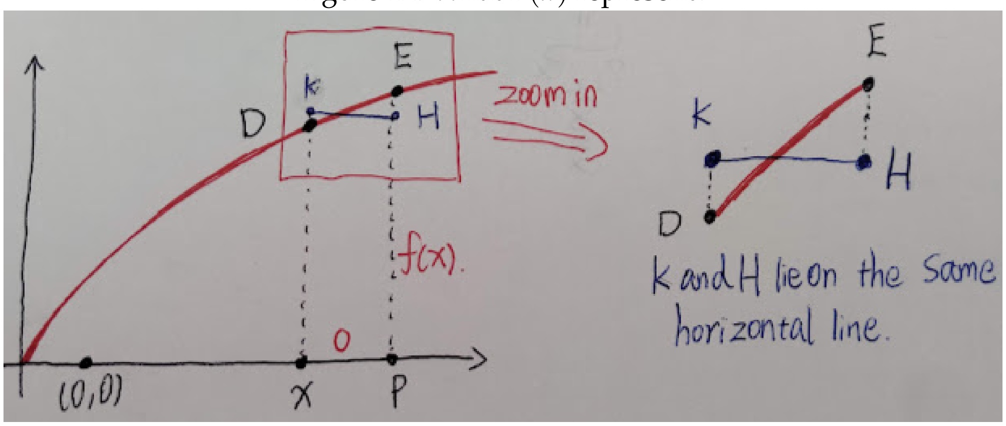


Figure 13: What is the area DXPE?

To calculate the area DXPE, Newton used a geometric approach. In Figure 13, Newton stated that we can always find a point above point D and named it point K. Simultaneously, below point E, we can always find a point and named it H. Newton required that points K and H must lie on the same horizontal line (be horizontally synchronized). He then discovered that the area of the rectangle KXPH is equal to the area of the curve DXPE. This geometric proof is now known as the Mean Value Theorem of integration[30].

As I mentioned above: $O$ is the distance between $X$ and $P$. Also $f(x)$ is the height. Newton's assertion is that when $O$ is small enough, the area DXPE is approximately equal to $O \cdot f(x)$. In other words, If $O$ is small enough, the increment of $F(x)$ with respect to $O$ is approximately equal to $O \cdot f(x)$. By combining this with the definition of the fluxion, Newton deduced the following[1]:

$$F(\dot{x}) = \frac{O \cdot f(x)}{O} = f(x)$$

$F(\dot{x})$ is the rate of change of $F(x)$, which also fluxion of $F(x)$. Therefore, Newton determine the Fluxion of $F(x)$ is indeed the curve $f(x)$. Actually, at this point, we have arrived at the first fundamental theorem of calculus[6].

$$F(\dot{x}) = \dot{x}\square f(x) = f(x) \longleftrightarrow \frac{d}{dx}F(x) = \frac{d}{dx}\int_0^x f(x)dx = f(x)$$

Following the first fundamental theorem of calculus, it becomes relatively easy for us to derive the Newton-Leibniz formula. Using the first fundamental theorem of calculus, we have the following hold. In Figure 14 and 15, I will use the Leibniz's notation because it's easy to understand:

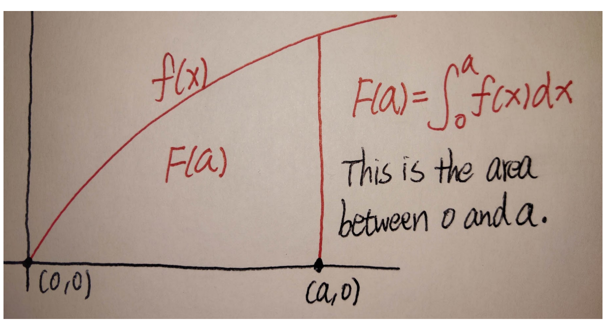


Figure 14: In Figure 14, The area between 0 and a is $F(a) = \int_0^a f(x)dx$

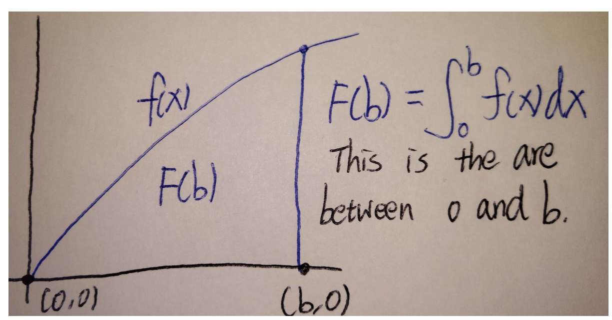


Figure 15: In Figure 15, The area between 0 and b is $F(b) = \int_0^b f(x)dx$

Let the red area be $F(a) = \int_0^a f(x)dx$, and the blue area be $F(b) = \int_0^b f(x)dx$. When Newton placed $F(a)$ and $F(b)$ on the same coordinate axis, he observe that the area under the curve from $(a, b)$ is equal to $F(b) - F(a)$. In Figure 16. This area is had been marking as the black area, which can be expressed as the definite integral: $\int_a^b f(x)dx$[1] [6] [30]. This is the Newton-Leibniz formula from Newton's perspective. The greatest significance of the Newton-Leibniz formula is its ability to greatly simplify calculations.

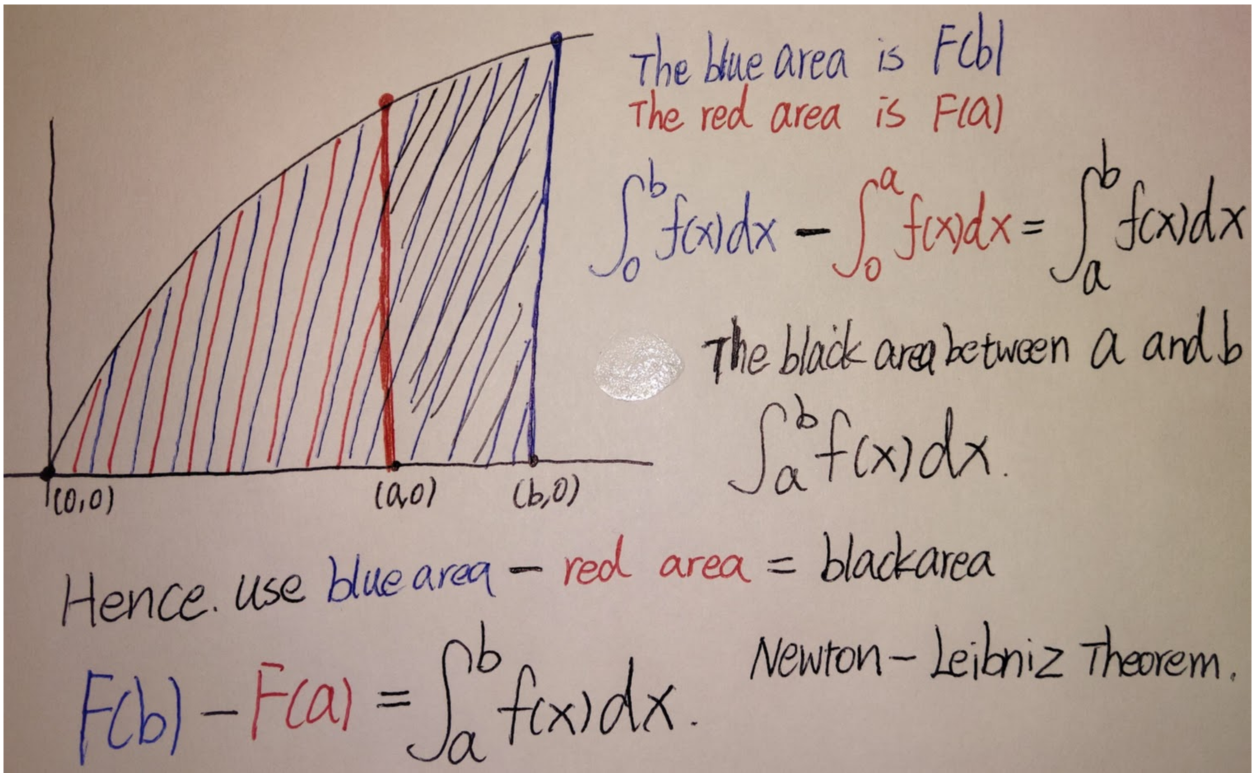


Figure 16: Newton-Leibniz formula from Newton's perspective

# 3 The principle of Leibniz's calculus

As I introduce Leibniz in Introduction part. Gottfried Wilhelm Leibniz is credited with independently developing both the integral and the differential in history of calculus. Leibniz's notation and approach to calculus are still widely used modern calculus. Leibniz published his work on calculus in the late 17th century, introducing the integral notation $\int$ (the integral sign) and the $\frac{d}{dx}$ notation for differentiation[1]. Unlike Newton first develop the differential, then integral. He first focused on the concept of the integral and emphasizing the summation of infinitesimal quantities. then he use integral led to the development of differential[1].

## 3.1 Infinitesimal

Before we go deeper, let me introduce to the concept of "infinitesimal," which is a fundamental and crucial concept in Leibniz's calculus. Leibniz developed his calculus based on the contemplation of geometry, especially the study of triangles. Leibniz named calculus the "Method of infinitesimals" and built it upon the concept of infinitesimals. At first, Leibniz introduced "moment "[1]. The term "moment" is derived from the Latin word "momentum". The definition of "momentum" in Latin refers to movement or impulse. Leibniz wants the "moment" to represent a small quantity or a momentary change in his calculus framework[4]. In most of his manuscripts and research, "moment" is used to represent a differential area and describe the notion of a small increment or infinitesimal quantity.

It can be seen that the infinitesimal plays an important role in Leibniz's calculus. Leibniz used "moment" to represent the infinitesimal[4]. However, he didn't show a precise definition or clear concept of the infinitesimal, which makes his understanding and expression often ambiguous and confusing. Although Leibniz worked hard to perfect the infinitesimal by introducing "terminata" and "interminata" and successively introduced the concepts of infinity and infinite small. Finally, the foundational explanation of calculus remained unclear until it was slowly clarified in the 19th century.

### 3.2 Integral

Leibniz's integral always revolves around a question:

How to find the area under the curve in the function graph?

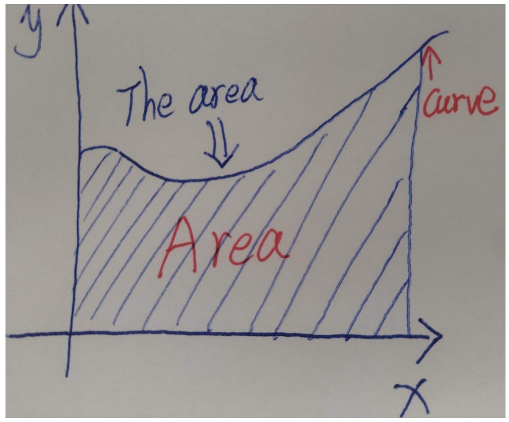


Figure 17: Area under the curve

The fundamental idea of Leibniz's integral could be understood as the addition of infinitesimals[1][13]. As we can see in Figure 18, let $f(x)$ be the curve. The area under the $f(x)$ had been separated by dividing into infinitesimal rectangles. One of the examples I show is the rectangles with $x_n$ and $x_{n-1}$ and the distance between $x_n$ and $x_{n-1}$ is named as $dx$, In the previous chapter 3.1, I introduce the term "moment"; therefore, for each infinitesimal rectangle, Leibniz used "moment" to represent the area of the rectangle[1].

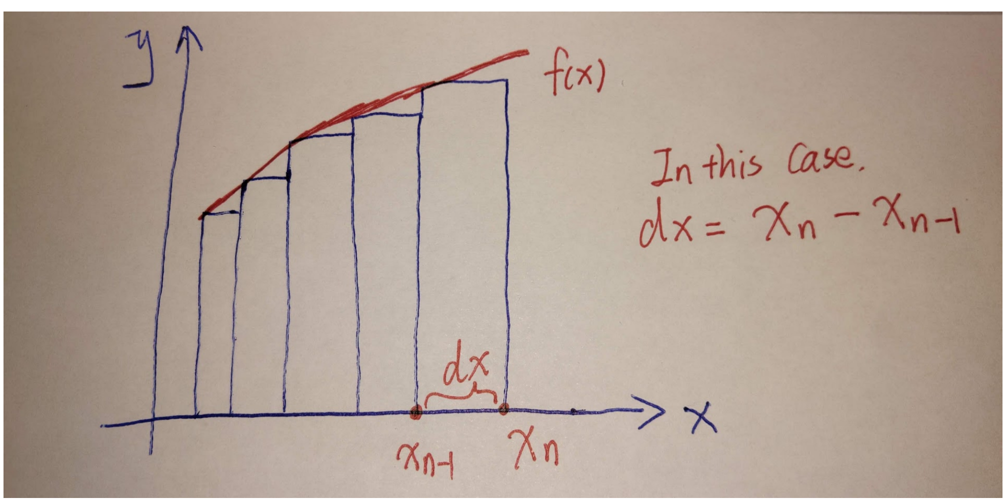


Figure 18: Fundamental idea of Leibniz's integral

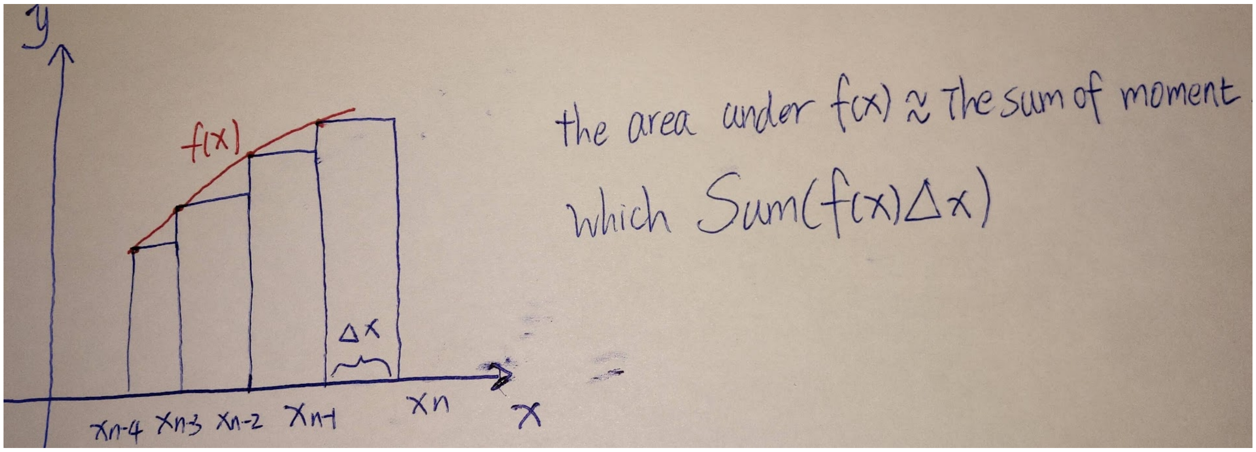


Figure 19: Fundamental idea of Leibniz's integral

Finally, In order to calculate the area under the curve $f(x)$, we need to add the rectangles area together. In Figure 19, Lebniz created a notation $Sum(f(x)\triangle x)$ represents a sum of rectangles and $Sum(f(x)\triangle x) = \int f(x)$[5] [13].

- $Sum()$ is the same as notation Sigma $\sum$, which is sum of multiple terms.[1] [5]
- $f(x)$ represents the function being evaluated at different values of x.[1] [5]
- $\triangle x$ symbol represents a small increment or change in the x, Easy way to understand is the width of each rectangle.[1] [5]

Leibniz developed integral notation $\int$[5] announcing the advent of advanced mathematics to everyone. It is a giant step in mathematical notation in the history of calculus, The notation had been used and presented in the modern calculus. Leibniz's integral is simple and straightforward, Unlike the Newton's integral with a lot of problem. Later, Cauchy and Riemann perfected the definition of the integral based on Leibniz's integral.

### 3.3 Differentiation

Differentiation refers to the process of finding the derivative of a function. When Leibniz began exploring derivatives, he realized that finding Tangent lines was useful for derivates[12]. He defined the tangent line as a straight line that connects two points on a curve that are infinitely close together. This infinitesimal distance can be expressed using differentials or differences between two adjacent variable values.[4]

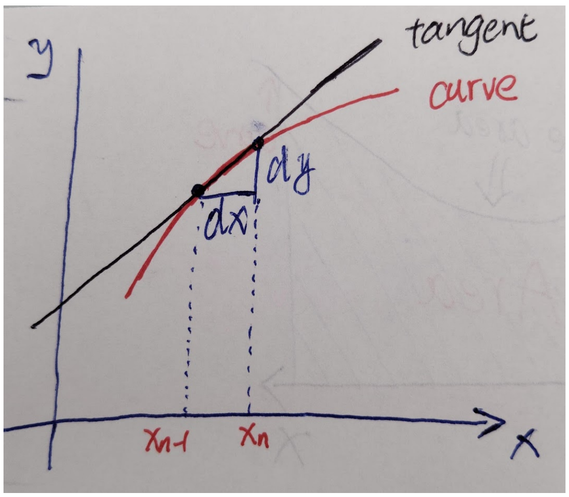


Figure 20: Tangent Line

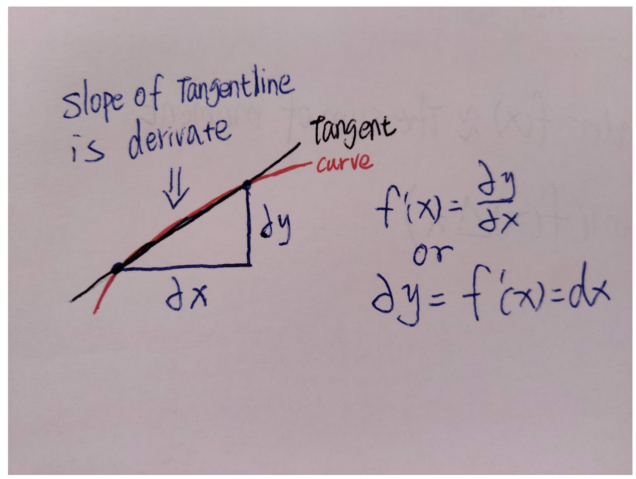


Figure 21: Leibniz's idea of differentials

Let $f(x)$ be the curve and $x_n$ and $x_{n-1}$ be two points. In chapter 3.2 Leibniz called the distance between $x_n$ and $x_{n-1}$ is $dx$. In Figure 20, $dy$ represents the incremental change in the curve corresponding to $dx$. Since the line connecting two points $x_n$ and $x_{n-1}$ is the tangent line, $dy$ also represents the incremental change in the Tangent line corresponding to $dx$[11][12].

In Figure 21, $dx$ and $dy$ are two independent infinitesimals that form the differentials triangle. Leibniz's idea is that: when we calculated the quotient of these infinitesimals to obtain the slope of the Tangent line. We can see that the Tangent line represents the hypotenuse of the triangle. The slope of the Tangent line is equivalent to the derivative. Hence, we have $f'(x) = \frac{dy}{dx}$ and $dy = f'(x)dx$[12][5]. Based on these concepts, Leibniz specifically named them differentials. The notation dx and dy shown above represent differentials.

### 3.4 Some Rule From Leibniz's Calculus

Leibniz made significant improvement to calculus and he developed a several fundamental rules that are still used in modern calculus[1][13][14]. These significant rules include the product rule, quotient rule, chain rule, and power rule. It's important for us to remeber those rule because they are the fundamental tools in modern calculus and are still widely used in modern mathematics and applications. Figure 22 provided the general formula for those rules, it allows us to find its derivative.

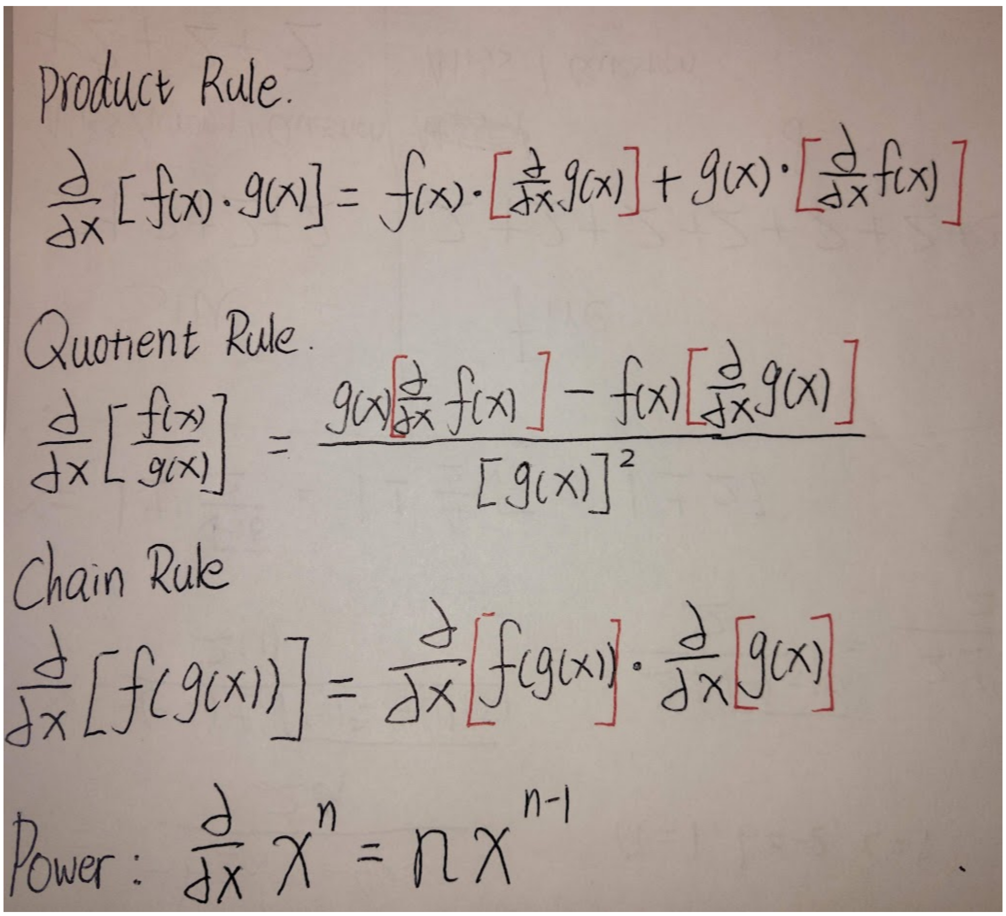


Figure 22: General formula in Leibniz's Calculus

### 3.5 Deriving the Newton-Leibniz Formula from Leibniz's Perspective and Method

During the 17th century, Newton and Leibniz both found out a significant formula independently. Although nobody sure Newton and Leibniz would love having their names associated together in a formula because of the horrible relationship between both of them during their lifetimes. The Mathematicians in later years decided to name the formula by both Newton's name and Leibniz's name, which is commonly known as Newton-Leibniz formula. The formula greatly simplifies the calculation of definite integrals and it's a powerful bridge connect between differential calculus and integral calculus. Hence, I personally acknowledge that the Newton-Leibniz formula is successful. Let's use the fact from chapter 3.1, 3.2 and 3.3 and take a deep look at how Leibniz derived the formula[9] [14].


Figure 23: what $f(b) - f(a)$ represents?

Given the follow Figure 23. Let $f(x)$ be the curve line and $a$,$b$ be two points, then Its function value is the definite integral of the function on the interval [a,b]. We first need to know what $f(b) - f(a)$ represents?[9] In the following, I will give an easy example and a general example.

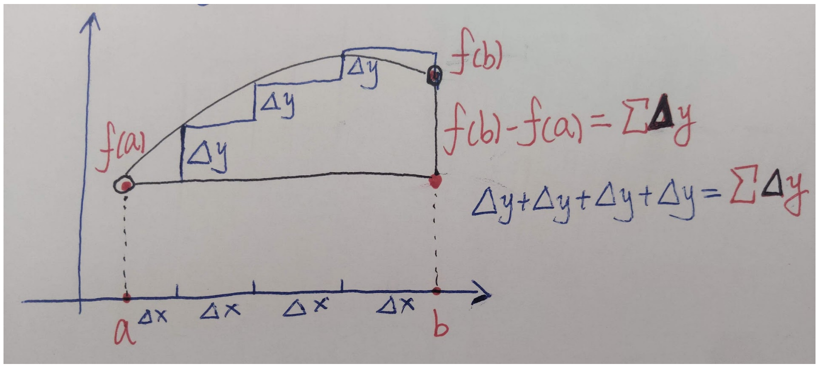


Figure 24: Example with four equal parts

To show what $f(b) - f(a)$ represents? The easy examples are Figure 24, Let's divide the interval [a,b] into four infinitesimals $\triangle x$[11]. Each $\triangle x$ symbol represents a small increment or change in the x. If we divide the interval [a, b] into four equal parts such that the sum of the whole (In Figure 24, $\sum \triangle y$) is equal to the sum of its parts (In Figure 24, $\triangle y + \triangle y + \triangle y + \triangle y$), Leibniz conclude that $f(b) - f(a) = \sum \triangle y$[1][6].

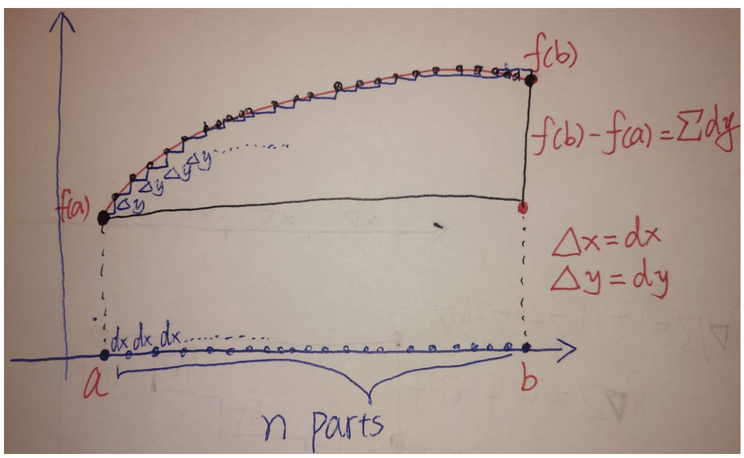


Figure 25: Example with $n$ equal parts

Let us expand the framework further and consider dividing the interval [a, b] into n equal parts[11]. In Chapter 3.2, Leibniz proved that the width of each rectangle must be an infinitesimal $\triangle x$, Leibniz also points out that $\triangle x = dx$. In other words, the distance corresponding to each interval $\triangle x$ becomes dx, and the corresponding $\triangle y$ is also denoted as $dy$. Leibniz had show: $f(b) - f(a) = \sum dy$. In chapter 3.3 Leibniz show $dy = f'(x)dx$. Hence, $f(b) - f(a) = \sum dy = \sum f'(x)dx = \int_a^b f'(x)dx$[1][6][14]. This is the Newton-Leibniz formula, which is the basic formula of calculus.

# 4 Conclusion

## 4.1 Leibniz's contributions

Leibniz's contributions to calculus were significant and foundational. He developed calculus based on geometric inspiration. He introduced the concept of "differential triangles" and quickly established numerous theorems. Through his study of differential triangles, Leibniz gradually realized the essence of finding the area under a curve and finding the tangent to a curve. He discovered the inverse relationship between differentiation and integration. With the concept of inverse relationship, He derived the Newton-Leibniz formula. Leibniz's famous masterpiece was the notation in calculus. He created the notation for differentials $(dx, dy)$[5] and the integral sign $\int$[5]. This notation greatly facilitated the expression and manipulation of mathematical ideas. Leibniz's calculus notation has been widely accepted and presented in modern calculus. To sum up, Leibniz's contributions to calculus revolutionized the field of mathematics. His concepts and notation provided a powerful framework for solving problems in calculus and laid the groundwork for further developments in the subject.[1]

## 4.2 Newton's contributions

Newton's contributions to calculus are significant and foundational. The origins of Newton's calculus lie in the field of kinematics. During Newton's time, the concepts of calculus were not precisely defined as they are in modern times. Therefore, Newton used the term "fluxions" and introduced the concept of instantaneous rates of change[2]. He used the concept of fluxions to describe the instantaneous rate of change of a function, similar to the concept of derivatives. Through continuous calculations, Newton developed a relatively complete algorithm for differentiation and integration in the 17th century, known as the Method of Fluxions[2]. This method, created by Newton, provided a framework for performing differentiation and integration operations on functions and was applied to solve mathematical and physical problems. The Method of Fluxions enabled the determination of tangents to curves, computation of areas, and solution of equations, among other applications. The fundamental ideas of fluxions build a strong framework for the development of calculus[1][2]. It stands as a significant step in the progression of calculus but It's also a legendary history in the history of mathematics and science.

### 4.3 The similarities and differences between Newton and Leibniz in calculus

Similarities

- Independent Development: Both Newton and Leibniz independently developed the foundations of calculus.
- Newton and Leibniz proved that differentiation and integration are inverse operations relationship, which used to developing the Newton-Leibniz formula.
- Use of the Concept of "Moment" and Infinitesimals: Both Newton and Leibniz based their calculus on the concept of "moment" and employed infinitesimals. However, neither of them provided a rigorous and clearlydefinition of infinitesimals.

Differences

- They had different starting points in developing calculus. Newton developed calculus based on his studies in kinematics and the study of motion. Leibniz, on the other hand, focused on studying the tangent lines of curves and the calculation of areas under curves.
- Newton first developed the concept of derivatives(which is fluxion) and later introduced the concept of integration. Leibniz, on the other hand, first developed the concept of integration and then introduced the concept of derivatives.
- Notation: Leibniz introduced infinitesimal differentials $(dx, dy)$ and the integral symbol $\int$, while Newton used his own notation $\dot{x}$ based on "fluxions" and "fluents."
- Newton was more concerned with establishing the calculus system and its fundamental methods. Leibniz focused on developing and promoting operational formulas and techniques within calculus.

### 4.4 Advancements in Calculus since Newton and Leibniz

After the contributions from Newton and Leibniz to calculus in the 17th century, calculus has become a significant versatile mathematical subject. Newton and Leibniz deriving the framework of calculus and successfully using it to solve many problems; however, calculus had inherent errors due to its unstable foundations. This also led 18th-century mathematicians to focus more on the practical applications of calculus and less on its rigor. Infinitesimals is a important example. In early calculus, mathematicians used infinitesimals to describe important concepts such as rates of change and differentials, but this raised many issues regarding the validity and consistency of infinitesimals. This lack of rigor caused controversies and criticisms within the mathematical community at that time. Here is some improvement after Newton and Leibniz.

- Mathematical Rigor and Precision:

  In the 19th and 20th centuries, mathematicians recognized the need to establish rigorous foundations for calculus. Augustin-Louis Cauchy[20] and Karl Weierstrass[21] led this endeavor by introducing the epsilon-delta method[19]. This approach provided a precise definition of limits, continuity, and convergence, solidifying the conceptual underpinnings of calculus and making it a more robust and reliable mathematical tool.

- Set Theory and Real Numbers:

  The introduction of set theory by Georg Cantor[22] had a profound impact on calculus. It provided a powerful framework for handling infinite processes and enabled a more systematic approach to the notion of real numbers. This development expanded the scope of calculus, facilitating deeper explorations into the continuity and differentiability of functions.

- Notion of Functions:

  The concept of a function underwent significant refinement and formalization. With the contributions of mathematicians like Karl Weierstras[21]s and Richard Dedekind[23], functions were defined more rigorously, leading to a deeper understanding of their properties and behaviors. This formalization played a crucial role in advancing various aspects of calculus."

- Limits and Derivatives:

  In the quest for mathematical rigor, mathematicians formalized the concepts of limits and derivatives. This rigorous treatment ensured that limits and derivatives were precisely defined and enabled a more rigorous analysis of functions and their rates of change[19].

- Riemann integral:

  The Riemann integral is relate to the real analysis. Riemann gave for the first time a precise definition of the integral of a function over a given interval. Some technical deficiencies of the Riemann integral can be repaired by the later Riemann-Stirges integral and Lebesgue integral[24].

- Riemann Sum and Partition:

  Riemann introduced the concept of a Riemann sum. He realized that by dividing the interval of integration into smaller subintervals, which also known as partitions[25]. and approximating the function within each subinterval, he could obtain better approximations of the total area under the curve. Riemann sums formed the basis for his integral definition and provided a systematic way to evaluate definite integrals[25].

- Advancements in Integration Theory:

  Mathematicians such as Henri Lebesgue[26] and Karl Weierstrass[21] made significant advancements in integration theory. Beyond the traditional Riemann integral, the development of Lebesgue integration addressed issues of convergence, allowing for a more comprehensive framework capable of handling a wider class of functions. This extended the power and applicability of calculus in various mathematical contexts.

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