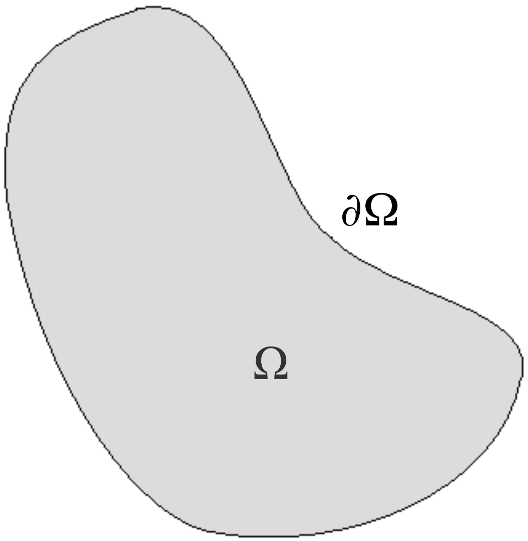
Deterministic boundary value problem

General boundary value problem

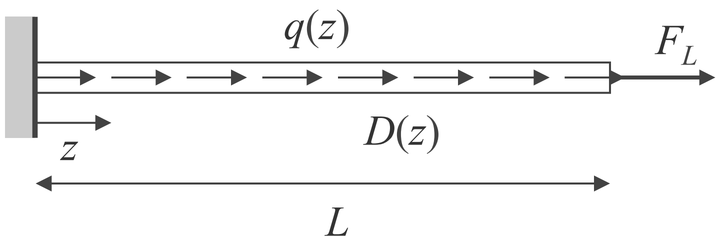


$$\mathbf{L}(\mathbf{z}) \cdot \mathbf{u}(\mathbf{z}) + \mathbf{f}(\mathbf{z}) = 0 \qquad \text{in } \Omega$$

$$\text{given} \qquad \mathbf{u}(\mathbf{z})\big|_{\Gamma_0} = 0, \nabla \mathbf{u}(\mathbf{z})\big|_{\Gamma_1} = \bar{\mathbf{t}} \qquad \Gamma_0 \cup \Gamma_1 = \partial\Omega$$

Deterministic boundary value problem

Static equilibrium of a bar



$$\frac{d}{dz}\left(D(z)\frac{du(z)}{dz}\right)+q(z)=0 \qquad \text{in } [0,L]$$

$$\text{given} \qquad u(0)=0, \left.\frac{du}{dz}\right|_L=\frac{F_L}{D(L)}$$

Stochastic boundary value problem

General boundary value problem

$$\mathbf{L}(\mathbf{z},\theta)\cdot\mathbf{u}(\mathbf{z},\theta)+\mathbf{f}(\mathbf{z},\theta)=0 \qquad \text{in } \Omega$$

$$\text{given} \qquad \mathbf{u}(\mathbf{z},\theta)\big|_{\Gamma_0}=0, \nabla\mathbf{u}(\mathbf{z},\theta)\big|_{\Gamma_1}=\bar{\mathbf{t}} \qquad \Gamma_0\cup\Gamma_1=\partial\Omega$$

Static equilibrium of a bar

$$\frac{d}{dz}\left(D(z,\theta)\frac{du(z,\theta)}{dz}\right)+q(z,\theta)=0 \qquad \text{in } [0,L]$$

$$\text{given} \qquad u(0,\theta)=0, \left.\frac{du}{dz}\right|_L=\frac{F_L(\theta)}{D(L,\theta)}$$

Stochastic boundary value problem

Static equilibrium of a bar

$$\frac{d}{dz}\left(D(z,\theta)\frac{du(z,\theta)}{dz}\right)+q(z,\theta)=0 \qquad \text{in } [0,L]$$

$$\text{given} \qquad u(0,\theta)=0, \left.\frac{du}{dz}\right|_L = \frac{F_L(\theta)}{D(L,\theta)}$$

- Choose a complete functional space Θ
- Define a subspace Θ_n of Θ spanned by a finite set of functions $\{N_i(z), i=1,\ldots,n\}$
- Approximate $u(z,\theta)$ as

$$u_h(z,\theta)=\sum_{i=1}^{n}u_i(\theta)N_i(z)$$

Galerkin method

Residual:

$$R\left(u_h(z,\theta),z\right) = \frac{d}{dz}\left(D(z,\theta)\frac{du_h(z,\theta)}{dz}\right) + q(z,\theta)$$

Inner product of the residual with any test function $v(z) \in \Theta_n$ that satisfies $v(0)=0$ vanishes

$$\Rightarrow \int_0^L v(z) R\left(u_h(z,\theta),z\right) dz = 0$$

$$\Rightarrow \ldots \Rightarrow \int_0^L \frac{dv(z)}{dz} D(z,\theta) \frac{du_h(z,\theta)}{dz} dz = \int_0^L v(z) q(z,\theta)\, dz + v(L) F_L(\theta)$$

Set:

$$u_h(z,\theta) = \sum_{i=1}^{n} u_i(\theta) N_i(z) \qquad v(z) = \sum_{i=1}^{n} v_i N_i(z)$$

...

Galerkin method

…

$$\mathbf{K}(\theta)\mathbf{u}(\theta) = \mathbf{F}(\theta)$$

where:

Stochastic stiffness matrix:

$$K_{ij}(\theta) = \int_0^L \frac{dN_i(z)}{dz} D(z,\theta) \frac{dN_j(z)}{dz} dz$$

Stochastic force vector:

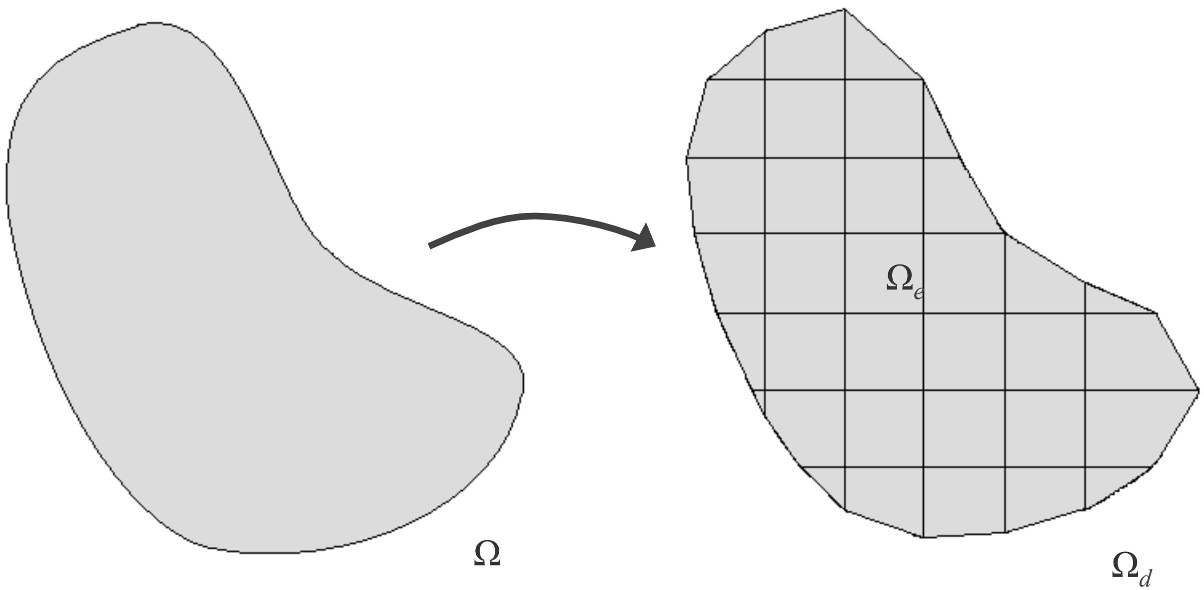
$$F_i(\theta) = \int_0^L N_i(z) q(z,\theta)\, dz + N_i(L) F_L(\theta)$$

Stochastic vector of random unknowns:

$$\mathbf{u}(\theta) = \left[u_1(\theta); u_2(\theta); \cdots; u_n(\theta)\right]$$

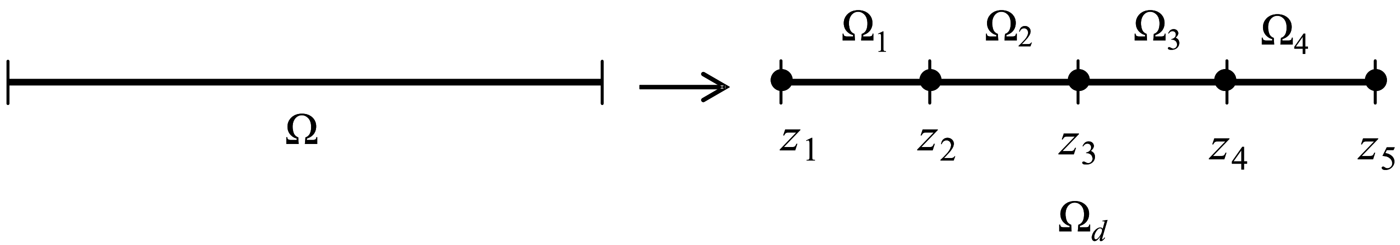
Finite element shape functions

Discretize the domain with a finite number of polygonal elements

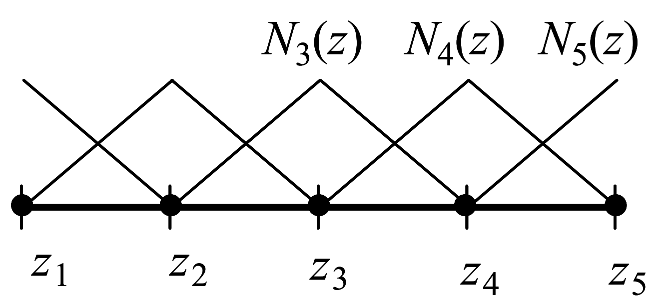


Finite element shape functions

Discretize the domain with a finite number of polygonal elements



Choose the basis functions as piece-wise linear polynomials that interpolate linearly between the nodes



Properties of finite element shape functions

- The i th shape function equals one at node i and zero at all other nodes

$$N_i(z) = \begin{cases} 1 & \text{for } z = z_i \\ 0 & \text{for } z = z_j,\, j \neq i \end{cases}$$

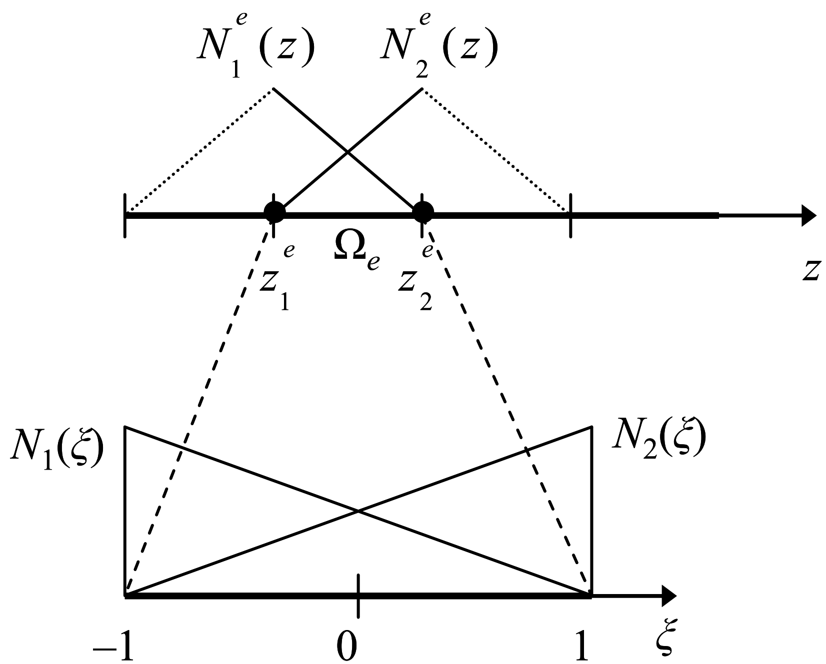
- Partition of unity

$$\sum_{i=1}^{n} N_i(z) = 1$$

Local (element) shape functions

- Define a linear mapping to a standard elemental domain:

$$\xi : [z_1^e, z_2^e] \rightarrow [-1,1]$$



- Local shape functions

$$N_1(\xi) = \frac{1}{2}(1-\xi) \qquad N_2(\xi) = \frac{1}{2}(1+\xi)$$

Element stiffness matrix and force vector

$$\mathbf{K}(\theta) = \sum_{e=1}^{n_e} \mathbf{K}^e(\theta) \qquad \mathbf{F}(\theta) = \sum_{e=1}^{n_e} \mathbf{F}^e(\theta)$$

Element stochastic stiffness matrix:

$$\mathbf{K}^e(\theta) = \frac{1}{2h^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\int_{-1}^{1} D(z,\theta)\,d\xi$$

Element stochastic force vector:

$$\mathbf{F}^e(\theta) = \frac{h^e}{2}\int_{-1}^{1} q(z,\theta)\begin{bmatrix} \dfrac{1-\xi}{2} \\ \dfrac{1+\xi}{2} \end{bmatrix} d\xi$$

Note: If e is the last element

$$F_2^e(\theta) = \frac{h^e}{2}\int_{-1}^{1} N_2(\xi)q(z,\theta)\,d\xi + F_L(\theta)$$

Discretization of random fields

Approximation of $D(z,\theta)$ and $q(z,\theta)$ with a finite number of random variables

$$D(z,\theta) \approx D(z,\mathbf{X}) \qquad q(z,\theta) \approx q(z,\mathbf{X})$$

Mean-square and variance discretization errors:

$$err_{\text{m-s}}(z) = \mathrm{E}\left[\left(D(z,\theta) - D(z,\mathbf{X})\right)^2\right] \qquad err_{\mathrm{Var}}(z) = \mathrm{Var}\left[D(z,\theta) - D(z,\mathbf{X})\right]$$

Global discretization errors:

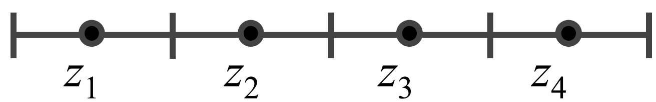
$$\overline{err}_{\text{m-s}} = \frac{1}{L}\int_0^L err_{\text{m-s}}(z)\,dz$$

$$err_{\text{m-s}} = \sup_{z\in\Omega} err_{\text{m-s}}(z)$$

Point discretization methods

- Choose a number of points in the domain
- Approximate the random field in terms of the random variables corresponding to these points

Midpoint method



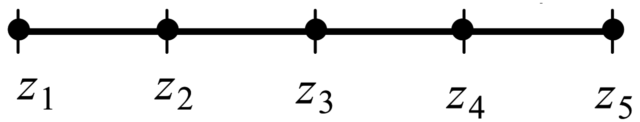
Random variables in the discretization: $X_i = D(z_i, \theta)$

Random field approximation: $D(z, \mathbf{X}) = X_i \qquad \text{for } z \in \Omega_i$

Point discretization methods

- Choose a number of points in the domain
- Approximate the random field in terms of the random variables corresponding to these points

Shape function method



Random variables in the discretization: $X_i = D(z_i, \theta)$

Random field approximation: $D(z, \mathbf{X}) = \sum_{i=1}^{n} X_i N_i(z)$

Note: The random field representing the displacements $u(z, \theta)$ is discretized by the shape function method

Point discretization methods

Example

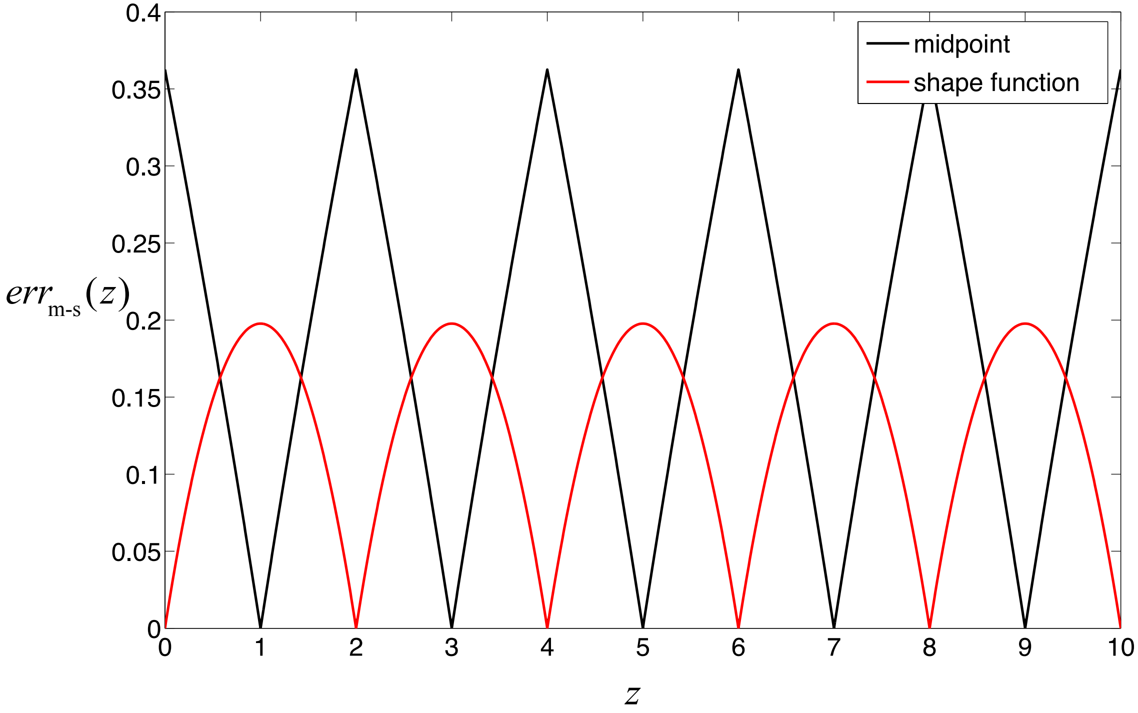
- 1D homogeneous Gaussian random field in $[0, 10]$
- Autocorrelation coefficient function: $\rho(\zeta) = \exp\left(-\frac{2|\zeta|}{\theta}\right)$

Point discretization methods

Example

Comparison of midpoint and shape function methods

Local point-wise error

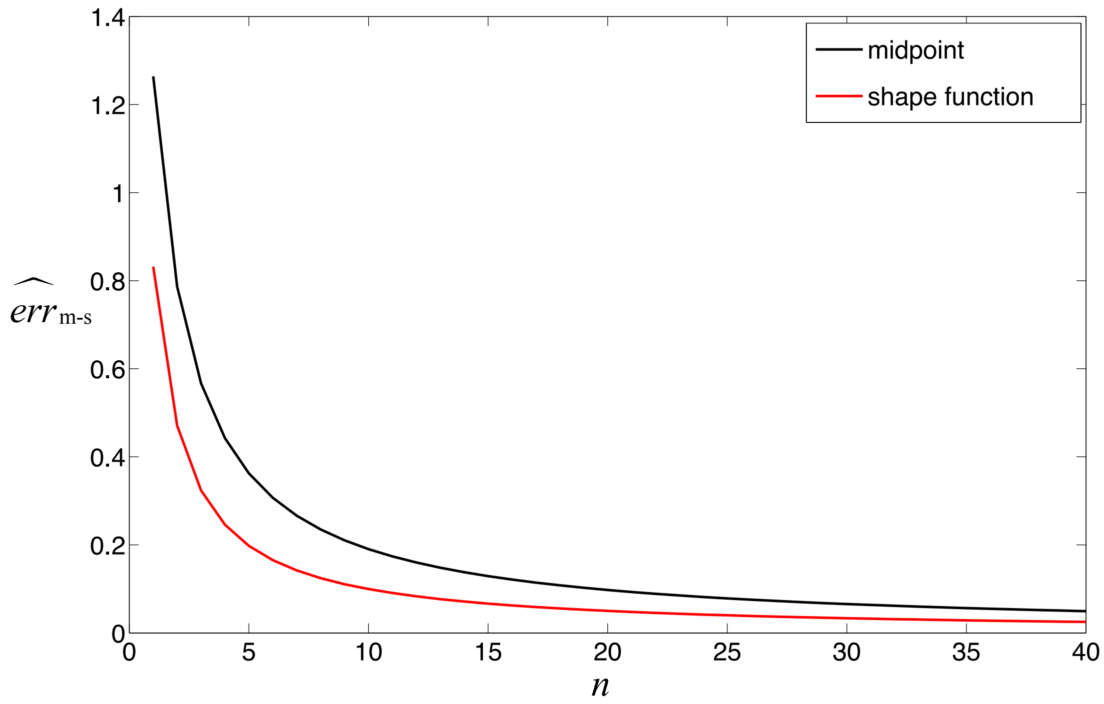


Point discretization methods

Example

Comparison of midpoint and shape function methods

Global error

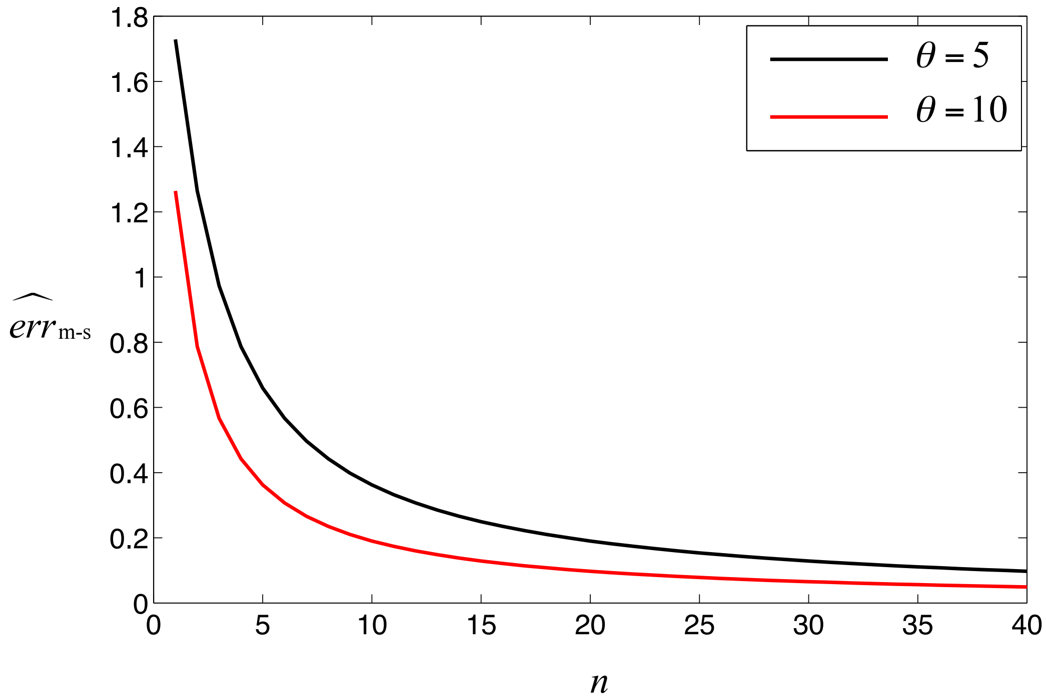


Point discretization methods

Example

Midpoint method: Influence of the scale of fluctuation θ

Global error



Karhunen-Loève expansion

Karhunen-Loève expansion of a second order random field:

$$D(z,\theta) = \mu_D(z) + \sum_{i=1}^{\infty} \sqrt{\lambda_i} \cdot \varphi_i(z) \cdot U_i(\theta)$$

Fredholm integral eigenvalue problem:

$$\int_{\Omega} \Gamma_{DD}(z_1, z_2)\varphi_i(z_2)\, dz_2 = \lambda_i \varphi_i(z_1) \qquad \forall i = 1, 2, \ldots$$

Orthogonality of eigenfunctions: $\left\langle \varphi_i(z), \varphi_j(z) \right\rangle = \int_{\Omega} \varphi_i(z)\varphi_j(z)\, dz = \delta_{ij}$

Spectral decomposition of the auto-covariance function:

$$\Gamma_{DD}(z_1, z_2) = \sum_{i=1}^{\infty} \lambda_i \varphi_i(z_1)\varphi_i(z_2)$$

Karhunen-Loève expansion

Random variables in the expansion

$$U_i(\theta) = \frac{1}{\sqrt{\lambda_i}} \int_{\Omega} \left(D(z,\theta) - \mu_D(z) \right) \varphi_i(z) \, dz$$

Properties of the random variables:

- They have zero mean: $\mathrm{E}\left[U_i\right] = 0$
- The are orthonormal (uncorrelated, unit standard deviation) $\mathrm{E}\left[U_i U_j\right] = \delta_{ij}$
- If $D(z,\theta)$ is Gaussian they are **independent standard normal**

Karhunen-Loève expansion

Truncated Karhunen-Loève expansion (after m largest eigenvalues and corresponding eigenfunctions):

$$D(z,\mathbf{U}) = \mu_D(z) + \sum_{i=1}^{m} \sqrt{\lambda_i} \cdot \varphi_i(z) \cdot U_i$$

The set of $\{\varphi_i(z)\}$ form a complete orthogonal basis in $L^2(\Omega)$ and are optimal with respect to the mean square truncation error

Karhunen-Loève expansion

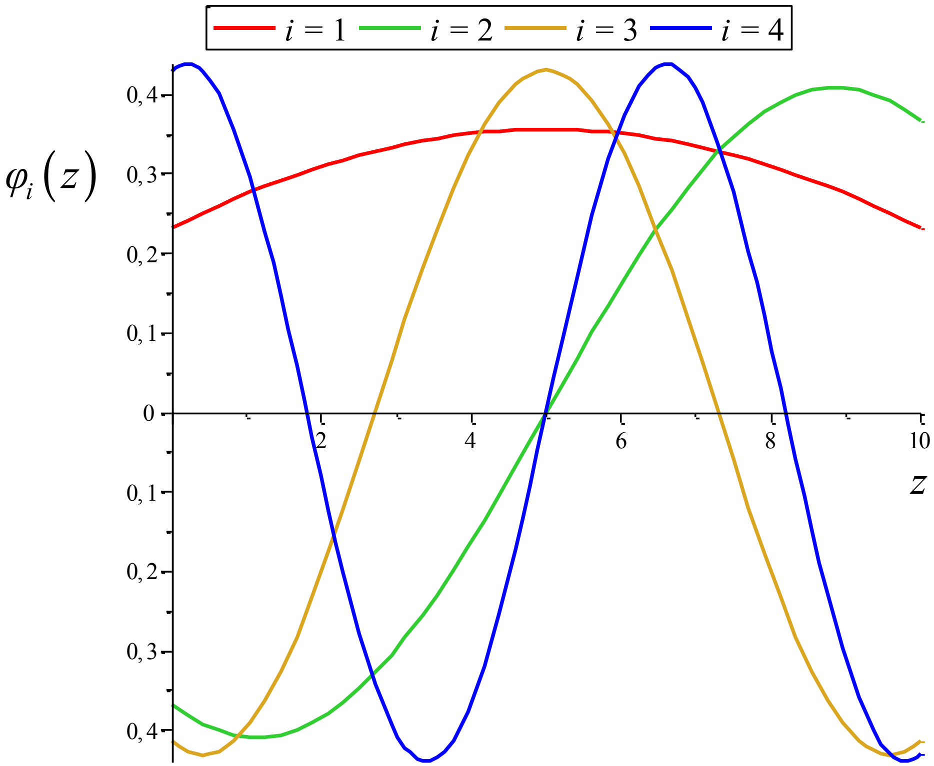
Example

- 1D homogeneous Gaussian random field in $[0, 10]$
- Autocorrelation coefficient function: $\rho(\zeta) = \exp\left(-\frac{2|\zeta|}{\theta}\right)$

Karhunen-Loève expansion

Example

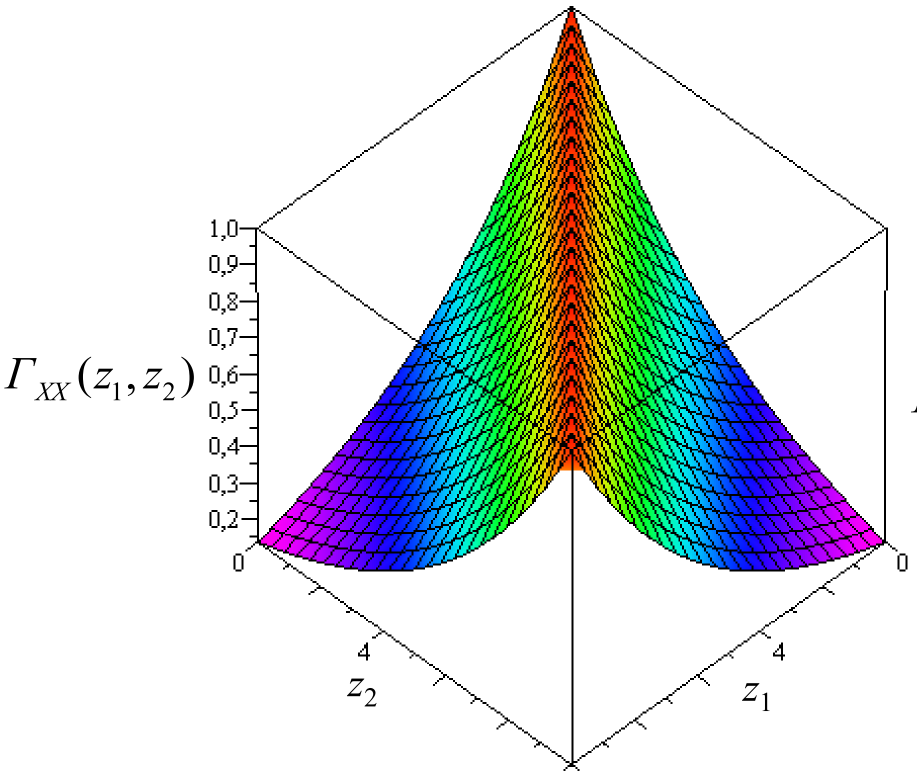
Eigenfunctions



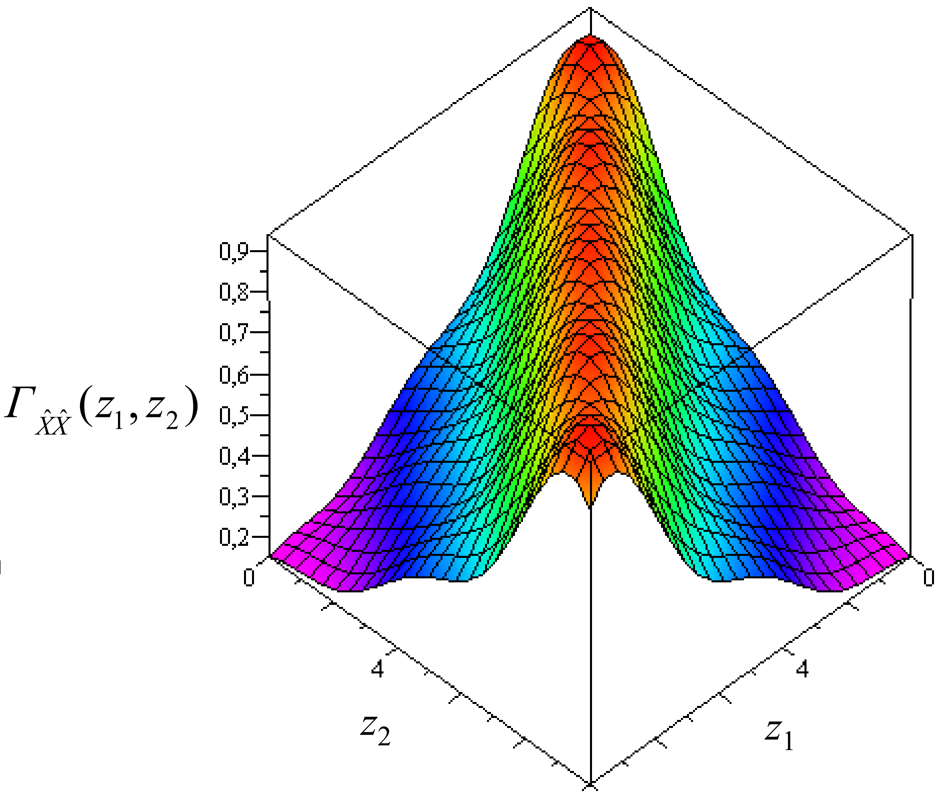
Karhunen-Loève expansion

Example

Original covariance



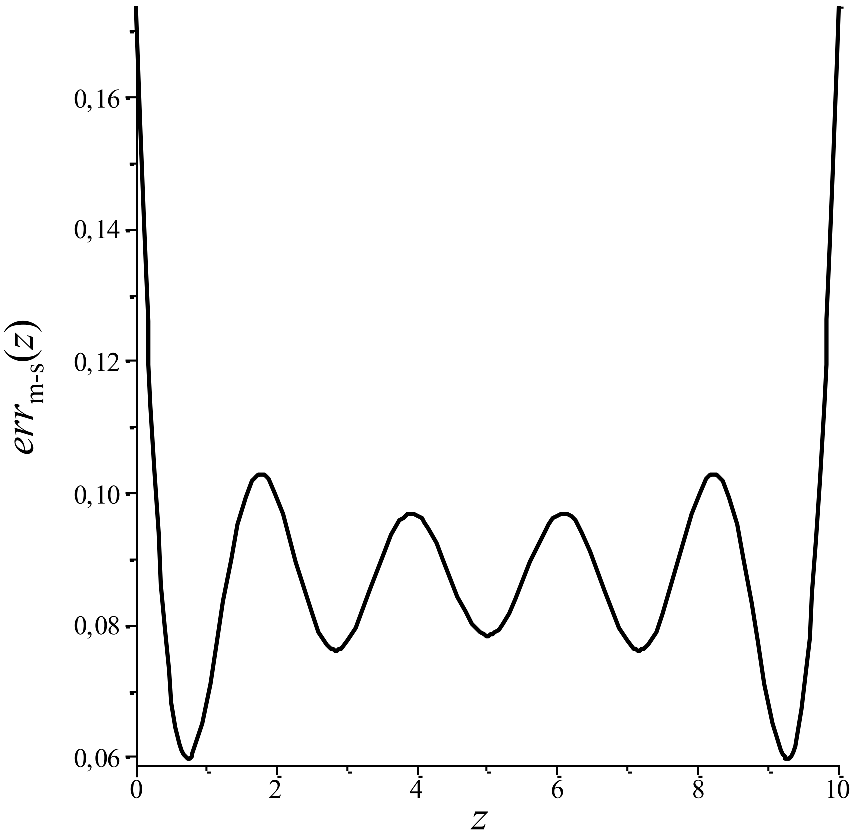
Approximation with 5 terms



Karhunen-Loève expansion

Example

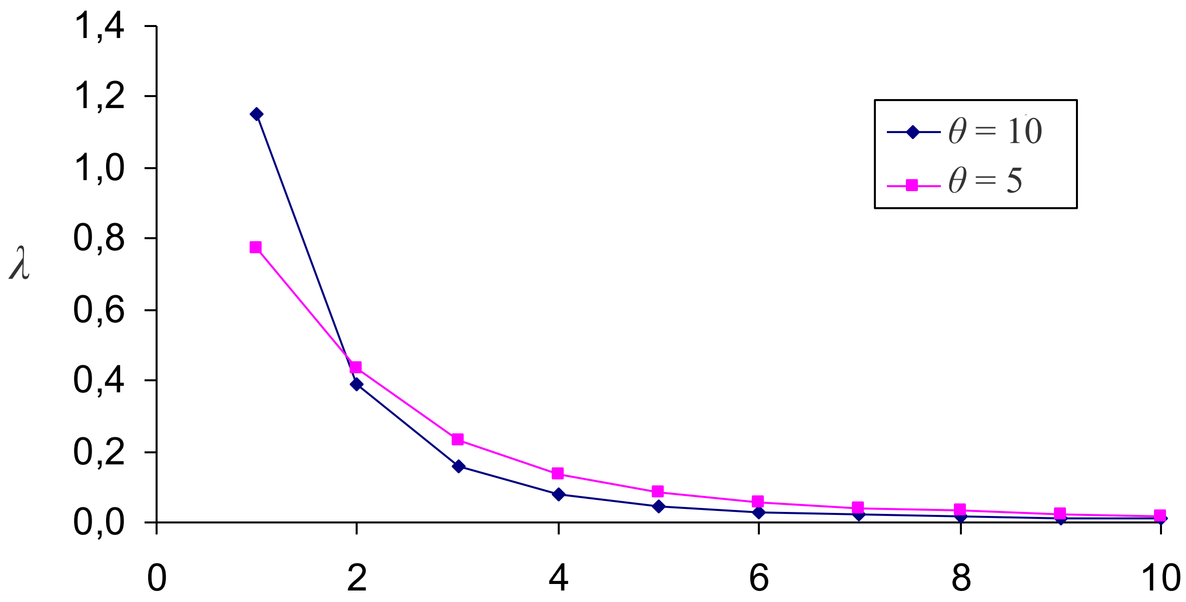
Local (point-wise) error with 5 terms



Karhunen-Loève expansion

Example

Eigenvalues – influence of scale of fluctuation θ



KL expansion of non-Gaussian random field

Marginal distribution of random field: $F_{D(z)}(x,z)$

- Express as a function of a standard Gaussian random field:

Set

$$F_{D(z)}(D(z,\theta),z) = \Phi(U(z,\theta))$$

$$\Rightarrow D(z,\theta) = F_{D(z)}^{-1}(\Phi(U(z,\theta)),z)$$

- Perform KL expansion of Gaussian random field:

$$U(z,\mathbf{U}) = \sum_{i=1}^{m} \sqrt{\lambda_i} \cdot \varphi_i(z) \cdot U_i$$

where

$$\int_{\Omega} \rho_{UU}(z_1,z_2)\varphi_i(z_2)\,dz_2 = \lambda_i \varphi_i(z_1)$$

…

KL expansion of non-Gaussian random field

…

$$\Rightarrow D(z,\mathbf{U}) = F_{D(z)}^{-1}\left[\Phi\left(\sum_{i=1}^{m}\sqrt{\lambda_i}\cdot\varphi_i(z)\cdot U_i\right),z\right]$$

Example: Lognormal random field

$$D(z,\theta) = \exp\left(\mu_U(z) + \sigma_U(z)U(z,\theta)\right)$$

KL representation:

$$D(z,\mathbf{U}) = \exp\left(\mu_U(z) + \sigma_U(z)\sum_{i=1}^{m}\sqrt{\lambda_i}\cdot\varphi_i(z)\cdot U_i\right)$$

Stochastic stiffness matrix and force vector

Express random field with their KL expansion:

$$D(z,\theta) \approx D(z,\mathbf{U}) \qquad\qquad q(z,\theta) \approx q(z,\mathbf{U})$$

Element stochastic stiffness matrix:

$$\mathbf{K}^e(\mathbf{U}) = \frac{1}{2h^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_{-1}^{1} D(z,\mathbf{U})\, d\xi$$

Element stochastic force vector:

$$\mathbf{F}^e(\mathbf{U}) = \frac{h^e}{2}\int_{-1}^{1} q(z,\mathbf{U}) \begin{bmatrix} \dfrac{1-\xi}{2} \\ \dfrac{1+\xi}{2} \end{bmatrix} d\xi$$

Assemble matrices:

$$\mathbf{K}(\mathbf{U})\mathbf{u}(\mathbf{U}) = \mathbf{F}(\mathbf{U})$$