Advanced Stochastic Finite Element Methods

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Advanced Stochastic Finite Element Methods

Elective course for MSc and PhD students

3 ECTS

Lecturer: Iason Papaioannou (iason.papaioannou@tum.de)

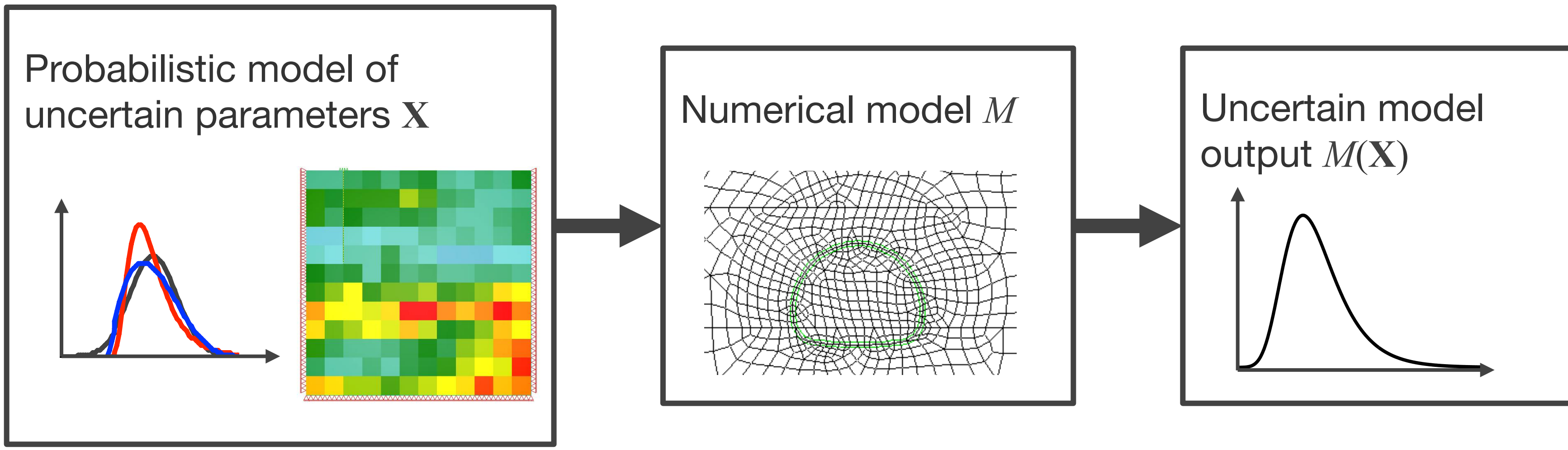
Guest lecturer: Elisabeth Ullmann (elisabeth.ullmann@ma.tum.de)

Course dates: June 6 – 10, 9am – 12:30pm and 14pm – 16pm

Lecture room: N2619

Course language: English

Motivation


Probabilistic model of uncertain parameters $\mathbf{X}$
Numerical model M
Uncertain model output $M(\mathbf{X})$

Lecture Contents

1. Advanced Monte Carlo methods
   a) Advanced sampling (LHS, QRS)
   b) Multi-level Monte Carlo
2. Spectral stochastic finite element methods
   a) Spectral representation of random variables – PC expansion
   b) Stochastic Galerkin method
   c) Stochastic collocation methods
   d) Sparse PC expansions
3. Stochastic finite elements and reliability
   a) Definition of the reliability problem
   b) Reliability analysis with PC expansions

Class schedule

Day	Lectures 9am – 12:30pm	Tutorial 2pm – 4pm	Lecture notes
Monday June 6	Introduction, Review of probability theory, linear finite elements, discretization of random fields	Discretization of random fields (the Karhunen-Loève expansion)	Chapters 1 – 4
Tuesday June 7	Monte Carlo simulation advanced sampling methods	Finite elements with random field inputs	Chapters 4 – 5
Wed. June 8	Polynomial chaos expansion Stochastic Galerkin method	Monte Carlo finite element methods	Chapter 6
Thursday June 9	Stochastic collocation methods Sparse PCE	PCE represention of finite element solution	Chapter 6
Friday June 10	9.00-10.30: Stochastic finite elements and reliability 11.00 – 12.30: Multilevel Monte Carlo method (lecture held by Prof. Ullmann)		

Lecture organization

- Lecture: White board/slides
- Homework-exercises
- Examination: Project work
  - Students must hand in a Matlab code and a short report
  - Oral presentation of results
- Prerequisites
  - Linear finite element method
  - Probability theory (random variables/random fields)
  - Basic knowledge of functional analysis is recommended

Literature

Lecture Notes
Stochastic Finite Element Methods
I Papaioannou
TU München

Numerical methods for stochastic computations: A spectral method approach
D Xiu
Princeton University Press, 2010

Stochastic finite elements and reliability: A state-of-the-art report
B Sudret, A Der Kiureghian
UC Berkeley, 2000

Stochastic finite elements: A spectral approach
RG Ghanem, PD Spanos
Dover Publications, 2004

Random variables

Random variable – Mathematical tool for modeling uncertain quantities

- Random variables map possible outcomes of an experiment (collected in the sample space S) to the real numbers
- A random variable is represented by a capital letter, e.g. X
- An outcome of a random variable is represented by a lower case letter, e.g. x

Random variables

Cumulative distribution function (CDF) – Function that returns the probability that a random variable X is smaller than or equal to an outcome x

$$F_X(x) = P(X \leq x)$$

Notes:

- The CDF is a non-decreasing function
- The CDF has limits $F_X(-\infty) = 0$, $F_X(\infty) = 1$
- $P(a < X \leq b) = F_X(b) - F_X(a)$

Discrete random variables

Random variables with discrete number of outcomes

Probability Mass Function (PMF) – Function that returns the probability that a discrete random variable X takes a specific value x

$$p_X(x) = P(X = x)$$

CDF of discrete random variable: $$F_X(x) = \sum_{x_i \leq x} p_X(x_i)$$

Normalization rule: $$\sum_{\forall x_i} p_X(x_i) = 1$$

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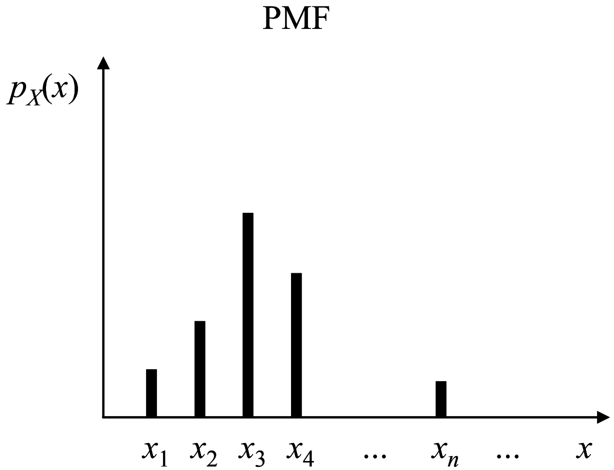

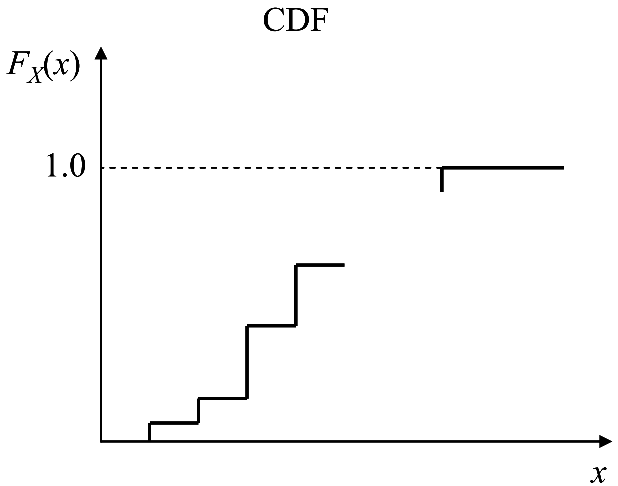

Continuous random variables

Random variables which can take any value within one or more intervals

Probability Density Function (PDF) – Function whose integral over an interval gives the probability that X takes a value within this interval

$$f_X(x)dx = P(x < X \leq x + dx)$$

CDF of continuous random variable: $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt \quad \Rightarrow f_X(x) = \frac{dF_X(x)}{dx}$

Normalization rule: $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

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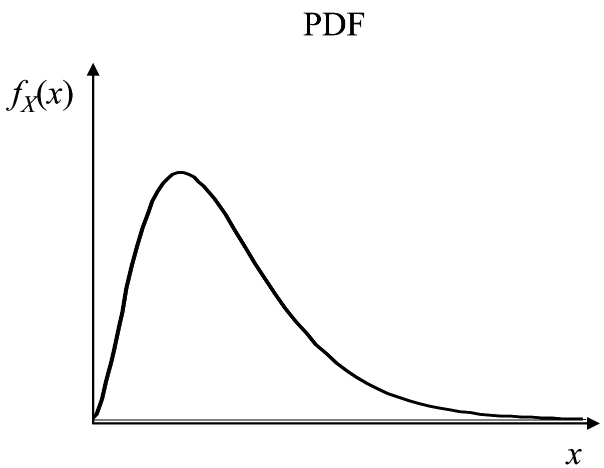

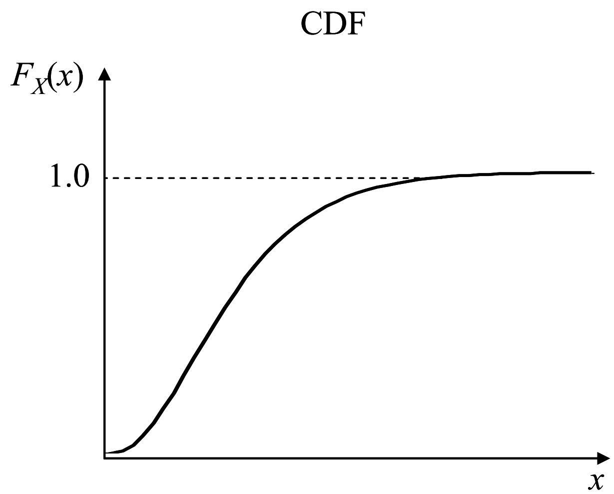

Expectations and moments

Expectation of a function

- Discrete random variable $$\mathrm{E}[g(X)] = \sum_{\forall x_i} g(x_i) \cdot p_X(x_i)$$
- Continuous random variable $$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\, dx$$

Expectations and moments

n-th moment of a random variable

$$\mu_n = \mathrm{E}[X^n]$$

Mean value

$$\mu_X = \mathrm{E}[X]$$

n-th central moment of a random variable

$$\mu_n' = \mathrm{E}[(X-\mu_X)^n]$$

Variance

$$\begin{aligned}\mathrm{Var}[X] &= \mathrm{E}[(X-\mu_X)^2] \\ &= \mathrm{E}[X^2]-\mathrm{E}[X]^2\end{aligned}$$

Standard deviation

$$\sigma_X = \sqrt{\mathrm{Var}[X]}$$

Coefficient of variation

$$\delta_X = \frac{\sigma_X}{|\mu_X|}, \qquad \mu_X \neq 0$$

Normal (Gaussian) distribution

Defined by two parameters, the mean and the standard deviation $X \sim N(\mu,\sigma)$

PDF: $f_X(x) = \frac{1}{\sigma}\varphi\left(\frac{x-\mu}{\sigma}\right)$ CDF: $F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$

$\varphi(u), \Phi(u)$: PDF, CDF of the **standard normal random variable** $U \sim N(0,1)$

$$\varphi(u) = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{u^2}{2}\right] \qquad \Phi(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{z^2}{2}\right] dz$$

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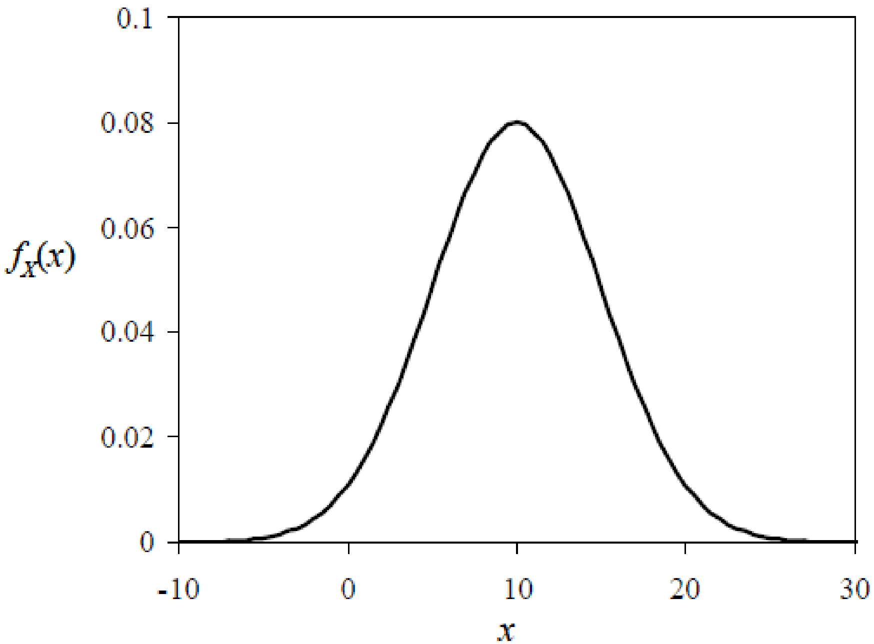
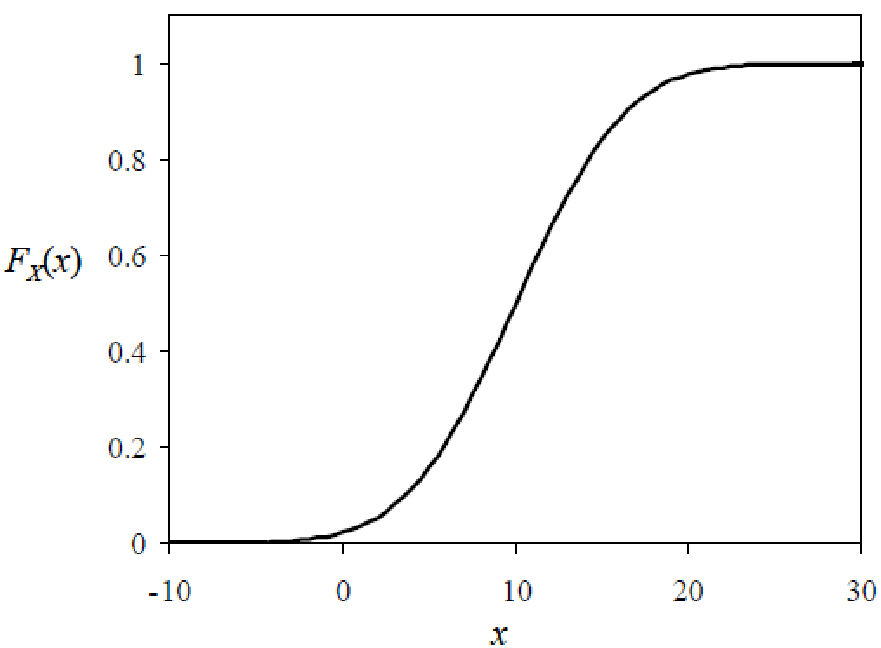


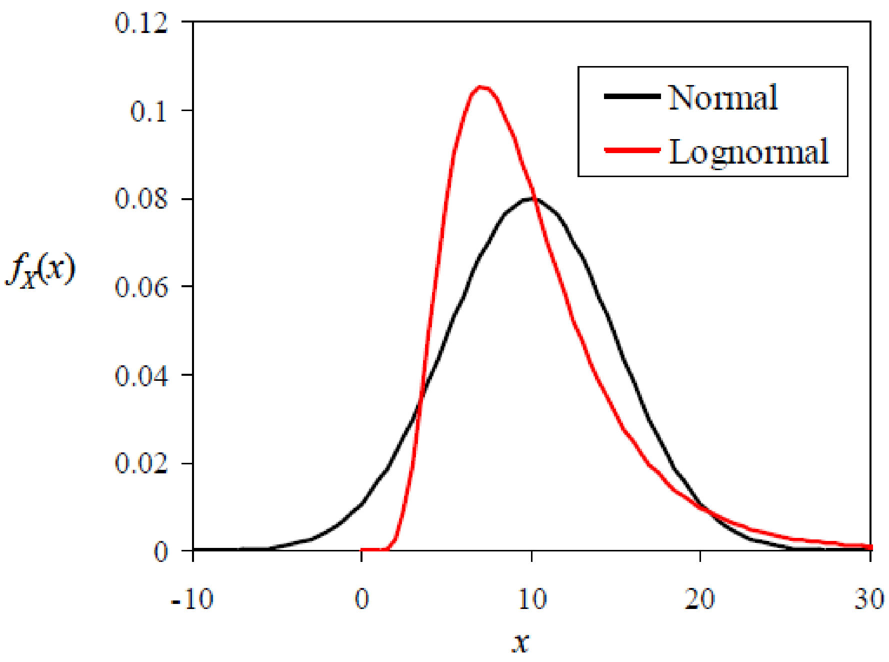


Lognormal distribution

The logarithm of a lognormal random variable is normal

$$X \sim LN(\mu,\sigma) \Rightarrow \ln(X) \sim N(\mu,\sigma)$$

Asymmetric distribution with strictly positive outcomes



Random vectors

Models of uncertain quantities that are observed simultaneously, e.g.

- Wave height and wave period
- Mechanical properties of the same material
- Load acting on structure and capacity of the structure

The individual random variables are gathered in a vector

$$\mathbf{X} = [X_1, X_2, \ldots, X_n]^\mathrm{T}$$

Random vectors

Joint CDF

e.g. for two random variables X, Y

$$F_{XY}(x, y) = P[(X \leq x) \cap (Y \leq y)]$$

Notes:

- The joint CDF is a non-decreasing function in each argument
- The joint CDF has limits $F_{XY}(-\infty, y) = 0$, $F_{XY}(x, -\infty) = 0$, $F_{XY}(\infty, \infty) = 1$

Random vectors

Discrete random vectors – Joint PMF

e.g. for two discrete random variables X, Y

$$p_{XY}(x, y) = P[(X = x) \cap (Y = y)]$$

Normalization rule: $$\sum_{\forall x_i} \sum_{\forall y_i} p_{XY}(x_i, y_i) = 1$$

Continuous random vectors – Joint PDF

e.g. for two continuous random variables X, Y

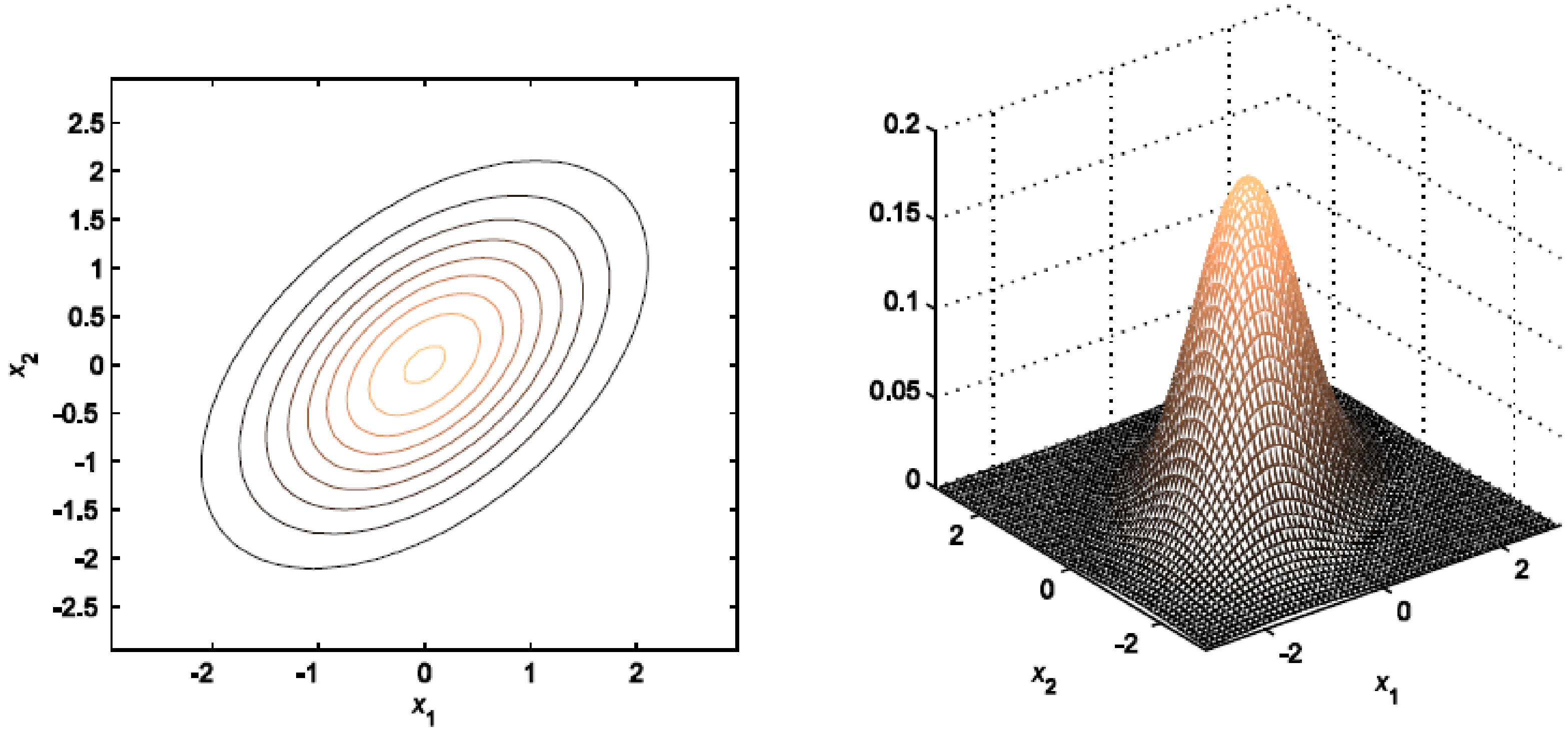
$$f_{XY}(x, y)dxdy = P[(x < X \leq x + dx) \cap (y < Y \leq y + dy)]$$

Normalization rule: $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx\, dy = 1$$

Random vectors

e.g. two continuous random variables

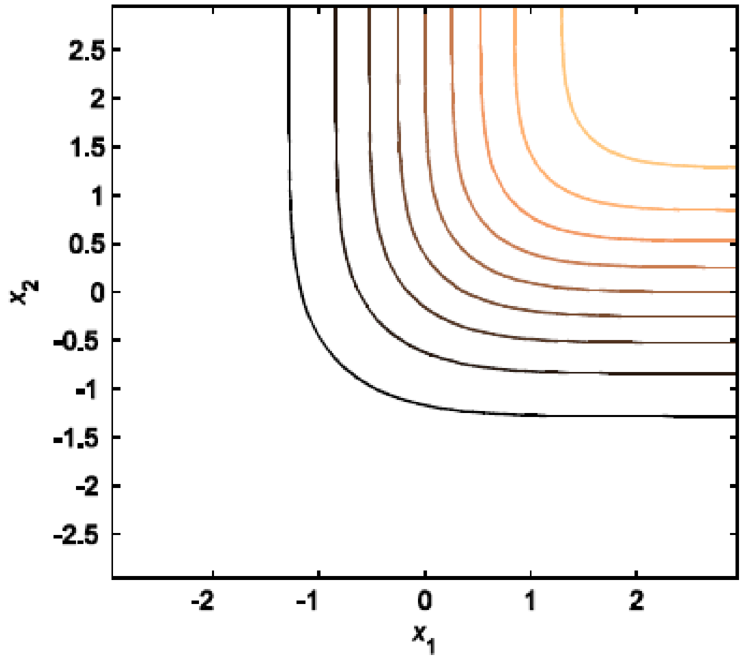
Joint PDF

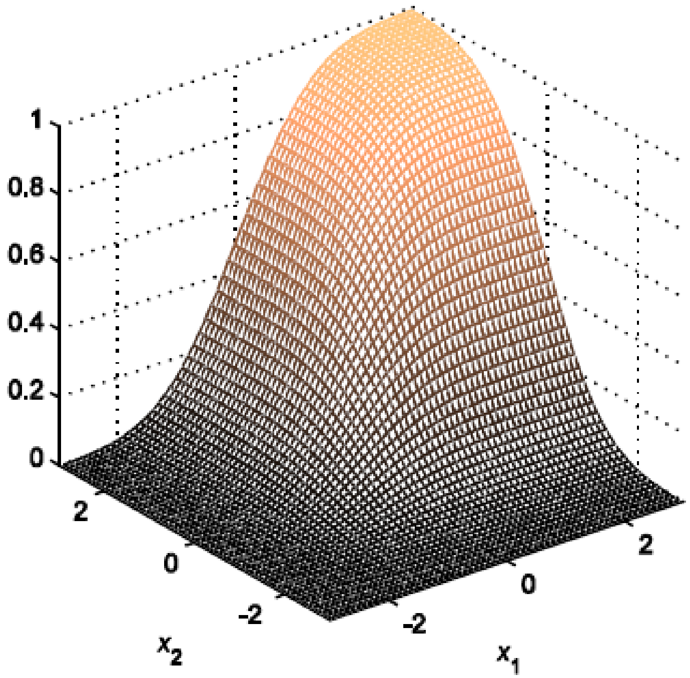


Random vectors

e.g. two continuous random variables

Joint CDF





Statistically independent random variables

Statistically independent events

$$P[(X = x) \cap (Y = y)] = P(X = x)P(Y = y)$$

For two S.I. discrete random variables X, Y

$$p_{XY}(x, y) = p_X(x)\, p_Y(y)$$

For two S.I. continuous random variables X, Y

$$f_{XY}(x, y) = f_X(x)\, f_Y(y)$$

Expectations – random vectors

Expectation of a function

- Discrete random variable $$\mathrm{E}[g(X,Y)] = \sum_{\forall x_i} \sum_{\forall y_i} g(x_i, y_i) \cdot p_{XY}(x_i, y_i)$$
- Continuous random variable $$\mathrm{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) \cdot f_{XY}(x,y)\, dx\, dy$$

Mean vector

$$\boldsymbol{\mu}_{\mathbf{X}} = \mathrm{E}[\mathbf{X}] = [\mu_1, \mu_2, \ldots, \mu_n]^{\mathrm{T}}$$

Covariance

$$\begin{aligned}\mathrm{Cov}[X,Y] &= \mathrm{E}[(X-\mu_X)(Y-\mu_Y)] \\ &= \mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y]\end{aligned}$$

Expectations – random vectors

Covariance matrix

$$\Sigma_{\mathbf{XX}} = \begin{bmatrix} \mathrm{Var}[X_1] & \mathrm{Cov}[X_1, X_2] & \dots & \mathrm{Cov}[X_1, X_n] \\ & \mathrm{Var}[X_2] & \dots & \mathrm{Cov}[X_2, X_n] \\ & & \ddots & \vdots \\ \textit{symmetric} & & & \mathrm{Var}[X_n] \end{bmatrix}$$

Correlation coefficient

$$\rho_{XY} = \frac{\mathrm{Cov}[X,Y]}{\sigma_X \sigma_Y}, \qquad -1 \le \rho_{XY} \le 1$$

Covariance and correlation coefficient measure the linear dependence between two random variables

Two random random variables are **uncorrelated** if $\rho_{XY} = 0$

Multinormal distribution

Defined through the mean vector and covariance matrix $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

Joint PDF: $$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}(\det \boldsymbol{\Sigma})^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]$$

Some properties of the multinormal distribution:

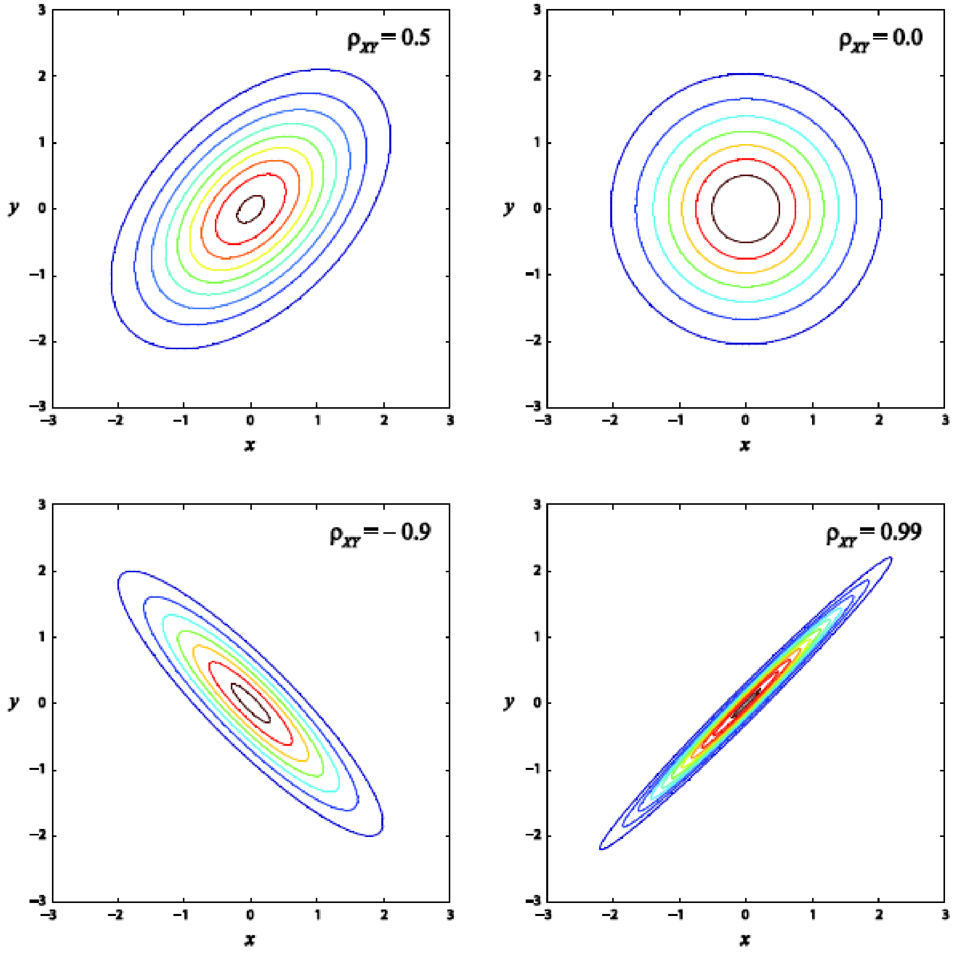
- Linear functions of normal random vectors are normal

$$\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \quad \Rightarrow \mathbf{AX} + \mathbf{b} \sim N(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^{\mathrm{T}})$$

- Uncorrelated normal random variables are also independent!

Multinormal distribution

Binormal distribution with varying correlation coefficients



Central limit theorem

Consider a set of i.i.d. random variables $X_1, X_2, \ldots, X_n$ with means μ and st. dev. σ

i.i.d.: Independent and identically distributed

The random variable

$$Y = \frac{1}{n}\sum_{i=1}^{n} X_i$$

converges in distribution to the normal distribution with mean μ and st. dev. $\sigma/n^{1/2}$

Transformation of random variables

Consider a random variables X with CDF $F_X(x)$ and a random variables Y with CDF $F_Y(y)$

Set

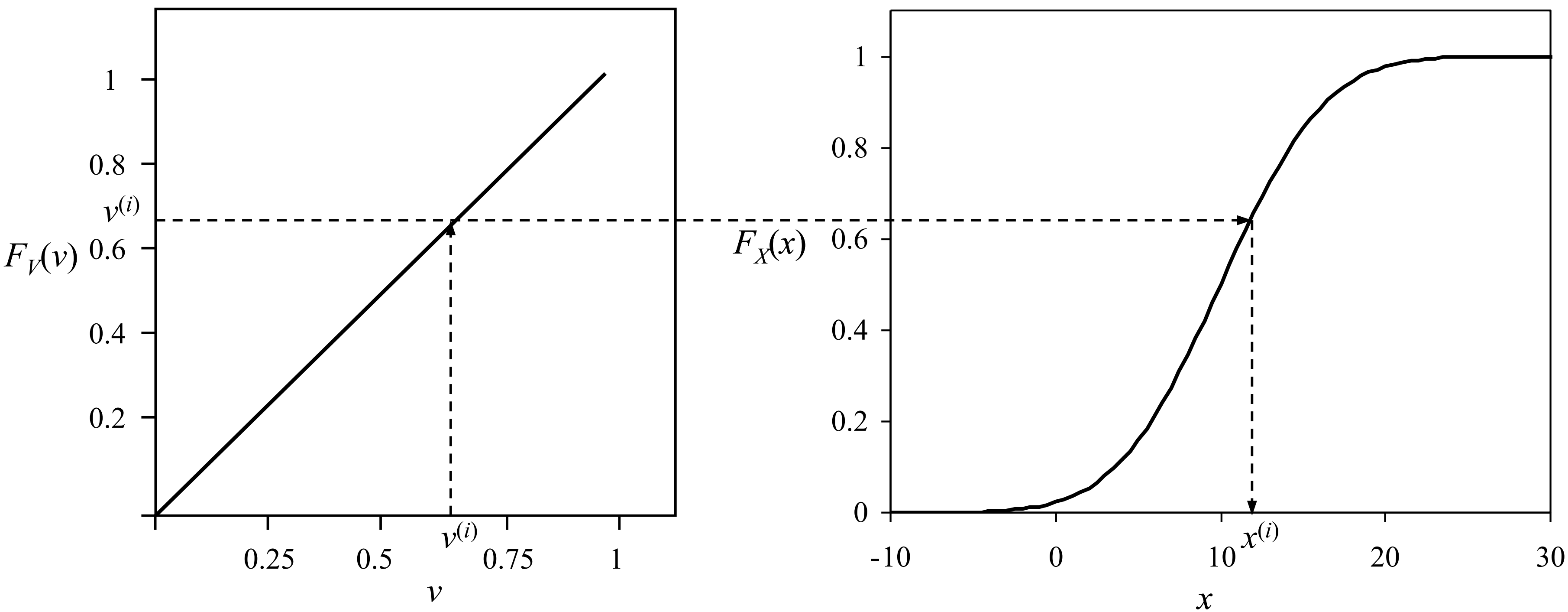
$$F_X(x) = F_Y(y)$$

$$\Rightarrow x = F_X^{-1}\left[F_Y(y)\right]$$

Isoprobabilistic transformation

Transformation of random variables

Sampling from a normal distribution



Random fields

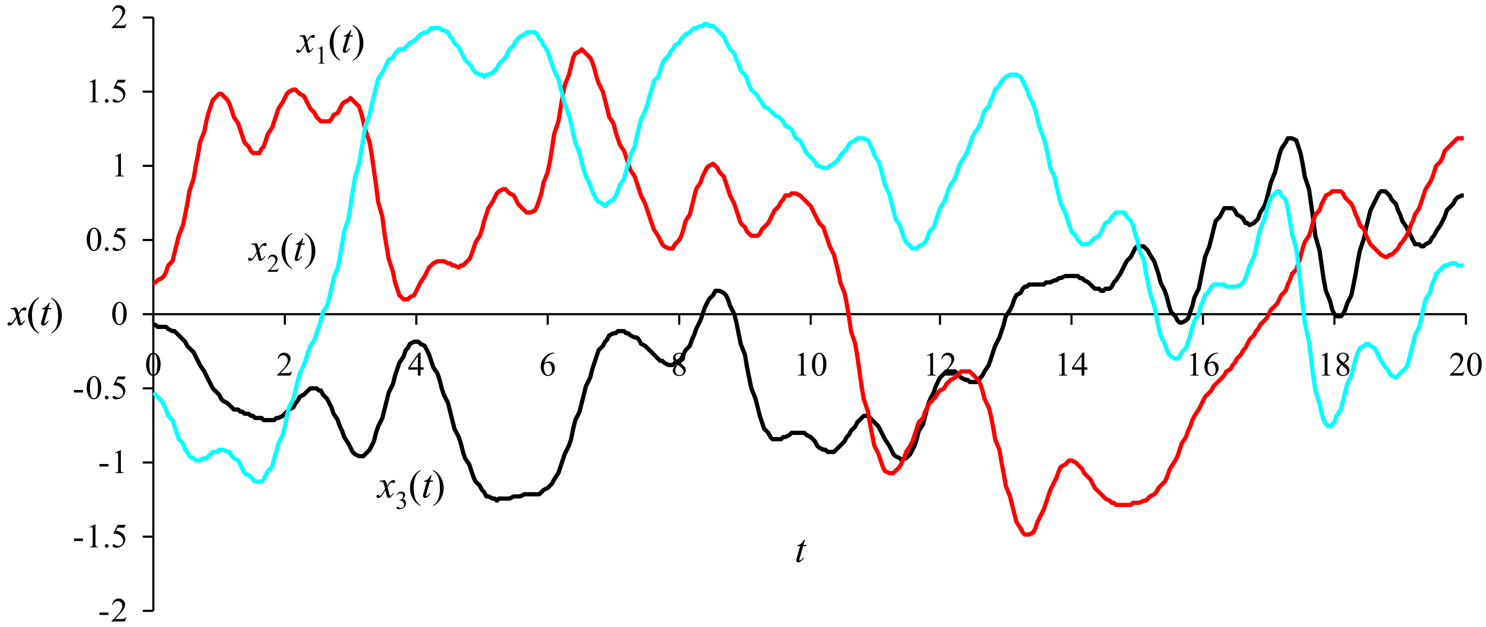
Random field – A collection of random variables indexed by a spatial parameter

Let $X(\mathbf{z})$ be a random field, where $\mathbf{z}$ is a location parameter

- For every $\mathbf{z}_i$, $X(\mathbf{z}_i)$ is a random variable
- A random outcome (realization or sample function) of the random field is a spatial function denoted $x(\mathbf{z})$

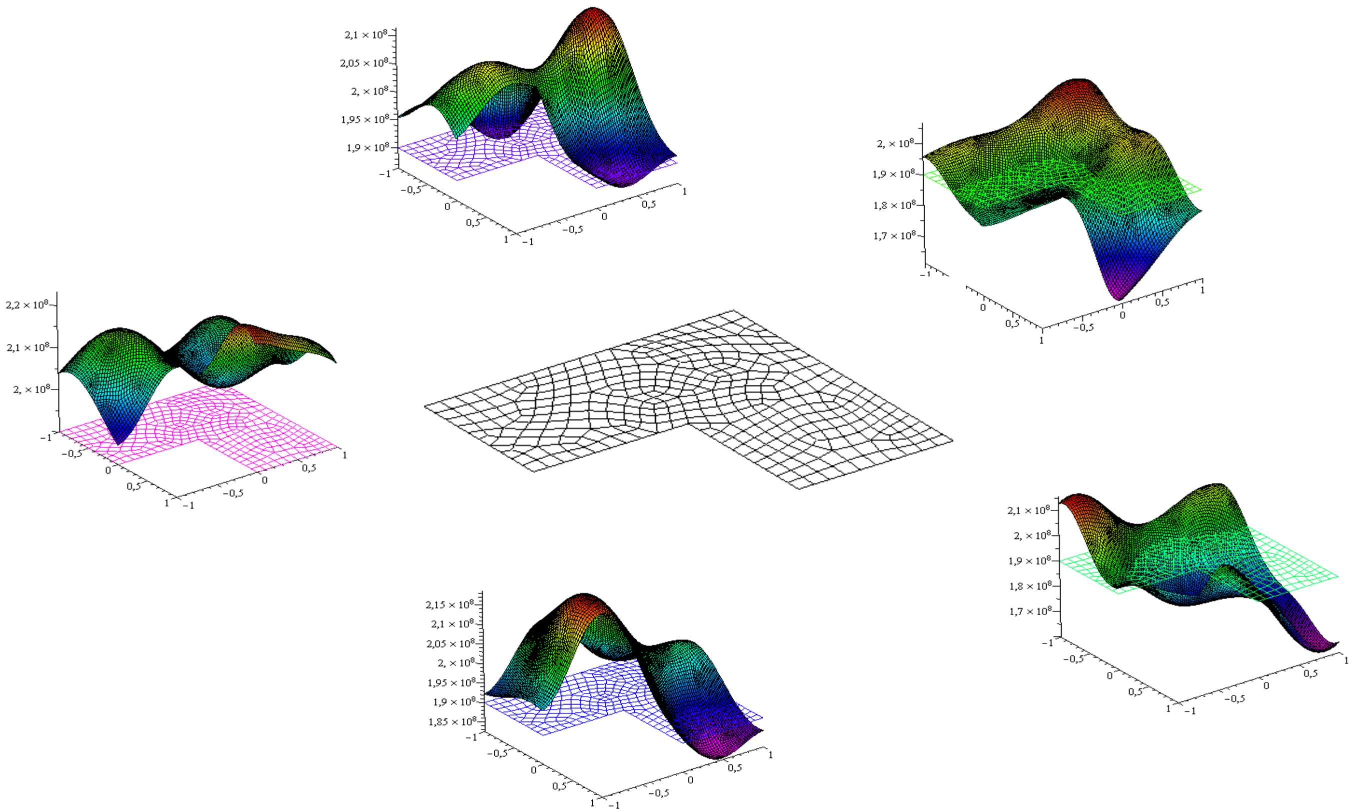
Random fields

Sample functions of a 1D random field



Random fields

Sample functions of a 2D random field



Random fields

A random field can be treated as a collection of random vectors

We can define finite-dimensional distributions, e.g.

- First order (marginal) PDF: $f_{X(\mathbf{z})}(x,\mathbf{z})$
- Second-order PDF: $f_{X(\mathbf{z}_1)X(\mathbf{z}_2)}(x_1,\mathbf{z}_1;x_2,\mathbf{z}_2)$

Moment functions of random fields

Mean function: $$\mu_X(\mathbf{z}) = \mathrm{E}[X(\mathbf{z})]$$

Variance function: $$\sigma_X^2(\mathbf{z}) = \mathrm{E}[(X(\mathbf{z}) - \mu_X(\mathbf{z}))^2]$$

Auto-covariance function: $$\begin{aligned}\Gamma_{XX}(\mathbf{z}_1,\mathbf{z}_2) &= \mathrm{Cov}[X(\mathbf{z}_1), X(\mathbf{z}_2)] \\ &= \mathrm{E}[(X(\mathbf{z}_1) - \mu_X(\mathbf{z}_1))(X(\mathbf{z}_2) - \mu_X(\mathbf{z}_2))]\end{aligned}$$

Auto-correlation coefficient function: $$\rho_{XX}(\mathbf{z}_1,\mathbf{z}_2) = \frac{\Gamma_{XX}(\mathbf{z}_1,\mathbf{z}_2)}{\sigma_X(\mathbf{z}_1)\sigma_X(\mathbf{z}_2)}$$

Homogeneous random field

Strictly homogeneous random field:

The joint distributions of $X(\mathbf{z}_1), X(\mathbf{z}_2), \ldots, X(\mathbf{z}_n)$ and $X(\mathbf{z}_1+\zeta), X(\mathbf{z}_2+\zeta), \ldots, X(\mathbf{z}_n+\zeta)$ are the same for any ζ

Weakly homogeneous random field:

- The mean function is constant: $\mu_X(\mathbf{z}) \rightarrow \mu_X$
- The auto-covariance function depends on the difference in location:

$$\Gamma_{XX}(\mathbf{z}_1, \mathbf{z}_2) \rightarrow \Gamma_{XX}(\mathbf{z}_1 - \mathbf{z}_2) = \Gamma_{XX}(\zeta)$$

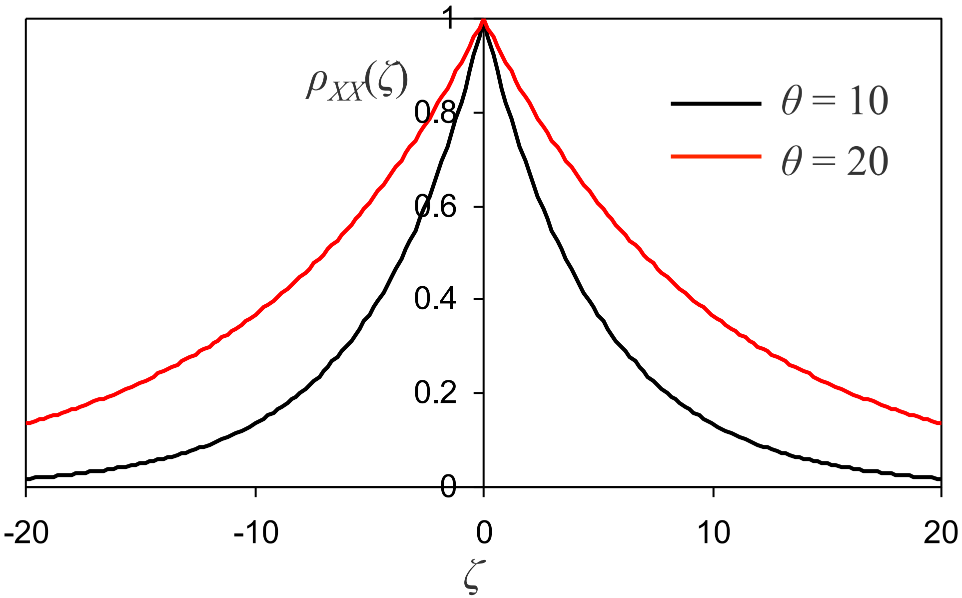
Note: A weakly homogeneous Gaussian random field is also strictly homogeneous

Homogeneous random field

Exponential correlation model

$$\rho(\zeta) = \exp\left(-\frac{2|\zeta|}{\theta}\right)$$

θ: Scale of fluctuation



Sample functions