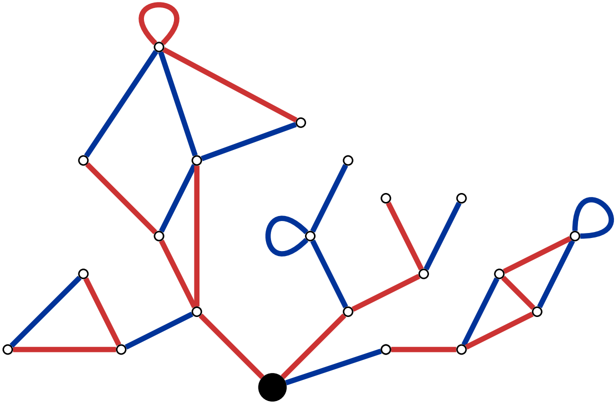
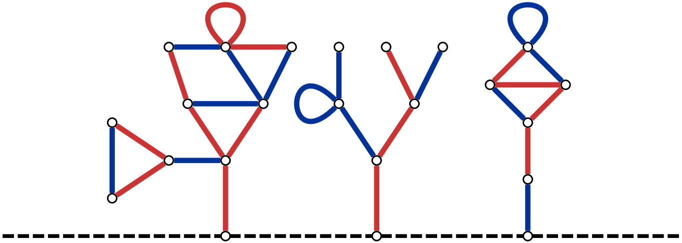
Combinatorial Game Theory & The Fuzzy Consequences

Isaac Beh

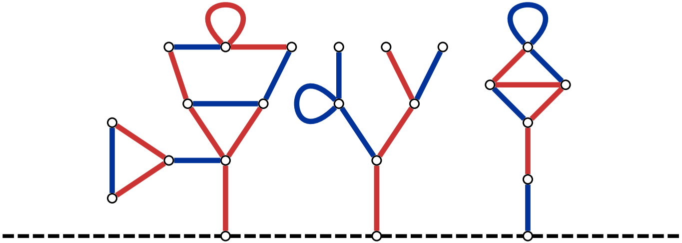
Introduction to Hackenbush



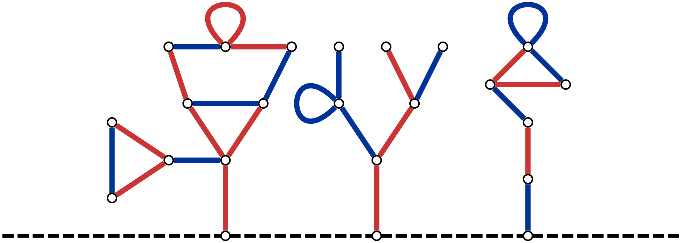
Introduction to Hackenbush



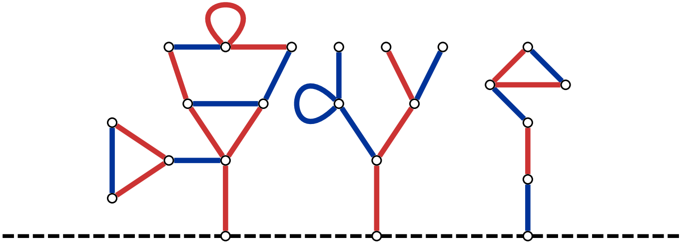
Introduction to Hackenbush



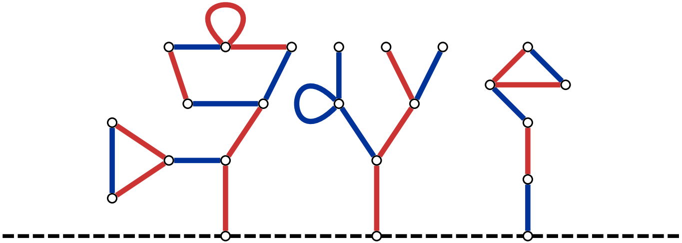
Introduction to Hackenbush



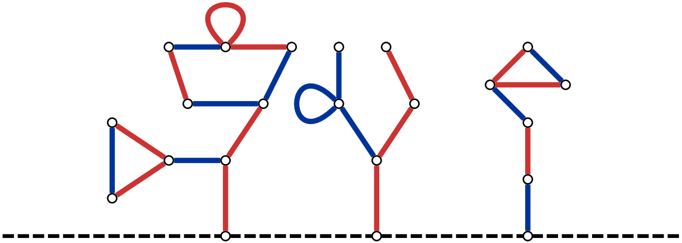
Introduction to Hackenbush



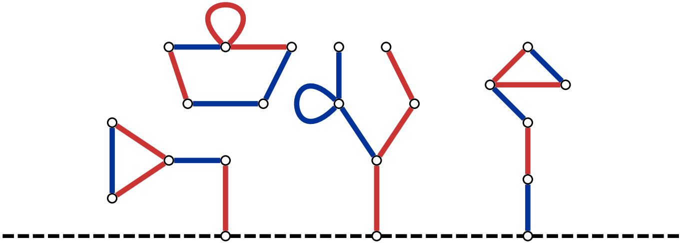
Introduction to Hackenbush



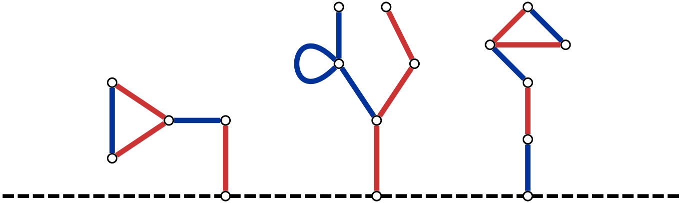
Introduction to Hackenbush



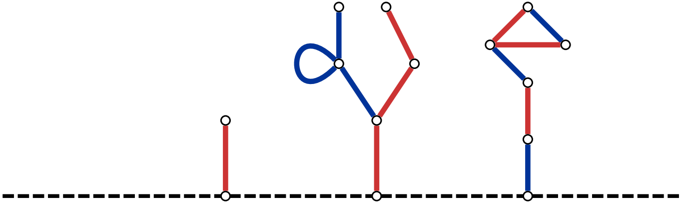
Introduction to Hackenbush



Introduction to Hackenbush



Introduction to Hackenbush



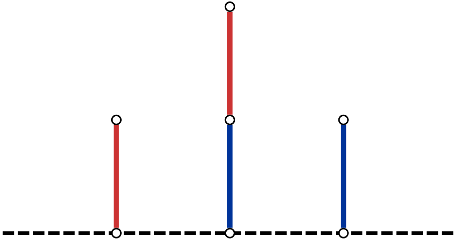
Introduction to Hackenbush

Introduction to Hackenbush

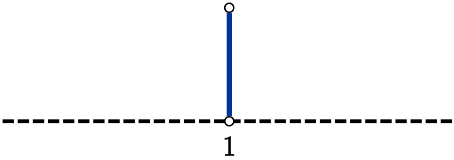
Introduction to Hackenbush

Red Wins

Basic Strategy & Intuition

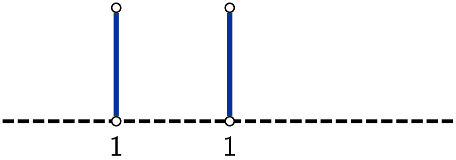


The Integers



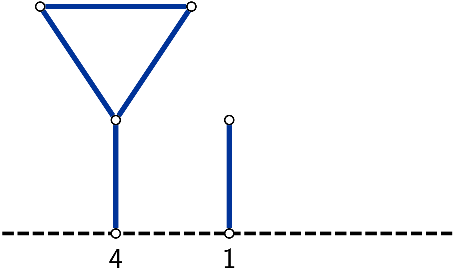
$$1$$

The Integers



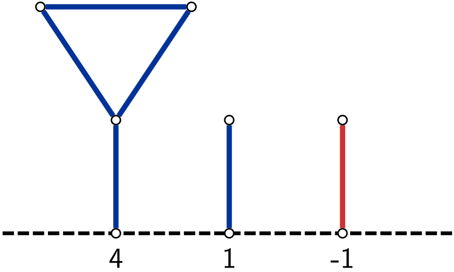
2

The Integers



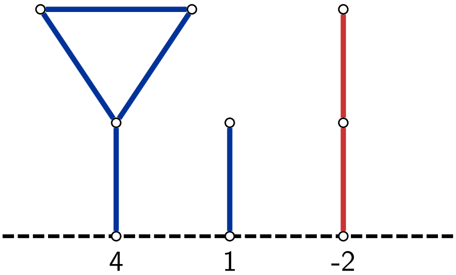
5

The Integers



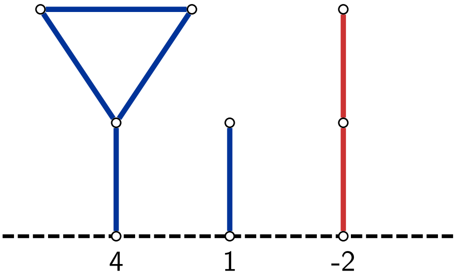
4

The Integers



3

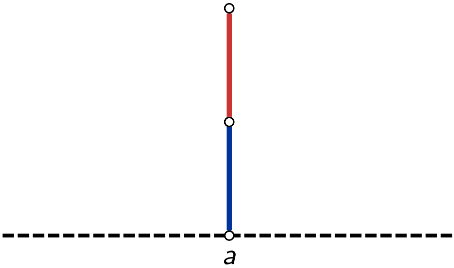
The Integers



3

$3 > 0$ so Blue wins

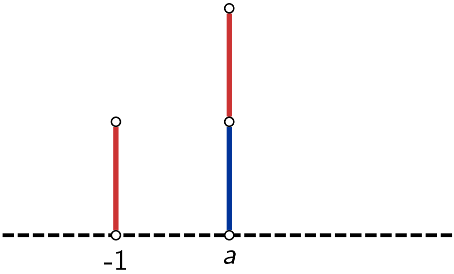
A Non-Integer Value



$$a$$

Blue wins, so $a > 0$

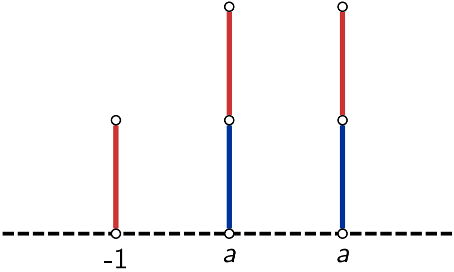
A Non-Integer Value



$$-1 + a$$

Red wins, so $-1 + a < 0$ and $a < 1$

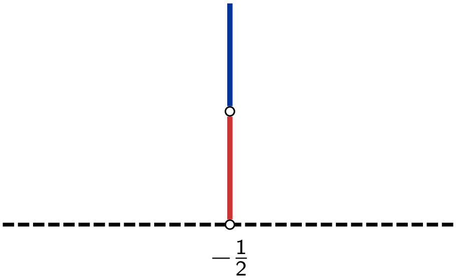
A Non-Integer Value



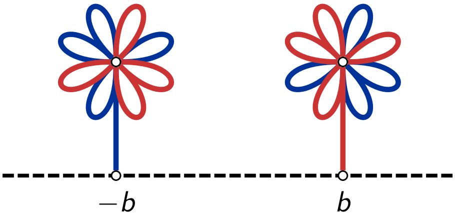
$$-1 + a + a$$

Second player wins, so $-1 + a + a = 0$ and $a = \frac{1}{2}$

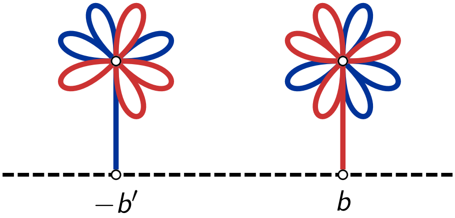
Tweedle-Dum Tweedle-Dee



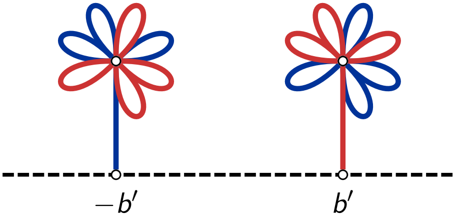
Tweedle-Dum Tweedle-Dee



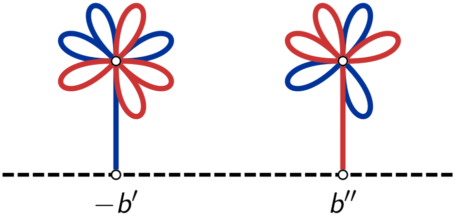
Tweedle-Dum Tweedle-Dee



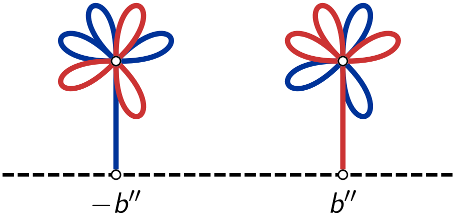
Tweedle-Dum Tweedle-Dee



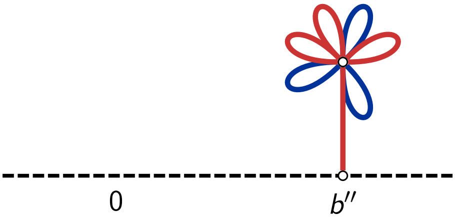
Tweedle-Dum Tweedle-Dee



Tweedle-Dum Tweedle-Dee



Tweedle-Dum Tweedle-Dee



Tweedle-Dum Tweedle-Dee

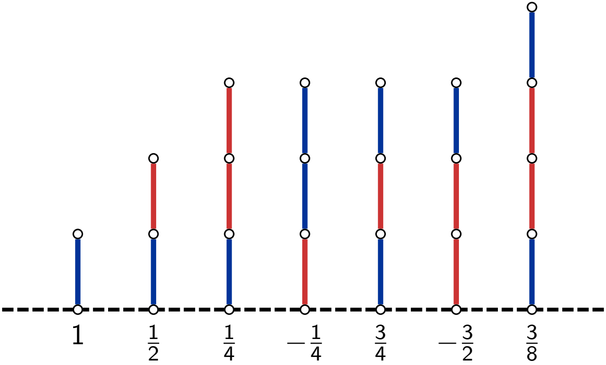
Second Player Wins

0 0

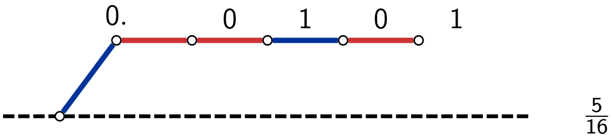
Tweedle-Dum Tweedle-Dee

$$-x + x = 0$$

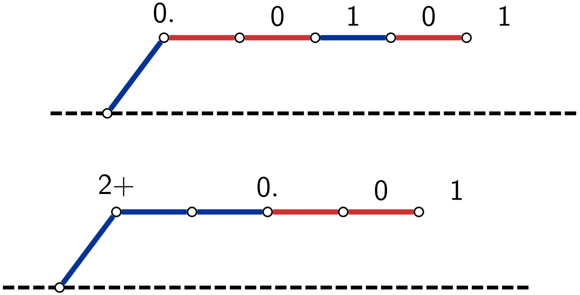
The Reals



The Reals



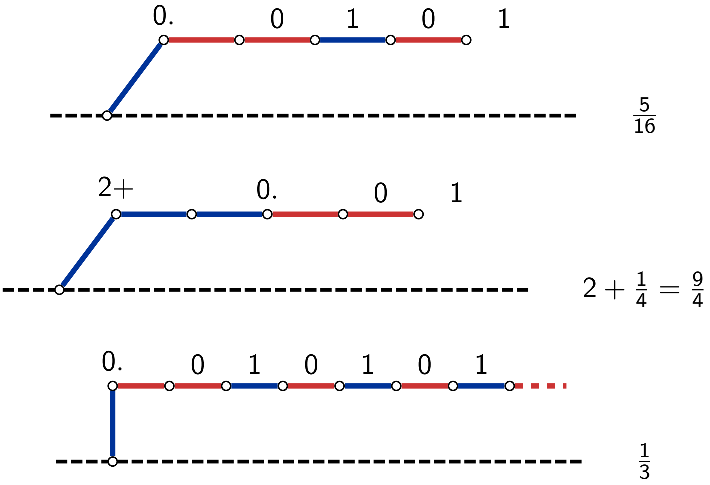
The Reals



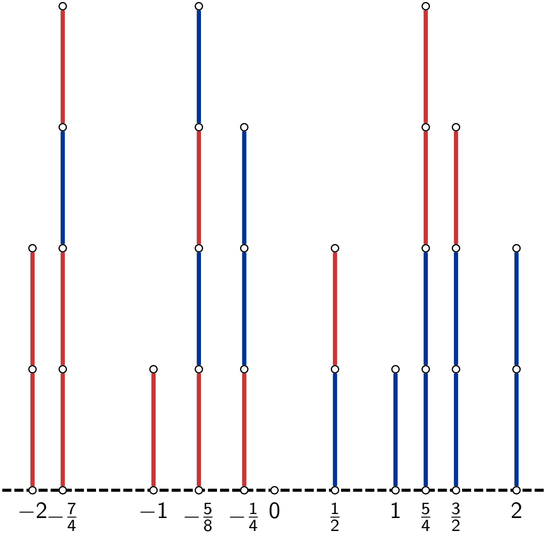
$\frac{5}{16}$

$2 + \frac{1}{4} = \frac{9}{4}$

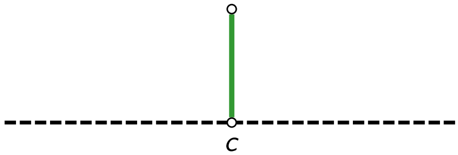
The Reals



The Reals

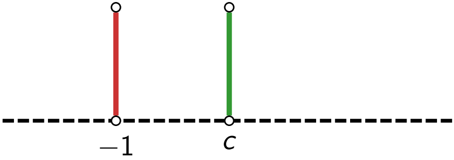


Raising the Temperature & Getting Fuzzy



c

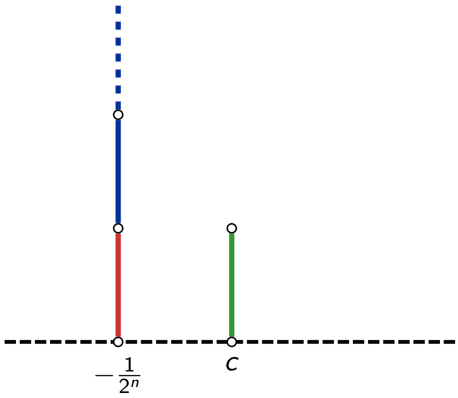
Raising the Temperature & Getting Fuzzy



$$-1 + c$$

Red wins, so $-1 + c < 0$ and $c < 1$

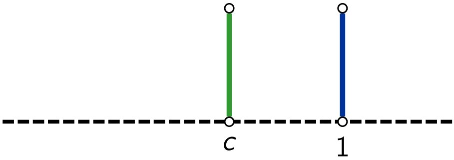
Raising the Temperature & Getting Fuzzy



$$-\frac{1}{2^n} + c$$

Red wins, so $-\frac{1}{2^n} + c < 0$ and $a < \frac{1}{2}$

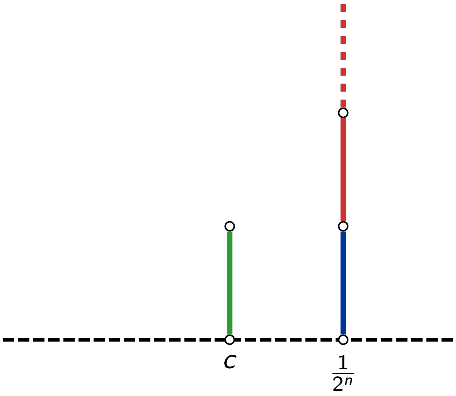
Raising the Temperature & Getting Fuzzy



$$c + 1$$

Blue wins, so $c + 1 > 0$ and $-1 < c$

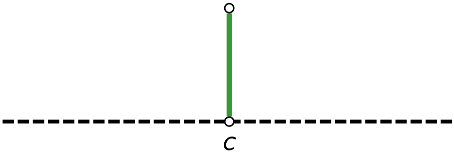
Raising the Temperature & Getting Fuzzy



$$c + \frac{1}{2^n}$$

Blue wins, so $c + \frac{1}{2^n} > 0$ and $-\frac{1}{2^n} < c$

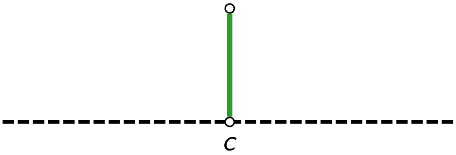
Raising the Temperature & Getting Fuzzy



$$C$$

$-\frac{1}{2^n} < c < \frac{1}{2^n}$ for all $n \in \mathbb{N}$

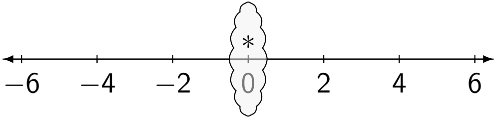
Raising the Temperature & Getting Fuzzy



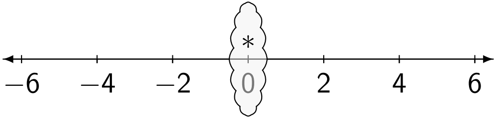
$$*$$

$$* \parallel 0$$

Raising the Temperature & Getting Fuzzy

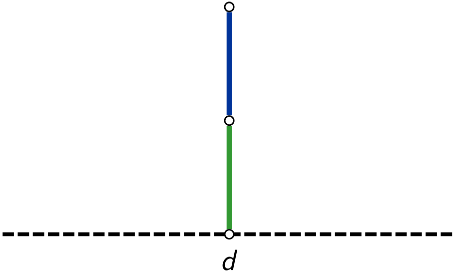


Raising the Temperature & Getting Fuzzy



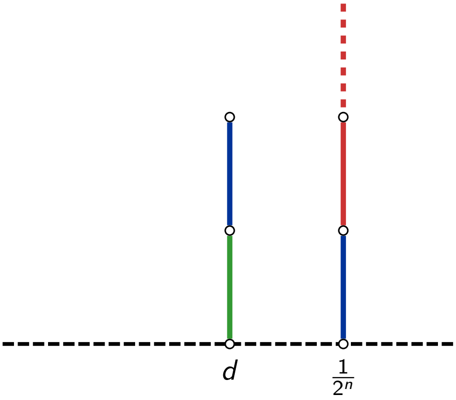
$A < B$	$B - A > 0$	or Blue always wins $B - A$
$A = B$	$B - A = 0$	or the second player wins $B - A$
$A > B$	$B - A < 0$	or Red always wins $B - A$
$A \parallel B$	$B - A \parallel 0$	or the first player wins $B - A$

Deeper into the Fuzz



d

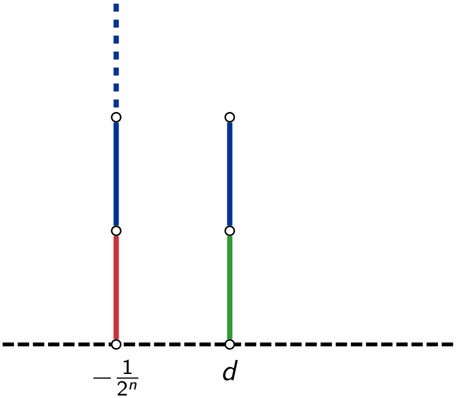
Deeper into the Fuzz



$$d + \frac{1}{2^n}$$

Blue wins, so $d + \frac{1}{2^n} > 0$ and $-\frac{1}{2^n} < d$

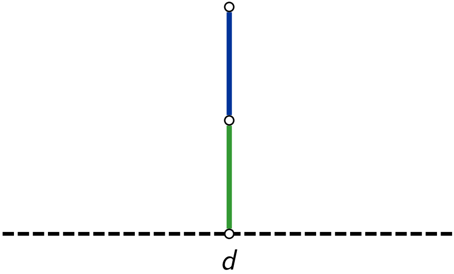
Deeper into the Fuzz



$$-\frac{1}{2^n} + d$$

Red wins, so $-\frac{1}{2^n} + d < 0$ and $d < \frac{1}{2^n}$

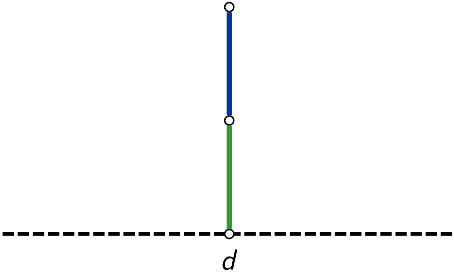
Deeper into the Fuzz



$$d$$

$$-\frac{1}{2^n} < d < \frac{1}{2^n} \text{ for all } n \in \mathbb{N}$$

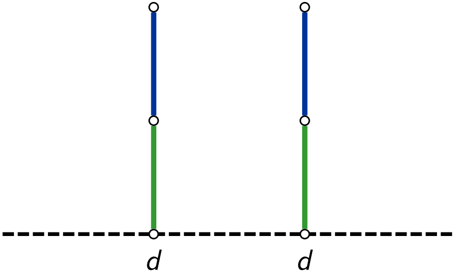
Deeper into the Fuzz



$$d$$

$$d \parallel 0$$

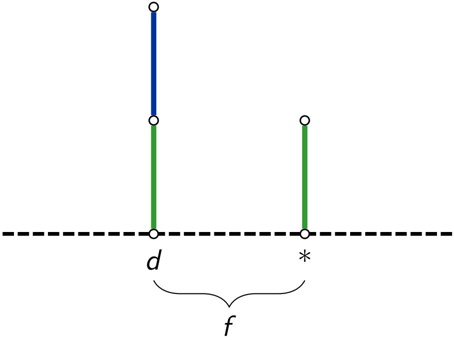
Deeper into the Fuzz



$$d + d$$

Blue wins, so $d + d > 0$ and thus $d \neq 0$.

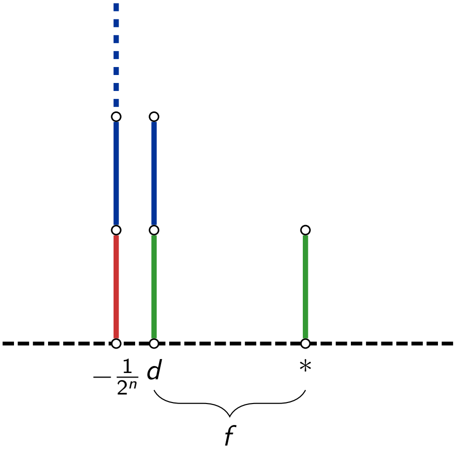
The Ups and Downs



$$f$$

Blue wins, so $f > 0$

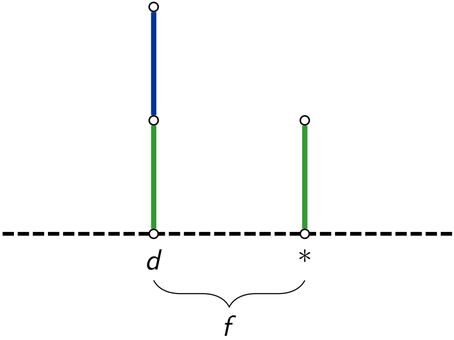
The Ups and Downs



$$-\frac{1}{2^n} + f$$

Red wins, so $-\frac{1}{2^n} + f < 0$ and $f < \frac{1}{2^n}$

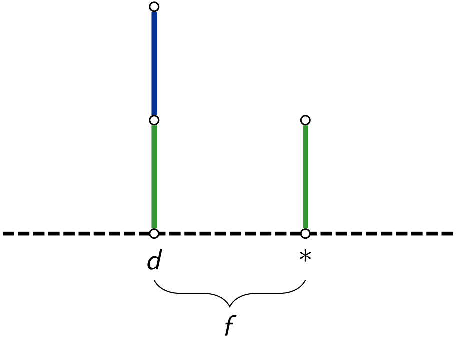
The Ups and Downs



$$f$$

$0 < f < \frac{1}{2^n}$ for all $n \in \mathbb{N}$

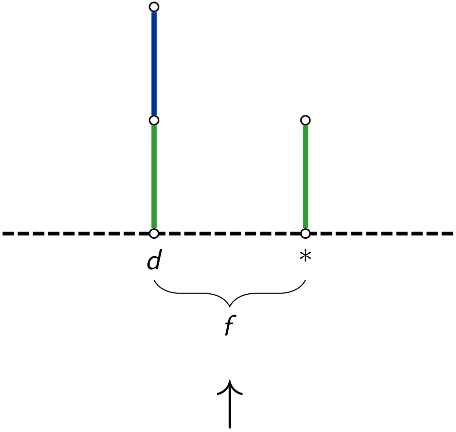
The Ups and Downs



$$\uparrow$$

$0 < \uparrow < \frac{1}{2^n}$ for all $n \in \mathbb{N}$

The Ups and Downs



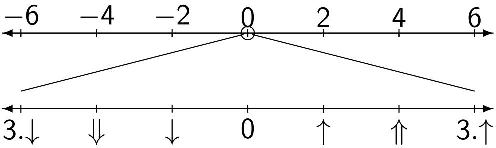
$$0 < \uparrow < \frac{1}{2^n} \text{ for all } n \in \mathbb{N}$$

The Ups and Downs

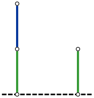
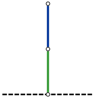
- $\downarrow := -\uparrow$
- $\Uparrow := \uparrow + \uparrow$ with $\Uparrow - \uparrow = \uparrow > 0$ so $\Uparrow > \uparrow$
- $\Downarrow := \downarrow + \downarrow$
- $n.\uparrow := \overbrace{\uparrow + \uparrow + \cdots + \uparrow}^{n \text{ times}}$
- $n.\downarrow := \overbrace{\downarrow + \downarrow + \cdots + \downarrow}^{n \text{ times}}.$

The Ups and Downs

- $\downarrow := -\uparrow$
- $\Uparrow := \uparrow + \uparrow$ with $\Uparrow - \uparrow = \uparrow > 0$ so $\Uparrow > \uparrow$
- $\Downarrow := \downarrow + \downarrow$
- $n.\uparrow := \overbrace{\uparrow + \uparrow + \cdots + \uparrow}^{n \text{ times}}$
- $n.\downarrow := \overbrace{\downarrow + \downarrow + \cdots + \downarrow}^{n \text{ times}}.$



More Fuzziness with $\uparrow *$

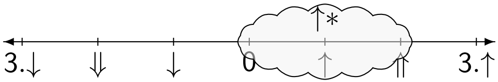
- $\uparrow * := \uparrow + * = \uparrow - * =$  $-$  $=$ 
- $\downarrow * := \downarrow + *$
- $n.\uparrow * := n.\uparrow + *$
- $n.\downarrow * := n.\downarrow + *$

More Fuzziness with $\uparrow *$

- $\uparrow * - \uparrow = * \parallel 0$, so $\uparrow * \parallel \uparrow$
- $\uparrow * - \downarrow = \uparrow + * + \uparrow = \Uparrow * > 0$, so $\uparrow * > \downarrow$
- $\uparrow * - \Uparrow = * - \uparrow = \downarrow * \parallel 0$, so $\uparrow * \parallel \Uparrow$
- $\uparrow * - 3.\uparrow = * - \Uparrow = \Downarrow * < 0$, so $\uparrow * < 3.\uparrow$

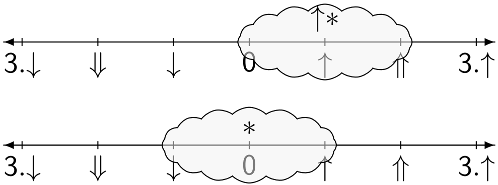
More Fuzziness with $\uparrow *$

- $\uparrow * - \uparrow = * \parallel 0$, so $\uparrow * \parallel \uparrow$
- $\uparrow * - \downarrow = \uparrow + * + \uparrow = \Uparrow * > 0$, so $\uparrow * > \downarrow$
- $\uparrow * - \Uparrow = * - \uparrow = \downarrow * \parallel 0$, so $\uparrow * \parallel \Uparrow$
- $\uparrow * - 3.\uparrow = * - \Uparrow = \Downarrow * < 0$, so $\uparrow * < 3.\uparrow$



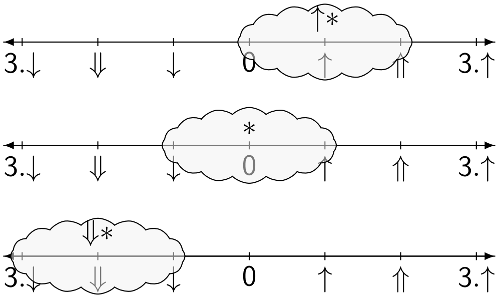
More Fuzziness with $\uparrow *$

- $\uparrow * - \uparrow = * \parallel 0$, so $\uparrow * \parallel \uparrow$
- $\uparrow * - \downarrow = \uparrow + * + \uparrow = \Uparrow * > 0$, so $\uparrow * > \downarrow$
- $\uparrow * - \Uparrow = * - \uparrow = \downarrow * \parallel 0$, so $\uparrow * \parallel \Uparrow$
- $\uparrow * - 3.\uparrow = * - \Uparrow = \Downarrow * < 0$, so $\uparrow * < 3.\uparrow$



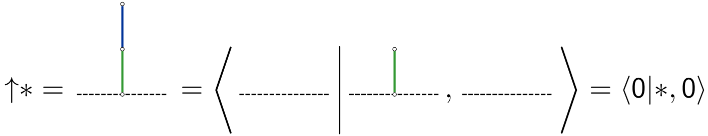
More Fuzziness with $\uparrow *$

- $\uparrow * - \uparrow = * \parallel 0$, so $\uparrow * \parallel \uparrow$
- $\uparrow * - \downarrow = \uparrow + * + \uparrow = \Uparrow * > 0$, so $\uparrow * > \downarrow$
- $\uparrow * - \Uparrow = * - \uparrow = \downarrow * \parallel 0$, so $\uparrow * \parallel \Uparrow$
- $\uparrow * - 3.\uparrow = * - \Uparrow = \Downarrow * < 0$, so $\uparrow * < 3.\uparrow$

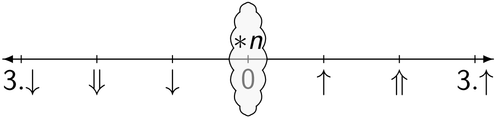


Other Strange Creatures

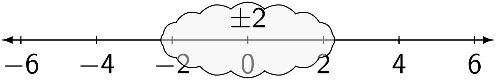
- Other games must be:
  - 2-player
  - Deterministic
  - No hidden information
  - Finite
- Games have braket notation like:

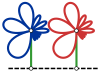


- $*n := \langle *(n-1), \ldots, *2, * | *(n-1), \ldots, *2, * \rangle$ and $*n + *m = *(n \text{ XOR } m)$



Other Strange Creatures

- 

- Hot games incentivise quick action. The temperature of the game is equal to the advantage of going first and is determined from its thermographs.
- The hottest components are the most attractive.
- Multiplication is distributive and commutative.
- Games are not closed under addition, e.g. 
- Finite games are closed and form a field.
- Infinite game results rely upon the set theory axioms.
- “Tiny” and “miny” ($+_G$ and $-_G$) are the smallest possible positive and negative values.
- Every small valued game has an atomic weight or uppitiness, which is how many $\uparrow$’s it is approximately equal to.

Applications

- Pleasure and aesthetic beauty
- Go
- Timetabling?

Further Reading/Viewing

- "Winning Ways for Your Mathematical Plays" by Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy
- HACKENBUSH: a window to a new world of math by Owen Maitzen
- Elwyn Berlekamp's channel
- `https://hackenbush.xyz/`