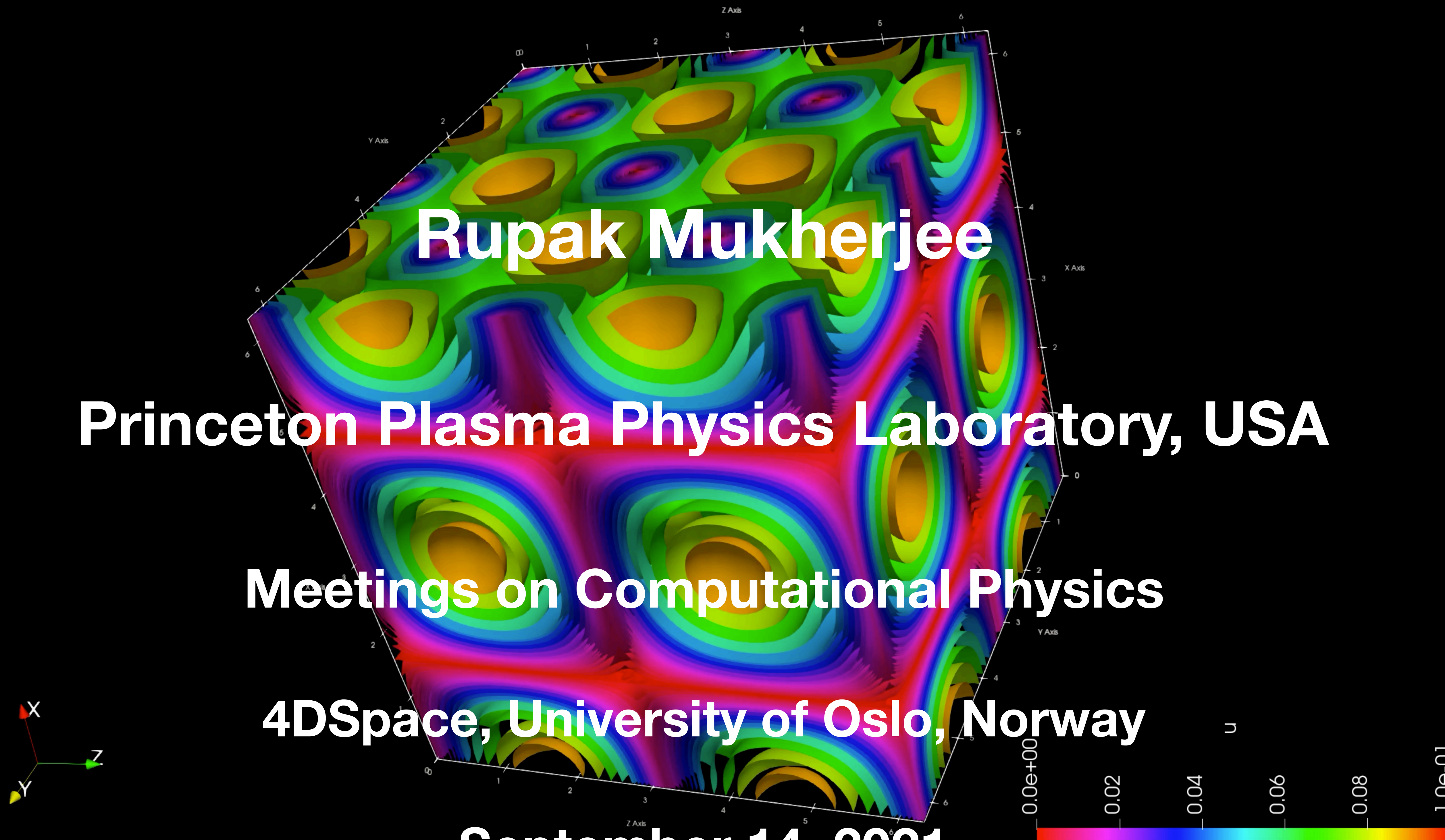
Alfven waves: _scatter, _gather



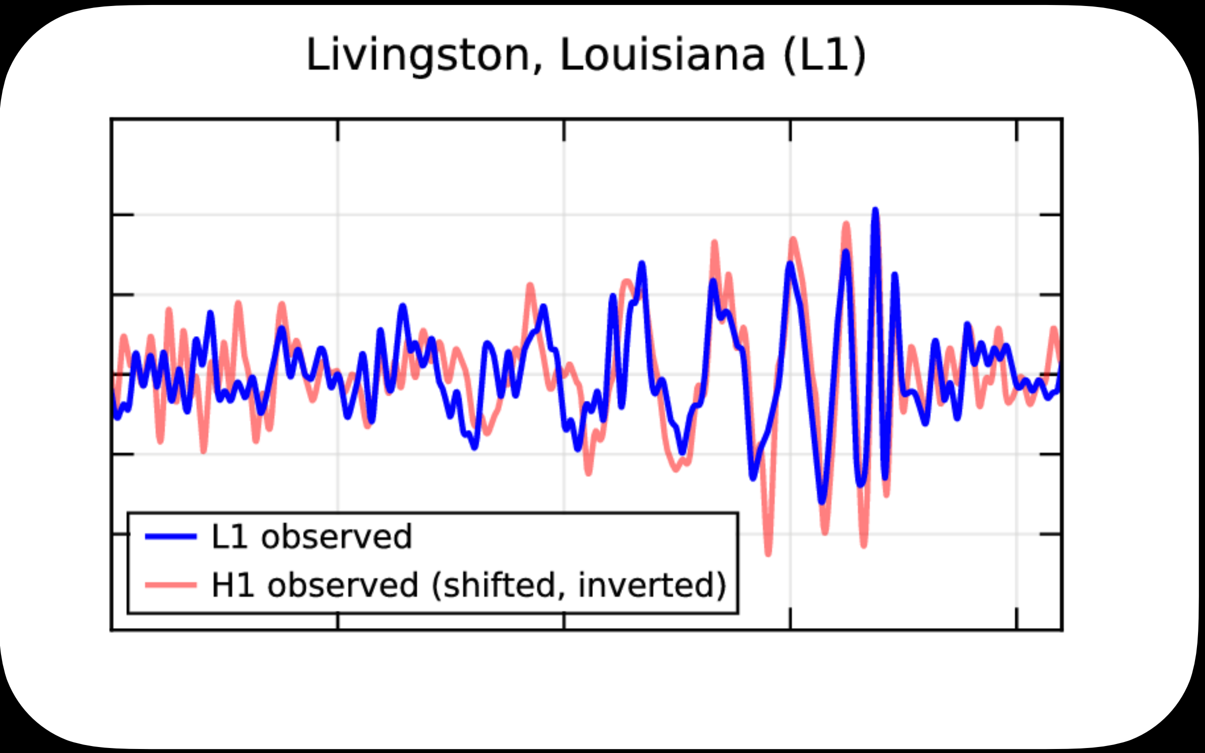
Rupak Mukherjee

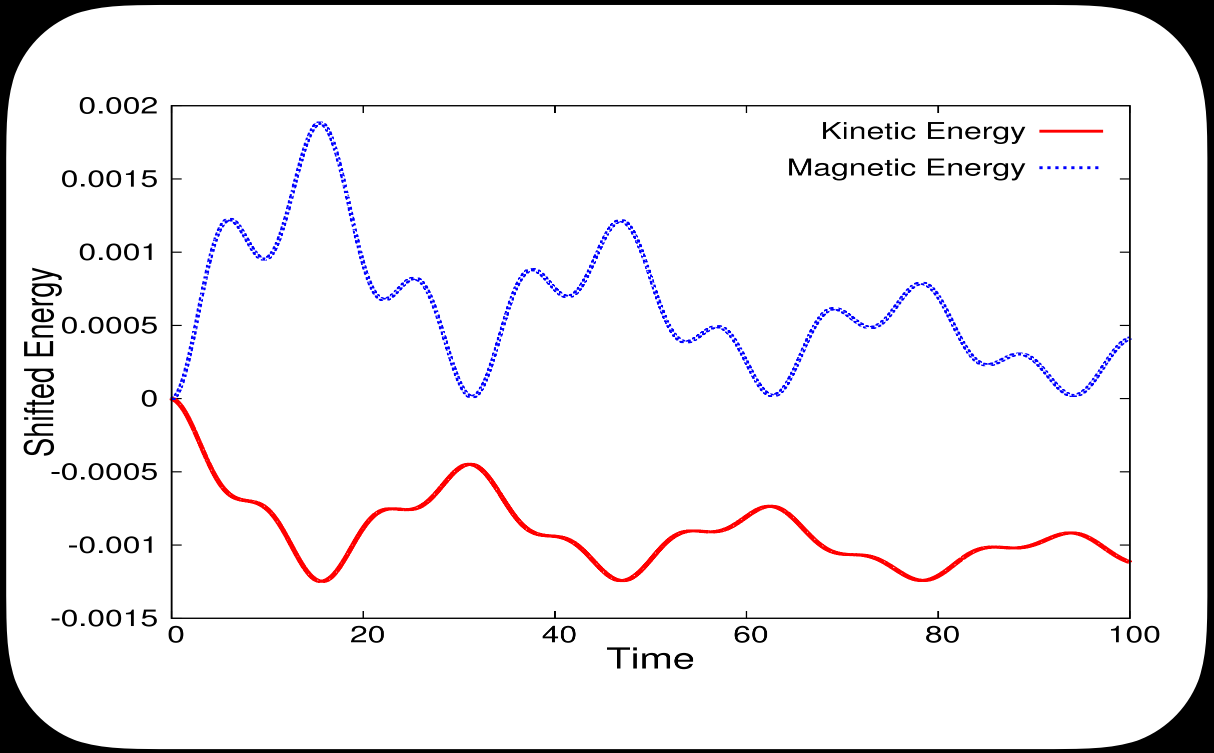
Princeton Plasma Physics Laboratory, USA

Meetings on Computational Physics

4DSpace, University of Oslo, Norway

September 14, 2021





GW150914

Alfven waves: _scatter, _gather

September 14, 20**15**

September 14, 20**21**

Sorry, there is no connection with Gravitational wave.

Hannes Alfvén, Cosmical Electrodynamics, 1963

Ohm's law (for conducting fluid): $\vec{E} = -\vec{u} \times \vec{B}$

Faraday's law of induction: $\dfrac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E} \quad = \vec{\nabla} \times (\vec{u} \times \vec{B})$

Split $\vec{B}$ into $\vec{B}_0 + \vec{b}$; such that $\vec{u} = -\vec{b}$ (**No** assumption of linearity)

[Ref: H Abdelhamid, Z Yoshida, Physics of Plasmas, **23**, 022105 (2016)]

Dust of vector calculus:

$$\vec{\nabla} \times (\vec{u} \times \vec{B}) = -\vec{\nabla} \times [\vec{b} \times (\vec{B_0} + \vec{b})]$$

$$= -\vec{\nabla} \times [\vec{b} \times \vec{b}] - \vec{\nabla} \times (\vec{b} \times \vec{B_0})$$

$$= -(\vec{B_0} \cdot \vec{\nabla})\vec{b}$$

$$\frac{\partial \vec{b}}{\partial t} + (\vec{B_0} \cdot \vec{\nabla})\vec{b} = 0 \Rightarrow$$ **Supports arbitrary amplitude!**

An incomplete review on Alfvén wave dissipation

For a homogeneous plasma in uniform constant magnetic field, dissipation of the *shear* Alfven wave due to resistive and viscous effects is very very small.
(because of the high values of R_m and R_e)

If an inhomogeneity of the magnetic field is in a direction (x) perpendicular to the direction of wave propagation (z) then $k_{||}$ is a function of x. Then one can get damping from

(1) phase mixing and,
(2) resonant absorption damping at a certain point.

For both cases ultimately the damping length goes as $R_m^{1/3}$ - a very long distance.

Faster dissipation can occur if the magnetic field is chaotic. Then the damping length is proportional to $log\ R_m$.
This was shown by Similon and Sudan, ApJ, **336**, 442 (1989) in a **2D** slab geometry.
Later works by Malara *et al* ApJ, 533, 523, (2000) extended it to **3D** and more realistic geometries. It was also shown for an *Arnold-Beltrami-Childress* field.

Slowly becoming computation-heavy. HPC came in action!

The problem: **Initialize a flow and magnetic field in a 3D periodic domain**

This configuration is 3D MHD unstable

It will trigger nonlinear Alfven wave

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Equipartition of energy - System will try to thermalize

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FPU(T) problem?

Why?

Is it a numerical bug?

'Time-crystal' in cold plasma?

*I will come back to these questions

Recur ✓

Does NOT recur ⊗

Taylor-Green Flow

$$u_x = A\ U_0\left[\cos(kx)\sin(ky)\cos(kz)\right]$$

$$u_y = -A\ U_0\left[\sin(kx)\cos(ky)\cos(kz)\right]$$

$$u_z = 0$$

Arnold-Beltrami-Childress Flow

$$u_x = U_0\left[A\sin(kz) + C\cos(ky)\right]$$

$$u_y = U_0\left[B\sin(kx) + A\cos(kz)\right]$$

$$u_z = U_0\left[C\sin(ky) + B\cos(kx)\right]$$

Roberts Flow

$$u_x = A\ U_0\sin(kz)$$

$$u_y = B\ U_0\sin(kx)$$

$$u_z = C\ U_0\sin(ky)$$

Cats-Eye Flow

$$u_x = U_0\ B\ \sin(ky)$$

$$u_y = U_0\ A\ \sin(kx)$$

$$u_z = U_0\left[A\cos(kx) - B\sin(ky)\right]$$

Single (No large scale electric field due to charge separation)

Fluid (Highly collisional, departure from Maxwellian is negligible)

MHD equations

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0$$

Continuity equation

$$\rho \frac{\partial \vec{u}}{\partial t} + \underbrace{\rho(\vec{u} \cdot \vec{\nabla})\vec{u}}_{\text{Convective derivative}} = \underbrace{\vec{j} \times \vec{B}}_{\text{Lorentz force}} - \underbrace{\vec{\nabla} P}_{\text{Compressible}} + \underbrace{\mu \nabla^2 \vec{u}}_{\text{Viscous}} + \cancelto{0}{\rho \vec{F}}$$

Momentum equation

$$\frac{d}{dt}\left(\frac{P}{\rho^\gamma}\right) = 0$$

Closure equation (Following results use $P = C_s^2 \rho$)

$$\vec{\nabla} \times \vec{B} = 4\pi \vec{j}$$

Ampère's law (Displacement current is relativistically small)

$$\frac{\partial \vec{B}}{\partial t} = -c \vec{\nabla} \times \vec{E}$$

Faraday's law of induction

$$\vec{E} + (\vec{u} \times \vec{B})/c = \underbrace{\eta \vec{j}}_{\text{Resistive}}$$

Ohm's law

Plasma parameters

R_e	R_m	M_A	M_s
10^3	10^3	1	0.1

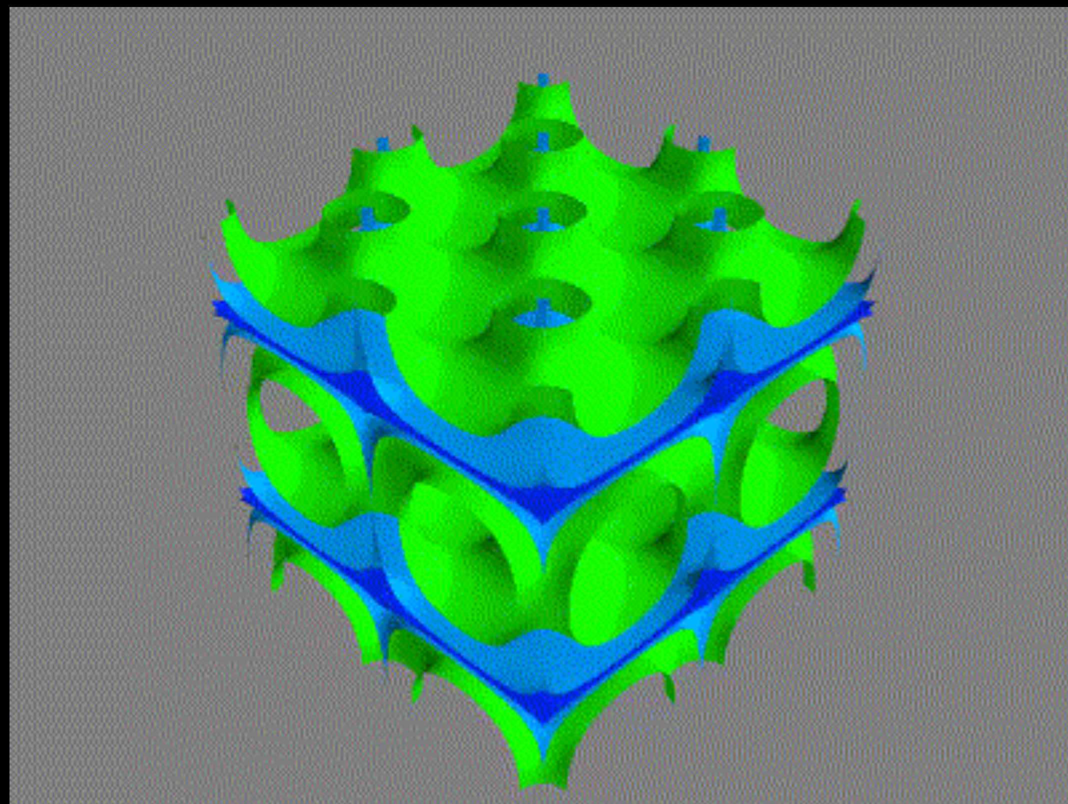
Simulation parameters

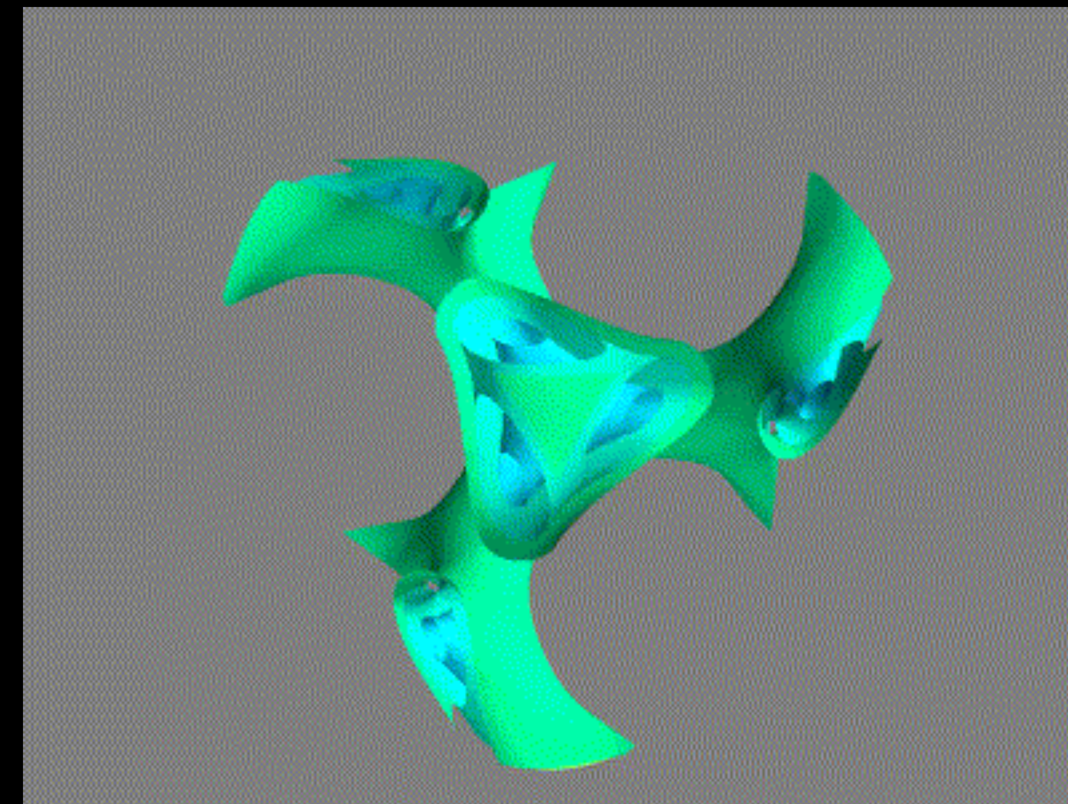
L	ρ_0	U_0	A = B = C	k
2π	1	0.1	1	1

Taylor-Green Flow **Recur**

ABC Flow **Does NOT recur**

Kinetic iso-surface

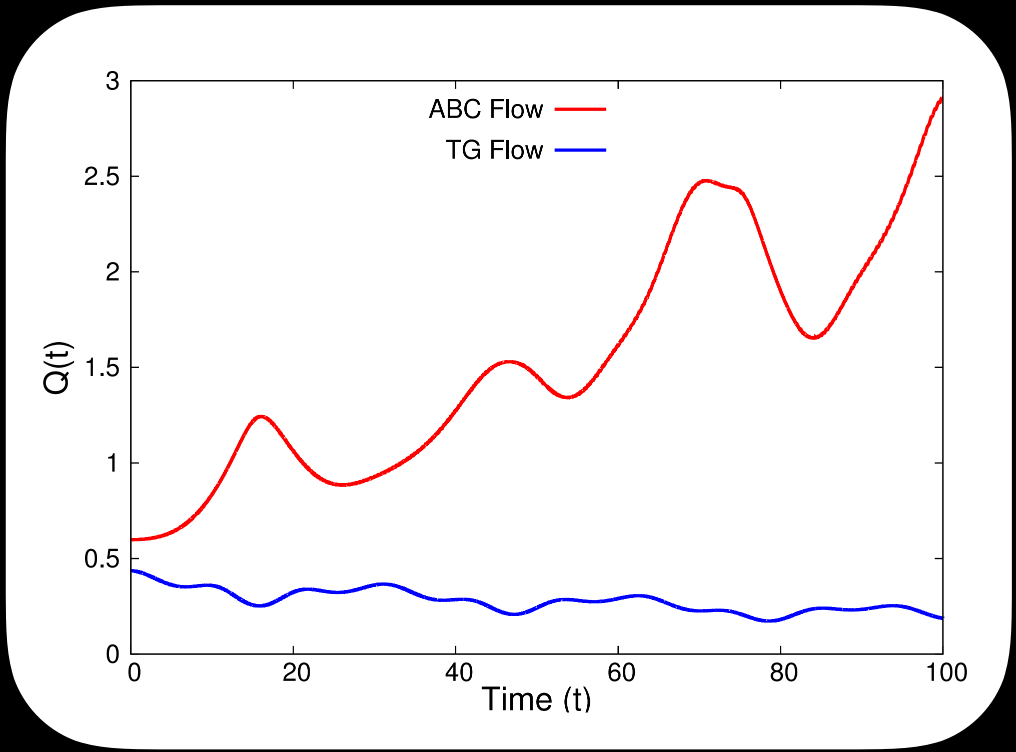




Magnetic iso-surface

The *best* explanation we have found!

$$Q(t) = \frac{\int_V \left[\left(\vec{\nabla} \times \vec{u}\right)^2 + \frac{1}{2}\left(\vec{\nabla} \times \vec{B}\right)^2\right] dV}{\int_V \left(|\vec{u}|^2 + \frac{1}{2}|\vec{B}|^2\right) dV} = \frac{\sum_k k^2 |c_k|^2}{\sum_k |c_k|^2}$$



Rayleigh quotient (modified), Q(t), measures the number of effective 'active degrees of freedom'.

[Thyagaraja, Phys. Fluids, **22** (11), 2093 (1979); Thyagaraja, Phys. Fluids, **24** (11), 1973]

Rule of Thumb!

If *Q(t)* is bounded, Recurrence CAN happen in HD. [Birkhoff, Chapter 7, Dynamical Systems, 1960]

For Taylor-Green flow, *Q(t)* is bounded.

For Arnold-Beltrami-Childress flow, *Q(t)* increases without bound.

Recurrence in experiment?	Experimentally, recurrence phenomena have been observed in shallow water waves, ocean waves, Couettee turbulence, and quantum dynamics. A detailed description of recurrence in dynamical systems in the sense of Poincare recurrence has been given by Katok and Hasselblatt.

Earlier observation of Recurrence?

Fermi-Pasta-Ulam (Tsingou), 1955	**(1D)**
Yuen & Ferguson, Phys Fluids, **21**, 2116 (1978)	**(2D)**
Thyagaraja, Phys Fluids, **22**, 2093 (1979)	**(2D)**
Thyagaraja, Phys Fluids, **24**, 1973 (1981)	**(2D)**

So what's new?

For the first time, we found recurrence in numerical simulations of the **three** dimensional **magnetohydrodynamic** equations.

RM, R Ganesh, A Sen, Physics of Plasmas, **26**, 022101 (2019) *[Editor's Pick]*

What does the 1950's theory say?

STUDIES OF NON LINEAR PROBLEMS

E. FERMI, J. PASTA, and S. ULAM
Document LA-1940 (May 1955).

We imagine a one-dimensional continuum with the ends kept fixed and with forces acting on the elements of this string. In addition to the usual linear term expressing the dependence of the force on the displacement of the element, this force contains higher order terms. For the purposes of numerical work this continuum is replaced by a finite number of points (at most 64 in our actual computation) so that the partial differential equation defining the motion of this string is replaced by a finite number of total differential equations. We have, therefore, a dynamical system of 64 particles with forces acting between neighbors with fixed end points. If x_i denotes the displacement of the i-th point from its original position, and α denotes the coefficient of the quadratic term in the force between the neighboring mass points and β that of the cubic term, the equations were either

$$\ddot{x}_i = (x_{i+1} + x_{i-1} - 2x_i) + \alpha\,[(x_{i+1} - x_i)^2 - (x_i - x_{i-1})^2] \qquad (i = 1, 2, \cdots, 64), \tag{1}$$

or

$$\ddot{x}_i = (x_{i+1} + x_{i-1} - 2x_i) + \beta\,[(x_{i+1} - x_i)^3 - (x_i - x_{i-1})^3] \qquad (i = 1, 2, \cdots, 64). \tag{2}$$

α and β were chosen so that at the maximum displacement the nonlinear term was small, e.g., of the order of one-tenth of the linear term. The correspond-

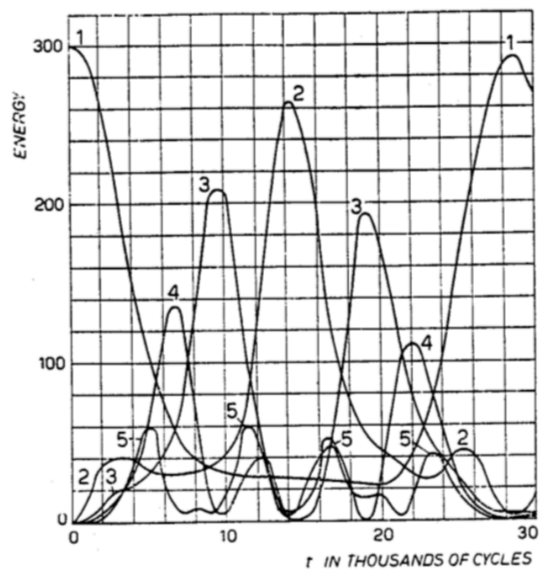
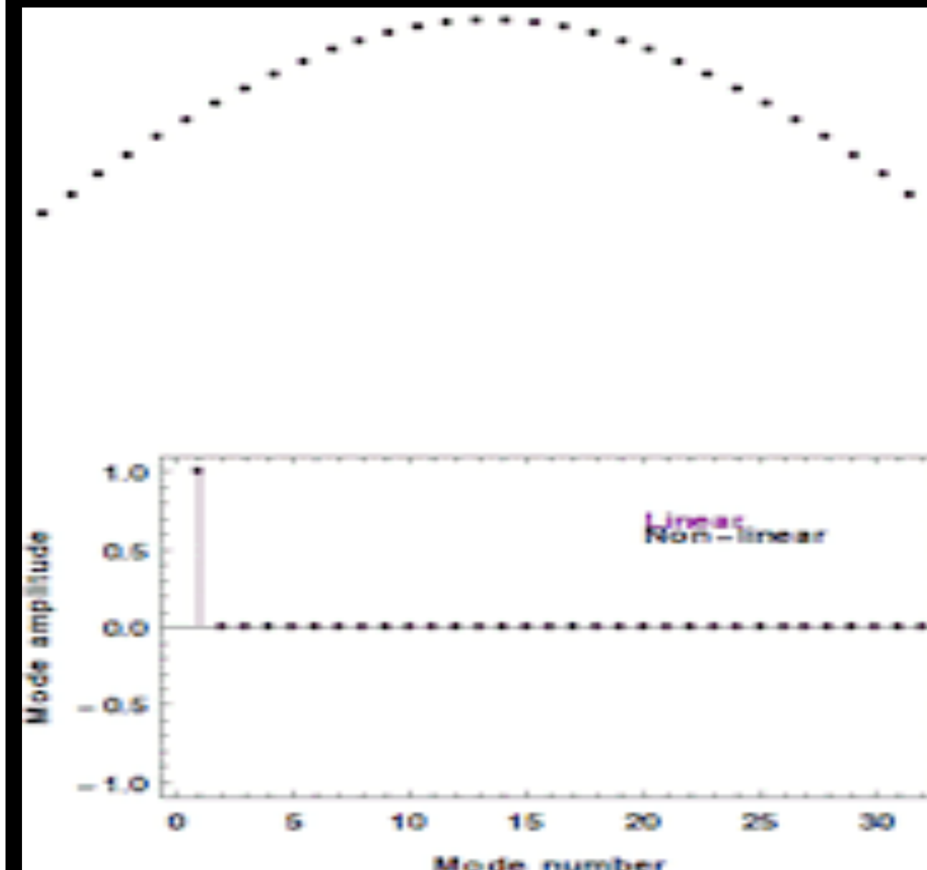


Fig. 1. – The quantity plotted is the energy (kinetic plus potential in each of the first five modes). The units for energy are arbitrary. $N = 32$; $\alpha = 1/4$; $\delta t^2 = 1/8$. The initial form of the string was a single sine wave. The higher modes never exceeded in energy 20 of our units. About 30,000 computation cycles were calculated.

From Wikipedia



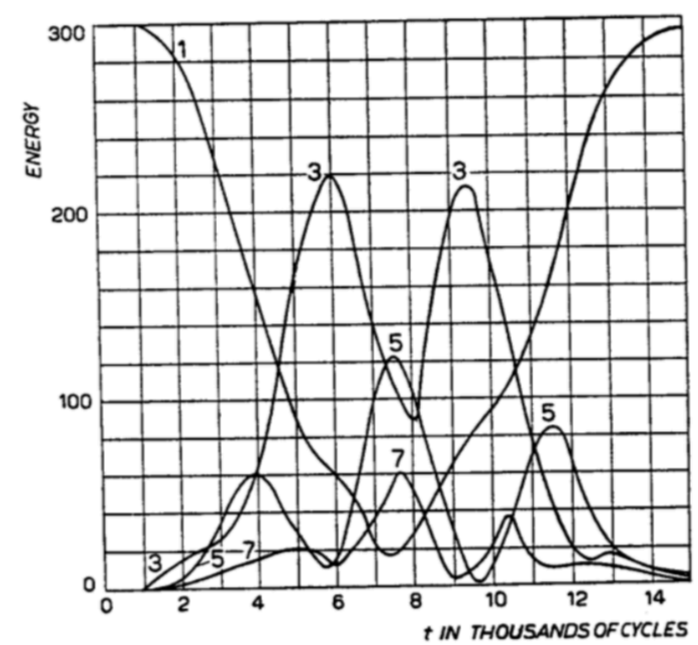


Fig. 4. – The initial configuration assumed was a single sine wave; the force had a cubic term with $\beta = 8$ and $\delta t^2 = 1/8$. Since a cubic force acts symmetrically (in contrast to a quadratic force), the string will forever keep its symmetry and the effective number of particles for the computation is $N = 16$. The even modes will have energy 0.

If one should look at the problem from the point of view of statistical mechanics, the situation could be described as follows: the phase space of a point representing our entire system has a great number of dimensions. Only a very small part of its volume is represented by the regions where only one or a few out of all possible Fourier modes have divided among themselves almost all the available energy. If our system with nonlinear forces acting between the neighboring points should serve as a good example of a transformation of the phase space which is ergodic or metrically transitive, then the trajectory of almost every point should be everywhere dense in the whole phase space. With overwhelming probability this should also be true of the point which at time $t = 0$ represents our initial configuration, and this point should spend most of its time in regions corresponding to the equipartition of energy among various degrees of freedom. As will be seen from the results this seems hardly the case. We have plotted (figs. 1 to 7) the ergodic sojourn times in certain subsets of our phase space. These may show a tendency to approach limits as guaranteed by the ergodic theorem. These limits, however, do not seem to correspond to equipartition even in the time average. Certainly, there seems to be very little, if any, tendency towards equipartition of energy among all degrees of freedom at a given time. In other words, the systems certainly do not show mixing. [2]

Document LA-1940.
Los Alamos National Laboratory.

Note: No magnetic field

What do the 1970's theories say?

Fermi–Pasta–Ulam recurrence in the two-space dimensional nonlinear Schrödinger equation

Henry C. Yuen

Department of Fluid Mechanics, TRW Defense and Space Systems Group, Redondo Beach, California 90278

Warren E. Ferguson, Jr.

Department of Mathematics, University of Arizona, Tucson, Arizona 85721

(Received 24 July 1978)

Numerical solutions of the two-space dimensional nonlinear Schrödinger equation with spatially periodic boundary conditions have been obtained. It has been found that the long-time evolution exhibits the Fermi–Pasta–Ulam recurrence phenomenon for a wide range of initial conditions.

It has been shown by Zakharov[1] that the evolution of a weakly nonlinear, deep-water, gravity wave train subjected to two-space dimensional modulation is governed by the following equation for the complex envelope A:

$$i\left(\frac{\partial A}{\partial t}+\frac{\omega_0}{2k_0}\frac{\partial A}{\partial x}\right)-\frac{\omega_0}{8k_0^2}\frac{\partial^2 A}{\partial x^2}+\frac{\omega_0}{4k_0^2}\frac{\partial^2 A}{\partial y^2}-\frac{1}{2}\omega_0 k_0^2|A|^2A=0\,, \tag{1}$$

where ω_0 and k_0 are the frequency and wavenumber of

Recurrent motions in certain continuum dynamical systems

A. Thyagaraja

Culham Laboratory, United Kingdom Atomic Energy Authority, Abingdon, Oxon, OX14 3DB, United Kingdom

(Received 28 February 1979)

Recent observations of Yuen and Ferguson on the dynamical structure of nonlinearly saturated, spatially periodic solutions of the nonlinear Schrödinger equation are given a simple analytical explanation. It is shown that many nonlinear systems can be described as effectively possessing a finite number of degrees of freedom even though the evolution equations formally relate to a continuum. Powerful theorems in general dynamics then lead to the existence of generally recurrent motions which need not, however, be quasi-periodic or even almost periodic. An explicit example is given to show that a system need not be conservative in order to exhibit recurrence. Explicit estimates of the effective number of degrees of freedom are given for the important nonlinear Schrödinger equation and the Korteweg–de Vries equation.

Note: Still no magnetic field

What do the 1980's theories say?

Thyagaraja, 1979

iltonian systems to general dynamical systems. The key to the phenomenon of recurrence is the concept of the effective number of degrees of freedom of the motion. It is shown that this quantity is a functional of the initial data in the particular case of nonlinear Schrödinger equation.

Thus, the initial conditions determine whether or not the motion of the system is recurrent in Birkhoff's sense in conservative systems such as those described by the nonlinear Schrödinger equation or the Korteweg–de Vries equations. In dissipative systems, any motion

The physical picture of the motion is relatively easy to understand. Although it is true that nonlinearity renders the modes with low n values unstable, and dispersion transfers the energy to shorter length scales, the "cascade" process does not go on forever. Thus, if the energy content of the sufficiently short wavelengths was small to begin with, it *remains* small for *all* time. Since energy is conserved overall, it is perpetually redistributed among a finite set of modes and the motion overall appears recurrent. In fact, this is precisely the phenomenon observed by Yuen and Ferguson.

ion or even conservative evolution equations. Stable, recurrent motions are common to all physical systems which can be characterized as having, effectively, a finite number of degrees of freedom.

Recurrent motions in certain continuum dynamical systems

A Thyagaraja, Phys Fluids, **22**, 2093 (1979)

Thyagaraja, 1981

discussed and clarified. The concept of recurrence has previously[1] been related to the capacity of the system to confine the energy effectively to a limited number of modes of oscillation. It is also shown to be intimately connected to the concepts of Lagrange stability and Poisson stability. In one dimension, recurrence is equivalent to the noncollapsing nature of the solutions whereas the situation is more complicated in higher dimensions, and no general criterion is as yet available for deciding which initial data lead to noncollapsing, recurrent motions.

Recurrence, dimensionality, and Lagrange stability of solutions of the nonlinear Schrödinger equation

A Thyagaraja, Phys Fluids, **24**, 1973 (1981)

Note: And still no magnetic field

And then a silence of more than 35 years...

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The surprise! Some initial conditions RECUR

FPU(T) problem?

Yes, kind of, but we have magnetic field

Why?

All the degrees of freedom are NOT active

Is it a numerical bug?

No, verified with multiple codes!

'Time-crystal' in cold plasma?

Yes!

First observation of recurrence with the simulation package

Turbulent Astroplasma Replicator Accessories (TARA)

(A MULTI-GPU 3D-MHD SOLVER)

https://rupakmukherjee.github.io/TARA/

DOI: 10.5281/zenodo.4682188

Later, we reproduced with PLUTO

And you are most welcome to cross-check with your own simulation suit...