Motion of a Pendulum with Horizontal Springs

UCLA Summer19C Phy4AL L9G8 Final Report

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Abstract

We have learned, from Physics 1, the motion of a physical pendulum and of a spring-mass system. In this lab, we intend to build and analyze a system that consists of a combination of the two. As usual, we are interested in the angular frequency, ω_c, of the combined oscillation. A theoretical prediction is made on ω_c, and an experiment is conducted to verify the prediction. Error analysis and future aspects of a more complicated system are as well stated. There are two parts in our experiment. Part 1 is about small angle approximation, where we fix the mass and only vary the angle of oscillations in order to find the best angle which is used in part 2. In part2, we use the angle found in part 1 and vary the mass. If the prediction is correct, the actual angular frequencies observed should agree with our predictions.

1 Theoretical background

- Small angle approximation:
  when angle θ is small, $sin(\theta) \approx \theta$.
- Differential equation:
  $\frac{d^2\theta}{dt^2} = -\alpha\theta \implies \theta = \theta_0 cos(\sqrt{\alpha}t + \phi)$, where angular frequency $\omega = \sqrt{\alpha}$.
- Physics equations for Tork:
  $\tau = F \times l$. $\sum \tau = I\alpha = I\frac{d^2\theta}{dt^2}$. $I = \sum ml^2$
  angular frequencies for pendulum and spring-mass: $\omega_p = \sqrt{\frac{g}{L}}, \omega_s = \sqrt{\frac{k}{m}}$.

2 Materials

- Two programmed Arduinos: one transmitter(with ultrasound sensor) and one receiver
- weights ($50g - 100g$) that have a total weight of at least $200g$
- two clamp stands
- two springs
- a string (approx. 1m)
- a metal rod (longer than the distance between stands)
- a blackboard
- tapes
- a container or a box

3 Setup and conceptual drawings

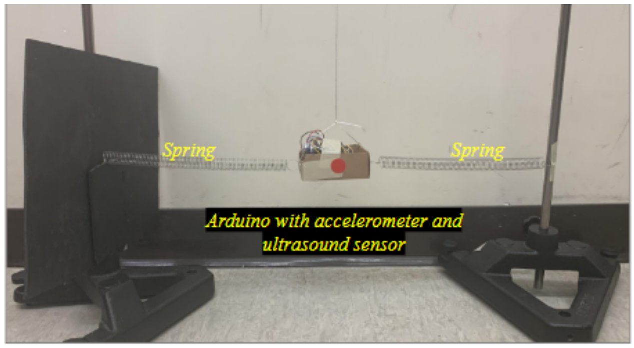
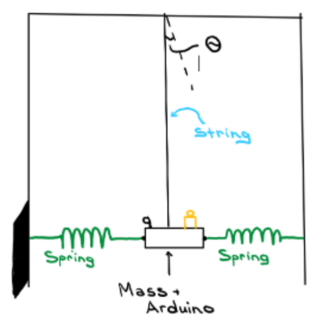


Figure 1: overview (setup)

(a) entire system



(b) apparatus

Figure 2: apparatus and free body diagram

We connect the transmitter (in a box) vertically to a string hanging from the rod above. In our case, the string is hung at approximately the middle of the rod so that both springs are stretched. The two springs are connected horizontally to the transmitter, and we expect to observe a horizontal motion of the transmitter.

4 Assumptions and predictions

In order to simplify the calculations and predictions, we assume that the spring is weightless and does not stretch (acting like a rod with no weight), and we do not consider the mass of the springs into oscillation when analyzing the pendulum system. Another important phenomenon that can appear is damping. When predicting, we ignore the damping effect because we hope to see a ”perfect” simple harmonic motion.

For convenience, let ”transmitter box” denote transmitter + box + additional weights added to the box, and let ”mass” denote the mass of the transmitter box. There are three forces acting on the transmitter box (weight, tension from the string, and springs' restoring force), which in turn produce tork. We take clockwise rotation as the positive direction for tork. Let θ be the angle the string deviates from vertical axis, L be the length of the string, $k_{eq} = k_1 + k_2$ be the equivalent spring constant of two springs connected in parallel, and m be the mass.

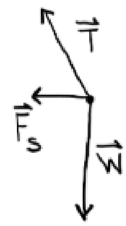


Figure 3: free body diagram

$$\tau_{net} = \sum \tau = \sum F_i \times l_i = TLsin(0) + WLsin(180 - \theta) + F_sLsin(90 - \theta)$$
$$= mgLsin(\theta) + k_{eq}Lsin(\theta)Lcos(\theta) = mgLsin(\theta) + \frac{1}{2}k_{eq}L^2sin(2\theta)$$

Taking small angle into consideration,

$$\tau_{net} = mgL\theta + k_{eq}L^2\theta = -I\alpha = -mL^2\alpha$$

$$\alpha = -(\frac{g}{L} + \frac{k_{eq}}{m})\theta$$

Applying the differential equation, we get $\omega_c = \sqrt{\frac{g}{L} + \frac{k_{eq}}{m}}$, or $\omega_c^2 = \omega_p^2 + \omega_s^2$. After measuring values for each variable, we get the following predictions:

Mass(kg)	0.18
String Length(m)	0.9065
k$_{eq}(N/m)$	7.50
Predicted $\omega_c(rad/s)$	7.24

Table 1: Part 1 prediction: small angle

	Mass(kg)	Predicted $\omega_c(rad/s)$
Trial 1	0.180	7.24
Trial 2	0.230	6.59
Trial 3	0.280	6.13
Trial 4	0.330	5.79
Trial 5	0.380	5.53

Table 2: Part 2 prediction: varied mass

5 Experiment and data taking

In prelab, we measured the equivalent spring constant using Hooke's law (see Appendix A). When running the experiments, we first use a meter stick to measure how much the center of mass deviates from its equilibrium position, which is proportional to the angle since $sin(\theta) = \frac{\Delta x}{L}$. Then we start collecting the ultrasound data while letting the transmitter oscillate freely. Notice that we orient the blackboard in a way such that the ultrasound emitter is facing it. In part 1, we vary the initial angle (or initial displacement from the equilibrium position) in each trial; in part 2, we fix the angle and add $50g$ weights to the transmitter box in each trial. For both parts, the ultrasound sensor sends the distance and time data to the receiver, and we use Python programming to best fit the data. Here is a closer look of the transmitter box:

Figure 4: ultrasound sensor

6 Results for part 1

Here is the plots from part 1 as well as the python data fitting results.

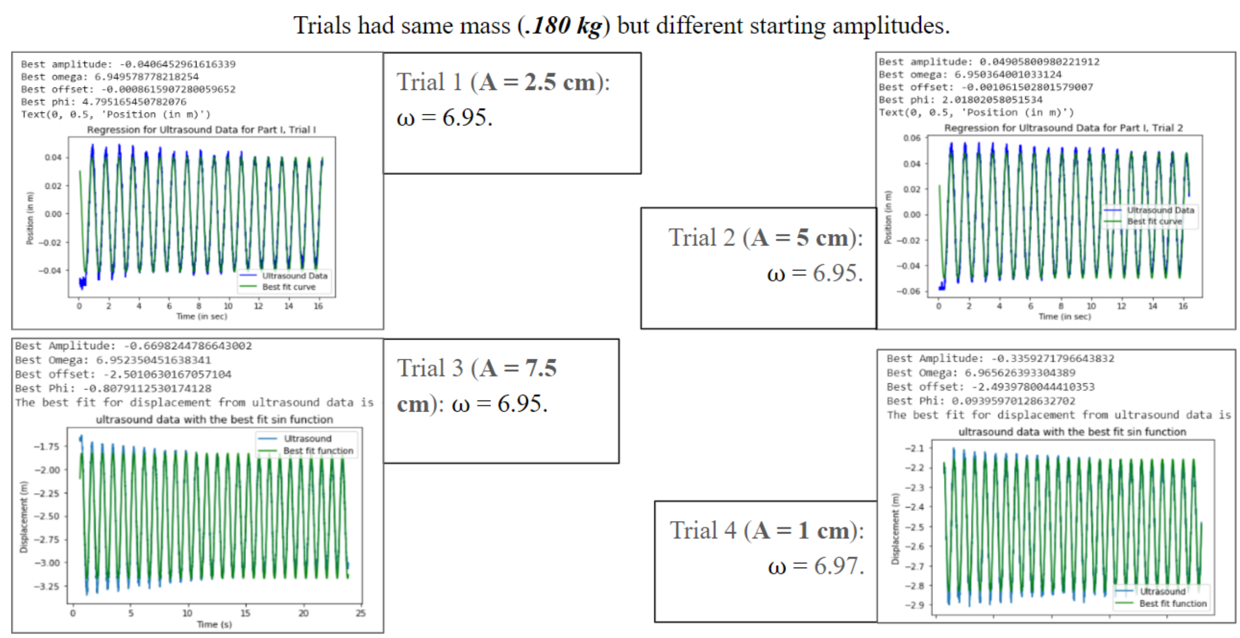


Figure 5: Part 1 data fitting

	Amplitude(cm)	Predicted ω_c	Observed ω_c	Error(%)
Trial 1	2.5	7.24	6.95	4.01
Trial 2	5	7.24	6.95	4.01
Trial 3	7.5	7.24	6.95	4.01
Trial 4	1.0	7.24	6.97	3.73

Table 3: Part 1 results: small angle

In part 1, we fix the mass to be $0.18kg$ and only vary the angle. When amplitude $A = 1cm$, we get the least error. We stopped trying smaller amplitude since that will make the oscillation too small to clearly observe. On one hand, it verifies the validity of small angle approximation; on the other hand, the noticeable error reflects the impact of some assumptions like negligible mass of string and springs.

7 Results for part 2

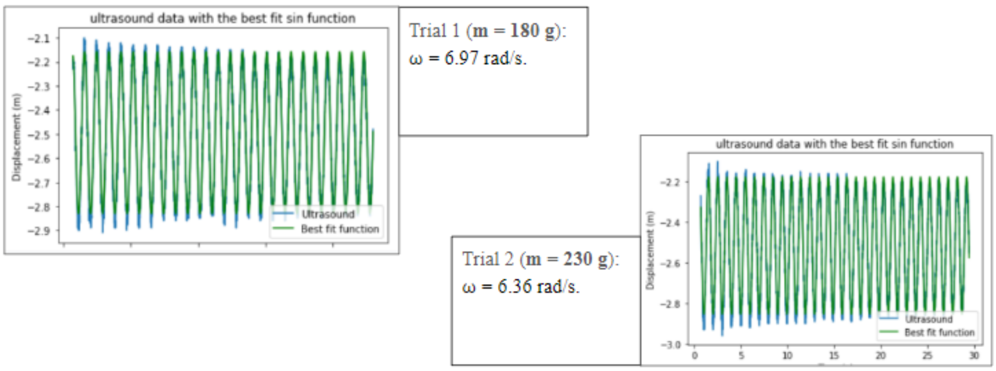


Figure 6: Part 2 data fitting: trials 1,2

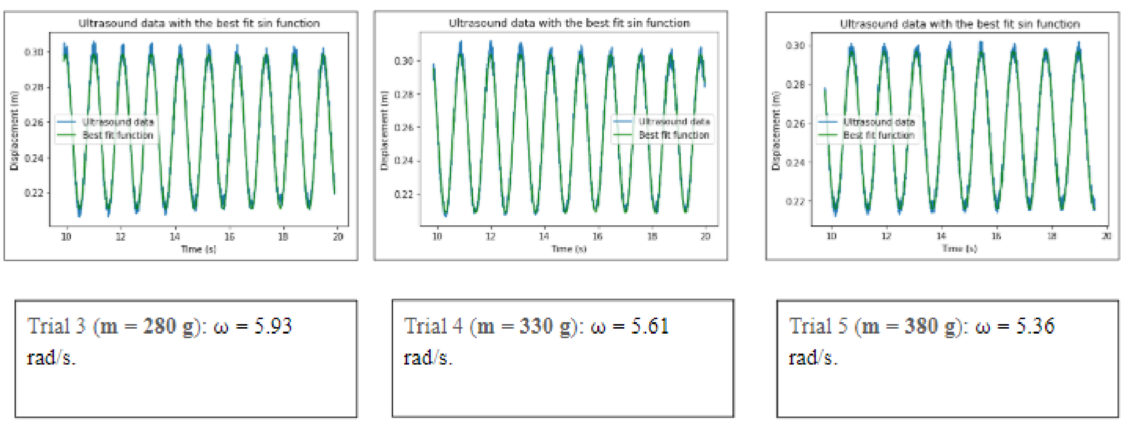


Figure 7: Part 2 data fitting: trials 3,4,5

	Mass(kg)	Predicted ω_c	Observed ω_c	Percent error(%)
Trial 1	0.180	7.24	6.97	3.7
Trial 2	0.230	6.59	6.36	3.5
Trial 3	0.280	6.13	5.93	3.3
Trial 4	0.330	5.79	5.61	3.1
Trial 5	0.380	5.53	5.36	3.1

Table 4: Part 2 prediction: varied mass

In part 2, we fix the angle to be the best we got from part 1, which is $1cm$ in amplitude ($\theta = sin^{-1}(\frac{0.01}{L})$). When the mass increases, we predict that the angular frequency will decrease, and our results have a trend of decreasing angular frequency. However, the error continues to exist.

8 Error analysis

Although the result in part 2 looks quite satisfying with a trend that matches well with our predictions, the 3% to 4% error rate appears in both parts. We wonder what might be the cause of it, and here are a few plausible ones that might cause the actual angular frequencies to be smaller than predicted ones:

- Although the damping effect reflected from our plots is small, it can't be completely ignored. Ignoring damping might cause the predicted value to be larger.
- The mass of the string and springs cannot be completely ignored because parts of the springs also oscillate with the transmitter box. This will cause the predicted mass to be smaller, which in turn causes a larger predicted value.
- The string might not act exactly like a rod, which is hard to verify.
- The prediction inherently uses the small angle approximation.
- The amplitudes in part 2 are not exactly $1cm$. Most of them are larger than $1cm$, which makes the small angle approximation less accurate.

9 Conclusion and improvements

The experiment is quite successful overall. It verifies our predictions in both part 1 and part 2. The result in part 1 shows that a smaller angle does give us a more accurate result which supports the small angle approximation technique. The result in part 2 shows that the angular frequency decreases when the mass of the object increases, which also agrees with our prediction. We are facing about 3% error in both parts, which can be explained by our assumptions as shown in section 8. The small error does suggest that the relation $\omega_c^2 = \omega_p^2 + \omega_s^2$ we derived in section 4 holds.

We can still make some improvement in the future. For instance, we only changed the mass in this project, which shows the effect from spring-mass system to the angular frequency of the combined system. We can also change the length of the string to see the effect from the pendulum system. We are also interested in taking damping and the mass of string and springs into consideration, which may produce a better result with a smaller error rate.

A more sophisticated version of combined systems is to replace the string of a physical pendulum by a spring, as shown in figure 8. Please see Appendix B for more details.

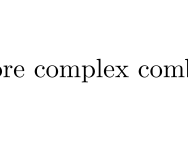


Figure 8: A more complex combined oscillation

10 Contribution

This is a group project so all of our group members put in considerable amount of effort. I mainly did the calculations for predictions and the experiment, including setup and video taking. I also did part of the Python data fitting and made part of the slides. I can never finish this project without my wonderful partners James Yoon and Zoe McDonald. Also, I want to give special thanks to our Instructor Dr. Katsushi Arisaka and Teaching Assistant Chandan Kittur, as well as UCLA Physics Department for providing us with such a great opportunity to design our own project.

11 Appendix

A Spring constants using Hooke's Law

According to Hooke's law, how much a spring stretches from its equilibrium point is linearly proportional to the stretching force acting on it. Thus, we put different masses under a vertical spring and measure the length of it. The plot should yield a straight line, and the value of slope is the desired spring constant.

Figure 9: Hooke's law

In experiment, we add $50g$ mass to the spring each time for five times and use Excel to do the linear fit. The spring constants for two springs are $3.7N/m$ and $3.8N/m$. Therefore, the equivalent spring constant when they are connected in parallel is $7.5N/m$.

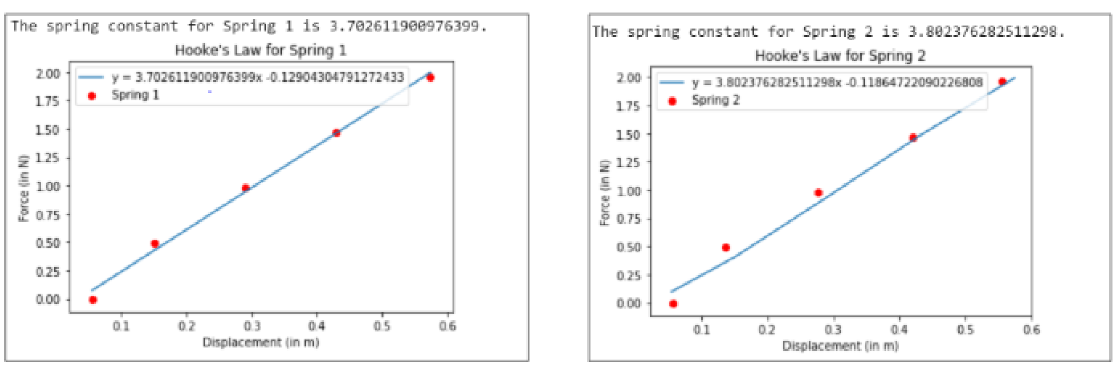


Figure 10: Hooke's law

B A complex version of combined oscillation

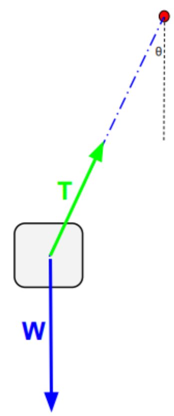


Figure 11: free body diagram

The difference here is that the length of the spring will change when θ changes. Still, we take the pendulum as the main oscillation. Let L be the length of the spring respect to θ, L_0 be the vertical equilibrium length of the spring without any mass attached, m be the mass. According Hooke's law, $L = L_0 + \frac{mg}{k}cos(\theta)$. Notice that the reason why we can directly approximate L like this is that the vertical oscillation must be really small when compared to horizontal oscillation (θ small). This is actually a quite strong assumption.

$$\tau_{net} = W \times L = mgLsin(\theta) = -I\alpha = -mL^2\alpha$$

$$\alpha = -\frac{g}{L}sin(\theta) = -\frac{g}{L_0 + \frac{mg}{k}cos(\theta)}sin(\theta) \approx -\frac{g}{L_0 + \frac{mg}{k}}\theta = -\frac{kg}{kL_0 + mg}\theta$$

Therefore, the angular frequency of the pendulum is $\sqrt{\frac{kg}{kL_0+mg}}$, and we can use this result to estimate the motion of the object. $\theta = \theta_0 cos(\sqrt{\frac{kg}{kL_0+mg}}t)$. At time t,

$$\big(x(t), y(t)\big) = \big(L_0 + \frac{mg}{k}cos(\theta)\big)\big(sin(\theta), cos(\theta)\big) = L_0\big(sin(\theta), cos(\theta)\big) + \frac{mg}{2k}\big(sin(2\theta), cos(2\theta) + 1\big)$$

$$= L_0\Big(sin\big(\theta_0 cos(\sqrt{\frac{kg}{kL_0 + mg}}t)\big), cos\big(\theta_0 cos(\sqrt{\frac{kg}{kL_0 + mg}}t)\big)\Big)$$

$$+\frac{mg}{2k}\Big(sin\big(2\theta_0 cos(\sqrt{\frac{kg}{kL_0 + mg}}t)\big), cos\big(2\theta_0 cos(\sqrt{\frac{kg}{kL_0 + mg}}t)\big) + 1\Big)$$

Notice that $L_0\big(sin(\theta), cos(\theta)\big)$ is the contribution of the equilibrium length of the spring to the motion, and $\frac{mg}{2k}\big(sin(2\theta), cos(2\theta) + 1\big)$ is the motion of oscillation.
Moreover, when equilibrium length is much smaller than the stretched length, or $\frac{L_0}{L} \approx 0$,

$$\alpha \approx -\frac{g}{\frac{mg}{k}}\theta = -\frac{k}{m}\theta$$

The angular frequency when L_0 is negligible is approximately $\sqrt{\frac{k}{m}}$, which used to be the angular frequency for spring-mass system. $\theta \approx \theta_0 cos(\sqrt{\frac{k}{m}}t)$,

$$\big(x(t), y(t)\big) \approx \frac{mg}{2k}\Big(sin\big(2\theta_0 cos(\sqrt{\frac{k}{m}}t)\big), cos\big(2\theta_0 cos(\sqrt{\frac{k}{m}}t)\big) + 1\Big)$$

(a) $\frac{L_0}{L} > 0$

(b) $\frac{L_0}{L} \approx 0$

Figure 12: Simplified motion of the systems