Mathcamp 2016 Qualifying Quiz Solutions

Applicant ID: 6360

References

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Solutions

1. It is the future, and you are working as an alchemist in the depths of the Earth's core. You work in the business of turning silver into gold and back. Each silver piece is worth 1 dollar; each gold piece is worth $\Phi = \frac{1+\sqrt{5}}{2}$ dollars.

   You start with a single piece of silver. On your first day of the job, you turn the silver piece into a gold piece. On each successive day, you turn each silver piece from the previous day into a gold piece, and each gold piece from the previous day into two pieces, one silver and one gold. So at the end of day 1, you have 1 gold piece; at the end of day 2, you have 1 gold piece and 1 silver piece; at the end of day 3, you have 2 gold pieces and 1 silver piece.Prove that at the end of day n, your treasure is worth Φ^n dollars for all positive integers n.

   Solution: $\phi = \frac{1+\sqrt{5}}{2}$ is one of the roots of the equation:

   $$x^2 - x - 1 = 0$$

   Then think about the problem. For every day $n(n \geq 1)$, the number of silver equals to the number of gold in day $n-1$, and the number of gold equals the number of gold in day $n-1$ add the number of silver in day $n-1$.

   So set the number of silver in day n $f(n)$, the number of gold in day n $g(n)$:

   $$g(n) = g(n-1) + f(n-1)$$
   $$f(n) = g(n-1)$$

   the total money of the day n is the sum of the money of gold and silver, each silver worth 1 and each gold worth ϕ, so total money equals to:

   $$f(n) + \phi g(n) = g(n-1) + \phi(g(n-1) + f(n-1))$$
   $$= \phi[f(n-1) + \phi g(n-1)] - (\phi^2 - \phi - 1)g(n-1)$$

   Because $\phi^2 - \phi - 1 = 0$,

   $$f(n) + \phi g(n) = \phi[f(n-1) + \phi g(n-1)]$$

   $f(n) + \phi g(n)$ is a geometric sequence which the first element is $f(1) + \phi g(1) = \phi$ and the common ratio is ϕ. So

   $$f(n) + \phi g(n) = \phi^n$$

□

2. Francisco and Savannah are playing a game with two tokens, which are placed on the squares of a rectangular grid of arbitrary size, as shown in either part of Figure 1. The two tokens must be in different rows and columns. The players take turns moving a token of their choice to a different square, satisfying the following constraints:

   - Tokens can never be moved upward or to the right.
   - The row ordering must be preserved: if one token is above the other, it must stay above the other.
   - The column ordering must be preserved: if one token is to the right of the other, it must stay to the right of the other.

   Francisco goes first. Whichever player has no legal moves (with either token) loses.

   (a) Suppose one token begins above and to the right of the other, as in Figure 1a. (The constraints require that the two tokens stay in that order.) For which starting positions does Francisco win and for which does Savannah win? What is the winning strategy in each case?

   Solution: (a):
   Let's assume the left-bottom one is called A and the right-upper one is called B. Every turn A can just go left or go down, and B can just go left or go down.

   Set the leftest and the lowest square be $(0,0)$, the initial position of A be (x_1,y_1), the initial position of B be (x_2,y_2). Anytime A doesn't reach $(0,0)$, it can always move, whatever goes left or goes down, because B is always on the left-up side. So we can just think about the situation when A reaches $(0,0)$. If at the time, B can still move, then B wins; otherwise, A wins. The types that B cannot move is when B will move to $(0,i)$ or $(i,0)(i \in Z)$. Because B moves first, so B can only be at $(i,j)(i \geq 1, j \geq 1, ij \geq 2)$ when A reaches $(0,0)$.

   Now define $\Delta x = x_2 - x_1, \Delta y = y_2 - y_1, \sigma = \Delta x + \Delta y$, then

   $$\sigma = \Delta x + \Delta y = x_2 + y_2 - (x_1 + y_1)$$

   Every time we move A, it can only be move left or move down, so one of the the x_1 and y_1 will decrease 1. The same, every time we move B, one of the the x_2 and y_2 will decease 1. But σ will remain the same and σ is a Invariant. So when A reaches $(0,0)$, σ will be the same as the initial situation.
   For all the situation that B wins: $i+j \geq 3$ when A is at$(0,0)$,

   $$\sigma = \Delta x + \Delta y = i + j - (0+0) = i + j \geq 3$$

   Because σ is an invariant. When $x_2 + y_2 - (x_1 + y_1) \geq 3$, B will win. Otherwise, $x_2 + y_2 - (x_1 + y_1) = 2$, A will win.

   □

   (b) Do the same analysis assuming one token begins below and to the right of the other, as in Figure 1b. (The constraints require that the two tokens stay in that order.)

   Solution: (b):
   Set the left-upper one be $A(x_1,y_1)$, the right-bottom one be $B(x_2,y_2)$, $\Delta x = x_2 - x_1, \Delta y = y_1 - y_2$.

For A, there's no B's limit in x-axis, but it has limit in y-axis by B, because when it moves down, it can't reach the same row as B. For B, it doesn't have limit in y-axis by A, but it has limit in x-axis caused by B because when it moves left it cannot reach the same column as A.

So it gives a sense that A uses Δx to beat B while B uses Δy to beat A. We define

$$\lambda = \Delta x - \Delta y$$

For every turn we have four possible moves:

(i):A moves left, B moves left. x_1 will decrease 1 and x_2 will decrease 1, Δx will remain the same and Δy will remain the same too. So $\lambda = \Delta x - \Delta y$ will remain the same.

(ii):A moves left, B moves down. x_1 will decrease 1 and y_2 will decrease 1, Δx will increase 1 and Δy will increase 1. But $\lambda = \Delta x - \Delta y$ will remain the same.

(iii):A moves down, B moves left. y_1 will decrease 1 and x_2 will decrease 1, Δx will decrease 1 and Δy will decrease 1. But $\lambda = \Delta x - \Delta y$ will remain the same.

$(iiii)$:A moves down, B moves down. y_1 will decrease 1 and y_2 will decrease 1, Δx will remain the same and Δy will remain the same too. So $\lambda = \Delta x - \Delta y$ will remain the same.

For all the possible moves, λ will always remain the same. So λ is an invariant.

Whatever how A and B move, λ will never change. Only Δx and Δy change. We can define the current position of $A(x_i, y_i)$, $B(x_j, y_j)$, $\Delta x_n = x_j - x_i$, $\Delta y_n = y_i - y_j$. At last, when one of the points can't move, it should be their closest status, which means Δx and Δy get to their minimum. So we can simplify the problem that A just moves down and B just moves left, because this is the fastest way to decrease both Δx and Δy.

When $\lambda = 0$. Because B moves first, after the last step of B, $\Delta x_n = 1$ but $\Delta y_n = 2$, so A can still move its last step. After that, B cannot move. A wins.

When $\lambda > 0$. When $\Delta y_n = 1$ that A cannot move, $\Delta x_n = \Delta y_n + \lambda > 2$, so B can still move to the left. B wins.

When $\lambda < 0$. When $\Delta x_n = 1$ that B cannot move, $\Delta y_n = \Delta x_n - \lambda > 1$, so A can still move to the left. A wins.

The conclusion is:

When $\lambda \le 0$, A wins;

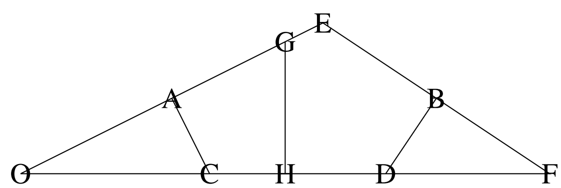
When $\lambda > 0$, B wins. $\square$

3. Given an arbitrary triangle cut out from paper, we can fold one of its sides in half, so that one corner overlaps with another. This makes a crease through the midpoint of that side, as in Figure 2a.

   Unfolding and repeating with the other two sides, we get two more creases, as in Figure 2b.

   (a) If two of the three creases have the same length, must the triangle be isosceles?

   Solution: (a):



Set $O(0,0)$, $F(a,0)$, $E(p,q)$.
Then $A(\frac{p}{2},\frac{q}{2})$, $B(\frac{p+a}{2},\frac{q}{2})$
$k_{AC} = -\frac{1}{k_{AE}} = -\frac{p}{q}$, $b_{AC} = \frac{p^2+q^2}{2q}$, so $y_{AC} = -\frac{p}{q}x + \frac{p^2+q^2}{2q}$
$y_{AC} = 0$, $x = \frac{p^2+q^2}{2p}$, so $C(\frac{p^2+q^2}{2p},0)$
$AC^2 = (\frac{p}{2} - \frac{p^2+q^2}{2p})^2 + \frac{q^2}{4} = \frac{q^2(p^2+q^2)}{4p^2}$

$k_{BD} = -\frac{1}{k_{EF}} = \frac{a-p}{q}$, $b_{BD} = \frac{q^2-a^2+p^2}{2q}$, so $y_{BD} = \frac{a-p}{q}x + \frac{q^2+p^2-a^2}{2q}$
$y_{BD} = 0$, $x = \frac{q^2}{2(p-a)} + \frac{p+a}{2}$
$BD^2 = \frac{q^2(p-a)^2+q^4}{4(p-a)^2}$

Use triangle similar, $\frac{q}{p} = \frac{2GH}{a}$, $GH = \frac{aq}{2p}$, $GH^2 = \frac{a^2q^2}{4p^2}$

If $AC^2 = BD^2$,
$\frac{q^2(p^2+q^2)}{4p^2} = \frac{q^2(p-a)^2+q^4}{4(p-a)^2}$
$(p-a)^2q^2 = p^2q^2$
$a = 2p$, so $OE = EF$.

If $GH^2 = AC^2$
$\frac{a^2q^2}{4p^2} = \frac{q^2(p^2+q^2)}{4p^2}$
$a^2 = p^2+q^2$, $OF^2 = OE^2$, so $OF = OE$

If $GH^2 = DB^2$
$\frac{a^2q^2}{4p^2} = \frac{q^2[(p-a)^2+q^2]}{4(p-a)^2}$
$\frac{a^2}{p^2} = \frac{(p-a)^2+q^2}{(p-a)^2}$,
$\frac{OF^2}{p^2} = \frac{EF^2}{(p-a)^2}$
Only when $a = 2p$, $OF = EF$, so it's uncertain.
We can give a counterexample here: if E is a right angle, let $OE = \sqrt{\frac{\sqrt{5}+1}{2}}EF, OF = \sqrt{\frac{\sqrt{5}+3}{2}}EF$,
then $BD = \frac{1}{2}OE = \frac{\sqrt{\frac{\sqrt{5}+1}{2}}}{2}EF$, $GH = \frac{OF}{2\sqrt{\frac{\sqrt{5}+1}{2}}}$, simplify it, we find $BD = GH$, but obviously

$OF \neq OE \neq EF$. Conclusion: If two of the three creases have the same length, the triangle may not be isosceles.

$\square$

(b) If all three creases have the same length, must the triangle be equilateral?

Solution: (b):
If all three creases have the same length, then $OE = OF$ and $OE = EF$, so $OE = OF = EF$ which shows the triangle is equilateral. $\square$

4. Drake is thinking of a positive integer x. He tells Misha the number of digits x has in base 2. He tells Ivy the number of digits x has in base 3. For example, if Drake thinks of $x = 11 = 1011_2 = 102_3$, he'll tell Misha "x has 4 digits in base 2" and he'll tell Ivy "x has 3 digits in base 3".

   (a) Drake alternates asking Misha and Ivy if they know x. They have the following conversation:

   MISHA: No, I don't know x.

   IVY: No, I don't know x.

   MISHA: Yes, now I know x.

   IVY: Yes, now I know x.

   What was x?

   Solution: (a):
   Set $s_2(x)$ represent the interval when there are x digits in base 2, $s_3(x)$ represent the interval when there are x digits in base 3.

$$(1)_2 \leq s_2(1) \leq (1)_2 \Rightarrow s_2(1) = 1$$

$$(10)_2 \leq s_2(2) \leq (11)_2 \Rightarrow 2 \leq s_2(2) \leq 3$$

$$(100)_2 \leq s_2(3) \leq (111)_2 \Rightarrow 4 \leq s_2(3) \leq 7$$

$$(1000)_2 \leq s_2(4) \leq (1111)_2 \Rightarrow 8 \leq s_2(4) \leq 15$$

$$2^{n-1} \leq s_2(n) \leq 2^n - 1$$

$$(1)_3 \leq s_3(1) \leq (2)_3 \Rightarrow 1 \leq s_3(1) \leq 2$$

$$(10)_3 \leq s_3(2) \leq (22)_3 \Rightarrow 3 \leq s_3(2) \leq 8$$

$$(100)_3 \leq s_3(3) \leq (222)_3 \Rightarrow 9 \leq s_3(3) \leq 26$$

$$(1000)_3 \leq s_3(4) \leq (2222)_3 \Rightarrow 27 \leq s_3(4) \leq 80$$

$$3^{n-1} \leq s_3(n) \leq 3^n - 1$$

Assume x has a digits in base 2 and b digits in base 3. We use (M) to represent what Misha knows, (I) to represent what Ivy knows.

$$2^{a-1} \leq x \leq 2^a - 1(M)$$

$$3^{b-1} \leq x \leq 2^b - 1(I)$$

After Misha says: No, I don't know x, we get:

$$2^{a-1} \leq x \leq 2^a - 1(M)$$

$$3^{b-1} \leq x \leq 2^b - 1(I)$$

$$x \geq 2(I)$$

(Because Ivy knows digits in base 2 is greater than 1)

After Ivy says: No, I don't know x, we get:

$$2^{a-1} \leq x \leq 2^a - 1(M)$$

$$x \geq 3(M)$$

(The digits of x in base 3 cannot be 1, because if it's 1, Ivy knows both $x \leq 2$ and $x \geq 2$, then she knows $x = 2$.)

$$3^{b-1} \leq x \leq 3^b - 1(I)$$

$$x \geq 2(I)$$

After that, Misha says: Yes, now I know x. This means the new condition $x \geq 3$ helps her to reduce the interval. So we get:

$$3 \leq x \leq 2^a - 1$$

only has one solution, so $a = 2, x = 3$.

Ivy can just think the ways as Misha, so she knows $x = 3$ too.

□

(b) Suppose Drake instead chooses some other functions f and g, tells Misha $f(x)$, and tells Ivy $g(x)$. Drake then alternates asking them if they know x until they both say "Yes". The functions f and g are common knowledge: you, Misha, and Ivy all know what they are. But of course, you don't know the particular numbers $f(x)$ and $g(x)$ that Drake tells Misha and Ivy.

Can Drake choose functions f and g such that you can always deduce x just by listening to Misha and Ivy's conversation?

Solution: (b):
Yes, she can. If she chooses two function $f(x)$ and $g(x)$, $f(1) = 1$, $f(2k) = f(2k+1) = k$, $g(2k-1) = g(2k) = k$. There are only two different x that make $f(x) = g(x)$, so you can always tell x by just listening to Misha and Ivy's conversation. $\square$

5. We call some positive integers *oddly nice* according to the following rules:

   - 1 is oddly nice.
   - An integer $n > 1$ is oddly nice if and only if an odd number of its proper divisors are oddly nice.

   Which numbers are oddly nice? If $s(n)$ is the number of oddly nice proper divisors of an integer n, what are all the possible values of $s(n)$? Prove your answer.

   Solution: Lemma 1: All the prime numbers are oddly nice.

   Proof:
   Prime numbers only have one proper divisor which is 1, so all of them are oddly nice.

   Lemma 2: If $n = \prod_{i=1}^{t} p_i$(p_i are prime numbers), then n is an oddly nice number and $s(n) = 2^t - 1$.

   Proof:
   Use the Second Mathematical Induction.
   When there's only one prime p_1, from Lemma 1 we know it is an oddly nice number and $s(p_1) = 1$.
   For a positive integer k, if $\forall t \in [1,k](t \in Z)$, $\prod_{i=1}^{t} p_i$ is an oddly nice number.
   For $k+1$, $n = \prod_{i=1}^{k+1} p_i$, the divisor number $d(n) = 2^{k+1}$. For all the divisors, they are the product of several prime numbers, so all of the products are oddly nice numbers. The proper divisor will not include n itself, so $s(n) = 2^{k+1} - 1$ which shows n is an oddly nice number.

   Lemma 3: $n = p^{\alpha}(\alpha \geq 2)$ is not an oddly nice number.

   Proof:
   Use Mathematical Induction.
   Start with $\alpha = 2$. n only has proper divisors 1, p. By using Lemma 1, p is an oddly nice prime. So $s(n) = 2$, n is not an oddly nice number.
   For $\alpha = k$, n is not an oddly nice number and $s(n) =$ an even number.
   For $\alpha = k+1$, n gets a new divisor p^k, but p^k is not an oddly nice number. So $s(p^{k+1}) = s(p^k) =$ an even number. So $s(p^{\alpha})$ is not an oddly nice number.

   Lemma 4: $n = pq^{\alpha}(\alpha \geq 2)$ is not an oddly nice number.

   Proof:
   Use Mathematical Induction.
   Start with $\alpha = 2$, n only has divisors $1, p, q, pq, q^2$. Using Lemma 1, 2 and 3, we know they oddly nice divisors are $1, p, q, pq$. So $s(pq^2) = 4$, which shows n is not an oddly nice number.
   For $\alpha = k$, n is not an oddly nice number, which implies $s(n) =$ an even number.
   For $\alpha = k+1$, $s(n)$ adds new divisors q^{k+1}, pq^k. Using Lemma 3, we know q^{k+1} is not an oddly nice number. We also know pq^{k+1} is not an oddly nice number. So $s(pq^{k+1}) = s(pq^k)$ which means it is not an oddly nice number.

   Lemma 5: If $n = \prod_{i=1}^{t} p_i^{\alpha_i}(\prod_{i=1}^{t} \alpha_i \geq 2)$ is not an oddly nice number, then $p_{t+1} \prod_{i=1}^{t} p_i^{\alpha_i}(\prod_{i=1}^{t} \alpha_i \geq 2)$ is also not an oddly nice number.

Proof:
When we add p_{t+1} into n, the only oddly nice numbers added are the combination of primes $p_1, p_2 \ldots, p_t$. Other things like $p_{t+1}p_i^{\alpha_i}$ are not oddly nice numbers because it has been proved before this proof to p_{t+1}. So $s(p_{t+1}\prod_{i=1}^{t} p_i^{\alpha_i}) = s(\prod_{i=1}^{t} p_i^{\alpha_i}) + \sum_{i=0}^{t}\binom{t}{i} = s(\prod_{i=1}^{t} p_i^{\alpha_i}) + 2^{t-1} =$ an even number.
So $p_{t+1}\prod_{i=1}^{t} p_i^{\alpha_i}$ is not an oddly nice number.

Lemma 6: If $p_t\prod_{i=1}^{t-1} p_i^{\alpha_i}(\prod_{i=1}^{t-1}\alpha_i \geq 2)$ is not an oddly nice number, then $n = \prod_{i=1}^{t} p_i^{\alpha_i}(\prod_{i=1}^{t}\alpha_i \geq 2)$ is not an oddly nice number.

Proof:
Use t-dimentional Mathematical Induction.
Define $f(\alpha_1, \alpha_2 \ldots, \alpha_t)$ the number of oddly nice numbers of n, $g(\alpha_1, \alpha_2 \ldots, \alpha_t)$ judge if n is an oddly nice number. If yes, then it equals 1, else it equals 0. From Lemma 5, we know that:
$\forall i \in [1,t]$, if $\alpha_i = 1, f(\alpha_1, \alpha_2 \ldots, \alpha_t)$ is an even number and
$g(\alpha_1, \alpha_2 \ldots, \alpha_t) = 0$
For $(\alpha_1, \alpha_2 \ldots, \alpha_t)$, $f(\alpha_1, \alpha_2 + 1 \ldots, \alpha_t + 1), f(\alpha_1 + 1, \alpha_2 \ldots, \alpha_t + 1), \ldots f(\alpha_1 + 1, \alpha_2 + 1 \ldots, \alpha_t)$ are not oddly nice numbers, so $f(\alpha_1, \alpha_2 + 1 \ldots, \alpha_t + 1), f(\alpha_1 + 1, \alpha_2 \ldots, \alpha_t + 1), \ldots f(\alpha_1 + 1, \alpha_2 + 1 \ldots, \alpha_t)$ are even numbers, and $g(\alpha_1, \alpha_2 \ldots + 1, \alpha_t + 1) = 0, g(\alpha_1 + 1, \alpha_2 \ldots, \alpha_t + 1) = 0, \ldots g(\alpha_1 + 1, \alpha_2 + 1 \ldots, \alpha_t) = 0$
In order to express easier, define:

$$f_s(\alpha_1) = f(\alpha_1, \alpha_2 + 1 \ldots, \alpha_t + 1)$$

$$f_s(\alpha_2) = f((\alpha_1 + 1, \alpha_2 \ldots, \alpha_t + 1))$$

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$$f_s(\alpha_t) = f(\alpha_1 + 1, \alpha_2 + 1 \ldots, \alpha_t)$$

$$g_s(\alpha_1) = g(\alpha_1, \alpha_2 + 1 \ldots, \alpha_t + 1)$$

$$g_s(\alpha_2) = g((\alpha_1 + 1, \alpha_2 \ldots, \alpha_t + 1))$$

$$.$$

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$$g_s(\alpha_t) = g(\alpha_1 + 1, \alpha_2 + 1 \ldots, \alpha_t)$$

So
$f((\alpha_1 + 1, \alpha_2 + 1 \ldots, \alpha_t + 1))$
$= f_s(\alpha_1) + g_s(\alpha_1) + f_s(\alpha_2) + g_s(\alpha_2) + \ldots + f_s(\alpha_t) + g_s(\alpha_t)$
$= \sum_{i=1}^{t} f_s(\alpha_i)$ which is an even number.

So $n = \prod_{i=1}^{t} p_i^{\alpha_i} (\prod_{i=1}^{t} \alpha_i \geq 2)$ is not an oddly nice number.

Lemma 7: $n = \prod_{i=1}^{t} p_i^{\alpha_i} (\prod_{i=1}^{t} \alpha_i \geq 2, t \in Z)$ cannot be an oddly nice number.

Proof:
Now we have two statements: $P(t) = p_t \prod_{i=1}^{t-1} p_i^{\alpha_i} (\prod_{i=1}^{t-1} \alpha_i \geq 2)$ is not an oddly nice number. $Q(t) = \prod_{i=1}^{t} p_i^{\alpha_i} (\prod_{i=1}^{t} \alpha_i \geq 2)$ is not an oddly nice number.
We can use Spiral Mathematical Induction.
$P(2)$ is true.
If $P(k)$ is true, we can use Lemma 6 to get $Q(k)$ is true. Because $Q(k)$ is true, we can use Lemma 5 to get $P(k+1)$ is true.
So both $P(t)$ and $Q(t)$ are true for all the integer t.
$Q(t)$ is $n = \prod_{i=1}^{t} p_i^{\alpha_i} (\prod_{i=1}^{t} \alpha_i \geq 2, t \in Z)$ cannot be an oddly nice number.

Conclusion: At last the only situation that a number can be an oddly nice number is Lemma 2, when the number can be expressed as $n = \prod_{i=1}^{t} p_i (p_i$ are prime numbers) and $s(n) = 2^t - 1$. $\square$

6. Waley starts with a list of all the positive integers in order. He can perform the following operations on it:

   - A 2-flip, which reverses pairs of elements, turning $1,2,3,4,5,6,\ldots$ into $2,1,4,3,6,5,\ldots$.
   - A 3-flip, which reverses triples of elements, turning $1,2,3,4,5,6,\ldots$ into $3,2,1,6,5,4,\ldots$.
   - More generally, an n-flip, for any integer $n > 1$: the list is split into groups of n consecutive terms, and then each group is reversed.

   Waley can perform any number of these operations, in any order. For instance, he can perform a 2-flip and then a 3-flip, which will first turn $1,2,3,4,5,6,7,8,\ldots$ into $2,1,4,3,6,5,8,7,\ldots$ and then into $4,1,2,5,6,3,10,7,8,\ldots$.

   If you give Waley a finite sequence of distinct positive integers, when can he put that sequence at the beginning of his list (in order)? You should find a strategy for Waley to follow whenever this can be done, and prove that all other sequences are not attainable.

   Solution:

   Define the position of the sequence $p_1, p_2, \ldots p_n$, the difference between the position of adjacent two element $\Delta p_1 = p_2 - p_1, \Delta p_2 = p_3 - p_2, \ldots \Delta p_{n-1} = p_n - p_{n-1}$

   Lemma 1: flip will never change the parity of Δp_i.

   Proof:
   If it is a $k-$flip, the whole list is divided into several parts, each part contains k element. If p_1 is in the t_1's part and p_2 is in the t_2's part. After flipping,

$$p_1{}' = (k - (p_1 - (t_1 - 1)k)) + (t_1 - 1)k = 2(t_1 - 1)k + k - p_1$$
$$p_2{}' = (k - (p_2 - (t_2 - 1)k)) + (t_2 - 1)k = 2(t_2 - 1)k + k - p_2$$
$$p_1{}' - p_2{}' = 2(t_1 - t_2)k + (p_2 - p_1)$$

   $2(t_1 - t_2)k$ is an even number, so $p_1{}' - p_2{}'$ has the same parity as $p_1 - p_2$, $\Delta p_i{}'$ has the same parity as Δp_i.

   Lemma 2: $\exists \Delta p_i = \textit{ an even number}$, then Waley cannot put the sequence at the beginning of the list(in order).

   Proof:
   we want to put the sequence at the beginning of the list, it means that $\forall \Delta p_i = 1$ at last. By using Lemma 1, 1 is an odd number, so $\forall \Delta p_i = \textit{ an odd number}$.

   Lemma 3: Any sequence with length 2 and $\Delta p_1 = \textit{ an odd number}$ can be done.

   Proof:
   Set the two position p_1, p_2, then p_1 and p_2 are always one odd one even.

   (i): If p_1 is odd and p_2 is even. We can directly use $2-$flip, then $p_1{}' = p_1 + 1$, $p_2{}' = p_2 - 1$, $\Delta p_1' = \Delta p_1 - 2$. In this situation, we reduce the difference between p_1 and p_2 by $2-$flip.

(ii): ifp_2 is odd and p_1 is even. We can use p_2-flip, then $p'_1 = 1$ and $p'_2 =$ an even number, then we can use (i) to reduce Δp_1.

After using the combination of (i) and (ii), at last we can reduce to $\Delta p_1 = 1$, which means p_{f1} and p_{f2} are adjacent. Now we can use $p_{f2}-$flip, which we can make p_{f1}, p_{f2} at the beginning of the list. If need, we can use another $2-$flip to adjust the order.

Lemma 4: Any sequence with any length of n with $\forall \Delta p_i =$ *an odd number* can be done.

Proof:
Use Mathematical Induction. From Lemma 3, we have known that any sequence with any length of 2, can be done.
If a sequence of n can be done,
For a sequence of $n+1$, we can first put the $p_1, p_2 \dots p_n$ at the beginning of the list, set the position $p_{n+1} = p_n + 2k + 1$ ($\Delta p_n = 2k+1$). What we want to do is to make $p_{n+1} = p_n + 1$.

First we use $n+2k-$flip, then we get a list:

$$\dots(2k\ members)p_n, p_{n-1}\dots, p1, \dots(n+2k-1\ members), p_{n+1}\dots$$

Then we use $2n+4k-$flip, we get a new list:

$$p_{n+1}, \dots(n+2k-1\ members), p_1, \dots, p_{n-1}, p_n, \dots(2k\ members)\dots$$

Next we use $n+k-$flip, we get a list:

$$\dots(n+k-1\ members), p_{n+1}, p_n, p_{n-1}, \dots p_1, \dots$$

At last we use $2n+k-$flip, we get a new list:

$$p_1, \dots, p_{n-1}, p_n, p_{n+1}, \dots$$

We can see that sequence $p_1, \dots, p_{n-1}, p_n, p_{n+1}$ is at the beginning of the list.

Conclusion: Any sequence with any length of n with $\forall \Delta p_i =$ *an odd number* can be put at the beginning of the list by using Lemma 4.

□