Mathcamp 2016 Qualifying Quiz Solutions

Applicant ID: 6406

References

Solutions

1. It is the future, and you are working as an alchemist in the depths of the Earth's core. You work in the business of turning silver into gold and back. Each silver piece is worth 1 dollar; each gold piece is worth $\Phi = \frac{1+\sqrt{5}}{2}$ dollars.

   You start with a single piece of silver. On your first day of the job, you turn the silver piece into a gold piece. On each successive day, you turn each silver piece from the previous day into a gold piece, and each gold piece from the previous day into two pieces, one silver and one gold. So at the end of day 1, you have 1 gold piece; at the end of day 2, you have 1 gold piece and 1 silver piece; at the end of day 3, you have 2 gold pieces and 1 silver piece.Prove that at the end of day n, your treasure is worth Φ^n dollars for all positive integers n.

   Solution: I use $G(n)$ to represent the amount of gold at the end of n^{th} day and I use $S(n)$ to represent the amount of silver at the end of n^{th} day. Then I write out the formula for $G(n)$ and $S(n)$:

$$G(1) = 1$$
$$S(1) = 0$$
$$G(n) = S(n-1) + G(n-1)$$
$$S(n) = G(n-1)$$

   Then I use $G(n-2)$ for $S(n-1)$ and I get:

$$G(n) = G(n-1) + G(n-2)$$
$$S(n) = G(n-1)$$

   As a result, both $G(n)$ and $S(n)$ are Fibonacci sequences. I use the formula for Fibonacci sequence,

$$G(n) = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}$$
$$S(n) = \frac{(\frac{1+\sqrt{5}}{2})^{n-1} - (\frac{1-\sqrt{5}}{2})^{n-1}}{\sqrt{5}}$$

   The total value we get at the end of day n is:

$$\begin{aligned} value &= \frac{1+\sqrt{5}}{2} \cdot \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}} + \frac{(\frac{1+\sqrt{5}}{2})^{n-1} - (\frac{1-\sqrt{5}}{2})^{n-1}}{\sqrt{5}} \\ &= \frac{(\frac{1+\sqrt{5}}{2})^2 \cdot (\frac{1+\sqrt{5}}{2})^{n-1} + (\frac{1-\sqrt{5}}{2})^{n-1} + (\frac{1+\sqrt{5}}{2})^{n-1} - (\frac{1-\sqrt{5}}{2})^{n-1}}{\sqrt{5}} \\ &= (\frac{1+\sqrt{5}}{2})^n \\ &= \phi^n \end{aligned}$$

$\square$

2. Francisco and Savannah are playing a game with two tokens, which are placed on the squares of a rectangular grid of arbitrary size, as shown in either part of Figure 1. The two tokens must be in different rows and columns. The players take turns moving a token of their choice to a different square, satisfying the following constraints:

   - Tokens can never be moved upward or to the right.
   - The row ordering must be preserved: if one token is above the other, it must stay above the other.
   - The column ordering must be preserved: if one token is to the right of the other, it must stay to the right of the other.

   Francisco goes first. Whichever player has no legal moves (with either token) loses.

   (a) Suppose one token begins above and to the right of the other, as in Figure 1a. (The constraints require that the two tokens stay in that order.) For which starting positions does Francisco win and for which does Savannah win? What is the winning strategy in each case?

   Solution: The game will end when Savannah reach the position $(1, 1)$. Let's assume that S begins at the position (x_1, y_1) and F begins at the position (x_2, y_2) and the grid's size is $A \times B$. During the whole game, the distance between F and S $\alpha = (x_2 + y_2) - (x_1 + y_1)$ does not change. Let's assume F will win the game. Since he begins first, α should greater than 2, or he will be on the same row or column with S.

   Here are the positions for F to win the game:

$$(x_2 + y_2) - (x_1 + y_1) > 2$$

$$1 \leq x_1 < x_2 \leq A$$

$$1 \leq y_1 < y_2 \leq B$$

   And S will win the game if F begins at other points. □

   (b) Do the same analysis assuming one token begins below and to the right of the other, as in Figure 1b. (The constraints require that the two tokens stay in that order.)

   Solution: The only difference between (a) and (b) is that α is changing. But if we make $\beta = (x2 - x1) - (y2 - y1)$, β does not change. We can analyze in this way, F needs $H = (x_1 - x_2 - 1)$ more steps to approach S and let him can't go horizontally. If the vertical distance for S to survive $V = (y_1 - y_2 - 1)$ is smaller than H, S will lose. So the position for F to win the game:

$$(x_1 - x_2 - 1) - (y_1 - y_2 - 1) > 0 \Leftrightarrow \beta > 0 \rightarrow x_2 + y_2 > x_1 + y_1$$

$$1 \leq x_1 < x_2 \leq A$$

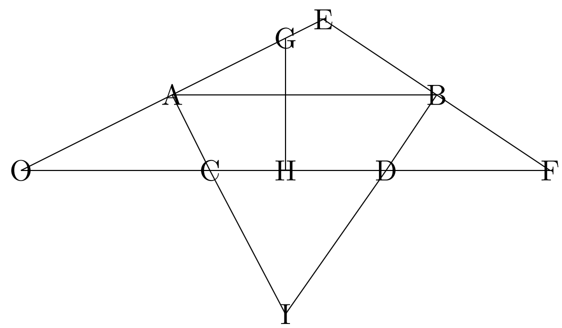
$$1 \leq y_2 < y_1 \leq B$$

   And S will win the game if F begins at other points. □

3. Given an arbitrary triangle cut out from paper, we can fold one of its sides in half, so that one corner overlaps with another. This makes a crease through the midpoint of that side, as in Figure 2a.

   Unfolding and repeating with the other two sides, we get two more creases, as in Figure 2b.

   (a) If two of the three creases have the same length, must the triangle be isosceles?



Solution:

First situation:

$$AC = BD$$

Since A, B are the midpoints of OE, EF,

$$AB \parallel OF$$

$$AB \parallel CD$$

Hence,

$$\triangle AIB \sim \triangle CID$$

We can easily get,

$$\frac{IC}{AC} = \frac{ID}{BD}$$

We know that $AC = BD$, so

$$CI = DI$$

$$AI = BI$$

$$\angle IAB = \angle IBA$$

Since $\angle EBI = \angle EAC = 90 degrees$,

$$\angle EBA = \angle EAB$$

Hence,

$$\angle EOF = \angle EFO$$

So, $EO = EF$, $\triangle OEF$ is isosceles when $AC = BD$.

Second situation:

$$AC = GH$$

We can easily get:

$$\triangle GHO \cong \triangle CAO (AAS)$$

Hence,

$$OA = OH$$

$$OE = OF$$

So, $\triangle OEF$ is isosceles when $AC = GH$.

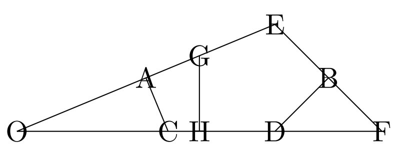
Third situation:

$$GH = BD$$

We can get a triangle that is not isosceles when $GH = BD$.
Here are the dimensions for the triangle:

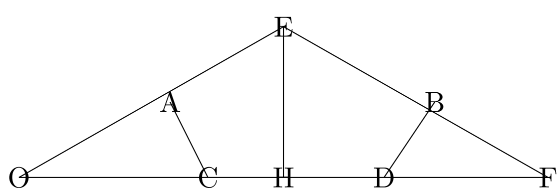
$$\angle EFO = \frac{\pi}{4}, EF = 2, OF = 2\sqrt{2} + 2, OE = 2\sqrt{2 + \sqrt{2}}.$$

At this time, $GH = BD = HD = 1$ and $AC = \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2}+1}$.
Here's the picture:



$\square$

(b) If all three creases have the same length, must the triangle be equilateral?



Solution:
It must be equilateral. Actually we can get it from (a). In this situation,

$$\triangle ACO \cong \triangle BDF \cong \triangle HEO \cong \triangle HEF$$

Hence,

$$OA = FB = OH = FH$$

$$OE = EF = OF$$

So, $\triangle OEF$ must be equilateral when $AC = EH = BD$. $\square$

4. Drake is thinking of a positive integer x. He tells Misha the number of digits x has in base 2. He tells Ivy the number of digits x has in base 3. For example, if Drake thinks of $x = 11 = 1011_2 = 102_3$, he'll tell Misha "x has 4 digits in base 2" and he'll tell Ivy "x has 3 digits in base 3".

   (a) Drake alternates asking Misha and Ivy if they know x. They have the following conversation:

   MISHA: No, I don't know x.

   IVY: No, I don't know x.

   MISHA: Yes, now I know x.

   IVY: Yes, now I know x.

   What was x?

   Solution: Let's assume Misha gets a and Ivy gets b. After Ivy hears Misha says she does not know x, she knows that a is larger than 1. So $x \geq 2$. After Misha hears Ivy's answer, she knows that $x \geq 3$ because if $x = 2$, Ivy should know the answer. Misha knows the answer immediately because the largest possible number for her is 3. So $x = 3$. now let's find a and b. We know that:

$$2^{a-1} \leq x \leq 2^a - 1$$

$$2^a - 1 = 3$$

$$a = 2$$

   In the same way,

$$3^{b-1} \leq x \leq 3^b - 1$$

$$3^{b-1} = 3$$

$$b = 2$$

   Conclusion: $a = 2, b = 2, x = 3$.

   □

   (b) Suppose Drake instead chooses some other functions f and g, tells Misha $f(x)$, and tells Ivy $g(x)$. Drake then alternates asking them if they know x until they both say "Yes". The functions f and g are common knowledge: you, Misha, and Ivy all know what they are. But of course, you don't know the particular numbers $f(x)$ and $g(x)$ that Drake tells Misha and Ivy.

   Can Drake choose functions f and g such that you can always deduce x just by listening to Misha and Ivy's conversation?

   Solution: Yes. We only need to divide the domain in to several intervals of length two. And they have same values. For example:

$$f(1) = 1$$

$$f(2x) = x^2$$

$$f(2x + 1) = x^2$$

$$g(2x - 1) = x^2$$

$$g(2x) = x^2$$

   □

5. We call some positive integers *oddly nice* according to the following rules:

   - 1 is oddly nice.
   - An integer $n > 1$ is oddly nice if and only if an odd number of its proper divisors are oddly nice.

   Which numbers are oddly nice? If $s(n)$ is the number of oddly nice proper divisors of an integer n, what are all the possible values of $s(n)$? Prove your answer.

Solution:
Lemma 1: 1 is oddly nice. If p is a prime, p is oddly nice. $p^2, p^3 ...$ are not oddly nice.
proof: p has one proper divisor 1, so it's oddly nice. $p^2, p^3 ...$ all have two proper divisors $1, p$, so they are not oddly nice.

lemma 2: Any number $n = p_1 \cdot p_2 \cdot ... \cdot p_m (p_i$ are different primes) is oddly nice. And $S(n) = 2^n - 1$.
proof: 1 is oddly nice. If $Q(m) = p_1 \cdot p_2 \cdot ... \cdot p_m (m$ goes from 1 to m)are oddly nice. $S(Q(m+1)) = \binom{m+1}{0} + \binom{m+1}{1} + ... + \binom{m+1}{m} = 2^{m+1} - 1$, which is odd. So $n = p_1 \cdot p_2 \cdot ... \cdot p_m (p_i$ are different primes) is oddly nice and all proper divisors of n are oddly nice. It has $2^m - 1$ oddly nice divisors.

Lemma 3: An oddly n will not be oddly nice anymore after timing $p^m (m \geq 2)$
proof: Let's assume that $Q(m)$ is oddly nice and it has $S(Q(m))$ oddly nice proper divisors and it has $G(Q(m)) = S(Q(m)) + 1$ oddly nice divisors. Obviously $G(Q(m))$ is even. Let's assume $Q(m) = {p_1}^{\alpha_1} \cdot {p_2}^{\alpha_2} \cdot ... \cdot {p_m}^{\alpha_m} (p_i$ are different primes). Let's assume p' is a prime that is not in the explanation of $Q(m)$. My goal is to find $G(Q(m) \cdot (p')^k)(k \geq 2)$.
Let's figure out what divisors are in p'^k: $p', p'^2, ...p'^k$ Let's say that p'^i can't have more divisors except itself because they're divisors of p'^n.
Then I count for $G(Q(m) \cdot p'), G(Q(m) \cdot p'^2)...G(Q(m) \cdot p'^k)$ separately.
$G(Q(m) \cdot p') = G(Q(m)) + G(Q(m)) = 2 \cdot G(Q(m))$ because after timing p', every oddly nice divisor in $G(Q(m))$ becomes a new oddly nice divisor.
$G(Q(m) \cdot p'^k) = G(Q(m))(k \geq 2)$ because p'^k itself is not oddly nice as we proved in Lemma 1, and it does not create new divisors.(Be careful that p'^k can't split into more divisors).
We get $G(Q(m) \cdot p'^k) = 2 \cdot G(Q(m))$, which is even. Now it comes a problem, we can't decide if $Q(m) \cdot p'^k$ is oddly nice. If $Q(m) \cdot p'^k$ is oddly nice, $S(Q(m) \cdot p'^k)$ will be odd and if $Q(m) \cdot p'^k$ is not oddly nice,$S(Q(m) \cdot p'^k)$ will be even. It looks right. But after thinking about it again, I find the problem. Here's what I find:
When we got $G(Q(m) \cdot p'^k)$ which is even, we supposed that $Q(m) \cdot p'^k$ did not create new oddly number. If it's an oddly number, $G(Q(m) \cdot p'^k)$ will increase one and it will be odd. $S(Q(m) \cdot p'^k) = G(Q(m) \cdot p'^k) - 1$ will be even. And we get that $Q(m) \cdot p'^k$ is not an oddly number. It is a contradiction. Hence, an oddly n will not be oddly nice anymore after timing $p^m (m \geq 2)$.

Then I have the last part to prove,

Lemma 4: If $Q(n)$ is not oddly nice, $Q(n) \cdot p^k (k \geq 2)$ is not oddly nice.
proof: I use Mathematical Induction to prove it. From Lemma 3 we know that $Q(n) \cdot p$ is not oddly nice, so it has even number oddly nice divisors. We need to prove that the number of oddly nice divisors in $Q(n) \cdot p^2$ is even. As we know in Lemma 1, p^2 itself is not oddly nice. So after oddly nice divisors in $Q(n) \cdot p$ time p^2 there's no oddly nice divisor created. Therefore, $Q(n) \cdot p^2$ is not oddly nice and it has the same number oddly nice divisors as $Q(n) \cdot p$.
Then I'm going to prove that if $Q(n) \cdot p^k (k > 2)$ is not oddly nice, $Q(n) \cdot p^{k+1}$ is not oddly nice

either. From Lemma1, p^{k+1} itself is not oddly nice. So after oddly nice divisors in $Q(n) \cdot p^k$ time p^{k+1} there's no oddly nice divisor created. Therefore, $Q(n) \cdot p^{k+1}(k > 2)$ is not oddly nice and it has the same number oddly nice divisors as $Q(n) \cdot p^k$.

And I get Lemma 5:
if p, q coprime, then pq is oddly nice if and only if p, q are both oddly nice. An oddly nice number times $p^k(k \geq 2)$ will be not oddly nice. And it will never become oddly nice again($Lemma4$). And the only way to make oddly nice numbers is to make $n = p_1 \cdot p_2 \cdot p_3 \cdot ...p_m$. I also find that the oddly nice divisors of n is a multiplicative function($lemma2$).($lemma2$) Every prime divisors of n has 2 oddly nice divisors, so n has $S(n) = 2^m - 1$ oddly nice proper divisors.

Conclusion: A number n is oddly nice if and only if when n can be written in $n = p_1 \cdot p_2 \cdot ... \cdot p_m(p_i$ are different primes). It shows that Lemma 2 is the only way for a number to be oddly nice. And $S(n) = 2^m - 1$. □

6. Waley starts with a list of all the positive integers in order. He can perform the following operations on it:

   - A 2-flip, which reverses pairs of elements, turning $1, 2, 3, 4, 5, 6, \ldots$ into $2, 1, 4, 3, 6, 5, \ldots$.
   - A 3-flip, which reverses triples of elements, turning $1, 2, 3, 4, 5, 6, \ldots$ into $3, 2, 1, 6, 5, 4, \ldots$.
   - More generally, an n-flip, for any integer $n > 1$: the list is split into groups of n consecutive terms, and then each group is reversed.

   Waley can perform any number of these operations, in any order. For instance, he can perform a 2-flip and then a 3-flip, which will first turn $1, 2, 3, 4, 5, 6, 7, 8, \ldots$ into $2, 1, 4, 3, 6, 5, 8, 7, \ldots$ and then into $4, 1, 2, 5, 6, 3, 10, 7, 8, \ldots$.

   If you give Waley a finite sequence of distinct positive integers, when can he put that sequence at the beginning of his list (in order)? You should find a strategy for Waley to follow whenever this can be done, and prove that all other sequences are not attainable.

   Solution: Let's assume the whole sequence is a and the sequence Waley wants to move to the front is b, and the i^th integer of b is on the position p_i.
   Lemma 1: Flip operations will not change the parity of $p_j - p_i$.
   proof: Before flipping, the distance between two numbers is just $p_j - p_i$. Let's assume it's a $k - flip$ and $b(i)$ is on the position α of its group and $b(j)$ is on the position β of its group. After flipping, $b(i)$ goes $k+1-2\alpha$ to the right and gets to $p_i + k + 1 - 2\alpha$. In the same way, $b(j)$ gets to $p_j + k + 1 - 2\beta$. The distance between them is $p_j - p_i + 2(\alpha - \beta)$ which has the same parity as $p_j - p_i$ because a number will not change its parity after adding an even number.

   From lemma 1 we get something really important:
   Lemma 2: Waley can't put $b(i)$ and $b(j)$ to the beginning of the sequence if $p_j - p_i$ is even. And he can put any $b(i)$ and $b(j)$ to the beginning of the sequence if $p_j - p_i$ is odd.

   Then we're going to find a strategy for Waley to always achieve his goal when the numbers he chooses satisfy lemma 1. I'm going to use mathematical induction to prove it.

   2-integer-strategy: When there're 2 integers. I will call them $a(k)$ and $a(\phi)$, $a(k)$ is $b(1)$ and $a(\phi)$ is $b(2)$. They are on the position k, ϕ in sequence a. Let's assume $k < \phi$. As we know in lemma 2, $\phi - k$ is odd. Let's assume $\varepsilon = \frac{\phi-k-1}{2}$, Here're the operations he can use,
   First, $k + 2\varepsilon - flip$, $a(k)$ goes to position $\phi - k$. $a(\phi)$ goes to position $3(k + 2\varepsilon) + 1 - \phi = 2\phi - 2$.
   Second, $2k + 4\varepsilon - flip$, $a(\phi - k)$ goes to position $\phi + k - 1$. $a(2\phi - 2)$ goes to position 1.
   Third, $k + \varepsilon - flip$, $a(\phi + k - 1)$ goes to position $\frac{\phi+k+1}{2}$, $a(1)$ goes to position $k + \varepsilon$.
   Then, $2k + \varepsilon - flip$, $a(\frac{\phi+k+1}{2})$ goes back to position k, $a(k + \varepsilon)$ goes to position $k + 1$.
   What we need to do next is to put these two integers to the front of the sequence.
   $k + 1 - flip$, $a(k)$ goes to position 2, $a(k + 1)$ goes to position 1.
   At last, $2 - flip$, $a(1)$ goes to position 2, $a(2)$ goes to position 1.
   Finally, $b(1)$ goes to position 1 and $b(2)$ goes to position 2.
   Therefore, 2 integers can be moved to the beginning of the sequence if they satisfy Lemma 2.

   $k(k \geq 3)$-integer-strategy: Then we're going to prove that if k things can be moved to the beginning of the sequence, $k + 1$ things can also be moved to the beginning of the sequence.
   Let's assume $b(1), b(2)...b(k)$ are the integers that are done and they are on the positions $1, 2, 3...k$. His goal is to make $b(k + 1)$ which is $a(\phi)$ to position $k + 1$. Let's assume $\varepsilon = \frac{\phi-k-1}{2}$. Here're the

operations he can use,
First, $k + 2\varepsilon - flip$, $a(i)(1 \le i \le k)$ goes to position $k + 2\varepsilon + 1 - i = \phi - i$. $a(\phi)$ goes to position $3(k + 2\varepsilon) + 1 - \phi = 2\phi - 2$.
Second, $2k + 4\varepsilon - flip$, $a(i)$ goes to position $2\phi - 1 - i$. $a(\phi - i)$ goes to position $\phi + i - 1$, $a(2\phi - 2)$ goes to position 1.
Third, $k + \varepsilon - flip$, $a(\phi + i - 1)$ goes to position $3(k + \varepsilon) + 1 - (\phi + i - 1) = \frac{\phi+3k+1}{2} - i$, $a(1)$ goes to position $k + \varepsilon + 1 - 1 = k + \varepsilon$.
At last, $2k + \varepsilon - flip$, $a(\frac{\phi+3k+1}{2} - i)$ goes to position $2k + \varepsilon + 1 - (\frac{\phi+3k+1}{2} - i) = i$, $a(k + \varepsilon)$ goes to position $2k + \varepsilon + 1 - (k + \varepsilon) = k + 1$.
Hence, he can move $b(k + 1)$ to position k if he can move $b(k)$ to position k.

Conclusion: Let's assume Waley wants to move a sequence b to the front of the sequence and the i^th integer in b is on the position p_i. Waley can achieve his movement if and only if every $p_{i+1} - p_i$ is odd. And he can use 2-integer-strategy to move 2 integers to the front of the sequence and k-integer-strategy to move more than 2 integers to the front of the sequence. $\square$