Exploring the Unreal

Understanding Complex Numbers and Functions

Amodvardhan Dubey

A Summer Study

June 19, 2025

Contents

Preface

We often encounter complex numbers in high school, but unlike real numbers, their functions are rarely explored in much depth. The closest one typically gets is Euler's formula — a result that seems almost magical. Yet the yearning for a more dynamic, calculus-like understanding of complex functions often remains unfulfilled.

This text began as an attempt to satisfy my own curiosity, but it has grown into something that I hope fellow students can use to discover the beauty of the complex realm for themselves.

This text was written alongside my study of *Fundamentals of Complex Analysis with Applications to Engineering, Science, and Mathematics* by Edward B. Saff and Arthur David Snider. As such, the sequence of topics closely follows that textbook. However, this is not a reproduction or substitute for the original. Rather, it is a concise, theoretically focused re-articulation reflecting my own understanding developed during study.

I am grateful to Tristan Needham's *Visual Complex Analysis*, LibreTexts and Wikipedia for further understanding, and Desmos and GeoGebra for helping me illustrate various ideas.

Chapter 1

Complex Numbers

1.1 Why i? : Bombelli's Leap

'Complex numbers' are expressions of the form $a+bi$ where $a, b \in \mathbb{R}$; we say two complex numbers are equal i.e. $a + bi = c + di$ if and only if $a = c$ and $b = d$. We shall discuss Bombelli's exploration of cubics as a way to motivate us to take complex numbers to be a bit less imaginary than we think. Consider the cubic;

$$x^3 = 3px + 2q \tag{1.1}$$

(Note: Any cubic $x^3 + ax^2 + bx + c = 0$ can be transformed into this form by substituting $x = y - \frac{a}{3}$)

Cardano gave a remarkable formula to solve such a cubic:

$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}. \tag{1.2}$$

Bombelli considered the case $x^3 = 15x + 4$ which yields

$$x = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}$$

Does this mean there is no real solution? No ! due to the monotonic nature of x^3 , it must intersect the line $15x + 4$ at some point on the Cartesian plane, from inspection we see $x = 4$ as a solution, then it is natural to ask : Is it possible to devise rules of algebra of complex numbers that allow us to deduce the solution? Here, Bombelli made a leap by hypothesizing $\sqrt[3]{2 + 11i} = 2 + ni$ and $\sqrt[3]{2 - 11i} = 2 - ni$, here he assumed complex numbers $z_1 = a + bi$ and $z_2 = c + di$ **add** according to the rule

$$\boldsymbol{z_1 + z_2 = (a + c) + (b + d)i} \tag{1.3}$$

Further, to recover the value of 'n' he equated $(2 \pm ni)^3 = 2 \pm 11i$ where he had to assume complex numbers **multiply** according to the rule (to obtain $n = 1$)

$$\boldsymbol{z_1 \cdot z_2 = (ac - bd) + (ad + bc)i} \tag{1.4}$$

Also, as every non zero complex number $z = a + bi$ has the multiplicative inverse $z^{-1} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$, we can define **division of complex numbers** as $\frac{z_1}{z_2} = z_1 * z_2^{-1}$ $(z_2 \neq 0)$.

1.2 Point Representation of Complex Numbers

Due to the similarity of $\mathbb{C}$ with $\mathbb{R}^2$ it is natural to represent complex numbers as 2D vectors, with the number $a + bi$ corresponding to the point (a, b) on the Cartesian plane, ie. the x-axis becomes the 'Real Axis' and the y-axis becomes the 'Imaginary Axis'.

Thus, one can represent the number in its polar form as

$$a + bi = r(cos\theta + isin\theta) = r\angle\theta$$

Here, $\boldsymbol{r} = \sqrt{a^2 + b^2}$ is called the 'modulus' or 'magnitude' of the number, it is the magnitude of the vector (a,b) or the distance of the number from the origin on the cartesian representation ; while $\boldsymbol{\theta}$ is called the argument ('arg') of the number, it is the angle the vector (a,b) makes with the positive x-axis. For any given argument θ_0 , $\theta = \theta_0 + 2k\pi$ where $k \in \mathbb{Z}$ is another valid value , we define the 'principal value' of the argument (denoted Arg) by convention as the one lying in the interval $(-\pi, \pi]$ (*any half open interval of length* 2π *suffices to determine a unique value of the argument*).
An elegant consequence of such representation is the geometric multiplication rule (easily obtained via application of angle-sum trigonometric identities):

$$(R\angle\theta)(r\angle\phi) = Rr\angle\theta + \phi \tag{1.5}$$

i.e. the on multiplication, the moduli of the numbers multiply as real numbers while the arguments add.

Another term is the **conjugate** of a complex number $z = x + yi$ being $\overline{z} = x - yi$ i.e. the reflection of the point/vector (x,y) about the real axis; it is evident $z \cdot \overline{z} = |z|^2$ ($|z|$ being the modulus of z)

1.3 The Complex Exponential

Now that we have discussed addition and multiplication of complex numbers, it is natural to wonder whether exponentiation can also be extended to the complex domain. Let us

assume that a function $f(z) = e^z$ can be defined with the following properties:
$e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$ and $\frac{df(z)}{dz} = e^z$, as with the real exponential.

(Note: Complex differentiation is defined analogously to real differentiation; it will be discussed in greater detail later.)

Thus, $e^{s+it} = e^s \cdot e^{it}$. We are then left to interpret e^{it}. Using the chain rule, we treat i as an ordinary constant to obtain $\frac{d}{dt}e^{it} = i \cdot e^{it}$. Thus, if e^{it} denotes the *position* of a point on the complex plane at time t, the derivative tells us its *velocity.*

We know for $t = 0$, $e^{i0} = e^0 = 1$, thus the initial position of the point is at $(1, 0)$. But we know multiplication by i involves rotation by $90°$ by the geometric multiplication rule. Thus, the velocity is cross-radial (anti-clockwise) and has the same magnitude as the position vector. Hence, the radial distance of the point, i.e., the modulus of position, which is 1, doesn't change.

This implies the velocity remains with constant unit magnitude. Thus, if t represents time, the expression e^{it} must correspond to the position of the point at time t given by $(\cos t, \sin t)$ under such motion. On the complex plane, this becomes

$$\boldsymbol{e^{it} = \cos t + i \sin t} \tag{1.6}$$

This is the **Euler's formula**, first derived by Leonard Euler using power series definitions of exponential and sinusoids.

1.3.1 n^th Roots

Combining the Euler's formula with polar representation we have $r\angle\theta = r \cdot e^{i\theta}$ and the assumed rule $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$, for natural values of n, we have $(r \cdot e^{i\theta})^n = r^n e^{ni\theta}$, thus for a complex number $z = re^{i\theta}$, an 'n$^{\text{th}}$ root' of z is given by;

$$\zeta = \sqrt[n]{r}e^{i\theta/n}$$

Where $\sqrt[n]{r}$ is the positive real root of r. Although adding integral multiples of 2π to the argument doesn't change the number z, it affects ζ . Thus

$$\zeta = \sqrt[n]{r}e^{i(\theta+2k\pi)/n}$$

where $k \in \mathbb{N}$ and $0 \leq k \leq n - 1$. The upper limit of k is at $(n - 1)$ as $k = n$ leads to adding 2π to the argument of the root which on the complex plane corresponds to the same point (number) and thus for $k \geq n$ we just get repetition.
Also while an equally valid choice is $-(n - 1) \leq k \leq 0$, it implies $1 \leq k + n \leq n$, but as discussed, adding n to k doesn't change the roots, we see the arguments correspond to again the same points as for the positive k (with $k = 0$ being replaced by $k = n$).

1.4 Planar Sets

As we define properties of real functions on 1D intervals, for complex functions we do so on 2D planar sets on the complex plane.

The simplest planar set is a 'neighborhood' of some complex number z_0, defined as the set of points satisfying the inequality

$$|z - z_0| < \epsilon$$

where ϵ is some positive number; it is called an **open disk or circular neighborhood** of z_0.

For some set S, a point $z_0 \in S$ is called an **interior point** of S if $\exists$ a circular neighborhood of z_0 which is entirely in S (i.e., is a subset of S).

If all points in a set are interior points, then it is an **open set**, e.g., any open disk. The set $|z| \leq 4$ is NOT an open set, as the boundary points with $|z| = 4$ are not interior points.

Formally, a point z_0 is called a **boundary point** of S if every neighborhood of the point contains at least one point in and one point not in S.

The set of all boundary points of a set is called, unsurprisingly, the boundary (or frontier) of the set. A set is said to be **closed** if it contains all its boundary points, e.g., a 'closed disk', which is the set of points satisfying the inequality $|z - z_0| \leq \epsilon$, where ϵ is a positive real number.

Let z_k with $k \in \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$ be points on the plane. Then the path joining successive z_k (z_1 to z_2, z_2 to z_3, and so on) by $n - 1$ segments forms a continuous path called a **polygonal path**.

If every point in a set is connected to every other point of the set by a polygonal path lying entirely in the set, then the set is said to be **connected** (intuitively, you can draw a path from one point to any other point in the set without lifting your pen or leaving the set). An open and connected set is called a **domain**. Let us look at a theorem involving such a set.

Theorem 1.1. *If $u : D \to \mathbb{R}$ is a function where $u(z) = u(x, y)$ if $z = x + iy$ and $D \subset \mathbb{C}$ is a domain, then*

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$$

implies $u(z)$ is constant on D.

Proof: As any polygonal path can be replaced by another polygonal path comprising only of segments parallel to the x- and y-axes, the value at one point on the domain must be the same as at any other point!

A set is called **bounded** if $\exists$ a positive number R such that $|z| < R$, and unbounded otherwise.

Also, a **region** is a union of a domain and any subset of its boundary points (notably, every domain is a region as the empty set is also a subset).

Let us end our discussion of planar sets with two fundamental results on open and closed sets:

Given $S \subseteq \mathbb{C}$, S contains none of its boundary points $\iff$ S is open.

Proof: ($\Rightarrow$) The neighborhood of every point in S must contain only points in S or only points not in S, as it cannot contain both. But the neighborhood of any point always contains the point itself; therefore, it can only contain points in S, i.e., every point in S is an interior point, thus S is open. ($\Leftarrow$) If S is open, it cannot have a boundary point by definition. Hence proved.

Given $S \subseteq \mathbb{C}$, S is closed $\iff$ $\mathbb{C} \setminus S$ is open.

Proof: Any boundary point of S must also be one of $\mathbb{C} \setminus S$, as the neighborhood of such points contains at least one point of both. But if S is closed (i.e., S has all these points), then $\mathbb{C} \setminus S$ has none of them, thus $\mathbb{C} \setminus S$ is open. Similarly, if $\mathbb{C} \setminus S$ is open, then it contains none of the boundary points (by the previous result), thus all the boundary points reside in S, making it closed.

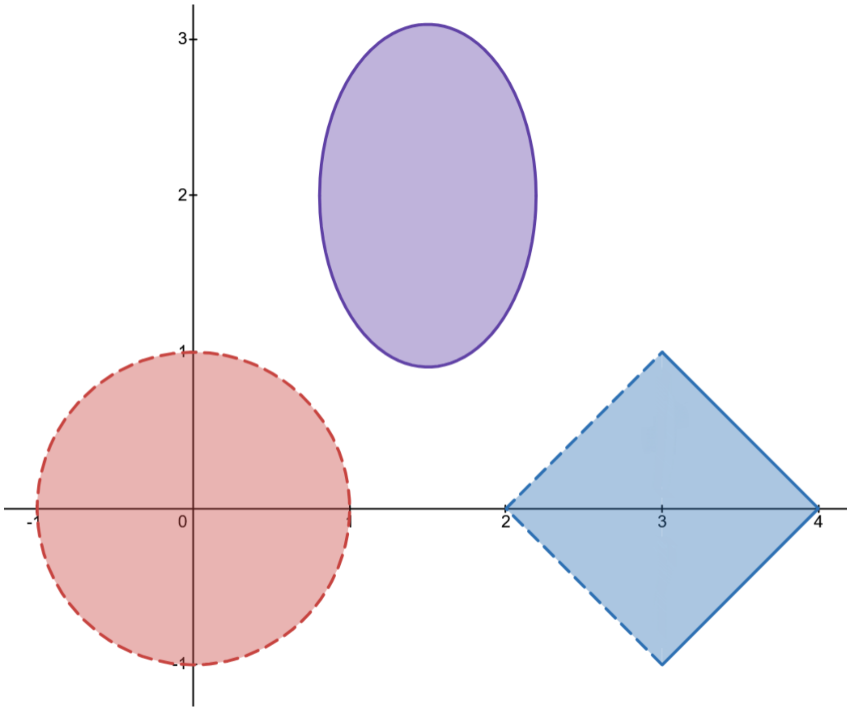

Figure 1.1: Three bounded and connected sets : open (red disk), closed (purple ellipse) and a region (blue square). *Boundary points form a solid line when included and dotted when excluded from the set.*

Chapter 2

Towards Complex Calculus

2.1 Functions of a Complex Variable

A function is a relation that maps an element from a set called the 'domain' to a unique element in another set called the 'codomain'.

Here we shall look at functions complex valued functions of a complex variable. The domain of a function written as an expression of the input is taken to be all points where the expression is defined unless stated otherwise . For instance the domain of $f(z) = 1/z$ is $\mathbb{C} \setminus \{0\}$.

If $z = x + iy$ then $f(z) = f(x, y)$, further we can decompose the image of z into a real part $u(x, y)$ and an imaginary part $v(x, y)$ to get$f(z) = u(x, y) + iv(x, y)$. Thus any such function can be viewed as a pair of real valued functions of two real variables (the x and y coordinates of a point on the complex plane).

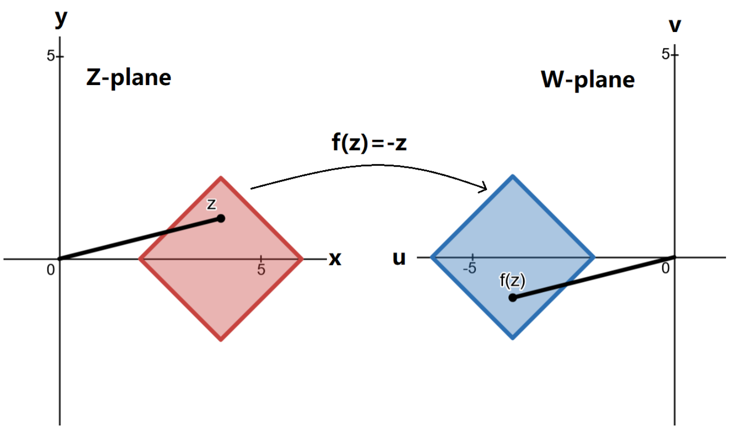

Figure 2.1: The function $f(z) = -z = e^{i\pi}z$ rotates the argument of input by 180° , thus transforming the red region (domain) to the blue one (range)

Visualizing complex functions similar to how we do real ones would require four orthogonal axes which is difficult if not impossible to imagine. Thus, a simpler visualization

is the domain being shown on a complex plane called the 'z plane' and the image (range) being shown on another plane called the 'w plane'.
However, no single representation is equally helpful for all types of functions. For instance, the function $f(z) = \frac{1}{z}$, also known as the **inversion function**, exhibits behavior that is difficult to capture clearly on the standard complex plane. To overcome this, a powerful technique known as **stereographic projection** provides a more elegant way to visualize such functions.

2.2 The Riemann Sphere

Stereographic Projection allows us to make a one-to-one correspondence between points on the complex xy plane and the surface of the unit sphere $x^2 + y^2 + z^2 = 1$ called the **Riemann Sphere**. This is achieved joining any point $z = x + iy$ to the North Pole $(0, 0, 1)$ of the sphere. The point of intersection of this line and the Riemann Sphere is unique and is identified as the 'projection' of z.

2.2.1 The Projection

The number $z = x + iy$ is represented by the point $(x, y, 0)$, the line joining it with the north pole is thus $(xt, yt, 1 - t)$ using parameter $t \in \mathbb{R}$. Thus, at the projection, $(xt)^2 + (yt)^2 + (1 - t)^2 = 1 \iff (|z|^2 + 1)t^2 - 2t = 0$ but $t \neq 0$ as it corresponds to the north pole $\Rightarrow t = \frac{2}{|z|^2+1}$.

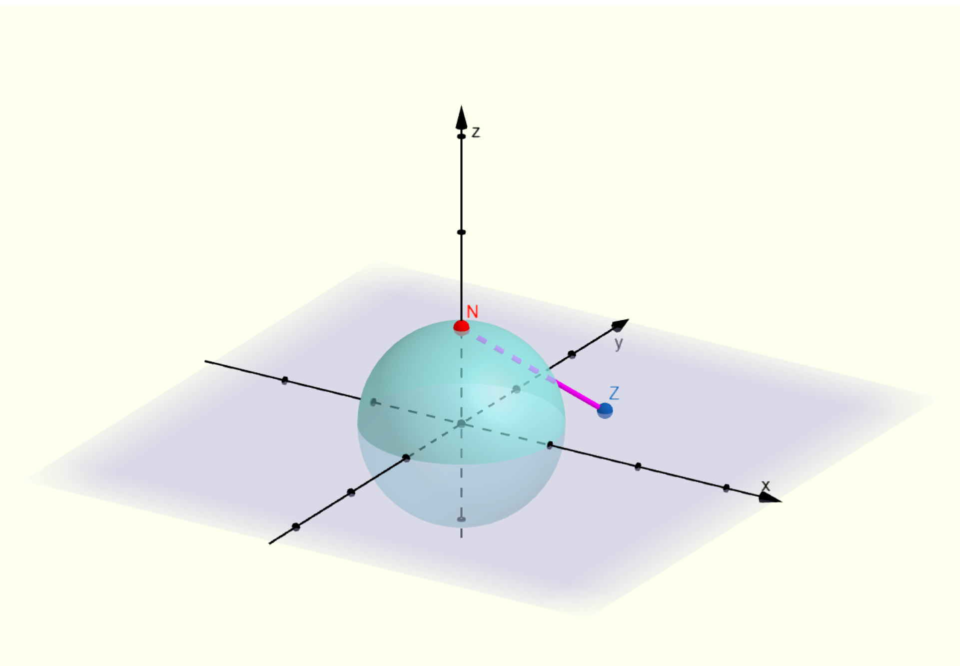

Figure 2.2: Projecting a number Z onto the sphere

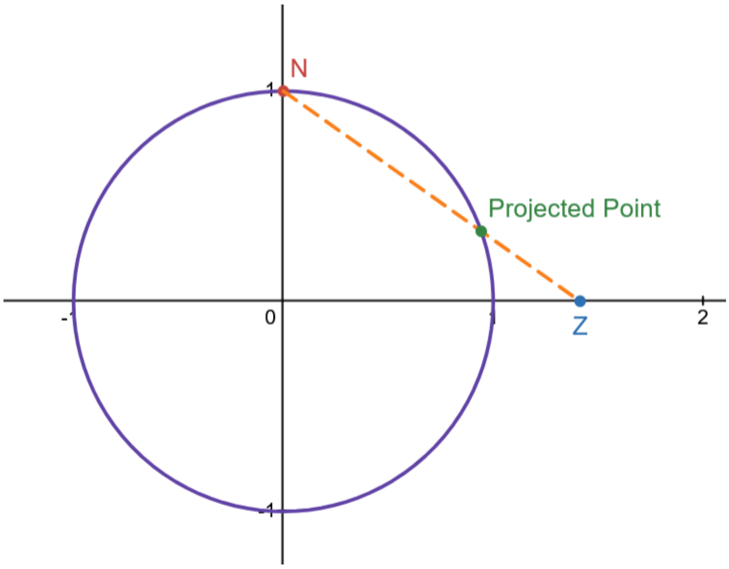

Figure 2.3: A cross section of above diagram through a plane containing N $(0, 0, 1)$, $Z = x + iy$ and the origin

Further since $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$, the projected point $(\hat{x}, \hat{y}, \hat{z})$ is given by:

$$\hat{x} = \frac{2\,\mathrm{Re}(z)}{|z|^2 + 1}, \qquad \hat{y} = \frac{2\,\mathrm{Im}(z)}{|z|^2 + 1}, \qquad \hat{z} = \frac{|z|^2 - 1}{|z|^2 + 1}$$

Similarly, for a known point on the sphere (x_1, x_2, x_3) we can find the corresponding preimage on the complex plane by $x = x_1/t$, $y = x_2/t$, $x_3 = 1 - t$ as

$$x = \frac{x_1}{1 - x_3} \qquad y = \frac{x_2}{1 - x_3}$$

2.2.2 The Point at Infinity

As we increase $|z|$, the projection on the sphere approaches the north pole $(0, 0, 1)$ but it remains an image to no finite $z \in \mathbb{C}$ under our projection. However, we can assign the north pole to the extended complex number ∞.
This turns our projection on the sphere a bijection with the **extended complex plane** defined as:

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

We can now define a **'neighborhood of infinity'** which is the preimage of an 'arctic cap' on the sphere defined as $x_3 > \epsilon$ for some $\epsilon \in (-1, 1)$; $|z| = \frac{\sqrt{x_1^2 + x_2^2}}{1 - x_3} = \frac{\sqrt{1 - x_3^2}}{1 - x_3}$, hence, given the increasing nature of the function $g(x) = \frac{\sqrt{1 - x^2}}{1 - x}$ as shown by calculus, on the complex plane, the 'arctic cap' corresponds to set of points satisfying:

$$|z| > R = g(\epsilon)$$

Thus the preimage is the set $\{z \in \mathbb{C} : |z| > R\} \cup \{\infty\}$ which is the exterior of a circle centered at the origin including the point at infinity.

2.2.3 Lines are Circles

Lines and circles are described by the common equation:

$$A(x^2 + y^2) + Bx + Cy + D = 0$$

Substituting x and y as expressions of x_1,x_2 and x_3 we get

$$A(x_1^2 + x_2^2) + Bx_1(1 - x_3) + Cx_2(1 - x_3) + D(1 - x_3)^2 = 0$$

Simplifying further using $x^2 + y^2 + z^2 = 1$,

$$A(1 - x_3^2) + Bx_1(1 - x_3) + Cx_2(1 - x_3) + D(1 - x_3)^2 = 0$$

Since $(1 - x_3) \neq 0$ for any projected point,

$$A(1 + x_3) + Bx_1 + Cx_2 + D(1 - x_3) = 0$$

However, as only single powers of x_1,x_2 and x_3 persist, this is the equation of a plane which we know can intersect the unit sphere only in circles.
Thus lines and circles are projected onto the sphere as circles. For lines, $A = 0$; so the equation becomes

$$Bx_1 + Cx_2 + D(1 - x_3) = 0$$

Note that the north pole $(0, 0, 1)$ is a solution to this equation, meaning the projection of a line passes through the projection of the point at infinity. Hence, using the Riemann Sphere, we interpret lines as not having ends in the same way circles don't, the line "touches infinity and comes back" !

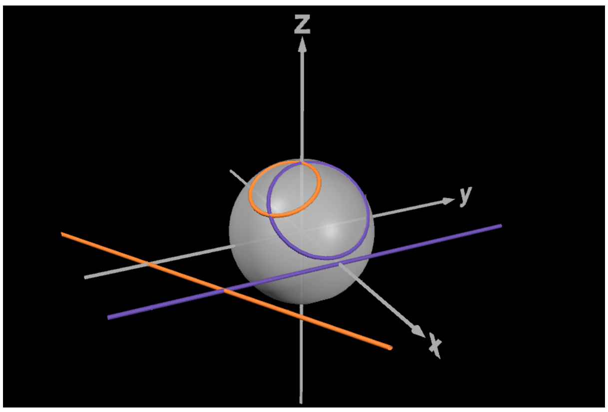

Figure 2.4: Two lines at different distances projected onto the Riemann Sphere.

2.2.4 The Inversion Function

Consider the limit $\lim_{x\to 0^+}\frac{1}{x}$ for real x, $\lim_{x\to 0^+}\frac{1}{x}=+\infty$ and $\lim_{x\to 0^-}\frac{1}{x}=-\infty$, $\therefore$ the limit doesn't exist.
However, in the previous section we discovered that a line on the complex plane (including the real axis) extend in both directions to meet at the same infinity, thus in $\hat{\mathbb{C}}$, the limit does exist ! : Consider any neighborhood of infinity $|z|>R$, then, we can always find a neighborhood of zero given by $|z-0|<R$ such that $f(z)=1/z$ always lies in that neighborhood of infinity (we will soon discuss limits in more detail).
Thus, although certain operations like $0\cdot\infty$ and $\infty-\infty$ remain undefined on $\hat{\mathbb{C}}$, we can still define division by ∞ and 0 in an intuitive manner taking inspiration from the limit as:

$$\frac{a}{0}=\infty \qquad \text{and} \qquad \frac{a}{\infty}=0 \quad \text{for } a\in\hat{\mathbb{C}}\setminus\{0,\infty\}$$

Using this definition, the inversion function becomes a bijection from the extended complex plane to itself.
Thus, every point on the Riemann Sphere, maps to some other unique point on the sphere; Given a $z\in\mathbb{C}$ let us find its image on the sphere:

$$P(1/z)=\left(\frac{2\,\mathrm{Re}(1/z)}{|1/z|^2+1},\frac{2\,\mathrm{Im}(1/z)}{|1/z|^2+1},\frac{|1/z|^2-1}{|1/z|^2+1}\right)$$

Now, $1/z=\bar{z}/(z\cdot\bar{z})=\bar{z}/|z|^2 \Rightarrow |z|^2\,\mathrm{Re}(1/z)=\mathrm{Re}(z)$ and $|z|^2\,\mathrm{Im}(1/z)=-\,\mathrm{Im}(z)$, hence

$$P(1/z)=\left(\frac{2\,\mathrm{Re}(z)}{|z|^2+1},-\frac{2\,\mathrm{Im}(z)}{|z|^2+1},\frac{1-|z|^2}{1+|z|^2}\right)$$

comparing with the projection of z given by:

$$P(z)=\left(\frac{2\,\mathrm{Re}(z)}{|z|^2+1},\frac{2\,\mathrm{Im}(z)}{|z|^2+1},\frac{|z|^2-1}{|z|^2+1}\right)$$

Thus, the y and z coordinates flip while the x coordinate remains same, that is, the **inversion function rotates the Riemann Sphere by 180° about the real axis** (much like how $f(z)=-z$ rotates the complex plane by 180°) !

In general, the functions of the form $M(z)=\frac{az+b}{cz+d}$ for $a,b,c,d\in\mathbb{C}$ and $ad-bc\neq 0$ called **Möbius transformations** are neatly visualized on the Riemann Sphere, as they can be broken into composition of the following maps:

- $z\to z+d/c$, a translation.
- $z\to 1/z$, inversion.
- $z\to -\frac{ad-bc}{c^2}z$, a scaling and rotation (via geometric multiplication rule).

- $z \to z + (a/c)$, another translation.

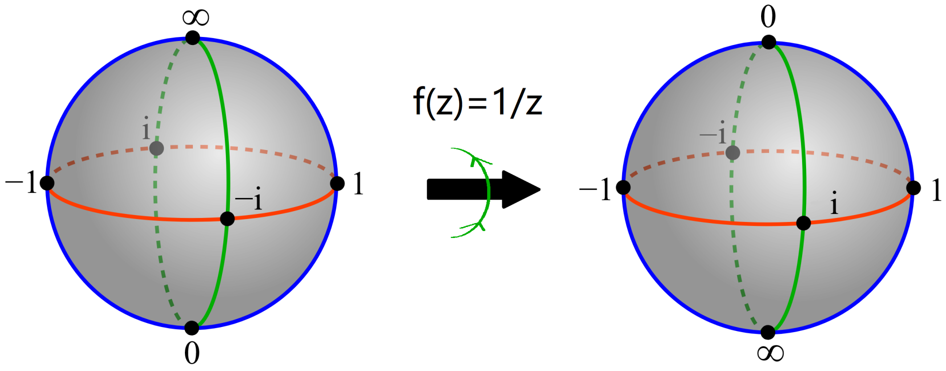

Figure 2.5: the inversion function 'inverting' the Riemann Sphere by rotating about the real axis. The sphere on the left is labelled by preimages of P(z) and on the right by preimages of P(1/z).

2.3 Limits and Continuity

2.3.1 Limit of a Sequence:

If a sequence of complex numbers $z_1, z_2, \ldots, z_n$ can get arbitrarily close to some number z_0 for large enough n, then it is said to be convergent with limit z_0.

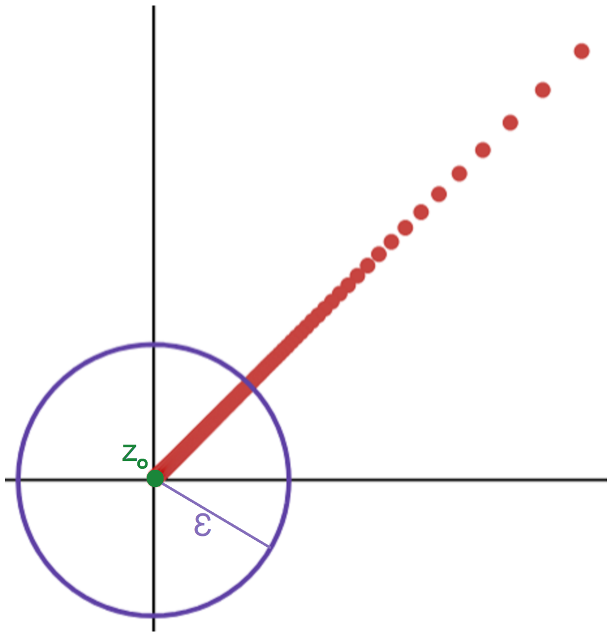

Figure 2.6: A sequence of points converging to the origin.

We can make this statement more precise as:

Definition 2.1. *A sequence of complex numbers z_n ($n \in \mathbb{N}$) is said to converge to or have limit z_0, that is*

$$\lim_{n\to\infty} z_n = z_0$$

if for any (real) $\epsilon > 0$, $\exists\ N \in \mathbb{N}$ such that

$$|z_n - z_0| < \epsilon \text{ for all } n > N$$

In words, this means that for every circular neighborhood of the 'limit', there exists a term beyond which all subsequent (possibly repeated) terms of the sequence lie within the neighborhood. For example, the sequence

$$z_n = \frac{an+b}{cn+d} \qquad \text{for} \qquad c \neq 0$$

has limit 'a/c' while the sequence

$$z_n = (i)^n$$

doesn't converge , as the terms keep jumping around $i, -1, -i, 1$ without settling near any single value.

2.3.2 Limit of a Function:

We can similarly define the limit of a complex function. We say w_0 is the limit of the function $f(z)$ at z_0 if for any (ϵ) neighborhood of w_0 we are able to find a (δ) neighborhood of z_0 (excluding z_0 itself since we are concerned only with the points in its vicinity) which maps to a planar set fully inside w_0's neighborhood.

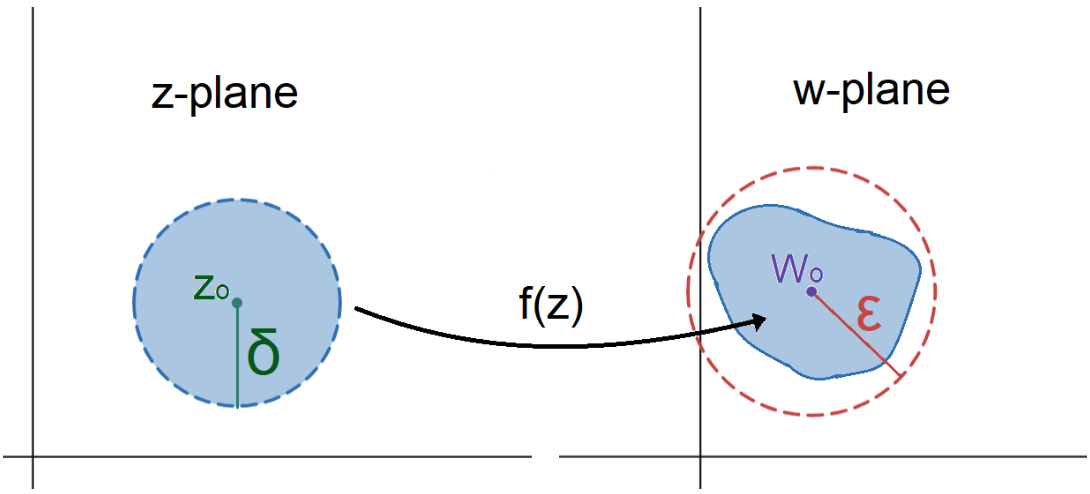

Figure 2.7: $f(z)$ transforming the δ neighborhood of Z_0 into a region lying inside the ϵ neighborhood of W_0

Formally:

Definition 2.2. *The limit of the function $f(z)$ is w_0 as z approaches z_0 , that is*

$$\lim_{z \to z_0} f(z) = w_0$$

if for any (real) $\epsilon > 0$, $\exists$ a δ such that

$$|f(z) - w_0| < \epsilon$$

whenever

$$0 < |z - z_0| < \delta$$

For example,

$$\lim_{z \to z_0} z^2 = z_0^2$$

Proof: $0 < |z - z_0| < \delta$.
Further, using triangle inequality ($|z_1 + z_2| \leq |z_1| + |z_2|$) $|z^2 - z_0^2| = |z - z_0||z - z_0 + 2z_0| \leq |z - z_0|(|z - z_0| + 2|z_0|) < \delta(\delta + 2|z_0|)$. Thus, we aim to find δ such that $\delta(\delta + 2|z_0|) \leq \epsilon$ for any given ϵ; that is, $\delta^2 + 2|z_0|\delta - \epsilon \leq 0$, choosing the bigger root by quadratic formula we obtain $0 < \delta \leq \sqrt{|z_0|^2 + \epsilon} - |z_0|$. Hence, we obtain $\delta = \sqrt{|z_0|^2 + \epsilon} - |z_0|$.

2.3.3 Relating Sequential and Functional Limits:

The limit of a sequence and that of a function are related by the following theorem:

Theorem 2.1.

$$\lim_{z \to z_0} f(z) = w_0$$

$$\Longleftrightarrow$$

For every sequence $\{z_n\}$ with $z_n \neq z_0$ and $\lim_{n\to\infty} z_n = z_0$: $\lim_{n\to\infty} f(z_n) = w_0$.

($\Rightarrow$): Assume $\lim_{z\to z_0} f(z) = w_0$, and let $\{z_n\}$ be a sequence with $z_n \neq z_0$ and $\lim_{n\to\infty} z_n = z_0$. Hence, given any $\epsilon > 0$, $\exists\, \delta > 0$ such that

$$0 < |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \epsilon.$$

Since z_n converges to z_0, $\exists\, N \in \mathbb{N}$ such that $|z_n - z_0| < \delta$ for all $n > N$. And since $z_n \neq z_0$, it follows that $0 < |z_n - z_0| < \delta$ for all $n > N$. $\Rightarrow |f(z_n) - w_0| < \epsilon$ for all $n > N$. $\therefore \lim_{n\to\infty} f(z_n) = w_0$. $\square$

($\Leftarrow$): Assume that for every sequence $\{z_n\}$ with $z_n \neq z_0$ and $\lim_{n\to\infty} z_n = z_0$, we have $\lim_{n\to\infty} f(z_n) = w_0$.

Suppose, for contradiction, that $\lim_{z\to z_0} f(z) \neq w_0$. This means there exists some $\epsilon_0 > 0$ such for every $\delta > 0$ we choose (however small), there is always some x with $0 < |x - z_0| < \delta$, but $|f(x) - w_0| \geq \epsilon_0$.

Now, define $\delta_k = 1/k$. Then for each δ_k, by the assumption above, there exists some point z_k such that

$$0 < |z_k - z_0| < \delta_k = \frac{1}{k}, \quad \text{and} \quad |f(z_k) - w_0| \geq \epsilon_0.$$

This gives us a sequence $\{z_k\}$ where each $z_k \neq z_0$, and since the distance from z_k to z_0 is less than $1/k$, the sequence clearly gets arbitrarily close to z_0 as k increases. In other words, $\{z_k\}$ is a sequence with $z_k \neq z_0$ and $z_k \to z_0$.

But then, by how we chose the z_k, we know that for every k, $|f(z_k) - w_0| \geq \epsilon_0$, which means the values $f(z_k)$ stay at least ϵ_0 away from w_0 (cannot get arbitrarily close) — so they cannot converge to w_0.

This contradicts our assumption that every such sequence $\{z_n\}$ must satisfy $f(z_n) \to w_0$. Hence, the assumption that the limit doesn't exist must be false. Therefore,

$$\lim_{z\to z_0} f(z) = w_0. \qquad \square$$

2.3.4 Properties of Limits

Theorem 2.2. *If* $\lim_{z\to z_0} f(z) = A$ *and* $\lim_{z\to z_0} g(z) = B$*, then:*

(i) $\lim\limits_{z\to z_0} (f(z) \pm g(z)) = A \pm B,$

(ii) $\lim\limits_{z\to z_0} f(z)g(z) = AB,$

(iii) $\lim\limits_{z\to z_0} \dfrac{f(z)}{g(z)} = \dfrac{A}{B}, \quad \text{if } B \neq 0.$

(iv) $\lim_{z\to z_0} C \cdot f(z) = C \cdot A$

The proofs of these properties are exactly the same way as they do in real analysis, because the key ingredients — like $|z_1 z_2| = |z_1||z_2|$ and the triangle inequality — still hold for complex numbers. To maintain focus, we omit the proofs here.

2.3.5 Continuity

Definition 2.3. *A function* $f(z)$ *is said to be continuous at* z_0 *if:* $lim_{z\to z_0} f(z) = f(z_0)$

Clearly, this is the same definition as for real functions.
A function is said to be **continuous on the set** S, if it is continuous for all points in S. We often think of continuity as the 'ability to draw the functional curve without lifting up our pen from the page', a similar analogy can be made for complex continuity: a continuous curve (without breaks, can be drawn by without lifting your pen on the z-plane) is mapped to another continuous curve by a function continuous in the superset of the curve (simply warped or bent, not broken). This follows from the fact that composition of continuous functions is continuous.

As continuity is just a restriction on the limit, the previously discussed properties of limits yield:

Theorem 2.3. *If $f(z)$ and $g(z)$ are continuous at z_0, then the functions $f(z) \pm g(z)$ and $f(z)g(z)$ are also continuous at z_0.*
Moreover, the quotient $\frac{f(z)}{g(z)}$ is continuous at z_0 provided $g(z_0) \neq 0$.

A consequence of these properties is that polynomials with complex coefficients are continuous on all of $\mathbb{C}$. Hence, rational functions, which are ratios of two polynomials, are continuous everywhere except at points where the denominator polynomial is zero.

2.3.6 Further Insight

Decomposing Complex Limits:

Let $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$, and $w_0 = u_0 + iv_0$. Then,

$$\lim_{z \to z_0} f(z) = w_0$$

if and only if

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} u(x, y) = u_0 \quad \text{and} \quad \lim_{\substack{x \to x_0 \\ y \to y_0}} v(x, y) = v_0.$$

Proof: ($\Rightarrow$): Let $\lim_{z \to z_0} f(z) = w_0$. Then for a given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(z) - w_0| < \epsilon, \quad \text{whenever } |z - z_0| < \delta.$$

Now,

$$\left|\overline{f(z)} - \overline{w_0}\right| = \left|\overline{f(z) - w_0}\right| = |f(z) - w_0| < \epsilon \quad \text{whenever } |z - z_0| < \delta.$$

Thus,

$$\lim_{z \to z_0} \overline{f(z)} = \overline{w_0}$$

Combining it with the known properties of limit we get:

$$\lim_{\substack{x\to x_0\\ y\to y_0}} u(x,y) = \lim_{z\to z_0}\left(\frac{f(z)+\overline{f(z)}}{2}\right) = \frac{w_0+\overline{w_0}}{2} = u_0,$$

$$\lim_{\substack{x\to x_0\\ y\to y_0}} v(x,y) = \lim_{z\to z_0}\left(\frac{f(z)-\overline{f(z)}}{2i}\right) = \frac{w_0-\overline{w_0}}{2i} = v_0.$$

$$\lim_{\substack{x\to x_0\\ y\to y_0}} u(x,y) = u_0, \quad \lim_{\substack{x\to x_0\\ y\to y_0}} v(x,y) = v_0.$$

($\Leftarrow$):

$$u_0 + iv_0 = \lim_{\substack{x\to x_0\\ y\to y_0}} u(x,y) + i\lim_{\substack{x\to x_0\\ y\to y_0}} v(x,y) = \lim_{z\to z_0}\left(\frac{f(z)+\overline{f(z)}}{2}\right) + i\lim_{z\to z_0}\left(\frac{f(z)-\overline{f(z)}}{2i}\right).$$

$$= \lim_{z\to z_0} f(z).$$

$$\Rightarrow \lim_{z\to z_0} f(z) = w_0.$$

(**Note:** We can write $\lim_{z\to z_0} u(z) = \lim_{\substack{x\to x_0\\ y\to y_0}} u(x,y)$ because limits in $\mathbb{C}$ correspond to limits in $\mathbb{R}^2$. Since

$$|z-z_0| = \sqrt{(x-x_0)^2+(y-y_0)^2},$$

the condition $|z-z_0|<\delta$ means (x,y) lies within a δ-neighborhood of (x_0,y_0), and vice versa. Thus, the complex limit $z\to z_0$ is equivalent to the multivariable limit $(x,y)\to(x_0,y_0)$, which generalizes naturally to higher dimensions. Importantly, the two-variable limit

$$\lim_{\substack{x\to x_0\\ y\to y_0}} u(x,y)$$

is not necessarily equal to the step-wise limits

$$\lim_{y\to y_0}\left(\lim_{x\to x_0} u(x,y)\right) \quad \text{or} \quad \lim_{x\to x_0}\left(\lim_{y\to y_0} u(x,y)\right).$$

For example, for

$$f(x,y) = \frac{xy}{x^2+y^2},$$

both iterated limits equal zero at $x_0=y_0=0$, but along the path $y=x$,

$$\lim_{x\to 0} f(x,x) = \frac{1}{2} \neq 0,$$

so the full two-variable limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.)

Continuity of Complex Exponential:

Using Euler's theorem $e^{z_0} = e^{x_0}(\cos y_0 + i\sin y_0)$. Utilizing insight from the last problem, we have $u(x,y) = e^x\cos(y)$ and $v(x,y) = e^x\sin(y)$, as they are product of continuous real functions, we have $\lim_{\substack{x\to x_0\\ y\to y_0}} u(x,y) = e^{x_0}(\cos y_0)$ and $\lim_{\substack{x\to x_0\\ y\to y_0}} v(x,y) = e^{x_0}(\sin y_0)$. Hence proved, $\lim_{z\to z_0} e^z = e^{z_0}$, that is, the complex exponential is continuous over the complex plane.

Limit tending to infinity:

We say $\lim_{n\to\infty} z_n = \infty$ if for every $R > 0$, $\exists\, N \in \mathbb{N}$ such that for all $n > N$, $|z_n| > R$. Similarly, for functions, $\lim_{z\to z_0} f(z) = \infty$ if for every $R > 0$, $\exists\, \delta > 0$ such that $|f(z)| > R$ whenever $0 < |z - z_0| < \delta$.

These definitions are natural considering the definition of a 'neighborhood of infinity' as introduced earlier in The Point at Infinity. The length of the segment joining the projections of two numbers z, w on the Riemann Sphere is called their '**Chordal distance/Chi Metric**' denoted as $\boldsymbol{\chi(z,w)}$. As discussed earlier, a neighborhood of infinity with a larger R maps to a smaller 'arctic cap' on the Riemann Sphere, thus an equivalent statement for the limit at infinity is that given an arbitrarily small 'arctic cap', the projections of z_n for $n > N$ or projections of $f(z)$ for $0 < |z - z_0| < \delta$ lie within the cap. In other words $\lim_{n\to\infty} \chi(z_n, \infty) = 0$ and $\lim_{z\to z_0} \chi(f(z), \infty) = 0$.

2.4 Complex Differentiation

2.4.1 Definition

With limits in place, we can define the complex derivative exactly like the real one:

Definition 2.4. *f be a complex-valued function defined in a neighborhood of z_0. Then the **derivative** of f at z_0 is given by*

$$\frac{df}{dz}(z_0) \equiv f'(z_0) := \lim_{\Delta z\to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z},$$

provided this limit exists. (Such an f is said to be differentiable at z_0.)

For real functions, we often test differentiable as the derivative approaching the same value from 'both' (left and right) sides. Similarly, for complex functions, the derivative must approach the same value from all directions to z_0. This is because if the derivative takes two different values depending on the direction of Δz, that means every δ neighborhood of z_0 maps to functional values arbitrarily close to both L_1 and L_2 (say) however this is not possible if it has a limit; for a small enough $\epsilon < |L_1 - L_2|$, the ϵ neighborhood of L_1 fully excludes values closer than $(|L_1 - L_2| - \epsilon)$ to L_2, that is it cannot get arbitrarily close to L_2 (similarly L_1)!

For example, the function $f(z) = \overline{z}$ if approached parallel to x-axis, then $\Delta z = \Delta x \Rightarrow$

$$\frac{\overline{z + \Delta z} - \overline{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x}{\Delta x} = 1$$

However when approaching parallel to y-axis, $\Delta z = i\Delta y \Rightarrow$

$$\frac{\overline{z + \Delta z} - \overline{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{-i\Delta y}{i\Delta y} = -1$$

Thus, the limit does not exist for any $z \in \mathbb{C}$, that is, the conjugation map is differentiable nowhere on the complex plane.

2.4.2 Properties

On the other hand, for a positive integer n, one can use binomial formula to get $\frac{d}{dz} z^n = nz^{n-1}$ just as for the real function $f(x) = x^n$.
This also implies the differentiability of polynomials by the usual derivative properties given below (derived just like for real case, from the properties of limits).

Theorem 2.4. *If f and g are differentiable at z, then*
$(f \pm g)'(z) = f'(z) \pm g'(z)$,
$\Big(c\,f\Big)'(z) = c\,f'(z)$ *(for any constant c),*
$(fg)'(z) = f(z)\,g'(z) + f'(z)\,g(z)$,
$\left(\frac{f}{g}\right)'(z) = \frac{g(z)\,f'(z) - f(z)\,g'(z)}{g(z)^2}$ *if* $g(z) \neq 0$.
Moreover, if g is differentiable at z and f is differentiable at $g(z)$, then the chain rule holds:

$$\frac{d}{dz}\,f\big(g(z)\big) = f'\big(g(z)\big)\,g'(z).$$

2.4.3 Analyticity

Definition 2.5. *A complex-valued function $f(z)$ is said to be* ***analytic*** *on an open set S if it has a derivative at every point of S.*

We also say $f(z)$ is **analytic at some point** z_0 if it is analytic in some neighborhood of z_0.

A point z_0 is called a '**singularity**' (or singular point) of $f(z)$ if it is not analytic at z_0 but there exists a punctured neighborhood $(0 < |z - z_0| < \delta)$ of it in which f is analytic.

For example, the roots of the denominator in a rational function. (This implies $f'(z_0)$ doesn't exist as it is the only way for it to be not analytic at z_0)
A function which is analytic over $\mathbb{C}$ is called **entire** (for example, polynomials).

2.4.4 Further Insight

Differentiability Implies Continuity

let $f(z)$ be differentiable at z_0, then as $\Delta z \to 0 \iff z \to z_0$:

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = L \text{ (say)}$$

Then by the property of limits on multiplication,

$$\lim_{z\to z_0}(f(z)-f(z_0)) = \lim_{z\to z_0}(f(z)) - f(z_0) = \lim_{z\to z_0} L(z-z_0) = 0$$

$$\Rightarrow \lim_{z\to z_0} f(z) = f(z_0)$$

Hence proved, f is continuous at z_0.

Complex Derivative as the 'Amplitwist'

Let $f(z)$ be differentiable on z_0. Then

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \left(\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right)(z-z_0)$$

Say $\lambda(z) = \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)$ then

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \lambda(z)(z-z_0)$$

Clearly, $\lim_{z\to z_0} \lambda(z) = 0$, thus in a small neighborhood of z_0:

$$f(z) \approx f(z_0) + f'(z_0)(z-z_0) \quad \text{(exact in the limit } z \to z_0)$$

If in polar form, $\boldsymbol{f'(z_0) = \rho e^{i\phi}}$, then we can interpret $f(z)$ as transforming tiny (infinitesimal) vectors $(z-z_0)$ emanating from z_0 to vectors whose magnitudes are scaled (amplified) by 'ρ':

$$\lim_{z\to z_0} |f(z)-f(z_0)| = \rho|z-z_0|$$

and their argument rotated (twisted) by 'ϕ'

$$\lim_{z\to z_0} \arg\{f(z)-f(z_0)\} = \arg\{z-z_0\} + \phi$$

(by the geometric multiplication rule 1.5) to obtain the transformed tiny vector

$$\lim_{z \to z_0} [f(z) - f(z_0)] = \rho e^{i\phi}(z - z_0)$$

which emanates from the point $f(z_0)$.
The real derivative is often understood as the slope of a local linear approximation of the function, similarly the complex derivative can be understood as the measure of the 'Amplitwist' of a local similarity transformation approximation of the function (shapes drawn in an infinitesimally small neighborhood of z_0 are scaled and rotated to similar shapes in the co-domain).

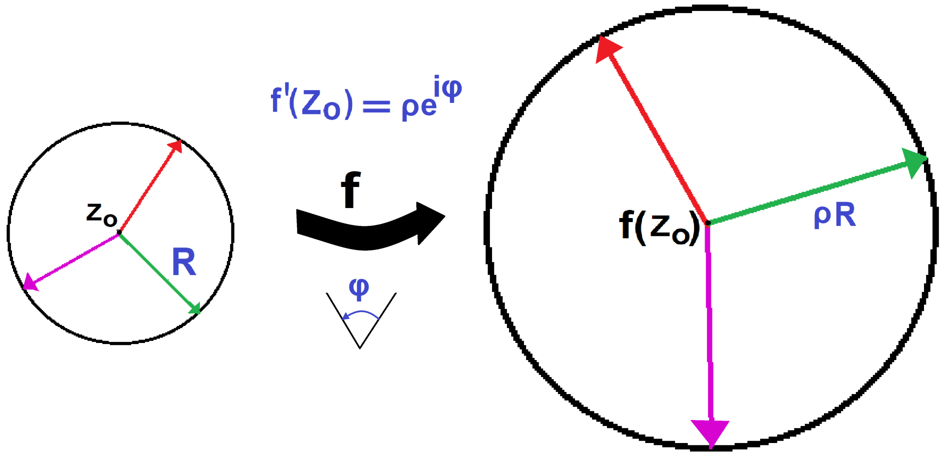

Figure 2.8: infinitesimal vectors ($R \to 0$) from z_0 being amplitwisted by f to infinitesimal vectors from $f(z_0)$

The Conformal Nature of Analytic Functions

If $|f'(z_0)| = \rho = 0$, the image of tiny shapes in an infinitesimal neighborhood of z_0 is evidently not transformed by a simple similarity transformation (Amplitwist), as the image vectors collapse to a single point at $f(z_0)$ (i.e., they are crushed to zero). These vectors no longer have a well-defined "twist", since the zero vector has no well-defined argument.
Let us consider a domain (containing z_0) where $|f'(z)| \neq 0$ and the function is differentiable at all points, that is, it is analytic at z_0; Then the function takes continuous curves with well defined tangents in the z-plane to other such curves in the w-plane, amplitwisted according to the non-zero derivative at each point.
Proof: Consider a continuous curve parameterized by $t \in (\alpha, \beta) \subseteq \mathbb{R}$ (for example the arc-length parameter, increasing monotonically along one direction of the curve) as $\gamma(t) = x(t) + iy(t)$, then the unit tangential vector at $t = t_0$ is given by

$$\lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{|\gamma(t) - \gamma(t_0)|}$$

The curve when transformed by the analytic function becomes $f(\gamma(t))$. Thus using the local similarity transformation approximation (allowed by analyticity of f), the unit tangent vector at $t = t_0$ becomes

$$\lim_{t\to t_0} \frac{f(\gamma(t)) - f(\gamma(t_0))}{|f(\gamma(t)) - f(\gamma(t_0))|} = \frac{f'(\gamma(t_0))}{|f'(\gamma(t_0))|}\frac{\gamma(t) - \gamma(t_0)}{|\gamma(t) - \gamma(t_0)|} \equiv \text{ tangent vector rotated by } \phi$$

Note: Often, the condition $\gamma'(t_0) \neq 0$ is also imposed to ensure that the curve doesn't "stop" in "time" (parameter t) and turn abruptly to a different direction making the tangent direction undefined at $\gamma(t_0)$. $\gamma'(t_0) = 0$ can allow a limit like above to exist (one endpoint kept fixed as $\gamma(t_0)$), but which could differ when the endpoints of the limiting secant approach $\gamma(t_0)$ in varying rates from both sides (t_0^- and t_0^+) making the tangent undefined.

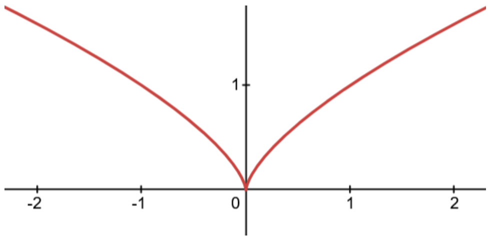

Figure 2.9: parametric curve (t^3, t^2) having a cusp at the origin, with $\gamma'(0) = 0$.

However, this means that at an intersection of two curves, both the angle and the direction of turning ('sense' of the angle) between intersecting curves are preserved. Such a mapping is called a **Conformal Map** ; it preserves the angles and orientations locally. Thus, we have a beautiful result:

Theorem 2.5. *Let f be analytic on a domain $D \subseteq \mathbb{C}$, and suppose that $f'(z) \neq 0$ for all $z \in D$. Then f is a conformal map on D.*

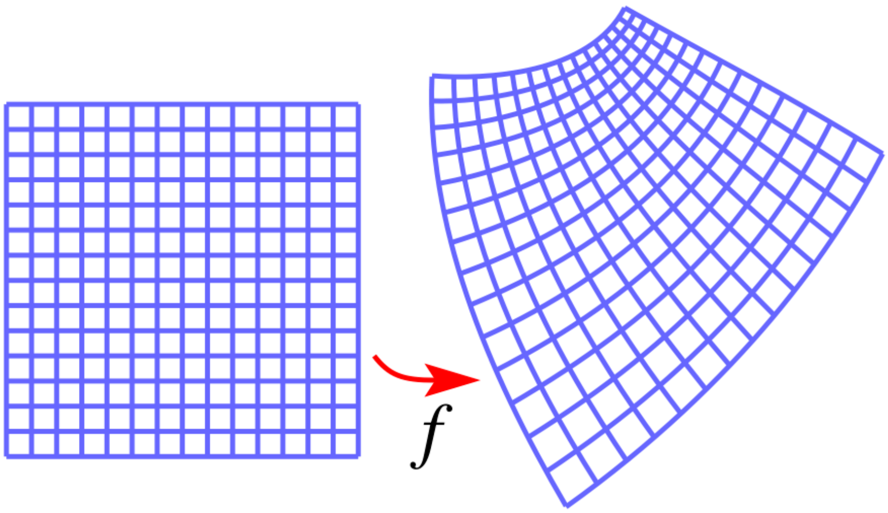

Figure 2.10: A conformal map f transforming a rectangular grid into a deformed grid such that the angle between two intersecting curves still remains 90°.

2.5 The Cauchy-Riemann Equations

If a function $f(z) = u(x,y) + iy(x,y)$ is differentiable at $z_0 = x_0 + iy_0$, the derivative of the function must be independent of the direction of approach of z_0. This suggests a restriction on how the real and complex parts of the function are related.

For horizontal approach, $\Delta z = \Delta x$

$$f'(z_0) = \lim_{\Delta x \to 0}\left[\frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x}\right] + i\lim_{\Delta x \to 0}\left[\frac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x}\right] = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

For vertical approach, $\Delta z = i\Delta y$

$$f'(z_0) = \lim_{\Delta y \to 0}\left[\frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{i\Delta y}\right] + i\lim_{\Delta y \to 0}\left[\frac{v(x_0, y_0+\Delta y) - v(x_0,y_0)}{i\Delta y}\right] = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Comparing the real and imaginary parts, we obtain the **Cauchy-Riemann Equations**:

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} \tag{2.1}$$

Hence, we have the following condition

Theorem 2.6. *The function $f(z) = u(x,y) + iv(x,y)$ is differentiable at $z_0 \Rightarrow$ the Cauchy-Riemann equations are satisfied at z_0*

This serves as a necessary condition that helps to rule out non-differentiable functions. Consider the function

$$f(z) = \begin{cases} \dfrac{xy}{z}, & z \neq 0 \\ 0, & z = 0, \end{cases}$$

Then $u = \frac{x^2y}{x^2+y^2}$ and $v = \frac{-xy^2}{x^2+y^2}$ At the origin, using $\lim h \to 0$,

$$\frac{\partial u}{\partial x} = \frac{\frac{h^2 * 0}{h^2}}{h} = 0, \qquad \frac{\partial u}{\partial y} = \frac{\frac{h * 0^2}{h^2}}{h} = 0$$

$$\frac{\partial v}{\partial x} = \frac{\frac{-h * 0^2}{h^2}}{h} = 0, \qquad \frac{\partial v}{\partial y} = \frac{\frac{-0 * h^2}{h^2}}{h} = 0$$

Thus, the CR conditions are satisfied. Let us check if the function is differentiable at the origin by approaching along the line $y = mx$:

$$\lim_{z \to 0}\frac{f(z) - 0}{z - 0} = \frac{mx^2}{(x+imx)^2} = \frac{m}{(1+im)^2}$$

That is, the limit depends on m, the slope of the direction of approach, so the derivative doesn't exist at the origin. We conclude that the Cauchy-Riemann equations are not sufficient to imply differentiability.
However, the continuity of partial derivatives of u and v allows us to state a sufficient condition:

Theorem 2.7. *Let the function $f(z) = u(x, y) + iv(x, y)$ be defined on an open set G (containing z_0). If the first partial derivatives of u and v are continuous at z_0 and satisfy the Cauchy-Riemann equations at z_0 then f is differentiable at z_0.*

Proof: The difference quotient is:

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{[u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)] + i[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)]}{\Delta x + i\Delta y}$$

Now,

$$u(x_0+\Delta x, y_0+\Delta y)-u(x_0, y_0) = [u(x_0+\Delta x, y_0+\Delta y)-u(x_0, y_0+\Delta y)]+[u(x_0, y_0+\Delta y)-u(x_0, y_0)]$$

By the Mean Value Theorem, there exists a x' between x_0 and $x_0 + \Delta x$ such that:

$$u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0 + \Delta y) = \Delta x \left[\frac{\partial u}{\partial x}(x', y_0 + \Delta y)\right]$$

Now as $\Delta z \to 0 \iff \Delta y \to 0$ and $\Delta x \to 0$, that is $x' \to x_0$; Thus, by continuity of partial derivative at z_0 (i.e. jointly continuous with respect to both x and y),

$$\frac{\partial u}{\partial x}(x', y_0 + \Delta y) = \frac{\partial u}{\partial x}(x_0, y_0) + \lambda_1$$

where $\lambda_1(x', y_0 + \Delta y) \to 0$ when $\Delta z \to 0$.
Similarly, we simplify the other terms including those of v using the same method to obtain:

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\Delta x\Big[\frac{\partial u}{\partial x} + \lambda_1 + i(\frac{\partial v}{\partial x} + \lambda_2)\Big] + \Delta y\Big[\frac{\partial u}{\partial y} + \lambda_3 + i(\frac{\partial v}{\partial y} + \lambda_4)\Big]}{\Delta x + i\Delta y}$$

Now, using Cauchy-Riemann equations to make all derivatives with respect to x,

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\Delta x\Big[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\Big] + \Delta y\Big[-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\Big]}{\Delta x + i\Delta y} + \frac{\lambda}{\Delta x + i\Delta y}$$

where $\lambda = \Delta x(\lambda_1 + i\lambda_2) + \Delta y(\lambda_3 + i\lambda_4)$.

Now, by triangle inequality,

$$0 \leq \left|\frac{\lambda}{\Delta x + i\Delta y}\right| \leq \left|\frac{\Delta x}{\Delta x + i\Delta y}\right| |\lambda_1 + i\lambda_2| + \left|\frac{\Delta y}{\Delta x + i\Delta y}\right| |\lambda_3 + i\lambda_4| \leq |\lambda_1 + i\lambda_2| + |\lambda_3 + i\lambda_4|$$

Thus, $\frac{\lambda}{\Delta x + i\Delta y} \to 0$ as $\Delta z \to 0$ (as $\lambda_1, \lambda_2, \lambda_3$ and λ_4 do)
Hence, we have the limit :

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\Delta x\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right] + i\Delta y\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]}{\Delta x + i\Delta y} = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right](x_0, y_0)$$

$\therefore$ f is differentiable at z_0.

2.5.1 Further Insight

When is an Analytic function constant?

If $f'(z) = 0$ for an analytic function f in some domain, by CR equations,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

thus $u =$ constant , $v =$ constant $\Rightarrow f = u + iv =$ constant.
However by CR equations, $\mathrm{Re}(f)$ or $\mathrm{Im}(f)$ alone being constant suffices as well . Also, consider $|f|$ to be constant, then

$$|f|^2 = u^2 + v^2 = \text{ constant}$$

This implies, with CR equations,

$$\frac{\partial |f|^2}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 2u\frac{\partial u}{\partial x} - 2v\frac{\partial u}{\partial y} = 0$$

$$\frac{\partial |f|^2}{\partial y} = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial u}{\partial x} = 0$$

Thus multiplying the equations by u and v respectively and adding,

$$2u^2\frac{\partial u}{\partial x} - 2uv\frac{\partial u}{\partial y} + 2uv\frac{\partial u}{\partial y} + 2v^2\frac{\partial u}{\partial x} = 2(u^2 + v^2)\frac{\partial u}{\partial x} = 0$$

Now, if $(u^2 + v^2) = 0 = |f|^2$ then $u = v = 0$ so f is constant (zero). Otherwise, $\partial u/\partial x = 0$ similarly from the equations above $\partial u/\partial y = 0 \Rightarrow u = \mathrm{Re}(f) =$ constant, but we know this implies f is constant. Thus, $|f| =$ constant $\Rightarrow f =$ constant (for analytic f).

2.6 Harmonic Functions

As we shall see in a later chapter, analytic functions have the special property that their partial derivatives of all orders exist and are continuous. Assuming this, the second order mixed partial derivatives of u and v are equal regardless of the order of differentiation (by continuity, the iterated limits of $\Delta x \to 0$ and $\Delta y \to 0$ can be applied in any order).

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\frac{\partial u}{\partial x}$$

Combining this with CR equations,

$$-\frac{\partial}{\partial x}\frac{\partial v}{\partial x} = \frac{\partial}{\partial y}\frac{\partial v}{\partial y} \Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

By starting with mixed derivative of v, we can show the same to hold true for u, that is, u and v satisfy the Laplace equation :

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Thus $\mathrm{Re}(f) = u$ and $\mathrm{Im}(f) = v$ are Harmonic Functions:

Definition 2.6. *A function $\mathbb{R}^2 \to \mathbb{R}$: ϕ is said to be* ***harmonic*** *in a domain D if, all its second-order partial derivatives are continuous in D and it satisfies the Laplace equation in D.*

Thus, we have the following theorem:

Theorem 2.8. *If a function f is analytic in a domain D, then $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are harmonic functions in D.*

For a given harmonic function $u(x, y)$ in some domain, we can find another function $v(x, y)$ such that $f = u + iv$ is analytic by solving for $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ via CR equations using known derivatives of u ; v is then called the **harmonic conjugate** of u. If v_1 and v_2 are two harmonic conjugates of u they can only differ by a constant as their derivatives are equal.

For example: consider the harmonic function $u = e^x \cos(y)$ then $\partial v/\partial x = -\partial u/\partial y = e^x \sin(y) \Rightarrow v = e^x \sin(y) + \psi(y)$ and $\partial v/\partial y = \partial u/\partial x = e^x \cos(y) \Rightarrow e^x \cos(y) + \psi'(y) = e^x \cos(y) \Rightarrow \psi'(y) = 0$, so $\psi(y)$ is constant. For simplicity, let $\psi(y) = 0$; Thus, we have the analytic function $f(z) = u + iv = e^x(\cos(y) + i\sin(y)) = e^z$.

2.6.1 Further Insight

Level Curves

Curves produced by equations of the kind $u(x, y) =$ constant or $v(x, y) =$ constant are called the **level curves** of u and v respectively.
At any given point on a level curve, the gradient of the function is normal to the level curve, since it points in the direction of steepest increase, while the level curve lies along direction where the function remains constant.
Consider the dot product of the gradients of u and v:

$$\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)$$

Using the Cauchy-Riemann equations to express the derivatives of v in terms of those of u, we get,

$$\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot \left(\frac{-\partial u}{\partial y}, \frac{\partial u}{\partial x}\right) = 0$$

As the dot product is zero, the gradients of u and v are orthogonal. Since the gradients are normal to the respective level curves, this implies that the level curves must intersect at right angles wherever they meet.
Thus, given the level curves of a harmonic function, the level curves of its harmonic conjugate can be constructed by moving normal to them. This geometric relationship visually encodes the Cauchy-Riemann conditions between u and v.

Harmonic functions often arise in physics, such as temperature in steady-state heat conduction or electrostatic potential; Thus, level curves can depict isotherms or equipotentials.

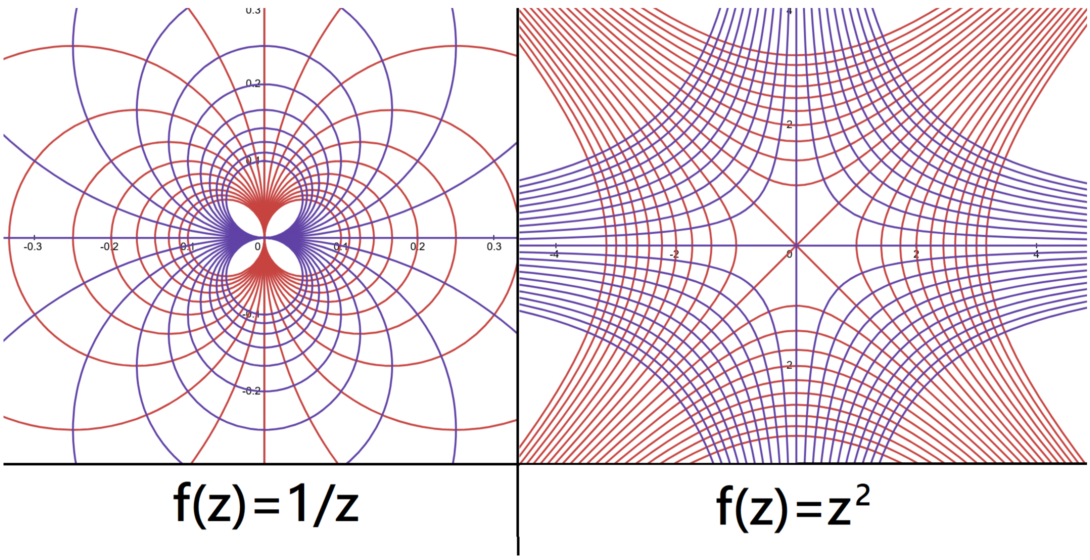

Figure 2.11: level curves of two analytic functions with red for Re(f) and blue for Im(f).

Chapter 3

Elementary Functions

Polynomials, trigonometric functions, logarithms and related expressions play a key role in both practical and theoretical applications. In this chapter we shall extend such familiar functions to the complex plane and explore their behavior.

3.1 Polynomials and Rational Functions

3.1.1 Polynomials

Polynomials are functions of the form

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

where $a_i \in \mathbb{C}$. The degree of a nonzero polynomial, denoted $\deg(f)$, is the largest $i \in \mathbb{N}$ for which $a_i \neq 0$. By convention, the degree of the zero polynomial $f(z) = 0$ is left undefined.

We often factor out a 'dividend' polynomial by a 'divisor' polynomial into unique 'quotient' and 'remainder' polynomials, this idea is formalized as the **Division Algorithm**:

Theorem 3.1. *Let $f(z)$ and $g(z)$ be polynomials where $g(z)$ is a nonzero polynomial. Then there exist unique polynomials $q(z)$,$r(z)$ such that*

$$f(z) = g(z)q(z) + r(z)$$

where either $\deg r(z) < \deg g(z)$ or $r(z)$ is the zero polynomial.

Proof: Let $\deg f(z) = n$ and $\deg g(z) = m$. If $f(z) = 0$ then $q(z) = r(z) = 0$ trivially. Also, if $n < m$, then $f(z) = 0 \cdot g(z) + f(z)$, i.e. $q(z) = 0$ and $r(z) = f(z)$. Thus we shall now consider $n \geq m$. Assume the theorem holds for all $\deg(f) < n$ for sake of induction.

Let

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

$$g(z) = b_m z^m + b_{m-1} z^{m-1} + \ldots + b_1 z + b_0$$

then the polynomial

$$h(z) = f(z) - \frac{a_n}{b_m} z^{n-m} g(z)$$

has degree $< n = \deg(f)$ thus, by inductive hypothesis, there exist $q'(z)$ and $r(z)$ such that

$$h(z) = q'(z)g(z) + r(z)$$

now let $q'(z) = q(z) - \frac{a_n}{b_m} z^{n-m}$, then

$$f(z) = q(z)g(z) + r(z)$$

Now that the existence of the polynomials are proven for $f(z)$, to prove uniqueness, suppose there exist two such representations:

$$f(z) = q_1(z)g(z) + r_1(z) = q_2(z)g(z) + r_2(z)$$

Then subtracting,

$$g(z)(q_1(z) - q_2(z)) = r_2(z) - r_1(z)$$

If $q_1(z) \neq q_2(z)$, then $\deg(g(z)(q_1(z) - q_2(z))) = \deg(r_2(z) - r_1(z)) \geq \deg(g)$, while $\deg(r_2) < \deg(g)$ and $\deg(r_1) < \deg(g)$, which is a contradiction. Thus $q_1 = q_2$ and $r_1 = r_2$. □

Now if the divisor polynomial is $(z - z_0)$ for some z_0, then $\deg(r) \leq 1$, that is, $r(z) =$ constant and the quotient is a polynomial of one less degree

$$p_n(z) = p_{n-1}(z)(z - z_0) + \text{ constant}$$

If z_0 is a zero of the polynomial (that is, $p_n(z_0) = 0$), then the remainder must be zero $\Rightarrow$ $(z - z_0)$ is a factor of the polynomial. Factoring out such terms is called 'deflating' the polynomial.

We know while the equation $z^2 + 1 = 0$ has no real solution, in complex domain, $z = i, -i$ are zeros of the polynomial. Can we be sure all complex polynomials have a zero ? If yes, the deflation can be continued to completely factor the polynomial into linear terms. Gauss proved exactly this, called **The Fundamental Theorem of Algebra**:

Theorem 3.2. *Every non-constant polynomial with complex coefficients has at least one zero in* $\mathbb{C}$.

We postpone its proof to the next chapter. With this, we can continue deflating $p_{n-1}, p_{n-2} \ldots$ to reach the final factorization as

$$p_n = a_n(z - z_1)(z - z_2)\ldots(z - z_n)$$

Thus any complex polynomial of degree n has exactly n roots (counting repetitions) , that is, at most n distinct roots. A root z_0 is said to have **multiplicity** k in $p_n(z)$ if

$$p_n(z) = (z - z_0)^k q(z)$$

where $q(z_0) \neq 0$.
We can write any polynomial of z in terms of a polynomial of $(z - z_0)$ for any z_0 by substituting $\zeta + z_0$ in place of z and expanding it to find the coefficients of $\zeta = (z - z_0)$. The constant term in this form must be $p_n(z_0)$ as all other terms become zero at $z = z_0$ due to the ζ factor. Similarly, the constant term in its derivative will be the coefficient of $(z - z_0)$ while higher terms will go to zero at z_0, so it will be $p'_n(z_0)$. Extending this further, as the k^{th} derivative of $(z - z_0)^k$ is $k!$; Thus, in the k^{th} derivative of $p_n(z)$ represented as polynomial of $(z - z_0)$, the coefficient of $(z - z_0)^k$ times $k!$ will remain while higher terms will go to zero at $z = z_0$.

$$\Rightarrow \quad p_n(z) = \sum_{k=0}^{n} a_k \frac{(z - z_0)^k}{k!} \quad \text{where} \quad a_k = p_n^{(k)}(z_0) \tag{3.1}$$

This is called the **Taylor form** of the polynomial centered at z_0. The standard form is the Taylor form centered at the origin, also called the **Maclaurin form**. It follows if z_0 is a root of multiplicity k then $p_n^{(i)}(z_0) = 0$ for all $i < k$ and $p_n^{(k)}(z_0) \neq 0$.

3.1.2 Rational Functions

Rational Functions are ratios of two polynomials. Consider the function with numerator degree m and denominator degree n, fully deflated and common zeros (same linear factors in numerator and denominator) cancelled:

$$R_{m.n} = \frac{a_m(z - z_1)(z - z_2)\ldots(z - z_m)}{b_n(z - \zeta_1)(z - \zeta_2)\ldots(z - \zeta_m)}$$

The remaining (non-common) zeros of the numerator are the zeros of the rational function. At the zeros the denominator, called **poles** of $(R_{m,n})$, the functional limit tends to infinity and the function becomes undefined.
If $m < n$ then we can represent the rational function as a sum of powers of its pole-factors, a representation called its **partial fraction decomposition**, as illustrated in

the following theorem:

Theorem 3.3. *If*

$$R_{m,n} = \frac{a_m z^m + a_{m-1}z^{m-1} + \ldots + a_1 z + a_0}{b_n(z-\zeta_1)^{d_1}(z-\zeta_2)^{d_2}\ldots(z-\zeta_r)^{d_r}}$$

where the poles ζ_i are distinct and $n = d_1 + d_2 + \ldots + d_r$ then there exists constants $A_j^{(i)}$ such that

$$R_{m,n} = \sum_{i=1}^{r}\sum_{j=0}^{d_i-1}\frac{A_j^{(i)}}{(z-\zeta_i)^{d_i-j}}$$

To prove this theorem, we aim to find expressions for the $A_t^{(s)}$'s (assuming the theorem holds). To do this, consider the function $f_s(z) = (z-\zeta_s)^{d_s}R(z)$, which removes the pole ζ_s from our rational function $R(z)$. Thus,

$$f_s(z) = \sum_{j=0}^{d_s-1} A_j^{(s)}(z-\zeta_s)^j + (z-\zeta_s)^{d_s}h(z) \quad \text{where } h(z) = \sum_{i=1,i\neq s}^{r}\sum_{j=0}^{d_i-1}\frac{A_j^{(i)}}{(z-\zeta_i)^{d_i-j}}$$

As the pole has been removed, $f_s(z)$ is differentiable at ζ_s. So let us differentiate $f_s(z)$ k times to isolate the desired $A_t^{(s)}$. Addressing the second term, each subsequent derivative introduces a term with one less power of $(z-\zeta_s)$ (apart from other higher power terms, by product rule), therefore as long as $k < d_s$, $\frac{d^k}{dz^k}[(z-\zeta_s)^{d_s}h(z)]$ vanishes at $z = \zeta_s$. Now for the first term, consider the fact:

$$\frac{d^k}{dz^k}[A_j^{(s)}(z-\zeta_s)^j] = \frac{j!}{(j-k)!}(z-\zeta_s)^{j-k} \qquad \text{for } j \leq k \text{ and zero otherwise.}$$

Thus, at $z = \zeta_s$, it gives zero for all $j \neq k$. But at $j = k$, $\frac{d^k}{dz^k}[A_j^{(s)}(z-\zeta_s)^j]\Big|_{z=\zeta_s} = k!$, using this we have:

$$\frac{d^k}{dz^k}f_s(z)\Big|_{z=\zeta_s} = A_k^{(s)}k!$$

Finally, changing dummy variables to simplify, we obtain:

$$\boxed{A_j^{(i)} = \frac{1}{j!}\frac{d^j}{dz^j}[(z-\zeta_i)^{d_i}R_{m,n}(z)]\Big|_{z=\zeta_i}} \tag{3.2}$$

Proof:

We now prove that the partial fraction decomposition exists using the above definition of $A_j^{(i)}$. First, we begin by claiming that when the ζ_1 terms are subtracted from $R_{m,n}(z)$, the resulting (following) expression has no pole at ζ_1.

$$R_{m,n}(z) - \sum_{j=0}^{d_1-1} \frac{A_j^{(1)}}{(z-\zeta_1)^{d_1-j}} = R_{m,n}(z) - \frac{1}{(z-\zeta_1)^{d_1}} \sum_{j=0}^{d_1-1} A_j^{(1)}(z-\zeta_1)^j$$

Let

$$T(z) = \sum_{j=0}^{d_1-1} A_j^{(1)}(z-\zeta_1)^j,$$

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m,$$

$$Q(z) = b_n(z-\zeta_2)^{d_2}(z-\zeta_3)^{d_3}\cdots(z-\zeta_r)^{d_r},$$

Then the difference becomes

$$\frac{P(z)}{(z-\zeta_1)^{d_1}Q(z)} - \frac{T(z)}{(z-\zeta_1)^{d_1}} = \frac{P-TQ}{(z-\zeta_1)^{d_1}Q}$$

Thus to not have poles at ζ_1, we need to prove that the polynomial $P - TQ$ has ζ_1 as a zero with multiplicity $\geq d_1$, or equivalently:

$$(P-TQ)^{(j)}(\zeta_1) = 0 \qquad \text{where } j = 0, 1, \ldots, d_1 - 1$$

For this, we first identify $A_j^{(1)}$ as coefficients of $T(z)$ in the Taylor form centered at ζ_1, thus, $A_j^{(1)} = T^{(j)}(\zeta_1)/j!$ but also by the above derived formula,

$$A_j^{(1)} = \frac{1}{j!}\frac{d^j}{dz^j} f_1(z)\Big|_{z=\zeta_s} \qquad \text{where } f_1(z) = (z-\zeta_1)^{d_1} R_{m,n}(z) = \frac{P(z)}{Q(z)}$$

This implies $T^{(j)}(\zeta_1) = f_1^{(j)}(\zeta_1)$, that is, $(f_1 - T)^{(j)}(\zeta_1) = 0$. Now, since $P - TQ = Q(f_1 - T)$ expanding $P - TQ)^{(j)}$ in terms of derivatives of Q and $(f_1 - T)$ we have proved $(P-TQ)^{(j)}(\zeta_1) = 0 \qquad$ where $j = 0, 1, \ldots, d_1 - 1$. Likewise, we can remove all other poles by the following difference:

$$R_{m,n}(z) - \sum_{i=1}^{r}\sum_{j=0}^{d_i-1} \frac{A_j^{(i)}}{(z-\zeta_i)^{d_i-j}}$$

However, this is now a rational function without any poles, that is, a polynomial. Notice that as all terms ($R_{m,n}$ and summation terms) are proper rationals (degree of denominator exceeds that of numerator), when z becomes unbounded, the expression approaches zero. But the only polynomial where $\lim_{z\to\infty} p_n(z) = 0$ is the zero polynomial.

$$\Rightarrow R_{m,n}(z) - \sum_{i=1}^{r}\sum_{j=0}^{d_i-1} \frac{A_j^{(i)}}{(z-\zeta_i)^{d_i-j}} = 0 \iff R_{m,n}(z) = \sum_{i=1}^{r}\sum_{j=0}^{d_i-1} \frac{A_j^{(i)}}{(z-\zeta_i)^{d_i-j}} \qquad \square$$

3.1.3 Further Insight

Real Partial Fractions

The reader may know partial fractions presented as an algorithm for integration of rational functions with real coefficients. Interestingly, we can recover them from our discussion of its complex counterpart. Consider a proper rational function with real coefficients $R_{m,n}$. Clearly, for a real pole ζ_s, the defining formula for $A_j^{(s)}$ yields a real value. Now consider a complex pole ζ_i then $\overline{\zeta_i}$ is also a pole of same multiplicity, say d_i. This is because

$$p_n^{(j)}(\overline{\zeta_i}) = \overline{p_n^{(j)}(\zeta_i)} \qquad \text{(as conjugation distributes over sums and products)}$$

so the conditions for the denominator $p_n(z)$ with real coefficients, $p_n^{(i)}(z_0) = 0$ for all $i < d_i$ and $p_n^{(d_i)}(z_0) \neq 0$ hold for both $z_0 = \zeta_i$ and $z_0 = \overline{\zeta_i}$. Further as

$$\overline{A_j^{(i)}} = \overline{\frac{1}{j!}\frac{d^j}{dz^j}[(z-\zeta_i)^{d_i}R_{m,n}(z)]\Big|_{z=\zeta_i}} = \frac{1}{j!}\frac{d^j}{dz^j}[(z-\overline{\zeta_i})^{d_i}R_{m,n}(z)]\Big|_{z=\overline{\zeta_i}}$$

(as $R_{m,n}$ has real coefficients, $\overline{R_{m,n}(z)} = R_{m,n}(\overline{z})$) In the partial fraction decomposition we can group together terms of the kind

$$\frac{A}{(z-\zeta_i)^j} + \frac{\overline{A}}{(z-\overline{\zeta_i})^j}$$

but as these are complex conjugates (for real z), this gives us a real rational term. To illustrate this with an example, consider the following function with an irreducible quadratic in the denominator (no real roots):

$$R(x) = \frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{px^2+qx+r}{(x-a)(x-\zeta)(x-\bar{\zeta})} = \frac{A}{x-a} + \frac{B}{x-\zeta} + \frac{\bar{B}}{x-\bar{\zeta}}$$

But

$$\frac{B}{x-\zeta} + \frac{\bar{B}}{x-\bar{\zeta}} = \frac{B(x-\bar{\zeta})+\bar{B}(x-\zeta)}{x^2+bx+c} = \frac{(B+\bar{B})x+(B\bar{\zeta}+\bar{B}\zeta)}{x^2+bx+c} = \frac{Cx+D}{x^2+bx+c}$$

where $C = 2\,\mathrm{Re}(B)$ and $D = 2\,\mathrm{Re}(B\bar{\zeta})$. So we have the final real partial fraction decomposition :

$$R(x) = \frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Cx+D}{x^2+bx+c}$$

3.2 The Exponential, Trigonometric and Hyperbolic Functions

3.2.1 The Exponential Function

For $z = x + iy$, $e^z = e^x(\cos(y) + i\sin(y))$ by Euler's formula is an entire function as

$$\frac{d}{dz}e^z = e^z$$

Also as $|e^z| = e^x$ it is never zero. Like the real exponential, it also satisfies the **Division Rule** : $e^{z_1}/e^{z_2} = e^{z_1 - z_2}$. Also due to the periodicity of real trigonometric functions, we can prove e^z is not one-one and satisfies the following theorem:

Theorem 3.4. *(i)* $e^z = 1 \iff z = 2k\pi i$ *where* k *is an integer.*

(ii) $e^{z_1} = e^{z_2} \iff z_1 = z_2 + 2k\pi i$ *where* k *is an integer.*

Proof: (i) If $e^z = 1 \Rightarrow |e^z| = e^x = 1 \Rightarrow x = 0$. Further from Euler's formula $\arg(e^z) = y$, but $\arg(1) = 2k\pi \Rightarrow y = 2k\pi$, thus $z = x + iy = 0 + 2k\pi i$.
(ii) $e^{z_1} = e^{z_2}$ then by division rule $e^{z_1 - z_2} = 1$ Then by (i), $z_1 - z_2 = 2k\pi i \Rightarrow z_1 = z_2 + 2k\pi i$ where $k \in \mathbb{Z}$.
It follows that e^z is **periodic** in the complex sense, with period $2\pi i$. $f(z)$ is periodic in some domain if there exists a non zero constant λ such that $f(z + \lambda) = f(z)$ for all z in the domain. λ is then referred to as the period of f. If we restrict the domain of e^z to any horizontal strip of the form $\{c < y = \text{Im}(z) \leq c + 2\pi |\ c \in \mathbb{R}\ \}$ then the function is one-one there, therefore such a strip is called a **fundamental region** for e^z.

3.2.2 Trigonometric Functions

From Euler's Formula, for real y,

$$e^{iy} = \cos(y) + i\sin(y)$$

$$e^{-iy} = \cos(y) - i\sin(y)$$

From this we obtain

$$\cos(y) = \frac{e^{iy} + e^{-iy}}{2} \quad \text{and} \quad \sin(y) = \frac{e^{iy} - e^{-iy}}{2i}$$

Now that we have obtained expressions for trigonometric functions in terms of the complex exponential, we can naturally extend their domain to the complex plane as follows

Definition 3.1. *Given any $z \in \mathbb{C}$ we define,*

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \qquad \textit{and} \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

Utilizing the known properties of e^z , we see that the following familiar trigonometric identities hold even for the complex argument:

- $\sin(z + 2\pi) = \sin z, \quad \cos(z + 2\pi) = \cos z$
- $\sin(-z) = -\sin z, \quad \cos(-z) = \cos z$
- $\sin^2 z + \cos^2 z = 1$
- $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1$
- $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$
- $\sin 2z = 2 \sin z \cos z, \quad \cos 2z = \cos^2 z - \sin^2 z$

$\sin(z)$ and $\cos(z)$ are entire, but the other trigonometric functions defined from them are not analytic at points where the denominator becomes zero:

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}$$

That is, $\tan(z)$ and $\sec(z)$ are analytic at $\mathbb{C} \setminus \{\frac{\pi}{2} + k\pi\}$ while $\cot(z)$ and $\csc(z)$ are analytic at $\mathbb{C} \setminus \{k\pi\}$ for $k \in \mathbb{Z}$. The usual derivative properties are satisfied as follows:

$$\begin{aligned} \frac{d}{dz} \sin z &= \cos z, & \frac{d}{dz} \cos z &= -\sin z, \\ \frac{d}{dz} \tan z &= \sec^2 z, & \frac{d}{dz} \sec z &= \sec z \tan z, \\ \frac{d}{dz} \cot z &= -\csc^2 z, & \frac{d}{dz} \csc z &= -\csc z \cot z. \end{aligned}$$

Not everything is the same however, while the real trigonometric functions are bounded ($|\sin(x)|, |\cos(x)| \leq 1$) consider the for real number y, $\cos(iy) = \frac{e^z + e^{-z}}{2} \Rightarrow$ the magnitude of $\cos(iy)$ is always greater than or equal to 1, unlike the cosine of a real number.

3.2.3 Hyperbolic Functions

Any function $f(z)$ can be broken into an odd and even part, say f_o and f_e respectively, so that

$$f(z) = f_e(z) + f_o(z).$$

Then,

$$f(-z) = f_e(z) - f_o(z).$$

Solving for f_e and f_o, we get:

$$f_e(z) = \frac{f(z) + f(-z)}{2}, \qquad f_o(z) = \frac{f(z) - f(-z)}{2}.$$

Applying this to $f(z) = e^z$, we find:

$$f_e(z) = \frac{e^z + e^{-z}}{2}, \qquad f_o(z) = \frac{e^z - e^{-z}}{2}.$$

Note that $(f_e(t))^2 - (f_o(t))^2 = 1$, so the parametrized curve $t \in \mathbb{R}$, $(f_e(t), f_o(t))$ lies on the unit hyperbola $x^2 - y^2 = 1$. Thus, we can draw an analogy, just as $(\cos t, \sin t)$ parametrizes the unit circle, $(f_e(t), f_o(t))$ parametrizes the unit hyperbola. This motivates the naming: $f_e(z)$ is called the hyperbolic cosine ($\cosh(z)$), and $f_o(z)$ is called the hyperbolic sine ($\sinh(z)$).
The parameter t can be thought of as a generalized angle, defined in such a way that the area enclosed by the curve (circle or hyperbola), the position vector $(\cos(t), \sin(t))$ or $(\cosh(t), \sinh(t))$, and the x-axis is equal to $t/2$.
Since their formulas are already written in terms of the exponential function, they extend directly to complex z as follows:

Definition 3.2. *Given any $z \in \mathbb{C}$, we define*

$$\cosh(z) = \frac{e^z + e^{-z}}{2}, \qquad \sinh(z) = \frac{e^z - e^{-z}}{2}.$$

In the complex perspective, we see a direct connection between the hyperbolic and trigonometric functions:

$$\cos(iz) = \cosh(z), \cosh(iz) = \cos(z) \qquad \sin(iz) = i\sinh(z), \sinh(iz) = i\sin(z)$$

Thus, identities of hyperbolic functions follow directly from those of trigonometric ones. For example,

$$\frac{d}{dz}\cosh(z) = \frac{d}{dz}\cos(iz) = -i\sin(iz) = -i \cdot i\sinh(z) = \sinh(z)$$

Similarly, $\frac{d}{dz}\sinh(z) = \cosh(z)$.
We define the remaining hyperbolic functions similarly to trigonometric ones:

$$\tanh z = \frac{\sinh z}{\cosh z}, \qquad \coth z = \frac{\cosh z}{\sinh z}, \qquad \text{sech } z = \frac{1}{\cosh z}, \qquad \text{csch } z = \frac{1}{\sinh z}.$$

3.3 The Logarithm

Having discussed the complex exponential, the natural next step is to discuss its inverse, the logarithm. However, for the function to have an inverse which is also a single valued function, it must be one-one (have a unique preimage for every image) which the real exponential is but the complex exponential is not.
Thus, the complex logarithm is a multiple valued function, returning all possible preimages.

$$\log(z) = w \iff e^w = z$$

To find its explicit form let $w = u + iv$ and $z = re^{i\theta}$ then,

$$e^{u+iv} = e^u e^{iv} = re^{i\theta}$$

Thus, $r = e^u$, so $u = \text{Log}(r)$ where $\text{Log}(r)$ is the single valued real logarithm of r and $v = \theta = \arg(z)$ (not defined for $r = 0$). If we denote the principal value of $\arg(z)$ by $Arg(z)$, $v = Arg(z) + 2k\pi$ where $k \in \mathbb{Z}$. Thus the argument function gives the multi-valuedness to the complex logarithm, which we can now define as follows:

Definition 3.3. *Given any $z \neq 0$, we define $\log(z)$ as the set*

$$\log(z) = \ln|z| + i \arg z$$

$$\log(z) = \text{Log}\,|z| + i\,\text{Arg}(z) + i2k\pi \qquad \textit{where } k = \pm1, \pm2, \pm3, \ldots$$

The usual properties hold by those of real logarithm and those of argument like $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$ and $\text{Arg}(z_1/z_2) = \text{Arg}(z_1) - \text{Arg}(z_2)$, but with the important subtlety that different k may be used for log of different numbers:

$$\log(z_1 \cdot z_2) = \log(z_1) + \log(z_2) \quad , \quad \log\left(\frac{z_1}{z_2}\right) = \log(z_1) - \log(z_2)$$

The line of discontinuities (jump of 2π) for a chosen interval for $\arg(z)$ to lie in is called the **branch cut** (the ray $\theta = \tau$ for the interval $(\tau, \tau + 2\pi]$). The principal value of the logarithm is thus inferred from that of the argument:

$$\text{Log}(z) = \text{Log}\,|z| + i\,\text{Arg}(z) \tag{3.3}$$

This is justified for any branch of $\text{Arg}(z)$ that includes 0, so that the logarithm of positive reals remains real. The usual branch cut is the non positive real axis (for $(-\pi, \pi]$). Thus, for all points other than the branch cut the logarithm is continuous and in fact differentiable as given in the following theorem:

Theorem 3.5. *The function* $\operatorname{Log}(z)$ *is analytic in the domain* D^* *which is the set of all points on complex plane except the ones lying on the non positive real axis. Its derivative given as:*

$$\frac{d}{dz}\operatorname{Log}(z) = \frac{1}{z} \qquad \text{for } z \in D^*$$

Proof: Let $w = \operatorname{Log}(z)$ and $w_0 = \operatorname{Log}(z_0)$ for $z_0 \in D^*$. We want to show

$$\lim_{z \to z_0} \frac{w - w_0}{z - z_0} = \frac{1}{z_0}$$

We know $z = e^w$ by analyticity of exponential function,

$$\lim_{w \to w_0} \frac{z - z_0}{w - w_0} = \frac{d}{dw} e^w \bigg|_{w = w_0} = e^{w_0} = z_0$$

Thus we need to show

$$\lim_{z \to z_0} \frac{w - w_0}{z - z_0} = \frac{1}{\lim_{w \to w_0} \frac{z - z_0}{w - w_0}}$$

This follows from the limit of composition of functions as $w \to w_0$ when $z \to z_0$ as $w = \operatorname{Log}(z)$ is continuous at z_0 and the fact that the earlier limit is non zero to apply the division property of limits (as $z_0 \neq 0$ since 0 lies on the branch cut) . Thus,

$$\lim_{z \to z_0} \frac{w - w_0}{z - z_0} = \frac{1}{\lim_{w \to w_0} \frac{z - z_0}{w - w_0}} = \frac{1}{z_0} \qquad \square$$

In view of Theorem 2.8, we have the corollaries:

Corollary 3.5.1. *The function* $\operatorname{Arg} z$ *is harmonic in the domain* D^**.*

Corollary 3.5.2. *The function* $\operatorname{Log}|z|$ *is harmonic in the domain* D^**.*

Choosing the principal interval for $\operatorname{Arg}(z)$ is fully arbitrary. So, for the interval $(\tau, 2\pi + \tau]$ the logarithm would be analytic everywhere except the ray $\theta = \tau$ and the origin. Thus, we can choose any desired **branch**, as defined below, to ensure analyticity of logarithm at any point not lying on the branch cut.

Definition 3.4. *A function* $F(z)$ *is a branch of some multi-valued function* $f(z)$ *in a domain* D *if for all* $z \in D$, $F(z)$ *is continuous and* $F(z) \in f(z)$*.*

Thus, Log and Arg are branches of log and arg respectively.

3.3.1 Further Insight

Boundary Value Problems using Log **and** Arg

Having discovered the harmonic nature of $\operatorname{Log}|z| = \operatorname{Log}\sqrt{x^2+y^2}$ and $\operatorname{Arg}(z)$ we can apply then to boundary value problems like electrostatic potential and temperature, using the uniqueness theorem of for harmonic functions (to be discussed in the next chapter).

For example, if the function is constant at same radial distances from the z axis then it must vary logarithmically with respect to the distance. Such a potential is seen for an infinitely long straight uniformly charged wire along the z axis.

Similarly, $Arg(z)$ can be thought of as the potential of a semi infinite parallel plate capacitor with infinitesimal separation whose interior is the branch cut, the potential 'spirals down' from the positive plate to the negative plate as we move cross radially and jumps abruptly as we pass through the interior, indicating an electric field approaching infinity inside. This gives us an approximation for the fringing fields of a charged capacitor very close to its edge.

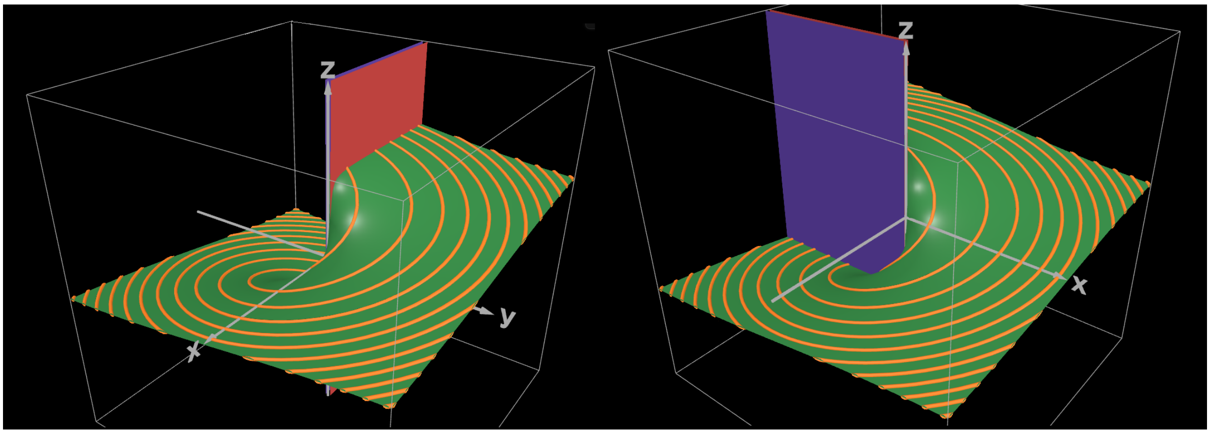

Figure 3.1: The potential (indicated as the height of green surface) falling as we move from the positive (red) plate to the negative (blue) plate in the semi-infinite capacitor.

Now say consider a 'wedge' being the region between θ_- and θ_+. Say $\phi(\theta_-) = \phi_-$ and $\phi(\theta_+) = \phi_+$ Then $\phi \propto \operatorname{Arg}(z) = \theta$ given that the choice of branch cut is made to lie outside the wedge to ensure harmonicity (so θ_- and θ_+ lie in the branch).
So taking $\phi = A\theta + B$ and solving for A and B we get,

$$\phi(\theta) = \left(\frac{\phi_+ - \phi_-}{\theta_+ - \theta_-}\right)\theta + \left(\frac{\phi_-\theta_+ - \phi_+\theta_-}{\theta_+ - \theta_-}\right)$$

Now consider the $\operatorname{Im}(z) > 0$ semi-infinite plane to be an infinite metallic sheet. Now we introduce a hot object touching the edge $-1 < x < 1$ at temperature T_h such that the non contact parts of the real axis (edge) remain at a constant temperature, taken 0 for simplicity. Then we can find the temperature $T(x, y)$ satisfying the Laplace equation by

considering the fact that at $z = 1$ and $z = -1$ it is essentially a $180°$ wedge.
So we superpose the shifted wedges to get

$$T = a\,\mathrm{Arg}(z+1) + b\,\mathrm{Arg}(z-1) + c$$

For $z = -3$ we have $0 = \pi a + \pi b + c$; $z = 0$ we have $T_h = 0 \cdot a + \pi b + c$; $z = +3$ we have $0 = a \cdot 0 + b \cdot 0 + c$.
Solving these, we get $c = 0$, $b = T_h/\pi$, $a = -T_h/\pi \Rightarrow$

$$T(x, y) = \frac{T_h}{\pi}(\mathrm{Arg}(z-1) - \mathrm{Arg}(z+1))$$

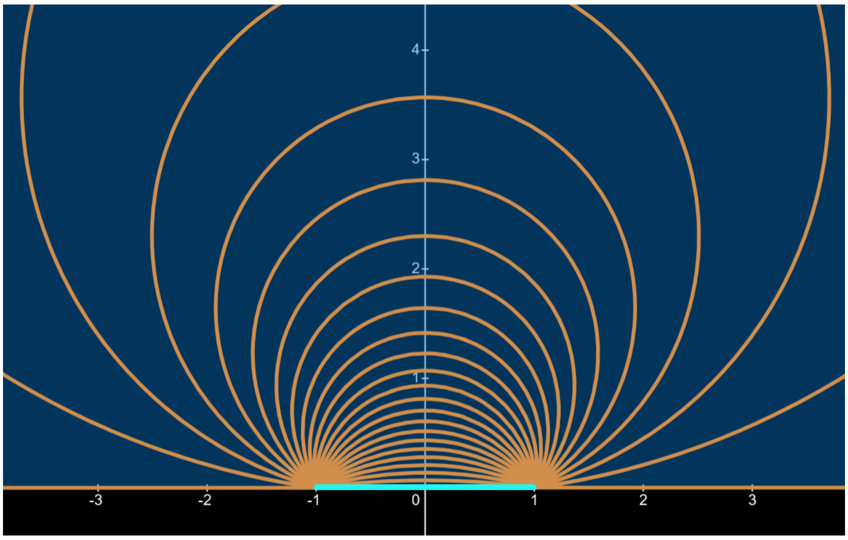

Figure 3.2: The isotherms of above deduced $T(x, y)$; The temperature drops more steeply near the slit source (cyan)

Chapter 4

Appendix

4.1 'Proving' Green's Theorem

4.2 Julia Sets and Newton's Method

4.3 Parallel LCR Circuit