Exploring the Unreal

Understanding Complex Numbers and Functions

Amodvardhan Dubey

A Summer Study

August 27, 2025

Contents

Preface

We often encounter complex numbers in high school, but unlike real numbers, their functions are rarely explored in much depth. The closest one typically gets is Euler's formula — a result that seems almost magical. Yet the yearning for a more dynamic, calculus-like understanding of complex functions often remains unfulfilled.
This text began as an attempt to satisfy my own curiosity, but it has grown into something that I hope fellow students can use to discover the beauty of the complex realm for themselves.
This text was written alongside my study of *Fundamentals of Complex Analysis with Applications to Engineering, Science, and Mathematics* by Edward B. Saff and Arthur David Snider. As such, the sequence of topics closely follows that textbook. However, this is not a reproduction or substitute for the original. Rather, it is a concise, theoretically focused re-articulation presented from the perspective of a student discovering these ideas.
I am grateful to Tristan Needham's *Visual Complex Analysis*, LibreTexts and Wikipedia for further understanding, and Desmos and GeoGebra for helping me illustrate various ideas.

Chapter 1

Complex Numbers

1.1 Why i? : Bombelli's Leap

'Complex numbers' are expressions of the form $a+bi$ where $a, b \in \mathbb{R}$; we say two complex numbers are equal i.e. $a+bi = c+di$ if and only if $a = c$ and $b = d$. We shall discuss Bombelli's exploration of cubics as a way to motivate us to take complex numbers to be a bit less imaginary than we think. Consider the cubic:

$$x^3 = 3px + 2q \tag{1.1}$$

(Note: Any cubic $x^3 + ax^2 + bx + c = 0$ can be transformed into this form by substituting $x = y - \frac{a}{3}$)

Cardano gave a remarkable formula to solve such a cubic:

$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}. \tag{1.2}$$

Bombelli considered the case $x^3 = 15x + 4$ which yields

$$x = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}$$

Does this mean there is no real solution? No ! due to the monotonic nature of x^3 , it must intersect the line $15x + 4$ at some point on the Cartesian plane. From inspection, we see $x = 4$ as a solution, then it is natural to ask : Is it possible to devise rules of algebra of complex numbers that allow us to deduce the solution? Here, Bombelli made a leap by hypothesizing $\sqrt[3]{2 + 11i} = 2 + ni$ and $\sqrt[3]{2 - 11i} = 2 - ni$, where he assumed complex numbers $z_1 = a + bi$ and $z_2 = c + di$ **add** according to the rule

$$\boldsymbol{z_1 + z_2 = (a + c) + (b + d)i} \tag{1.3}$$

Further, to recover the value of ‘n’ he equated $(2 \pm ni)^3 = 2 \pm 11i$ where he had to assume complex numbers **multiply** according to the rule (to obtain $n = 1$)

$$\boldsymbol{z_1 \cdot z_2 = (ac - bd) + (ad + bc)i} \tag{1.4}$$

Also, as every non zero complex number $z = a + bi$ has the multiplicative inverse $z^{-1} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$, we can define **division of complex numbers** as $\frac{z_1}{z_2} = z_1 * z_2^{-1}$ $(z_2 \neq 0)$.

1.2 Point Representation of Complex Numbers

Due to the similarity of $\mathbb{C}$ with $\mathbb{R}^2$ it is natural to represent complex numbers as 2D vectors, with the number $a + bi$ corresponding to the point (a, b) on the Cartesian plane, ie. the x-axis becomes the ‘Real Axis’ and the y-axis becomes the ‘Imaginary Axis’.

Thus, one can represent the number in its polar form as

$$a + bi = r(cos\theta + isin\theta) = r\angle\theta$$

Here, $\boldsymbol{r} = \sqrt{a^2 + b^2}$ is called the ‘modulus’ or ‘magnitude’ of the number, it is the magnitude of the vector (a,b) or the distance of the number from the origin on the cartesian representation ; while $\boldsymbol{\theta}$ is called the argument (‘arg’) of the number, it is the angle the vector (a,b) makes with the positive x-axis. For any given argument θ_0 , $\theta = \theta_0 + 2k\pi$ where $k \in \mathbb{Z}$ is another valid value , we define the ‘principal value’ of the argument (denoted Arg) by convention as the one lying in the interval $(-\pi, \pi]$ (*any half open interval of length* 2π *suffices to determine a unique value of the argument*).
An elegant consequence of such representation is the geometric multiplication rule (easily obtained via application of angle-sum trigonometric identities):

$$(R\angle\theta)(r\angle\phi) = Rr\angle\theta + \phi \tag{1.5}$$

i.e. on multiplication, the moduli of the numbers multiply as real numbers while the arguments add.

Another term is the **conjugate** of a complex number $z = x + yi$ being $\overline{z} = x - yi$ i.e. the reflection of the point/vector (x,y) about the real axis; it is evident $z \cdot \overline{z} = |z|^2$ ($|z|$ being the modulus of z)

1.3 The Complex Exponential

Now that we have discussed addition and multiplication of complex numbers, it is natural to wonder whether exponentiation can also be extended to the complex domain. Let us assume that a function $f(z) = e^z$ can be defined with the following properties: $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$ and $\frac{df(z)}{dz} = e^z$, as with the real exponential.

(Note: Complex differentiation is defined analogously to real differentiation; it will be discussed in greater detail later.)

Thus, $e^{s+it} = e^s \cdot e^{it}$. We are then left to interpret e^{it}. Using the chain rule, we treat i as an ordinary constant to obtain $\frac{d}{dt}e^{it} = i \cdot e^{it}$. Thus, if e^{it} denotes the *position* of a point on the complex plane at time t, the derivative tells us its *velocity.*

We know for $t = 0$, $e^{i0} = e^0 = 1$, thus the initial position of the point is at $(1, 0)$. But we know multiplication by i involves rotation by $90°$ by the geometric multiplication rule. Thus, the velocity is cross-radial (anti-clockwise) and has the same magnitude as the position vector. Hence, the radial distance of the point, i.e., the modulus of position, which is 1, doesn't change.

This implies the velocity remains with constant unit magnitude. Thus, if t represents time, the expression e^{it} must correspond to the position of the point at time t given by $(\cos t, \sin t)$ under such motion. On the complex plane, this becomes

$$\boldsymbol{e^{it} = \cos t + i \sin t} \tag{1.6}$$

This is the **Euler's formula**, first derived by Leonard Euler using power series definitions of exponential and sinusoids.

1.3.1 n$^{\text{th}}$ Roots

Combining the Euler's formula with polar representation we have $r\angle\theta = r \cdot e^{i\theta}$ and the assumed rule $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$, for natural values of n, we have $(r \cdot e^{i\theta})^n = r^n e^{ni\theta}$, thus for a complex number $z = re^{i\theta}$, an 'n$^{\text{th}}$ root' of z is given by;

$$\zeta = \sqrt[n]{r}e^{i\theta/n}$$

Where $\sqrt[n]{r}$ is the positive real root of r. Although adding integral multiples of 2π to the argument doesn't change the number z, it affects ζ . Thus

$$\zeta = \sqrt[n]{r}e^{i(\theta+2k\pi)/n}$$

where $k \in \mathbb{N}$ and $0 \leq k \leq n - 1$. The upper limit of k is at $(n - 1)$ as $k = n$ leads to adding 2π to the argument of the root which on the complex plane corresponds to the same point (number) and thus for $k \geq n$ we just get repetition.
Also while an equally valid choice is $-(n - 1) \leq k \leq 0$, it implies $1 \leq k + n \leq n$, but as discussed, adding n to k doesn't change the roots, we see the arguments correspond to again the same points as for the positive k (with $k = 0$ being replaced by $k = n$).

1.4 Planar Sets

As we define properties of real functions on 1D intervals, for complex functions we do so on 2D planar sets on the complex plane.

The simplest planar set is a 'neighborhood' of some complex number z_0, defined as the set of points satisfying the inequality

$$|z - z_0| < \epsilon$$

where ϵ is some positive number; it is called an **open disk or circular neighborhood** of z_0.

For some set S, a point $z_0 \in S$ is called an **interior point** of S if $\exists$ a circular neighborhood of z_0 which is entirely in S (i.e., is a subset of S).

If all points in a set are interior points, then it is an **open set**, e.g., any open disk. The set $|z| \leq 4$ is NOT an open set, as the boundary points with $|z| = 4$ are not interior points.

Formally, a point z_0 is called a **boundary point** of S if every neighborhood of the point contains at least one point in and one point not in S.

The set of all boundary points of a set is called, unsurprisingly, the boundary (or frontier) of the set. A set is said to be **closed** if it contains all its boundary points, e.g., a 'closed disk', which is the set of points satisfying the inequality $|z - z_0| \leq \epsilon$, where ϵ is a positive real number.

Let z_k with $k \in \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$ be points on the plane. Then the path joining successive z_k (z_1 to z_2, z_2 to z_3, and so on) by $n - 1$ segments forms a continuous path called a **polygonal path**.

If every point in a set is connected to every other point of the set by a polygonal path lying entirely in the set, then the set is said to be **connected** (intuitively, you can draw a path from one point to any other point in the set without lifting your pen or leaving the set). An open and connected set is called a **domain**. Let us look at a theorem involving such a set.

Theorem 1.1. *If $u : D \to \mathbb{R}$ is a function where $u(z) = u(x, y)$ if $z = x + iy$ and $D \subset \mathbb{C}$ is a domain, then*

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$$

implies $u(z)$ is constant on D.

Proof: As any polygonal path can be replaced by another polygonal path comprising only of segments parallel to the x- and y-axes, the value at one point on the domain must be the same as at any other point!

A set is called **bounded** if $\exists$ a positive number R such that $|z| < R$, and unbounded otherwise. A set that is both closed and bounded is said to be **compact**. Also, a union of a domain and any subset of its boundary points is called a **region** (notably, every domain is a region as the empty set is also a subset).

Let us end our discussion of planar sets with two fundamental results on open and closed sets:

Given $S \subseteq \mathbb{C}$, S contains none of its boundary points $\iff$ S is open.

Proof: ($\Rightarrow$) The neighborhood of every point in S must contain only points in S or only points not in S, as it cannot contain both. But the neighborhood of any point always contains the point itself; therefore, it can only contain points in S, i.e., every point in S is an interior point, thus S is open. ($\Leftarrow$) If S is open, it cannot have a boundary point by definition. Hence proved.

Given $S \subseteq \mathbb{C}$, S is closed $\iff$ $\mathbb{C} \setminus S$ is open.

Proof: Any boundary point of S must also be one of $\mathbb{C} \setminus S$, as the neighborhood of such points contains at least one point of both. But if S is closed (i.e., S has all these points), then $\mathbb{C} \setminus S$ has none of them, thus $\mathbb{C} \setminus S$ is open. Similarly, if $\mathbb{C} \setminus S$ is open, then it contains none of the boundary points (by the previous result), thus all the boundary points reside in S, making it closed.

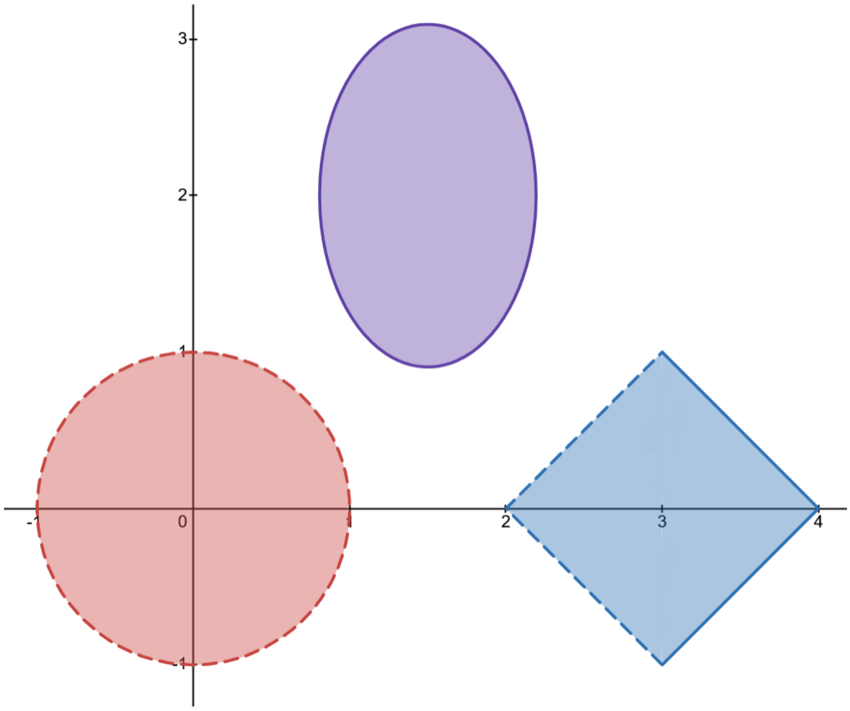


Figure 1.1: Three bounded and connected sets : open (red disk), closed (purple ellipse) and a region (blue square). *Boundary points form a solid line when included and dotted when excluded from the set.*

Chapter 2

Towards Complex Calculus

2.1 Functions of a Complex Variable

A function is a relation that maps an element from a set called the 'domain' to a unique element in another set called the 'codomain'.

Here we shall look at functions complex valued functions of a complex variable. The domain of a function written as an expression of the input is taken to be all points where the expression is defined unless stated otherwise . For instance the domain of $f(z) = 1/z$ is $\mathbb{C} \setminus \{0\}$.

If $z = x + iy$ then $f(z) = f(x, y)$, further we can decompose the image of z into a real part $u(x, y)$ and an imaginary part $v(x, y)$ to get$f(z) = u(x, y) + iv(x, y)$. Thus any such function can be viewed as a pair of real valued functions of two real variables (the x and y coordinates of a point on the complex plane).

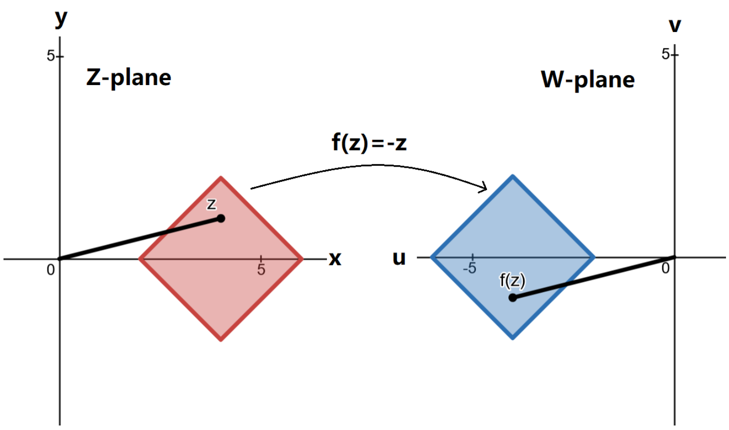


Figure 2.1: The function $f(z) = -z = e^{i\pi}z$ rotates the argument of input by 180° , thus transforming the red region (domain) to the blue one (range)

Visualizing complex functions similar to how we do real ones would require four orthogonal axes which is difficult if not impossible to imagine. Thus, a simpler visualization

is the domain being shown on a complex plane called the 'z plane' and the image (range) being shown on another plane called the 'w plane'.
However, no single representation is equally helpful for all types of functions. For instance, the function $f(z) = \frac{1}{z}$, also known as the **inversion function**, exhibits behavior that is difficult to capture clearly on the standard complex plane. To overcome this, a powerful technique known as **stereographic projection** provides a more elegant way to visualize such functions.

2.2 The Riemann Sphere

Stereographic Projection allows us to make a one-to-one correspondence between points on the complex xy plane and the surface of the unit sphere $x^2 + y^2 + z^2 = 1$ called the **Riemann Sphere**. This is achieved joining any point $z = x + iy$ to the North Pole $(0, 0, 1)$ of the sphere. The point of intersection of this line and the Riemann Sphere is unique and is identified as the 'projection' of z.

2.2.1 The Projection

The number $z = x + iy$ is represented by the point $(x, y, 0)$, the line joining it with the north pole is thus $(xt, yt, 1 - t)$ using parameter $t \in \mathbb{R}$. Thus, at the projection, $(xt)^2 + (yt)^2 + (1 - t)^2 = 1 \iff (|z|^2 + 1)t^2 - 2t = 0$ but $t \neq 0$ as it corresponds to the north pole $\Rightarrow t = \frac{2}{|z|^2+1}$.

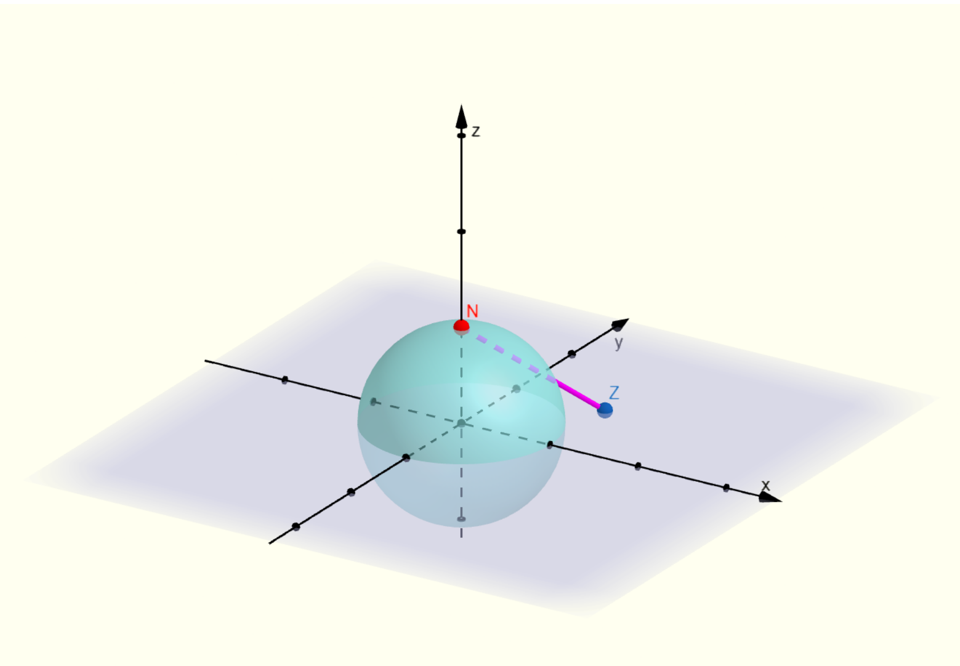


Figure 2.2: Projecting a number Z onto the sphere

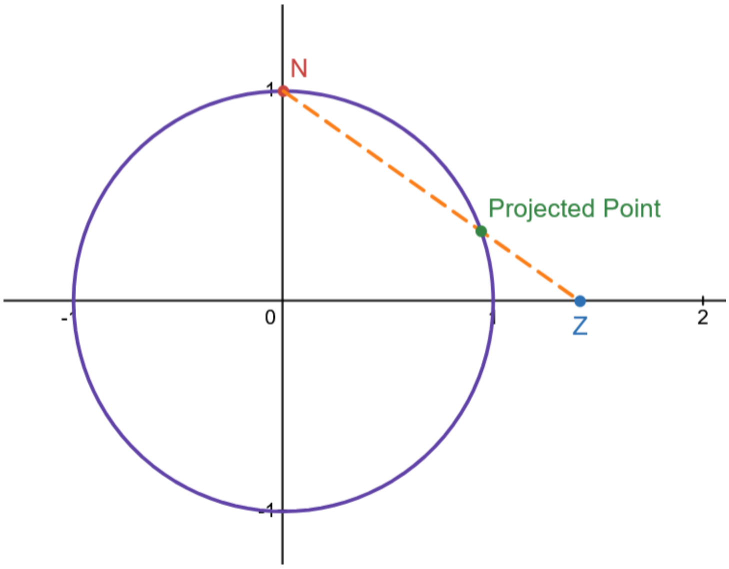


Figure 2.3: A cross section of above diagram through a plane containing N $(0, 0, 1)$, $Z = x + iy$ and the origin

Further since $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$, the projected point $(\hat{x}, \hat{y}, \hat{z})$ is given by:

$$\hat{x} = \frac{2\,\mathrm{Re}(z)}{|z|^2+1}, \qquad \hat{y} = \frac{2\,\mathrm{Im}(z)}{|z|^2+1}, \qquad \hat{z} = \frac{|z|^2-1}{|z|^2+1}$$

Similarly, for a known point on the sphere (x_1, x_2, x_3) we can find the corresponding preimage on the complex plane by $x = x_1/t$, $y = x_2/t$, $x_3 = 1 - t$ as

$$x = \frac{x_1}{1-x_3} \qquad y = \frac{x_2}{1-x_3}$$

2.2.2 The Point at Infinity

As we increase $|z|$, the projection on the sphere approaches the north pole $(0, 0, 1)$ but it remains an image to no finite $z \in \mathbb{C}$ under our projection. However, we can assign the north pole to the extended complex number ∞.
This turns our projection on the sphere a bijection with the **extended complex plane** defined as:

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

We can now define a **'neighborhood of infinity'** which is the preimage of an 'arctic cap' on the sphere defined as $x_3 > \epsilon$ for some $\epsilon \in (-1, 1)$; $|z| = \frac{\sqrt{x_1^2+x_2^2}}{1-x_3} = \frac{\sqrt{1-x_3^2}}{1-x_3}$, hence, given the increasing nature of the function $g(x) = \frac{\sqrt{1-x^2}}{1-x}$ as shown by calculus, on the complex plane, the 'arctic cap' corresponds to set of points satisfying:

$$|z| > R = g(\epsilon)$$

Thus the preimage is the set $\{z \in \mathbb{C} : |z| > R\} \cup \{\infty\}$ which is the exterior of a circle centered at the origin including the point at infinity.

2.2.3 Lines are Circles

Lines and circles are described by the common equation:

$$A(x^2 + y^2) + Bx + Cy + D = 0$$

Substituting x and y as expressions of x_1,x_2 and x_3 we get

$$A(x_1^2 + x_2^2) + Bx_1(1 - x_3) + Cx_2(1 - x_3) + D(1 - x_3)^2 = 0$$

Simplifying further using $x^2 + y^2 + z^2 = 1$,

$$A(1 - x_3^2) + Bx_1(1 - x_3) + Cx_2(1 - x_3) + D(1 - x_3)^2 = 0$$

Since $(1 - x_3) \neq 0$ for any projected point,

$$A(1 + x_3) + Bx_1 + Cx_2 + D(1 - x_3) = 0$$

However, as only single powers of x_1,x_2 and x_3 persist, this is the equation of a plane which we know can intersect the unit sphere only in circles.
Thus lines and circles are projected onto the sphere as circles. For lines, $A = 0$; so the equation becomes

$$Bx_1 + Cx_2 + D(1 - x_3) = 0$$

Note that the north pole $(0, 0, 1)$ is a solution to this equation, meaning the projection of a line passes through the projection of the point at infinity. Hence, using the Riemann Sphere, we interpret lines as not having ends in the same way circles don't, the line "touches infinity and comes back" !

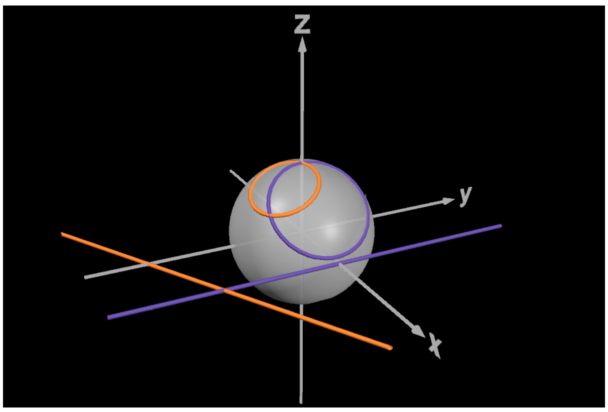


Figure 2.4: Two lines at different distances projected onto the Riemann Sphere.

2.2.4 The Inversion Function

Consider the limit $\lim_{x\to 0^+} \frac{1}{x}$ for real x, $\lim_{x\to 0^+} \frac{1}{x} = +\infty$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$, $\therefore$ the limit doesn't exist.

However, in the previous section we discovered that a line on the complex plane (including the real axis) extend in both directions to meet at the same infinity, thus in $\hat{\mathbb{C}}$, the limit does exist ! : Consider any neighborhood of infinity $|z| > R$, then, we can always find a neighborhood of zero given by $|z - 0| < R$ such that $f(z) = 1/z$ always lies in that neighborhood of infinity (we will soon discuss limits in more detail).

Thus, although certain operations like $0 \cdot \infty$ and $\infty - \infty$ remain undefined on $\hat{\mathbb{C}}$, we can still define division by ∞ and 0 in an intuitive manner taking inspiration from the limit as:

$$\frac{a}{0} = \infty \qquad \text{and} \qquad \frac{a}{\infty} = 0 \quad \text{for } a \in \hat{\mathbb{C}} \setminus \{0, \infty\}$$

Using this definition, the inversion function becomes a bijection from the extended complex plane to itself.

Thus, every point on the Riemann Sphere, maps to some other unique point on the sphere; Given a $z \in \mathbb{C}$ let us find its image on the sphere:

$$P(1/z) = \left(\frac{2\,\mathrm{Re}(1/z)}{|1/z|^2+1}, \frac{2\,\mathrm{Im}(1/z)}{|1/z|^2+1}, \frac{|1/z|^2-1}{|1/z|^2+1}\right)$$

Now, $1/z = \bar{z}/(z\cdot\bar{z}) = \bar{z}/|z|^2 \Rightarrow |z|^2\,\mathrm{Re}(1/z) = \mathrm{Re}(z)$ and $|z|^2\,\mathrm{Im}(1/z) = -\,\mathrm{Im}(z)$, hence

$$P(1/z) = \left(\frac{2\,\mathrm{Re}(z)}{|z|^2+1}, -\frac{2\,\mathrm{Im}(z)}{|z|^2+1}, \frac{1-|z|^2}{1+|z|^2}\right)$$

comparing with the projection of z given by:

$$P(z) = \left(\frac{2\,\mathrm{Re}(z)}{|z|^2+1}, \frac{2\,\mathrm{Im}(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)$$

Thus, the y and z coordinates flip while the x coordinate remains same, that is, the **inversion function rotates the Riemann Sphere by 180° about the real axis** (much like how $f(z) = -z$ rotates the complex plane by 180°) !

In general, the functions of the form $M(z) = \frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$ called **Möbius transformations** are neatly visualized on the Riemann Sphere, as they can be broken into composition of the following maps:

- $z \to z + d/c$, a translation.
- $z \to 1/z$, inversion.
- $z \to -\frac{ad-bc}{c^2} z$, a scaling and rotation (via geometric multiplication rule).

- $z \to z + (a/c)$, another translation.

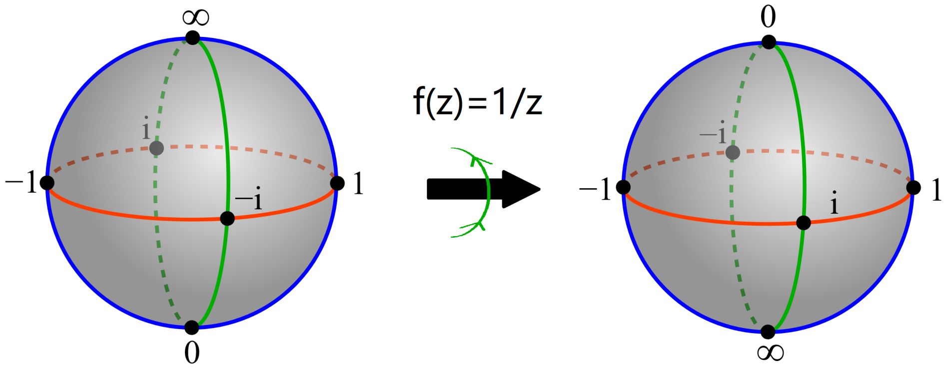


Figure 2.5: the inversion function 'inverting' the Riemann Sphere by rotating about the real axis. The sphere on the left is labelled by preimages of P(z) and on the right by preimages of P(1/z).

2.3 Limits and Continuity

2.3.1 Limit of a Sequence:

If a sequence of complex numbers $z_1, z_2, \ldots, z_n$ can get arbitrarily close to some number z_0 for large enough n, then it is said to be convergent with limit z_0.

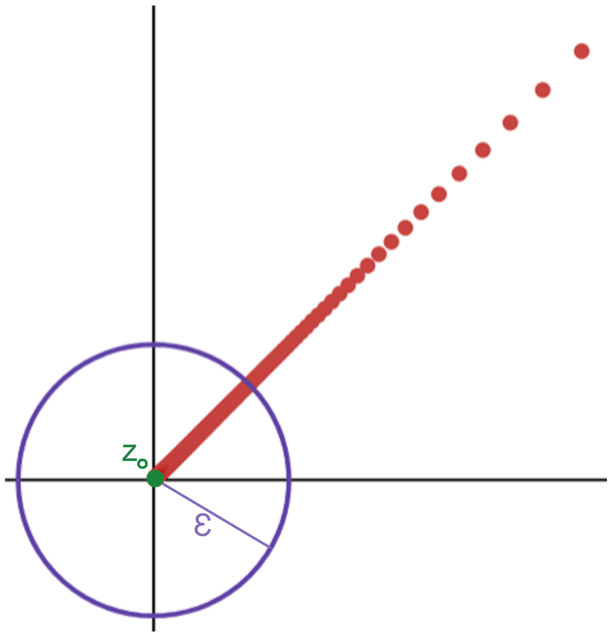


Figure 2.6: A sequence of points converging to the origin.

We can make this statement more precise as:

Definition 2.1. *A sequence of complex numbers z_n ($n \in \mathbb{N}$) is said to converge to or have limit z_0, that is*

$$\lim_{n\to\infty} z_n = z_0$$

if for any (real) $\epsilon > 0$, $\exists\ N \in \mathbb{N}$ such that

$$|z_n - z_0| < \epsilon \text{ for all } n > N$$

In words, this means that for every circular neighborhood of the 'limit', there exists a term beyond which all subsequent (possibly repeated) terms of the sequence lie within the neighborhood. For example, the sequence

$$z_n = \frac{an+b}{cn+d} \qquad \text{for} \qquad c \neq 0$$

has limit 'a/c' while the sequence

$$z_n = (i)^n$$

doesn't converge , as the terms keep jumping around $i, -1, -i, 1$ without settling near any single value.

2.3.2 Limit of a Function:

We can similarly define the limit of a complex function. We say w_0 is the limit of the function $f(z)$ at z_0 if for any (ϵ) neighborhood of w_0 we are able to find a (δ) neighborhood of z_0 (excluding z_0 itself since we are concerned only with the points in its vicinity) which maps to a planar set fully inside w_0's neighborhood.

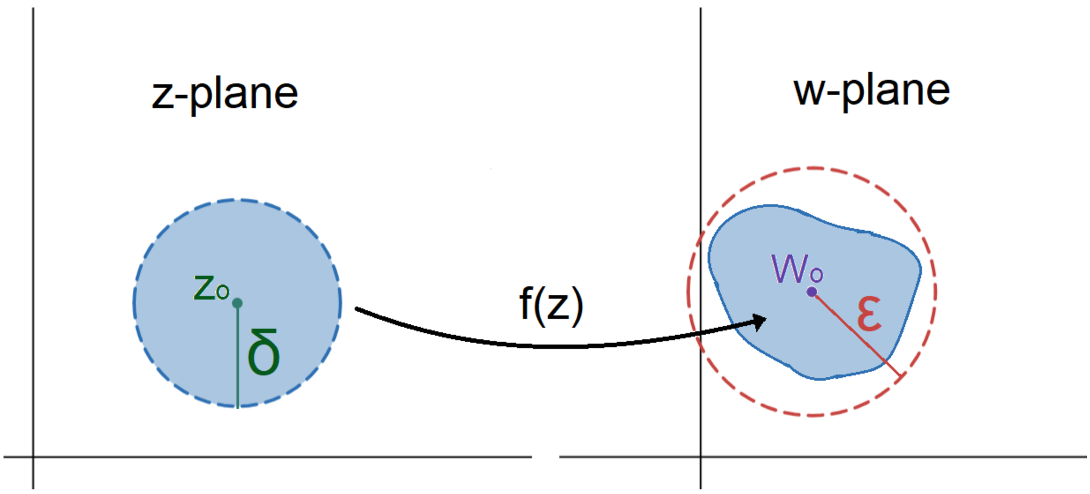


Figure 2.7: $f(z)$ transforming the δ neighborhood of Z_0 into a region lying inside the ϵ neighborhood of W_0

Formally:

Definition 2.2. *The limit of the function* $f(z)$ *is* w_0 *as* z *approaches* z_0 *, that is*

$$\lim_{z\to z_0} f(z) = w_0$$

if for any (real) $\epsilon > 0$, $\exists$ *a* δ *such that*

$$|f(z) - w_0| < \epsilon$$

whenever

$$0 < |z - z_0| < \delta$$

For example,

$$\lim_{z\to z_0} z^2 = z_0^2$$

Proof: $0 < |z - z_0| < \delta$.
Further, using triangle inequality ($|z_1 + z_2| \leq |z_1| + |z_2|$) $|z^2 - z_0^2| = |z - z_0||z - z_0 + 2z_0| \leq |z - z_0|(|z - z_0| + 2|z_0|) < \delta(\delta + 2|z_0|)$. Thus, we aim to find δ such that $\delta(\delta + 2|z_0|) \leq \epsilon$ for any given ϵ; that is, $\delta^2 + 2|z_0|\delta - \epsilon \leq 0$, choosing the bigger root by quadratic formula we obtain $0 < \delta \leq \sqrt{|z_0|^2 + \epsilon} - |z_0|$. Hence, we obtain $\delta = \sqrt{|z_0|^2 + \epsilon} - |z_0|$.

2.3.3 Relating Sequential and Functional Limits:

The limit of a sequence and that of a function are related by the following theorem:

Theorem 2.1.

$$\lim_{z\to z_0} f(z) = w_0$$

$$\Longleftrightarrow$$

For every sequence $\{z_n\}$ *with* $z_n \neq z_0$ *and* $\lim_{n\to\infty} z_n = z_0$ *:* $\lim_{n\to\infty} f(z_n) = w_0$.

($\Rightarrow$): Assume $\lim_{z\to z_0} f(z) = w_0$, and let $\{z_n\}$ be a sequence with $z_n \neq z_0$ and $\lim_{n\to\infty} z_n = z_0$. Hence, given any $\epsilon > 0$, $\exists\, \delta > 0$ such that

$$0 < |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \epsilon.$$

Since z_n converges to z_0, $\exists\, N \in \mathbb{N}$ such that $|z_n - z_0| < \delta$ for all $n > N$. And since $z_n \neq z_0$, it follows that $0 < |z_n - z_0| < \delta$ for all $n > N$. $\Rightarrow |f(z_n) - w_0| < \epsilon$ for all $n > N$. $\therefore \lim_{n\to\infty} f(z_n) = w_0$. $\square$

($\Leftarrow$): Assume that for every sequence $\{z_n\}$ with $z_n \neq z_0$ and $\lim_{n\to\infty} z_n = z_0$, we have $\lim_{n\to\infty} f(z_n) = w_0$.

Suppose, for contradiction, that $\lim_{z\to z_0} f(z) \neq w_0$. This means there exists some $\epsilon_0 > 0$ such for every $\delta > 0$ we choose (however small), there is always some x with $0 < |x - z_0| < \delta$, but $|f(x) - w_0| \geq \epsilon_0$.

Now, define $\delta_k = 1/k$. Then for each $\delta_k < \delta$ ($k > 1/\delta$), by the assumption above, there exists some point z_k such that

$$0 < |z_k - z_0| < \delta_k = \frac{1}{k}, \quad \text{and} \quad |f(z_k) - w_0| \geq \epsilon_0.$$

This gives us a sequence $\{z_k\}$ where each $z_k \neq z_0$, and since the distance from z_k to z_0 is less than $1/k$, the sequence clearly gets arbitrarily close to z_0 as k increases. In other words, $\{z_k\}$ is a sequence with $z_k \neq z_0$ and $z_k \to z_0$.

But then, by how we chose the z_k, we know that for every $k > 1/\delta$, $|f(z_k) - w_0| \geq \epsilon_0$, which means the values $f(z_k)$, for $k > \lceil 1/\delta \rceil$, stay at least ϵ_0 away from w_0 (cannot get arbitrarily close) — so they cannot converge to w_0.

This contradicts our assumption that every such sequence $\{z_n\}$ must satisfy $f(z_n) \to w_0$. Hence, the assumption that the limit doesn't exist must be false. Therefore,

$$\lim_{z\to z_0} f(z) = w_0. \qquad \square$$

2.3.4 Properties of Limits

Theorem 2.2. *If* $\lim_{z\to z_0} f(z) = A$ *and* $\lim_{z\to z_0} g(z) = B$, *then:*

(i) $\displaystyle\lim_{z\to z_0} (f(z) \pm g(z)) = A \pm B$,

(ii) $\displaystyle\lim_{z\to z_0} f(z)g(z) = AB$,

(iii) $\displaystyle\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{A}{B}, \quad \text{if } B \neq 0.$

(iv) $\lim_{z\to z_0} C \cdot f(z) = C \cdot A, \quad C \in \mathbb{C}$

The proofs of these properties follow exactly the same way as they do in real analysis, because the key ingredients — like $|z_1 z_2| = |z_1||z_2|$ and the triangle inequality — still hold for complex numbers. To maintain focus, we omit the proofs here.

2.3.5 Continuity

Definition 2.3. *A function* $f(z)$ *is said to be continuous at* z_0 *if:* $\lim_{z\to z_0} f(z) = f(z_0)$

Clearly, this is the same definition as for real functions.
A function is said to be **continuous on the set** S, if it is continuous for all points in S. We often think of continuity as the 'ability to draw the functional curve without lifting up our pen from the page', a similar analogy can be made for complex continuity: a continuous curve (without breaks, can be drawn by without lifting your pen on the z-plane) is mapped to another continuous curve by a function continuous in any domain containing the curve (simply warped or bent, not broken). This follows from the fact that composition of continuous functions is continuous.

As continuity is just a restriction on the limit, the previously discussed properties of limits yield:

Theorem 2.3. *If $f(z)$ and $g(z)$ are continuous at z_0, then the functions $f(z) \pm g(z)$ and $f(z)g(z)$ are also continuous at z_0.*
Moreover, the quotient $\frac{f(z)}{g(z)}$ is continuous at z_0 provided $g(z_0) \neq 0$.

A consequence of these properties is that polynomials with complex coefficients are continuous on all of $\mathbb{C}$. Hence, rational functions, which are ratios of two polynomials, are continuous everywhere except at points where the denominator polynomial is zero.

2.3.6 Further Insight

Decomposing Complex Limits:

Let $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$, and $w_0 = u_0 + iv_0$. Then,

$$\lim_{z \to z_0} f(z) = w_0$$

if and only if

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} u(x, y) = u_0 \quad \text{and} \quad \lim_{\substack{x \to x_0 \\ y \to y_0}} v(x, y) = v_0.$$

Proof: ($\Rightarrow$): Let $\lim_{z \to z_0} f(z) = w_0$. Then for a given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(z) - w_0| < \epsilon, \quad \text{whenever } |z - z_0| < \delta.$$

Now,

$$\left|\overline{f(z)} - \overline{w_0}\right| = \left|\overline{f(z) - w_0}\right| = |f(z) - w_0| < \epsilon \quad \text{whenever } |z - z_0| < \delta.$$

Thus,

$$\lim_{z \to z_0} \overline{f(z)} = \overline{w_0}$$

Combining it with the known properties of limit we get:

$$\lim_{\substack{x\to x_0\\ y\to y_0}} u(x,y) = \lim_{z\to z_0}\left(\frac{f(z)+\overline{f(z)}}{2}\right) = \frac{w_0+\overline{w_0}}{2} = u_0,$$

$$\lim_{\substack{x\to x_0\\ y\to y_0}} v(x,y) = \lim_{z\to z_0}\left(\frac{f(z)-\overline{f(z)}}{2i}\right) = \frac{w_0-\overline{w_0}}{2i} = v_0.$$

$$\lim_{\substack{x\to x_0\\ y\to y_0}} u(x,y) = u_0, \quad \lim_{\substack{x\to x_0\\ y\to y_0}} v(x,y) = v_0.$$

($\Leftarrow$):

$$u_0 + iv_0 = \lim_{\substack{x\to x_0\\ y\to y_0}} u(x,y) + i\lim_{\substack{x\to x_0\\ y\to y_0}} v(x,y) = \lim_{z\to z_0}\left(\frac{f(z)+\overline{f(z)}}{2}\right) + i\lim_{z\to z_0}\left(\frac{f(z)-\overline{f(z)}}{2i}\right).$$

$$= \lim_{z\to z_0} f(z).$$

$$\Rightarrow \lim_{z\to z_0} f(z) = w_0.$$

Note: We can write $\lim_{z\to z_0} u(z) = \lim_{\substack{x\to x_0\\ y\to y_0}} u(x,y)$ because limits in $\mathbb{C}$ correspond to limits in $\mathbb{R}^2$. Since

$$|z-z_0| = \sqrt{(x-x_0)^2+(y-y_0)^2},$$

the condition $|z-z_0|<\delta$ means (x,y) lies within a δ-neighborhood of (x_0,y_0), and vice versa. Thus, the complex limit $z\to z_0$ is equivalent to the multivariable limit $(x,y)\to(x_0,y_0)$, which generalizes naturally to higher dimensions. Importantly, the two-variable limit

$$\lim_{\substack{x\to x_0\\ y\to y_0}} u(x,y)$$

is not necessarily equal to the step-wise limits

$$\lim_{y\to y_0}\left(\lim_{x\to x_0} u(x,y)\right) \quad \text{or} \quad \lim_{x\to x_0}\left(\lim_{y\to y_0} u(x,y)\right).$$

For example, for

$$f(x,y) = \frac{xy}{x^2+y^2},$$

both iterated limits equal zero at $x_0=y_0=0$, but along the path $y=x$,

$$\lim_{x\to 0} f(x,x) = \frac{1}{2} \neq 0,$$

so the full two-variable limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Continuity of Complex Exponential:

Using Euler's theorem $e^{z_0} = e^{x_0}(\cos y_0 + i \sin y_0)$. Utilizing insight from the last problem, we have $u(x,y) = e^x \cos(y)$ and $v(x,y) = e^x \sin(y)$, as they are product of continuous real functions, we have $\lim_{\substack{x \to x_0 \\ y \to y_0}} u(x,y) = e^{x_0}(\cos y_0)$ and $\lim_{\substack{x \to x_0 \\ y \to y_0}} v(x,y) = e^{x_0}(\sin y_0)$. Hence proved, $\lim_{z \to z_0} e^z = e^{z_0}$, that is, the complex exponential is continuous over the complex plane.

Limit tending to infinity:

We say $\lim_{n \to \infty} z_n = \infty$ if for every $R > 0$, $\exists\, N \in \mathbb{N}$ such that for all $n > N$, $|z_n| > R$. Similarly, for functions, $\lim_{z \to z_0} f(z) = \infty$ if for every $R > 0$, $\exists\, \delta > 0$ such that $|f(z)| > R$ whenever $0 < |z - z_0| < \delta$.
These definitions are natural considering the definition of a 'neighborhood of infinity' as introduced earlier in The Point at Infinity. The length of the segment joining the projections of two numbers z, w on the Riemann Sphere is called their '**Chordal distance/Chi Metric**' denoted as $\boldsymbol{\chi(z,w)}$. As discussed earlier, a neighborhood of infinity with a larger R maps to a smaller 'arctic cap' on the Riemann Sphere. Thus, an equivalent statement for $\lim_{z \to z_0} f(z) = \infty$ is that given an arbitrarily small 'arctic cap', the projections of z_n for $n > N$ or projections of $f(z)$ for $0 < |z - z_0| < \delta$ lie within the cap. In other words $\lim_{n \to \infty} \chi(z_n, \infty) = 0$ and $\lim_{z \to z_0} \chi(f(z), \infty) = 0$.

2.4 Complex Differentiation

2.4.1 Definition

With limits in place, we can define the complex derivative exactly like the real one:

Definition 2.4. *f be a complex-valued function defined in a neighborhood of z_0. Then the* ***derivative*** *of f at z_0 is given by*

$$\frac{df}{dz}(z_0) \equiv f'(z_0) := \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z},$$

provided this limit exists. (Such an f is said to be differentiable at z_0.)

For real functions, we often test differentiability as the derivative approaching the same value from 'both' (left and right) sides. Similarly, for complex functions, the derivative must approach the same value from all directions to z_0. This is because if the derivative takes two different values depending on the direction of Δz, that means every δ neighborhood of z_0 maps to functional values arbitrarily close to both L_1 and L_2 (say) however this is not possible if it has a limit; for a small enough $\epsilon < |L_1 - L_2|$, the ϵ neighborhood of L_1 fully excludes values closer than $(|L_1 - L_2| - \epsilon)$ to L_2, that is it cannot get arbitrarily close to L_2 (similarly L_1)!

For example, the function $f(z) = \overline{z}$ if approached parallel to x-axis, then $\Delta z = \Delta x \Rightarrow$

$$\frac{\overline{z + \Delta z} - \overline{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x}{\Delta x} = 1$$

However when approaching parallel to y-axis, $\Delta z = i\Delta y \Rightarrow$

$$\frac{\overline{z + \Delta z} - \overline{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{-i\Delta y}{i\Delta y} = -1$$

Thus, the limit does not exist for any $z \in \mathbb{C}$, that is, the conjugation map is differentiable nowhere on the complex plane.

2.4.2 Properties

On the other hand, for a positive integer n, one can use binomial formula to get $\frac{d}{dz} z^n = nz^{n-1}$ just as for the real function $f(x) = x^n$.

This also implies the differentiability of polynomials by the usual derivative properties given below (derived just like for real case, from the properties of limits).

Theorem 2.4. *If f and g are differentiable at z, then*
$(f \pm g)'(z) = f'(z) \pm g'(z)$,
$\Big(c\,f\Big)'(z) = c\,f'(z)$ *(for any constant c),*
$(fg)'(z) = f(z)\,g'(z) + f'(z)\,g(z)$,
$\left(\frac{f}{g}\right)'(z) = \frac{g(z)\,f'(z) - f(z)\,g'(z)}{g(z)^2}$ *if $g(z) \neq 0$.*
Moreover, if g is differentiable at z and f is differentiable at $g(z)$, then the chain rule holds:

$$\frac{d}{dz} f\Big(g(z)\Big) = f'\Big(g(z)\Big)\,g'(z).$$

2.4.3 Analyticity

Definition 2.5. *A complex-valued function $f(z)$ is said to be **analytic** on an open set S if it has a derivative at every point of S.*

We also say $f(z)$ is **analytic at some point** z_0 if it is analytic in some neighborhood of z_0.

A point z_0 is called a '**singularity**' (or singular point) of $f(z)$ if it is not analytic at z_0 but there exists a punctured neighborhood $(0 < |z - z_0| < \delta)$ of it in which f is analytic.

For example, the roots of the denominator in a rational function. (This implies $f'(z_0)$ doesn't exist as it is the only way for it to be not analytic at z_0)
A function which is analytic over $\mathbb{C}$ is called **entire** (for example, polynomials).

2.4.4 Further Insight

Differentiability Implies Continuity

let $f(z)$ be differentiable at z_0, then as $\Delta z \to 0 \iff z \to z_0$:

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = L \text{ (say)}$$

Then by the property of limits on multiplication,

$$\lim_{z\to z_0}(f(z) - f(z_0)) = \lim_{z\to z_0}(f(z)) - f(z_0) = \lim_{z\to z_0} L(z - z_0) = 0$$

$$\Rightarrow \lim_{z\to z_0} f(z) = f(z_0)$$

Hence proved, f is continuous at z_0.

Complex Derivative as the 'Amplitwist'

Let $f(z)$ be differentiable on z_0. Then

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \left(\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right)(z - z_0)$$

Say $\lambda(z) = \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)$ then

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \lambda(z)(z - z_0)$$

Clearly, $\lim_{z\to z_0} \lambda(z) = 0$, thus in a small neighborhood of z_0:

$$f(z) \approx f(z_0) + f'(z_0)(z - z_0) \quad \text{(exact in the limit } z \to z_0)$$

If in polar form, $\boldsymbol{f'(z_0) = \rho e^{i\phi}}$, then we can interpret $f(z)$ as transforming tiny (infinitesimal) vectors $(z - z_0)$ emanating from z_0 to vectors whose magnitudes are scaled (amplified) by 'ρ':

$$\lim_{z\to z_0} |f(z) - f(z_0)| = \rho|z - z_0|$$

and their argument rotated (twisted) by 'ϕ'

$$\lim_{z\to z_0} \arg\{f(z) - f(z_0)\} = \arg\{z - z_0\} + \phi$$

(by the geometric multiplication rule 1.5) to obtain the transformed tiny vector

$$\lim_{z \to z_0} [f(z) - f(z_0)] = \rho e^{i\phi}(z - z_0)$$

which emanates from the point $f(z_0)$.
The real derivative is often understood as the slope of a local linear approximation of the function, similarly the complex derivative can be understood as the measure of the 'Amplitwist' of a local similarity transformation approximation of the function (shapes drawn in an infinitesimally small neighborhood of z_0 are scaled and rotated to similar shapes in the co-domain).

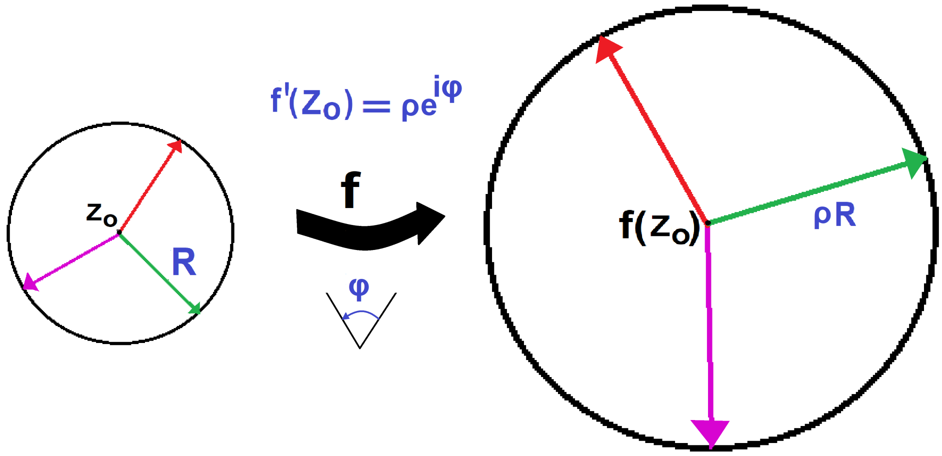


Figure 2.8: infinitesimal vectors ($R \to 0$) from z_0 being amplitwisted by f to infinitesimal vectors from $f(z_0)$

The Conformal Nature of Analytic Functions

If $|f'(z_0)| = \rho = 0$, the image of tiny shapes in an infinitesimal neighborhood of z_0 is evidently not transformed by a simple similarity transformation (Amplitwist), as the image vectors collapse to a single point at $f(z_0)$ (i.e., they are crushed to zero). These vectors no longer have a well-defined "twist", since the zero vector has no well-defined argument.
Let us consider a domain (containing z_0) where $|f'(z)| \neq 0$ and the function is differentiable at all points, that is, analytic at z_0; Then the function takes continuous curves with well defined tangents in the z-plane to other such curves in the w-plane, amplitwisted according to the non-zero derivative at each point.
<u>*Proof*</u>: Consider a continuous curve parameterized by $t \in (\alpha, \beta) \subseteq \mathbb{R}$ (for example the arc-length parameter, increasing monotonically along one direction of the curve) as $\gamma(t) = x(t) + iy(t)$, then the unit tangential vector at $t = t_0$ is given by

$$\lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{|\gamma(t) - \gamma(t_0)|}$$

The curve when transformed by the analytic function becomes $f(\gamma(t))$. Thus using the local similarity transformation approximation (allowed by analyticity of f), the unit tangent vector at $t = t_0$ becomes

$$\lim_{t\to t_0} \frac{f(\gamma(t)) - f(\gamma(t_0))}{|f(\gamma(t)) - f(\gamma(t_0))|} = \frac{f'(\gamma(t_0))}{|f'(\gamma(t_0))|} \frac{\gamma(t) - \gamma(t_0)}{|\gamma(t) - \gamma(t_0)|} \equiv \text{ tangent vector rotated by } \phi$$

Note: Often, the condition $\gamma'(t_0) \neq 0$ is also imposed to ensure that the curve doesn't "stop" in "time" (parameter t) and turn abruptly to a different direction making the tangent direction undefined at $\gamma(t_0)$. $\gamma'(t_0) = 0$ can allow a limit like above to exist (one endpoint kept fixed as $\gamma(t_0)$), but which could differ when the endpoints of the limiting secant approach $\gamma(t_0)$ in varying rates from both sides (t_0^- and t_0^+) making the tangent undefined.

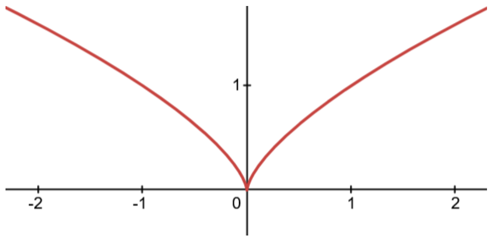


Figure 2.9: parametric curve (t^3, t^2) having a cusp at the origin, with $\gamma'(0) = 0$.

However, this means that at an intersection of two curves, both the angle and the direction of turning ('sense' of the angle) between intersecting curves are preserved. Such a mapping is called a **Conformal Map** ; it preserves the angles and orientations locally. Thus, we have a beautiful result:

Theorem 2.5. *Let f be analytic on a domain $D \subseteq \mathbb{C}$, and suppose that $f'(z) \neq 0$ for all $z \in D$. Then f is a conformal map on D.*

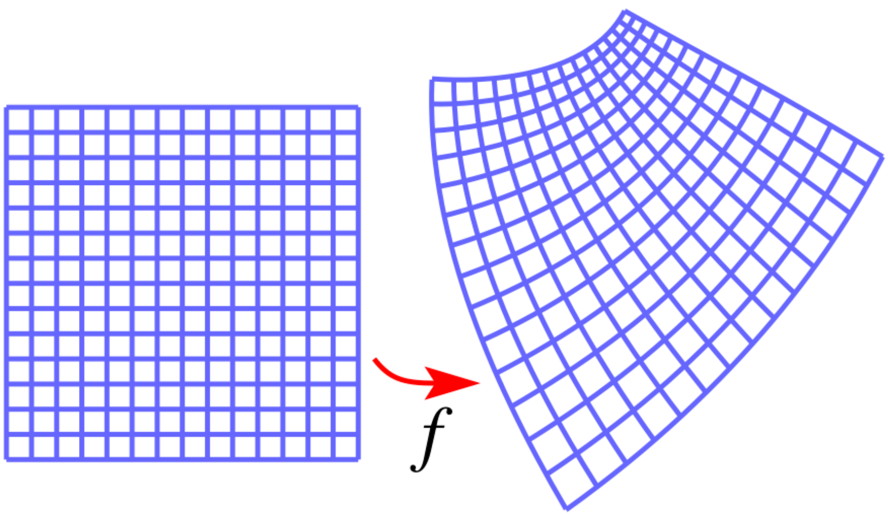


Figure 2.10: A conformal map f transforming a rectangular grid into a deformed grid such that the angle between two intersecting curves still remains 90°.

2.5 The Cauchy-Riemann Equations

If a function $f(z) = u(x, y) + iy(x, y)$ is differentiable at $z_0 = x_0 + iy_0$, the derivative of the function must be independent of the direction of approach of z_0. This suggests a restriction on how the real and complex parts of the function are related.

For horizontal approach, $\Delta z = \Delta x$

$$f'(z_0) = \lim_{\Delta x \to 0} \left[\frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} \right] + i \lim_{\Delta x \to 0} \left[\frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \right] = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

For vertical approach, $\Delta z = i\Delta y$

$$f'(z_0) = \lim_{\Delta y \to 0} \left[\frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} \right] + i \lim_{\Delta y \to 0} \left[\frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y} \right] = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

Comparing the real and imaginary parts, we obtain the **Cauchy-Riemann Equations**:

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} \tag{2.1}$$

Hence, we have the following condition

Theorem 2.6. *The function $f(z) = u(x, y) + iv(x, y)$ is differentiable at $z_0 \Rightarrow$ the Cauchy-Riemann equations are satisfied at z_0*

This serves as a necessary condition that helps to rule out non-differentiable functions. Consider the function

$$f(z) = \begin{cases} \dfrac{xy}{z}, & z \neq 0 \\ 0, & z = 0, \end{cases}$$

Then $u = \frac{x^2 y}{x^2+y^2}$ and $v = \frac{-xy^2}{x^2+y^2}$ At the origin, using $\lim h \to 0$,

$$\frac{\partial u}{\partial x} = \frac{\frac{h^2 * 0}{h^2}}{h} = 0, \qquad \frac{\partial u}{\partial y} = \frac{\frac{h * 0^2}{h^2}}{h} = 0$$

$$\frac{\partial v}{\partial x} = \frac{\frac{-h * 0^2}{h^2}}{h} = 0, \qquad \frac{\partial v}{\partial y} = \frac{\frac{-0 * h^2}{h^2}}{h} = 0$$

Thus, the CR conditions are satisfied. Let us check if the function is differentiable at the origin by approaching along the line $y = mx$:

$$\lim_{z \to 0} \frac{f(z) - 0}{z - 0} = \frac{mx^2}{(x + imx)^2} = \frac{m}{(1 + im)^2}$$

That is, the limit depends on m, the slope of the direction of approach, so the derivative doesn't exist at the origin. We conclude that the Cauchy-Riemann equations are not sufficient to imply differentiability.
However, the continuity of partial derivatives of u and v allows us to state a sufficient condition:

Theorem 2.7. *Let the function $f(z) = u(x, y) + iv(x, y)$ be defined on an open set G (containing z_0). If the first partial derivatives of u and v are continuous at z_0 and satisfy the Cauchy-Riemann equations at z_0 then f is differentiable at z_0.*

Proof: The difference quotient is:

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{[u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)] + i[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)]}{\Delta x + i\Delta y}$$

Now,

$$u(x_0+\Delta x, y_0+\Delta y)-u(x_0, y_0) = [u(x_0+\Delta x, y_0+\Delta y)-u(x_0, y_0+\Delta y)]+[u(x_0, y_0+\Delta y)-u(x_0, y_0)]$$

By the Mean Value Theorem, there exists a x' between x_0 and $x_0 + \Delta x$ such that:

$$u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0 + \Delta y) = \Delta x \left[\frac{\partial u}{\partial x}(x', y_0 + \Delta y)\right]$$

Now as $\Delta z \to 0 \iff \Delta y \to 0$ and $\Delta x \to 0$, that is $x' \to x_0$; Thus, by continuity of partial derivative at z_0 (i.e. jointly continuous with respect to both x and y),

$$\frac{\partial u}{\partial x}(x', y_0 + \Delta y) = \frac{\partial u}{\partial x}(x_0, y_0) + \lambda_1$$

where $\lambda_1(x', y_0 + \Delta y) \to 0$ when $\Delta z \to 0$.
Similarly, we simplify the other terms including those of v using the same method to obtain:

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\Delta x\left[\frac{\partial u}{\partial x} + \lambda_1 + i(\frac{\partial v}{\partial x} + \lambda_2)\right] + \Delta y\left[\frac{\partial u}{\partial y} + \lambda_3 + i(\frac{\partial v}{\partial y} + \lambda_4)\right]}{\Delta x + i\Delta y}$$

Now, using Cauchy-Riemann equations to make all derivatives with respect to x,

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\Delta x\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right] + \Delta y\left[-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right]}{\Delta x + i\Delta y} + \frac{\lambda}{\Delta x + i\Delta y}$$

where $\lambda = \Delta x(\lambda_1 + i\lambda_2) + \Delta y(\lambda_3 + i\lambda_4)$.

Now, by triangle inequality,

$$0 \le \left|\frac{\lambda}{\Delta x + i\Delta y}\right| \le \left|\frac{\Delta x}{\Delta x + i\Delta y}\right| |\lambda_1 + i\lambda_2| + \left|\frac{\Delta y}{\Delta x + i\Delta y}\right| |\lambda_3 + i\lambda_4| \le |\lambda_1 + i\lambda_2| + |\lambda_3 + i\lambda_4|$$

Thus, $\frac{\lambda}{\Delta x + i\Delta y} \to 0$ as $\Delta z \to 0$ (as $\lambda_1, \lambda_2, \lambda_3$ and λ_4 do)
Hence, we have the limit :

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\Delta x\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right] + i\Delta y\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]}{\Delta x + i\Delta y} = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right](x_0, y_0)$$

$\therefore f$ is differentiable at z_0.

2.5.1 Further Insight

When is an Analytic function constant?

If $f'(z) = 0$ for an analytic function f in some domain, by CR equations,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

thus $u =$ constant , $v =$ constant $\Rightarrow f = u + iv =$ constant.
However by CR equations, $\text{Re}(f)$ or $\text{Im}(f)$ alone being constant suffices as well . Also, consider $|f|$ to be constant, then

$$|f|^2 = u^2 + v^2 = \text{ constant}$$

This implies, with CR equations,

$$\frac{\partial |f|^2}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 2u\frac{\partial u}{\partial x} - 2v\frac{\partial u}{\partial y} = 0$$

$$\frac{\partial |f|^2}{\partial y} = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial u}{\partial x} = 0$$

Thus multiplying the equations by u and v respectively and adding,

$$2u^2\frac{\partial u}{\partial x} - 2uv\frac{\partial u}{\partial y} + 2uv\frac{\partial u}{\partial y} + 2v^2\frac{\partial u}{\partial x} = 2(u^2 + v^2)\frac{\partial u}{\partial x} = 0$$

Now, if $(u^2 + v^2) = 0 = |f|^2$ then $u = v = 0$ so f is constant (zero). Otherwise, $\partial u/\partial x = 0$ similarly from the equations above $\partial u/\partial y = 0 \Rightarrow u = \text{Re}(f) =$ constant, but we know this implies f is constant. Thus, $|f| =$ constant $\Rightarrow f =$ constant (for analytic f).

2.6 Harmonic Functions

As we shall see in a later chapter, real and imaginary parts of analytic functions have the special property that their partial derivatives of all orders exist and are continuous. Assuming this, the second order mixed partial derivatives of u and v are equal regardless of the order of differentiation (by continuity, the iterated limits of $\Delta x \to 0$ and $\Delta y \to 0$ can be applied in any order).

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\frac{\partial u}{\partial x}$$

Combining this with CR equations,

$$-\frac{\partial}{\partial x}\frac{\partial v}{\partial x} = \frac{\partial}{\partial y}\frac{\partial v}{\partial y} \Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

By starting with mixed derivative of v, we can show the same to hold true for u, that is, u and v satisfy the Laplace equation :

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Thus $\mathrm{Re}(f) = u$ and $\mathrm{Im}(f) = v$ are Harmonic Functions:

Definition 2.6. *A function $\mathbb{R}^2 \to \mathbb{R}$: ϕ is said to be* ***harmonic*** *in a domain D if, all its second-order partial derivatives are continuous in D and it satisfies the Laplace equation in D.*

Thus, we have the following theorem:

Theorem 2.8. *If a function f is analytic in a domain D, then* $\mathrm{Re}(f)$ *and* $\mathrm{Im}(f)$ *are harmonic functions in D.*

For a given harmonic function $u(x, y)$ in some domain, we can find another function $v(x, y)$ such that $f = u + iv$ is analytic by solving for $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ via CR equations using known derivatives of u ; v is then called the **harmonic conjugate** of u. If v_1 and v_2 are two harmonic conjugates of u they can only differ by a constant as their derivatives are equal.
For example: consider the harmonic function $u = e^x \cos(y)$ then $\partial v/\partial x = -\partial u/\partial y = e^x \sin(y) \Rightarrow v = e^x \sin(y) + \psi(y)$ and $\partial v/\partial y = \partial u/\partial x = e^x \cos(y) \Rightarrow e^x \cos(y) + \psi'(y) = e^x \cos(y) \Rightarrow \psi'(y) = 0$, so $\psi(y)$ is constant. For simplicity, let $\psi(y) = 0$; Thus, we have the analytic function $f(z) = u + iv = e^x(\cos(y) + i \sin(y)) = e^z$.

2.6.1 Further Insight

Level Curves

Curves produced by equations of the kind $u(x, y) = \text{constant}$ or $v(x, y) = \text{constant}$ are called the **level curves** of u and v respectively.
At any given point on a level curve, the gradient of the function is normal to the level curve, since it points in the direction of steepest increase, while the level curve lies along direction where the function remains constant.
Consider the dot product of the gradients of u and v:

$$\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)$$

Using the Cauchy-Riemann equations to express the derivatives of v in terms of those of u, we get,

$$\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot \left(\frac{-\partial u}{\partial y}, \frac{\partial u}{\partial x}\right) = 0$$

As the dot product is zero, the gradients of u and v are orthogonal. Since the gradients are normal to the respective level curves, this implies that the level curves must intersect at right angles wherever they meet.
Thus, given the level curves of a harmonic function, the level curves of its harmonic conjugate can be constructed by moving normal to them. This geometric relationship visually encodes the Cauchy-Riemann conditions between u and v.

Harmonic functions often arise in physics, such as temperature in steady-state heat conduction or electrostatic potential; Thus, level curves can depict isotherms or equipotentials.

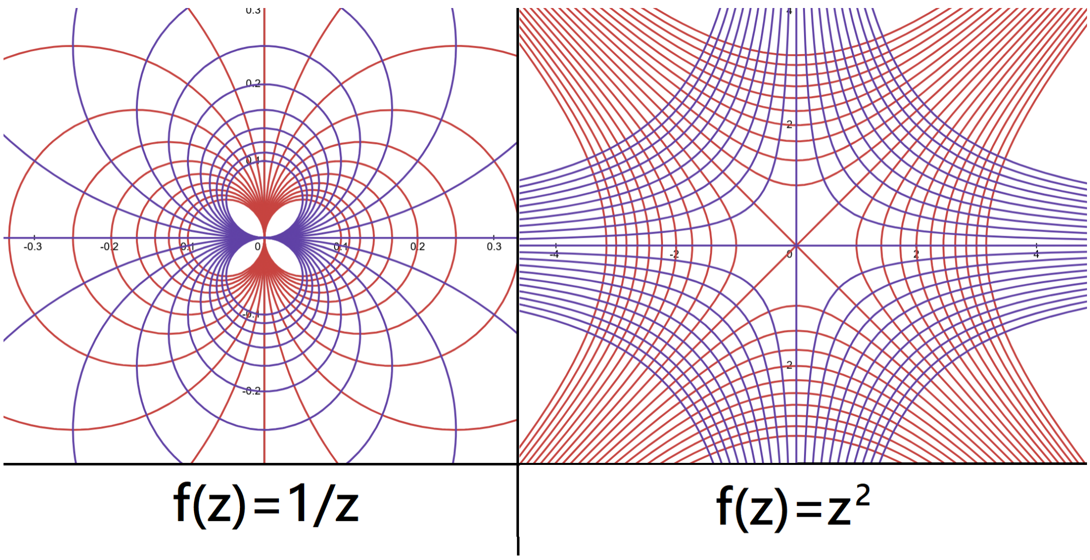


Figure 2.11: level curves of two analytic functions with red for $\mathrm{Re}(f)$ and blue for $\mathrm{Im}(f)$.

Chapter 3

Elementary Functions

Polynomials, trigonometric functions, logarithms and related expressions play a key role in both practical and theoretical applications. In this chapter we shall extend such familiar functions to the complex plane and explore their behavior.

3.1 Polynomials and Rational Functions

3.1.1 Polynomials

Polynomials are functions of the form

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

where $a_i \in \mathbb{C}$. The degree of a nonzero polynomial, denoted $\deg(f)$, is the largest $i \in \mathbb{N}$ for which $a_i \neq 0$. By convention, the degree of the zero polynomial $f(z) = 0$ is left undefined.

We often factor out a 'dividend' polynomial by a 'divisor' polynomial into unique 'quotient' and 'remainder' polynomials, this idea is formalized as the **Division Algorithm**:

Theorem 3.1. *Let $f(z)$ and $g(z)$ be polynomials where $g(z)$ is a nonzero polynomial. Then there exist unique polynomials $q(z)$,$r(z)$ such that*

$$f(z) = g(z)q(z) + r(z)$$

where either $\deg r(z) < \deg g(z)$ or $r(z)$ is the zero polynomial.

Proof: Let $\deg f(z) = n$ and $\deg g(z) = m$. If $f(z) = 0$ then $q(z) = r(z) = 0$ trivially. Also, if $n < m$, then $f(z) = 0 \cdot g(z) + f(z)$, i.e. $q(z) = 0$ and $r(z) = f(z)$. Thus we shall now consider $n \geq m$. Assume the theorem holds for all $\deg(f) < n$ for sake of induction.

Let

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

$$g(z) = b_m z^m + b_{m-1} z^{m-1} + \ldots + b_1 z + b_0$$

then the polynomial

$$h(z) = f(z) - \frac{a_n}{b_m} z^{n-m} g(z)$$

has degree $< n = \deg(f)$ thus, by inductive hypothesis, there exist $q'(z)$ and $r(z)$ such that

$$h(z) = q'(z)g(z) + r(z)$$

now let $q'(z) = q(z) - \frac{a_n}{b_m} z^{n-m}$, then

$$f(z) = q(z)g(z) + r(z)$$

Now that the existence of the polynomials are proven for $f(z)$, to prove uniqueness, suppose there exist two such representations:

$$f(z) = q_1(z)g(z) + r_1(z) = q_2(z)g(z) + r_2(z)$$

Then subtracting,

$$g(z)(q_1(z) - q_2(z)) = r_2(z) - r_1(z)$$

If $q_1(z) \neq q_2(z)$, then $\deg(g(z)(q_1(z) - q_2(z))) = \deg(r_2(z) - r_1(z)) \geq \deg(g)$, while $\deg(r_2) < \deg(g)$ and $\deg(r_1) < \deg(g)$, which is a contradiction. Thus $q_1 = q_2$ and $r_1 = r_2$. $\square$

Now if the divisor polynomial is $(z - z_0)$ for some z_0, then $\deg(r) < 1$, that is, $r(z) =$ constant and the quotient is a polynomial of one less degree

$$p_n(z) = p_{n-1}(z)(z - z_0) + \text{ constant}$$

If z_0 is a zero of the polynomial (that is, $p_n(z_0) = 0$), then the remainder must be zero $\Rightarrow$ $(z - z_0)$ is a factor of the polynomial. Factoring out such terms is called 'deflating' the polynomial.

We know while the equation $z^2 + 1 = 0$ has no real solution, in complex domain, $z = i, -i$ are zeros of the polynomial. Can we be sure all complex polynomials have a zero ? If yes, the deflation can be continued to completely factor the polynomial into linear terms. Gauss proved exactly this, called **The Fundamental Theorem of Algebra**:

Theorem 3.2. *Every non-constant polynomial with complex coefficients has at least one zero in* $\mathbb{C}$.

We postpone its proof to the next chapter. With this, we can continue deflating $p_{n-1}, p_{n-2}\dots$ to reach the final factorization as

$$p_n = a_n(z - z_1)(z - z_2)\dots(z - z_n)$$

Thus any complex polynomial of degree n has exactly n roots (counting repetitions) , that is, at most n distinct roots. A root z_0 is said to have **multiplicity** k in $p_n(z)$ if

$$p_n(z) = (z - z_0)^k q(z)$$

where $q(z_0) \neq 0$.
We can write any polynomial of z in terms of a polynomial of $(z - z_0)$ for any z_0 by substituting $\zeta + z_0$ in place of z and expanding it to find the coefficients of $\zeta = (z - z_0)$. The constant term in this form must be $p_n(z_0)$ as all other terms become zero at $z = z_0$ due to the ζ factor. Similarly, the constant term in its derivative will be the coefficient of $(z-z_0)$ while higher terms will go to zero at z_0, so it will be $p'_n(z_0)$. Extending this further, as the k^{th} derivative of $(z - z_0)^k$ is $k!$; Thus, in the k^{th} derivative of $p_n(z)$ represented as polynomial of $(z - z_0)$, the coefficient of $(z - z_0)^k$ times $k!$ will remain while higher terms will go to zero at $z = z_0$.

$$\Rightarrow \quad p_n(z) = \sum_{k=0}^{n} a_k \frac{(z - z_0)^k}{k!} \quad \text{where} \quad a_k = p_n^{(k)}(z_0) \tag{3.1}$$

This is called the **Taylor form** of the polynomial centered at z_0. The standard form is the Taylor form centered at the origin, also called the **Maclaurin form**. It follows if z_0 is a root of multiplicity k then $p_n^{(i)}(z_0) = 0$ for all $i < k$ and $p_n^{(k)}(z_0) \neq 0$.

3.1.2 Rational Functions

Rational Functions are ratios of two polynomials. Consider the function with numerator degree m and denominator degree n, fully deflated and common zeros (same linear factors in numerator and denominator) cancelled:

$$R_{m.n} = \frac{a_m(z - z_1)(z - z_2)\dots(z - z_m)}{b_n(z - \zeta_1)(z - \zeta_2)\dots(z - \zeta_m)}$$

The remaining (non-common) zeros of the numerator are the zeros of the rational function. At the zeros the denominator, called **poles** of $(R_{m,n})$, the functional limit tends to infinity and the function becomes undefined.
If $m < n$ then we can represent the rational function as a sum of powers of its pole-factors, a representation called its **partial fraction decomposition**, as illustrated in

the following theorem:

Theorem 3.3. *If*

$$R_{m,n} = \frac{a_m z^m + a_{m-1}z^{m-1} + \ldots + a_1 z + a_0}{b_n(z-\zeta_1)^{d_1}(z-\zeta_2)^{d_2}\ldots(z-\zeta_r)^{d_r}}$$

where the poles ζ_i *are distinct and* $n = d_1 + d_2 + \ldots + d_r$ *then there exists constants* $A_j^{(i)}$ *such that*

$$R_{m,n} = \sum_{i=1}^{r}\sum_{j=0}^{d_i-1} \frac{A_j^{(i)}}{(z-\zeta_i)^{d_i-j}}$$

To prove this theorem, we aim to find expressions for the $A_t^{(s)}$'s (assuming the theorem holds). To do this, consider the function $f_s(z) = (z-\zeta_s)^{d_s}R(z)$, which removes the pole ζ_s from our rational function $R(z)$. Thus,

$$f_s(z) = \sum_{j=0}^{d_s-1} A_j^{(s)}(z-\zeta_s)^j + (z-\zeta_s)^{d_s}h(z) \quad \text{where } h(z) = \sum_{i=1, i\neq s}^{r}\sum_{j=0}^{d_i-1}\frac{A_j^{(i)}}{(z-\zeta_i)^{d_i-j}}$$

As the pole has been removed, $f_s(z)$ is differentiable at ζ_s. So let us differentiate $f_s(z)$ k times to isolate the desired $A_t^{(s)}$. Addressing the second term, each subsequent derivative introduces a term with one less power of $(z-\zeta_s)$ (apart from other higher power terms, by product rule), therefore as long as $k < d_s$, $\frac{d^k}{dz^k}[(z-\zeta_s)^{d_s}h(z)]$ vanishes at $z = \zeta_s$. Now for the first term, consider the fact:

$$\frac{d^k}{dz^k}[A_j^{(s)}(z-\zeta_s)^j] = \frac{j!}{(j-k)!}(z-\zeta_s)^{j-k} \qquad \text{for } j \leq k \text{ and zero otherwise.}$$

Thus, at $z = \zeta_s$, it gives zero for all $j \neq k$. But at $j = k$, $\frac{d^k}{dz^k}[A_j^{(s)}(z-\zeta_s)^j]\Big|_{z=\zeta_s} = k!$, using this we have:

$$\frac{d^k}{dz^k}f_s(z)\Big|_{z=\zeta_s} = A_k^{(s)}k!$$

Finally, changing dummy variables to simplify, we obtain:

$$\boxed{A_j^{(i)} = \frac{1}{j!}\frac{d^j}{dz^j}[(z-\zeta_i)^{d_i}R_{m,n}(z)]\Big|_{z=\zeta_i}} \tag{3.2}$$

Proof:

We now prove that the partial fraction decomposition exists using the above definition of $A_j^{(i)}$. First, we begin by claiming that when the ζ_1 terms are subtracted from $R_{m,n}(z)$, the resulting (following) expression has no pole at ζ_1.

$$R_{m,n}(z) - \sum_{j=0}^{d_1-1} \frac{A_j^{(1)}}{(z-\zeta_1)^{d_1-j}} = R_{m,n}(z) - \frac{1}{(z-\zeta_1)^{d_1}} \sum_{j=0}^{d_1-1} A_j^{(1)}(z-\zeta_1)^j$$

Let

$$T(z) = \sum_{j=0}^{d_1-1} A_j^{(1)}(z-\zeta_1)^j,$$

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m,$$

$$Q(z) = b_n (z-\zeta_2)^{d_2}(z-\zeta_3)^{d_3}\cdots(z-\zeta_r)^{d_r},$$

Then the difference becomes

$$\frac{P(z)}{(z-\zeta_1)^{d_1} Q(z)} - \frac{T(z)}{(z-\zeta_1)^{d_1}} = \frac{P-TQ}{(z-\zeta_1)^{d_1} Q}$$

Thus to not have poles at ζ_1, we need to prove that the polynomial $P - TQ$ has ζ_1 as a zero with multiplicity $\geq d_1$, or equivalently:

$$(P-TQ)^{(j)}(\zeta_1) = 0 \qquad \text{where } j = 0, 1, \ldots, d_1 - 1$$

For this, we first identify $A_j^{(1)}$ as coefficients of $T(z)$ in the Taylor form centered at ζ_1, thus, $A_j^{(1)} = T^{(j)}(\zeta_1)/j!$ but also by the above derived formula,

$$A_j^{(1)} = \frac{1}{j!}\frac{d^j}{dz^j} f_1(z)\Big|_{z=\zeta_s} \qquad \text{where } f_1(z) = (z-\zeta_1)^{d_1} R_{m,n}(z) = \frac{P(z)}{Q(z)}$$

This implies $T^{(j)}(\zeta_1) = f_1^{(j)}(\zeta_1)$, that is, $(f_1 - T)^{(j)}(\zeta_1) = 0$. Now, since $P - TQ = Q(f_1 - T)$ expanding $P - TQ)^{(j)}$ in terms of derivatives of Q and $(f_1 - T)$ we have proved $(P-TQ)^{(j)}(\zeta_1) = 0 \qquad \text{where } j = 0, 1, \ldots, d_1 - 1$. Likewise, we can remove all other poles by the following difference:

$$R_{m,n}(z) - \sum_{i=1}^{r} \sum_{j=0}^{d_i-1} \frac{A_j^{(i)}}{(z-\zeta_i)^{d_i-j}}$$

However, this is now a rational function without any poles, that is, a polynomial. Notice that as all terms ($R_{m,n}$ and summation terms) are proper rationals (degree of denominator exceeds that of numerator), when z becomes unbounded, the expression approaches zero. But the only polynomial where $\lim_{z\to\infty} p_n(z) = 0$ is the zero polynomial.

$$\Rightarrow R_{m,n}(z) - \sum_{i=1}^{r} \sum_{j=0}^{d_i-1} \frac{A_j^{(i)}}{(z-\zeta_i)^{d_i-j}} = 0 \iff R_{m,n}(z) = \sum_{i=1}^{r} \sum_{j=0}^{d_i-1} \frac{A_j^{(i)}}{(z-\zeta_i)^{d_i-j}} \qquad \square$$

3.1.3 Further Insight

Real Partial Fractions

The reader may know partial fractions presented as an algorithm for integration of rational functions with real coefficients. Interestingly, we can recover them from our discussion of its complex counterpart. Consider a proper rational function with real coefficients $R_{m,n}$. Clearly, for a real pole ζ_s, the defining formula for $A_j^{(s)}$ yields a real value. Now consider a complex pole ζ_i then $\overline{\zeta_i}$ is also a pole of same multiplicity, say d_i. This is because

$$p_n^{(j)}(\overline{\zeta_i}) = \overline{p_n^{(j)}(\zeta_i)} \qquad \text{(as conjugation distributes over sums and products)}$$

so the conditions for the denominator $p_n(z)$ with real coefficients, $p_n^{(i)}(z_0) = 0$ for all $i < d_i$ and $p_n^{(d_i)}(z_0) \neq 0$ hold for both $z_0 = \zeta_i$ and $z_0 = \overline{\zeta_i}$. Further as

$$\overline{A_j^{(i)}} = \overline{\frac{1}{j!}\frac{d^j}{dz^j}[(z-\zeta_i)^{d_i} R_{m,n}(z)]\Big|_{z=\zeta_i}} = \frac{1}{j!}\frac{d^j}{dz^j}[(z-\overline{\zeta_i})^{d_i} R_{m,n}(z)]\Big|_{z=\overline{\zeta_i}}$$

(as $R_{m,n}$ has real coefficients, $\overline{R_{m,n}(z)} = R_{m,n}(\overline{z})$, we combine this with the conjugation-derivative relation given in the Appendix) In the partial fraction decomposition we can group together terms of the kind

$$\frac{A}{(z-\zeta_i)^j} + \frac{\overline{A}}{(z-\overline{\zeta_i})^j}$$

but as these are complex conjugates (for real z), this gives us a real rational term. To illustrate this with an example, consider the following function with an irreducible quadratic in the denominator (no real roots):

$$R(x) = \frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{px^2+qx+r}{(x-a)(x-\zeta)(x-\bar{\zeta})} = \frac{A}{x-a} + \frac{B}{x-\zeta} + \frac{\bar{B}}{x-\bar{\zeta}}$$

But

$$\frac{B}{x-\zeta} + \frac{\bar{B}}{x-\bar{\zeta}} = \frac{B(x-\bar{\zeta}) + \bar{B}(x-\zeta)}{x^2+bx+c} = \frac{(B+\bar{B})x + (B\bar{\zeta}+\bar{B}\zeta)}{x^2+bx+c} = \frac{Cx+D}{x^2+bx+c}$$

where $C = 2\,\mathrm{Re}(B)$ and $D = 2\,\mathrm{Re}(B\bar{\zeta})$. So we have the final real partial fraction decomposition :

$$R(x) = \frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Cx+D}{x^2+bx+c}$$

3.2 The Exponential, Trigonometric and Hyperbolic Functions

3.2.1 The Exponential Function

For $z = x + iy$, $e^z = e^x(\cos(y) + i\sin(y))$ by Euler's formula is an entire function as

$$\frac{d}{dz}e^z = e^z$$

As $|e^z| = e^x$ it is never zero. Like the real exponential, it also satisfies the **Division Rule** : $e^{z_1}/e^{z_2} = e^{z_1-z_2}$. Due to the periodicity of real trigonometric functions, we can prove e^z is not one-one and satisfies the following theorem:

Theorem 3.4. *(i)* $e^z = 1 \iff z = 2k\pi i$ *where* k *is an integer.*

(ii) $e^{z_1} = e^{z_2} \iff z_1 = z_2 + 2k\pi i$ *where* k *is an integer.*

Proof: (i) If $e^z = 1 \Rightarrow |e^z| = e^x = 1 \Rightarrow x = 0$. Further from Euler's formula $\arg(e^z) = y$, but $\arg(1) = 2k\pi \Rightarrow y = 2k\pi$, thus $z = x + iy = 0 + 2k\pi i$.
(ii) $e^{z_1} = e^{z_2}$ then by division rule $e^{z_1-z_2} = 1$ Then by (i), $z_1 - z_2 = 2k\pi i \Rightarrow z_1 = z_2 + 2k\pi i$ where $k \in \mathbb{Z}$.
It follows that e^z is **periodic** in the complex sense, with period $2\pi i$. $f(z)$ is periodic in some domain if there exists a non zero constant λ such that $f(z + \lambda) = f(z)$ for all z in the domain. λ is then referred to as the period of f. If we restrict the domain of e^z to any horizontal strip of the form $\{c < y = \mathrm{Im}(z) \leq c + 2\pi |\ c \in \mathbb{R}\ \}$ then the function is one-one there, therefore such a strip is called a **fundamental region** for e^z.

3.2.2 Trigonometric Functions

From Euler's Formula, for real y,

$$e^{iy} = \cos(y) + i\sin(y)$$

$$e^{-iy} = \cos(y) - i\sin(y)$$

From this we obtain

$$\cos(y) = \frac{e^{iy} + e^{-iy}}{2} \quad \text{and} \quad \sin(y) = \frac{e^{iy} - e^{-iy}}{2i}$$

Now that we have obtained expressions for trigonometric functions in terms of the complex exponential, we can naturally extend their domain to the complex plane as follows

Definition 3.1. *Given any $z \in \mathbb{C}$ we define,*

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \qquad \textit{and} \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

Utilizing the known properties of e^z , we see that the following familiar trigonometric identities hold even for the complex argument:

- $\sin(z + 2\pi) = \sin z, \quad \cos(z + 2\pi) = \cos z$
- $\sin(-z) = -\sin z, \quad \cos(-z) = \cos z$
- $\sin^2 z + \cos^2 z = 1$
- $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1$
- $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$
- $\sin 2z = 2 \sin z \cos z, \quad \cos 2z = \cos^2 z - \sin^2 z$

$\sin(z)$ and $\cos(z)$ are entire, but the other trigonometric functions defined from them are not analytic at points where the denominator becomes zero:

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}$$

That is, $\tan(z)$ and $\sec(z)$ are analytic at $\mathbb{C} \setminus \{\frac{\pi}{2} + k\pi\}$ while $\cot(z)$ and $\csc(z)$ are analytic at $\mathbb{C} \setminus \{k\pi\}$ for $k \in \mathbb{Z}$. The usual derivative properties are satisfied as follows:

$$\begin{aligned}
\frac{d}{dz} \sin z &= \cos z, & \frac{d}{dz} \cos z &= -\sin z, \\
\frac{d}{dz} \tan z &= \sec^2 z, & \frac{d}{dz} \sec z &= \sec z \tan z, \\
\frac{d}{dz} \cot z &= -\csc^2 z, & \frac{d}{dz} \csc z &= -\csc z \cot z.
\end{aligned}$$

Not everything is the same however, while the real trigonometric functions are bounded ($|\sin(x)|, |\cos(x)| \leq 1$) consider the for real number y, $\cos(iy) = \frac{e^z + e^{-z}}{2} \Rightarrow$ the magnitude of $\cos(iy)$ is always greater than or equal to 1, unlike the cosine of a real number.

3.2.3 Hyperbolic Functions

Any function $f(z)$ can be broken into an odd and even part, say f_o and f_e respectively, so that

$$f(z) = f_e(z) + f_o(z).$$

Then,

$$f(-z) = f_e(z) - f_o(z).$$

Solving for f_e and f_o, we get:

$$f_e(z) = \frac{f(z) + f(-z)}{2}, \qquad f_o(z) = \frac{f(z) - f(-z)}{2}.$$

Applying this to $f(z) = e^z$, we find:

$$f_e(z) = \frac{e^z + e^{-z}}{2}, \qquad f_o(z) = \frac{e^z - e^{-z}}{2}.$$

Note that $(f_e(t))^2 - (f_o(t))^2 = 1$, so the parametrized curve $t \in \mathbb{R}$, $(f_e(t), f_o(t))$ lies on the unit hyperbola $x^2 - y^2 = 1$. Thus, we can draw an analogy, just as $(\cos t, \sin t)$ parametrizes the unit circle, $(f_e(t), f_o(t))$ parametrizes the unit hyperbola. This motivates the naming: $f_e(z)$ is called the hyperbolic cosine ($\cosh(z)$), and $f_o(z)$ is called the hyperbolic sine ($\sinh(z)$).
The parameter t can be thought of as a generalized angle, defined in such a way that the area enclosed by the curve (circle or hyperbola), the position vector $(\cos(t), \sin(t))$ or $(\cosh(t), \sinh(t))$, and the x-axis is equal to $t/2$.
Since their formulas are already written in terms of the exponential function, they extend directly to complex z as follows:

Definition 3.2. *Given any $z \in \mathbb{C}$, we define*

$$\cosh(z) = \frac{e^z + e^{-z}}{2}, \qquad \sinh(z) = \frac{e^z - e^{-z}}{2}.$$

In the complex perspective, we see a direct connection between the hyperbolic and trigonometric functions:

$$\cos(iz) = \cosh(z), \cosh(iz) = \cos(z) \qquad \sin(iz) = i\sinh(z), \sinh(iz) = i\sin(z)$$

Thus, identities of hyperbolic functions follow directly from those of trigonometric ones. For example,

$$\frac{d}{dz}\cosh(z) = \frac{d}{dz}\cos(iz) = -i\sin(iz) = -i \cdot i\sinh(z) = \sinh(z)$$

Similarly, $\frac{d}{dz}\sinh(z) = \cosh(z)$.
We define the remaining hyperbolic functions similarly to trigonometric ones:

$$\tanh z = \frac{\sinh z}{\cosh z}, \qquad \coth z = \frac{\cosh z}{\sinh z}, \qquad \text{sech } z = \frac{1}{\cosh z}, \qquad \text{csch } z = \frac{1}{\sinh z}.$$

3.3 The Logarithm

Having discussed the complex exponential, the natural next step is to discuss its inverse, the logarithm. However, for the function to have an inverse which is also a single valued function, it must be one-one (have a unique preimage for every image) which the real exponential is but the complex exponential is not.
Thus, the complex logarithm is a multiple valued function, returning all possible preimages.

$$\log(z) = w \iff e^w = z$$

To find its explicit form let $w = u + iv$ and $z = re^{i\theta}$ then,

$$e^{u+iv} = e^u e^{iv} = re^{i\theta}$$

Thus, $r = e^u$, so $u = \text{Log}(r)$ where $\text{Log}(r)$ is the single valued real logarithm of r and $v = \theta = \arg(z)$ (not defined for $r = 0$). If we denote the principal value of $\arg(z)$ by $\text{Arg}(z)$, $v = \text{Arg}(z) + 2k\pi$ where $k \in \mathbb{Z}$. Thus, the argument function gives multi-valuedness to the complex logarithm, which we can now define as follows:

Definition 3.3. *Given any $z \neq 0$, we define $\log(z)$ as the set*

$$\log(z) = \ln|z| + i\arg z$$

$$\log(z) = \text{Log}\,|z| + i\,\text{Arg}(z) + i2k\pi \qquad \textit{where } k = \pm 1, \pm 2, \pm 3, \ldots$$

The usual properties hold by those of real logarithm and those of argument like $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$ and $\text{Arg}(z_1/z_2) = \text{Arg}(z_1) - \text{Arg}(z_2)$, but with the important subtlety that different k may be used for log of different numbers:

$$\log(z_1 \cdot z_2) = \log(z_1) + \log(z_2) \quad , \quad \log\left(\frac{z_1}{z_2}\right) = \log(z_1) - \log(z_2)$$

The line of discontinuities (jump of 2π) for a chosen interval for $\arg(z)$ to lie in is called the **branch cut** (the ray $\theta = \tau$ for the interval $(\tau, \tau + 2\pi]$). The principal value of the logarithm is thus inferred from that of the argument:

$$\text{Log}(z) = \text{Log}\,|z| + i\,\text{Arg}(z) \tag{3.3}$$

This is justified for any branch of $\text{Arg}(z)$ that includes 0, so that the logarithm of positive reals remains real. The usual branch cut is the non positive real axis (for $(-\pi, \pi]$). Thus, for all points other than the branch cut the logarithm is continuous and in fact differentiable as given in the following theorem:

Theorem 3.5. *The function* $\text{Log}(z)$ *is analytic in the domain* D^* *which is the set of all points on complex plane except the ones lying on the non positive real axis. Its derivative given as:*

$$\frac{d}{dz}\text{Log}(z) = \frac{1}{z} \qquad \textit{for } z \in D^*$$

Proof: Let $w = \text{Log}(z)$ and $w_0 = \text{Log}(z_0)$ for $z_0 \in D^*$. We want to show

$$\lim_{z \to z_0} \frac{w - w_0}{z - z_0} = \frac{1}{z_0}$$

We know $z = e^w$ by analyticity of exponential function,

$$\lim_{w \to w_0} \frac{z - z_0}{w - w_0} = \frac{d}{dw} e^w \bigg|_{w = w_0} = e^{w_0} = z_0$$

Thus we need to show

$$\lim_{z \to z_0} \frac{w - w_0}{z - z_0} = \frac{1}{\lim_{w \to w_0} \frac{z - z_0}{w - w_0}}$$

This follows from the limit of composition of functions as $w \to w_0$ when $z \to z_0$ as $w = \text{Log}(z)$ is continuous at z_0 and the fact that the earlier limit is non zero to apply the division property of limits (as $z_0 \neq 0$ since 0 lies on the branch cut) . Thus,

$$\lim_{z \to z_0} \frac{w - w_0}{z - z_0} = \frac{1}{\lim_{w \to w_0} \frac{z - z_0}{w - w_0}} = \frac{1}{z_0} \qquad \square$$

In view of Theorem 2.8, we have the corollaries:

Corollary 3.5.1. *The function* $\text{Arg}\, z$ *is harmonic in the domain* D^**.*

Corollary 3.5.2. *The function* $\text{Log}\,|z|$ *is harmonic in the domain* D^**.*

Choosing the principal interval for $\text{Arg}(z)$ is fully arbitrary. So, for the interval $(\tau, 2\pi + \tau]$ the logarithm would be analytic everywhere except the ray $\theta = \tau$ and the origin. Thus, we can choose any desired **branch**, as defined below, to ensure analyticity of logarithm at any point not lying on the branch cut.

Definition 3.4. *A function* $F(z)$ *is a branch of some multi-valued function* $f(z)$ *in a domain* D *if for all* $z \in D$*,* $F(z)$ *is continuous and* $F(z) \in f(z)$*.*

Thus, Log and Arg are branches of log and arg respectively.

3.3.1 Further Insight

Boundary Value Problems using Log **and** Arg

Having discovered the harmonic nature of $\text{Log}\,|z| = \text{Log}\sqrt{x^2+y^2}$ and $\text{Arg}(z)$ we can apply then to boundary value problems like electrostatic potential and temperature, using the uniqueness theorem of for harmonic functions (to be discussed in the next chapter).

For example, if the function is constant at same radial distances from the z axis then it must vary logarithmically with respect to the distance. Such a potential is seen for an infinitely long straight uniformly charged wire along the z axis.

Similarly, $Arg(z)$ can be thought of as the potential of a semi infinite parallel plate capacitor with infinitesimal separation whose interior is the branch cut, the potential 'spirals down' from the positive plate to the negative plate as we move cross radially and jumps abruptly as we pass through the interior, indicating an electric field approaching infinity inside. This gives us an approximation for the fringing fields of a charged capacitor very close to its edge.

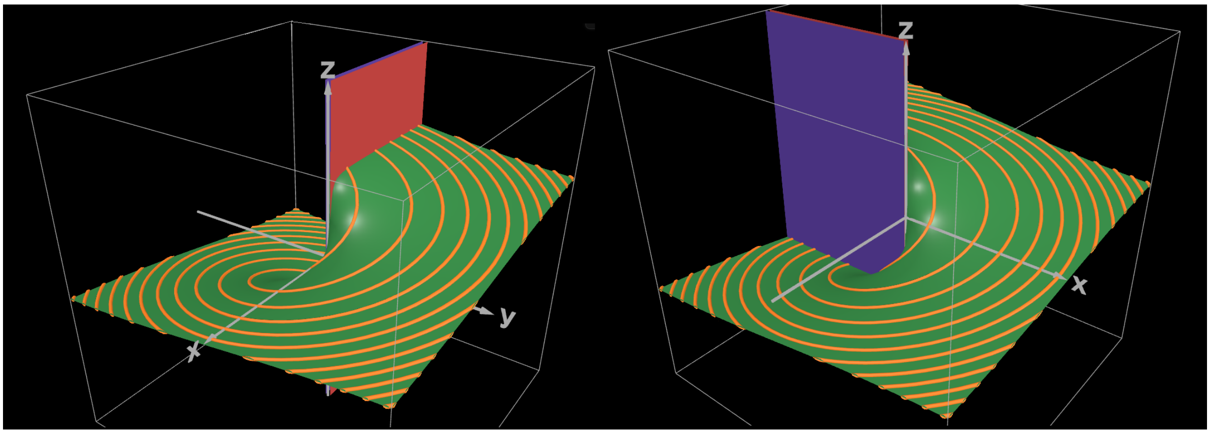


Figure 3.1: The potential (indicated as the height of green surface) falling as we move from the positive (red) plate to the negative (blue) plate in the semi-infinite capacitor.

Now say consider a 'wedge' being the region between θ_- and θ_+. Say $\phi(\theta_-) = \phi_-$ and $\phi(\theta_+) = \phi_+$ Then $\phi \propto \text{Arg}(z) = \theta$ given that the choice of branch cut is made to lie outside the wedge to ensure harmonicity (so θ_- and θ_+ lie in the branch).
So taking $\phi = A\theta + B$ and solving for A and B we get,

$$\phi(\theta) = \left(\frac{\phi_+ - \phi_-}{\theta_+ - \theta_-}\right)\theta + \left(\frac{\phi_-\theta_+ - \phi_+\theta_-}{\theta_+ - \theta_-}\right)$$

Now consider the $\text{Im}(z) > 0$ semi-infinite plane to be an infinite metallic sheet. Now we introduce a hot object touching the edge $-1 < x < 1$ at temperature T_h such that the non contact parts of the real axis (edge) remain at a constant temperature, taken 0 for simplicity. Then we can find the temperature $T(x, y)$ satisfying the Laplace equation by

considering the fact that at $z = 1$ and $z = -1$ it is essentially a 180° wedge.
So we superpose the shifted wedges to get

$$T = a\,\mathrm{Arg}(z+1) + b\,\mathrm{Arg}(z-1) + c$$

For $z = -3$ we have $0 = \pi a + \pi b + c$; $z = 0$ we have $T_h = 0 \cdot a + \pi b + c$; $z = +3$ we have $0 = a \cdot 0 + b \cdot 0 + c$.
Solving these, we get $c = 0$, $b = T_h/\pi$, $a = -T_h/\pi \Rightarrow$

$$T(x,y) = \frac{T_h}{\pi}(\mathrm{Arg}(z-1) - \mathrm{Arg}(z+1))$$

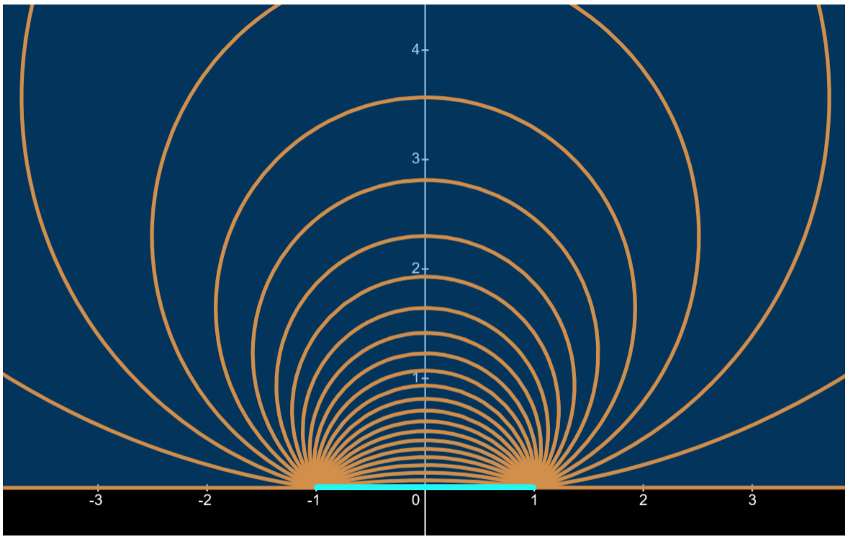


Figure 3.2: The isotherms of above deduced $T(x,y)$; The temperature drops more steeply near the slit source (cyan).

3.4 Complex Exponentiation and Inverse Trigonometric Functions

3.4.1 Complex Exponentiation

Equipped with the complex logarithm, we can motivate a definition for complex powers of a complex number taking inspiration from the property which holds for $n \in \mathbb{Z}$:

$$z^n = (e^{\log(z)})^n = e^{n \log(z)}$$

Definition 3.5. *If $\alpha \in \mathbb{C}$ and $z \neq 0$ then we define the exponent z^α as:*

$$z^\alpha = e^{\alpha \log(z)}$$

Naturally, complex exponentiation inherits multivaluedness from the complex logarithm.

$$z^\alpha = e^{\alpha(\mathrm{Log}(|z|)+i\,\mathrm{Arg}(z)+2ki\pi)}$$

Suppose two branches $k = k_1$ and $k = k_2$ give the same value of z^α, then

$$e^{2\alpha k_1 i\pi} = e^{2\alpha k_2 i\pi} \quad \Rightarrow \quad 2\alpha k_1 i\pi = 2\alpha k_2 i\pi + 2mi\pi \qquad \Rightarrow \alpha = m/(k_1 - k_2)$$

This is only possible if α is a rational number, otherwise all branches give infinite distinct values for the exponent. Consider the case where $\alpha = m/n$ where $m, n \in \mathbb{Z}$ and $n > 0$,

$$z^\alpha = e^{\alpha(\mathrm{Log}(|z|)+i\,\mathrm{Arg}(z))} \cdot e^{2\frac{mk}{n}i\pi}$$

and let the $d = \gcd(m, n)$ (greatest common divisor) then as we increase k from 0, (mk/n) first becomes an integer again at $k = n/d$ so $k = 0, 1, \ldots, (n/d - 1)$ give n/d distinct values for the exponential (rest are repeated, as argued in n^{th} Roots). So, if m and n are co-prime, $z^{m/n}$ gives the n distinct n^{th} roots of z^m . However, if $d = \gcd(m, n) > 1$ then we only get a subset of size n/d of those roots.
To summarize:

- Integer exponents produce a single value.
- Rational exponents yield finitely many values (specifically, $\frac{n}{\gcd(m,n)}$ where $\alpha = \frac{m}{n}$).
- Non-rational exponents lead to infinitely many distinct values.

The principal branch of z^α is simply the principal branch of the logarithm used, that is $e^{\alpha\,\mathrm{Log}(z)}$. So, using chain rule, the principal value of z^α has derivative :

$$\frac{d}{dz}e^{\alpha\,\mathrm{Log}(z)} = \alpha z^\alpha \frac{1}{z} = \alpha z^{\alpha-1} \qquad (z^\alpha \text{ and } z^{\alpha-1} \text{ assumed to have same branch of } \log(z))$$

in the slit domain $D^* = \mathbb{C} \setminus (-\infty, 0]$.

3.4.2 Inverse Trigonometric Functions

As we have defined trigonometric functions explicitly by exponentials, it's natural that we can also give their inverses as a logarithm. For instance, starting with:

$$\sin(w) = z \quad \Rightarrow \quad \frac{e^{iw} - e^{-iw}}{2i} = z$$

$$\Rightarrow \quad e^{2iw} - 2iz\, e^{iw} - 1 = 0$$

Solving for e^{iw} using the quadratic formula:

$$e^{iw} = iz + \sqrt{1 - z^2}$$

Taking logarithm to retrieve the argument of sine as a function of its value :

$$\boxed{\sin^{-1}(z) = -i \log\left(iz + \sqrt{1 - z^2}\right)}$$

Similarly, for $\cos(w) = z$,

$$\frac{e^{iw} + e^{-iw}}{2} = z \Rightarrow e^{2iw} - 2z\, e^{iw} + 1 = 0 \Rightarrow e^{iw} = z + \sqrt{z^2 - 1}$$

$$\boxed{\cos^{-1}(z) = -i \log\left(z + \sqrt{z^2 - 1}\right)}$$

And for $\tan(w) = z$,

$$\frac{e^{iw} - e^{-iw}}{i(e^{iw} + e^{-iw})} = z \Rightarrow (1 - iz)e^{2iw} = 1 + iz \Rightarrow e^{2iw} = \frac{1 + iz}{1 - iz}$$

$$\boxed{\tan^{-1}(z) = \frac{i}{2} \log\left(\frac{1 - iz}{1 + iz}\right) \quad (z \neq \pm i)}$$

Using the principal logarithm (including for the square root in $\sin^{-1}$ and $\cos^{-1}$) we can define the principal branch of the inverse trigonometric functions, leading to their well-known ranges. For example, when $z \in [-1, 1] \subset \mathbb{R}$,

$$\mathrm{Sin}^{\text{-}1}(z) = -i \operatorname{Log}\left[iz + e^{\frac{1}{2}\operatorname{Log}(1-z^2)}\right] = -i \operatorname{Log}\left|iz + e^{\frac{1}{2}\operatorname{Log}(1-z^2)}\right| + \operatorname{Arg}\left(iz + e^{\frac{1}{2}\operatorname{Log}(1-z^2)}\right)$$

But,

$$\left|iz + e^{\frac{1}{2}\operatorname{Log}(1-z^2)}\right| = \sqrt{z^2 + 1 - z^2} = 1 \quad \Rightarrow \quad \operatorname{Log}|\cdots| = 0$$

and

$$\operatorname{Re}\left(iz + e^{\frac{1}{2}\operatorname{Log}(1-z^2)}\right) = e^{\frac{1}{2}\operatorname{Log}(1-z^2)} > 0$$

so $iz + e^{\frac{1}{2}\operatorname{Log}(1-z^2)}$ lies in the right half-plane. Therefore,

$$-\frac{\pi}{2} < \mathrm{Sin}^{\text{-}1}(z) = \operatorname{Arg}\left(iz + e^{\frac{1}{2}\operatorname{Log}(1-z^2)}\right) < \frac{\pi}{2}$$

We also observe that the derivatives for the principal values (obtained by chain rule) are also same as ones in real calculus:

$$\frac{d}{dz}\,\text{Sin}^{\text{-1}}(z) = \frac{-i}{iz + (1-z^2)^{1/2}}\left[i + \frac{-2z}{2(1-z^2)^{1/2}}\right] = \frac{1}{iz + (1-z^2)^{1/2}}\left[1 + \frac{iz}{(1-z^2)^{1/2}}\right]$$

$$\Rightarrow \quad \frac{d}{dz}\,\text{Sin}^{\text{-1}}(z) = \frac{1}{(1-z^2)^{1/2}} \qquad (z \neq \pm 1)$$

Similarly,

$$\Rightarrow \quad \frac{d}{dz}\,\text{Cos}^{\text{-1}}(z) = \frac{-1}{(1-z^2)^{1/2}} \qquad (z \neq \pm 1)$$

Note that the derivative depends on the branch of square root (usually taken by principal logarithm, giving the positive root for positive real numbers). Doing the same for $\tan^{-1}$,

$$\frac{d}{dz}\tan^{-1}(z) = \frac{i}{2}\left[\frac{1+iz}{1-iz}\right]\frac{(1+iz)(-i)-(1-iz)(i)}{(1+iz)^2} = \frac{i}{2}\left[\frac{1}{1-iz}\right]\frac{-2i}{(1+iz)}$$

$$\Rightarrow \quad \frac{d}{dz}\tan^{-1}(z) = \frac{1}{1+z^2} \qquad (z \neq \pm i)$$

Unlike arcsin and arccos, the inverse tangent does not involve a square root and is analytic on most of $\mathbb{C}$, except for the branch cut on imaginary axis for $|\operatorname{Im}(z)| \geq 1$, where the logarithm's argument becomes real and non-positive. However, the derivative itself, wherever defined for a given branch is independent of the branch of the logarithm chosen.

Further, we can also find the inverse hyperbolic functions by relating them with the inverse trigonometric functions as follows:

$$z = \sin(w) = \frac{1}{i}\sinh(iw) \Rightarrow w = \arcsin(z) = \frac{1}{i}\sinh^{-1}(iz) \Rightarrow \sinh^{-1}(z) = i\arcsin(-iz)$$

$$z = \cos(w) = \cosh(iw) \Rightarrow w = \arccos(z) = \frac{1}{i}\cosh^{-1}(z)) \Rightarrow \cosh^{-1}(z) = i\arccos(z)$$

$$z = \tan(w) = \frac{1}{i}\tanh(iw) \Rightarrow w = \arctan(z) = \frac{1}{i}\tanh^{-1}(iz)) \Rightarrow \tanh^{-1}(z) = i\arctan(-iz)$$

Thus, we have:

$$\sinh^{-1}(z) = \log\left[z + (z^2+1)^{1/2}\right] \tag{3.4}$$

$$\cosh^{-1}(z) = \log\left[z + (z^2-1)^{1/2}\right] \tag{3.5}$$

$$\tanh^{-1}(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \qquad (z \neq \pm 1) \tag{3.6}$$

3.4.3 Further Insight

Real vs Complex Exponentiation

One can easily prove from the definition that the following properties hold for principal ($\mathrm{Arg}(z) \in (-\pi, \pi]$) exponentiation:

- $z^0 = 1$ for $z \neq 0$
- $\frac{z^\alpha}{z^\beta} = z^{\alpha-\beta}$
- $z^\alpha z^\beta = z^{\alpha+\beta}$

While these are same as familiar identities in real exponentiation, not everything is the same. For instance for real z, 1^z is always 1, however, for complex z,

$$1^z = e^{z\log(1)} = e^{2kiz\pi} \qquad \text{where } k \in \mathbb{Z}$$

so $1^z = 1$ only if $(kz) \in \mathbb{Z}$ but it can take any value $m \neq 0$ by setting $z = \log(m)/2ki\pi$ for $k \neq 0$. Though for the principal value, $k = 0$ so $1^z = 1$.
Another property familiar from the real case, namely $z^\alpha \cdot w^\alpha = (zw)^\alpha$, does **not** generally hold for $z, w \in \mathbb{C}$, even when all exponents use the **same branch** of the logarithm. For example, using the **principal branch**, we have:

$$z^\alpha \cdot w^\alpha = e^{\alpha(\mathrm{Log}\,|z|+i\,\mathrm{Arg}(z))} \cdot e^{\alpha(\mathrm{Log}\,|w|+i\,\mathrm{Arg}(w))} = e^{\alpha(\mathrm{Log}\,|zw|+i(\mathrm{Arg}(z)+\mathrm{Arg}(w)))}$$

By geometric multiplication rule 1.5, we have

$$\mathrm{Arg}(z) + \mathrm{Arg}(w) = \mathrm{Arg}(zw) + 2k\pi \quad \text{for some } k \in \mathbb{Z}$$

So,

$$z^\alpha \cdot w^\alpha = (zw)^\alpha \cdot e^{2\pi i\alpha k}$$

Thus, for the equality to hold, it is necessary that $e^{2\pi i\alpha k} = 1$, which is **not** generally true unless $\alpha \in \mathbb{Q}$ with specific conditions on k.

For instance, let $z = w = -1 + i$, so $\mathrm{Arg}(z) = 3\pi/4$, and hence $\mathrm{Arg}(z) + \mathrm{Arg}(w) = 3\pi/2$. Now $(zw) = (-1+i)^2 = -2i$, which has argument $\mathrm{Arg}(-2i) = -\pi/2$, so we find

$$\mathrm{Arg}(z) + \mathrm{Arg}(w) = 3\pi/2 = \mathrm{Arg}(zw) + 2\pi$$

i.e., $k = 1$. If we take $\alpha = 1/2$, then $e^{2\pi i\alpha k} = e^{i\pi} = -1$, and thus $z^\alpha \cdot w^\alpha = -(zw)^\alpha$. The identity fails in this case.

Chapter 4

Complex Integration

Having discussed limits, continuity, differentiability and elementary functions in complex calculus, it is natural to turn towards integration. In real calculus, (definite) integration is carried out along segments of the real axis, but with the plane at our disposal, we shall consider integration along arbitrary curves on the plane. Let us start by first making the idea of a 'curve' on the plane more concrete.

4.1 Contours

If one draws a curve γ on the plane in a time interval $a \leq t \leq b$ then the points on the curve $z = x + iy$ can be formalized as a function of the 'time' at which they were drawn, $z(t) = x(t) + iy(t)$. $z(t)$ is then called a parametrization of γ.

4.1.1 Smooth Curves

The simplest curves, continuous with no self intersections and well defined tangents at all points, (as discussed in Section 2.4.4) are called **smooth curves**. Here we want the pen to move with well defined velocities without any sudden change ($z'(t)$ must be finite and continuous). As a reminder, we also require the derivative to be non-zero everywhere to avoid the possibility of a cusp:

$$z'(t) = (dx/dt, dy/dt) = x'(t) + iy'(t) \neq 0$$

To prevent self-intersections we require $z(t)$ to be one-to-one (no point is traced more than once). However for **smooth closed curves** we allow the endpoints to coincide. In contrast, a smooth curve having distinct endpoints is called a **smooth arc**. Thus, we define smooth curves as the following:
(Note that the derivatives at the ends $t = a$ and $t = b$ correspond to right and left hand derivatives respectively.)

Definition 4.1. *A point set γ in the complex plane is said to be a* ***smooth arc*** *if it is the parametrized by some continuous complex-valued function $z(t)$, $t \in [a, b] \subset \mathbb{R}$, that satisfies the following conditions:*

(i) *$z(t)$ has continuous derivatives on $[a, b]$,*

(ii) *$z'(t) \neq 0$ on $[a, b]$,*

(iii) *$z(t)$ is one-to-one on $[a, b]$*

A point set γ in the complex plane is said to be a ***smooth closed curve*** *if it satisfies (i) and (ii) along with*

(iii') *$z(t)$ is one-to-one on the half open interval $[a, b)$ with*
$z(a) = z(b)$ and $z'(a) = z'(b)$

4.1.2 Directed Curves

The parametrization of a smooth arc by 'time' t gives a natural ordering to the points, where $z(t_1)$ precedes $z(t_2)$ if $t_1 < t_2$. Naturally, we have two possible orderings: one where $z(a)$ is the initial point ($z(a)$ precedes $z(b)$) and one where $z(b)$ is the initial point ($z(b)$ precedes $z(a)$). This ordering can be visually represented by arrows pointing in the direction of $z'(t)$ (tangent/velocity vector).

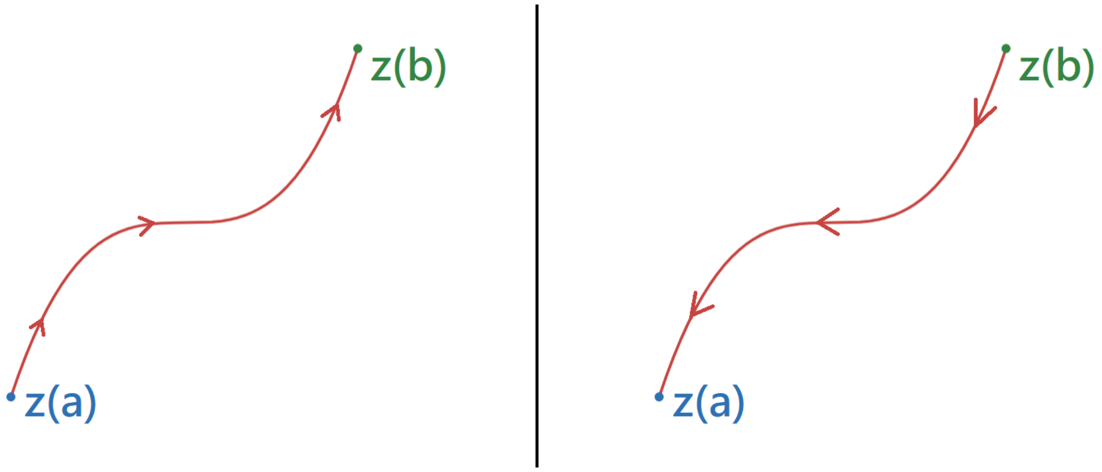


Figure 4.1: Directed smooth arcs.

Given a parametrization $z(t)$, with one ordering on $t \in [a, b]$, a reverse directed parametrization is easily constructed as $z(-t)$ on $t \in [-b, -a]$ (including for contours). For smooth closed curves, any point on the curve can serve as the initial point. Given a suitable parametrization with $t \in [a, b]$, the two possible directions of traversal are determined by the direction of unit tangent vector $z'(t)/|z'(t)|$, each assigning an ordering to the points where $z(a) = z(b)$ is the initial (and terminal) point and $z(t_1)$ precedes $z(t_2)$ if $a \leq t_1 < t_2 < b$. The only other ordering for the same initial point is the reverse: $z(t_2)$ precedes $z(t_1)$.

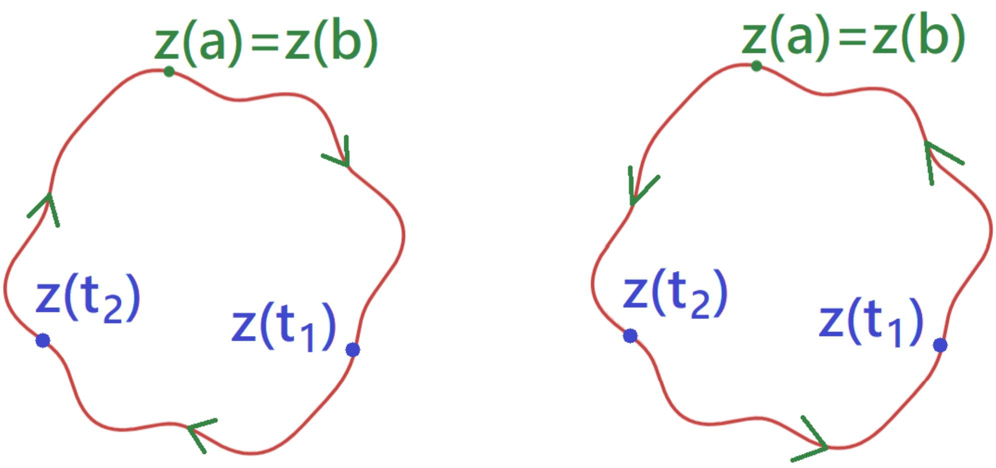


Figure 4.2: Directed smooth closed curves.

Directed smooth curves is the blanket term for directed smooth arcs and directed smooth closed curves.

4.1.3 General Curves for Integration

More generally, the curves we integrate over are called **Contours**. They are formed by joining directed smooth curves end-to-end, giving a continuous and directed path that may have cusps, corners, and self intersections. The undirected point set underlying a contour is called a **piecewise smooth curve**. They can be intuited as any curve one can draw on a plane without lifting up their pen from start to finish. Formally we define them as follows:

Definition 4.2. *A* ***contour*** Γ *is either a single point* z_0 *or a finite sequence of directed smooth curves* $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ *such that the terminal point of* γ_k *coincides with the initial point of* γ_{k+1} *for each* $k = 1, 2, \ldots, n-1$*. Thus, one can write*

$$\Gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n.$$

The contour inherits its direction from its component curves. If z_1 and z_2 lie on the same component γ_k then they are ordered by the direction of γ_k. If they lie on different components $z_1 \in \gamma_i$ and $z_2 \in \gamma_j$ then z_1 precedes z_2 if $i < j$. However, there is a caveat due to the allowance of self-intersections, if z_i is a point of self-intersection on the contour, then its occurrences (in the different intersecting smooth components) must be ordered separately.

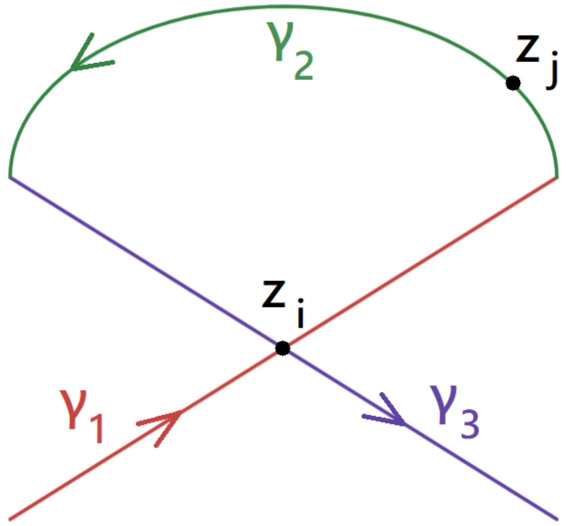


Figure 4.3: A contour made of three smooth components. The first occurrence of the intersection z_i on γ_1 precedes z_j while the second occurrence on γ_3 succeeds it.

A point contour $\Gamma = z_0$ is parametrized simply by the constant z_0. Other contours can by parametrized simply by making a piecewise function from parametrizations of their components. Formally, $z(t)$ for $t \in [a, b]$ is an admissible parametrization of the contour $\Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ if $[a, b]$ can be partitioned into n subintervals $[\tau_0, \tau_1], [\tau_1, \tau_2], \ldots, [\tau_{n-1}, \tau_n]$ where

$$a = \tau_0 < \tau_1 < \ldots < \tau_{n-1} < \tau_n = b$$

such that on each subinterval $[\tau_{i-1}, \tau_i]$ $z(t)$ is an admissible parametrization of the smooth component γ_i consistent with its direction. $z'(t)$ may have jump discontinuities at points of joining τ_i.

The initial and final points of Γ are then $z(a)$ and $z(b)$ respectively as for smooth curves. The contour made by reversing directions of all components of Γ is called the **opposite contour**, denoted as $-\Gamma$. A contour with $z(a) = z(b)$ is said to be a **closed contour**. When the only multiple point of a closed contour is its initial-terminal point ($z(t)$ is one-to-one on $[a, b)$), the curve is called a **simple closed contour**. The direction along a simple closed contour can also be characterized in view of the **Jordan Curve Theorem**:

Theorem 4.1. *Any simple closed contour separates the plane into two domains which it is the boundary of. One of these domains, called the interior, is bounded while the other, called the exterior, is unbounded.*

Proof: By the looks of it. □

(A proof is well beyond the scope of this text; we shall accept it as being intuitively obvious.)

Now if one imagines standing on and facing in the 'direction' of the curve, then the curve is said to be positively oriented if the interior lies on our left (anti-clockwise direction) and negatively oriented if the interior lies on the right (clockwise direction).

We conclude this section by discussing the length of curves. For a smooth curve parametrized by $z(t)$ for $t \in [a, b]$, let $s(t)$ denote length of the arc traversed along the curve from $z(a)$ to $z(t)$. Then $ds = |dz|$ (the length of the vector $z(t) - z(t_0)$ approaches the arc length between $z(t_0)$ and $z(t)$ as $t \to t_0$) that is,

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Hence, the length of the curve $s(b) - s(a)$ is found by the fundamental theorem of calculus to be:

$$L(\gamma) = \text{ length of } \gamma = \int_a^b \frac{ds}{dt} dt = \int_a^b \left|\frac{dz}{dt}\right| dt$$

The length of a contour is simply the sum of its component smooth curves:

$$L(\Gamma) = \sum_{i=1}^{n} L(\gamma_i)$$

4.2 Contour Integrals

4.2.1 Riemann Integrals on Smooth Curves

In real calculus we define definite integral of a function f over an interval $[a, b]$ by the intuitive idea of finding the area under the curve $(x, f(x))$ by summing up infinitesimal rectangles via the limit of the sum $\sum_{k=1}^{n} f(c_k)\Delta x_k$, called a **Riemann Sum**.

Let us try to extend the definition to complex functions, starting with smooth curves. Consider a function f defined along a directed smooth curve γ with α and β as its initial and final points respectively. We now define a **partition** $\mathcal{P}_n$ of γ for $n \in \mathbb{N}$ as a finite sequence of points $z_0, z_1, \ldots, z_n$ where $z_0 = \alpha$, $z_n = \beta$ and z_i precedes z_j whenever $i < j$. The length of the largest arc along γ between two successive points z_k and z_{k+1} in the partition is called the **mesh** of the partition, denoted by $\mu(\mathcal{P}_n)$. The mesh serves as a measure of how fine the subdivision is; consequently if the mesh is small, successive points must be close to each other and the number of subdivisions n large.

Now let $\{c_1, c_2, \ldots, c_n\}$ be any subset of points on γ such that c_k lies on the arc from z_{k-1} to z_k along γ. Then the following sum $S(\mathcal{P}_n)$ is called a Riemann Sum of f

corresponding to the partition $\mathcal{P}_n$:

$$\boxed{S(\mathcal{P}_n) = \sum_{k=1}^{n} f(c_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} f(c_k)\Delta z_k \qquad \text{where } \Delta z_k = (z_k - z_{k-1})}$$

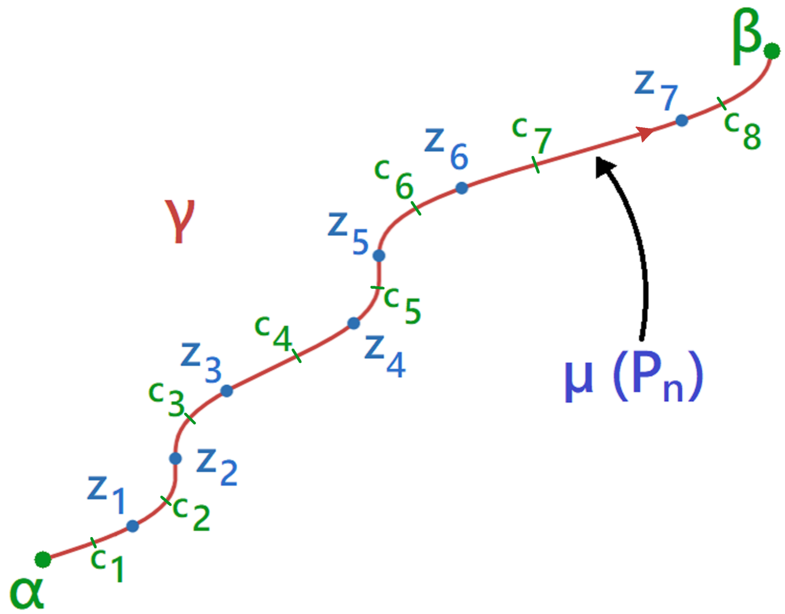


Figure 4.4: A partitioned curve. $(n = 8)$

We can now define the integral of a complex function:

Definition 4.3. *Let f be a complex valued function defined on a directed smooth curve γ. f is said to be* ***integrable*** *along γ if there exists a complex number L such that it is the limit of any sequence of Riemann sums*

$$S(\mathcal{P}_1), S(\mathcal{P}_2), \ldots, S(\mathcal{P}_n), \ldots$$

corresponding to any sequence of partitions satisfying $\lim_{n\to\infty} \mu(\mathcal{P}_n) = 0$. That is,

$$\lim_{n\to\infty} S(\mathcal{P}_n) = L \qquad \textit{whenever} \qquad \lim_{n\to\infty} \mu(\mathcal{P}_n) = 0$$

Then, L is called the **integral** of f over γ, we write:

$$L \quad = \quad \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta z_k \quad = \quad \int_\gamma f(z)dz \tag{4.1}$$

The familiar properties of real integrals also hold in the complex case under this definition:

$$\int_\gamma [f(z) \pm g(z)]\,dz = \int_\gamma f(z)\,dz \pm \int_\gamma g(z)\,dz$$

$$\int_\gamma c\,f(z)\,dz = c\int_\gamma f(z)\,dz \qquad \text{(for any complex constant } c\text{)}$$

$$\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz \qquad (-\gamma = \text{ opposite curve of } \gamma)$$

We are yet to figure a way to compute these integrals. It would be convenient if we could break it down into two real integrals like we do when composing complex limits. To do this, first let us consider the integral of a complex function f along a segment $[a, b]$ of the real line, that is, $z(t) = t$. Then, as $dt = dz$ in the Riemann sum and $f(t) = u(t) + iv(t)$, we have (assuming integrability of u, v and f):

$$\int_\gamma f(z)dz = \int_a^b f(t)dt = \int_a^b [u(t) + iv(t)]dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

However, in the last expression we only require u and v to be Riemann integrable in the real sense. We know from real analysis that continuous real valued functions are integrable along the real line (intuitively too, we expect the area under the curve of a continuous function to be well defined). From this follows:A continuous complex function must be integrable along a segment of the real axis, as we know its real and imaginary parts are continuous (see 2.3.6). This leads to an extension of the fundamental theorem of calculus for integrals along the real intervals, as if $f(t)$ has anti-derivative (with respect to real variable t) $F(t) = U(t) + iV(t)$, we have $U'(t) = u(t)$ and $V'(t) = v(t)$:

Theorem 4.2. *If the complex-valued function f is continuous on $[a, b]$ and $F'(t) = f(t)$ for all $t \in [a, b] \subset \mathbb{R}$, then*

$$\int_a^b f(t)\,dt = F(b) - F(a)\,.$$

Thus, if we are able to represent the complex integral of f along a smooth curve γ as an integral of some other complex function along the real line, we are done. The following theorem allows exactly that:

Theorem 4.3. *Let f be a function integrable on the directed smooth curve γ with parametrization $z(t)$ of corresponding direction, then:*

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt.$$

Proof: (This proof is not fully precise, but we focus on the intuitive idea here) Let

$z_k = z(t_k)$ where $z(t)$ is an admissible parametrization of γ, $t_k \in [a, b]$ and $z_k \in \mathcal{P}_n$, then

$$S(\mathcal{P}_n) = \sum_{k=1}^{n} f(c_k)\Delta z_k = \sum_{k=1}^{n} f(c_k)\frac{\Delta z_k}{\Delta t_k}\Delta t_k \qquad \text{where } \Delta t_k = (t_k - t_{k-1})$$

In the limit $lim_{n\to\infty}\mu(\mathcal{P}_n) = 0$ the fraction $\Delta z_k/\Delta t_k$ approaches the tangent/velocity vector $z'(c_k)$. So in the limit of the sum we have:

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta z_k = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)z'(c_k)\Delta t_k = \int_a^b f(z(t))z'(t)\,dt \qquad \square$$

We are now equipped to compute integrals along any smooth curve by converting them into integrals along real intervals. As all parametrizations give the same limit, we have the corollary:

Corollary 4.3.1. *If f is integrable on a directed smooth curve γ with $z_1(t)$ for $t \in [a, b]$ and $z_2(t)$ for $t \in [c, d]$ as its two admissible parametrizations consistent with the direction, then*

$$\int_a^b f(z_1(t))z_1'(t)\,dt = \int_c^d f(z_2(t))z_2'(t)\,dt$$

Further, in light of Theorem 4.2: If $f(t)$ is continuous on γ (and hence on $[a, b]$) then $f(t)z'(t)$ is integrable on the real interval $[a, b]$ (as $z'(t)$ is continuous for smooth curves). Thus, we may as well call it an alternate definition for $\int_\gamma f(z)dz$, under which continuous functions are integrable.

Theorem 4.4. *If f is continuous on the directed smooth curve γ , then f is integrable along γ.*

Thus, in the preceding theorem one may swap 'integrable' with 'continuous' as this restriction still allows for almost all the applications.

4.2.2 The General Case

Definition 4.4. *Let Γ be a contour consisting of the directed smooth curves $(\gamma_1, \gamma_2, \ldots, \gamma_n)$, and let f be a function continuous on Γ. Then the **contour integral** of f along Γ is denoted by $\int_\Gamma f(z)\,dz$ and is defined as:*

$$\int_\Gamma f(z)\,dz := \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \cdots + \int_{\gamma_n} f(z)\,dz.$$

Thus, we generalize integration to contours as the sum of integrals over its smooth directed components. If Γ is a point contour, we define:

$$\int_\Gamma f(z)\,dz = 0$$

The properties of the integral discussed for smooth curves (linearity and reversal of orientation) naturally extend to the general case. If $z(t)$ is a parametrization of Γ for $t \in [a, b]$ such that it is an admissible parametrization of corresponding direction of γ_k in the subinterval $[\tau_{k-1}, \tau_k] \subset [a, b]$ then we have:

$$\int_\Gamma f(z)\,dz = \sum_{k=1}^{n} \int_{\tau_{k-1}}^{\tau_k} f(z(t))z'(t)\,dt := \int_a^b f(z(t))z'(t)\,dt \tag{4.2}$$

As each term in the summation is independent of the parametrization (by Corollary 4.3.1), so is the entire sum. For a closed contour, this means any initial-terminal point gives the same closed loop integral. This is easily seen when the initial-terminal point coincides with the initial point of some γ_k. However, if the initial-terminal point lies in between the endpoints on a smooth component, we must break that component into two parts to define suitable intervals for admissible parametrization. This increases the number of smooth components by one. But this is no anomaly, as the different decomposition of the contour still corresponds to the same set of points and orientation.

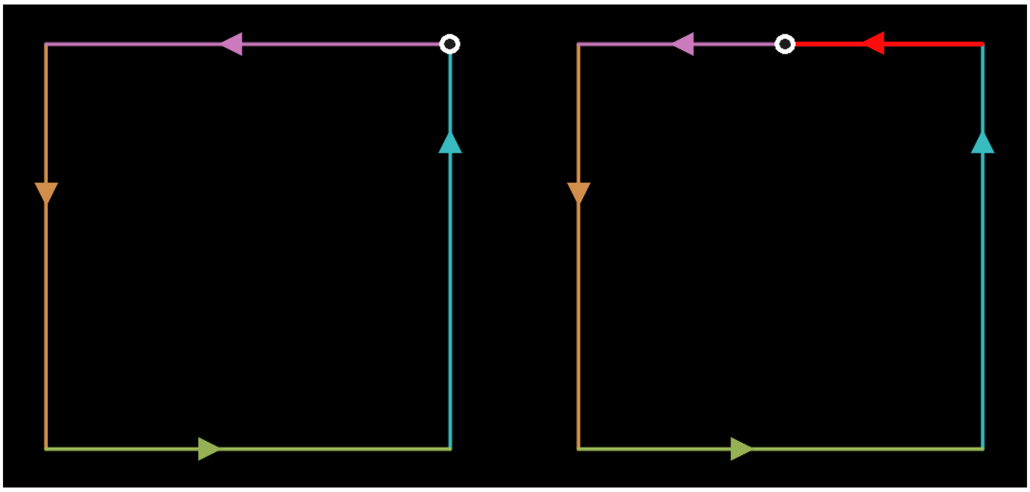


Figure 4.5: A positively oriented square contour decomposed into smooth components (colored segments) depending on the choice of initial-terminal point (indicated by the white circle).

In many applications, we do not require the integral explicitly, and an upper bound on its magnitude suffices. To find it, let $f(z) \leq M$ on a smooth directed curve γ, then using the generalized triangle inequality on a Riemann Sum, we get

$$\left|\sum_{k=1}^{n} f(c_k)\Delta z_k\right| \leq \sum_{k=1}^{n} |f(c_k)|\,|\Delta z_k| \leq M \sum_{k=1}^{n} |\Delta z_k|$$

Since each chord Δz_k is the shortest path joining z_{k-1} and z_k, the sum of their lengths cannot be greater than the total curve length $L(\gamma)$, so we have

$$\left|\sum_{k=1}^{n} f(c_k)\Delta z_k\right| \leq M \cdot L(\gamma)$$

In the limit $\mu(\mathcal{P}_n) \to 0$, the sum becomes an integral giving:

$$\left|\int_\gamma f(z)\,dz\right| \leq M \cdot L(\gamma)$$

Applying this to each smooth component of the contour Γ, and using the generalized triangle inequality on the contour integral $\int_\Gamma f(z)\,dz$, we obtain the following theorem:

Theorem 4.5. *If f is continuous on the contour Γ and $|f(z)| \leq M$ for all z on Γ, then*

$$\left|\int_\Gamma f(z)\,dz\right| \leq M \cdot L(\Gamma)$$

The smallest valid choice of M is $\max_{z\in\Gamma} |f(z)|$.

4.3 Independence of Path

Theorem 4.2 allows us to compute integrals along the real line by simply evaluating the difference of the anti-derivatives at the two endpoints of a path. However, quite elegantly, the **Fundamental Theorem of Calculus** extends to the complex plane for any contour, implying the path-independence of an integral from one point to another:

Theorem 4.6. *Let $f(z)$ be a function continuous on a domain D with an anti-derivative $F(z)$ throughout D; that is $dF/dz = f(z)$ for all $z \in D$. Then for any contour Γ lying in D, with initial point α and terminal point β, we have*

$$\int_\Gamma f(z)dz = F(\beta) - F(\alpha)$$

Proof: From an admissible parametrization $z(t)$ for $t \in [a,b]$ of $\Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, there exists $[\tau_{k-1}, \tau_k] \subset [a,b]$ such that:

$$\int_{\gamma_k} f(z)dz = \int_{\tau_{k-1}}^{\tau_k} f(z(t))z'(t)dt$$

But from chain rule, we know:

$$\frac{d[F(z(t))]}{dt} = \frac{dF}{dz}\frac{dz}{dt} = f(z)z'(t) \qquad \text{for } t \in [\tau_{k-1}, \tau_k]$$

Thus, using Theorem 4.2, we get:

$$\int_{\tau_{k-1}}^{\tau_k} f(z(t))z'(t)dt = \int_{\tau_{k-1}}^{\tau_k} \frac{d[F(z(t))]}{dt}dt = F(z(\tau_k)) - F(z(\tau_{k-1}))$$

Then from equation 4.2, the contour integral becomes:

$$\int_\Gamma f(z)dz = \sum_{k=1}^{n}[F(z(\tau_k)) - F(z(\tau_{k-1}))] = \sum_{k=1}^{n} F(z(\tau_k)) - \sum_{k=1}^{n} F(z(\tau_{k-1}))$$

Simplifying the summation,

$$\sum_{k=1}^{n} F(z(\tau_k)) - \sum_{k=1}^{n} F(z(\tau_{k-1})) = F(z(\tau_n)) + \left[\sum_{k=1}^{n-1} F(z(\tau_k)) - \sum_{k=2}^{n} F(z(\tau_{k-1}))\right] - F(z(\tau_0))$$

Thus, the summation terms cancel and the terms of $z(\tau_0) = \alpha$ and $z(\tau_n) = \beta$ remain:

$$\int_\Gamma f(z)dz = F(\beta) - F(\alpha) \qquad \square$$

As the endpoints of a closed contour lying in D coincide ($\alpha = \beta$), we get the result:

Corollary 4.6.1. *If f is continuous in a domain D and has an anti-derivative throughout D, then $\int_\Gamma f(z)dz = 0$ for all loops Γ lying in D.*

In fact, the implications in the properties discussed so far exist in both directions, illustrated in the following theorem about a somewhat non-trivial equivalence of three statements:

Theorem 4.7. *Let $f(z)$ be a function continuous on a domain D. Then the following are equivalent:*

(i) f has an antiderivative in D

(ii) If Γ is a loop (closed contour) lying in D, then $\int_\Gamma f(z)dz = 0$

(iii) The contour integrals of f are independent of path in D.

Proof: From the previous theorem we know (i) implies (ii) , so we want to prove (ii) implies (iii) and (iii) implies (i). $(ii) \Rightarrow (iii)$: If we have two contours Γ_1 and Γ_2 having same initial and terminal points, then the contour Γ formed by joining Γ_1 and $-\Gamma_2$ is closed, this gives:

$$\int_\Gamma f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{-\Gamma_2} f(z)dz = \int_{\Gamma_1} f(z)dz - \int_{\Gamma_2} f(z)dz$$

But by our assumption of (ii), $\int_\Gamma f(z)dz = 0 \Rightarrow \int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz$, so (iii) is implied.

$\underline{(iii) \Rightarrow (i)}$: To find an antiderivative $F(z)$ of $f(z)$, we define it as the integral along any contour $\Gamma \in D$ from a fixed point $z_0 \in D$ to z, taking inspiration from Theorem 4.6 (the existence of at least one such Γ is guaranteed by the connected property of domains):

$$F(z) = \int_\Gamma f(z)dz \text{ (the constant } F(z_0) \text{ is omitted as it doesn't affect the derivative)}$$

This is a well defined function due to the integral being path independent by assumption of (iii). Then, let $(z+\Delta z)$ be a point near z such that the straight line contour $\Delta\Gamma$ from z to $(z+\Delta z)$ lies in D.

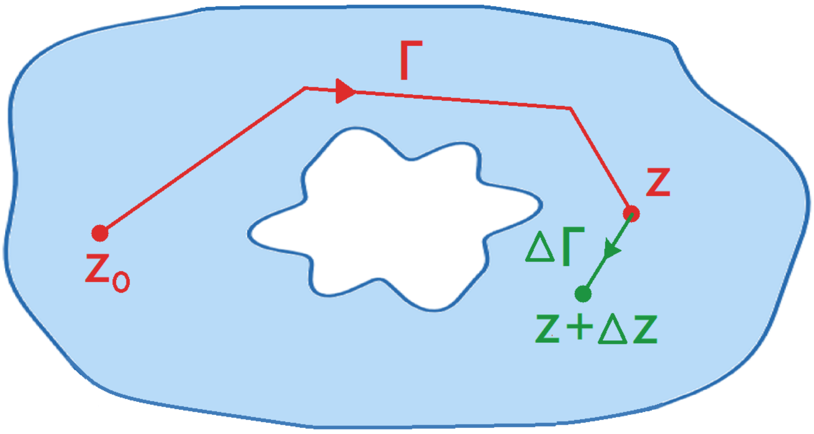


Figure 4.6: The blue-shaded region depicts D with red Γ from z_0 to z and green $\Delta\Gamma$ from z to $z+\Delta z$

Then, by definition of $F(z)$,

$$F(z+\Delta z) = \int_{\Gamma+\Delta\Gamma} f(z)dz = F(z) + \int_{\Delta\Gamma} f(z)dz$$

Parametrizing $\Delta\Gamma$ by $z(t) = z + t\Delta z$ for $t \in [0,1]$ we get:

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} = \int_0^1 f(z+t\Delta z)dt$$

We seek to prove the limit of the above expression as $\Delta z \to 0$ is $f(z)$. That is, for all $\epsilon > 0$, we want a $\delta > 0$ such that,

$$\left| \int_0^1 f(z+t\Delta z)dt - f(z) \right| < \epsilon$$

whenever $|\Delta z| \in (0, \delta)$. We now set an upper bound to the difference using Theorem 4.5.

Choose $M > \max_{t\in[0,1]} |f(z+t\Delta z) - f(z)|$ to get:

$$\left|\int_0^1 f(z+t\Delta z)dt - f(z)\right| = \left|\int_0^1 [f(z+t\Delta z) - f(z)]dt\right| < M \cdot L(\Delta\Gamma) = M\,|\Delta z|$$

Hence, choosing $\delta = \epsilon/M$ we see the desired inequality holds (Note that such a choice of M is made possible by the continuity of $f(z)$ which ensures its boundedness in D). Thus, $F(z)$ exists and is the antiderivative of $f(z)$ in D, so (i) is implied. □

4.3.1 Further Insight

Extending Real Techniques to Complex Integration

Given initial and terminal points α and β of any contour Γ lie in a domain where $f(z)$ is continuous, we can use a notation for integrals much more similar to the real case in view of the discussed path independence:

$$\int_\alpha^\beta f(z)\,dz := \int_\Gamma f(z)\,dz$$

Integration by Parts: Then, if the two functions $f(z)$ and $g(z)$ have continuous first derivatives, using the product rule with Theorem 4.6 gives:

$$\int_\alpha^\beta f'(z)g(z)\,dz = f(z)g(z)|_\alpha^\beta - \int_\alpha^\beta f(z)g'(z)\,dz,$$

Integration by Substitution: On some domain, let $f(z)$ be continuous on a contour with antiderivative $F(z)$ and $u(z)$ be a function with continuous first derivative. Then, we know the antiderivative of $f(u(z))u'(z)$ to be $F(u(z))$; this gives:

$$\int_\alpha^\beta f(u(z))u'(z)\,dz = F(u(\beta)) - F(u(\alpha))$$

Assuming $u(\alpha)$ and $u(\beta)$ lie in a domain where $f(z)$ is continuous with antiderivative $F(z)$, we can write,

$$\int_{u(\alpha)}^{u(\beta)} f(u)du = F(u(\beta)) - F(u(\alpha)) = \int_\alpha^\beta f(u(z))u'(z)\,dz$$

Examples: $\underline{\int_\pi^i e^z\cos(z)dz}$: Using integration by parts with $f'(z) = e^z$ and $g(z) = \cos(z)$,

$$\int_\pi^i e^z\cos(z)dz = e^z\cos(z)|_\pi^i + \int_\pi^i e^z\sin(z)dz = e^z\cos(z)|_\pi^i + e^z\sin(z)|_\pi^i - \int_\pi^i e^z\cos(z)dz$$

$$\Rightarrow \int_\pi^i e^z\cos(z)dz = \frac{e^z}{2}(\cos(z)+\sin(z))\bigg|_\pi^i = \frac{1}{2}[e^i(\cos(i)+\sin(i)) + e^\pi]$$

$\underline{\int_\pi^i \sin^2(z)\cos(z)dz}$: Using substitution $u(z) = \sin(z)$, $f(u) = u^2$ we get:

$$\int_\pi^i \sin^2(z)\cos(z)dz = \int_{\sin(\pi)}^{\sin(i)} u^2 du = \frac{u^3}{3}\bigg|_{\sin(\pi)}^{\sin(i)} = \frac{1}{3}\sin^3(i)$$

$\underline{\int_1^{1+i} \frac{1}{1+z^2}dz}$: We split the integrand using partial fractions and apply the substitutions $u = (z+i)$ and $u = (z-i)$ to solve each part:

$$\frac{1}{1+z^2} = \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]$$

$$\begin{aligned}\int_1^{1+i} \frac{1}{1+z^2}dz &= \frac{1}{2i}\left[\mathrm{Log}(z-i)\bigg|_1^{1+i} - \mathrm{Log}(z+i)\bigg|_1^{1+i}\right] \\ &= \frac{1}{2i}[-\mathrm{Log}(1-i) - \mathrm{Log}(2i+1) + \mathrm{Log}(1+i)]\end{aligned}$$

At each step, we can use the identities $\mathrm{Log}(a/b) = \mathrm{Log}(a) - \mathrm{Log}(b)$ and $\mathrm{Log}(a \cdot b) = \mathrm{Log}(a) + \mathrm{Log}(b)$ as the arguments stay within $(-\pi, \pi]$:

$$\frac{1}{2i}[-\mathrm{Log}(1-i) - \mathrm{Log}(2i+1) + \mathrm{Log}(1+i)] = \frac{1}{2i}[\mathrm{Log}(i+1) - \mathrm{Log}[(2i+1)(1-i)]]$$

$$= \frac{1}{2i}[\mathrm{Log}(i+1) - \mathrm{Log}(3+i)] = \boxed{\frac{1}{2i}\mathrm{Log}\left(\frac{1+i}{3+i}\right)}$$

However, since we know $\arctan(z)$ is an antiderivative of the integrand, we may also use:

$$\int_1^{1+i} \frac{1}{1+z^2}dz = \tan^{-1}(1+i) - \tan^{-1}(1)$$

$$= \frac{1}{2i}\left[\mathrm{Log}\left(\frac{1+i(1+i)}{1-i(1+i)}\right) - \mathrm{Log}\left(\frac{1+i}{1-i}\right)\right] = \boxed{\frac{1}{2i}\mathrm{Log}\left(\frac{1+i}{3+i}\right)}$$

So, both approaches lead to the same result. While in real calculus the rational function $1/(1+z^2)$ is treated as a mysterious special case due to the lack of real roots of the denominator, complex partial fractions allow us to deal with it just like we would deal with a function like $1/(1-z^2)$. Further, as the antiderivatives of a function can only differ by a constant, it shows the fundamentally logarithmic nature of the inverse tangent; even if unknown prior.

4.4 Cauchy's Integral Theorem

We shall explore this topic with two alternate lenses, one of deformation of contours and the other of vector analysis, adopting the dual-approach as in Saff and Snider's text. We

begin with the vector analysis approach, which may feel more familiar to the reader who has encountered vector fields previously (e.g. in physics).
In the last section we showed if $f(z)$ is continuous and possesses an (analytic) antiderivative in a domain, its closed loop integral must vanish. Here, we shall see how this property relates with the analyticity of $f(z)$.

4.4.1 The Vector Analysis Approach

Let $\mathbf{V}$ be a two dimensional vector field on the complex plane, that is, it assigns a 2D vector (V_1, V_2) to points (x, y):

$$\mathbf{V}(x,y) = (V_1(x,y), V_2(x,y))$$

The **line integral** of $\mathbf{V}$ along some contour Γ parametrized by $z(t) = x(t) + iy(t)$ for $t \in [a,b]$ is given as

$$\int_\Gamma V_1 dx + V_2 dy = \int_a^b \left[V_1(x(t), y(t)) \frac{dx}{dt} + V_2 \frac{dy}{dt} \right] dt$$

It can be physically interpreted as the work done on a particle by the force $\mathbf{V}$ as it traverses the path Γ. We may represent a contour integral in terms of line integrals as follows,

$$\begin{aligned}
\int_\Gamma f(z)\,dz &= \int_a^b f(z(t)) \frac{dz(t)}{dt}\,dt \\
&= \int_a^b \left[u(x(t),y(t)) + i\,v(x(t),y(t))\right] \left(\frac{dx}{dt} + i\,\frac{dy}{dt} \right) dt \\
&= \int_a^b \left[u(x(t),y(t)) \frac{dx}{dt} - v(x(t),y(t)) \frac{dy}{dt} \right] dt \\
&\quad + i \int_a^b \left[v(x(t),y(t)) \frac{dx}{dt} + u(x(t),y(t)) \frac{dy}{dt} \right] dt;
\end{aligned}$$

that is,

$$\int_\Gamma f(z)\,dz = \int_\Gamma (u\,dx - v\,dy) + i \int_\Gamma (v\,dx + u\,dy) \tag{4.3}$$

So, $\mathrm{Re}(\int_\Gamma f(z)\,dz)$ = line integral of vector field $\overline{f} = (u, -v)$ while $\mathrm{Im}(\int_\Gamma f(z)\,dz)$ = line integral of vector field $i \cdot \overline{f} = (v, u)$.

If $f(z)$ is analytic, we know the derivatives of u and v are intimately linked by the Cauchy-Riemann equations. Thus, we shall use **Green's Theorem** to proceed which relates the line integral of a vector field along the boundary of a domain to an **area integral** in its interior, involving the derivatives of its components. However, for a single closed contour Γ to be the boundary of a domain, it must be simple, that is, have no self-intersections. Moreover, the domain must be devoid of holes, since the edges of such holes would also comprise the boundary. Such a domain is called a simply-connected domain and can be formalized as:

Definition 4.5. *A domain D is called a* ***simply connected domain*** *if for any simple closed contour Γ lying in D, the interior of Γ lies wholly in D.*

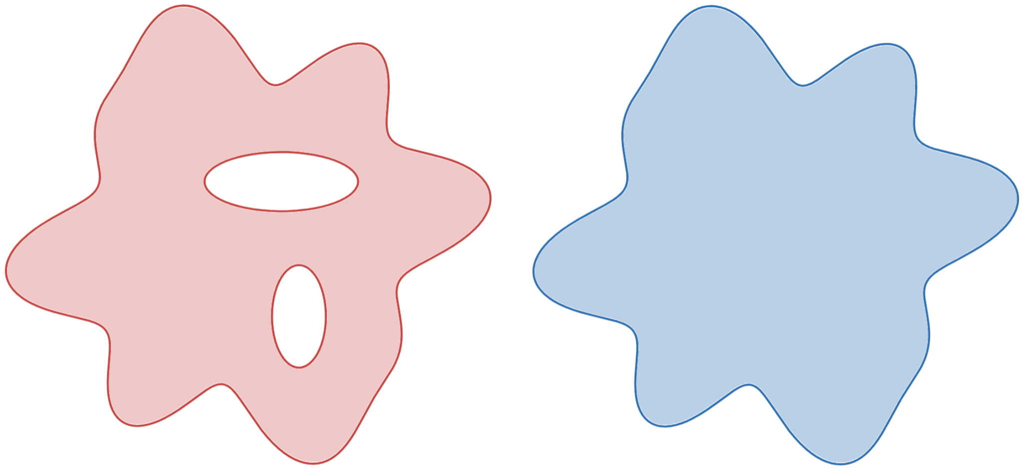


Figure 4.7: One can construct loops within the red set that encircle its elliptical holes, so the red set is **not** simply connected. However, the blue set, obtained by filling in the holes, **is** simply connected.

Green's Theorem, in our context can then be stated as follows,

Theorem 4.8. *Let $\boldsymbol{V}(x, y) = (V_1(x, y), V_2(x, y))$ be a continuously differentiable vector field defined on a simply connected domain D, and let Γ be a positively oriented simple closed contour lying in D. Then the line integral of $\boldsymbol{V}$ around Γ equals the integral of $(\partial V_2/\partial x - \partial V_1/\partial y)$ with respect to area over the interior of Γ, D'. That is,*

$$\int_\Gamma V_1 dx + V_2 dy = \iint_{D'} \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right) dxdy$$

A heuristic argument for it is given in the Appendix: 'Proving' Green's Theorem. We may now apply it to the line integrals in $\int_\Gamma f(z)\, dz$:

$$\begin{aligned} \int_\Gamma f(z)\, dz &= \int_\Gamma (u\, dx - v\, dy) + i \int_\Gamma (v\, dx + u\, dy) \\ &= \iint_{D'} \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy + i \iint_{D'} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dxdy \end{aligned}$$

However, by Cauchy-Riemann equations, the integrands in the area integral vanish ! so the loop integral is zero. Note that we require $f'(z)$ to be continuous (so partials of u

and v are continuous) to apply the theorem. In most practical scenarios, we can verify the continuity of $f'(z)$ directly.
The mathematician Edouard Goursat showed that the integral vanishes even without the assumption of $f'(z)$ to be continuous. Also, integration over any closed loop can be decomposed into sum of integrals over simple closed ones by 'snipping' the contour at self intersections. In view of these generalizations, we obtain the Cauchy-Goursat Theorem or **Cauchy's Integral Theorem** :

Theorem 4.9. *If f is analytic in a simply connected domain D and Γ is any loop (closed contour) in D, then*

$$\int_{\Gamma} f(z)dz = 0$$

Combining this with Theorem 4.7 yields:

Theorem 4.10. *In a simply connected domain, an analytic function has an antiderivative, its contour integrals are independent of path, and its loop integrals vanish.*

However, when the function is not analytic in a simply connected domain the integral may or may not vanish as illustrated in the following example.
Example 1: Find $\int_C (z-z_0)^n\, dz$ for negative integers n where O is a positively oriented circle of radius r centered at z_0.

Solution: If $n \neq -1$, the existence of antiderivative $(z-z_0)^{n+1}/(n+1)$ along the closed curve implies the integral is zero by FTC (complex). For $n=-1$, let us use the parametrization $z(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$ to get:

$$\int_O \frac{1}{z-z_0}\,dz = \int_0^{2\pi} \frac{ire^{it}}{z_0 + re^{it} - z_0}\,dt = i\int_0^{2\pi} dt = 2\pi i$$

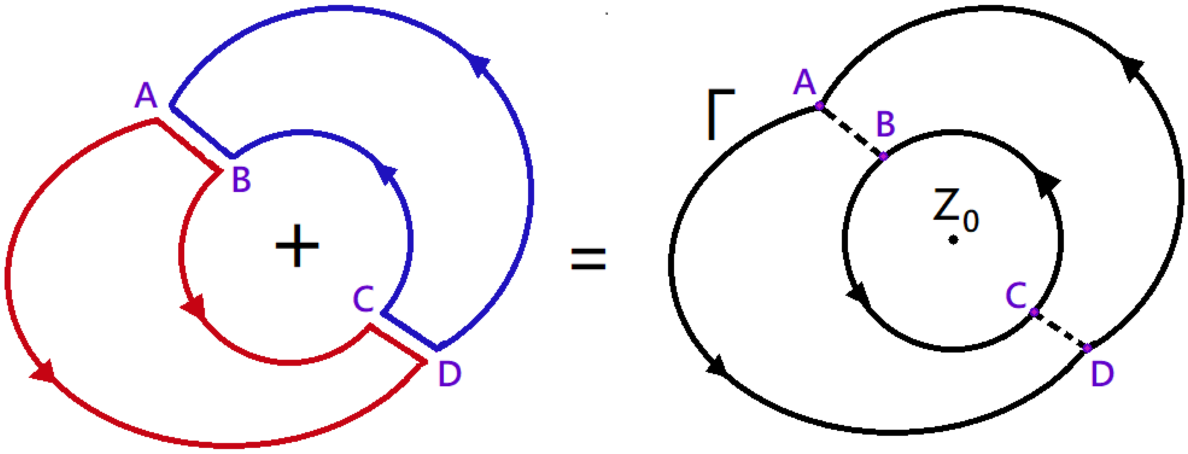
Example 2: Generalize the above integral for any positively oriented simple closed contour Γ not passing through the singularity z_0.

Solution: For $n \neq -1$ the same argument holds to prove the integral vanishes. For $n=-1$, the function is analytic at all points except the singularity which gives two cases: **(i)** z_0 lies outside the loop, then the function is analytic in the interior of the loop, hence, the integral is zero. **(ii)** If z_0 lies in side the loop, we make a circular contour centered at z_0 lying in the interior of Γ. By the previous example we know the integral around the circle is $2\pi i$. We join two points A and D on Γ to C and D on the circle O respectively. This splits Γ into two contours AD and DA and the circle into BC and CB labeled according to the direction of transit (illustrated below). In the region between AD and $ABCD$ (circular arc + two segments), $1/(z-z_0)$ is analytic, thus by Theorem 4.10, the path independence implies:

$$\int_{AD} \frac{1}{z-z_0} dz = \int_{ABCD} \frac{1}{z-z_0} dz = \left(\int_{AB} + \int_{BC} + \int_{CD} \right) \frac{1}{z-z_0} dz$$

Similarly for DA and $DCBA$:

$$\int_{DA} \frac{1}{z-z_0} dz = \int_{DCBA} \frac{1}{z-z_0} dz = \left(\int_{DC} + \int_{CB} + \int_{BA} \right) \frac{1}{z-z_0} dz$$



Adding together these equations, AD and DA combine to Γ, BC and CB combine to O and the oppositely traversed line segments cancel:

$$\int_{\Gamma} \frac{1}{z-z_0} dz = \left(\int_{DA} + \int_{AD} \right) \frac{1}{z-z_0} dz = \left(\int_{BC} + \int_{CB} \right) \frac{1}{z-z_0} dz = \int_{O} \frac{1}{z-z_0} dz$$

But as we know the integral along O from previous example, we conclude:

$$\int_{\Gamma} \frac{1}{z-z_0} dz = 2\pi i$$

Example 3: Generalize the result in previous example to any proper rational function along any simple closed curve Γ not passing through its poles where D' is its interior.

Solution: We use the partial fraction decomposition of the rational function as discussed in Chapter 3, then:

$$\int_{\Gamma} R_{m,n} dz = \sum_{i=1}^{r} \sum_{j=0}^{d_i-1} \int_{\Gamma} \frac{A_j^{(i)}}{(z-\zeta_i)^{d_i-j}} dz$$

Now, applying result of previous example, we get:

$$\int_{\Gamma} R_{m,n} dz = \sum_{\substack{k=1 \\ \zeta_k \in D'}}^{r} 2\pi i A_{d_k-1}^{(k)}$$

4.4.2 The Deformation of Contours Approach

One can imagine contours being elastic bands with indicated orientation, which can be shrunk or stretched to arbitrary degrees. We say a loop Γ_0 can be continuously deformed into the loop Γ_1 in a domain D if we can move Γ_0 continuously in the plane, without ever leaving D, such that it coincides with Γ_1 exactly (in position and orientation).

As an illustration, let D be the set where $|z| \leq 1$, Γ_0 be the unit circle and Γ_1 be a square inscribed in Γ_0 whose diagonals lie along the axes, then one can shrink Γ_0, bending its parts near the axes into corners until it coincides with Γ_1, as shown below.

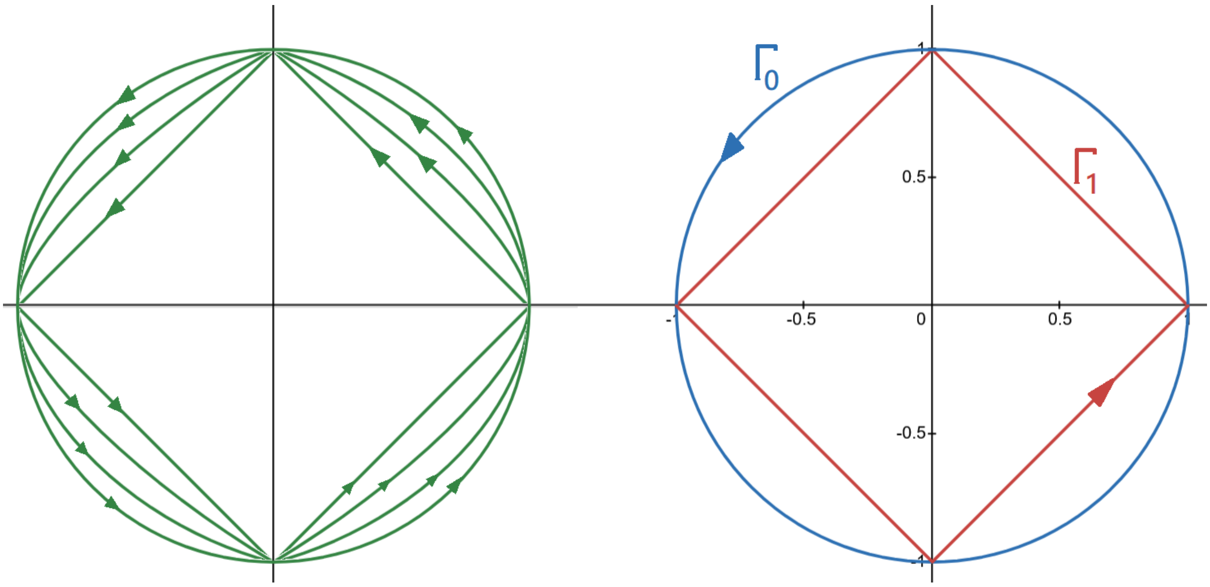


Figure 4.8: The (blue) circle is deformed into the (red) square, the intermediate stages (Γ_s) shown in green.

We can make this notion of deformation more precise as follows:

Definition 4.6. *The loop Γ_0 is said to be continuously deformable to the loop Γ_1 in the domain D if there exists a function $z(s,t)$ continuous on the unit square $(s,t) \in [0,1] \times [0,1]$, that satisfies the following conditions:*

(i) For each $s \in [0,1]$, $z(s,t)$ parametrizes a loop lying in D

(ii) Γ_0 is parametrized by $z(0,t)$.

(iii) Γ_1 is parametrized by $z(1,t)$.

Thus, for two curves to be continuously deformable into one another, they must belong to a continuum of curves, obtained by varying s, at the extremes of which lie our original curves where t parametrizes every such curve in the continuum for a given s.

The symmetry of the continuously deformable relation can be seen by the fact that if

$z(s,t)$ transforms Γ_0 to γ_1 then $z(1-s,t)$ does it in reverse.

Example 1: Let D be the entire complex plane. Find a deformation function to shrink any loop in the plane to the point contour $z = 0$.

Solution: Let the loop be Γ_0 parametrized by $z_0(t)$. Then we have the transformation function:

$$z(s,t) = (1-s)z_0(t)$$

Figure 4.9: Loop Γ_0 being shrunk to the origin, as in Example 1.

The property of the complex plane as a set that allows us to do this, is its 'simple connectedness':

Definition 4.7. *Any domain D possessing the property that every loop in D can be continuously deformed in D to a point is called a* ***simply connected domain****.*

It is worth noting that this definition is equivalent to the Definition 4.5 provided in the previous section. We can give a heuristic argument as follows:

- Def 4.7 $\Rightarrow$ Def 4.5 : If every closed contour in D can be shrunk to a point then so can simple ones. However, for a simple closed contour to shrink fully within D, its interior must also lie within D, otherwise, a "hole" (a part not in D) would obstruct the deformation. Think of a loop encircling a rigid rod protruding from the plane; it cannot pass through the rod and is trapped around it, unable to collapse further.

- Def 4.5 $\Rightarrow$ Def 4.7 : If the interior of a simple closed curve lies wholly in D, we can approximate it arbitrarily well by a polygon P, and tile the polygon with triangles as we do in the Appendix: 'Proving' Green's Theorem. Then each side of the polygonal boundary P can be gradually collapsed into the adjacent sides by introducing a hinge at one of its vertices (say, an endpoint of the hypotenuse) and collapse the triangle by shrinking the segment opposite to the hinge, as illustrated below. Then we repeat the steps with the new 'boundary triangles'. With each collapse, the curve shrinks smaller, ultimately shrinking to a point in its former interior.

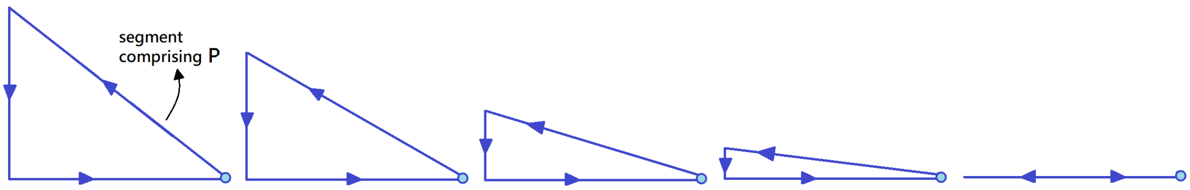


Figure 4.10: The blue point at the vertex acts as a hinge, allowing us to fold and collapse the opposite segment.

For a general closed contour, we first break it at its self-intersection points into simple closed components. Each of these components can then be individually shrunk to its respective self-intersection point. Once all segments are collapsed this way, we are left with a simple loop, which can be shrunk to a point as above. □

Interesting scenarios can happen when the domain is not simply connected:

(i) If $D = \mathbb{C} \setminus \{0\}$ and Γ_0 is the unit circle centered at the origin looping two times, parametrized as $z(t) = e^{4\pi i t}$ for $t \in [0, 1]$, then one cannot deform it into a circle looping around only once, as it is analogous to a rubber band wound twice around a stick, the only way to undo a loop is by moving it vertically along the stick but only movement normal to the stick (in the plane) is permitted.

(ii) Extending this analogy, we can say that two simple loops can be deformed into one another if and only if their interior contains the same holes (if any); that is, two elastic loops can be stretched to coincide with one another only if they are wound around the same sticks (if any).

Scenario **(ii)** is particularly useful for us, in view of the following theorem:

Theorem 4.11. *Let f be a function analytic in a domain D containing the loops Γ_0 and Γ_1. If these loops can be continuously deformed into one another in D, then*

$$\int_{\Gamma_0} f(z)\, dz = \int_{\Gamma_1} f(z)\, dz$$

This is known as the **Deformation Invariance Theorem**. A full proof of it requires topological concepts beyond this text. However, we can present a *Proof of a Weak Version:* With the assumption of $f'(z)$ being continuous and the deformation function $z(s,t)$ having continuous second-order partial derivatives, we define:

$$I(s) = \int_{\Gamma_s} f(z)\,dz = \int_0^1 f(z(s,t))\frac{\partial z(s,t)}{\partial t}\,dt$$

which is the line integral of f along any curve Γ_s parametrized by $z(s,t)$. Then by *Leibniz Rule for Integrals* we differentiate under the integral sign (which can be intuited as the distribution of differentiation over a sum when applied to a Riemann Sum):

$$\frac{dI(s)}{ds} = \int_0^1 f'(z(s,t))\frac{\partial z}{\partial s}\cdot\frac{\partial z}{\partial t} + f(z(s,t))\frac{\partial^2 z}{\partial s \partial t}\,dt$$

But by assumed continuity of the mixed partials, we see the integrand also equals:

$$\frac{d}{dt}\left[f(z(s,t))\frac{\partial z}{\partial s}\right] = f'(z(s,t))\frac{\partial z}{\partial s}\cdot\frac{\partial z}{\partial t} + f(z(s,t))\frac{\partial^2 z}{\partial t \partial s}$$

Thus,

$$\frac{dI(s)}{ds} = \int_0^1 \frac{d}{dt}\left[f(z(s,t))\frac{\partial z}{\partial s}\right]dt \qquad = f(z(s,1))\frac{\partial z}{\partial s}(s,1) - f(z(s,0))\frac{\partial z}{\partial s}(s,0) \tag{4.4}$$

But as Γ_s are loops, the initial terminal points are same, that is, $(s,0) = (s,1)$ so $dI(s)/ds = 0 \iff I(s)$ is constant. In particular, $I(0) = I(1)$, that is,

$$\int_{\Gamma_0} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz \qquad \square$$

An immediate consequence is the **Cauchy's Integral Theorem** as given in the previous section.

Proof: Any loop in a simply connected domain can be shrunk to a point. But we know, the integral of a continuous function (as analyticity implies continuity) over a shrinking loop converges to zero by Theorem 4.5.

Example 2: Generalize the result of Example 1 in the previous section for $n = -1$ and any positively oriented simple closed contour containing z_0 in its interior.

Solution: Since any simple closed curve and the circle centered at z_0 share the same hole, they are continuously deformable into one another (the existence of a deformation is intuitively assumed as in scenario **(ii)**). So, from the Deformation Invariance theorem, they must have the same integral, which we have already found for the circle to be $2\pi i$. We see that deformation allows a quick solution, without the need of making clunky connections as we did without it in the previous section.

4.5 Cauchy's Integral Formula and Its Consequences

4.5.1 The Formula

Consider a function $f(z)$ analytic inside and on a simple closed contour Γ. Cauchy's Integral theorem theorem tells us $\int_\Gamma f(z)\,dz$ must be zero. However if we instead integrate the function $f(z)/(z-z_0)$ around the same loop, then we can consider the integral around an infinitesimal circle centered at z_0, as we know it yields the same integral, evident by continuous deformation in $\{\text{interior}\} \setminus \{z_0\}$ or as explained in Example 2 of The Vector Analysis Approach. However, in a small enough circle, the continuity of $f(z)$ allows us to approximate it as the constant $f(z_0)$ on the circle, then the integral reduces to

$$f(z_0) \int_{C_r} \frac{1}{z-z_0}\,dz = 2\pi i f(z_0)$$

where C_r is the circle $|z-z_0| = r$. This result is called the **Cauchy's Integral Formula**:

Theorem 4.12. *Let Γ be a simple closed positively oriented contour. If f is analytic in some simply connected domain D containing Γ and z_0 is any point inside Γ, then*

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-z_0}\,dz$$

Proof: To make the above argument rigorous, we simply write:

$$\int_\Gamma \frac{f(z)}{z-z_0}\,dz = \int_{C_r} \frac{f(z)}{z-z_0}\,dz = \int_{C_r} \frac{f(z_0)}{z-z_0}\,dz + \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

We can now bound the second term above by Theorem 4.5 where M_r is $\max\{|f(z)-f(z_0)|\}$ for $z \in C_r$,

$$\int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\,dz \leq \frac{M_r}{r} L(C_r) = \frac{2\pi M_r r}{r} = 2\pi M_r$$

However by continuity, we know as r approaches 0, M_r vanishes, so the second term doesn't contribute to the integral. □

Example 1: Find the following integral for any simple closed Γ whose interior excludes -2 and includes 2:

$$\int_\Gamma \frac{\cos(z)}{z^2-4}$$

Solution: We know the function $\cos(z)/(z+2)$ is analytic in the interior so we write it in a modified form and apply the Cauchy's Integral Formula:

$$\int_\Gamma \frac{\cos(z)/(z+2)}{z-2} = 2\pi i \left(\frac{\cos(2)}{2+2}\right) = \frac{\pi \cos(2) i}{2}$$

An elegant consequence of this theorem is that if one knows the values of $f(z)$ on a closed loop in the interior of which it is known to be analytic, the value $f(z_0)$ for any z_0 in the interior is uniquely determined. That is, the behavior of a function analytic in a region is uniquely determined by its values on the boundary.

4.5.2 Differentiating The Formula

A slight change in notation, from z_0 to z and z to ζ, motivates the idea of differentiating under the integral sign to obtain $f'(z)$:

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \Rightarrow \quad f'(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta$$

This result is indeed valid, and in attempting to prove it, we uncover a more general result where $f(z)$ only needs to be continuous and Γ can be any contour:

Theorem 4.13. *Let g be continuous on the contour Γ, and for each z not on Γ, define*

$$G(z) := \int_\Gamma \frac{g(\zeta)}{\zeta - z}\, d\zeta.$$

Then the function G is analytic at each point not on Γ, and its derivative is given by

$$G'(z) = \int_\Gamma \frac{g(\zeta)}{(\zeta - z)^2}\, d\zeta.$$

Proof: We want to prove:

$$\lim_{\Delta z \to 0} \frac{G(z + \Delta z) - G(z)}{\Delta z} = \int_\Gamma \frac{g(\zeta)}{(\zeta - z)^2}\, d\zeta.$$

or equivalently, that the difference J approaches zero in limit $\Delta z \to 0$:

$$J = \int_\Gamma \frac{G(z + \Delta z) - G(z)}{\Delta z} - \frac{g(\zeta)}{(\zeta - z)^2}\, d\zeta.$$

Substituting the expression for $G(z)$, we get:

$$\frac{G(z + \Delta z) - G(z)}{\Delta z} = \frac{1}{\Delta z}\int_\Gamma \left[\frac{1}{\zeta - (z + \Delta z)} - \frac{1}{\zeta - z}\right] g(\zeta)\, d\zeta$$

$$= \int_\Gamma \frac{g(\zeta)}{(\zeta - z - \Delta z)(\zeta - z)}\, d\zeta$$

Thus, the difference becomes:

$$\begin{aligned} J &= \int_\Gamma \frac{g(\zeta)\,d\zeta}{(\zeta - z - \Delta z)(\zeta - z)} - \int_\Gamma \frac{g(\zeta)\,d\zeta}{(\zeta - z)^2} \\ &= \Delta z \int_\Gamma \frac{g(\zeta)\,d\zeta}{(\zeta - z - \Delta z)(\zeta - z)^2}. \end{aligned}$$

To show $J \to 0$ when $\Delta z \to 0$, we must set an upper bound to the integral. So, let $M = \max\{g(\zeta)\}$ for $\zeta \in \Gamma$ and let d be the minimum (straight line) distance from z to Γ, then $|z - \zeta| \geq d$. Since we are free to take $|\Delta z|$ arbitrarily small, let us assume $|\Delta z| < (d/2)$. Then, by the triangle inequality, we get:

$$|\zeta - z - \Delta z| \geq |\zeta - z| - |\Delta z| > d - \frac{d}{2} = \frac{d}{2}$$

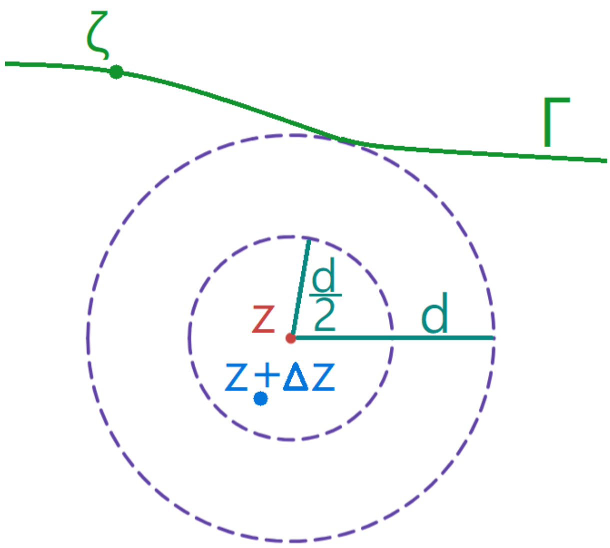


Figure 4.11: Illustration of the bounds used above.

Thus, we have the bound on integral by Theorem 4.5:

$$\left| \int_\Gamma \frac{g(\zeta)\,d\zeta}{(\zeta - z - \Delta z)(\zeta - z)^2} \right| \leq \frac{ML(\Gamma)}{(d/2) \cdot (d^2)} = \frac{2ML(\Gamma)}{d^3}$$

The magnitude of J then satisfies:

$$|J| = \left| \Delta z \int_\Gamma \frac{g(\zeta)\,d\zeta}{(\zeta - z - \Delta z)(\zeta - z)^2} \right| \leq 2\frac{|\Delta z| ML(\Gamma)}{d^3}$$

Since the bound goes to zero as $|\Delta z| \to 0$, we conclude that $G'(z)$ exists and is given by the stated formula. $\square$

4.5.3 Differentiating Again?

There seems to be no reason to stop at the first derivative! Differentiating n times under the integral sign suggests the following theorem, for which we have already proven the base case ($n = 1$).

Theorem 4.14. *Let g be continuous on the contour Γ, and for each z not on Γ, define*

$$G(z) := \int_\Gamma \frac{g(\zeta)}{\zeta - z}\, d\zeta.$$

Then the function G is analytic at each point not on Γ, and its derivative is given by

$$G^{(n)}(z) = n! \int_\Gamma \frac{g(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta.$$

Proof of Inductive Step: Suppose the formula holds for some positive integer n. We want to show it holds for $n+1$:

$$G^{(n+1)}(z) = (n+1)! \int_\Gamma \frac{g(\zeta)}{(\zeta - z)^{n+2}}\, d\zeta.$$

Let us denote

$$G_n(z) := \int_\Gamma \frac{g(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta.$$

Then by the inductive hypothesis,

$$G^{(n)}(z) = n! G_n(z).$$

To find $G^{(n+1)}(z)$, we differentiate $G_n(z)$. Firstly,

$$G_n(z + \Delta z) - G_n(z) = \int_\Gamma g(\zeta) \left[\frac{1}{(\zeta - z - \Delta z)^{n+1}} - \frac{1}{(\zeta - z)^{n+1}} \right] d\zeta$$

Simplifying using the binomial theorem,

$$\begin{aligned} \frac{1}{(\zeta - z - \Delta z)^{n+1}} - \frac{1}{(\zeta - z)^{n+1}} &= \frac{(\zeta - z)^{n+1} - (\zeta - z - \Delta z)^{n+1}}{(\zeta - z - \Delta z)^{n+1}(\zeta - z)^{n+1}} \\ &= \sum_{k=1}^{n+1} \binom{n+1}{k} \frac{-(-\Delta z)^k}{(\zeta - z - \Delta z)^k (\zeta - z)^{n+1}} \end{aligned}$$

Therefore,

$$\frac{G_n(z + \Delta z) - G_n(z)}{\Delta z} = \sum_{k=1}^{n+1} \binom{n+1}{k} (\Delta z)^{k-1} \int_\Gamma \frac{(-1)^{1+k} g(\zeta)}{(\zeta - z - \Delta z)^k (\zeta - z)^{n+1}} d\zeta.$$

As $\Delta z \to 0$, all terms with $k \neq 1$ vanish since they have a factor of $(\Delta z)^{k-1} \to 0$ (by reasoning similar to the base case). So, the only surviving term is:

$$\lim_{\Delta z \to 0} \frac{G_n(z+\Delta z) - G_n(z)}{\Delta z} = \lim_{\Delta z \to 0} \binom{n+1}{1} \int_\Gamma \frac{g(\zeta)}{(\zeta - z - \Delta z)^{n+1}(\zeta - z)} d\zeta$$

Thus, by continuity of the integrand on points not on the contour ($\zeta \neq z$):

$$G_n'(z) = (n+1) \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+2}}\, d\zeta.$$

Finally we have:

$$G^{(n+1)}(z) = n! G_n'(z) = n!(n+1) \int_\Gamma \frac{g(\zeta)}{(\zeta - z)^{n+2}}\, d\zeta = (n+1)! \int_\Gamma \frac{g(\zeta)}{(\zeta - z)^{n+2}}\, d\zeta. \qquad \square$$

Let $f(z)$ be analytic at a point z_0, that is, differentiable in a neighborhood $|z - z_0| < r$ whose simple closed positively oriented circular boundary is C_r. Then, by the Cauchy's Integral Formula, we can substitute $G(z) = f(z) = 2\pi i g(z) =$ and $\Gamma = C_r$ to obtain the derivative of $f(z)$ of any order in the neighborhood! Thus, a wonderful consequence of this theorem is the following pair of theorems: the infinite differentiability of a function differentiable once in a neighborhood and a **generalized Cauchy Integral Formula**.

Theorem 4.15. *If f is analytic in a domain D then all its derivatives $f', f'', \ldots, f^{(n)}, \ldots$ exist and are analytic in D.*

Theorem 4.16. *If f is analytic inside and on the simple closed positively oriented contour Γ and if z is any point inside Γ, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

Applying Cauchy-Riemann equations to real and imaginary parts of $f^{(n)}$, one can obtain partials derivatives of u and v of any order (even mixed) where $f = u + iv$. Thus, we have the following corollary:

Corollary 4.16.1. *If $f = u + iv$ is analytic in a domain D, then all partial derivatives of u and v exist and are continuous in D.*

For example, using $f'(z) = (\partial u/\partial x) + i(\partial v/\partial x) = (\partial v/\partial y) - i(\partial u/\partial y)$,

$$f''(z) = \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} - i\frac{\partial^2 u}{\partial x \partial y}$$

$$f''(z) = \frac{\partial^2 v}{\partial y \partial x} - i\frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2} - i\frac{\partial^2 v}{\partial y^2}$$

By continuity of $f''(z)$, all second order partial derivatives of u and v are continuous. Thus, our argument in Harmonic Functions has been validated.

We conclude this section with another theorem. We know if the loop integral of a continuous function f is zero for all loops, then it must have an antiderivative, but the antiderivative is analytic by definition, and hence is its derivative, f.

Theorem 4.17. *If f is continuous in a domain D and if*

$$\int_\Gamma f(z)\,dz = 0$$

for every closed contour Γ in D, then f is analytic in D.

This is a converse of Cauchy's Integral Theorem, called **Morera's Theorem**.

4.6 Bounds for Analytic Functions

Applying Theorem 4.5 to the generalized Cauchy formulae discussed above, we obtain the **Cauchy estimates** for the derivatives of an analytic function.

Theorem 4.18. *Let f be analytic inside and on a circle C_R of radius R centered about z_0. If $|f(z)| \le M$ for all z on C_R, then the derivatives of f at z_0 satisfy*

$$\left|f^{(n)}(z_0)\right| \le \frac{n!M}{R^n} \qquad n \in \mathbb{N}$$

Proof: Let C_R be a positively oriented simple closed contour. Then the generalized Cauchy formula gives:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_R} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta$$

For $\zeta \in C_R$, the integrand is bounded by M/R^{n+1} and $L(C_R) = 2\pi R$. Thus, Theorem 4.5 gives:

$$\left|f^{(n)}(z_0)\right| \le \frac{n!M}{2\pi R^{n+1}} \cdot 2\pi R = \frac{n!M}{R^n} \qquad \square$$

This result has several nontrivial consequences. For instance, suppose $f(z)$ is analytic and bounded by some number M over the whole plane $\mathbb{C}$. Then for $n = 1$, we can make R arbitrarily large to set an arbitrarily small bound to $|f'(z_0)|$, that is, $f'(z_0) = 0$ for all $z_0 \in \mathbb{C}$; this implies f must be constant. This result is called **Liouville's Theorem**:

Theorem 4.19. *The only bounded entire functions are the constant functions.*

Now consider the non-constant polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ where $a_n \neq 0$. Then we have:

$$P(z) = z^n(a_n + a_{n-1}/z + \ldots + a_1/z^{n-1} + a_0/z^n)$$

Clearly, we have

$$P(z)/z^n \to a_n \quad \text{as} \qquad |z| \to \infty$$

As promised in Section 3.1.1, this fact allows us to present a

Proof of the Fundamental Theorem of Algebra: Suppose $P(z)$ defined as above has no zeros. Then, by division rule of differentiation, it follows $f(z) = 1/P(z)$ is entire. Further, from the limit discussed above, there exists a R such that, for any $\epsilon > 0$

$$\left| a_n - \frac{P(z)}{z^n} \right| < \epsilon \text{ whenever } |z| > R$$

From triangle inequality this gives,

$$|a_n| - \left|\frac{P(z)}{z^n}\right| < \epsilon \quad \Rightarrow \quad \left|\frac{P(z)}{z^n}\right| > |a_n| - \epsilon = \frac{|a_n|}{2} \qquad \text{(by choosing } \epsilon = |a_n|/2)$$

Thus, outside the closed disk $|z| \leq R$, $f(z)$ is bounded as:

$$|f(z)| = \left|\frac{1}{P(z)}\right| < \frac{2}{|a_n z^n|} < \frac{2}{|a_n| R^n}$$

Inside the disk $|z| \leq R$, we have $f(z)$ continuous in a compact set, so it cannot unbounded.

Thus $f(z)$ is entire and bounded in the complex plane. Then, by Liouville's Theorem, $f(z)$ and hence $P(z)$ must be constant. So, we have proved the only polynomials with no zeros are constants $\Rightarrow$ all non-constant polynomials must have at least one zero. □

Let us parameterize C_r by $z(t) = Re^{it}$ for $t \in [0, 2\pi]$. Applying this to Cauchy's formula yields:

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{Re^{it}} iRe^{it}\, dt,$$

$$\Rightarrow \boxed{f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{it})\, dt.}$$

Thus, $f(z_0)$ equals the average value of the function around the circle C_R centered at z_0 (provided f is analytic inside the disk); this is known as the **mean-value property**. Taking the modulus and applying triangle inequality inside the integral, we get:

$$|f(z_0)| = \left|\frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{it})\, dt\right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + Re^{it})|\, dt.$$

Now, consider the case where $|f(z_0)|$ is the maximum value of $|f(z)|$ in a disk centered

at z_0. Then, for a C_R lying in the disk, values on it must be less than or equal to $|f(z_0)|$. However, the above inequality forces all values to be equal, as if some values are lesser, there is no way to compensate for the deficit to ensure the average stays $\geq |f(z_0)|$.
This leads us to the following lemma:

Lemma 4.1. *Suppose that f is analytic in a disk centered at z_0 and that the maximum value of $|f(z)|$ over this disk is $|f(z_0)|$. Then $|f(z)|$ is constant in the disk.*

Proof: Let C_R: $z_0 + Re^{it}$ for $t \in [0, 2\pi]$ be a circle lying in the disk. $|f(z_0)|$ being maximum implies $|f(z)| \leq |f(z_0)|$ for all $z \in C_r$. Assume, for contradiction, that $|f(z)|$ is non-constant; then there exists a point $z_1 = z_0 + Re^{it_1}$ on some C_r such that $|f(z_1)| < |f(z_0)|$. Since both $z(t) = (z_0 + Re^{it})$ and f are continuous functions, the composition $f(z_0 + Re^{it})$ is a continuous function for $t \in [0, 2\pi]$. Thus, by continuity at t_1, there must exist an interval $I = [t_1 - \delta, t_1 + \delta]$ such that for any $\epsilon > 0$:

$$|f(z_0 + Re^{it}) - f(z_1)| < \epsilon \qquad \text{whenever } t \in I$$

Choosing $2\epsilon = |f(z_0)| - |f(z_1)| > 0$, triangle inequality yields,

$$|f(z_0 + Re^{it})| < \epsilon + |f(z_1)| = |f(z_0)| - \epsilon$$

Further, as $|f(z_0 + Re^{it})| \leq |f(z_0)|$ for $t \in [0, 2\pi] \setminus I$, splitting the integral inequality over I and its complement in $[0, 2\pi]$, we get:

$$|f(z_0)| \leq \frac{1}{2\pi}\left(\int_I (|f(z_0)| - \epsilon)\, dt + \int_{[0,2\pi]\setminus I} |f(z_0)|\, dt\right)$$

$$\Rightarrow |f(z_0)| \leq \frac{1}{2\pi}(2\delta(|f(z_0)| - \epsilon) + (2\pi - 2\delta)|f(z_0)|) = |f(z_0)| - \frac{\delta\epsilon}{\pi}$$

But, $|f(z_0)| > |f(z_0)| - \delta\epsilon/\pi$, so we have a contradiction. Our assumption is falsified. □

We may now generalize the idea of the lemma to obtain the following version of the **maximum modulus principle**.

Theorem 4.20. *If f is analytic in a domain D and $|f(z)|$ achieves its maximum value at a point z_0 in D, then f is constant in D.*

Proof: Consider any point z' in D, then by connectedness of domains, there exists a polygonal path (contour made of finite number of line segments) from z_0 to z'. We now make a circle C_0 of radius R_0 centered at z_0 such that the disk enclosed D_0 (say) lies in D. By the previous lemma, D_0 must have $|f(z)| = |f(z_0)|$. Now, we make a chain of circles C_k with interiors $D_k \subset D$ with centers z_k lying on the polygonal path such that $k \in \mathbb{N}$, $z_k \in D_{k-1}$, and z_{k+1} succeeding z_k on the directed polygonal path from z_0 to z. Since each center is constructed to lie in the previous disk, starting from D_1, we have $|f(z_1)| = |f(z_0)|$. Then, again by previous lemma $|f(z)| = |f(z_0)|$ for all $z \in D_1$; continuing this process for each subsequent disk, we ultimately reach z' with a finite chain of disks (whose existence is guaranteed by the compactness of the polygonal path)$\Rightarrow$ $|f(z')| = |f(z_0)|$. Thus, we conclude $|f(z)|$ is the constant $|f(z_0)|$ for all points $z' \in D$ and hence f itself is constant in D. $\square$

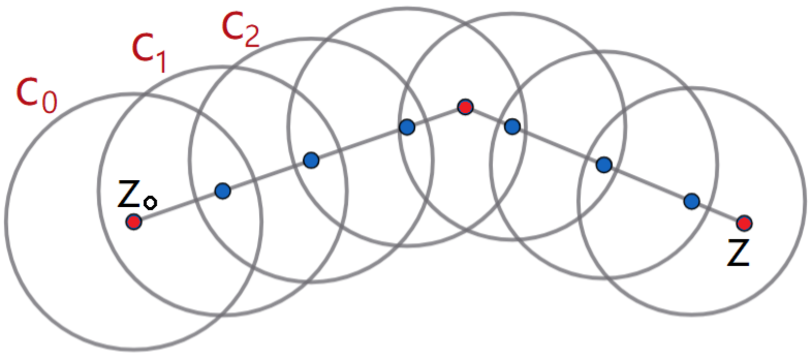


Figure 4.12: Illustration of the disk-chain used above.

We now additionally assume that D is bounded and that f is continuous on the boundary of D. Then, as $|f(z)|$ is continuous on a compact set, it must attain a maximum (and minimum) in this set. As we have shown, the maximum cannot occur in the interior unless f is constant. In any case, we can now state a special case of the above theorem:

Theorem 4.21. *A function analytic in a bounded domain and continuous up to and including its boundary attains its maximum modulus on the boundary.*

By applying this maximum modulus principle to $1/f(z)$, we obtain the analogous **minimum modulus principle**:

Theorem 4.22. *A function analytic in a bounded domain and continuous up to and including its boundary either attains its minimum modulus on the boundary, or has a zero in the domain.*

Proof: Assume $|f(z)|$ does not attain its minimum on the boundary. Then it must attain the minimum inside D. That is, $\exists\, b \in D$ such that:

$$|f(b)| \le |f(z)| \quad \text{for all } z \in D$$

Suppose $f(b) \neq 0$, then $f(z)$ is strictly non-zero in the domain $\Rightarrow$ $1/f(z)$ is analytic in D $\Rightarrow$ by maximum modulus principle, $|1/f(z)|$ has maximum at boundary $\Rightarrow$ $|f(z)|$

attains its minimum on the boundary. However, this contradicts our initial assumption $\Rightarrow f(b) = 0$, that is, f has a zero in D. $\square$

4.7 Applications to Harmonic Functions

4.7.1 The Analytic-Harmonic Duality

We can now use the tools built in this chapter to elaborate on the nature of Harmonic Functions, first discussed at the end of Chapter 2. We begin with proving the existence of harmonic conjugates in simply connected regions, to allow the application of these tools.

Theorem 4.23. *Let ϕ be a function harmonic on a simply connected domain D. Then there exists an analytic function f such that $\phi = \mathrm{Re}(f)$.*

Proof: If such an f exists, then Cauchy-Riemann equations yield:

$$f'(z) = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}$$

To show f exists, we aim to find the antiderivative of $g(z) = \partial\phi/\partial x - i\partial\phi/\partial y$. The harmonicity of ϕ (which assumes the continuity of its second partials) gives:

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) \quad , \quad \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right)$$

Thus, by Theorem 2.7, $g(z)$ is analytic on D. Further, Theorem 4.10 tells us $g(z)$ has an (analytic) antiderivative in D, say $G = u + iv$. This gives:

$$\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}$$

As u and ϕ have the same derivatives, we conclude they only differ by a constant in the domain, that is, $\phi(x, y) = u(x, y) + c$ for some (real) constant c. Then, defining $f(z) = G(z) + c = (u + c) + iv = \phi + iv$ gives us an analytic function of the desired kind. $\square$

Now, if $f(z) = \phi + iv$ is the "analytic completion" of ϕ in the simply connected domain D. Then, the function e^f is analytic with its modulus given by:

$$|e^{\phi+iv}| = |e^{\phi}||e^{iv}| = e^{\phi}$$

Due to the monotonically increasing nature of the real exponential, the maximum points of ϕ coincide with the maximum modulus points of e^f. Also, as the minimum points of ϕ are same as the maximum points of $-\phi$ we can state the following versions of the

maximum-minimum principle for harmonic functions:

Theorem 4.24. *If ϕ is harmonic in a simply connected domain D and $\phi(z)$ achieves its maximum or minimum value at some point z_0 in D, then ϕ is constant in D.*

Theorem 4.25. *A function harmonic in a bounded simply connected domain and continuous up to and including the boundary attains its maximum and minimum on the boundary.*

As a disk is a simply connected domain, starting with a disk containing z_0 allows us to use the same disk chain argument as for the maximum-modulus principle (Theorem 4.20) to extend these principles to even not simply connected domains. We adopt this generalization going forward.

As discussed in Boundary Value Problems using Log and Arg, in many physical applications we aim to find a harmonic solution in a domain satisfying values specified at the boundary, also known as a **Dirichlet Problem**. However, for such a solution to be physically meaningful, we must be sure a deduced solution is the only solution possible. This is guaranteed by the following uniqueness theorem:

Theorem 4.26. *Let $\phi_1(x, y)$ and $\phi_2(x, y)$ each be harmonic in a bounded domain D and continuous on D and its boundary. Furthermore, suppose that $\phi_1 = \phi_2$ on the boundary of D. Then $\phi_1 = \phi_2$ throughout D.*

Proof The harmonic function $\phi = \phi_1 - \phi_2$ must attain its minimum and maximum value at the boundary. As ϕ vanishes at the boundary, both the maximum and minimum are zero, so $\phi = 0 = \phi_1 - \phi_2$. □

4.7.2 Poisson Integral Formula

The Cauchy Integral formula tells us the analytic function in a domain for its values specified at the boundary. Thus, separating its real part may allow us to get an exact formula for the harmonic solution in the Dirichlet problem!

We begin with D being a disk of radius R whose positively oriented simple closed boundary is $C_R : |z| = R$. Let f be analytic on the disk and C_R. The Cauchy Integral formula for $|z| < R$ gives:

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

To extract $\mathrm{Re}(f)$, we consider the following function of ζ analytic inside and on C_R:

$$f(\zeta)\bar{z}/(R^2 - \zeta\bar{z})$$

Then, by the Cauchy Integral theorem,

$$\frac{1}{2\pi i}\int_{C_R}\frac{f(\zeta)\bar{z}}{R^2-\zeta\bar{z}}\,d\zeta = 0.$$

Adding this to the original integral simplifies it as:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{C_R}\left(\frac{1}{\zeta-z}+\frac{\bar{z}}{R^2-\zeta\bar{z}}\right)f(\zeta)\,d\zeta \\ &= \frac{1}{2\pi i}\int_{C_R}\frac{R^2-|z|^2}{(\zeta-z)(R^2-\zeta\bar{z})}f(\zeta)\,d\zeta. \end{aligned}$$

Parametrizing C_R by $z(t)=Re^{it}$ for $t\in[0,2\pi]$,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_0^{2\pi}\frac{R^2-|z|^2}{(Re^{it}-z)(R^2-Re^{it}\bar{z})}f(Re^{it})Rie^{it}dt \\ &= \frac{R^2-|z|^2}{2\pi}\int_0^{2\pi}\frac{f(Re^{it})}{(Re^{it}-z)(Re^{-it}-\bar{z})}dt \\ &= \frac{R^2-|z|^2}{2\pi}\int_0^{2\pi}\frac{f(Re^{it})}{|Re^{it}-z|^2}dt. \end{aligned}$$

Now, we identify ϕ as $\mathrm{Re}(f)$ and write $z=re^{i\theta}$ (polar form of z) to obtain the **Poisson Integral Formula**:

Theorem 4.27. *Let ϕ be harmonic in a domain containing the disk $|z|\le R$. Then for $0\le r<R$, we have*

$$\phi(re^{i\theta}) = \frac{R^2-r^2}{2\pi}\int_0^{2\pi}\frac{\phi(Re^{it})}{R^2+r^2-2rR\cos(t-\theta)}dt.$$

While the applications of this formula may seem limited, the result holds in much greater generality than discussed here. In particular, the boundary function $\phi(Re^{it})$ can be any real-valued piecewise continuous function of t, with only finitely many jump discontinuities. Moreover, the Riemann Mapping Theorem guarantees that any simply connected domain (other than $\mathbb{C}$ itself) can be conformally mapped onto the open unit disk. This is especially useful, as conformal maps preserve harmonicity. Although we do not explore these ideas in detail, the interested reader may refer to Saff and Snider's text for further insight.

4.7.3 Liouville's Theorem for Harmonic Functions

Let $\phi(z)$ be a harmonic function bounded above by some value M over the entire plane $\mathbb{C} \cong \mathbb{R}^2$, that is, $\phi(z) \leq M$ for all $z \in \mathbb{C}$. Let $f = \phi + i\psi$ be the "analytic completion" of ϕ. Then, the composition of entire functions $g(z) = e^{f(z)}$ is also entire. We notice that $g(z)$ is also bound as:

$$|g(z)| = |e^{f(z)}| = e^{\phi} \leq e^M$$

Thus, by Liouville's Theorem, $g(z)$ must be a constant. That is, $g'(z) = f'(z)e^{f(z)} = 0$. Since $e^{f(z)}$ is never zero, it implies $f'(z) = 0$, and hence f is constant. But as $\phi = \text{Re}(f)$, ϕ too must be a constant! A similar argument applies if ϕ is bound from below ($M \leq \phi(z)$) by choosing $g(z) = e^{-f(z)}$. This gives us the Liouville's Theorem for harmonic functions:

Theorem 4.28. *If ϕ is harmonic in the whole plane and bounded from above or below there, then ϕ is constant.*

Thus, non-constant functions harmonic in the whole plane $\mathbb{R}^2$ are unbounded from both above and below.

This also reveals a stronger form of the Liouville's Theorem for entire functions: instead of requiring a bound on the modulus, even a one-sided boundedness of either one of its components (real or imaginary) suffices to prove it is constant ! [To handle a bound on $\text{Im}(f) = \psi$, we can consider $-if(z) = \psi - i\phi$, so that $\psi = \text{Re}(-if(z))$]

Chapter 5

Series Representations for Analytic Functions

5.1 Sequences and Series

In Chapter 2, we defined the limit of a sequence, which formalized the idea of a sequence of numbers "approaching" a value. Consider the sequence given by $z_n = 1/2^n$ for $n \in \mathbb{N}$. Clearly, the sequence has the limit 0. However, if we consider the sums of first few terms of the sequence,

$$\frac{1}{2}+\frac{1}{4}=\frac{3}{4}, \qquad \frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{7}{8}, \qquad \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{15}{16}, \quad \ldots$$

We observe this sequence of sums has the limit 1. Thus, for certain sequences, we see the notion of sum of its infinitely many terms becomes well defined:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1$$

Thus, we formalize the notion and nomenclature of such infinite sums in Definition 5.1.

Clearly, an equivalent way to show a sum converges to S is to show the sequence of remainders at n$^{\text{th}}$ partial sum, $S - \sum_{j=0}^{n} c_j$ converges to zero. We use this method to obtain the infinite sum for the simple but useful **geometric series** $\sum_{j=0}^{\infty} c^j$.

Lemma 5.1. *The series $\sum_{j=0}^{\infty} c^j$ converges to $1/(1-c)$ if $|c| < 1$; that is,*

$$1+c+c^2+c^3+\ldots=\frac{1}{1-c} \qquad \text{if } |c|<1.$$

Proof: We first find an expression for the n$^{\text{th}}$ partial sum:

$$S_n - cS_n = 1 - c^{n+1} \Rightarrow S_n = \frac{1-c^{n+1}}{1-c}$$

This gives us the remainder:

$$S_n - \frac{1}{1-c} = \frac{-c^{n+1}}{1-c}$$

Since $|c| < 1$, we have $c^n \to 0$ as $n \to \infty$. As the remainder is a constant multiple of this limit to zero, it too tends to zero. □

Definition 5.1. *A* ***series*** *is a formal expression of the form $c_0 + c_1 + c_2 + \cdots$, or equivalently $\sum_{j=0}^{\infty} c_j$, where the* ***terms*** *c_j are complex numbers. The* ***nth partial sum*** *of the series, usually denoted S_n, is the sum of the first $n+1$ terms, that is,*

$$S_n := \sum_{j=0}^{n} c_j.$$

If the sequence of partial sums $\{S_n\}_{n=0}^{\infty}$ has a limit S, the series is said to ***converge****, or* ***sum****, to S, and we write*

$$S = \sum_{j=0}^{\infty} c_j.$$

A series that does not converge is said to ***diverge****.*

If the terms of a series are bounded by those of another series whose convergence is known, we intuitively expect the original series to converge as well. This method of determining convergence is called the **Comparison Test**:

Theorem 5.1. *Suppose the terms c_j satisfy the inequality*

$$|c_j| \leq M_j$$

for all integers j beyond some index J. Then, if the series $\sum_{j=0}^{\infty} M_j$ converges, so does $\sum_{j=0}^{\infty} c_j$.

We postpone its proof to section 5.4 Mathematical Theory of Convergence.

Example 1: Show that the series $\sum_{j=0}^{\infty}(3+2i)/(j+1)^j$ converges.

Solution: For all $j \geq 3$, we see the terms are bound as:

$$\left|\frac{3+2i}{(j+1)^j}\right| \leq \frac{4}{2^j}$$

As the series $\sum_{j=0}^{\infty} 4/2^j$ is convergent, the comparison test establishes the convergence.

We say a series $\sum_{j=0}^{\infty} c_j$ is **absolutely convergent** if the series $\sum_{j=0}^{\infty} |c_j|$ converges. A trivial application of the comparison test shows absolutely convergent series are convergent.

If the ratio of consecutive terms in a series approaches a value, we can think of it as behaving like a geometric series to determine convergence. This is the **Ratio Test**:

Theorem 5.2. *Suppose the terms of the series $\sum_{j=0}^{\infty} c_j$ have the property that the ratios $|c_{j+1}/c_j|$ approach the limit L as $j \to \infty$. Then the series converges if $L < 1$.*

Proof: Since the limit exists, for any given $\epsilon > 0$, there exists a $J \in \mathbb{N}$ such that for all $j > J$,

$$\left| \left| \frac{c_{j+1}}{c_j} \right| - L \right| < \epsilon.$$

Using the triangle inequality and the fact that $L = |L|$ (as it is the limit of a modulus),

$$\left| \frac{c_{j+1}}{c_j} \right| \leq L + \left| \left| \frac{c_{j+1}}{c_j} \right| - L \right| < L + \epsilon.$$

Now choose $\epsilon = (1 - L)/2$, so that $(L + \epsilon) < 1$. It then follows that for all $j > J$,

$$|c_{j+1}| < |c_j|(L + \epsilon).$$

By induction, for all $k > J$, we get

$$|c_k| \leq |c_J|(L + \epsilon)^{k-J}.$$

Thus, $|c_k|$ are bound by terms of the geometric series:

$$\frac{|c_J|}{(L + \epsilon)^J} \sum_{k=0}^{\infty} (L + \epsilon)^k$$

which is convergent as $(L + \epsilon) < 1$. The comparison test then allows us to conclude the convergence of the original series. $\square$

Example 2: Show that the series $\sum_{j=0}^{\infty} 4^j/j!$ converges.

Solution:

$$\left| \frac{c_{j+1}}{c_j} \right| = \frac{4^{j+1}}{(j+1)!} \cdot \frac{j!}{4^j} = \frac{4}{j+1}$$

The ratio approaches zero as $j \to \infty$; therefore we conclude the convergence of the series by the ratio test.

In our context, we shall deal with sequences and series whose terms are functions of the complex variable z. For example, the sequence $z_n = (z/2i)^n$ converges to 0 for $|z| < 2 = |2i|$ and diverges for $|z| > 2$.

To approximate a function effectively with sequences or series, we require a stronger notion of convergence. For example: the sequence $F_n(x) = x^n$ for real $x \in [0, 1)$ converges to $F(x) = 0$ for all x in the interval for large enough n. However, the functions always retain $\lim_{x\to 1^-} F_n(x) = 1$ and thus never become a good approximation of $F(x) = 0$ throughout the interval $[0, 1)$ at once. We say the convergence is **pointwise**, instead of **uniform**, as defined below:

Definition 5.2. *The sequence $\{F_n(z)\}_{n=1}^{\infty}$ is said to* ***converge uniformly*** *to $F(z)$ on the set T if for any $\epsilon > 0$ there exists an integer N such that when $n > N$,*

$$|F(z) - F_n(z)| < \epsilon \qquad \text{for all } z \in T.$$

Accordingly, the series $\sum_{j=0}^{\infty} f_j(z)$ converges uniformly to $f(z)$ on T if the sequence of its partial sums converges uniformly to $f(z)$ there.

The strength of uniform convergence lies in the fact that a single N works for all $z \in T$. In pointwise convergence, however, N is found for each $z \in T$ without the certainty of one N working for all simultaneously. Hence, uniform convergence implies pointwise convergence.

Example 3: Show that the series $\sum_{j=0}^{\infty}(z/z_0)^j$ is uniformly convergent in any disk $|z| \le r$ for $r < |z_0|$.

Solution: Since $(r/|z_0|) < 1$, the series $\sum_{j=0}^{\infty}(r/|z_0|)^j$ converges by Lemma 5.1. Comparing the remainder $\sum_{j=0}^{\infty} c_j - S$ for both series:

$$\left|\frac{(z/z_0)^{n+1}}{1 - (z/z_0)}\right| \le \frac{(r/|z_0|)^{n+1}}{1 - (r/|z_0|)}$$

The bound on the right is independent of z and can be made arbitrarily small by choosing N large enough (by the deduced convergence of the $(r/|z_0|)$ series). Hence, the same N works for all $|z| \le r$, confirming uniform convergence.

5.1.1 Further Insight

Let us now explore some less fundamental but useful results from the preceding theory.

Vanishing Difference of Consecutive Terms (Sequence)

Consider a convergent sequence $\{z_n\}_{n=1}^{\infty}$ with limit L. Then for any $\epsilon' > 0$ we have an N such that for all $n > N$:

$$|z_n - L| < \epsilon'$$

Then for all $n > N + 1$, triangle inequality yields:

$$|z_n - z_{n-1}| \leq |z_n - L| + |L - z_{n-1}| < 2\epsilon'$$

Given any $\epsilon > 0$, one can choose $\epsilon' = \epsilon/2$ to show $\lim_{n\to\infty}(z_n - z_{n-1}) = 0$, that is, the difference of consecutive terms of a convergent sequence approaches zero.

Vanishing Terms of a Convergent Series

Consider a convergent series $\sum_{j=0}^{\infty} c_j$ with sum S. Then by the above result, the difference of consecutive partial sums approaches zero, that is, $\lim_{n\to\infty}(S_n - S_{n-1}) = 0$. But $(S_n - S_{n-1}) = c_n$ so $\lim_{n\to\infty} c_n = 0$ that is, the terms of a convergent series must approach zero.

Diverging Geometric Series

Lemma 5.1 discussed the convergence of the geometric series $\sum_{j=0}^{\infty} c^j$ when $|c| < 1$. But when $|c| \geq 1$, the series must diverge as the terms don't vanish (which is a necessary condition by above result). To prove this, let $|c| \geq 1$ and assume for sake of contradiction that the terms approach zero, then for any $\epsilon > 0$, for all $n > N$ for some N,

$$|c^n - 0| < \epsilon$$

Choosing $\epsilon = 1$ results in a contradiction as $|c| \geq 1 \Rightarrow |c|^n \geq 1$, hence proved.

Ratio Test for Divergence

We discussed the ratio test when $L < 1$ for $\lim_{n\to\infty} |c_{j+1}/c_j| = L$. The above result motivates the idea of the series $\sum_{j=0}^{\infty} c_j$ being divergent when $L > 1$. To prove this, first consider the implication of the limit: There exists $J \in \mathbb{N}$ such that for all $j > J$,

$$\left|\left|\frac{c_{j+1}}{c_j}\right| - L\right| < \epsilon.$$

Using triangle inequality and choosing $\epsilon = (L-1)/2 > 0$, so that $(L - \epsilon) > 1$, we get:

$$\left|\frac{c_{j+1}}{c_j}\right| \geq L - \left|L - \left|\frac{c_{j+1}}{c_j}\right|\right| > L - \epsilon = 1 + \frac{L-1}{2} > 1.$$

Hence for all $j > J$,

$$|c_{j+1}| > |c_j|.$$

Thus, $|c_j|$ for $j > J$ has the lower bound:

$$|c_j| \geq |c_{J+1}| > 0 \qquad \text{for all } j > J.$$

Thus, $|c_j|$ cannot be made arbitrarily small by increasing j, which rules out the convergence of the series by the necessary condition of vanishing terms discussed earlier in this section.

5.2 Taylor Series

Suppose we wish to find a polynomial approximation $p_n(z)$ of an analytic function f near a point z. To do this, we say that the functional value and all derivatives upto an arbitrary number n of p_n and f at same at z_0. Then the Taylor form of $p_n(z)$ as discussed in 3.1.1 Polynomials gives:

$$p_n(z) = \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

As we can make n arbitrarily large, we can say the sequence $\{p_n\}_{n=1}^{\infty}$ approaches $f(z)$ in a neighborhood of z_0. We observe p_n looks like the n^{th} partial sum of a series, as discussed below.

Definition 5.3. *If f is analytic at z_0, then the series*

$$f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2}(z - z_0)^2 + \ldots = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z_0)(z - z_0)^k$$

is the ***Taylor series*** *for f around z_0. When $z_0 = 0$, it is also known as the* ***Maclaurin series*** *for f.*

Theorem 5.3. *If f is analytic in the disk $|z - z_0| < R$, then the Taylor series converges to $f(z)$ for all z in this disk. Furthermore, the convergence of the series is uniform in any closed subdisk $|z - z_0| \leq R' < R$.*

Proof: Proving uniform convergence for $|z - z_0| \leq R' < R$ for all $R' < R$ suffices to imply the pointwise convergence for all z in the open disk $|z - z_0| < R$. Let C be the positively oriented circle $|z - z_0| = (R + R')/2$, so that for any z in the disk enclosed by C, Cauchy's integral formula yields:

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

We manipulate the integrand into a form suggesting a geometric series with powers of $(z - z_0)/(\zeta - z_0)$ which is known to be convergent as $|z - z_0|/|\zeta - z_0| < 1$. We use the formula $S_n + c^{n+1}/(1-c) = 1/(1-c)$ with $c = (z-z_0)/(\zeta - z_0)$:

$$\begin{aligned}\frac{1}{\zeta - z} &= \frac{1}{(\zeta - z_0) - (z - z_0)} \\ &= \frac{1}{1 - \frac{z-z_0}{\zeta - z_0}} \cdot \frac{1}{\zeta - z_0} \\ &= \left[1 + \frac{z-z_0}{\zeta - z_0} + \frac{(z-z_0)^2}{(\zeta-z_0)^2} + \cdots + \frac{(z-z_0)^n}{(\zeta-z_0)^n} + \frac{(z-z_0)^{n+1}}{(\zeta-z_0)^{n+1}} \cdot \frac{1}{1-\frac{z-z_0}{\zeta-z_0}}\right] \frac{1}{\zeta - z_0}.\end{aligned}$$

Then the Cauchy's integral formula yields:

$$f(z) = \sum_{k=0}^{n} \frac{(z-z_0)^k}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta + T_n(z)$$

where

$$T_n(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)(z-z_0)^{n+1}}{(\zeta - z)(\zeta - z_0)^{n+1}}\, d\zeta$$

Now the formula for Cauchy estimates (Theorem 4.16) transforms the terms in the summation:

$$\frac{(z-z_0)^k}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta = \frac{f^{(k)}(z_0)(z-z_0)^k}{k!}$$

So these terms form exactly the n^{th} partial sum S_n of the Taylor series. Thus, the remainder $|f(z) - S_n|$ is $|T_n(z)|$. We now show it can be made arbitrarily small for all $|z - z_0| \le R'$ by choosing a sufficiently large n. To do this, we consider:

$$|z - z_0| \le R', \quad |\zeta - z_0| = \frac{R+R'}{2}, \quad \frac{|z-z_0|}{|\zeta - z_0|} \le \frac{2R'}{R+R'}$$

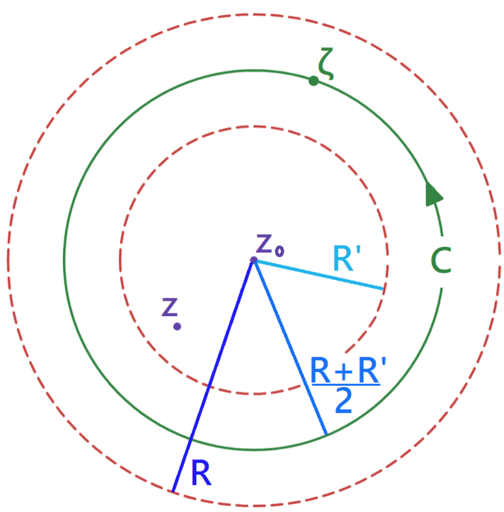


Figure 5.1: Uniform convergence in closed subdisks.

Also, by triangle inequality,

$$\frac{R-R'}{2} = \frac{R+R'}{2} - R' \leq |\zeta - z_0| - |z - z_0| \leq |\zeta - z|$$

Finally, considering $L(C) = \pi(R+R')$, Theorem 4.5 allows us to bound the remainder:

$$|T_n(z)| \leq \frac{1}{2\pi} \cdot \max_{\zeta \in C} |f(\zeta)| \frac{2}{R-R'} \left(\frac{2R'}{R+R'}\right)^{n+1} 2\pi \left(\frac{R+R'}{2}\right).$$

The right-hand side can be made smaller than any $\epsilon > 0$ independently of z by making n big enough as $2R' < (R+R')$. □

Example 1: Find the Taylor series and convergence properties of $\mathrm{Log}(z)$, $1/(1-z)$ and e^z around $z_0 = 1$,$z_0 = 0$ and $z_0 = 0$ respectively.
Solution: The consecutive derivatives of $\mathrm{Log}(z)$ are $z^{-1}, -z^{-2}, 2z^{-3}, \ldots$ which suggests the following formula (provable by induction):

$$\frac{d^j \,\mathrm{Log}(z)}{dz^j} = (-1)^{j+1}(j-1)! z^{-j}$$

Thus, the Taylor series of $\mathrm{Log}(z)$ centered at $z_0 = 1$ is

$$\mathrm{Log}(z) = 0 + \sum_{j=1}^{\infty} \frac{(-1)^{j+1}(z-1)^j}{j}$$

which holds for the disk $|z-1| < 1$ as it is the largest disk centered at $z_0 = 1$ excluding the branch cut.

Similarly, for $1/(1-z)$,

$$\frac{d^j (1-z)^- 1}{dz^j} = j!(1-z)^{-j-1}$$

Thus, for the disk $|z| < 1$ (as it is undefined $z = 1$) we get:

$$\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j$$

This is also obvious by the geometric series discussed in Lemma 5.1.

Now, for e^z, all derivatives are e^z, so for the disk $|z| < R$ for an arbitrarily large R, so the Taylor series around $z = 0$ is:

$$e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!} \qquad \square$$

5.2.1 Differentiating the Series

Given how simple the derivatives of polynomials are, it would be helpful if term-wise differentiation of such a series of f allows us to obtain the series for f'. Let us test this possibility on the above derived expansions:

$$\frac{d}{dz}e^z = \sum_{j=0}^{\infty} \frac{d}{dz}\frac{z^j}{j!} = \sum_{j=1}^{\infty} \frac{z^{j-1}}{(j-1)!}$$

We notice that only the form has changed but the infinite sum remains the same, giving the same series as for e^z. Similarly for $\mathrm{Log}(z)$,

$$\frac{d\,\mathrm{Log}(z)}{dz} = \sum_{j=1}^{\infty} \frac{d}{dz}\frac{(-1)^{j+1}(z-1)^j}{j} = \sum_{j=1}^{\infty}(-1)^{j-1}(z-1)^{j-1}$$

A change in index shows equivalence to the geometric series $\sum_{j=0}^{\infty}(1-z)^j$ which converges to:

$$\frac{1}{1-(1-z)} = \frac{1}{z}$$

for $|1-z| < 1$, confirming that the derivative can indeed be obtained this way. The following theorem confirms the validity of this method in general.

Theorem 5.4. *If f is analytic at z_0 , the Taylor series for f' around z_0 can be obtained by termwise differentiation of the Taylor series for f around z_0 and converges in the same disk as the series for f.*

Proof: If f is analytic in a disk $|z-z_0| < R$, by Theorem 4.15, we know f' is too. Then, as the j^{th} derivative of f' is the $(j+1)^{\text{th}}$ derivative of f, we obtain the Taylor series for f' as:

$$f'(z) = \sum_{k=0}^{\infty} \frac{f^{(k+1)}(z_0)(z-z_0)^k}{k!}$$

Now, termwise differentiation of the series for $f(z)$ gives:

$$\sum_{k=0}^{\infty} \frac{d}{dz}\frac{f^{(k)}(z_0)(z-z_0)^k}{k!} = \sum_{k=1}^{\infty} \frac{d}{dz}\frac{f^{(k)}(z_0)(z-z_0)^{k-1}}{(k-1)!}$$

A change in index shows this is identical to the series derived for f'. □

Combined with the infinitely differentiable property of the analytic function f, this inductively implies that the Taylor series for $f^{(n)}$ is obtained by termwise differentiation of the Taylor series of f, n times (converging in the same disk as f).

5.2.2 Sums and Products of Series

Given functions f and g analytic at z_0, we know $(cf)' = c(f')$ and $(f \pm g)' = (f' \pm g')$. Thus, we have the following theorem:

Theorem 5.5. *Let f and g be analytic functions with Taylor series*

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j \quad \text{and} \quad g(z) = \sum_{j=0}^{\infty} b_j (z - z_0)^j$$

around the point z_0 that is, $a_j = f^{(j)}(z_0)/j!$ and $b_j = g^{(j)}(z_0)/j!$. Then

- *The Taylor series for $cf(z)$, c being a constant, is*

$$\sum_{j=0}^{\infty} c a_j (z - z_0)^j.$$

- *The Taylor series for $f(z) \pm g(z)$ is*

$$\sum_{j=0}^{\infty} (a_j \pm b_j)(z - z_0)^j.$$

Where the disk of convergence for $f \pm g$ is at least as big as the smaller of the convergence disks for f and g (as the largest disk where both f and g are analytic has $f \pm g$ analytic).

Example 2: Find the Maclaurin series for the function $\cos(z) + i\sin(z)$.
Solution: The Maclaurin series for $\sin(z)$ and $\cos(z)$ are found to be:

$$\sin z = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} z^{2j+1} \qquad \cos z = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} z^{2j}$$

Thus, by the previous theorem the desired series is given as:

$$\cos(z) + i\sin(z) = \sum_{j=0}^{\infty} \frac{(i)^{2j}}{(2j)!} z^j + \sum_{j=0}^{\infty} \frac{(i)^{2j+1}}{(2j+1)!} z^j$$

Observing the even-odd pattern in the series, we change indexes to combine it to a single sum as:

$$\sum_{j=0}^{\infty} \frac{(iz)^j}{j!}$$

Notice that this is the series for e^{iz} as shown in Example 1. The process of finding Taylor series when limited to the real line, applied to the functions $e^x, \sin(x), \cos(x)$ naturally gives an alternate way to derive Euler's formula. □

To get a series for the product of two analytic functions f and g we are motivated to use the distributive property of multiplication:

$$[a_0 + a_1 z + a_2 z^2 + a_3 z^3 \ldots][b_0 + b_1 z + b_2 z^2 + b_3 z^3 \ldots]$$

$$= a_0 b_0 + (a_0 b_1 + b_1 a_0) z + (a_2 b_0 + a_1 b_1 + a_0 b_2) z^2 + (a_3 b_0 + a_2 b_1 + a_1 b_2 + a_0 b_3) z^3 + \ldots$$

Thus, we define a product of two Taylor series as:

Definition 5.4. *The **Cauchy product** of two Taylor series $\sum_{j=0}^{\infty} a_j (z - z_0)^j$ and $\sum_{j=0}^{\infty} b_j (z - z_0)^j$ is defined to be the (formal) series $\sum_{j=0}^{\infty} c_j (z - z_0)^j$ where*

$$c_j = \sum_{l=0}^{j} a_{j-l} b_l$$

Theorem 5.6. *Let f and g be analytic functions with Taylor series*

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j \quad \text{and} \quad g(z) = \sum_{j=0}^{\infty} b_j (z - z_0)^j$$

around the point z_0 that is, $a_j = f^{(j)}(z_0)/j!$ and $b_j = g^{(j)}(z_0)/j!$. Then the Taylor series for the product function fg is given by the Cauchy product of the two series.

Proof: We observe the following pattern:

$$(fg)' = fg' + f'g$$

$$(fg)'' = fg'' + 2f'g' + f''g$$

$$(fg)''' = fg''' + 3f''g' + 3f'g'' + f'''g$$

Thus, we see a Pascal's Triangle or Binomial coefficients emerging, suggesting the following General Leibniz rule:

$$(fg)^{(j)} = \sum_{l=0}^{j} \binom{j}{l} f^{(j-l)} g^{(l)} = \sum_{l=0}^{j} j! \frac{f^{(j-l)}}{(j-l)!} \frac{g^{(l)}}{l!}$$

To prove it by induction, given that is holds for $j = 1$, let us show it holds for $(fg)^{(j+1)}$

assuming it holds for $(fg)^{(j)}$:

$$\begin{aligned}
(fg)^{(j+1)} &= \left[\sum_{l=0}^{j}\binom{j}{l}f^{(j-l)}g^{(l)}\right]' \\
&= \sum_{l=0}^{j}\binom{j}{l}f^{(j+1-l)}g^{(l)} + \sum_{l=0}^{j}\binom{j}{l}f^{(j-l)}g^{(l+1)} \\
&= \binom{j}{0}f^{(j+1)}g^{(0)} + \sum_{l=1}^{j}\binom{j}{l}f^{(j+1-l)}g^{(l)} + \sum_{l=1}^{j}\binom{j}{l-1}f^{(j+1-l)}g^{(l)} + \binom{j}{j}f^{(0)}g^{(j+1)} \\
&= \binom{j+1}{0}f^{(j+1)}g^{(0)} + \left(\sum_{l=1}^{j}\left[\binom{j}{l-1}+\binom{j}{l}\right]f^{(j+1-l)}g^{(l)}\right) + \binom{j+1}{j+1}f^{(0)}g^{(j+1)} \\
&= \binom{j+1}{0}f^{(j+1)}g^{(0)} + \sum_{l=1}^{j}\binom{j+1}{l}f^{(j+1-l)}g^{(l)} + \binom{j+1}{j+1}f^{(0)}g^{(j+1)} \\
&= \sum_{l=0}^{j+1}\binom{j+1}{l}f^{(j+1-l)}g^{(l)}.
\end{aligned}$$

Thus, as the coefficients of $(z-z_0)^j$ in the Taylor series for (fg) is given by $(fg)^{(j)}/j!$, identifying $a_{j-l} = f^{(j-l)}/(j-l)!$ and $b_l = g^{(l)}/l!$ we see that it is precisely the expression we stated for the c_j in the Cauchy product. $\square$

Example 3: Find the Maclaurin series for $\sin(z)\cos(z)$.
Solution: The Maclaurin series for $\sin(z)$ and $\cos(z)$ are found in the previous example, we use the forms:

$$\sin(z) = \sum_{\substack{j=0 \\ j \text{ is odd}}}^{\infty}\frac{(-1)^{(j-1)/2}z^j}{j!} \qquad \cos(z) = \sum_{\substack{j=0 \\ j \text{ is even}}}^{\infty}\frac{(-1)^{(j/2)}z^j}{j!}$$

Thus, $a_j = (-1)^{(j-1)/2}/j!$ for odd j and zero otherwise and $b_j = (-1)^{(j/2)}/j!$ for even j and zero otherwise. The Cauchy product coefficients are then:

$$c_j = \sum_{\substack{l=0 \\ (j-l) \text{ is odd}, l \text{ is even}}}^{j}\frac{(-1)^{(j-l-1)/2}}{(j-l)!}\cdot\frac{(-1)^{(l/2)}}{l!}$$

and zero otherwise. Equivalently, set $l = 2k$ (even) and $j = 2m+1$ (odd) so that $j - l = 2(m-k)+1$ is odd. Then the sum becomes

$$c_{2m+1} = \sum_{k=0}^{m}\frac{(-1)^{m-k}}{(2(m-k)+1)!}\cdot\frac{(-1)^k}{(2k)!}.$$

Combining exponents, $(-1)^{m-k} \cdot (-1)^k = (-1)^m$, so

$$c_{2m+1} = \frac{(-1)^m}{(2m+1)!} \sum_{k=0}^{m} \binom{2m+1}{2k}$$

To evaluate the combinatorial sum, we use Binomial formula as

$$\sum_{r=0}^{2m+1} \binom{2m+1}{r} = (1+1)^{2m+1} = 2^{2m+1}, \qquad \sum_{r=0}^{2m+1} \binom{2m+1}{r} (-1)^r = (1-1)^{2m+1} = 0.$$

Adding and subtracting these two equations isolates the sum over even and odd r:

$$S_{\text{even}} + S_{\text{odd}} = 2^{2m+1}, S_{\text{even}} - S_{\text{odd}} = 0 \Rightarrow S_{\text{even}} = S_{\text{odd}} = 2^{2m},$$

Hence

$$S_{\text{even}} \sum_{k=0}^{m} \binom{2m+1}{2k} = 2^{2m} = 4^m.$$

Thus,

$$c_{2m+1} = \frac{(-1)^m \, 4^m}{(2m+1)!}$$

Finally this gives

$$\sin z \cos z = \sum_{m=0}^{\infty} \frac{(-1)^m \, 4^m \, z^{2m+1}}{(2m+1)!} = \frac{1}{2} \sum_{m=0}^{\infty} \frac{(-1)^m \, (2z)^{2m+1}}{(2m+1)!}$$

This series is precisely of $\sin(2z)/2$! Verifying the known trigonometric identity by the Cauchy product. □

Example 4: Given Taylor series expansions of analytic functions f and g as:

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j \quad \text{and} \quad g(z) = \sum_{j=0}^{\infty} b_j (z - z_0)^j$$

find the Taylor series around z_0 for the function obtained by their division, $h(z) = f(z)/g(z)$ wherever convergent.

Solution: Any disk centered at z_0 which does not contain any zeros of $g(z)$ has $h(z)$ analytic, telling us the existence of a convergent Taylor series for it. Thus, denoting the series as $h(z) = \sum_{j=0}^{\infty} c_j (z - z_0)^j$, we use the fact $h(z)g(z) = f(z)$ with the previous theorem to get:

$$a_j = \sum_{l=0}^{j} c_{j-l} b_l$$

For $j = 0$, we get the $c_0 = a_0/b_0$. Then for higher terms, we build a recursive formula for

c_j as:

$$a_j = c_j b_0 + \sum_{l=1}^{j} c_{j-l} b_l \Rightarrow \boxed{c_j = \frac{1}{b_0}\left(a_j - \sum_{l=1}^{j} c_{j-l} b_l\right)}$$

As we have derived a recursive formula that uniquely determines coefficients satisfying the Cauchy product property, the existence of the Taylor series for $h(z) = f(z)/g(z)$ implies these coefficients must form a series converging to $h(z)$.

Example 5: Find the first few terms of the Maclaurin series of $\tan(z)$.
Solution: Using the above recursive formula from Example 4 with the known series for $\sin(z)$ and $\cos(z)$,

$$a_0 = 0,\ a_1 = 1,\ a_2 = 0,\ a_3 = -\tfrac{1}{6},\ a_4 = 0,\ a_5 = \tfrac{1}{120},\ a_6 = 0, \ldots$$

$$b_0 = 1,\ b_1 = 0,\ b_2 = -\tfrac{1}{2},\ b_3 = 0,\ b_4 = \tfrac{1}{24},\ b_5 = 0,\ b_6 = -\tfrac{1}{720}, \ldots$$

yields,

$$\begin{aligned}
c_0 &= 0, \\
c_1 &= 1, \\
c_2 &= 0, \\
c_3 &= -\tfrac{1}{6} - (0 \cdot 0 + 1 \cdot (-\tfrac{1}{2}) + 0 \cdot 0) = \tfrac{1}{3}, \\
c_4 &= 0, \\
c_5 &= \tfrac{1}{120} - (0 \cdot 0 + \tfrac{1}{3} \cdot (-\tfrac{1}{2}) + 0 \cdot 0 + 1 \cdot \tfrac{1}{24} + 0 \cdot 0) = \tfrac{2}{15},
\end{aligned}$$

Hence, the Maclaurin series for $\tan(z)$ up to z^5 is

$$\tan(z) = z + \frac{z^3}{3} + \frac{2z^5}{15} + \ldots \quad \square$$

5.3 Power Series

Taylor series appears to be a special case of a more general series:

Definition 5.5. *A series of the form $\sum_{j=0}^{\infty} a_j (z - z_0)^j$ is called a **power series**. The constants a_j are the **coefficients** of the power series.*

What values of z allow the convergence of the series? Is the sum an analytic function? If yes, are the coefficients unique? Such questions arise naturally and thus, are our focus in this section.

5.3.1 Convergence

The following result settles the issue of convergence, much like the criteria of convergence for Taylor series.

Theorem 5.7. *For any power series $\sum_{j=0}^{\infty} a_j(z-z_0)^j$ there is exists $R \in [0,\infty] \subset \mathbb{R} \cup \{\infty\}$ which depends only on the coefficients $\{a_j\}$, such that*

- *the series converges for $|z-z_0| < R$,*
- *the series converges uniformly in any closed subdisk $|z-z_0| \leq R' < R$,*
- *the series diverges for $|z-z_0| > R$.*

*The number R is called the **radius of convergence** of the power series.*

Note that $R = 0$ corresponds to convergence at only z_0 (with limit zero, trivially) and $R = \infty$ corresponds to convergence for all $z \in \mathbb{C}$.

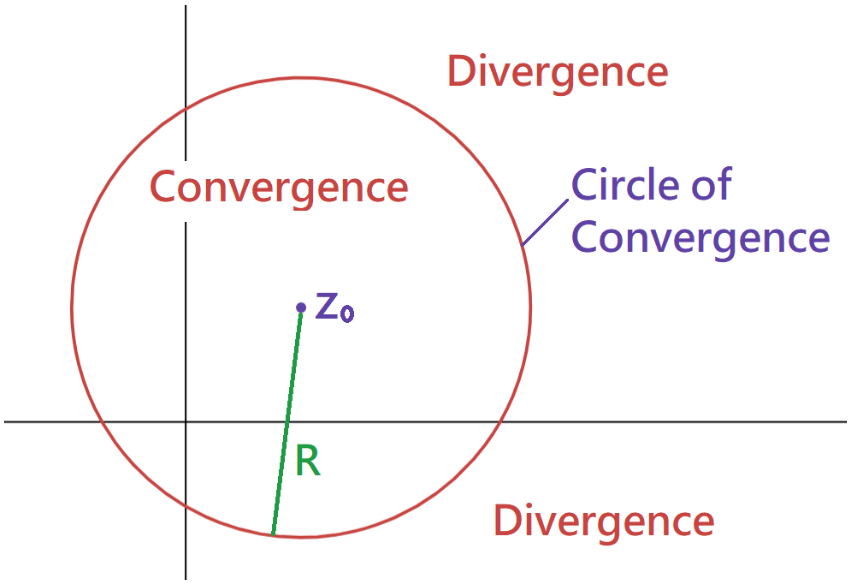


Figure 5.2: Circle of convergence.

We defer its proof to section 5.4 Mathematical Theory of Convergence. The essential reasoning in the proof is captured in the following lemma:

Lemma 5.2. *If the series $\sum_{j=0}^{\infty} a_j z^j$ converges at a point z_0 with $|z_0| = r$, then it converges at every point inside the disk $|z| < r$.*

Proof: The convergence of the series implies the modulus of the terms must be bounded by some constant M (as the sequence of terms approach zero, as discussed in section

5.1.1):

$$|a_j z_0^j| = |a_j| r^j \leq M \qquad (\text{for } j = 0, 1, 2, \ldots)$$

Thus, for $|z| < r$, we have

$$|a_j z^j| = |a_j| r^j \left(\frac{|z|}{r}\right)^j \leq M \left(\frac{|z|}{r}\right)^j$$

As the terms of the series $\sum_{j=0}^{\infty} a_j z^j$ are bounded by the geometric series $M \sum_{j=0}^{\infty} (|z|/r)^j$ which is known to be convergent by lemma 5.1, comparison test proves the lemma. □

The property of uniform convergence in subdisks in Theorem 5.7 is a powerful feature as seen in the following three results.

Lemma 5.3. *Let f_n be a sequence of functions continuous on a set $T \subset \mathbb{C}$ and converging uniformly to f on T. Then f is also continuous on T.*

Proof: Firstly, by the assumed uniform convergence, for any $\epsilon > 0$, there exists a natural number N so large that $|f(z) - f_N(z)| < \epsilon/3$ for all $z \in T$. Secondly, the continuity of the terms implies the existence of a $\delta > 0$ such that $|f_N(z) - f_N(z_0)| < \epsilon/3$ for any $z_0 \in T$ when $|z - z_0| < \delta$. Finally, triangle inequality implies:

$$|f(z) - f(z_0)| \leq |f(z) - f_N(z)| + |f_N(z) - f_N(z_0)| + |f_N(z_0) - f(z_0)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

whenever $|z - z_0| < \delta$. Thus,$f(z)$ is continuous at all $z_0 \in T$. □

The continuity of f implies its integrability. In fact, the integral of f is given by the limit of the sequence of integrals of f_n:

Theorem 5.8. *Let f_n be a sequence of functions continuous on a set $T \in \mathbb{C}$ containing the contour Γ, and suppose that f_n converges uniformly to f on T. Then the sequence $\int_\Gamma f_n(z)dz$ converges to $\int_\Gamma f(z)dz$*

Proof: By uniform convergence choose a natural number N such that for any $\epsilon > 0$, $|f_n(z) - f(z)| < \epsilon/L(\Gamma)$ for all $n > N$. Then, Theorem 4.5 yields:

$$\left|\int_\Gamma f(z)dz - \int_\Gamma f_n(z)dz\right| = \left|\int_\Gamma [f_n(z) - f(z)]dz\right| < \frac{\epsilon}{L(\Gamma)} \cdot L(\Gamma) = \epsilon \qquad \square$$

Combining these results with Morera's theorem (Theorem 4.17) and Cauchy's integral theorem, we obtain the following theorem.

Theorem 5.9. *Let f_n be a sequence of functions analytic in a simply connected domain D and converging uniformly to f in D. Then f is analytic in D.*

Proof: Lemma 5.3 allows us to conclude f is continuous and hence integrable. Cauchy's integral theorem tells us that for any contour Γ lying in D, $\int_\Gamma f_n(z) = 0$. Theorem 5.8 then tells us that $\int_\Gamma f(z) = 0$. But by Morera's theorem this implies the analyticity of f in D. □

We can in fact extend this to any domain, as any point in a domain lies in an open disk contained in the domain, which is simply connected.

5.3.2 Power Series = Taylor Series?

The partial sums of a power series are polynomials, and hence analytic. Then, as they converge uniformly in any closed subdisk in the circle of convergence, Theorem 5.9 tells us that the limiting function must be analytic in such a subdisk. But as any point in the circle of convergence lies inside a subdisk, we can state the following:

Theorem 5.10. *A power series sums to a function that is analytic at every point inside its circle of convergence.*

Further, theorems 5.7 and 5.8 allow us to integrate power series termwise along contours lying inside its circle of convergence. Using this, we recognize any power series (with $R > 0$) as a Taylor series for the limiting function.

Theorem 5.11. *If $\sum_{j=0}^\infty a_j(z - z_0)^j$ converges to $f(z)$ in some circular neighborhood of z_0 (that is, the radius of its circle of convergence is nonzero), then*

$$a_j = \frac{f^{(j)}(z_0)}{j!} \qquad (j = 0, 1, 2, \ldots).$$

Consequently, $\sum_{j=0}^\infty a_j(z - z_0)^j$ is the Taylor expansion of $f(z)$ around z_0.

Proof: Let C be a positively oriented circle centered at z_0 lying inside the circle of convergence. Then, by theorem 5.10, the analyticity of f allows the generalized Cauchy's integral formula:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$$

We now substitute $f(\zeta) = \sum_{j=0}^{\infty} a_j(\zeta - z_0)$ and integrate termwise by Theorem 5.8. Since,

$$\int_C \frac{(\zeta - z_0)^j}{(\zeta - z_0)^{n+1}}\, d\zeta = \begin{cases} 2\pi i & \text{if } j = n, \\ 0 & \text{otherwise.} \end{cases}$$

The only surviving term is $n!a_n$, so $f^{(n)}(z_0) = n!a_n \Rightarrow a_n = f^{(n)}(z_0)$. This completes the proof and shows the uniqueness of a power series (Taylor) representation of a function. □

Are the largest disks of convergence of the power series and Taylor series same? Firstly, the "Taylor disk" cannot extend beyond the circle of convergence as the power series is known to diverge there. Secondly, as the function is analytic inside the circle of convergence, its Taylor series must converge there. Hence, the disks are indeed the same. Further, the circle of convergence must have some point where the limit function is not analytic (or isn't well defined), as if not, there exist points arbitrarily close but outside the circle where the power series converges, which contradicts the definition of the radius of convergence.

Example 1: Find the solution function f analytic at the origin to the following differential equation with initial values:

$$\frac{d^2 f}{dz^2} + 4f = 0 \qquad f(0) = 1, f'(0) = 1$$

Solution: Assuming the existence of f till we obtain it, we have a power series representation in some neighborhood of the origin:

$$f(z) = \sum_{j=0}^{\infty} a_j z^j$$

$f(0) = 1$ yields $a_0 = 1$. Then, as we have discussed the validity of termwise differentiation of Taylor series, we have

$$f'(z) = \sum_{j=1}^{\infty} j a_j z^{j-1} \Rightarrow f''(z) = \sum_{j=2}^{\infty} j(j-1)a_j z^{j-2} = \sum_{j=0}^{\infty} (j+2)(j+1)a_{j+2} z^j$$

$f'(0) = 1$ yields $a_1 = 1$. Putting these in the differential equation,

$$\sum_{j=0}^{\infty} [(j+2)(j+1)a_{j+2} z^j + 4a_j z^j] = 0$$

As zero is an entire function, the coefficients of its power series expansion must be uniquely zero, so

$$a_{j+2} = \frac{-4a_j}{(j+1)(j+2)}$$

For even (j), we write $j = 2k$. Then, recursively applying the formula k times for the first few terms suggests the following formula which can be proved by induction:

$$a_{2(k+1)} = \frac{(-1)^{k+1}4^{k+1}}{(2k+2)!}$$

which gives the series,

$$a_0 + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}4^{k+1}}{(2k+2)!} z^{2k+2} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!}(2z)^{2k}$$

which is known to converge to the entire function $\cos(2z)$. Similarly, for odd j, we write $j = 2k+1$ to get:

$$a_{2k+3} = \frac{(-1)^{k+1}4^{k+1}}{(2k+3)!}$$

which gives the series,

$$a_1 z + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}4^{k+1}}{(2k+3)!} z^{2k+3} = z + \sum_{k=1}^{\infty} \frac{(-1)^k 4^k}{(2k)!}(z)^{2k+1}$$

which is known to converge to the entire function $\sin(2z)/2$. Thus, the final series for f, with both odd and even j, converges to the analytic function:

$$f(z) = \cos(2z) + \frac{\sin(2z)}{2}$$

which has been found to uniquely satisfy the given problem. $\square$

5.4 Mathematical Theory of Convergence

In this section, we discuss postponed proofs and elaborate on related concepts to complete the discussion on convergence.

5.4.1 The Cauchy Criterion

A **Cauchy sequence** is a sequence where the difference of further enough terms can be made arbitrarily small. This suggests that these further terms must be approaching a value to give a vanishing difference. The following theorem shows this equivalence between a Cauchy sequence and a sequence converging to a limit.

Theorem 5.12. *A necessary and sufficient condition for the sequence of complex numbers $\{A_n\}_{n=1}^{\infty}$ to converge is the following: For any $\epsilon > 0$ there exists an integer N such that $|A_n - A_m| < \epsilon$ for every pair of integers m and n satisfying $m > N, n > N$.*

Proof: (Necessity) If $\{A_n\}_{n=1}^{\infty}$ converges, given any $\epsilon > 0$, there exists N such that for all $n > N$, $|A_n - L| < \epsilon/2$. Then triangle inequality for $m, n > N$ gives:

$$|A_n - A_m| \leq |A_n - L| + |L - A_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

(Sufficiency) The convergence of real valued Cauchy sequences is obtained by an axiomatic construction of real numbers, to define the concept of irrational numbers, real exponentiation and so on. However, we omit this discussion as it would be too long of a detour for our purposes. For a complex Cauchy sequence $\{z_n\}_{n=1}^{\infty}$, we decompose it into real and imaginary parts $z_n = x_n + iy_n$ to get:

$$|z_n - z_m| < \epsilon' \Rightarrow \left|\sqrt{(x_n - x_m)^2 + (y_n - y_m)^2}\right| < \epsilon' \Rightarrow |x_n - x_m| < \epsilon', |y_n - y_m| < \epsilon'$$

Thus, x_n and y_n are also Cauchy sequences whose convergence is inferred from their real valuedness. Then, if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, we have $\lim_{n\to\infty} z_n = x + iy$ as for any given ϵ, choosing $\epsilon' = \epsilon/2$ gives:

$$|z_n - x - iy| \leq |x_n - x| + |iy_n - iy| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \square$$

Corollary 5.12.1. *If $\{A_n\}_{n=1}^{\infty}$ is a Cauchy sequence and N is chosen so that $|A_n - A_m| < \epsilon$ for every m and n greater than N, then each A_n with $n > N$ is within ϵ of the limit.*

Proof: By previous theorem, we know the Cauchy sequences have a limit, say $\lim_{n\to\infty} A_n = A$. Then, for any $\delta > 0$, there exists M such that for all $m > M$, $|A_m - A| < \delta$. Finally, (choosing $m > \max\{M, N\}$) triangle inequality yields:

$$|A_n - A| \leq |A_n - A_m| + |A_m - A| < \epsilon + \delta$$

As δ is any positive number, $|A_n - A| - \epsilon \leq \delta$ implies

$$|A_n - A| - \epsilon \leq 0 \Rightarrow |A_n - A| \leq \epsilon \quad \square$$

Corollary 5.12.2. *A necessary and sufficient condition for the series $\sum_{j=0}^{\infty} c_j$ to converge is the following: For any $\epsilon > 0$ there exists an N such that $\left|\sum_{j=n+1}^{m} c_j\right| < \epsilon$ for every pair of integers m and n satisfying $m > n > N$.*

This is the Cauchy criterion applied to the sequence of partial sums of a series. Such a series is called a **Cauchy series**.

5.4.2 The Weierstrass M-test

Equipped with Cauchy criterion, we are ready to prove the comparison test. In fact, we can prove a more general theorem called the **Weierstrass M-test**, which talks about the uniform convergence of a series whose terms are functions of a complex variable:

Theorem 5.13. *Suppose $\sum_{j=0}^{\infty} M_j$ is a convergent series with real non-negative terms and suppose, for all z in some set T and for all j greater than some number J, that $|f_j(z)| \leq M_j$. Then the series $\sum_{j=0}^{\infty} f_j(z)$ converges uniformly on T.*

Proof: By the Cauchy criterion, we know M_j's form a Cauchy series, so for some $J \in \mathbb{N}$, $m > n > J$ and $\epsilon > 0$ we have $\sum_{n+1}^{m} M_j < \epsilon$. But by the given conditions, $f_j(z)$'s also form a Cauchy series as:

$$\left|\sum_{n+1}^{m} f_j(z)\right| \leq \sum_{n+1}^{m} |f_j(z)| \leq \sum_{n+1}^{m} M_j < \epsilon$$

Hence $\sum_{j=0}^{\infty} f_j(z)$ converges for all $z \in T$ to some function $F(z)$(say). By Corollary 5.12.1, we know this convergence to be uniform, as

$$\left|F(z) - \sum_{j=0}^{n} f_j(z)\right| \leq \epsilon \qquad \text{for } z \in T \qquad \text{and } n > J \quad \square$$

The comparison test is now only a special case, where we simply have $f_j(z)$'s to be constant functions.

5.4.3 Convergence of Power Series

To prove Theorem 5.7, we aim to determine an expression for R (the radius of convergence). To do this, we first consider the resemblance of the power series $\sum_{j=0}^{\infty} a_j(z - z_0)^j$ with a geometric series, by saying its "common ratio" to have magnitude $\sqrt[n]{|a_n(z - z_0)^n|} = |z - z_0| \sqrt[n]{|a_n|}$. However, since a_j's may fluctuate, to assign a singular value, we introduce the notion of the "smallest eventual upper bound" or $\limsup$ (limit superior) of a real sequence:

Definition 5.6. *The **upper limit** of a sequence of real numbers $\{x_n\}_{n=1}^{\infty}$, denoted $\limsup x_n$ is defined to be the smallest number ℓ such that, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ so that for all $n > N$,*

$$x_n < \ell + \epsilon$$

If every $\ell \in \mathbb{R}$ satisfies this property, then we say $\limsup x_n = -\infty$; if no such number exists, then we say $\limsup x_n = \infty$.

If x_n converges to a limit L, then we simply have $\limsup x_n = L$. However, all real sequences have an upper limit by the above definition, convergent or not; for example, $\limsup(-1)^n = 1$, $\limsup(n) = \infty$, $\limsup(-n) = -\infty$.

Thus, we set the pseudo-"common ratio" of the power series to be $|z-z_0| \limsup \sqrt[n]{|a_n|}$. We know a geometric series converges when the common ratio has magnitude less than one and diverges when greater than one (ref. Section 5.1), this gives us a reasonable guess for R:

$$\boxed{R = \frac{1}{\limsup \sqrt[n]{|a_n|}}}$$

as $|z-z_0| \limsup \sqrt[n]{|a_n|} < 1$ for $|z-z_0| < R$ and $|z-z_0| \limsup \sqrt[n]{|a_n|} > 1$ for $|z-z_0| > R$. We use the usual conventions $1/0 = \infty$ and $1/\infty = 0$. This is known as the **Cauchy-Hadamard formula**. Let us see if our heuristically motivated formula holds rigorously.

Proof of Theorem 5.7: First we consider the case $\limsup \sqrt[n]{|a_n|} = \infty$, that is, $R = 0$. We have convergence only at $|z - z_0| = 0$ trivially (zero) and for any point $|z - z_0| > 0$, the series diverges as the terms of the series don't approach zero by the known lim sup, violating a necessary condition for convergence (as discussed in Section 5.1.1). To see this more clearly, let us assume convergence at a point with $|z-z_0| = r > 0$, as the terms approach zero, there exists an N such that for all $n > N$, we have

$$|a_n(z - z_0)^n| = |a_n|r^n < 1 \Rightarrow \sqrt[n]{|a_n|} < \frac{1}{r}$$

But this gives the candidate $1/r$ for $\limsup \sqrt[n]{|a_n|}$, contradicting $\limsup \sqrt[n]{|a_n|} = \infty$.

Next, we consider the case $R > 0$ (including $R = \infty$) where we seek to prove uniform convergence in subdisks $|z - z_0| \leq R' < R$ (which implies pointwise convergence in $|z - z_0| < R$). We consider a number k such that:

$$\frac{1}{R} = \limsup \sqrt[n]{|a_n|} < k < \frac{1}{R'}$$

Now, there exists an N such that for all $n > N$, $\sqrt[n]{|a_n|} < k$ (by definition of upper limit).

So, the terms of the power series with $j > N$ are bound as:

$$|a_j(z-z_0)^j| = \left(\sqrt[j]{|a_j|}|z-z_0|\right)^j < (kR')^j$$

However, as $kR' < 1$, we have the terms to be eventually bound by the convergent geometric series $\sum_{j=0}^{\infty}(kR')^j$. The M-test then implies the desired uniform convergence in the subdisk $|z-z_0| \le R'$. □

5.4.4 Further Insight

Bessel Functions

Bessel functions are solutions of the following differential equation, called the Bessel's equation of order n:

$$\frac{d^2f(z)}{dz^2} + \frac{1}{z}\frac{df(z)}{dz} + \left(1 - \frac{n^2}{z^2}\right)f(z) = 0$$

These functions often arise in physical systems with circular or cylindrical symmetry, for example, diffraction of plane-waves through a circular aperture (Airy disks) or vibrational modes on a circular drum membrane.

We are interested to find a solution that is analytic at the origin with a non-negative integer order 'n'. We express the Maclaurin series of this solution as $\sum_{j=0}^{\infty} a_j z^{j+r}$, where a_0 is the first non-zero coefficient (of z^r) in the series (the form is still general, as r is any non-negative integer). Substituting in Bessel's equation, we get:

$$\sum_{j=0}^{\infty} a_j(j+r)(j+r-1)z^{j+r-2} + \sum_{j=0}^{\infty}(j+r)a_j z^{j+r-2} + \sum_{j=0}^{\infty} a_j z^{j+r} - \sum_{j=0}^{\infty} n^2 a_j z^{j+r-2} = 0$$

$$\Rightarrow \sum_{j=0}^{\infty} a_j[(j+r)^2 - n^2]z^{j+r-2} + \sum_{j=0}^{\infty} a_j z^{j+r} = 0$$

In the first summation, $j = 0$ and $j = 1$ give the equations,

$$a_0[r^2 - n^2] = 0 \Rightarrow r = \pm n \text{ (as } a_0 \neq 0) \quad , \quad a_1[1 + r^2 - 2r - n^2] \Rightarrow a_1[1 \pm 2n] = 0 \Rightarrow a_1 = 0$$

We choose $r = n$ and discard $r = -n$ as the terms z^{j+r} for $j < n$ cannot be a part of a Maclaurin series. This gives,

$$\sum_{j=0}^{\infty}(a_{j+2}(j+2)(j+2+2n) + a_j)z^{j+n} = 0$$

Thus, we get a recursive formula for the coefficients,

$$a_{j+2} = \frac{-a_j}{(j+2)(j+2+2n)} \Rightarrow a_j = \frac{-a_{j-2}}{(j)(j+2n)}$$

For odd j, we have $a_j = 0$ as $a_1 = 0$, while for even j, we substitute $j = 2k$ to obtain a general formula (by induction):

$$a_{2k} = \frac{-a_{2(k-1)}}{2^2(k)(k+n)} \Rightarrow a_{2k} = \frac{(-1)^k n! a_0}{k!(k+n)!2^{2k}}$$

Since multiplying the solution by a constant still yields a valid solution, we are free to choose a_0. To obtain a clean looking form, we set $n!2^n a_0 = 1 \Rightarrow a_0 = 1/(n!2^n)$. This gives us the **Bessel function of the first kind of order $\boldsymbol{n}$**, denoted $J_n(z)$:

$$f(z) = \boxed{J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \cdot \left(\frac{z}{2}\right)^{2k+n}}$$

We are now left with the task of finding the disk of convergence of this solution. To do this, we present the following lemma,

Lemma 5.4. $\limsup \sqrt[n]{n!} = \infty$

Proof: Given any positive integer B, we examine the following ratio, assuming $n > 2B$:

$$\frac{n!}{B^n} = \frac{(2B)!}{B^{2B}} \prod_{k=1}^{n-2B} \frac{(2B+k)}{B} > \frac{(2B)!}{(2B)^{2B}} 2^n$$

Further, as there exists an n such that $n > (2B)^{2B}/(2B)!$ and $2^n > n$ (easily shown by induction $\forall n \in \mathbb{N}$), we get $2^n > (2B)^{2B}/(2B)!$, then

$$\frac{n!}{B^n} > \frac{(2B)!}{(2B)^{2B}} 2^n > 1 \Rightarrow n! > B^n \Rightarrow \sqrt[n]{n!} > B$$

Finally, given any $\epsilon > 0$, we know there exists a positive integer B such that $B > \epsilon$. As the inequalities hold for any natural number m greater than n as well, for all $m \geq n$ (for some $\epsilon > 0$), we get:

$$\sqrt[m]{m!} > \epsilon \qquad \square$$

Note that the existence of B and n are given by the Archimedean property of real numbers and the lemma is proved as $\sqrt[n]{n!}$ eventually exceeds every positive number (instead of staying below some $l + \epsilon$).

Now, as $a_j = 0$ for odd j in the solution, we only look at the subsequence of even j's:

$$\sqrt[2k]{|a_{2k}|} = \sqrt[2k]{\left| \frac{(-1)^k n! a_0}{k!(k+n)!2^{2k}} \right|} = \frac{1}{2} \sqrt[2k]{\left| \frac{n! a_0}{k!(k+n)!} \right|}$$

By above lemma, given any $A > 0$, we know there exists $m \in \mathbb{N}$ such that $\forall k \geq m$:

$$\sqrt[k]{k!} > A \Rightarrow \sqrt[k]{(k+n)!} > A \text{ (since } (k+n)! > k!)$$

$$\Rightarrow \sqrt[k]{k!(k+n)!} > A^2 \Rightarrow \sqrt[2k]{k!(k+n)!} > A \Rightarrow \frac{1}{\sqrt[2k]{k!(k+n)!}} < \frac{1}{A}$$

Now, by Archimedean property, we can choose natural number A to set an arbitrarily small bound to get:

$$\lim_{k\to\infty} \frac{1}{\sqrt[2k]{k!(k+n)!}} = 0$$

Further, as the numerator term $\sqrt[2k]{M}$ with $M = |n!a_0|$ is bound as $0 < M^{1/2k} \leq \max\{M, 1\}$, combined with the blowing up of the denominator, we get:

$$\lim_{k\to\infty} \sqrt[2k]{|a_{2k}|} = 0 \Rightarrow \limsup \sqrt[n]{|a_n|} = 0 \Rightarrow R = \infty$$

Thus, the Bessel functions of the first kind of non-negative integer order converge everywhere on $\mathbb{C}$ and so are **entire**.

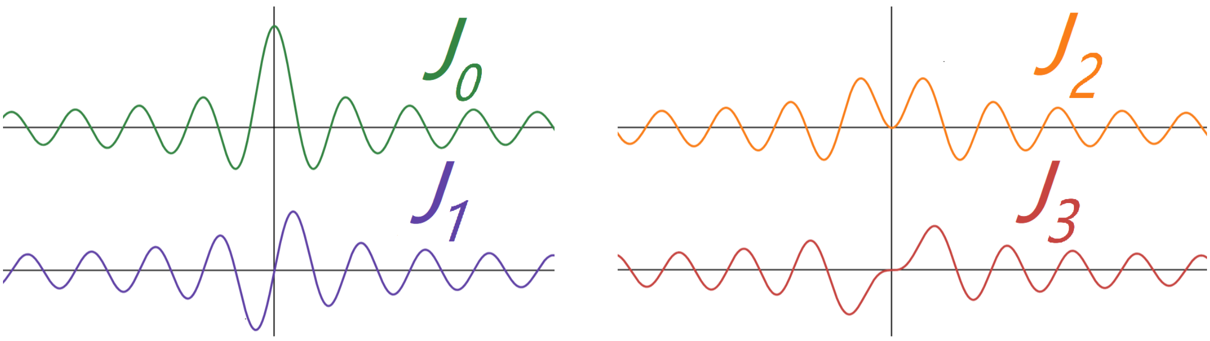


Figure 5.3: First four Bessel functions of non-negative integer order, showing oscillatory nature along the real line.

The Fibonacci sequence

The Fibonacci sequence, $1, 1, 2, 3, 5, 8, 13, \cdots$ arises in surprisingly many natural patterns. Its terms are defined by the following recursive relation:

$$a_0 = a_1 = 1$$

$$a_n = a_{n-1} + a_{n-2} \quad (n > 1)$$

Here, we intend to find a closed form expression for these numbers. We know that the coefficients of a power series can be found by differentiating the function it converges to.

Taking inspiration from this fact, define

$$f(z) := a_0 + a_1 z + a_2 z^2 + \cdots$$

Hypothesizing this series is convergent in some neighborhood of zero, we notice that

$$f(z) = a_0 + a_1 z + \sum_{j=0}^{\infty} a_{j+2} z^{j+2} = 1 + a_0 z + \sum_{j=0}^{\infty} (a_{j+1} + a_j) z^{j+2}$$

$$\Rightarrow f(z) = 1 + a_0 z + z \sum_{j=1}^{\infty} a_j z^j + z^2 \sum_{j=0}^{\infty} a_j z^j = 1 + z f(z) + z^2 f(z)$$

Rearranging this functional equation, we obtain $f(z)$ to be a rational function:

$$f(z) = \frac{1}{1 - z - z^2}$$

As $f(z)$ is analytic at zero, we may follow our steps in reverse to verify that the coefficients are indeed the Fibonacci numbers, validating our convergence hypothesis. Using the quadratic formula, $f(z)$ has the following poles:

$$x = \frac{-1 + \sqrt{5}}{2}, \quad y = \frac{-1 - \sqrt{5}}{2}$$

Thus, we use partial fraction decomposition to get:

$$f(z) = \frac{-1}{(z-x)(z-y)} = \frac{[(z-x) - (z-y)]}{(x-y)(z-x)(z-y)} = \frac{1}{(x-y)} \left[\frac{1}{(z-y)} - \frac{1}{(z-x)} \right]$$

Differentiating j times, we get:

$$f^{(j)}(z) = \frac{1}{(x-y)} \left[\frac{j!(-1)^j}{(z-y)^{j+1}} - \frac{j!(-1)^j}{(z-x)^{j+1}} \right]$$

Using the formula for Taylor coefficients, we may now find an explicit expression for the a_j's

$$a_j = \frac{f^{(j)}(0)}{j!} = \frac{1}{(x-y)} \left[\frac{(-1)^j}{(-y)^{j+1}} - \frac{(-1)^j}{(-x)^{j+1}} \right]$$

$$\Rightarrow a_j = \frac{1}{(x-y)} \left[\frac{y^{j+1} - x^{j+1}}{(xy)^{j+1}} \right]$$

Now, we have

$$(x-y) = \sqrt{5}, \qquad xy = \left(\frac{-1 - \sqrt{5}}{2} \right) \left(\frac{-1 + \sqrt{5}}{2} \right) = \frac{1-5}{4} = -1$$

Substituting in the expression for a_j, we get

$$a_j = \frac{1}{\sqrt{5}} \left[(-y)^{j+1} - (-x)^{j+1}\right]$$

Hence, the Fibonacci numbers can be represented by the following expression, called **Binet's formula**:

$$\boxed{a_j = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^{j+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{j+1}\right]}$$

Using this, we have the ratio of consecutive Fibonacci numbers to be:

$$\frac{a_{j+1}}{a_j} = \left[\frac{(-y)^{j+2} - (-x)^{j+2}}{(-y)^{j+1} - (-x)^{j+1}}\right] = (-y)\left[\frac{1-(x/y)^{j+2}}{1-(x/y)^{j+1}}\right]$$

Now as $|x/y| = (\sqrt{5}-1)/(\sqrt{5}+1) < 1$, we have $\lim_{j\to\infty}(x/y)^j = 0$. The ratio of consecutive Fibonacci numbers therefore has the limit:

$$\boxed{\lim_{j\to\infty} \frac{a_{j+1}}{a_j} = (-y) = \frac{1+\sqrt{5}}{2} = \phi}$$

The number $\phi = 1.61803\ldots$ is called the **Golden Ratio**.

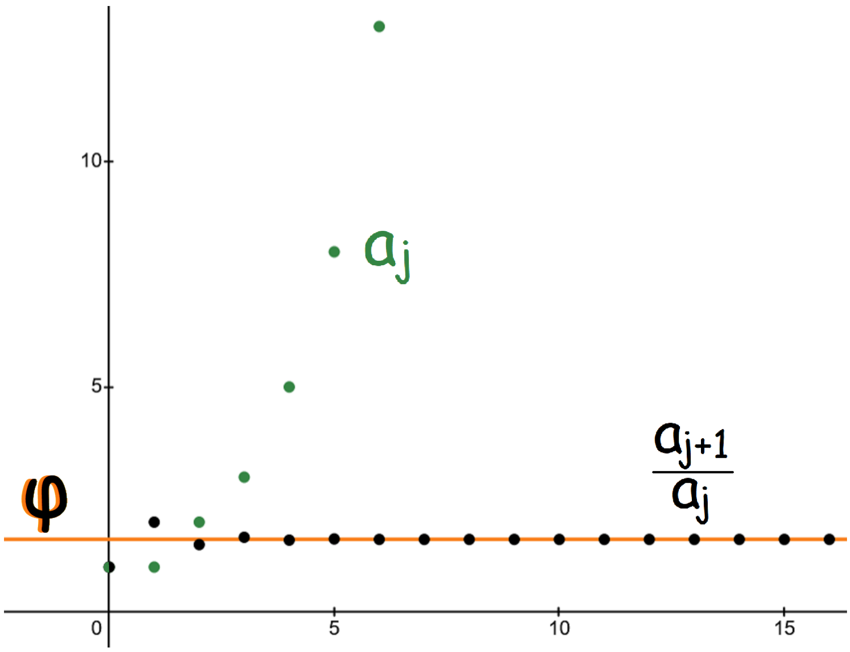


Figure 5.4: Fibonacci numbers (green) grow (roughly) like powers of the golden ratio ϕ, while the ratio of consecutive terms (black) converges to ϕ.

The Riemann Zeta function

The Riemann Zeta function is defined (wherever convergent) on the complex plane as:

$$\zeta(z) := \sum_{j=1}^{\infty} \frac{1}{j^z}$$

where j^z denotes the principle value of the exponent, i.e. $\exp(z \operatorname{Log} j)$. Here, we intend to find the domain over which this function is well defined (convergent). To apply our usual method of comparison with a geometric series, we make 'sum groups' of terms of length 2^k ($k \in \{0\} \cup \mathbb{N}$) as there is no direct geometric trend in the Riemann zeta function.

$$B_k := \sum_{j=2^k}^{2^{k+1}-1} \frac{1}{j^z}$$

Notice that $|j^z| = j^\lambda$, where $\lambda = \mathrm{Re}(z)$. To understand convergence, we first consider z along the real line ($z = \lambda \in \mathbb{R}$). For real z, the function j^z increases with j because $\exp(z \log j)$ is an increasing function of j when z is real. This allows us to bound each 'sum group' B_k as follows:

$$|B_k| = \left| \sum_{j=2^k}^{2^{k+1}-1} \frac{1}{j^z} \right| \leq \sum_{j=2^k}^{2^{k+1}-1} \frac{1}{j^\lambda} \leq \sum_{j=2^k}^{2^{k+1}-1} \frac{1}{(2^k)^\lambda} = 2^k \cdot (2^k)^{-\lambda} = 2^{k(1-\lambda)}$$

To ensure convergence of the series $\sum_k B_k$, we require that the bounding geometric sequence has the common ratio less than 1:

$$2^{1-\lambda} < 1 \quad \Rightarrow \quad \lambda = \mathrm{Re}(z) > 1$$

Hence, the sum of groups converges to S (say) for points with real part exceeding 1, i.e. $\forall \epsilon > 0$, $\exists\, l \in \mathbb{N}$ such that whenever $n > l$, we have

$$\left| S - \sum_{k=0}^{n} B_k \right| < \epsilon$$

To deduce from this the convergence of our original series, we again deal with only real values of z first so that the partial sums are increasing (positive terms), we may then easily extend it for non-real values of z as $|j^z| = j^\lambda$ where $\lambda = \mathrm{Re}(z)$. This increasing nature implies that the partial sums of the original series eventually lie in between two partial sums of the groups which lie inside the epsilon neighborhood of S:

$$(S - \epsilon) < \sum_{k=0}^{n} B_k = \sum_{j=1}^{2^{n+1}-1} \frac{1}{j^z} < (S + \epsilon)$$

For instance, take $N = 2^{l+2} - 1$ then for all $n > N$, we have

$$\sum_{k=0}^{m} B_k = \sum_{j=1}^{2^{m+1}-1} \frac{1}{j^z} \leq \left(\sum_{j=1}^{n} \frac{1}{j^z} \right) \leq \sum_{j=1}^{2^{m+2}-1} \frac{1}{j^z} = \sum_{k=0}^{m+1} B_k$$

where, for each n, we find $m \in \{l+1, l+2, \ldots\}$ such that $2^{m+1} - 1 \leq n \leq 2^{m+2}$. As $m > l$, we have

$$(S - \epsilon) < \sum_{k=0}^{m} B_k \leq \sum_{j=1}^{n} \frac{1}{j^z} \leq \sum_{k=0}^{m+1} B_k < (S + \epsilon)$$

$$\Rightarrow \left| S - \sum_{j=1}^{n} \frac{1}{j^z} \right| < \epsilon \qquad \text{(for all } n > N \text{, assuming real } z\text{)}$$

Hence, for values greater than 1 along the real line, the function converges.

Using the Weierstrass M-test, we may also conclude this convergence to be uniform in domains of the form $D = \{Re(z) > \gamma \,|\, \gamma \in \mathbb{R}\,, \gamma > 1\}$ as the series $\sum_{j=1}^{\infty} 1/j^{\gamma}$ is known to be convergent by the discussion above and the terms of $\zeta(z)$ are bound by it:

$$\left| \frac{1}{j^z} \right| = \frac{1}{j^{\mathrm{Re}(z)}} < \frac{1}{j^{\gamma}}$$

Further, as the exponential terms are analytic for $\mathrm{Re}(z) > 1$, in view of Theorem 5.9 and the uniform convergence in the simply connected domain D discussed above, $\zeta(z)$ is also analytic in D. Hence, **the Riemann zeta function is analytic at all points with Re(z) > 1** (by choosing γ arbitrarily close to 1).

We conclude this section by exploring the convergence of $\zeta(z)$ for $\mathrm{Re}(z) \leq 1$. To start, let us consider the very boundary of our analyzed region by taking $z = 1$:

$$\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

We know that setting an upper bound using B_k's will not work as we were forced to take $\lambda > 1$. This motivates us to set a lower bound instead (which may help us to show divergence):

$$B_k = \sum_{j=2^k}^{2^{k+1}-1} \frac{1}{j} \geq 2^k \cdot \left(\frac{1}{2^{k+1} - 1} \right) = \frac{1}{2} + \frac{1/2}{2^{k+1} - 1} > \frac{1}{2}$$

Thus, the series $\sum_k B_k$ must be divergent as the series $\sum_{j=0}^{\infty}(1/2)$ is divergent (otherwise, the comparison test would lead to the incorrect conclusion of $\sum_{j=0}^{\infty}(1/2)$ being convergent). Thus, the original series, **$\boldsymbol{\zeta(1)}$ too must be divergent**. To be more precise, assume the convergence of $\zeta(1)$ (to S, say) for sake of contradiction, then $\forall\, \epsilon > 0$,

$\exists\, N \in \mathbb{N}$ such that whenever $n > N$, we have

$$\left| S - \sum_{j=1}^{n} \frac{1}{j} \right| < \epsilon$$

As $(2^{N+1} - 1) > N$, this includes the values $n \in \{(2^{N+1} - 1), (2^{N+2} - 1), (2^{N+3} - 1), \ldots\}$, so for all $i > (N - 1)$ we have:

$$\left| S - \sum_{j=1}^{2^{i+1}-1} \frac{1}{j} \right| = \left| S - \sum_{k=0}^{i} B_k \right| < \epsilon$$

Hence, assuming the convergence of $\zeta(1)$ has led to the conclusion that $\sum_k B_k$ too is convergent, contradicting its above derived divergence.

Finally, for points on the real line where $z = \mathrm{Re}(z) = \lambda < 1$, since $\dfrac{1}{j^\lambda} > \dfrac{1}{j}$ for $\lambda < 1$, the series

$$\sum_{j=1}^{\infty} \frac{1}{j^\lambda}$$

diverges by comparison with $\zeta(1)$. Hence, **$\boldsymbol{\zeta(\lambda)}$ diverges along the real line for $\boldsymbol{\lambda \leq 1}$**. In fact, divergence holds for the entire half-plane $\mathrm{Re}(z) \leq 1$. While a complete proof of divergence for non-real z is more involved (so we omit it here), we can at least **rule out absolute convergence** in this half plane using the following lower bound:

$$|1/j^z| = 1/j^\lambda \geq 1/j \qquad (\lambda = \mathrm{Re}(z))$$

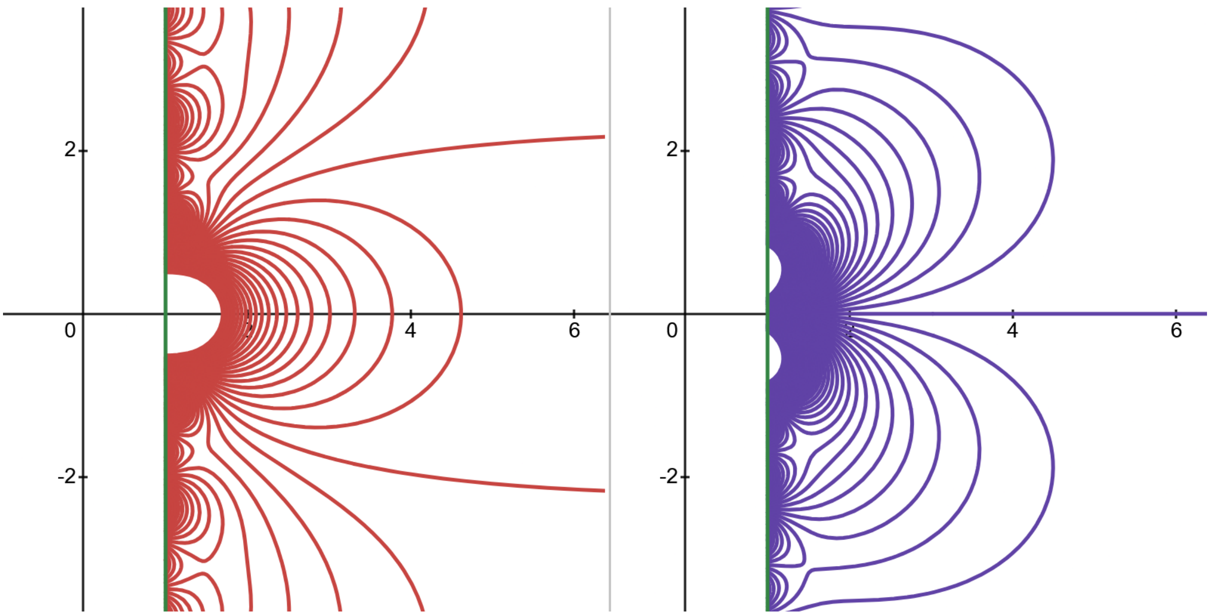


Figure 5.5: Level curves of real (red) and imaginary (blue) parts of $\zeta(z)$ (approximated by sum of first 100 terms).

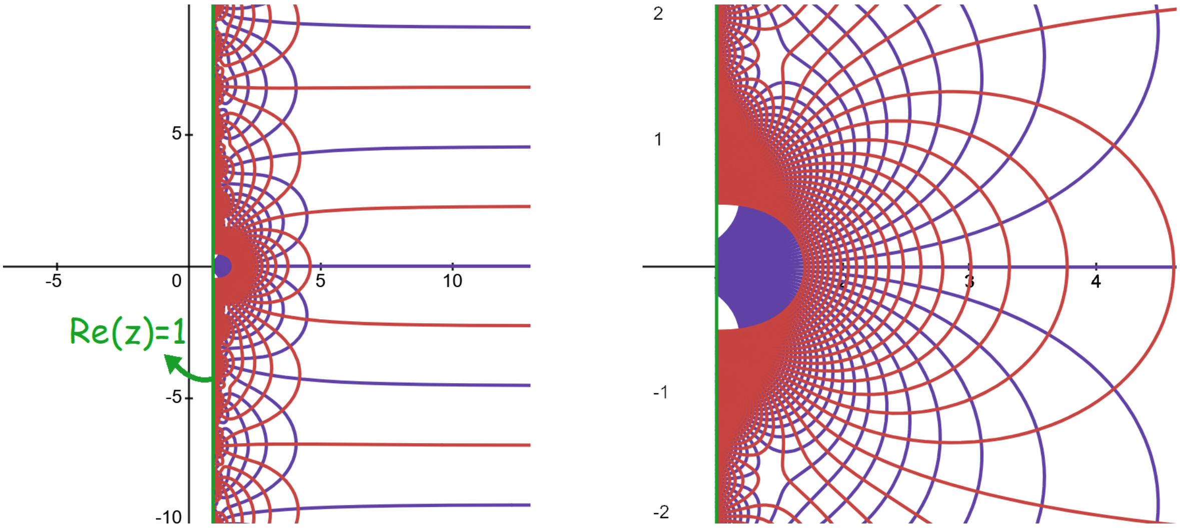


Figure 5.6: The real and imaginary level curves in previous figure superimposed to show them intersecting perpendicularly.

It is important to note that the above visualization is **extremely crude** as for points near the line $\mathrm{Re}(z) = 1$, the terms for $j > 100$ in the series become significant enough to cause large errors.

Abel's Limit Theorem

Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be a function analytic in the disk $|z| < 1$. If the series $\sum_{j=0}^{\infty} a_j$ converges to some value S, Abel's limit theorem says:

$$\lim_{r \to 1^-} f(r) = S$$

Although this may seem intuitive, the result is not trivial as the convergence of a series of functions in a domain doesn't guarantee a limit when approaching a boundary point. Therefore, we aim to prove, given any $\epsilon > 0$, we aim to find $\delta > 0$ such that whenever $r \in (1 - \delta, 1)$, we have:

$$|f(r) - S| < \epsilon$$

Let S_n denote the partial sum $\sum_{j=0}^{n} a_j$. Then, we have:

$$T_{n+1} := S - S_n = \sum_{j=0}^{\infty} a_j - \sum_{j=0}^{n} a_j = \sum_{j=n+1}^{\infty} a_j$$

Let this be the 'n^{th} tail' of the series with sum S. We know the sequence of tails converges to zero as $\lim_{n\to\infty}(S - S_n) = S - \lim_{n\to\infty} S_n = S - S = 0$. In light of this, we observe:

$$|f(r) - S| \leq |f(r) - S_n| + |S_n - S|$$

As $|S_n - S| = |T_{n+1}|$ can be made arbitrarily small (limit of tails), choose N_1 such that $|S_n - S| < \epsilon/4$ whenever $n > N_1$. Now, we have:

$$|f(r) - S_n| = \left|\sum_{j=n+1}^{\infty} a_j r^j + \sum_{j=1}^{n} a_j (r^j - 1)\right| \le \left|\sum_{j=n+1}^{\infty} a_j r^j\right| + \left|\sum_{j=1}^{n} a_j (r^j - 1)\right|$$

As the second term is simply a polynomial in r, from its continuity at $r = 1$, we know $\exists \delta(n) > 0$ such that when $r \in (1 - \delta_1(n), 1)$, we have:

$$\left|0 - \sum_{j=1}^{n} a_j (r^j - 1)\right| < \epsilon/4$$

Thus, when $n > N_1$ and $r \in (1 - \delta(n), 1)$, we have:

$$|f(r) - S| < \left|\sum_{j=n+1}^{\infty} a_j r^j\right| + \frac{2\epsilon}{4}$$

To bound the remaining term, let us represent a_j's as the difference of consecutive tails (as we have already bounded the tails):

$$\left|\sum_{j=n+1}^{\infty} a_j r^j\right| = \left|\sum_{j=n+1}^{\infty} \left(\sum_{k=j}^{\infty} a_k - \sum_{k=j+1}^{\infty} a_k\right) r^j\right| = \left|\sum_{j=n+1}^{\infty} r^j \left(\sum_{k=j}^{\infty} a_k\right) - \sum_{j=n+1}^{\infty} r^j \left(\sum_{k=j+1}^{\infty} a_k\right)\right|$$

$$= \left|\left(\sum_{k=n+1}^{\infty} a_k\right) r^{n+1} + \sum_{j=n+2}^{\infty} r^j \left(\sum_{k=j}^{\infty} a_k\right) - \sum_{j=n+1}^{\infty} r^j \left(\sum_{k=j+1}^{\infty} a_k\right)\right|$$

$$\Rightarrow \left|\sum_{j=n+1}^{\infty} a_j r^j\right| = \left|\left(\sum_{k=n+1}^{\infty} a_k\right) r^{n+1} + \sum_{j=n+2}^{\infty} \left(\sum_{k=j}^{\infty} a_k\right) (r^j - r^{j-1})\right|$$

Now, we use triangle inequality and the bound on $|T_{n+1}| < \epsilon/4$ for $n > N_1$ to get:

$$\left|\sum_{j=n+1}^{\infty} a_j r^j\right| \le |T_{n+1}| r^{n+1} + \sum_{j=n+2}^{\infty} |T_j| |r^{j-1}(r - 1)| < \frac{\epsilon \cdot r^{n+1}}{4} + \frac{\epsilon}{4} \left(\sum_{j=n+2}^{\infty} r^{j-1}(1 - r)\right)$$

where we used $|r - 1| = (1 - r)$ as $0 < r < 1$. Since the geometric series in brackets evaluates to r^{n+1}, this gives us the final bound as:

$$\left|\sum_{j=n+1}^{\infty} a_j r^j\right| < \left(\frac{\epsilon}{4} + \frac{\epsilon}{4}\right) r^{n+1} < \frac{2\epsilon}{4}$$

Hence, by choosing $n > N_1$ and $r \in (1 - \delta(n))$ we have:

$$|f(r) - S| < \left(\frac{2\epsilon}{4} + \frac{2\epsilon}{4}\right) = \epsilon \qquad \square$$

Having completed our proof of Abel's Limit Theorem, let us employ it to evaluate the following series:

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Firstly, we need to show the series is convergent. To do this, we observe the even partial sums have the form:

$$S_{2k} = \sum_{j=1}^{k} \frac{1}{j(j+1)}$$

We also observe that its terms are bound as:

$$\frac{1}{j(j+1)} < \frac{1}{j^2}$$

As the series $\sum 1/j^2$ is the Riemann Zeta function evaluated at $z = 2$ (proved to be convergent in previous section), this implies that the sequence of even partial sums converges by the comparison test. Further, as the odd partial sums can be written as:

$$S_{2k-1} = S_{2k} + \frac{1}{2k}$$

In the limit $k \to \infty$, we have

$$\lim_{k\to\infty} S_{2k-1} = \lim_{k\to\infty} S_{2k} + 0$$

As both the even and odd partial sums converge to the same limit, we conclude S exists. To find S, we observe that the series equals the Maclaurin series of $\mathrm{Log}(1+z)$ evaluated at $z = 1$ (a point lying on the circle of convergence not inside it). Hence, Abel's Limit Theorem gives:

$$\mathrm{Log}(1+z) = \sum_{j=0}^{\infty} \frac{(-1)^{j+1} z^j}{j} \Rightarrow \boxed{S = \mathrm{Log}(2)}$$

Appendix

A.1 The Conjugation-Derivative Relation

If $f(z)$ is a function differentiable n times at a point z_0, then

$$\overline{f^{(n)}(z_0)} = g^{(n)}(\overline{z_0}) \qquad \text{where } g(z) = \overline{f(\overline{z})}$$

Base case (n=1):

$$g'(\overline{z_0}) = \lim_{h\to 0} \frac{g(\overline{z_0}+h) - g(\overline{z_0})}{h} = \lim_{h\to 0} \frac{\overline{[f(z_0+\overline{h}) - f(z_0)]}}{h} = \lim_{\overline{h}\to 0} \overline{\left[\frac{f(z_0+\overline{h}) - f(z_0)}{\overline{h}}\right]} = \overline{f'(z_0)}$$

Inductive Step: Assume the formula holds for n=k, we need to show it for n=k+1.

$$g^{(k+1)}(\overline{z_0}) = \lim_{h\to 0} \frac{g^{(k)}(\overline{z_0}+h) - g^{(k)}(\overline{z_0})}{h} = \lim_{h\to 0} \frac{\overline{[f^{(k)}(z_0+\overline{h}) - f^{(k)}(z_0)]}}{h}$$

$$= \lim_{\overline{h}\to 0} \overline{\left[\frac{f^{(k)}(z_0+\overline{h}) - f^{(k)}(z_0)}{\overline{h}}\right]} = \overline{(f^{(k)})'(z_0)} = \overline{f^{(k+1)}(z_0)}$$

Hence proved.

A.2 'Proving' Green's Theorem

A.2.1 Curl(V)

The curl of a vector field (in 2D) at a point is defined as the positive circulation per unit area, where circulation refers to the line integral along a simple closed contour. That is, for a positively oriented simple closed contour C enclosing an area A,

$$\text{Curl}(\mathbf{V})(x_0, y_0) = \lim_{A\to 0} \frac{1}{A} \oint_C V_1 dx + V_2 dy$$

Here, in the limit $A \to 0$, the area enclosed by the loop shrinks to lie in an arbitrarily small neighborhood of (x_0, y_0).

For our purposes, we need to find an expression for Curl$(\mathbf{V})(x_0, y_0)$ for a right-angled triangle whose perpendicular sides are parallel to the x and y axes. Consider the triangle shown below:

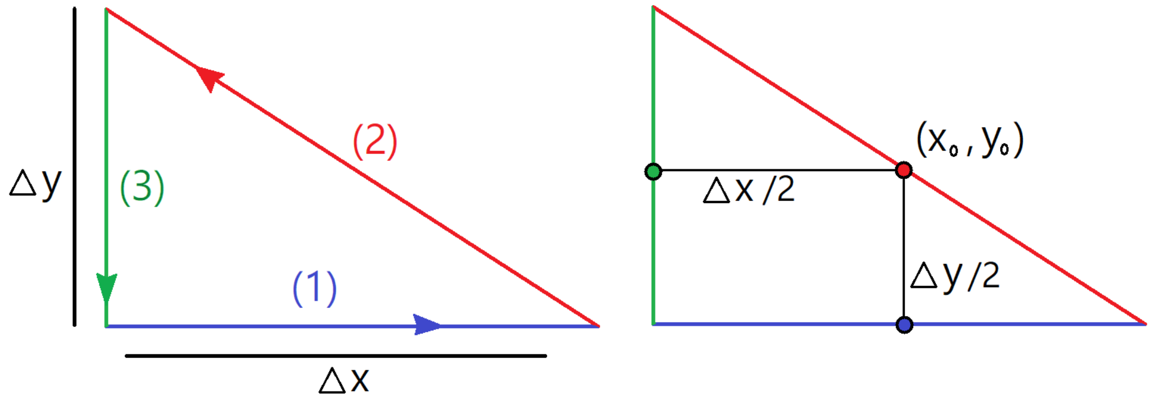


Figure 6.1: Limiting triangle.

We place the point (x_0, y_0) at the midpoint of the hypotenuse of the triangle. Label the sides as (1), (2), and (3); with (1) being the base parallel to the x-axis, (2) the hypotenuse, and (3) the vertical leg. So we break the line integral around the triangle into three parts. Since the triangle is small, we approximate the line integral over each segment by taking the vector field $\mathbf{V} = (V_1, V_2)$ to be constant along that segment, and equal to its value at the midpoint of the segment.

For segment **(1)**: The midpoint lies at $(x_0,\, y_0 - \frac{\Delta y}{2})$. Along this side, $dy = 0$, so the integral reduces to $V_1\, dx$ so we approximate:

$$\int_{(1)} V_1\, dx + V_2\, dy \approx V_1\left(x_0,\, y_0 - \tfrac{\Delta y}{2}\right) \Delta x.$$

For segment **(2)**: the hypotenuse rising diagonally to the left, the midpoint is precisely (x_0, y_0) by our setup. Along this segment, we have both $dx = -\Delta x$ and $dy = \Delta y$. So the integral becomes:

$$\int_{(2)} V_1\, dx + V_2\, dy \approx V_1(x_0, y_0)\,(-\Delta x) + V_2(x_0, y_0)\,(\Delta y)\,.$$

For segment **(3)**, the vertical leg going downward along the y-axis, $dx = 0$ and $dy = -\Delta y$. The midpoint lies at $(x_0 - \frac{\Delta x}{2},\, y_0)$. Hence:

$$\int_{(3)} V_1\, dx + V_2\, dy \approx V_2\left(x_0 - \tfrac{\Delta x}{2},\, y_0\right)(-\Delta y).$$

Finally, we sum the contributions from each segment and divide it by the area of the triangle, $A = \Delta x \Delta y/2$ to get:

$$\frac{1}{A}\oint_C V_1 dx + V_2 dy \approx \frac{V_1\left(x_0,\, y_0 - \frac{\Delta y}{2}\right) - V_1(x_0, y_0)}{\Delta y/2} + \frac{V_2(x_0, y_0) - V_2\left(x_0 - \frac{\Delta x}{2},\, y_0\right)}{\Delta x/2}$$

Taking the limit $\Delta x \to 0$ and $\Delta y \to 0$, the approximation becomes an equality, by the continuity of $\mathbf{V}$(x,y). :

$$\mathrm{Curl}(\mathbf{V})(x_0, y_0) = \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)(x_0, y_0)$$

A.2.2 Tiling the Contour

Now, we partition our simple closed contour Γ with interior domain D' and join subsequent points in the partition to approximate the contour with a polygon P. Since P is a polygon, we can tile it completely with 'N' arbitrarily small right angled triangles of the kind we discussed (grid-aligned), as shown in the figure below:

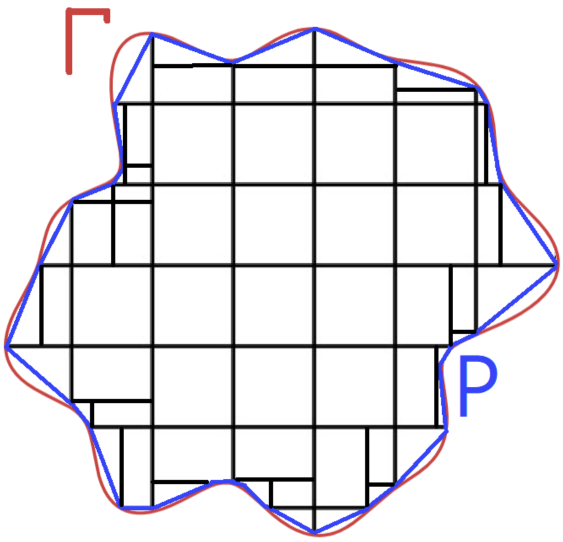


Figure 6.2: The (red) contour Γ being approximated by the (blue) polygon P, tiled with right angled triangles and rectangles.

Let the l^{th} small triangle C_l be a positively oriented contour enclosing the area A_l. When we sum the positive circulations along all these triangular tiles, the segments common to adjacent triangles don't contribute to the sum as each triangle integrates along the same line segment in opposite directions, as shown below:

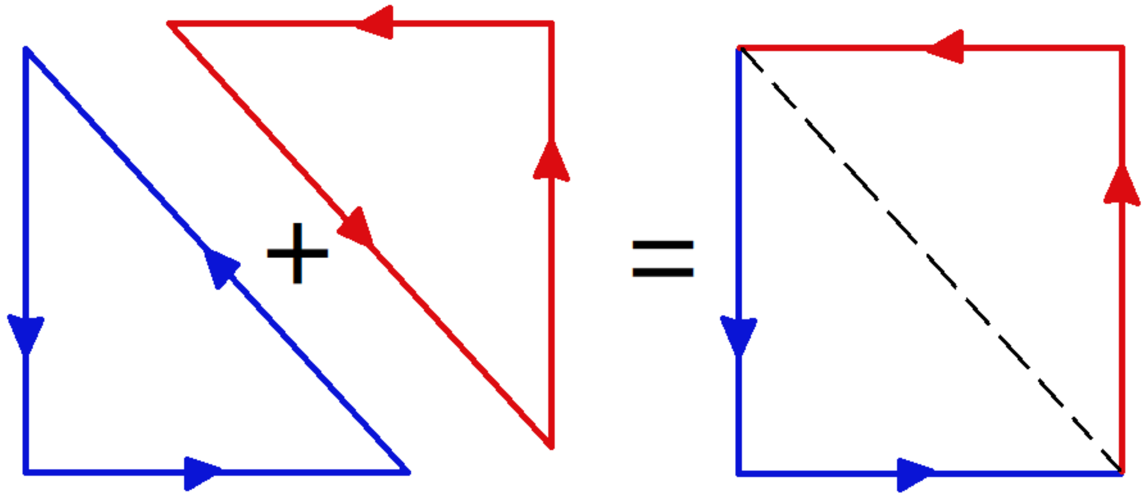


Figure 6.3: Line integrals along common segments between triangles cancel on sum.

But the only edges that are not common between adjacent (triangular) tiles are the edges of the polygon P itself, therefore,

$$\int_P V_1 dx + V_2 dy = \sum_{l=1}^{N} \int_{C_l} V_1 dx + V_2 dy$$

In the limit $N \to \infty$ and $A_l \to 0$ (tiling done by infinitely many infinitesimal triangles), P approximates Γ arbitrarily well and we get :

$$\int_\Gamma V_1 dx + V_2 dy = \lim_{\substack{N\to\infty \\ A_l\to 0}} \int_P V_1 dx + V_2 dy = \lim_{\substack{N\to\infty \\ A_l\to 0}} \sum_{l=1}^{N} \int_{C_l} V_1 dx + V_2 dy$$

But in the limit, substituting the expression for Curl($\mathbf{V}$), we obtain its area integral across D' by summing all tiles:

$$\lim_{\substack{N\to\infty \\ A_l\to 0}} \sum_{l=1}^{N} \left[\frac{1}{A_l} \int_{C_l} V_1 dx + V_2 dy \right] A_l = \lim_{\substack{N\to\infty \\ A_l\to 0}} \sum_{l=1}^{N} \mathrm{Curl}(\mathbf{V}) A_l := \iint_{D'} \mathrm{Curl}(\mathbf{V}) dx dy$$

Note that the conversion to area integral requires the integrand Curl($\mathbf{V}$) which is satisfied by the assumption of $\mathbf{V}$ to be continuously differentiable. Equating the different expressions for the limit of the sum, we obtain the expression for Green’s Theorem.

$$\int_\Gamma V_1 dx + V_2 dy = \iint_{D'} \mathrm{Curl}(\mathbf{V}) dx dy \qquad \square$$

A.3 Credits

- The Riemann sphere image is adapted from a work by GKFX (from Wikipedia, Riemann Sphere), licensed under CC BY-SA 3.0. Edits were made to indicate the transformation of the sphere by the inversion function.

- The conformal map illustration is based on a public domain image by Oleg Alexandrov, available on Wikimedia Commons: `https://commons.wikimedia.org/wiki/File:Conformal_map.svg`.