Sangaku13

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Abstract

This paper presents a problem and then seeks to provide its soultion. The reader should be familiar with algeba, specifically completing the square, and finding the slope of an angle bisector given two slopes. Also the reader should be familiar with calculs, specifiacally finding the tangent. Before the presentation of the problem, a few notes.

Finding the Problem

While taking a college calculus course we were presented with a difficult problem whle learning about sequences. Given two circles with centers $(1,1)$ and $(-1,1)$ both with radius 1 and the line $y=0$; what is the sequence that produces the Nth circle such that the all of them are tangentally contaiend with in the space created between the two circle and the line $y=0$?

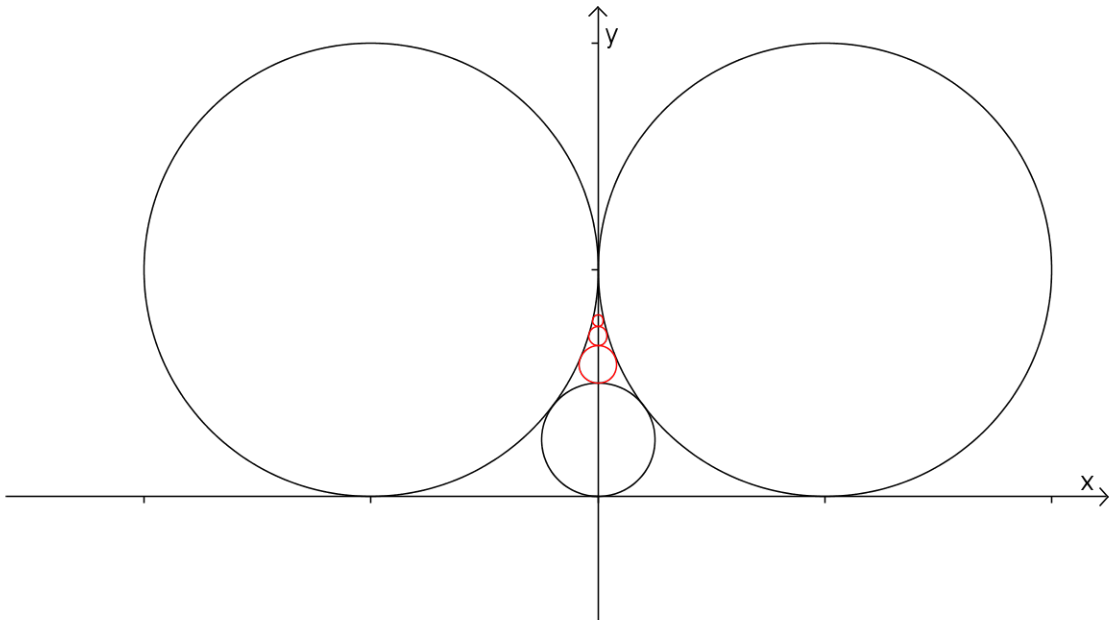

Figure 1: Inspiration of Problem

I experianced great success with the poblem which drove me to become interested in these types of poblems. My teacher described the problem as being similiar to Sangaku. Sankaku is the name given to the rigourous geometry problems adorned in the temples of the shinto shrines during the edo period of japan. More over Sangaku can be thought of as form of meditation . Because ofJapans isolation during this time, the country was cut off from some of the techniques of calculus that were being developed during that time by Lebinez and Newton. This knowledge would have surley expidited the process of reaching solutions to some of the problems presented. The problems offten involved finding the area of a set of circles that satisfied certain condition such as being the incircle to a triangle. Here are some pictures of Sankagu:

Figure 2: san

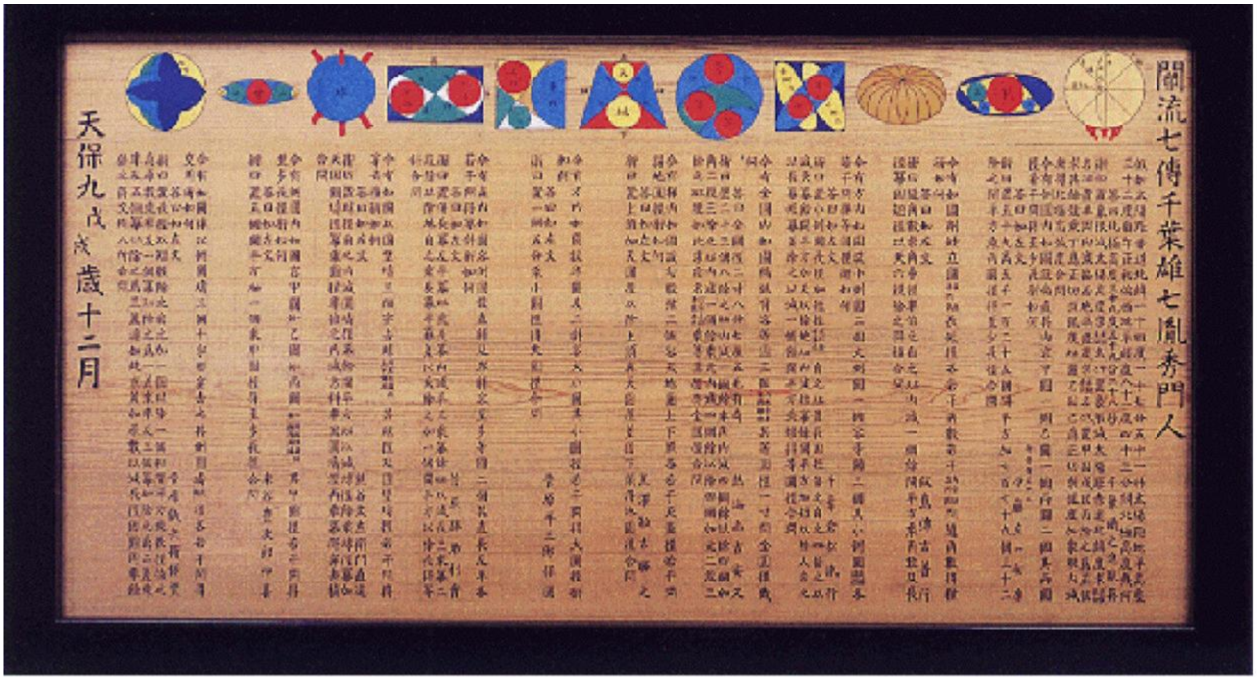

Figure 3: san

My interest in these problems grew. I became fasinated with the pure beauty , elegance and overall aesthetics of how each problem never seemingly ceased to please my eyes. I felt a deep significance eminating from these geometric structures, calling me, telling me there was some type of hidden code or pattern attributed with all of them. In some respect I felt a gemoetric similarity in configuration between crop circles design and sangaku design. Going over many pictures of crop circles patterns I soon became convinced that there was indeed some type of correlation. Many sankagu problems and crop circles have indices of concentric circles, and circles with co-linear origins, sharing tangents to one another. Here are a few examples of just how closely Sangaku and Crop-Circles can be seen as related to one another, if not by gemoetry alone.

Figure 4: crop

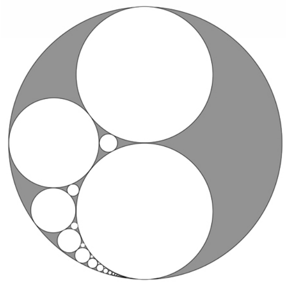

Figure 5: 1788 Sangaku, "The Pappus Chain"

Figure 6: Stonehendge, July 7^{th},1996

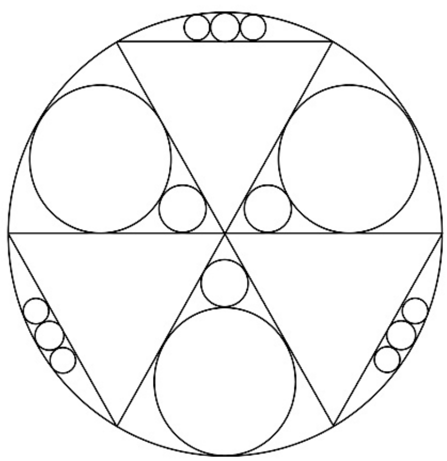

Figure 7: .

Figure 8:

The Problem

Given a curve $y = cos(x)$ and a circle with center$(0, 1/2)$ and radius$1/2$; What is the sequence that produces the Nth circle such that it is tangentally contained with-in the space created by the $cos(x)$, $y = 0$,and the previous circle?

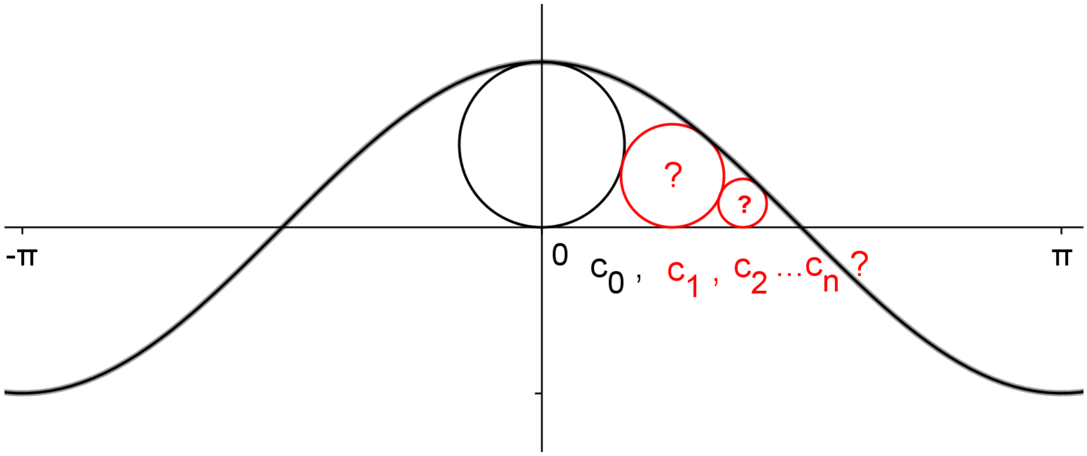

Figure 9: A Pecular Problem

Notation

Let the function $y = cos(x)$ be deoneted by $f(x)$ Let each circle in the list be denoted by C_n and its corresponding origin $O_n = (x_n, y_n)$ with $O_0 = (0, 1/2)$

Finding a Solution

I began by first considering the idea of the incircle and its application to this particular problem. The next circle would have to be an incircle to the triangle who's vertices are formed by the intersections of three lines: the tangent line to the $cos(x)$ at a specific point, t, for time , the tangent line to the previous circle at a specific point, a where both the previous circle and the next circle touch, and the line $y = 0$. Attempting to find an incenter to the incircle I continued further by focusing on the polygon vertices of the pedal triangle taking the incenter as the pedal point. In so doing I realized the following:

Three Realization

[1] One vertice of the pedal triangle would have to exist at the point $(t, cos(t))$.

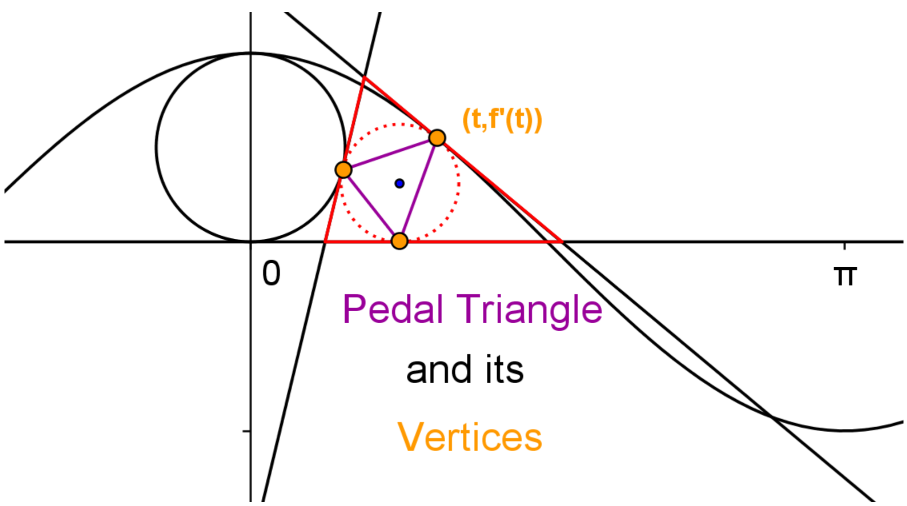

Figure 10:

[2] The intersection of the perpendicular bisector of the tangent line to the cosine function at the point $(t, cos(t))$, with the angle bisector formed by the line $y = 0$ and the tangent line to the cosine function at the point $(t, cos(t))$, will give the coordinates to the incenter.

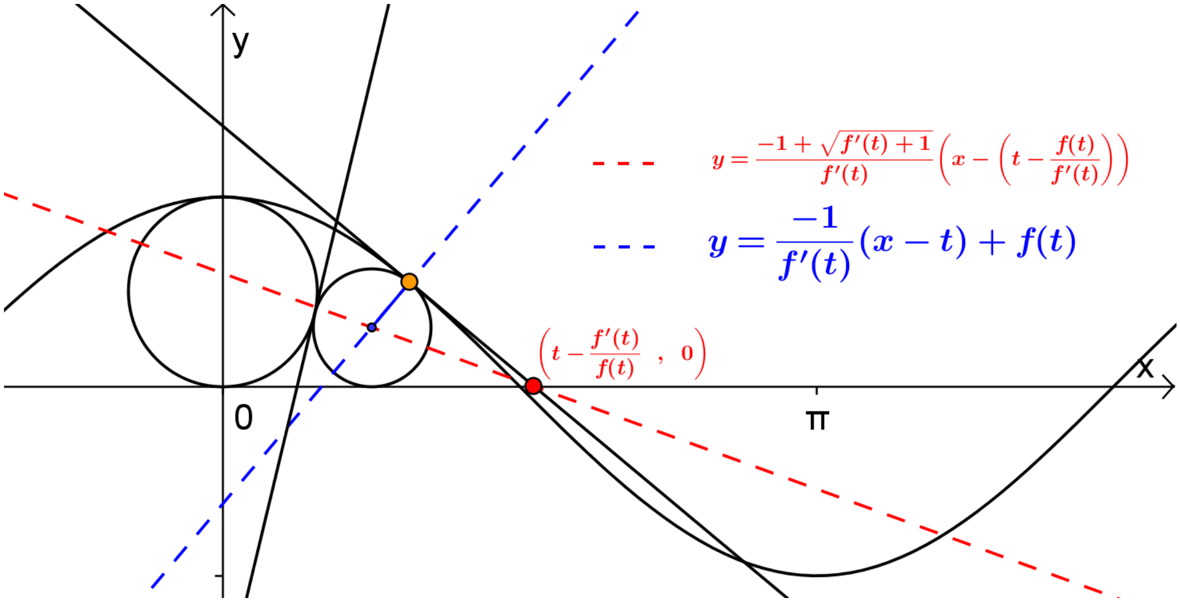

Figure 11: Realization2

[3] The distance from the center of next circle to the center of previous circle minus the radial length of the previous circle must to be equivilent to radial length of the next circle.

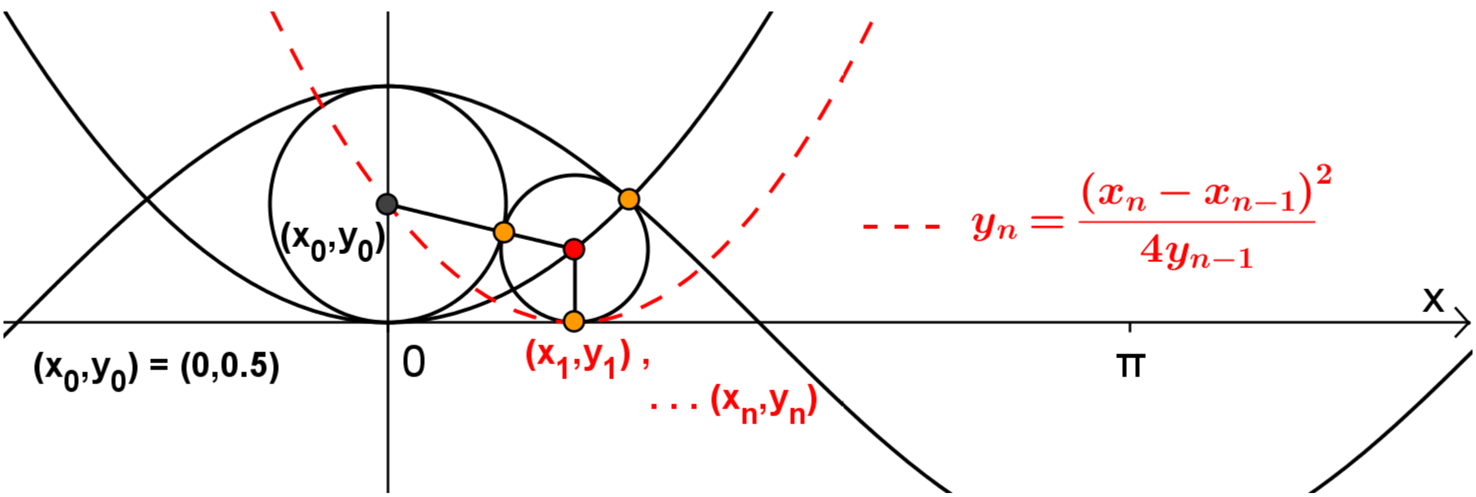

Figure 12: Realization3

this is a reference to [1]

These three realization helped me come up with a system of three equation that eventually helped me find a satisfactuary method of finding a solution.

let the cosine function be denoted by:

$$f(x) = cos(x)$$

It's tangent line with respect to time as it's paremeter is given by the equation:

$$y = f'(t)(x - t) + f(t)$$

And it's perpendicular bisector is given simply by taking the negative reciprocal of the slope

$$y = \frac{-1}{f'(t)}(x-t) + f(t) \tag{1}$$

The intersection of 1 with the interior angle bisector formed by the two lines $y = 0$ and $y = f'(t)(x-t) + f(t)$ will give the coordinates to the incenter. Given two slopes m_1 and m_2, the slope of the angle bisector, m_a, is given by the equation:

$$m_a = \frac{(m_1 m_2 - 1) + \sqrt{(m_1^2+1)(m_2^2+1)}}{m_1 + m_2} \tag{2}$$

given that m_1 is $f'(t)$ and $m_2 = 0$, substitutiting these values into 2 gives the following equation:

$$m_a = \frac{-1 + \sqrt{f'(t)^2 + 1}}{f'(t)} \tag{3}$$

m_a represents the slope of the interior angle bisector of the line $y = 0$ and the tangent line to the cosine function evaluated at $(t, cos(t))$. Reacalling formula of a line in x-intercept form:

$$y = m(x-b) \tag{4}$$

Where m represents the slope of the line and b the x-intercept because when $x = b$, $y = 0$. Plugging in m_a from 3 into m from 4 it is only neccessay to find b to find the x-intercept of the angle bisector. Because the tangent line to the cosine function at the point $(t, cos(t))$ shares the same x-intercept as the interior angle bisector wih the line $y = 0$ we can find b by solving ????? for $y = 0$ which results insomething thing quite interesting.

$$\begin{aligned} 0 &= f'(t)(x-t) + f(t) \\ -f(t) &= f'(x)(x-t) \\ \frac{-f(t)}{f'(t)} &= x - t \\ t - \frac{f(t)}{f'(t)} &= x \end{aligned} \tag{5}$$

Therefor by 5 when:

$$x = t - \frac{f(t)}{f'(t)} \tag{6}$$

$y = 0$ which which gives the x-intercept b to commplete our eqation of the angle bisector. Does look 6 familiar? Recall newtons method:

$$x_{n+1} = x_n + \frac{f(x_n)}{f'(x_n)}$$

Later more modifications will be made to the equation that will show even more striking resemblence to Newtons's Method. Plugging in 6 into 4 for $x = b$ the equation for the angle bisector becomes :

$$y = \frac{-1 + \sqrt{f'(t) + 1}}{f'(t)} \left(x - \left(t - \frac{f(t)}{f'(t)} \right) \right) \tag{7}$$

Recalling realizaton 2) , we are conserned with the intersection 1 with 7.

$$
\begin{aligned}
\frac{-1}{f'(t)}(x-t)+f(t) &= \frac{-1+\sqrt{f'(t)+1}}{f'(t)}\left(x-\left(t-\frac{f(t)}{f'(t)}\right)\right)\\
x\left(\frac{-1+\sqrt{f'(t)^2+1}}{f'(t)}+\frac{1}{f'(t)}\right) &= \frac{-1+\sqrt{f'(t)+1}}{f'(t)}\left(t-\frac{f(t)}{f'(t)}\right)+\frac{t}{f'(t)}+f(t)\\
x\left(\frac{\sqrt{f'(t)^2+1}}{f'(t)}\right) &= -\frac{f(t)}{f'(t)}\left(\frac{-1+\sqrt{f'(t)^2+1}}{f'(t)}\right)+t\left(\frac{-1+\sqrt{f'(t)^2+1}}{f'(t)}+\frac{1}{f'(t)}\right)+f(t)\\
x\left(\frac{\sqrt{f'(t)^2+1}}{f'(t)}\right) &= -\frac{f(t)}{f'(t)}\left(\frac{-1+\sqrt{f'(t)^2+1}}{f'(t)}\right)+t\left(\frac{\sqrt{f'(t)^2+1}}{f'(t)}\right)+f(t)\\
x\left(\sqrt{f'(t)^2+1}\right) &= f(t)\left(\frac{-1+\sqrt{f'(t)^2+1}}{f'(t)}\right)+t\left(\sqrt{f'(t)^2+1}\right)+[f(t)f'(t)]\\
x &= -\frac{f(t)}{f'(t)}\left(\frac{-1+\sqrt{f'(t)^2+1}}{\sqrt{f'(t)^2+1}}\right)+t+\frac{f(t)f'(t)}{\sqrt{f'(t)^2+1}}\\
x &= \frac{f(t)}{\sqrt{f'(t)^2+1}}\left(\frac{1-\sqrt{f'(t)^2+1}}{f'(t)}+f'(t)\right)+t \qquad (8)
\end{aligned}
$$

With 8 and 1 we have a two of the tree equations neccessary to complete our system. Reffering to realization 3 :

3) The distance from the center of next circle to the center of previous circle minus the radial length of the previous circle must to be equivilent to radial length of the next circle

Distance formula is given here by:

$$d=\sqrt{(x_0-x_1)^2+(y_0-y_1)^2} \qquad (9)$$

substituting the conditions mandated by 3ii , 9 can be rewriten as:

$$r_n=\sqrt{(h_{n-1}-h_n)^2+(k_{n-1}-k_n)^2}-r_{n-1} \qquad (10)$$

Where $n=1,2,3...\infty$, r_n represents the n^{th}-radius to the n^{th}-circle, and (h_n,k_n) the coordinates to the n^{th}-origin to each circle. realizing that in this special case that $r_n=k_n$ enables us to rewrite 10 as:

$$k_n=\sqrt{(h_{n-1}-h_n)^2+(k_{n-1}-k_n)^2}-k_{n-1} \qquad (11)$$

solving 11 for k_n:

$$
\begin{aligned}
k_{n-1}+k_n &= \sqrt{(h_{n-1}-h_n)^2+(k_{n-1}-k_n)^2}\\
(k_{n-1}+k_n)^2 &= (h_{n-1}-h_n)^2+(k_{n-1}-k_n)^2\\
(k_{n-1}+k_n)^2-(k_{n-1}-k_n)^2 &= (h_{n-1}-h_n)^2\\
k_{n-1}^2+2k_{n-1}k_n+k_n^2-k_{n-1}^2+2k_{n-1}k_n-k_n^2 &= (h_{n-1}-h_n)^2\\
4k_{n_1}k_n &= (h_{n-1}-h_n)^2\\
k_n &= \frac{(h_{n-1}-h_n)^2}{4k_{n-1}} \qquad (12)
\end{aligned}
$$

With a few modification to 12 , 8 and 1 the resulting system can be formed

$$
\begin{aligned}
y_n &= -\frac{1}{f'(t_n)}(x_n-t_n)+f(t_n) \qquad (13)\\
x_n &= \frac{f(t_n)}{\sqrt{f'(t_n)^2+1}}\left(\frac{1-\sqrt{f'(t_n)^2+1}}{f'(t_n)}+f'(t_n)\right)+t \qquad (14)\\
y_n &= \frac{(x_{n-1}-x_n)^2}{4y_{n-1}} \qquad (15)
\end{aligned}
$$

setting 13 equal to 15 and solving for x_n :

$$\begin{aligned}
\frac{(x_{n-1}-x_n)^2}{4y_{n-1}} &= -\frac{1}{f'(t_n)}(x_n-t_n)+f(t_n) \\
x_{n-1}^2-2x_{n-1}x_n+x_n^2 &= \frac{-4y_{n-1}}{f'(t_n)}(x_n-t_n)+4y_{n-1}f(t_n) \\
x_{n-1}^2-2x_{n-1}x_n+\frac{4y_{n-1}x_n}{f'(t_n)} &= -x_{n-1}^2+\frac{4y_{n-1}t_n}{f'(t_n)}+4y_{n-1}f(t_n) \\
x_n^2-2x_n\left(x_{n-1}-\frac{2y_{n-1}}{f'(t_n)}\right) &= -x_{n-1}^2+\frac{4y_{n-1}t_n}{f'(t_n)}+4y_{n-1}f(t_n) \\
x_n^2-2x_n\left(x_{n-1}-\frac{2y_{n-1}}{f'(t_n)}\right)+\left(x_{n-1}-\frac{2y_{n-1}}{f'(t_n)}\right)^2 &= \left(x_{n-1}-\frac{2y_{n-1}}{f'(t_n)}\right)^2-x_{n-1}^2+\frac{4y_{n-1}t_n}{f'(t_n)}+4y_{n-1}f(t_n) \\
\left(x_n-\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}\right)\right)^2 &= \left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}\right)^2-x_{n-1}^2+\frac{4y_{n-1}t_n}{f'(t_n)}+4y_{n-1}f(t_n) \\
x_n-\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}\right) &= \sqrt{\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}\right)^2-x_{n-1}^2+\frac{4y_{n-1}t_n}{f'(t_n)}+4y_{n-1}f(t_n)}
\end{aligned}$$

$$x_n = x_{n-1}-\frac{2y_{n-1}}{f'(t)}+\sqrt{\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}\right)^2-x_{n-1}^2+\frac{4y_{n-1}t_n}{f'(t_n)}+4y_{n-1}f(t_n)} \tag{16}$$

for sake of shortening the length of equation 14 let :

$$M_n = \frac{f(t_n)}{\sqrt{f'(t_n)^2+1}}\left(\frac{1-\sqrt{f'(t_n)^2+1}}{f'(t_n)}+f'(t_n)\right)$$

equation 14 then becomes:

$$x_n = M_n + t_n \tag{17}$$

Now setting 17 equal to 16 and solving for t_n

$$\begin{aligned}
M_n+t_n &= x_{n-1}-\frac{2y_{n-1}}{f'(t)}+\sqrt{\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}\right)^2-x_{n-1}^2+\frac{4y_{n-1}t_n}{f'(t_n)}+4y_{n-1}f(t_n)} \\
M_n+t_n-\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}\right) &= \sqrt{\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}\right)^2-x_{n-1}^2+\frac{4y_{n-1}t_n}{f'(t_n)}+4y_{n-1}f(t_n)} \\
\left(t_n-\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}-M_n\right)\right)^2 &= \left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}\right)^2-x_{n-1}^2+\frac{4y_{n-1}t_n}{f'(t_n)}+4y_{n-1}f(t_n)
\end{aligned}$$

$$t_n^2-2t_n\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}-M_n\right)+\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}-M_n\right)^2 = \left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}\right)^2-x_{n-1}^2+\frac{4y_{n-1}t_n}{f'(t_n)}+4y_{n-1}f(t_n)$$

$$\begin{aligned}
t_n^2-2t_n\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}-M_n\right)-\frac{4y_{n-1}t_n}{f'(t_n)} &= -\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}-M_n\right)^2+\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}\right)^2-x_{n-1}^2+4y_{n-1}f(t_n) \\
t_n^2-2t_n\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}-M_n+\frac{2y_{n-1}}{f'(t)}\right) &= -\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}-M_n\right)^2+\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}\right)^2-x_{n-1}^2+4y_{n-1}f(t_n)
\end{aligned}$$

$$t_n^2-2t_n\left(x_{n-1}-M_n\right)=-\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}-M_n\right)^2+\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}\right)^2-x_{n-1}^2+4y_{n-1}f(t_n)$$

$$t_n^2-2t_n\left(x_{n-1}-M_n\right)+\left(x_{n-1}-M_n\right)^2=\left(x_{n-1}-M_n\right)^2-\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}-M_n\right)^2+\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}\right)^2-x_{n-1}^2+4y_{n-1}f(t_n)$$

$$\left(t_n-\left(x_{n-1}-M_n\right)\right)^2=\left(x_{n-1}-M_n\right)^2-\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}-M_n\right)^2+\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}\right)^2-x_{n-1}^2+4y_{n-1}f(t_n)$$

$$t_n-\left(x_{n-1}-M_n\right)=\sqrt{\left(x_{n-1}-M_n\right)^2-\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}-M_n\right)^2+\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}\right)^2-x_{n-1}^2+4y_{n-1}f(t_n)}$$

$$t_n=\left(x_{n-1}-M_n\right)+\sqrt{\left(x_{n-1}-M_n\right)^2-\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}-M_n\right)^2+\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}\right)^2-x_{n-1}^2+4y_{n-1}f(t_n)} \tag{18}$$

with 12 , 16 , and 18 we have three equations that when used correctly will derrive accurate solutions to (x_n, y_n)

$$t_n=\left(x_{n-1}-M_n\right)+\sqrt{\left(x_{n-1}-M_n\right)^2-\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}-M_n\right)^2+\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}\right)^2-x_{n-1}^2+4y_{n-1}f(t_n)} \tag{19}$$

$$x_n=x_{n-1}-\frac{2y_{n-1}}{f'(t)}+\sqrt{\left(x_{n-1}-\frac{2y_{n-1}}{f'(t)}\right)^2-x_{n-1}^2+\frac{4y_{n-1}t_n}{f'(t_n)}+4y_{n-1}f(t_n)} \tag{20}$$

$$y_n=\frac{\left(x_{n-1}-x_n\right)^2}{4y_{n-1}} \tag{21}$$

Because 19 is not an explicitly defined function there lies a problem with-in solving 19 for t. At this point being completely stuck as to finding a means of solving for t explicitly, I decided to randomly guess values of t that would satisfy 19 with given intial condition $(x_0, y_0) = (0, 1/2)$ and esstimate the accuraccy by refferencing visually with a graph. Surprisingly I found my first guess $t = 0$ to be quite visually satisfacutory interms of accuraccy, however upon further inspection I found that this was clearly not a soultion.

Still searching for a solution to the problem, I began to think of other problems that all readdy had soultions that would be similiar to the problem I was working on. The first such problem that came to mind was the solution to$x = cos(x)$. The number x that satisfies this equation is known as the The Dotty number. The solution to $x = cos(x)$ is found using a method known as fixed point interation, as seen as having many uses, recalling Newton's Method for example. I began to think I too could use fixed point itteration to solve my problem.

After itterating my intial guess of $t = 1$ for the right-hand side of 19 I found that t eventually became fixed at $t = 0.98838367032952$. Realizing at this point my solution was to be obtained numerically instead of analytically I abondoned pencil and paper and set forth on the computer. I decided to use free math software called Geogebra. In this way I hoped that if others wanted to replicate my results it could be done in a way that was free of cost.

The following is the construction protocol that I have found to best replicate a desired solution set of circles.

Step	Input
1	$N = 5$
2	$f(x) = cos(x)$
3	$a:\ y = x$
4	$M_n(x) = \frac{f(x)}{\sqrt{f'(x)^2+1}}\left(\frac{1-\sqrt{f'(x)^2+1}}{f'(x)}\right)$
5	$h:$ Curve[$M_n(t)+t, \frac{-1}{f'(t)}\left(M_n(t)\right)+f(t)$,t,0,pi/2]
6	$T_n(h,k,x) = h - M_n(x) - \sqrt{(h-M_n(x)\ )^2+(h-\frac{2k}{f'(x)}-M_n(x)\ )^2+(h-\frac{2k}{f'(x)})^2-h^2+4kf(x)}$
7	Execute[Join[{"O_{1} = $(0,1/2)$" , "O_{2} = h(intersect[T_n(O_{1}(1,0),O_{1}(0,1),x),a , $(1,1)$](1,0))" ,Sequence[O_{" + $(i+2)$ + "} = h(intersect[T_n(O_{+$(i+1)$+}(1,0),O_{+$(i+1)$+}$(0,1),x)$,a,$(pi/2,pi/2)$](1,0))",i,1,N]]]
8	C_n = {Circle[O_1,O_1(0,1)], Circle[O_2,O_2(0,1)], . . . Circle[O_n,O_n(0,1)] }

This code produces, although very accurte, only 5 circles that are approximatley with-in the solution set. Although the number of circles n can be changed to be a higher number, I still have not found a method to find the n^{th} circle. For these two main reasons I am still unsatisfied with my results, and it pains me and gives me great pleasure to say that my work on this problem is still not done. Before my conclussion I would like add a few notes on where I believe the proper path to the solution is to be found. This problem can be represented as a system of differential equation. The recursive nature of obtaining solutions reminded me of delay differential equations.Therefore I thought, perhaps the system can rearranged to better fit a convention that is more analyzable using calculus. As I mentioned before the Dotty Number was a key element of meditation that gave me a break through as to a means of obtaining a solution. As before my thinking was "if this can be solved so can this", and so to is it again. I beilieve that if the solution to $x = cos(x)$, the Dotty number, can be expressed explicitly interms of the Lambert W. Function, so too can the solution to this problem.

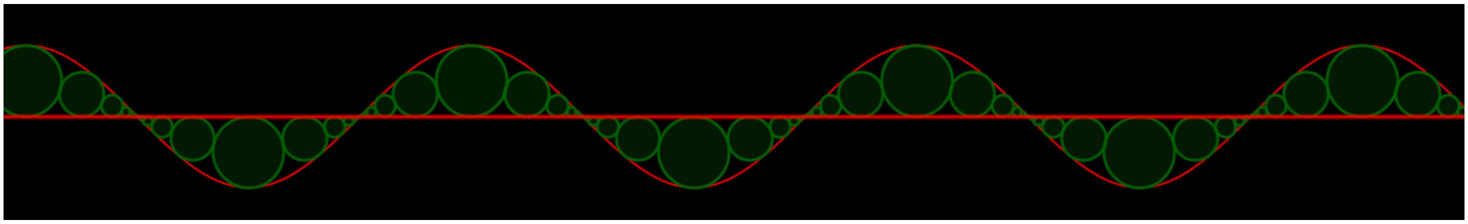

Figure 13: please work

$$
\begin{array}{c}
t_n \\
\downarrow \\
x_n \\
\downarrow \\
y_n
\end{array}
\tag{22}
$$