

# Dual vibration configuration interaction (DVCI). A novel factorisation of molecular Hamiltonian for high performance infrared spectrum computation.

Romain Garnier<sup>a,\*</sup>

<sup>a</sup>*Queen's University of Belfast*

---

## Abstract

Here is presented an original program based on molecular Schrödinger equations. It is dedicated to target specific states of infrared vibrational spectrum in a very precise way with a minimal usage of memory. An eigensolver combined with a new probing technique accumulate information along the iterations so that desired eigenpairs rapidly tend towards the variational limit. Basis set is augmented from the maximal components of residual vectors that usually require the construction of a big matrix block that here is bypassed with a new factorisation of the Hamiltonian. The latest borrows the mathematical concept of duality and the second quantization formalism of quantum theory.

*Keywords:* Vibration Configuration Interaction, Infrared spectrum, Iterative eigensolver, Residual error minimization, Duality, Second quantization.

---

## PROGRAM SUMMARY

*Program Title:* Dual vibration configuration interaction (DVCI)

*Licensing provisions:* GNU General Public License 3.

*Programming languages:* C/C++/Fortran.

*Supplementary materials:*

1. The sources of the code grouped in folder *DualVCI.zip* also available at <https://github.com/4Rom1/DualVCI>
2. The input files of examples treated in section 6.

*Nature of problem:* High computational cost in vibration configuration interaction methods [1, 2], coming from the necessity to solve a large eigenvalue problem to acquire a good precision. The dimension of the matrix exponentially increases with the size of the studied molecule.

---

\*E-mail address: rom1{dot}garnier{at}yahoo{dot}fr

*Solution method: The  $A_k$  decomposition [3] completed by a meaningful error evaluation namely the residue  $\|HX - EX\|$  minimised along iterations for specific targets given in input. The approximation space is generated in the same time as the residual vectors computed on the fly thanks to an adapted choice of excitations shaped on the Hamiltonian operator.*

## 1. Introduction

Nowadays, many devices are able to supply high quality spectrum measurements. However the interpretation of the samples remains a difficult task because the numerical accuracy that is possible to obtain is very limited for medium to large molecules ( $> 6$  atoms). In a typical resolution of vibrational Schrödinger equations in the Born-Oppenheimer frame, one can question the correctness of the model from two principal angles. First, the quality of the results will be affected by the one of the Potential Energy Surface (PES). This last point aside, there remains the validity of the numerical solutions when the PES is provided. That is the focus of the proposed method. A prior analysis of the potential energy supplies the harmonic states that are not able to correctly describe complex combinations of translational and rotational motions of the nucleus. Although still accurately limited, the VSCF method [4, 5] alone allows a better representation. With the idea that these first estimations can be combined to give a more authentic description, the cheapest technique remains the perturbation theory [6, 7, 8] that is known to struggle with strong resonances conventionally encountered in molecular spectroscopy [9, 10]. Vibration configuration interaction (VCI) method [1, 2, 11] permits a better precision with a much higher computational expense coming from the size of the variational space that is tenfold with the number of atoms of the molecule. To avoid this bottleneck, the  $A_k$  decomposition [3] initially designed for electronic structure calculation and firstly introduced in the VCI context with the Vibrational Multi Reference Configuration Interaction (VMRCI) [12], has been able to identify relevant sub-blocks of the Hamiltonian matrix thereby reducing the size of the system. Similar construction was used with Adaptive-VCI (A-VCI) [13, 14] to access the smallest eigenvalues with a very good precision. These last approaches showed promising results in term of size reduction, but still require to a posteriori determine a big matrix block designed to improve the accuracy of the solutions. In the present work, the non zeros of this sub-block are never collected and the number of operation to perform proper Matrix Vector Products (MVPs) is highly reduced thanks to a new factorisation of the Hamiltonian. Beside, the expense of RAM is even more diminished because there is the possibility to constraint numerical accuracy on a few specified targets. In next sections are presented the context and the state of the art. The concept of duality intervenes all along the paper each time an association is made between a group of objects especially in section 4 explaining the theoretical aspects of the algorithm. Thereafter a description of the successive processed operations are followed by benchmarks. In this section, the method can provide comparable quality (i.e less than  $1 \text{ cm}^{-1}$  error) of a full A-VCI calculation, but with a memory consumption scaled down by more than a factor 15. Next, the list of input parameters of the program serves as a user manual. At the end, the conclusion also mentions a list of possible future developments.

## 2. Context

For a molecule composed of  $N_A$  atoms, one considers NM dimensionless normal coordinates [15]  $\mathbf{q} = (q_1, q_2, q_3, \dots, q_{\text{NM}})$ , with  $\text{NM} = 3 * N_A - 6$ . The corresponding harmonic frequencies are designated by  $(\nu_1, \nu_2, \nu_3, \dots, \nu_{\text{NM}})$ . The model is based on the Watson Hamiltonian [16] with zero rotational angular momentum ( $J=0$ ). Its vibrational part contains the PES that is a multidimensional function known only for few points calculated by a first electronic resolution. It is represented in here as a Sum Of Product (SOP). For combinations of monomial degrees  $\mathbf{c} \in \mathbb{N}^{\text{NM}}$ , attributed to force constants  $K_{\mathbf{c}}$ , the PES of maximal degree DP has the general form

$$\mathcal{U}_{\mathcal{K}}(\mathbf{q}) = \sum_{\|\mathbf{c}\|_1 \leq \text{DP}} K_{\mathbf{c}} \prod_{n=1}^{\text{NM}} q_n^{c_n}, \quad (1)$$

and the vibrational part writes

$$\mathcal{H}_{\text{vib}}(\mathbf{q}) = -\frac{1}{2} \sum_{n=1}^{\text{NM}} \nu_n \frac{\partial^2}{\partial q_n^2} + \mathcal{U}_{\mathcal{K}}(\mathbf{q}). \quad (2)$$

The rotational part consists of the *Coriolis corrections*

$$\begin{aligned} \mathcal{H}_{\text{rot}}(\mathbf{q}) &= \mathcal{C}_{ijkl}(\mathbf{q}) - \frac{1}{8} \sum_{\alpha=1}^3 \mu_{\alpha\alpha}, \\ \mathcal{C}_{ijkl}(\mathbf{q}) &= -\frac{1}{2} \sum_{i < j} \sum_{k < l} Z_{ijkl} \left( \sqrt{\frac{\nu_j}{\nu_i}} q_i \frac{\partial}{\partial q_j} - \sqrt{\frac{\nu_i}{\nu_j}} q_j \frac{\partial}{\partial q_i} \right) \left( \sqrt{\frac{\nu_l}{\nu_k}} q_k \frac{\partial}{\partial q_l} - \sqrt{\frac{\nu_k}{\nu_l}} q_l \frac{\partial}{\partial q_k} \right) \\ Z_{ijkl} &= \sum_{(\alpha, \beta) \in (x, y, z)} \mu_{\alpha, \beta} \zeta_{ij}^{\alpha} \zeta_{kl}^{\beta}, \end{aligned} \quad (3)$$

where  $\mu_{\alpha, \beta}$  is the inverse of the moment of inertia and coefficients  $(\zeta_{ij}^{\alpha}, \zeta_{kl}^{\beta})$  are calculated according to the method of Meal and Polo [17]. The *Watson term*  $-\frac{1}{8} \sum_{\alpha=1}^3 \mu_{\alpha\alpha}$  modifies the eigenvalues by a constant so that the anhamornic frequencies calculated are not affected, then it will be omitted in the rest of the paper. In regard to the wave function of the total Hamiltonian  $\mathcal{H}$ , it is a linear combination of basis set elements belonging to an Ansatz  $B$

$$\Psi(\mathbf{q}) = \sum_{\mathbf{b} \in B} x_{\mathbf{b}} \Phi_{\mathbf{b}}(\mathbf{q}), \quad (4)$$

each one writing as a product of one dimensional harmonic oscillators

$$\Phi_{\mathbf{b}}(\mathbf{q}) = \phi_{b_1}(q_1) \dots \phi_{b_n}(q_n) \dots \phi_{b_{\text{NM}}}(q_{\text{NM}}), \quad (5)$$

solution of the equation

$$\begin{aligned} \mathcal{H}_0(\mathbf{q}) \Phi_{\mathbf{b}}(\mathbf{q}) &= E_{\mathbf{b}} \Phi_{\mathbf{b}}(\mathbf{q}), \\ E_{\mathbf{b}} &= \sum_{n=1}^{\text{NM}} \nu_n (b_n + 1/2), \\ \mathcal{H}_0(\mathbf{q}) &= \sum_{n=1}^{\text{NM}} \frac{\nu_n}{2} \left( -\frac{\partial^2}{\partial q_n^2} + q_n^2 \right). \end{aligned} \quad (6)$$

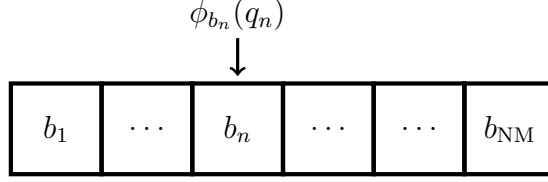


Figure 1: Multi index array identified with a basis function.  
Each index is an Hermite function degree corresponding to an harmonic quantum level.

Implicitly,  $\Phi_{\mathbf{b}}(\mathbf{q})$  is assimilated to the multi-index  $\mathbf{b}$  (cf figure 1), and recognized by an integer coinciding with a pointer address when stored in memory. The algorithm could also possibly work with optimized basis set [4, 5], but the efficiency will be impoverished because DVCi fully uses harmonic oscillator properties. Afterwards, classical variational formulation leads to the eigenvalue problem

$$H\mathbf{X} = E\mathbf{X}, \quad (7)$$

with matrix coefficients of  $H$  built from the integrals<sup>1</sup>

$$\langle \Phi_{\mathbf{s}} | \mathcal{H} | \Phi_{\mathbf{b}} \rangle = \int_{\mathbb{R}^{NM}} \Phi_{\mathbf{s}}(\mathbf{q}) (\mathcal{H}_{vib} + \mathcal{H}_{rot})(\mathbf{q}) \Phi_{\mathbf{b}}(\mathbf{q}) \quad (8)$$

$$\forall (\mathbf{s}, \mathbf{b}) \in B \times B.$$

*The curse of dimensionality.* The number of basis functions exponentially increases with the number of nucleus when adopting a brute force variational method. As example, for a 12-d normal coordinate system, the dimension of the configuration space will be  $11^{12}$  for a maximal quantum level equal to 10 in each direction. To solve the associated eigenvalue problem, one needs to manipulate items with the same size, and a multiplication by 8 gives about 251 terabytes of memory for a double precision vector.

### 3. State of the Art

To significantly reduce memory usage, matrix entries may not be stocked and computed with pruning conditions [18, 19, 20]. It consists in finding proper weights function  $\alpha_n$  calculated on harmonic frequency criteria, and a maximal quantum level  $\bar{d}$  to define the VCI space

$$\text{VCI}_{\alpha}(\bar{d}) = \{\mathbf{b} \in \mathbb{N}^{NM}, \sum_{n=1}^{NM} \alpha_n(b_n) \leq \bar{d}\}. \quad (9)$$

For example  $\alpha_n(b_n) = \lfloor \nu_n / \nu_{min} + 0.5 \rfloor * b_n$  [21]. When possible, symmetry properties may also be employed to separate basis set into groups of functions belonging to different irreducible representations. New and prospering works on tensorial factorisation relying upon alternating least square minimisation [22] permit a drastic reduction of RAM and time expenditure. In the present context, we can notice the efficacy of the Hierarchical Reduced-Rank Block

<sup>1</sup>Cf Appendix.

Power Method (HRRBPM) [23, 24] and the tensor train factorisation[25]. However, in this case the efficiency directly depends on the number of terms of the PES that should be small enough to observe accurate results with a low computational cost.

In the field of basis selection techniques, one widely uses combination of VCI and perturbation criteria with variation-perturbation theory [26, 27, 28, 29, 30, 31, 32, 33, 34]. For an member  $\Phi_{\mathbf{b}}$  of a growing subspace  $B$ , selected basis functions  $\Phi_{\mathbf{s}}$  should verify the perturbation criterion

$$\left| \frac{\langle \Phi_{\mathbf{s}} | \mathcal{H} - \mathcal{H}_0 | \Phi_{\mathbf{b}} \rangle^2}{E_{\mathbf{b}} - E_{\mathbf{s}}} \right| > \varepsilon_{\text{VP}}, \quad (10)$$

where  $\varepsilon_{\text{VP}}$  is a given threshold depending on the accuracy one wants to reach.  $(E_{\mathbf{s}}, \Phi_{\mathbf{s}})$  and  $(E_{\mathbf{b}}, \Phi_{\mathbf{b}})$  are eigen-pairs of  $\mathcal{H}_0$  relying upon the nature of the basis functions that are used. It typically designs the harmonic part (6) or the sum of single-mode VSCF operators [35, 36, 37, 38].

An other relevant criteria for subspace selection is the residue. It is widely adopted in Davidson like methods [39, 40, 41, 42] and has recently been implemented in A-VCI [13, 14] sharing same structure than the  $A_k$  decomposition [3]. In the  $A_k$  theory, one considers primary and secondary spaces respectively called  $B$  and  $B_s$ . In the whole space  $B \oplus B_s$ , Hamiltonian matrix writes:

$$H = \begin{pmatrix} H_B & H_{\text{SB}}^T \\ H_{\text{SB}} & H_s \end{pmatrix}, \quad (11)$$

where the different sub-blocks combine  $B$  and  $B_s$  in the following way

$$\begin{aligned} [H_B] &= [\langle \Phi_{\mathbf{s}} | \mathcal{H} | \Phi_{\mathbf{b}} \rangle]_{(\mathbf{s}, \mathbf{b}) \in B \times B}, \\ [H_{\text{SB}}] &= [\langle \Phi_{\mathbf{s}} | \mathcal{H} | \Phi_{\mathbf{b}} \rangle]_{(\mathbf{s}, \mathbf{b}) \in B_s \times B}, \\ [H_s] &= [\langle \Phi_{\mathbf{s}} | \mathcal{H} | \Phi_{\mathbf{b}} \rangle]_{(\mathbf{s}, \mathbf{b}) \in B_s \times B_s}. \end{aligned} \quad (12)$$

For the A-VCI  $B_s$  is included in  $\mathcal{H}_{\text{vib}}(B) \setminus B$ , the complement of  $B$  in its image by  $\mathcal{H}_{\text{vib}}(2)$ , and for VMRCI  $B_s \subset \text{STDQ}(B) \setminus B$  where STDQ refer to "Simple", "Double", "Triple" and "Quadruple" excitations. One can remark the use of the inclusion instead of equality because additional truncation on secondary space may be added to diminish the memory usage. For example one can cut down the maximal Harmonic energy as for the A-VCI, or only consider STD excitations in the case of VMRCI. The A-VCI is dedicated to compute the first eigenvalues of the Hamiltonian whereas VMRCI is focussed on any given initial state generally the fundamentals, first overtones, or combination bands. For an eigenpair  $(E, \mathbf{X})$  of  $H_B$  (11), and  $\tilde{\mathbf{X}} = (\mathbf{X}, \mathbf{0}_{B_s})$  the zero padded array of  $\mathbf{X}$  in  $B \oplus B_s$ , the residual vector writes

$$\mathbf{Y}_s = H\tilde{\mathbf{X}} - E\tilde{\mathbf{X}} = (\mathbf{0}_B, H_{\text{SB}}\mathbf{X})^T. \quad (13)$$

It might easily be related to the correction vector  $\mathbf{X}_s$  obtained in the  $A_k$  theory<sup>2</sup>

$$\mathbf{X}_s = (\mathbf{0}_B, (E.I_s - \text{diag}(H_s))^{-1} H_{\text{SB}} \mathbf{X})^T. \quad (14)$$

In an iterative process, A-VCI and VMRCI both consider respectively maximal components of vectors  $\mathbf{Y}_s$  and  $\mathbf{X}_s$  to enrich primary space  $B^3$ . Since the non null coordinates of  $(\mathbf{Y}_s, \mathbf{X}_s)$  are always assigned to secondary space  $B_s$ , the chosen ones will be nullified at next iteration. The expected effect is to minimize the errors between eigenpairs computed in  $B$  and the ones that we would have calculated in  $B \oplus B_s$ .

#### 4. Theory and algorithm

In this part, we will see that the general algorithm is made from an enhanced mix of the previously recalled methods and a new theoretical approach to the construction of the Hamiltonian. Here are the main features:

- For the generation of  $B_s$ , in addition to the usual ways of expansion previously reminded, the method is capable to calibrate an optimal choice of excitations according to the analysis of the force field and rotational terms.
- The precision is focused on a given choice of targets, which allows an effective compression of the information and a significant gain of memory.
- The presented factorisation relieves the occupation of memory by performing on-the-fly operations for the MVP  $H_{\text{SB}} \mathbf{X}$  (13) that are not compensated by a systematic augmentation in execution time.

##### 4.1. The local factorisation

Considering by  $\hat{q}_n^+$  and  $\hat{q}_n^-$  the creation (or raising) and annihilation (or lowering) operators[43, 44], acting on Hermite functions in the following manner

$$\begin{aligned} \hat{a}_n^+ |\phi_{b_n}\rangle &= \sqrt{b_n + 1} |\phi_{b_n+1}\rangle, \\ \hat{a}_n^- |\phi_{b_n}\rangle &= \sqrt{b_n} |\phi_{b_n-1}\rangle, \end{aligned} \quad (15)$$

the position and derivative may be written under second quantized form as

$$\begin{aligned} q_n &= \frac{1}{\sqrt{2}}(\hat{a}_n^- + \hat{a}_n^+), \\ \frac{\partial}{\partial q_n} &= \frac{1}{\sqrt{2}}(\hat{a}_n^- - \hat{a}_n^+) . \end{aligned} \quad (16)$$

The local factorisation intends to answer the question:

Are there common factors in front of each product of excitations once any derivative or

---

<sup>2</sup> $\text{diag}(H_s)$ : Diagonal of matrix  $H_s$ .  $E.I_s$  : scalar  $E$  multiplied by identity matrix  $I_s$ .  
See second part of the sum equation (9) in the paper [3], with  $E = \tilde{\lambda}$ .

<sup>3</sup>For the VMRCI, see the correction of the energies equation (19) in the paper [12].

polynomial has been expressed in term of operators (16) and incorporated into  $\mathcal{H}$ ?<sup>4</sup>

We can, in fact, treat it from another angle by noticing from elementary properties of Hermite functions<sup>5</sup> that the paired elements  $\langle \Phi_{\mathbf{s}} | \mathcal{H}_{vib} | \Phi_{\mathbf{b}} \rangle \neq 0$  involve only the local force constants<sup>6</sup>

$$\text{LFK}(\mathbf{s} - \mathbf{b}) = \left\{ \begin{array}{l} \mathbf{c} \in \mathbb{N}^{\text{NM}}, K_{\mathbf{c}} \neq 0, \\ \forall n \in \{1, \dots, \text{NM}\}, K_{\mathbf{c}} \neq \nu_n/2, \\ \exists t_n \in \mathbb{N}, |s_n - b_n| = c_n - 2t_n \end{array} \right\}. \quad (17)$$

In the same way, for a group of four canonical vectors ( $\mathbf{1}_i, \mathbf{1}_j, \mathbf{1}_k, \mathbf{1}_l$ ) of lengths NM with entry 1 respectively on position  $i, j, k, l$ , the local *Coriolis* interactions  $\langle \Phi_{\mathbf{s}} | \mathcal{H}_{rot} | \Phi_{\mathbf{b}} \rangle \neq 0$  contain the combinations

$$\text{LCI}(\mathbf{s} - \mathbf{b}) = \left\{ \begin{array}{l} \mathbf{c}^{ijkl} = \mathbf{1}_i + \mathbf{1}_j + \mathbf{1}_k + \mathbf{1}_l, \\ (i, j, k, l) \in \{1, \dots, \text{NM}\}^4, \\ i < j, k < l, \forall n \in \{1, \dots, \text{NM}\}, \\ \exists t_n \in \mathbb{N}, |s_n - b_n| = c_n^{ijkl} - 2t_n \end{array} \right\}. \quad (18)$$

These sets are gathered in the local force field written

$$\text{LFF}(\mathbf{s} - \mathbf{b}) = \text{LFK}(\mathbf{s} - \mathbf{b}) \cup \text{LCI}(\mathbf{s} - \mathbf{b}), \quad (19)$$

and defined no matter the sign of the differences  $(s_n - b_n)$ ,  $n \in \{1, \dots, \text{NM}\}$ . With definitions (17),(18), the program builds the set of natural numbers associated to the non void local force fields<sup>7</sup>

$$\text{LFF}^* = \{\mathbf{e} \in \mathbb{N}^{\text{NM}}, \text{LFF}(\mathbf{e}) \neq \emptyset\} \cup \{\mathbf{0}_{\text{NM}}\}. \quad (20)$$

From expressions (16), a creation is always accompanied by annihilation. Then, the dual of  $\mathcal{H}$  consisting in its intrinsic sum of product of excitations, is constructed from the positive and negative multi-increments

$$\pm \text{LFF}^* = \{(\pm e_1, \dots, \pm e_{\text{NM}}) \in \mathbb{Z}^{\text{NM}}, (e_1, \dots, e_{\text{NM}}) \in \text{LFF}^*\}, \quad (21)$$

and writes<sup>8</sup>

$$\mathcal{H}^* = \sum_{\mathbf{e} \in \pm \text{LFF}^*} \prod_{n=1}^{\text{NM}} \hat{a}_n^{\pm e_n}. \quad (22)$$

The spaces  $B, B_s$  are growing up together along the repetitions of the main loop. One denotes by  $A$  the set of added basis functions in  $B$  from one iteration to the other.  $B_s$  is completed by browsing the image  $\mathcal{H}^*(A)$ . In the meantime the MVP  $H_{\text{sb}} \mathbf{X}$  is partially calculated for

<sup>4</sup>The product is non commutative, but to apprehend the problem, one needs to picture it as if it was.

<sup>5</sup>Cf Appendix.

<sup>6</sup> $K_{\mathbf{c}} \neq \nu_n/2$  because it is included in the harmonic part (6) and will be accounted when  $\mathbf{s} = \mathbf{b}$ .

<sup>7</sup>The zero excitation  $\mathbf{0}_{\text{NM}} = (0, \dots, 0) \in \mathbb{N}^{\text{NM}}$  is systematically included.

<sup>8</sup> $\pm e_n$  is abusively employed to indicate the presence of a + or - sign in front of  $e_n$ .

the couples  $(\mathbf{b}, \mathbf{b} + \mathbf{e}) \in (A \times \mathcal{H}^*(A) \setminus B)$  as <sup>9</sup>

$$\begin{aligned} (\widetilde{H_{\text{sb}}\mathbf{X}})_{\mathbf{b}+\mathbf{e}} &= \sum_{\mathbf{b} \in A} \left[ \sum_{\mathbf{c} \in \{\text{LFK}(\mathbf{e})\}} K_{\mathbf{c}} \prod_{n=1}^{\text{NM}} \langle \phi_{b_n+e_n} | q_n^{c_n} | \phi_{b_n} \rangle \right. \\ &\quad \left. + \sum_{(i,j,k,l) \in \{\text{LCI}(\mathbf{e})\}} \langle \Phi_{\mathbf{b}+\mathbf{e}} | \mathcal{C}_{ijkl} | \Phi_{\mathbf{b}} \rangle \right] x_{\mathbf{b}}. \end{aligned} \quad (23)$$

The other part of the sum ( $\mathbf{b} \in B \setminus A$ ) and other components are then completed as explained in the next section. When setting the parameter DoGraph $\diamond$ 23 to zero, one also has the possibility to directly compute the whole vector  $H_{\text{sb}}\mathbf{X}$  by fetching  $\mathcal{H}^*(B)$  instead of  $\mathcal{H}^*(A)$  in (23). In that case it takes additional execution time because many tests are required to locate the addresses of the members of  $B_s$ . However, it is balanced by a smaller usage of RAM.

In supplement to the classical truncations, the excitations of  $\mathcal{H}^*$  can be selected with ThrKX $\diamond$ 9, defining a threshold on the sum of the force constants contained in each local force field. The purpose is to avoid the runaway of  $B_s$  size and incorporate only the most contributive basis functions to the residue.

#### 4.2. Complementary storage

Solving an eigenvalue problem usually requires a significant amount of MVPs, then the non null coefficients of  $H_{\text{b}}$  (12) might rather be collected than evaluated on the fly. So far, for any related method, the complexity to determine a non null matrix entry is at least of order  $O(\text{NM} * \text{NPES})$ , where NPES designates the total number of force constant in the PES. When using the local force field for the evaluation of  $\langle \Phi_{\text{s}} | \mathcal{H} | \Phi_{\text{b}} \rangle$ , it is enough to fetch

$$(\mathbf{s} - \mathbf{b})^+ = (|s_1 - b_1|, \dots, |s_{\text{NM}} - b_{\text{NM}}|) \text{ into LFF}^*(20), \quad (24)$$

calculate the terms in brackets equation (23) for  $\mathbf{e} = \mathbf{s} - \mathbf{b}$  and add the harmonic energy  $E_{\text{b}}$  when  $\mathbf{b} = \mathbf{s}$ . The localization of the position of  $(\mathbf{s} - \mathbf{b})^+$  in LFF\* is performed with a binary search [45] costing  $O(\text{NM} * \log(|\text{LFF}^*|))$  operations. Consequently, the total complexity is

$$O(\text{NM} * [|\text{LFF}(\mathbf{s} - \mathbf{b})| + \log(|\text{LFF}^*|)]), \quad (25)$$

and one can easily check that  $|\text{LFF}(\mathbf{s} - \mathbf{b})| + \log(|\text{LFF}^*|) \ll \text{NPES}$ . Indeed for  $\mathbf{e} \neq \mathbf{0}_{\text{NM}}$ <sup>10</sup>, the cardinal of LFF( $\mathbf{e}$ )(19) is always very much lower than NPES. The worst case being  $\text{NPES}/2 * \text{NM}$  happening only for a PES with no crossings. It therefore appears that the local factorisation consumes less operations than traditional methods for the determination of matrix coefficients, and for a small amount of MVPs, will be capable to integrate into a calculation on the fly without jeopardizing the execution time. Nevertheless, it will be all the more accelerated as the address of the paired elements is known in advance. This technique is actually employed for the residual block when the parameter DoGraph is strictly positif,

<sup>9</sup> $\mathcal{C}_{ijkl}$  is defined equation (3).

<sup>10</sup>The case  $\mathbf{e} = \mathbf{0}_{\text{NM}}$  intervenes only for the construction of the diagonal elements of  $H_{\text{b}}$ .



by keeping only the pointers on the non zeros of  $H_{\text{sb}}$ , thus directly accessible to complete the MVP (23) for the missing parts:

$$(H_{\text{sb}}\mathbf{X})_{\mathbf{s}} = (\widetilde{H_{\text{sb}}\mathbf{X}})_{\mathbf{s}} + \sum_{\mathbf{b} \in (B \setminus A)} \langle \Phi_{\mathbf{s}} | \mathcal{H} | \Phi_{\mathbf{b}} \rangle x_{\mathbf{b}}, \quad \forall \mathbf{s} \in \mathcal{H}^*(B) \setminus B. \quad (26)$$

#### 4.3. Initial space construction

Let's consider the ordered eigenvalues of block  $H_{\mathbf{b}}$  built at iteration  $\mathbf{i}$

$$E_0^{\mathbf{i}} \leq \dots \leq E_{\ell}^{\mathbf{i}} \leq E_{\ell+1}^{\mathbf{i}} \leq \dots \quad (27)$$

One can demonstrate with the Poincaré separation theorem[46] that each  $E_{\ell}^{\mathbf{i}}$  is a decreasing sequence of  $\mathbf{i}$ . Consequently, the minimization of the differences

$$E_{\ell}^{\mathbf{i}} - E_{\ell}^{\mathbf{i}+1} \quad (28)$$

could be effective only if no spectral hole is introduced in the interval holding targets at step zero where the eigenvalues are simple harmonic energies (6). Also, in order to integrate the perturbation effects, the initial space writes

$$B = \{\mathbf{b} / E_{\mathbf{b}} \leq E_{\text{max}} * \kappa\}, \quad (29)$$

where  $^{11}E_{\text{max}} = \text{MaxFreq} + E_0$ ,  $\text{MaxFreq} \diamond 17$  is the maximal tracked frequency and  $\kappa \diamond 18$  an empirical elongation factor accounting the maximal gap between an harmonic and a converged energy number  $\ell$ . Its default value is set to 1.2, but it is automatically increased in agreement with the anharmonicity growing up as we get away from the smallest eigenvalue. To avoid timing issues caused by the great number of combination band<sup>12</sup> to be tested, the initial space is calculated by recursively applying one simple raising excitations starting from  $\mathbf{0}_{\text{NM}}$  until size consistency.

## 5. The iterative process

The maximal size of the arrays used for  $B, B_{\mathbf{s}}, H_{\mathbf{b}}$  and  $H_{\text{sb}}$  is determined with the allocated memory controlled by the parameter `Memory`  $\diamond 19$ . At any iteration, the set `{Targ}` refers to eigenvectors of  $H_{\mathbf{b}}$  having one component larger than `ThrCoor`  $\diamond 14$  and assigned to targets given by the parameter `TargetState`  $\diamond 13$ . After a prior construction of the objects `{LFF(e), e ∈ LFF*}`(19), the sequence of successive main steps decomposes as

<1> Build the initial subspace (29).

<2> Compute the eigenpairs  $(E_{\ell}, \mathbf{X}_{\ell})$  of  $H_{\mathbf{b}}$  with the Implicitly Restarted Lanczos Method of ARPACK [47, 48]. The upper limit of calculated eigenvalues is chosen with `MaxEv`  $\diamond 24$ .

---

<sup>11</sup> $E_0 = \frac{1}{2} \sum_{n=1}^{\text{NM}} \nu_n.$   
<sup>12</sup> $\sum_n^{\text{NM}} b_n \nu_n.$

<3> Evaluate the residual vectors on the fly  $H_{\text{SB}}\mathbf{X}_\ell$ ,  $\ell \in \{\text{Targ}\}$  and secondary basis set  $B_s$  as explained in section 4. The expansion of  $B_s$  can additionally be reduced with an elimination of the less contributing excitations of  $\mathcal{H}^*$  via the parameter ThrKX $\diamond$ 9, by restricting the maximal harmonic energy with Freq0Max $\diamond$ 8 or the maximal quantum level MaxQLevel $\diamond$ 7. Biggest components (in absolute value) of residual vectors are then employed to select basis functions to be added for next iteration through the inputs NAdd  $\diamond$ 10, EtaComp  $\diamond$ 11 and MaxAdd  $\diamond$ 12.

<4> Go back to step <2> as long as

$$\max_{\ell \in \{\text{Targ}\}} \frac{\|H_{\text{SB}}\mathbf{X}_\ell\|}{E_\ell} > \text{EpsRez} \diamond 5.$$

This criterion can be fulfilled only if enough memory has been allocated at the beginning. Otherwise, the algorithm will stop until maximal number of basis functions has been reached. It also trivially appears that increasing the targets does also augment the required memory to make them converge at once.

## 6. Benchmarks

All the calculus are done on a 64 bits, 2.70GHz quad core processor (model Intel i5-3340M) with 8 Gigabytes of RAM. No parallel process is used in here. DVCi program runs on a single CPU so that the actual computational time is the same as the CPU wall time.

### 6.1. CH<sub>3</sub>CN: Acetonitrile

Methods are compared on molecule CH<sub>3</sub>CN with the same PES as the one used in Ref. [19, 49, 24, 13] that was initially introduced by Bégué and al. [50], computed at CCSDT/cc-pVTZ level for harmonic frequencies and B3LYP/cc-pVTZ for higher order terms. This PES counts 311 terms (12 quadratic, 108 cubic and 191 quartic). The benchmark results are taken from Avila et Carrington [19] where symmetry has been employed to separate the full VCI space into two smaller subspaces. This PES has a small number of non null derivatives regarding the size of the molecule. The reference space defined as in [19] is the pruned basis set

$$\left\{ \begin{array}{l} \mathbf{b} \in \mathbb{N}^{12}, \sum_{n=1}^{12} \alpha_n b_n \leq 27, \\ \alpha_1 = 3, \alpha_2 = 4, \alpha_3 = 3, \alpha_4 = 3, \alpha_5 = \alpha_6 = 3, \\ \alpha_7 = \alpha_8 = 4, \alpha_9 = \alpha_{10} = 3, \alpha_{11} = \alpha_{12} = 1. \end{array} \right\} \quad (30)$$

It contains 743 103 basis functions and the fundamental harmonic frequencies are

$$\begin{aligned} \nu_1 &= 3\,065, \nu_2 = 2\,297, \nu_3 = 1\,413, \nu_4 = 920, \\ \nu_5 &= \nu_6 = 3\,149, \nu_7 = \nu_8 = 1\,487, \\ \nu_9 &= \nu_{10} = 1\,061, \nu_{11} = \nu_{12} = 361 \text{ (cm}^{-1}\text{)}. \end{aligned}$$

*Results.*

Table 1: Acetonitrile anharmonic fundamental frequencies, its numbers in parentheses are the positions of the eigenvalues counting from 0.

Assignment (Component)	Frequency (number)	Relative Residue	Absolute error Ref-Here	Experimental values
$\nu_0(0.97)$	9837.43(0)	0.0015	?	
$\nu_{11}(0.97)$	361.11(1)	0.0034	-0.1198	362 [51]
$\nu_{12}(0.97)$	361.17(2)	0.0039	-0.1779	
$\nu_4(0.95)$	900.76(6)	0.0032	-0.1001	920 [51]
$\nu_9(0.97)$	1034.25(7)	0.0033	-0.1202	1041 [51]
$\nu_{10}(0.97)$	1034.35(8)	0.0041	-0.2229	
$\nu_3(0.74), \nu_9 + \nu_{11}(0.44),$ $\nu_{10} + \nu_{12}(0.44)$	1389.17(15)	0.0038	-0.1980	1385 [51]
$\nu_3(0.62), \nu_9 + \nu_{11}(0.52),$ $\nu_{10} + \nu_{12}(0.52)$	1397.94(17)	0.0042	-0.2485	1402 [52]
$\nu_7(0.97)$	1483.33(20)	0.0031	-0.1034	1450 [52]
$\nu_8(0.97)$	1483.46(21)	0.0040	-0.2341	
$\nu_2(0.90)$	2250.94(70)	0.0037	-0.1923	2267 [51]
$\nu_1(0.60), 2\nu_7(0.51),$ $2\nu_8(0.51)$	2947.42(187)	0.0043	-0.3665	
$\nu_1(0.67), 2\nu_7(0.45),$ $2\nu_8(0.45)$	2981.01(193)	0.0037	-0.2316	2954 [51]
$\nu_5(0.93)$	3049.12(218)	0.0039	?	3009 [51]
$\nu_6(0.93)$	3049.16(219)	0.0040	?	

*Performances summary.*Table 2: Performances summary on Acetonitrile fundamental targets. The CPU wall time is in second with the total number of iterations showed in parenthesis. EtaComp=3(cf $\diamond$ 11), NAdd=50 (cf  $\diamond$ 10), ThrKX=10<sup>-15</sup>.

Final size of $B$	Final size of $B_s$	Final $nnz(H_B)$	Final $nnz(H_{SB})$	CPU Wall time(s) (Iterations)	Memory usage (Mo)
6169	590840	247662	3213078	352(4)	115.6

The complexity of this problem comes from the fact that the fundamental anharmonic frequencies are dispatched far away from the extremity of the spectrum. In the recent work of Ondunlami et al[14], the A-VCI didn't catch up all of them when computing the first 238 eigenvalues. Their best declared results parallelized on a 24-core Intel Xeon E5-

2680 processors running at 2.8 GHz shows a maximal absolute error (relatively to the same reference) equal to  $0.305 \text{ cm}^{-1}$  on the first 121 eigenvalues with a time equal 5637 seconds and a final basis set of 86 238 elements. In the HRRBPM of Thomas et Carrington[24] all the anharmonic frequencies before  $2209 \text{ cm}^{-1}$  were computed with a error lower than  $0.38 \text{ cm}^{-1}$ , 3.2 hours cpu time and 115.6 MB on a single Intel Core i7-4770 processor running at 3.4 GHz. In here all the fundamentals are computed with 115.6 megabytes memory for a maximal error lower than  $0.37 \text{ cm}^{-1}$  and a time of 5 minutes 52 seconds on a single CPU running at 2.70GHz.

### 6.2. C<sub>2</sub>H<sub>4</sub> : Ethylene

The potential energy surface is originally the sixth order curvilinear symmetry-adapted coordinates of Delahaye & al [53] transformed into the sextic normal coordinate force field with PyPES [54]. In there work, point symmetry group D<sub>2h</sub> of C<sub>2</sub>H<sub>4</sub>, has been exploited to divide VCI matrix into 8 symmetry blocks of respective dimension 106 889, 101 265, 100 366, 105 518, 105 643, 101 145, 101 255, 105 697. No symmetry assumption is applied in here. Reference calculation is taken from PyPES benchmarks[54] and consists on the 8-degrees excitation space ( $\sum_{n=1}^{12} b_n \leq 8$ ) containing 125 270 basis functions.

*Harmonic Frequencies and derivative orders of the PES.*

$$\begin{aligned} \nu_1 : 825.0, & \quad \nu_2 : 950.2, & \quad \nu_3 : 966.4, \\ \nu_4 : 1050.8, & \quad \nu_5 : 1246.8, & \quad \nu_6 : 1369.4, \\ \nu_7 : 1478.5, & \quad \nu_8 : 1672.6, & \quad \nu_9 : 3140.9, \\ \nu_{10} : 3156.8, & \quad \nu_{11} : 3222.9, & \quad \nu_{12} : 3248.7. \end{aligned}$$

Derivative order	2	3	4	5	6
Number of terms	45	147	290	642	1732

The second derivatives contain the harmonic and non null crossed terms.

*Results.*

Table 3: Ethylene anharmonic fundamental frequencies, its numbers indicated in brackets are the position of the corresponding eigenvalue.

Assignment (Component)	Frequency (number)	Relative Residue	Absolute error Ref-Here	Experimental values [55]
$\nu_0(0.98)$	11017.05(0)	0.0035	?	
$\nu_1(0.98)$	823.71(1)	0.0053	-0.0116	825.93
$\nu_2(0.98)$	935.42(2)	0.0056	-0.1234	939.86
$\nu_3(0.98)$	950.85(3)	0.0057	-0.1523	948.9
$\nu_4(0.98)$	1026.03(4)	0.0055	-0.1310	1025.69
$\nu_5(0.98)$	1225.05(5)	0.0056	-0.1468	1222
$\nu_6(0.97)$	1342.91(6)	0.0045	-0.0083	1343.54
$\nu_7(0.98)$	1442.02(7)	0.0055	-0.2236	1442.47

$\nu_8(0.90)$	1625.48(8)	0.0053	-0.0774	1625.4
$\nu_9(0.86)$	2985.80(60)	0.0060	-0.2960	2983.937
$\nu_{10}(0.80)$	3019.05(63)	0.0056	0.1493	3022.03
$\nu_{11}(0.91)$	3079.66(65)	0.0056	-0.2618	3083.36
$\nu_{12}(0.93)$	3101.60(70)	0.0054	-0.2998	3104.89

*Performance summary.*

Table 4: Performances summary on ethylene fundamental targets. The CPU wall time is in second with the total number of iterations indicated in parentheses. EtaComp=2(cf◊11), NAdd=100 (cf ◊10).

Final size of $B$	Final size of $B_s$	Final $nnz(H_B)$	Final $nnz(H_{SB})$	CPU Wall time(s) (Iterations)	Memory usage (Mo)
19498	3674267	7491787	66159422	2238 (6)	656

The CPU wall time is equal to 37 minutes and 18 seconds. Even if the number of normal coordinates is the same as in the previous system, the memory requirement and the CPU time are significantly increased due to the higher number of terms in the PES. Note that an error lower than 1 cm<sup>-1</sup> is already observable from iteration 5 with 13039 elements of basis set.

### 6.3. C<sub>2</sub>H<sub>4</sub>O : Ethylene oxide

Here is computed the fundamentals of a 15-d Hamiltonian system where the PES is the one of Bégué and al [33] calculated at the CCSD(T)/cc-pVTZ level for harmonic frequencies and B3LYP/6-31+G(d,p) for the other terms. Apart from the harmonic force constants, the PES contains 180 cubic and 445 quartic terms. The reference results are taken from the A-VCI[14] where final basis set contains 7 118 214 elements.

*Harmonic frequency in cm<sup>-1</sup>.*

$$\begin{aligned}
&\nu_1 : 3117.9, \nu_2 : 1549.1, \nu_3 : 1300.1, \nu_4 : 1157.9, \nu_5 : 899.6, \\
&\nu_6 : 3196.6, \nu_7 : 1176.0, \nu_8 : 1052.2, \nu_9 : 3109.5, \nu_{10} : 1512.3, \\
&\nu_{11} : 1156.8, \nu_{12} : 850.2, \nu_{13} : 3211.3, \nu_{14} : 1175.0, \nu_{15} : 815.5,
\end{aligned}$$

Correspondence with the ones listed in [14]

$$\begin{aligned}
&\nu_1 \equiv \omega_{13}, \nu_2 \equiv \omega_{11}, \nu_3 \equiv \omega_9, \nu_4 \equiv \omega_6, \nu_5 \equiv \omega_3, \\
&\nu_6 \equiv \omega_{14}, \nu_7 \equiv \omega_8, \nu_8 \equiv \omega_4, \nu_9 \equiv \omega_{12}, \nu_{10} \equiv \omega_{10}, \\
&\nu_{11} \equiv \omega_5, \nu_{12} \equiv \omega_2, \nu_{13} \equiv \omega_{15}, \nu_{14} \equiv \omega_7, \nu_{15} \equiv \omega_1.
\end{aligned}$$

*Results.*

Table 5: Ethylene oxide anharmonic fundamental frequencies for five groups of targets separated by an horizontal bar. For comparison purpose, the zero point energy has the same value than in [14] (i.e 12461.473 cm<sup>-1</sup>). The numbers of the frequencies are the positions of the corresponding eigenvalues counting from 0.

Assignment (Component)	Frequency (number)	Relative Residue	Error Ref-Here
$\nu_0(0.98)$	12461.47(0)	0.0034	0.0010
$\nu_{15}(0.97)$	792.96(1)	0.0045	-0.3305
$\nu_{12}(0.96)$	822.19(2)	0.0041	-0.2802
$\nu_5(0.97)$	878.51(3)	0.0038	-0.2350
$\nu_8(0.97)$	1017.47(4)	0.0044	-0.3319
$\nu_4(0.96)$	1121.47(5)	0.0042	-0.2993
$\nu_{11}(0.97)$	1123.92(6)	0.0042	-0.2962
$\nu_{14}(0.97)$	1146.03(7)	0.0042	-0.3077
$\nu_7(0.97)$	1148.19(8)	0.0037	-0.2299
$\nu_3(0.94)$	1271.17(9)	0.0047	-0.3909
$\nu_{10}(0.97)$	1467.58(10)	0.0037	-0.2402
$\nu_2(0.94)$	1495.49(11)	0.0042	-0.3334
$\nu_9(0.64), \nu_2 + \nu_{10}(0.52)$	2906.77(95)	0.0041	-0.4562
$\nu_9(0.52), \nu_2 + \nu_{10}(0.63)$	2989.70(111)	0.0044	-0.4552
$\nu_1(0.45), 2\nu_{10}(0.62), \nu_8 + \nu_{11} + \nu_{15}(0.44)$	2916.94(99)	0.0040	-0.2640
$\nu_1(0.52), 2\nu_{10}(0.62)$	2952.86(103)	0.0047	-0.4654
$\nu_6(0.85)$	3025.71(116)	0.0036	-0.3801
$\nu_{13}(0.80)$	3037.31(118)	0.0038	-0.411

*Performances summary.*

Table 6: Performances summary on ethylene oxide. Each tracked state(s) are separated by an horizontal bar. The CPU wall time is in second with the total number of iterations showed in parenthesis. NAdd=100 for all targets except for  $\nu_{13}$  and  $\nu_6$  where NAdd=300. EtaComp=3 and ThrKX= 10<sup>-15</sup> in any case.

Target(s)	Final size of $B$	Final size of $B_s$	Final $nnz(H_B)$	Final $nnz(H_{SB})$	Wall time(s) (Iterations)	Memory usage (Mo)
$\nu_{15}, \nu_{12}$ $\nu_5, \nu_8,$ $\nu_4, \nu_{11},$ $\nu_{14}, \nu_7,$ $\nu_3, \nu_{10}$	83346	11182617	5396776	80959806	11823(7)	1197.3
$\nu_9$	133128	11840777	13352241	126500214	21507(16)	1383.6
$\nu_1$	121180	11788217	11420147	117896340	21430 (16)	1347.2
$\nu_6$	84554	8374499	7341322	79908192	11299(13)	922.5
$\nu_{13}$	119740	11137472	11041759	113491483	22014(14)	1291.3

Total	541948	54323582	48647958	518756035	88073	6141.9
-------	--------	----------	----------	-----------	-------	--------

The total CPU wall time is then 1 day 27 minutes and 53 seconds, and the total memory usage is 6.142 Gigabytes. The maximal absolute error on eigenvalues does not go over 0.47 cm<sup>-1</sup> giving a maximal relative error lower than  $4 * 10^{-5}$ . As a matter of comparison the reference calculation were done with a total memory usage of 128 gigabytes and 3 days time on a 24 cores computer meaning that the CPU wall time is much larger. Less accurate results (4-5 cm<sup>-1</sup> error on higher frequencies) are achieved with HRRBPM [23] with a memory usage of 14.6 gigabytes and a CPU wall time of 8.7 days.

#### 6.4. C<sub>3</sub>H<sub>3</sub>NO : Oxazole

The PES was constructed using the the adaptive density-guided approach (ADGA) introduced by Sparta et al [56, 57, 58]. The force constants, equilibrium geometry and normal coordinates were extracted from Madsen et al [59]. In their work they describe the construction of oxazole PES at CCSD(T)/cc-pVTZ level for the one-mode part and MP2/cc-pVTZ for the two-mode part. The three-mode part is extrapolated from the two-mode surface using MP2/cc-pVTZ gradients. The number of terms is 146 for the one mode, 4786 for the two modes and 4335 for the three modes couplings.

*Harmonic frequencies.*

$$\begin{aligned}
&\nu_1 : 603.8\text{cm}^{-1}, \nu_2 : 644.2\text{cm}^{-1}, \nu_3 : 748.9\text{cm}^{-1}, \\
&\nu_4 : 832.2\text{cm}^{-1}, \nu_5 : 849.7\text{cm}^{-1}, \nu_6 : 901.8\text{cm}^{-1}, \\
&\nu_7 : 913.8\text{cm}^{-1}, \nu_8 : 1071.8\text{cm}^{-1}, \nu_9 : 1109.2\text{cm}^{-1}, \\
&\nu_{10} : 1106.9\text{cm}^{-1}, \nu_{11} : 1181.2\text{cm}^{-1}, \nu_{12} : 1263.5\text{cm}^{-1}, \\
&\nu_{13} : 1348.3\text{cm}^{-1}, \nu_{14} : 1533.4\text{cm}^{-1}, \nu_{15} : 1570.4\text{cm}^{-1}, \\
&\nu_{16} : 3275.6\text{cm}^{-1}, \nu_{17} : 3286.0\text{cm}^{-1}, \nu_{18} : 3309.5\text{cm}^{-1}.
\end{aligned}$$

Table 7: Oxazole anharmonic frequencies for fundamental targets separated by an horizontal bar.

Eigenvalue number	Frequency	Relative Residue	Assignment (component)	Experimental values [60]
0	12559.9032	0.0046	$\nu_0(0.95)$	12457.5
1	592.8494	0.0074	$\nu_1(0.95)$	607 ( $A''$ )
2	631.2242	0.0075	$\nu_2(0.95)$	647 ( $A''$ )
3	727.3345	0.0072	$\nu_3(0.93)$	750 ( $A''$ )
4	795.0331	0.0077	$\nu_4(0.90)$	830( $A''$ )
5	827.7750	0.0070	$\nu_5(0.90)$	854 ( $A''$ )
6	884.4607	0.0074	$\nu_6(0.92)$	899 ( $A''$ )
7	894.7569	0.0071	$\nu_7(0.91)$	907 ( $A''$ )
8	1031.4634	0.0064	$\nu_8(0.90)$	1046 ( $A'$ )

9	1063.2165	0.0067	$\nu_{10}(0.82)$	1078 ( $A'$ )
10	1075.3896	0.0069	$\nu_9(0.82)$	1086 ( $A'$ )
11	1123.9852	0.0066	$\nu_{11}(0.89)$	1139 ( $A'$ )
13	1217.7686	0.0066	$\nu_{12}(0.94)$	1252 ( $A'$ )
16	1302.4880	0.0078	$\nu_{13}(0.90)$	1324 ( $A'$ )
24	1481.3806	0.0079	$\nu_{14}(0.86)$	1504 ( $A'$ )
27	1521.2312	0.0074	$\nu_{15}(0.91)$	1537 ( $A'$ )
585	3125.9955	0.0086	$\nu_{16}(0.76)$	3141 ( $A'$ )
608	3146.9603	0.0098	$\nu_{17}(0.82)$	3144 ( $A'$ )
618	3159.5778	0.0087	$\nu_{18}(0.81)$	3170 ( $A'$ )

### Performances summary.

Table 8: Performances summary on Oxazole molecule. Each screened states are separated by an horizontal bar. The CPU wall time is in second with the total number of iterations indicated in parenthesis. In both cases, NAdd=200, EtaComp=3, Freq0Max=20000, ThrKX=1. EpsRez=(0.008,0.01) respectively for each group.

Target(s)	Final size of $B$	Final size of $B_s$	Final $nnz(H_B)$	Final $nnz(H_{SB})$	CPU Wall time(s) (Iterations)	Memory usage (Mo)
$\nu_0 - \nu_{15}$	145820	27468841	87344774	299972877	74665(10)	5424.3
$\nu_{16} - \nu_{18}$	143916	29724836	60788880	285617698	170658(19)	4933
Total	289736	57193677	148133654	585590575	245323	10357.3

The total cpu wall time is 2 days 20 hours 8 minutes 43 seconds. A significantly bigger computational time for the second group of targets  $\nu_{16}, \nu_{17}, \nu_{18}$  principally comes from the additional number of iterations. To a lesser extent, there is also the constraint to calculate the eigenvalues starting from the extremities of the spectrum as in Lanczos algorithm. A Davidson eigensolver [40, 41] or polynomial filtering techniques [61, 62, 63, 42] could be more adapted.

## 7. Input parameters

*Presentation.* The key words are case insensitive and should start at the beginning of each line of the input file. No specific order of apparition is required. Comments are indicated with the symbols '/' or '@'. The potential energy file should also be present in the directory where is executed DVCi (cf PESType=3). A minimal input file looks like :

```

NMode 6           / Number of normal coordinates
PESType 1         / PES type of coefficients
OutName N2H2      / Extension name for output files.
PESName N2H2_PES.in / Name of the file of the PES.
Memory 80         / Maximal allocated memory in mega octets.
```



*Detailed list of the key words.*

◇1 NMode

Designs the number of mass weighted normal coordinates.

NMode =  $3 * N_A - 6$ . Where  $N_A$  stands for the number of atoms of the molecule.

◇2 DoRot

Indicate if rotational corrections should be added to the Hamiltonian.

In the PES file, the section to be filled should start and end with words

COORDINATES ENDCOOR and each different field must be separated by an exclamation mark.

- If = 0  
then  $\mathcal{H} = \mathcal{H}_{vib}(\mathbf{q})(2)$ . If no section COORDINATES is written in the PES file then DoRot is automatically set to 0.
- If > 0 and even  
then  $\mathcal{H} = \mathcal{H}_{vib}(\mathbf{q}) + \mathcal{H}_{rot}(\mathbf{q})$  (3). The equilibrium geometry in Bohr, atomic masses in electron rest mass, and normal coordinate eigenvectors should be indicated in the section COORDINATES like in the following example where the values has been extracted from Madsen & al[59].

COORDINATES

! Equilibrium geometry (bohr)

+1.188227703663817e+00	+1.871902685046820e+00	+5.700041831365050e-16 /C
-1.366889823622856e+00	+1.691283181574289e+00	+5.673619333132633e-16 /C
-1.998417882238274e+00	-8.244795338089052e-01	-2.155218936400323e-16 /O
+2.684304163103964e-01	-2.022882150581233e+00	-6.432553562388595e-16 /C
+2.229151382700317e+00	-5.583543727400242e-01	-2.215072662048790e-16 /N
+2.352687276664275e+00	+3.539113911722189e+00	+8.792950593981537e-16 /H
+2.239397718366979e-01	-4.057044206724038e+00	-1.063440276037351e-15 /H
-2.901750851724292e+00	+3.020857567289711e+00	+7.990414399553623e-16 /H

! Masses(me)

21874.6618172 /C
21874.6618172 /C
29156.9456749 /O
21874.6618172 /C
25526.0423547 /N
1837.15264562 /H
1837.15264562 /H
1837.15264562 /H

! Mode0: X

Y

Z

+3.231683769903235e-17	+6.115580033151089e-17	+4.025196381179206e-01
+2.743657157380967e-17	+4.590882496215692e-17	-5.186966592183442e-01
+1.991732529600559e-17	+5.710133055001405e-17	+5.169293296045854e-01
-7.257396408556236e-17	+7.324385585074392e-17	-1.938605777276555e-01
-9.053243042419710e-17	+2.384482767132475e-18	-1.671562436764069e-01
+3.416768858945461e-17	-1.546457080242301e-17	+2.188043144086224e-01
-1.950243964435382e-17	-1.457271158617074e-16	-2.002715238252209e-01

```

-5.645471816646615e-17 -3.099000577024135e-17 -3.849787518532105e-01
! Mode1
-4.471096934999722e-17 -2.133222140486714e-16 +3.245666277578175e-01
-7.195147510789707e-18 -1.776354539431898e-16 -3.130286914609319e-02
-1.161675776799252e-16 -2.465033916085211e-16 -2.713644979495499e-01
-7.304974442800625e-17 +1.130401139043093e-16 +4.555118862530116e-01
+1.980788418374012e-17 -1.509309233138445e-16 -5.878836462126088e-01
+3.871182981501824e-17 +3.894914167531566e-17 +2.788008333658181e-01
-1.775227773837671e-18 +1.687750614789954e-16 +4.342956786268967e-01
-9.998627049911543e-18 -1.727718015508009e-16 -2.443444074954252e-02
! Mode2
-1.362124303938439e-16 +1.648858093263572e-17 +7.985191937307874e-02
+9.223533160690561e-17 -1.520416449194739e-16 +4.427598081406853e-01
-3.106447728140722e-16 -2.024361724665231e-16 -7.334352062590022e-02
+3.248565849336364e-17 +2.181702164863730e-17 -1.148459584461199e-01
+3.231452029113849e-17 +1.995473253650141e-16 -3.854511885546462e-02
-1.704399688795554e-16 +6.014500621640033e-17 -3.252722203195085e-01
+1.114007103292430e-16 +8.683057684777304e-18 +1.582562080820018e-01
+3.567043855971248e-16 +2.236379927154288e-16 -8.041677585822095e-01
! Mode3
-8.530348729807672e-17 +5.064459244391696e-16 +2.099386275547177e-01
-2.860512641462178e-17 +2.849372440086422e-16 -1.949978814638483e-02
+9.671777362477948e-16 +3.228924378351529e-16 -1.437578533863218e-01
-3.535112014919970e-16 -6.871677835537049e-16 +4.816962590153900e-01
-6.328554966609436e-16 +1.463685147682383e-16 -1.649351614432971e-01
+3.642789510512844e-16 +2.417462616289285e-18 -4.324245782423197e-01
-5.058760423068888e-17 -3.092952891160942e-16 -6.990373338704934e-01
-6.384663222086937e-16 -5.923791764375996e-16 -3.242248209522854e-04
. . .
. . .
. . .
ENDCOORD

```

In accordance with the Wilson method[15], the normal coordinates are built from a set of  $3N_A - 6$  eigenvectors  $(Q_{i\alpha})_{i \in \{1, \dots, N_A\}, \alpha \in \{1, 2, 3\}}$  of the Hessian matrix

$$\left( \frac{1}{\sqrt{m_i} \sqrt{m_j}} \frac{\partial^2 \mathcal{U}_K}{\partial x_{i\alpha} \partial x_{j\beta}} \right)_{(i,j) \in \{1, \dots, N_A\}^2, (\alpha, \beta) \in \{1, 2, 3\}^2} \quad (31)$$

derived from mass weighted displacements ( $m_i$  mass of nucleus  $i$ )

$$\Delta y_{i\alpha} = \sqrt{m_i} (x_{i\alpha} - Xeq_{i\alpha}), \quad i \in \{1, \dots, N_A\}, \quad \alpha \in \{1, 2, 3\},$$

at the equilibrium geometry  $\mathbf{Xeq} \in \mathbb{R}^{3N_A}$ . The corresponding eigenvalues are the harmonic frequencies.

- If  $> 0$  and odd

then  $\mathcal{H} = \mathcal{H}_{vib}(\mathbf{q}) + \mathcal{H}_{rot}(\mathbf{q})$  and the non mass weighted normal coordinate eigenvectors

$$(Q_{i\alpha} / \sqrt{m_i})_{i \in \{1, \dots, N_A\}, \alpha \in \{1, 2, 3\}}$$

should be written instead of the classical ones.

### ◇3 PESType

Format of data's for the SOP PES.

- If PESType = 0 then the force constants  $K_c$  are expressed for dimensionless normal coordinates ( $q_n = Q_n/\sqrt{\nu_n}$ ) and supplied in  $\text{cm}^{-1}$ . Regarding the format of the PES it starts and ends with the key words FORCEFIELD ENDF. For NM normal coordinates, NM integers should be shown before the actual value of the force constant:

```
FORCEFIELD
2 0 0 0 0 0 , 664.213550943134237
4 0 0 0 0 0 , 4.335282791860437
6 0 0 0 0 0 , -0.471897107116644
0 2 0 0 0 0 , 675.140549094012272
0 4 0 0 0 0 , 7.072599090662686
2 0 1 0 1 0 , -12.115726349955748
2 0 1 0 2 0 , 1.215724777835937
2 0 1 0 3 0 , 0.161839021650279
2 0 1 0 1 2 , 0.800398391535361
2 0 1 0 0 2 , 0.662431742378760
2 0 0 1 0 0 , 11.492641524810505
2 0 0 2 0 0 , 0.117652856544075
2 0 0 3 0 0 , -0.819356186630299
. . . . . , .
. . . . . , .
. . . . . , .
```

ENDF

Here it means that the first term is the one in front of  $q_1^2$  namely  $\nu_1/2$ , and for the last showed line 2 0 0 3 0 0, we are dealing with the force constant  $K_{2,0,0,3,0,0} = -0.819356186630299 \text{ cm}^{-1}$  in agreement with the monomial  $q_1^2 q_4^3$ .

- If PESType  $\geq 1$  then the derivatives are provided in atomic units, and for a Taylor expansion around the equilibrium position we have the correspondence

$$K_{c_1 \dots c_{\text{NM}}} = \frac{1}{c_1! c_2! \dots c_{\text{NM}}! \prod_{n=1}^{\text{NM}} \sqrt{\nu_n}^{c_n}} \frac{\partial^{c_1, \dots, c_{\text{NM}}} \mathcal{U}_{\mathcal{K}}}{\partial Q_1^{c_1} \dots \partial Q_{\text{NM}}^{c_{\text{NM}}}} * \text{HartreeToCM}, \quad (32)$$

where HartreeToCM is a converting factor from Hartree to  $\text{cm}^{-1}$  and  $\nu_n = \sqrt{\frac{\partial^2 \mathcal{U}_{\mathcal{K}}}{\partial Q_n^2}}$ . The format of PES file is the same as the one supplied by the PyPES[64] library namely:

```
FORCEFIELD
[0,0,0,0,3,4 , 1.86861408859e-11],
[0,0,0,0,4 , -3.47804520495e-09],
[0,0,0,0,4,4 , 3.2867316136e-10],
```

```

[0,0,0,0,5,5 , 3.53998391655e-10] ,
[0,0,1,1 , 6.40569999931e-09] ,
[0,0,1,1,1,1 , -3.21892597605e-11] ,
[0,0,1,1,1,5 , -4.73206802979e-12] ,
[0,0,1,1,2 , 1.16300104731e-11] ,
[0,0,1,1,2,2 , -5.81559507099e-12] ,
[0,0,1,1,2,3 , 9.06462603669e-12] ,
[0,0,1,1,2,4 , -2.30405700341e-12] ,
[0,0,1,1,3 , -1.89925822496e-10] ,
[0,0,1,1,3,3 , -1.6879336773e-11] ,
[0,0,1,1,3,4 , 1.08730049094e-11] ,
. . . . . , .
. . . . . .
. . . . . .
ENDFF
    
```

The repetitions are to be associated with a derivative order when coordinates are numbered starting from zero. For example the first line means

$$\frac{\partial^4}{\partial Q_1^4} \frac{\partial}{\partial Q_4} \frac{\partial}{\partial Q_5} \mathcal{U}_{\mathcal{K}} = 1.86861408859 * 10^{-11} \text{ a.u.}$$

#### ◊4 ThrPES

Threshold for PES force constants or derivatives. Default value is  $10^{-20}$ .

#### ◊5 EpsRez

For eigenvectors of  $H_{\text{b}}$  (12)  $\mathbf{X}_{\ell}$ ,  $\ell \in \{\text{Targ}\}$ , it is the maximal accepted relative residue

$$\max_{\ell \in \{\text{Targ}\}} \frac{\|H_{\text{sb}} \mathbf{X}_{\ell}\|}{E_{\ell}}$$

before the algorithm stop. They are built from the MVPs (26). The default value is  $6 * 10^{-3}$ .

#### ◊6 ThrMat

Minimal allowed absolute value of coefficients of  $H_{\text{b}}$ . Default is  $10^{-20}$ . The matrix coefficients are computed with the full operator  $\mathcal{H} = \mathcal{H}_{\text{vib}} + \mathcal{H}_{\text{rot}}$ . If DoRot=0, only  $\mathcal{H}_{\text{vib}}$  will be considered.

#### ◊7 MaxQLevel

This is the common maximal quantum level for the whole space  $B \oplus B_{\text{s}}$ . It increases together with distances between nucleus in motion and then should carefully be chosen conforming to the spacial region where the potential energy is still correctly represented and has no more than one local minimum. Each level on normal coordinate  $n$  will be adjusted with Freq0Max◊8 as followed:

$$\text{MaxQLevel}(n) = \lfloor \frac{\text{Freq0Max}}{\nu_n} \rfloor$$

◊8 Freq0Max

Maximal allowed harmonic frequency in  $B \oplus B_s$ . Default value is 30000.

◊9 ThrKX

In operator  $\mathcal{H}^*(22)$  only the increments  $\mathbf{e} \in \text{LFF}^*$  verifying <sup>13</sup>

$$\sum_{\mathbf{c} \in \{\text{LFK}(\mathbf{e})\}} |K_{\mathbf{c}}| + \sum_{(i,j,k,l) \in \{\text{LCI}(\mathbf{e})\}} |Z_{ijkl}| > \text{ThrKX},$$

will be acceptable to generate the secondary space  $B_s$  and residual vectors. It should be strictly positive. Default value is 1.

◊10 NAdd

It is the minimal number of basis functions per non converged target states to be added for next iteration. They are chosen from maximal components (in absolute value) of the residual vectors (26)

$$\{(H_{\text{sb}}\mathbf{X}_{\ell})_{\mathbf{s}}, \ell \in \text{NotConv}, \mathbf{s} \in B_s\}, \quad (33)$$

where NotConv designs the set of non converged eigenpairs

$$\text{NotConv} = \{\ell \in \{\text{Targ}\}, \frac{\|H_{\text{sb}}\mathbf{X}_{\ell}\|}{|E_{\ell}|} > \text{EpsRez}\}. \quad (34)$$

To accelerate convergence, NAdd is multiplied by  $i + 1$ , where  $i$  designates the iteration number.

◊11 EtaComp

The new added basis functions are selected from all the components of the residual vectors (26) greater than

$$\frac{1}{\text{EtaComp} * \text{NNotConv}} \sum_{\ell \in \text{NotConv}} \|H_{\text{sb}}\mathbf{X}_{\ell}\|_{\infty}$$

where NotConv (34) and NNotConv respectively stand for the set of non converged tracked eigenpairs of  $H_b$  and its cardinal. EtaComp should be greater than one, this guaranty that at list one component per non converged residual vector will be picked up.

◊12 MaxAdd

Limit for number of basis functions to add at each iteration. Default value is 1000.

◊13 TargetState

It indicates the maximal component of the eigenvectors of  $H_b$  that should be assigned to the targets matching with a multi-index array (cf figure 1). Except for  $\mathbf{0}_{\text{NM}}$  symbolized by 0(1), only its non zeros should be indicated with the characters  $d(n)$  separated by

---

<sup>13</sup>  $Z_{ijkl}$  is defined equation (3)

a comma, where  $d$  stands for the degree of the Hermite function and  $n$  the normal coordinate. Alternatively TargetState can be followed by the label 'Fund' if the targets are the fundamentals and the ground state i.e  $1(n)$ ,  $n = \{1, \dots, NM\}$  and  $0(1)$ . If  $0(1)$  is not part of the targets then the zero point energy should be provided in  $\text{cm}^{-1}$  via the parameter GroundState  $\diamond 16$ .

$\diamond 14$  ThrCoor

Any eigenvector coordinate of  $H_b$  bigger (in absolute value) than this threshold and assigned to one of the targets, will be integrated into the iterative process and have its residual vector (26) calculated. In output, will be showed only the assignments of components larger than ThrCoor.

$\diamond 15$  AddTarget

Sometimes, different eigenvectors point to the same maximal components. Then the actual number of targets is bigger than the one specified by the user. So it allocates additional arrays to correct this increasing. The default value is 2.

$\diamond 16$  GroundState

Zero point energy required when it is not calculated (i.e not part of the targets). It can also be adopted as reference to printout the anharmonic frequencies.

$\diamond 17$  MinFreq, MaxFreq

Frequencies in  $\text{cm}^{-1}$  specified to make converge all the eigenvalues within the interval

$$[\text{MinFreq} + \text{GroundState}, \text{MaxFreq} + \text{GroundState}]$$

when no target is indicated. If MinFreq is greater than zero, the value of GroundState should be supplied, else it will be computed. If MaxFreq is not given in input, then it will be set to the maximal harmonic frequency of tracked states for the initial subspace construction and to Freq0Max $\diamond 8$  afterwards. Default values are  $[-100, 4000]$ .

$\diamond 18$  Kappa

Elongation factor for the construction of the initial subspace estimating the global deviation between converged eigenvalues and associated harmonic energies<sup>14</sup>. Its default value is 1.2 but it is automatically augmented to 1.3 when maximal target frequency is greater than  $3000 \text{ cm}^{-1}$ .

$\diamond 19$  Memory

Total allocated memory in megabytes for the slots occupied by the eigensolver, the matrices ( $H_b$ ,  $H_{sb}$ ), the multi indexes ( $B$ ,  $B_s$ ), the PES, the local force fields and corresponding positive increments  $\{\text{LFF}(\mathbf{e}), \mathbf{e} \in \text{LFF}^*\}$ (19). This value will be used to set up the upper limit of basis functions SizeActMax that is appraised taking into account the array shrinkage factors KNREZ $\diamond 21$ , KNNZ $\diamond 20$  and KNZREZ $\diamond 22$ . The total effective usage of memory is generally slightly bigger (about 5%) due to local array allocation.

---

<sup>14</sup>Association has to do with an harmonic energy and an eigenvalue having the same position  $\ell$  in (27).

◇20 KNNZ

Sparsity factor for  $H_b$ . The maximal number of non zero coefficients in  $H_b$  will be

$$\text{NNZActMax} = \text{KNNZ} * \text{SizeActMax} * \text{NXDualHTrunc}.$$

Where NXDualHTrunc is the number of excitations in  $\mathcal{H}^*$  after truncation with ThrKX◇9. Default value is 0.03. Should be in  $]0,1]$ .

◇21 KNREZ

Multiplicative factor of the maximal size of the residual space

$$\text{SizeRezMax} = \text{KNREZ} * \text{SizeActMax} * (\text{NXDualHTruncPos} - 1).$$

Where NXDualHTruncPos-1 is the number of raising excitations in operator  $\mathcal{H}^*$  after truncation with ThrKX◇9 (the first excitation being zero). Default value is 0.2. Should be in  $]0,1]$ .

◇22 KNZREZ

Shrinking factor for the maximal number of pointers on the non zeros of  $H_{sb}$

$$\text{NNZRezMax} = \text{KNZREZ} * \text{SizeActMax} * \text{NXDualHTrunc}.$$

Where NXDualHTrunc is the number of excitations in operator  $\mathcal{H}^*$  after truncation with ThrKX◇9. KNZREZ will be settled to zero when DoGraph=0.

◇23 DoGraph

- If= 0 The MVPs  $H_{sb}\mathbf{X}_\ell$ ,  $\ell \in \{\text{Targ}\}$  are fully calculated by browsing  $\mathcal{H}^*(B)$  instead of  $\mathcal{H}^*(A)$  in (23).
- If> 0 The row indexes and column pointers of the coupled elements of  $H_{sb}$  are stored in CSC<sup>15</sup> format to complete  $\widehat{H_{sb}}\mathbf{X}_\ell$ ,  $\ell \in \{\text{Targ}\}$  in (23) for the missing entries (26). This option necessary increases memory requirement. The expense is about 40% greater in memory and 40% smaller in CPU time compared with DoGraph=0.

Default value is 1.

◇24 MaxEV

Maximum eigenvalues to be computed when counted from the smallest one. This number is adjusted to the size of the initial subspace minus one when it is actually larger than the latter. The eigensolver uses the Mode 1 and option WHICH='LM' of ARPACK subroutine DSAUPD. The greatest magnitude eigenvalues are computed on the shifted matrix

$$H'_b = H_b - \text{Shift} * I_b, \text{ Shift} = \sum_{n=1}^{\text{NM}} \left[ \frac{1}{2} + \text{MaxQLevel}(n) * \nu_n \right],$$

where  $I_b$  designates the identity matrix. Default value is 30.

---

<sup>15</sup>Compressed Sparse Column

## ◇25 DeltaNev

Reduce the number of wanted eigenvalues as

$$\text{MaxEV} = \text{Min}(\text{MaxEV}, \text{MaxScreen} + \text{DeltaNev}),$$

where MaxScreen is the higher position of the targets that tends to decrease with iterations. The purpose is to lighten the computational effort on the eigensolver. Default value is 1000.

## ◇26 MAXNCV

This is the maximal number of Lanczos basis vectors generated at each iteration in DSAUPD subroutine. Default value is 2\*MaxEv◇24.

## ◇27 Tol

Stopping criterion for the relative accuracy of the Ritz values in DSAUPD subroutine. Default value is  $10^{-8}$ .

## ◇28 RefName

Name of the input text file holding a floating point number at the beginning of each line to compare with the final results. The printed error is the difference between one of this value and the closest calculated frequency. Then it should manually be corrected when this correspondence is not true.

## ◇29 Verbose

When non equal to zero, it allows to print additional informations such as intermediate CPU times, position of targets in initial space, the center of mass, the moment of inertia and the characteristics of the dual operator.

## ◇30 OutName

Extension for output file names created when PrintOut  $\neq$  0 (cf◇31).

## ◇31 PrintOut

- If  $\neq$  0: All the final basis set and the components of screened states with coinciding eigenvalue number in parentheses will be printed in the file OutName-FinalBasis.out. The eigenvectors in binary format are saved in the file OutName-Vectors.bin. These informations could be employed to compute infrared or Raman intensities in the final basis set.
- If = 0: No additional output file is created.

Default value is 0.

## 8. Conclusion

In this work has been presented a new algorithm to track specific states of molecular spectrum approaching the variational limit with a minimal usage of memory. Harmonic oscillator properties together with second quantization formulation were adopted to build



a novel assemblage of structures available for dynamic subspace enrichment. The resulting code has shown challenging performances and could obviously be applied for bigger systems than the ones studied in here. For future improvements, one could use the final basis set and wave functions to compute infrared and Raman intensities. Also remains the possibility to adapt the method for different implementations of internal coordinates already available in a software like TROVE [65].

## Acknowledgments

I would like to thank professor T.Carrington for providing me the potential energy surface of ethylene oxide.

## Appendix : Hermite function analytical formulas

In one dimension, Hermite functions verify

$$\int_{\mathbb{R}} \psi_b(q) q^{d_1} \frac{\partial^{d_2}}{\partial q^{d_2}} \psi_{b+e}(q) dq \neq 0 \text{ if } \exists t \in \mathbb{N}, |e| = d_1 + d_2 - 2t, \quad (35)$$

as well as for the switched product  $\frac{\partial^{d_1}}{\partial q^{d_1}} q^{d_2}$ .

It is easily demonstrable with recurrence relations [66, 67]

$$\begin{aligned} \psi'_b(q) &= \sqrt{\frac{b}{2}} \psi_{b-1}(q) - \sqrt{\frac{b+1}{2}} \psi_{b+1}(q), \\ q \psi_b(q) &= \sqrt{\frac{b}{2}} \psi_{b-1}(q) + \sqrt{\frac{b+1}{2}} \psi_{b+1}(q), \end{aligned} \quad (36)$$

and can directly be related to the definition of operators (16). For the coefficients

$$\langle [q^d] \rangle_{b,s} = \langle \psi_b(q) | q^d | \psi_s(q) \rangle, \quad (b, s) \in \{0, \dots, \text{Dim}\}^2, \quad d \geq 1,$$

the following property applies

$$\langle [q]^d \rangle_{b,s} = \langle [q^d] \rangle_{b,s}, \quad \forall (b, s) \in \{0, \dots, \text{Dim} - d + 1\}^2.$$

where

$$[q] = \begin{bmatrix} 0 & \sqrt{\frac{1}{2}} & 0 & \dots & \dots & 0 \\ \sqrt{\frac{1}{2}} & 0 & \sqrt{\frac{2}{2}} & \ddots & \ddots & \vdots \\ 0 & \sqrt{\frac{2}{2}} & 0 & \sqrt{\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \sqrt{\frac{3}{2}} & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 & \sqrt{\frac{\text{Dim}}{2}} \\ 0 & \dots & \dots & 0 & \sqrt{\frac{\text{Dim}}{2}} & 0 \end{bmatrix}$$

is the well known Jacobi matrix constructed with Hermite function recurrence relations (36).

- 
- [1] K. Christoffel, J. Bowman, Investigations of self-consistent field, scf ci and virtual state-configuration interaction vibrational energies for a model three-mode system, *Chem. Phys. Lett.* 85 (2) (1982) 220–224. doi:10.1016/0009-2614(82)80335-7.
- [2] T. Thompson, D. Truhlar, Scf ci calculations for vibrational eigenvalues and wavefunctions of systems exhibiting fermi resonance, *Chem. Phys. Lett.* 75 (1) (1980) 87–90. doi:10.1016/0009-2614(80)80470-2.
- [3] Z. Gershgorin, I. Shavitt, An application of perturbation theory ideas in configuration interaction calculations, *International Journal of Quantum Chemistry* 2 (6) (1968) 751–759. doi:10.1002/qua.560020603.
- [4] J. Bowman, K. Christoffel, F. Tobin, Application of scf-si theory to vibrational motion in polyatomic molecules, *J. Phys. Chem.* 83 (8) (1979) 905–920. doi:10.1021/j100471a005.
- [5] J. M. Bowman, The Self-Consistent-Field Approach to Polyatomic Vibrations, *Acc. Chem. Res.* 19 (12) (1986) 202–208. doi:10.1021/ar00127a002.
- [6] O. Christiansen, Møller-Plesset perturbation theory for vibrational wave functions, *Journal of Chemical Physics* 119 (12) (2003) 5773–5781. doi:10.1063/1.1601593.
- [7] K. Yagi, H. Karasawa, S. Hirata, K. Hirao, First-principles quantum calculations on the infrared spectrum and vibrational dynamics of the guanine-cytosine base pair, *ChemPhysChem* 10 (9-10) (2009) 1442–1444. doi:10.1002/cphc.200900234.
- [8] I. Respondek, D. M. Benoit, Fast degenerate correlation-corrected vibrational self-consistent field calculations of the vibrational spectrum of 4-mercaptopyridine, *Journal of Chemical Physics* 131 (5). doi:10.1063/1.3193708.
- [9] M. Herman, D. S. Perry, Molecular spectroscopy and dynamics: a polyad-based perspective, *Physical Chemistry Chemical Physics* 15 (25) (2013) 9970. doi:10.1039/c3cp50463h.
- [10] R. Roth, J. Langhammer, Pade-resummed high-order perturbation theory for nuclear structure calculations, *Physics Letters B* 683 (4-5) (2009) 6. arXiv:0910.3650, doi:10.1016/j.physletb.2009.12.046.
- [11] O. Christiansen, Vibrational structure theory: new vibrational wave function methods for calculation of anharmonic vibrational energies and vibrational contributions to molecular properties, *Phys. Chem. Chem. Phys.* 9 (23) (2007) 2942–2953. doi:10.1039/B618764A.
- [12] F. Pfeiffer, G. Rauhut, Multi-reference vibration correlation methods, *Journal of Chemical Physics* 140 (6), and references therein. doi:10.1063/1.4865098.
- [13] R. Garnier, M. Odunlami, V. L. Bris, D. Bégué, I. Baraille, O. Coulaud, Adaptive vibrational configuration interaction (A-VCI): A posteriori error estimation to efficiently compute anharmonic IR spectra, *The Journal of Chemical Physics* 144 (20) (2016) 204123. doi:10.1063/1.4952414.

- 
- [14] M. Odunlami, V. L. Bris, D. Bégué, I. Baraille, O. Coulaud, A-VCI: A flexible method to efficiently compute vibrational spectra, *The Journal of Chemical Physics* 146 (21) (2017) 214108. doi:10.1063/1.4984266.
- [15] E. B. Wilson, J. C. Decius, P. C. Cross, *Molecular Vibrations. The Theory of Infrared and Raman Vibrational Spectra*, McGraw Hill, New York London, 1955.
- [16] J. K. Watson, Simplification of the molecular vibration-rotation hamiltonian, *Molecular Physics* 15 (5) (1968) 479–490. doi:10.1080/00268976800101381.
- [17] J. H. Meal, S. R. Polo, VibrationRotation Interaction in Polyatomic Molecules. II. The Determination of Coriolis Coupling Coefficients, *The Journal of Chemical Physics* 24 (6) (1956) 1126. doi:10.1063/1.1742729.
- [18] G. Avila, T. Carrington, Solving the Schroedinger equation using Smolyak interpolants, *Journal of Chemical Physics* 139 (13). doi:10.1063/1.4821348.
- [19] G. Avila, T. Carrington Jr., Using nonproduct quadrature grids to solve the vibrational schrödinger equation in 12d, *J. Chem. Phys.* 134 (5) (2011) 054126. doi:10.1063/1.3549817.
- [20] G. Avila, T. Carrington, Using a pruned basis, a non-product quadrature grid, and the exact Watson normal-coordinate kinetic energy operator to solve the vibrational Schrödinger equation for C<sub>2</sub>H<sub>4</sub>, *Journal of Chemical Physics* 135 (6). doi:10.1063/1.3617249.
- [21] J. Brown, T. Carrington, Using an expanding nondirect product harmonic basis with an iterative eigensolver to compute vibrational energy levels with as many as seven atoms, *Journal of Chemical Physics* 145 (14). doi:10.1063/1.4963916.
- [22] G. Beylkin, M. J. Mohlenkamp, Algorithms for numerical analysis in high dimensions 26 (6) (2005) 2133–2159. doi:10.1137/040604959.
- [23] P. S. Thomas, T. C. Jr, An intertwined method for making low-rank , sum-of-product basis functions that makes it possible to compute vibrational spectra of molecules with more than 10 atoms 204110 (2016) 1–40. doi:10.1063/1.4983695.
- [24] P. S. Thomas, T. Carrington Jr., Using nested contractions and a hierarchical tensor format to compute vibrational spectra of molecules with seven atoms, *J. Phys. Chem. A* 119 (52) (2015) 13074–13091. doi:10.1021/acs.jpca.5b10015.
- [25] M. Rakhuba, I. Oseledets, Calculating vibrational spectra of molecules using tensor train decomposition 124101. doi:10.1063/1.4962420.
- [26] G. Rauhut, Configuration selection as a route towards efficient vibrational configuration interaction calculations, *J. Chem. Phys.* 127 (18) (2007) 184109. doi:10.1063/1.2790016.

- 
- [27] Y. Scribano, D. Benoit, Iterative active-space selection for vibrational configuration interaction calculations using a reduced-coupling vscf basis, *Chem. Phys. Lett.* 458 (4-6) (2008) 384–387. doi:10.1016/j.cplett.2008.05.001.
- [28] M. Neff, G. Rauhut, Toward large scale vibrational configuration interaction calculations, *J. Chem. Phys.* 131 (12). doi:10.1063/1.3243862.
- [29] G. Rauhut, T. Hrenar, A combined variational and perturbational study on the vibrational spectrum of P2F4, *Chemical Physics* 346 (1-3) (2008) 160–166. doi:10.1016/j.chemphys.2008.01.039.
- [30] I. Baraille, C. Larrieu, A. Dargelos, M. Chaillet, Calculation of non-fundamental ir frequencies and intensities at the anharmonic level. i. the overtone, combination and difference bands of diazomethane, h2cn2, *Chem. Phys.* 273 (2-3) (2001) 91–101. doi:10.1016/S0301-0104(01)00489-X.
- [31] P. Carbonnière, A. Dargelos, C. Pouchan, The vci-p code: An iterative variation-perturbation scheme for efficient computations of anharmonic vibrational levels and ir intensities of polyatomic molecules, *Theor. Chem. Acc.* 125 (3-6) (2010) 543–554. doi:10.1007/s00214-009-0689-7.
- [32] C. Pouchan, K. Zaki, Ab initio configuration interaction determination of the overtone vibrations of methyleneimine in the region 28003200 cm<sup>[sup 1]</sup>, *The Journal of Chemical Physics* 107 (2) (1997) 342. doi:10.1063/1.474395.
- [33] D. Bégué, N. Gohaud, C. Pouchan, P. Cassam-Chenai, J. Liévin, A comparison of two methods for selecting vibrational configuration interaction spaces on a heptatomic system: Ethylene oxide, *J. Chem. Phys.* 127 (16) (2007) 164115. doi:10.1063/1.2795711.
- [34] M. Sibaev, D. L. Crittenden, Balancing accuracy and efficiency in selecting vibrational configuration interaction basis states using vibrational perturbation theory, *Journal of Chemical Physics* 145 (6). doi:10.1063/1.4960600.
- [35] G. Chaban, J. Jung, R. Gerber, Anharmonic vibrational spectroscopy of glycine:testing of ab initio and empirical potentials, *J. Phys. Chem. A* 104 (44) (2000) 10035–10044. doi:10.1021/jp002297t.
- [36] G. M. Chaban, J. O. Jung, R. B. Gerber, Ab initio calculation of anharmonic vibrational states of polyatomic systems: Electronic structure combined with vibrational self-consistent field, *The Journal of Chemical Physics* 111 (5) (1999) 1823–1829. doi:10.1063/1.479452.
- [37] J. O. Jung, R. B. Gerber, Vibrational wave functions and spectroscopy of (h2o)<sub>n</sub>, n=2,3,4,5: Vibrational selfconsistent field with correlation corrections, *J. Chem. Phys.* 105 (23) (1996) 10332–10348. doi:http://dx.doi.org/10.1063/1.472960.

- 
- [38] T. K. Roy, R. B. Gerber, Vibrational self-consistent field calculations for spectroscopy of biological molecules: new algorithmic developments and applications., *Physical chemistry chemical physics : PCCP* 15 (24) (2013) 9468–92. doi:10.1039/c3cp50739d.
- [39] E. Davidson, The iterative calculation of a few of the lowest eigenvalues and corresponding eigenvectors of large real-symmetric matrices, *J. Comput. Phys.* 17 (1) (1975) 87–94. doi:10.1016/0021-9991(75)90065-0.
- [40] F. Ribeiro, C. Iung, C. Leforestier, A Jacobi-Wilson description coupled to a block-Davidson algorithm: An efficient scheme to calculate highly excited vibrational levels, *Journal of Chemical Physics* 123 (5) (2005) 0–10. doi:10.1063/1.1997129.
- [41] F. Ribeiro, C. Iung, C. Leforestier, Calculation of highly excited vibrational levels: A prediagonalized davidson scheme, *Chem. Phys. Lett.* 362 (3-4) (2002) 199–204. doi:10.1016/S0009-2614(02)00905-3.
- [42] Y. Zhou, Y. Saad, A ChebyshevDavidson Algorithm for Large Symmetric Eigenproblems, *SIAM Journal on Matrix Analysis and Applications* 29 (3) (2007) 954–971. doi:10.1137/050630404.
- [43] O. Christiansen, Vibrational coupled cluster theory., *The Journal of chemical physics* 120 (5) (2004) 2149–2159. doi:10.1063/1.1637579.
- [44] C. Cohen-Tannoudji, B. Diu, F. Laloë, *Quantum mechanics, Quantum Mechanics*, Wiley, 1977.
- [45] L. F. Williams, Jr., A modification to the half-interval search (binary search) method, in: *Proceedings of the 14th Annual Southeast Regional Conference, ACM-SE 14*, ACM, New York, NY, USA, 1976, pp. 95–101. doi:10.1145/503561.503582.
- [46] E. Ziegel, *Matrix Differential Calculus With Applications in Statistics and Econometrics*, *Technometrics* 31 (4) (1989) 501–502. doi:10.1080/00401706.1989.10488622.
- [47] C. Lanczos, An iteration method for the solution of the eigenvalue problem of linear differential and integral operators, *J. Res. Natl. Bur. Stand.* 45 (4) (1950) 255. doi:10.6028/jres.045.026.
- [48] R. B. Lehoucq, D. C. Sorensen, C. Yang, *ARPACK Users ’ Guide : Solution of Large Scale Eigenvalue Problems with Implicitly Restarted Arnoldi Methods .*, *Communication* 6 (1998) 147. doi:10.1137/1.9780898719628.
- [49] A. Leclerc, T. Carrington, Calculating vibrational spectra with sum of product basis functions without storing full-dimensional vectors or matrices, *J. Chem. Phys.* 140 (17) (2014) 174111. doi:10.1063/1.4871981.
- [50] D. Begue, P. Carbonniere, C. Pouchan, Calculations of vibrational energy levels by using a hybrid ab initio and dft quartic force field: Application to acetonitrile, *J. Phys. Chem. A* 109 (20) (2005) 4611–4616. doi:10.1021/jp0406114.

- 
- [51] T. Shimanouchi, Tables of molecular vibrational frequencies. Consolidated volume II, Journal of Physical and Chemical Reference Data 6 (3) (1977) 993–1102. doi:10.1063/1.555560.
- [52] R. Paso, R. Anttila, M. Koivusaari, The Infrared Spectrum of Methyl Cyanide Between 1240 and 1650 cm<sup>-1</sup>: The Coupled Band System  $\nu_3$ ,  $\nu_{16}$ , and  $\nu_7 + \nu_{28}$ , Journal of Molecular Spectroscopy 165 (2) (1994) 470–480. doi:10.1006/jmsp.1994.1150.
- [53] T. Delahaye, A. Nikitin, M. Rey, P. G. Szalay, V. G. Tyuterev, A new accurate ground-state potential energy surface of ethylene and predictions for rotational and vibrational energy levels, Journal of Chemical Physics 141 (10) (2014) 0–16. doi:10.1063/1.4894419.
- [54] M. Sibaev, D. L. Crittenden, The PyPES library of high quality semi-global potential energy surfaces, Journal of Computational Chemistry 36 (29) (2015) 2200–2207. doi:10.1002/jcc.24192.
- [55] R. Georges, M. Bach, M. Herman, The vibrational energy pattern in ethylene (12C<sub>2</sub>H<sub>4</sub>), Molecular Physics 97 (1-2) (1999) 279–292. doi:10.1080/00268979909482829.
- [56] M. Sparta, M. B. Hansen, E. Matito, D. Toffoli, O. Christiansen, Using electronic energy derivative information in automated potential energy surface construction for vibrational calculations, Journal of Chemical Theory and Computation 6 (10) (2010) 3162–3175. doi:10.1021/ct100229f.
- [57] M. Sparta, I.-M. Høyvik, D. Toffoli, O. Christiansen, Potential Energy Surfaces for Vibrational Structure Calculations from a Multiresolution Adaptive Density-Guided Approach: Implementation and Test Calculations, The Journal of Physical Chemistry A 113 (30) (2009) 8712–8723. doi:10.1021/jp9035315.
- [58] M. Sparta, D. Toffoli, O. Christiansen, An adaptive density-guided approach for the generation of potential energy surfaces of polyatomic molecules, Theoretical Chemistry Accounts 123 (5) (2009) 413–429. doi:10.1007/s00214-009-0532-1.
- [59] N. K. Madsen, I. H. Godtliebsen, O. Christiansen, Efficient algorithms for solving the non-linear vibrational coupled-cluster equations using full and decomposed tensors, The Journal of Chemical Physics 146 (13) (2017) 134110. doi:10.1063/1.4979498.
- [60] C. Pouchan, S. Senez, J. Raymond, H. Sauvaire, Étude expérimentale et théorique des vibrations moléculaires de l’isoxazole, Journal de Chimie Physique 71 (1974) 525–532. doi:10.1051/jcp/1974710525.
- [61] D. Sorensen, C. Yang, Accelerating the Lanczos algorithm via polynomial spectral transformations, TR97-29, Dept. of Computational and Applied Mathematics, Rice University, Houston, TX 0047.
- [62] H. O. Karlsson, Calculation of highly excited vibrational states using a Richardson-Leja-Davidson scheme, Journal of Chemical Physics 126 (8). doi:10.1063/1.2646409.

- [63] H. Fang, Y. Saad, A filtered Lanczos procedure for extreme and interior eigenvalue problems, *SIAM J. SCI. COMPUT.* 34 (4) (2012) 2220–2246. doi:10.1137/110836535.
- [64] M. Sibaev, D. Crittenden, *PyPES Extensible Library Manual* (2016).  
URL <https://sourceforge.net/projects/pypes-lib-ext/files/>
- [65] S. N. Yurchenko, W. Thiel, P. Jensen, Theoretical ROVibrational Energies (TROVE): A robust numerical approach to the calculation of rovibrational energies for polyatomic molecules, *Journal of Molecular Spectroscopy* 245 (2) (2007) 126–140. doi:10.1016/j.jms.2007.07.009.
- [66] I. S. Gradshteyn, I. M. Ryzhik, *Table of integrals, series, and products*, seventh Edition, Elsevier/Academic Press, Amsterdam, 2007.
- [67] G. Szegő, *Orthogonal Polynomials*, no. vol.~23 in American Mathematical Society colloquium publications, American mathematical society, 1939.