TOPOLOGICAL REALIZATION OF ALGEBRAS OF QUASI-INVARIANTS, I

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ABSTRACT. This is the first in a series of papers, where we introduce and study topological spaces that realize the algebras of quasi-invariants of finite reflection groups. Our result can be viewed as a generalization of a well-known theorem of A. Borel that realizes the ring of invariant polynomials a Weyl group W as a cohomology ring of the classifying space BG of the associated Lie group G. In the present paper, we state our realization problem for the algebras of quasi-invariants of Weyl groups and give its solution in the rank one case (for G = SU(2)). We call the resulting G-spaces $F_m(G,T)$ the m-quasi-flag manifolds and their Borel homotopy quotients $X_m(G,T)$ the spaces of m-quasi-invariants. We compute the equivariant K-theory and the equivariant (complex analytic) elliptic cohomology of these spaces and identify them with exponential and elliptic quasi-invariants of W. We also extend our construction of spaces quasi-invariants to a certain class of finite loop spaces ΩB of homotopy type of \mathbb{S}^3 originally introduced by D. L. Rector [Rec71a]. We study the cochain spectra $C^*(X_m,k)$ associated to the spaces of quasi-invariants and show that these are Gorenstein commutative ring spectra in the sense of Dwyer, Greenlees and Iyengar [DGI06].

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1. Introduction

Quasi-invariants are natural generalizations of classical invariant polynomials of finite reflection groups. In the case of Coxeter groups, they first appeared in mathematical physics — in the work of O. Chalykh and A. Veselov [CV90, CV93] in the early 1990s, and since then have found applications in many other areas: most notably, representation theory, algebraic geometry and combinatorics (see [FV02], [EG02], [Cha02], [BEG03], [FV03], [GW06], [BM08], [Tsu10], [BC11], [BEF20], [Gri21]). For arbitrary (complex) reflection groups, quasi-invariants were introduced in [BC11]. This last paper developed a general approach to quasi-invariants in the context of representation theory of rational double affine Hecke algebras, extending and refining the earlier results of [BEG03] in the Coxeter case. We will use [BC11] as our main reference on algebras of quasi-invariants; in particular, we will follow the notation and conventions of that paper in the present work.

We begin by recalling the definition of quasi-invariants in the Coxeter case. Let W be a finite real reflection group acting in its reflection representation V. Denote by $\mathcal{A} := \{H\}$ the set of reflection hyperplanes of W in V and write $s_H \in W$ for the reflection operator in H. The group W acts naturally on the polynomial algebra $\mathbb{C}[V]$ and, since the s_H 's generate W, the invariant polynomials $p \in \mathbb{C}[V]^W$ are determined by the equations

$$(1.1) s_H(p) = p$$

for all $H \in \mathcal{A}$. To define quasi-invariants we modify ('weaken') the equations (1.1) in the following way. For each reflection hyperplane $H \in \mathcal{A}$, we choose a linear form $\alpha_H \in V^*$ such that $H = \operatorname{Ker}(\alpha_H)$ and fix a non-negative integer $m_H \in \mathbb{Z}_+$, assuming that $m_{w(H)} = m_H$ for all $w \in W$. In other words, we choose a system of roots of W in V^* , which (abusing notation) we still denote by \mathcal{A} , and fix a W-invariant function $m : \mathcal{A} \to \mathbb{Z}_+$, $H \mapsto m_H$, which values we will refer to as multiplicities of hyperplanes (or roots) in \mathcal{A} . Now, with these extra data in hand, we replace the equations (1.1) by the following congruences in $\mathbb{C}[V]$:

$$(1.2) s_H(p) \equiv p \mod \langle \alpha_H \rangle^{2m_H}$$

where $\langle \alpha_H \rangle$ denotes the principal ideal in $\mathbb{C}[V]$ generated by the form α_H . For each $H \in \mathcal{A}$, the congruence (1.2) simply means that the polynomial $s_H(p) - p$ is divisible in $\mathbb{C}[V]$ by the power of the linear form α_H determined by the value of the multiplicity function m. It is easy to see that the set of all polynomials satisfying (1.2) (for m fixed) forms a graded subalgebra in $\mathbb{C}[V]$, which we denote $Q_m(W)$. Following [CV90], we call $Q_m(W)$ the algebra W-quasi-invariant polynomials of multiplicity m. Note that, for m = 0, we have $Q_0(W) = \mathbb{C}[V]$, while $\mathbb{C}[V]^W \subseteq Q_m(W) \subseteq \mathbb{C}[V]$ in general. Thus, for varying m, the quasi-invariants interpolate between the W-invariants and all polynomials.

Despite its simple definition, the algebras $Q_m(W)$ have a complicated structure: they do not seem to admit a good combinatorial description, nor do they have a natural presentation in terms of generators and relations. Nevertheless, these algebras possess many remarkable properties, such as Gorenstein duality (see Theorem 2.3), and are closely related to some fundamental objects in representation theory, such as Dunkl operators and double affine Hecke algebras (see [BEG03, BC11]).

The goal of the present work is to give a topological realization of the algebras of quasi-invariants as (equivariant) cohomology rings of certain spaces naturally attached to compact connected Lie groups. Our main result can be viewed as a generalization of a well-known theorem of A. Borel [Bor53] that realizes the algebra of invariant polynomials of a Weyl group W as the cohomology ring of the classifying space BG of the associated Lie group G. As the algebras $Q_m(W)$ are defined over C, we should clarify what we really mean by "topological realization". It is a fundamental consequence of Quillen's rational homotopy theory [Qui69] that every reduced, locally finite, graded commutative algebra A defined over a field k of characteristic zero is topologically realizable, i.e. $A \cong H^*(X,k)$ for some (simply-connected) space X. When equipped with cohomological grading, the algebras $Q_m(W)$ have all the above-listed properties (cf. Lemma 2.2); hence, the natural question: For which values of m the $Q_m(W)$'s are realizable, has an immediate answer: for all m. A more interesting (and much less obvious) question is whether one can realize quasi-invariants topologically as a diagram of algebras $\{Q_m(W)\}\$ (indexed by m) together with natural structure that these algebras carry (e.g., W-action). It is one of the objectives of this work to formulate a realization problem for the algebras of quasi-invariants in a precise (axiomatic) form by selecting a list of the desired properties. In the present paper, we state this problem for the classical Weyl groups (i.e., the crystallographic Coxeter groups over \mathbb{R} or \mathbb{C}) in terms of classifying spaces of compact Lie groups (see Section 2.4); in our subsequent paper, we will try to formulate a p-local version of the realization problem for algebras of quasi-invariants of non-crystallographic (in fact, non-Coxeter) groups defined over the p-adic numbers in terms of p-compact groups.

We now give a general overview of our work, our problems and motivation.

Quasi-invariants and cohomology theories. In mathematical physics, quasi-invariants naturally arise in three different flavors: rational (polynomial), trigonometric (exponential) and elliptic. Having in hand topological spaces $X_m(G,T)$ that realize the algebras $Q_m(W)$, it is natural to expect that the above three types of quasi-invariants correspond to three basic cohomology theories evaluated at $X_m(G,T)$: namely, the ordinary (singular) cohomology, topological K-theory and elliptic cohomology. We will show that this is indeed the case: in fact, quasi-invariants can be defined for an arbitrary (complex-oriented generalized) cohomology theory, though in general their properties have yet to be studied.

Quasi-flag manifolds. For a compact connected Lie group G, our spaces of quasi-invariants can be naturally realized as Borel homotopy quotients of certain G-spaces $F_m(G,T)$:

$$X_m(G,T) = EG \times_G F_m(G,T)$$

We call $F_m(G,T)$ the m-quasi-flag manifold of G as in the special case m=0, we have $F_0(G,T)=G/T$, the classical flag manifold. We remark that, in general, the spaces $F_m(G,T)$ are defined only as G-equivariant homotopy types, although our construction provides some natural models for them as finite G-CW complexes. By restricting the action of the Lie group G on $F_m(G,T)$ to its maximal torus $T \subseteq G$, it is natural to ask for T-equivariant cohomology (resp., T-equivariant K-theory, elliptic cohomology, . . .) of $F_m(G,T)$. The T-equivariant cohomology is related to the G-equivariant one by the well-known general formula

(1.3)
$$H_G^*(F_m, \mathbb{C}) \cong H_T^*(F_m, \mathbb{C})^W,$$

where W is the Weyl group associated to (G,T). Since $H_G^*(F_m,\mathbb{C})=H^*(X_m,\mathbb{C})\cong Q_m(W)$, formula (1.3) shows that the W-quasi-invariants can be, in fact, realized as W-invariants: $Q_m(W) \cong$ $H_T^*(F_m,\mathbb{C})^W$ in the graded commutative algebras $H_T^*(F_m,\mathbb{C})$. The latter algebras come equipped with natural $H_T^*(\mathrm{pt},\mathbb{C})$ -module structure induced by the canonical map $F_m \to \mathrm{pt}$. Identifying $H_T^*(\mathrm{pt},\mathbb{C})\cong\mathbb{C}[V]$ and taking onto account the W-action, we can view $H_T^*(F_m,\mathbb{C})$ as modules over the crossed product algebra $\mathbb{C}[V] \rtimes W$. We will show that these $\mathbb{C}[V] \rtimes W$ -modules coincide — up to a half-integer shift of multiplicities — with the modules of CW-valued quasi-invariants, $\mathbf{Q}_{m+\frac{1}{2}}(W)$, introduced and studied in [BC11]. As observed in [BC11], for integer m, the action of $\mathbb{C}[V] \times W$ on $\mathbb{Q}_m(W)$ naturally extends to the rational double affine Hecke (a.k.a. Cherednik) algebra $\mathcal{H}_m(W)$ associated to (W,m). We will show that the topological construction of the quasi-flag manifolds $F_m(G,T)$ generalizes to half-integer values of m, although at the expense of producing spaces equipped only with T-action. By [BC11], we get then an action of $\mathcal{H}_{m+1}(W)$ on the T-equivariant cohomology of $F_{m+\frac{1}{2}}(G,T)$. This phenomenon seems to generalize to other cohomology theories, defining, in particular, an action of trigonometric (resp., non-degenerate) DAHA on T-equivariant K-theoretic (resp., elliptic) quasi-invariants. Constructing these actions algebraically and giving them a topological explanation is an interesting problem that we leave for the future.

Topological refinements. The realization of algebras of quasi-invariants raises many natural questions regarding topological analogues ('refinements') of basic properties that these algebras possess. A general framework to deal with such questions is provided by stable homotopy theory. Indeed, our spaces of quasi-invariants $X_m(G,T)$ are closely related to the classifying spaces of compact Lie groups, and the latter have been studied extensively in recent years by means of

stable homotopy theory (see, e.g., [DGI06], [BG14], [Gre18], [Gre20], [BCHV21]). From this perspective, the main object of study is the mapping spectrum

$$(1.4) C^*(X,k) := \operatorname{Map}(\Sigma^{\infty} X_+, Hk)$$

called the *cochain spectrum* of a topological space X. As its notation suggests, $C^*(X,k)$ is a commutative ring spectrum that — for an arbitrary commutative ring k — plays the same role as the usual (differential graded) k-algebra of cochains on X in the case when k is a field of characteristic zero. In particular, the (stable) homotopy groups of the spectrum (1.4) are isomorphic to the singular cohomology groups of the space X:

$$\pi_{-i}[C^*(X,k)] \cong H^i(X,k)$$

The ring spectrum (1.4) thus refines (in a homotopy-theoretic sense) the cohomology ring $H^*(X, k)$. For example, if G is a compact connected Lie group and k is a field of characteristic 0, the Borel Theorem mentioned above identifies $H^*(BG, k)$ with the algebra $k[V]^W$ of invariant polynomials of W. The cochain spectrum $C^*(BG, k)$ of the classifying space BG can thus be viewed as a refinement of the algebra $k[V]^W$. In the same manner, we will regard the cochain spectra $C^*(X_m(G,T),k)$ of our spaces $X_m(G,T)$ as homotopy-theoretic refinements of the algebras of quasi-invariants $Q_m(W)$. The point is that the known algebraic properties of $Q_m(W)$ should have topological analogues for $C^*(X_m(G,T),k)$. For example, one of the main theorems about quasi-invariants (see Theorem 2.3) says that the (graded) algebras $Q_m(W)$ defined over $\mathbb C$ are Gorenstein if W is a Coxeter group. It is therefore natural to expect that the corresponding ring spectra $C^*(X_m(G,T),k)$ are also Gorenstein — but now in a topological sense [DGI06] and over an arbitrary field k. We will show that this expectation is indeed correct, at least in the rank one case (see Theorem 7.1 and Theorem 7.2), and the spectra of quasi-invariants have a number of other interesting properties. Our results are only first steps in this direction, and many natural questions motivated, in particular, by representation theory have yet to be answered.

Homotopy Lie groups. The spaces of quasi-invariants of compact Lie groups, $X_m(G,T)$, can be constructed functorially in a purely homotopy-theoretic way. In the rank one case, we use to this end the so-called fibre-cofibre construction — a classical (though not very well-known) construction in homotopy theory introduced by T. Ganea [Gan65]. A generalization of Ganea's construction allows us to define the analogues of $X_m(G,T)$ for certain finite loop spaces closely related to compact Lie groups, and perhaps most interestingly, for p-compact groups — p-local analogues of finite loop spaces also known as homotopy Lie groups. In this last case, the classical Weyl groups are replaced by pseudo-reflection groups defined over the field \mathbb{Q}_p of p-adic numbers. It is well known that all such pseudo-reflection groups can be realized as complex reflection groups (see [CE74]), and we thus provide realizations of algebras of quasi-invariants of complex reflection groups defined in [BC11], albeit in a p-local setting. The simplest exotic examples are the rank one p-compact groups $\hat{\mathbb{S}}_p^{2n-1}$, called the Sullivan spheres, whose 'Weyl groups' are the cyclic groups $W = \mathbb{Z}/n$ of order n > 2 such that $n \mid (p-1)$. These examples are already quite rich: we will treat them in a separate paper.

We divide our work into three parts. The present paper (Part I) focuses entirely on the 'global' rank one case: here, we define and study the spaces of quasi-invariants for the Lie group G = SU(2) and for a certain class of finite loop spaces ΩB of homotopy type of \mathbb{S}^3 known as *Rector spaces*. In Part II, we formulate a p-local version of the realization problem for algebras of quasi-invariants defined over \mathbb{Q}_p and give its solution in the 'local' rank one case: namely, for the p-compact groups associated with Sullivan spheres $\hat{\mathbb{S}}_p^{2n-1}$. In Part III, we then use the spaces introduced in Part I

and Part II as 'building blocks' for constructing spaces of quasi-invariants for arbitrary compact connected Lie groups and for 'generic' p-compact groups related to Clark-Ewing spaces.

Contents of the present paper. We now describe in more detail the results of the present paper. In Section 2, after reviewing basic facts about quasi-invariants, we state our realization problem for Weyl groups in the classical framework of compact connected Lie groups. As mentioned above, we take an axiomatic approach: the properties that we choose to characterize the topological spaces of quasi-invariants are modeled on properties of algebraic varieties of quasi-invariants introduced and studied in [BEG03]. In fact, our main axioms (QI_1) – (QI_5) in Section 2.4 are natural homotopy-theoretic analogues of basic geometric properties of the varieties of quasi-invariants listed in Section 2.2.

In Section 3, we give a solution of our realization problem for G = SU(2) (see Theorem 3.9). To this end, as mentioned above, we employ the Ganea fibre-cofibre construction. This construction plays an important role in abstract homotopy theory (specifically, in the theory of LS-categories and related work on the celebrated Ganea Conjecture in algebraic topology, see e.g. [CLOT03] and Example 3.2 below). However, we could not find any applications of it in Lie theory or classical homotopy theory of compact Lie groups (perhaps, with the exception of the simple (folklore) Example 3.3). We therefore regard Proposition 3.7 and Theorem 3.9 that describe the Ganea tower of the Borel maximal torus fibration of a compact connected Lie group as original contributions of the present paper. The G-spaces $F_m(G,T)$ that we call the m-quasi-flag manifolds of G are defined to be the homotopy fibres of iterated (level m) fibrations in this Ganea tower (see Definition 3.10). In Section 3.4 and Section 3.5, we describe some basic properties of the G-spaces $F_m(G,T)$. First, we compute the T-equivariant cohomology of $F_m(G,T)$ (see Proposition 3.14) and identify it with a module of 'nonsymmetric' ($\mathbb{C}W$ -valued) quasi-invariants (see Corollary 3.16). In this way, we provide a topological interpretation of generalized quasi-invariants introduced in [BC11]. Then, in Section 3.5, we define natural analogues of the classical Demazure (divided difference) operators for our quasi-flag manifolds $F_m(G,T)$. Our construction is purely topological (see Proposition 3.19): it generalizes the Bressler-Evans construction of the divided difference operators for the classical flag manifolds $F_0(G,T)$ given in [BE90].

In Section 4, we extend our topological construction of spaces of quasi-invariants to a large class of finite loop spaces ΩB called the *Rector spaces* (or fake Lie groups of type SU(2)). These remarkable loop spaces were originally constructed in [Rec71a] as examples of nonstandard ('exotic') deloopings of \mathbb{S}^3 . Our construction does not apply to all Rector spaces, but only to those that accept homotopically nontrivial maps from \mathbb{CP}^{∞} . These last spaces admit a beautiful arithmetic characterization discovered by D. Yau in [Yau02]. We show that the 'fake' spaces of quasi-invariants, $X_m(\Omega B, T)$, associated to the Rector-Yau spaces have the same *rational* cohomology as our 'genuine' spaces of quasi-invariants, $X_m(G,T)$, constructed in Section 3 (see Theorem 4.7); however, in general, they are homotopically non-equivalent (see Corollary 5.12).

In Section 5, we compute the G-equivariant (topological) K-theory $K_G^*(F_m)$ of the spaces $F_m = F_m(G,T)$ and identify it with $\mathcal{Q}_m(W)$, the exponential quasi-invariants of the Weyl group $W = \mathbb{Z}/2\mathbb{Z}$ (see Theorem 5.6). Then, we relate $K_G^*(F_m)$ to the (completed) G-equivariant cohomology $\widehat{H}_G^*(F_m,\mathbb{Q}) := \prod_{k=0}^{\infty} H_G^k(F_m,\mathbb{Q})$ by constructing explicitly the G-equivariant Chern character map

We show that (1.5) factors through the natural map $K_G^*(F_m) \to K^*(X_m)$ to the Borel G-equivariant K-theory $K^*(X_m) = K^*(EG \times_G F_m)$ of F_m , inducing an isomorphism upon rationalization (see Proposition 5.8): $K^*(X_m)_{\mathbb{Q}} \cong \widehat{H}_G^*(F_m, \mathbb{Q}) \cong \widehat{Q}_m(W)$. In this way, we link

topologically the exponential and the usual (polynomial) quasi-invariants of W. In Section 5, we also compute the K-theory of 'fake' spaces of quasi-invariants associated to the Rector-Yau loop spaces ΩB (see Theorem 5.10). The result of this computation has an important consequence — Corollary 5.12 — that provides a numerical K-theoretic invariant N_B distinguishing the spaces $X_m(\Omega B, T)$ up to homotopy equivalence for different B's.

In Section 6, we compute the T-equivariant $\mathcal{E}ll_T^*(F_m)$ and G-equivariant $\mathcal{E}ll_G^*(F_m)$ complex analytic elliptic cohomology of F_m (see Theorem 6.3 and Theorem 6.6, respectively). We express the result in two ways: geometrically (as coherent sheaves on a given Tate elliptic curve E) and analytically (in terms of Θ -functions and q-difference equations). We also compute the spaces (graded modules) of global sections of the elliptic cohomology sheaves of F_m with twisted coefficients:

$$\mathrm{Ell}_T^*(E,\mathcal{L}) := \bigoplus_{n=0}^\infty H_\mathrm{an}^0(E,\,\mathcal{E}ll_T^*(F_m) \otimes \mathcal{L}^n) \quad \text{and} \quad \mathrm{Ell}_G^*(E,\mathcal{L}) := \mathrm{Ell}_T^*(E,\mathcal{L})^W \;,$$

where \mathcal{L}^n stands for the *n*-th tensor power of the *Looijenga bundle* \mathcal{L} on the elliptic curve E, a canonical W-equivariant line bundle originally introduced and studied in [Loo77]. This computation (see Theorem 6.7) is inspired by results of [Gan14], and technically, it is perhaps the most interesting cohomological computation of the paper.

Finally, in Section 7, we prove that our spaces of quasi-invariants $X_m(G,T)$ are Gorenstein in the sense of stable homotopy theory: more precisely, the associated commutative ring spectra $C^*(X_m,k)$ (see (1.4)) are orientable Gorenstein (relative to k) and satisfies the Gorenstein duality of shift a=1-4m (see Theorem 7.1). This result should be viewed as a homotopy-theoretic analogue of Theorem 2.3 on Gorenstein property of algebras of quasi-invariants. We also prove the analogous result (see Theorem 7.2) for the 'fake' spaces of quasi-invariants $X_m(\Omega B, T)$, although under the additional assumption that $k=\mathbb{F}_p$ for some prime p.

This work brings together ideas and techniques from parts of algebra and topology that are (still) fairly distant from each other. To make it accessible to readers with different background we included two appendices. In Appendix A, we briefly review Milnor's classical construction of classifying spaces of topological groups in terms of iterated joins. As it should be clear from results of Section 3, our construction of spaces of quasi-invariants can be viewed as a generalization of Milnor's construction. In Appendix B, we collect basic definitions from stable homotopy theory concerning regularity and duality properties of commutative ring spectra. This material is needed to understand our motivation and results in Section 7 that were greatly inspired by the beautiful paper [DGI06]. All in all, we tried to give references to all essential facts that we are using, even when these facts are considered to be obvious or well known by experts.

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2. Realization problem

In this section, we state our topological realization problem for algebras of quasi-invariant polynomials of Weyl groups in terms of classifying spaces of compact connected Lie groups.

2.1. Quasi-invariants of finite reflection groups. We recall the general definition of quasiinvariants from [BC11]. Let V be a finite-dimensional vector space over \mathbb{C} , and let W be a finite subgroup of GL(V) generated by pseudoreflections. We recall that an element $s \in GL(V)$ is a pseudoreflection if it has finite order $n_s > 1$ and acts as the identity on some hyperplane H_s in V. We let $\mathcal{A} = \{H_s\}$ denote the set of all hyperplanes corresponding to the pseudoreflections of W and observe that W acts naturally on A by permutations. The (pointwise) stabilizer W_H of each $H \in \mathcal{A}$ in W is a cyclic subgroup of order $n_H \geq 2$ that depends only on the orbit of H in \mathcal{A} . The characters of W_H then also form a cyclic group of order n_H generated by the determinant character det: $GL(V) \to \mathbb{C}^*$ of GL(V) restricted to W_H . We write

$$e_{H,i} := \frac{1}{n_H} \sum_{w \in W_H} (\det w)^{-i} w , \quad i = 0, 1, \dots, n_H - 1 ,$$

for the corresponding idempotents in the group algebra $\mathbb{C}W_H \subseteq \mathbb{C}W$.

Now, let $\mathbb{C}[V] = \operatorname{Sym}_{\mathbb{C}}(V^*)$ denote the polynomial algebra of V. This algebra carries a natural W-action (extending the linear action of W on V^*) and can thus be viewed as a $\mathbb{C}W$ -module. We can then characterize the invariant polynomials $p \in \mathbb{C}[V]^W$ by the equations

(2.1)
$$e_{H,-i}(p) = 0, \quad i = 1, \dots, n_H - 1,$$

which hold for all hyperplanes $H \in \mathcal{A}$. To define quasi-invariants we relax the equations (2.1) in the following way (cf. (1.2)). For each hyperplane $H \in \mathcal{A}$, we fix a linear form $\alpha_H \in V^*$, such that $H = \text{Ker}(\alpha_H)$, and choose $n_H - 1$ positive integers $\{m_{H,i}\}_{i=1,\dots,n_H-1}$ which we refer to as multiplicities of H. We assume that $m_{H,i} = m_{H',i}$ for each i whenever H and H' are in the same orbit of W in \mathcal{A} . We write $\mathcal{M}(W) := \{m_{H,i} \in \mathbb{Z}_+ : i = 1, \ldots, n_H - 1\}_{[H] \in \mathcal{A}/W}$ for the set of all such multiplicities regarding them as functions on the set A/W of W-orbits in A.

Definition 2.1 ([BC11]). A polynomial $p \in \mathbb{C}[V]$ is called a W-quasi-invariant of multiplicity $m = \{m_{H,i}\} \in \mathcal{M}(W)$ if it satisfies the conditions

(2.2)
$$e_{H,-i}(p) \equiv 0 \mod \langle \alpha_H \rangle^{n_H m_{H,i}}, \quad i = 1, \dots, n_H - 1,$$

for all $H \in \mathcal{A}$. We write $Q_m(W)$ for the subspace of all such polynomials in $\mathbb{C}[V]$.

In general, Q(W) is not an algebra: for arbitrary W and $m \in \mathcal{M}(W)$, the subspace of quasiinvariant polynomials may not be closed under multiplication in $\mathbb{C}[V]$. In Part II of our work, we will give necessary and sufficient conditions (on W and m) that ensure the multiplicativity property of $Q_m(W)$. In the present paper, we simply restrict our attention to Coxeter groups, i.e. the finite subgroups W of GL(V) generated by real reflections. In this case the conditions (2.2) are equivalent to (1.2) and the above definition of quasi-invariants reduces to the original definition of Chalykh and Veselov [CV90] given in the Introduction.

Thus, from now on, we assume that W is a real finite reflection group, V being its (complexified) reflection representation.

The next lemma collects some elementary properties of quasi-invariants that follow easily from the definition (see, e.g., [BEG03]).

Lemma 2.2. Let W be an arbitrary Coxeter group. Then, for any $m \in \mathcal{M}(W)$,

- (1) $\mathbb{C}[V]^W \subset Q_m(W) \subseteq \mathbb{C}[V]$ with $Q_0(W) = \mathbb{C}[V]$ and $\cap_m Q_m(W) = \mathbb{C}[V]^W$.
- (2) $Q_m(W)$ is a graded subalgebra of $\mathbb{C}[V]$ stable under the action of W. (3) $Q_m(W)$ is a finite module over $\mathbb{C}[V]^W$ and hence a finitely generated \mathbb{C} -subalgebra of $\mathbb{C}[V]$.

We may think of quasi-invariants of W as a family of subalgebras of $\mathbb{C}[V]$ interpolating between the W-invariants and all polynomials. To make this more precise we will identify the set $\mathcal{M}(W)$ of multiplicities on \mathcal{A} with the set of W-invariant functions $m: \mathcal{A} \to \mathbb{Z}_+$ and put on this set the following natural partial order¹:

$$m' \ge m \quad \stackrel{\text{def}}{\iff} \quad m'_{\alpha} \ge m_{\alpha} , \ \forall \, \alpha \in \mathcal{A} ,$$

The algebras of W-quasi-invariants of varying multiplicities then form a contravariant diagram of shape $\mathcal{M}(W)$ — a functor $\mathcal{M}(W)^{\mathrm{op}} \to \mathtt{CommAlg}_{\mathbb{C}}$ with values in the category of commutative algebras — that we simply depict as a filtration on $\mathbb{C}[V]$:

$$\mathbb{C}[V] = Q_0(W) \supseteq \dots \supseteq Q_m(W) \supseteq Q_{m'}(W) \supseteq \dots \supseteq \mathbb{C}[V]^W$$

The most interesting algebraic property of quasi-invariants is given by the following theorem, the proof of which (unlike the proof of Lemma 2.2) is not elementary.

Theorem 2.3 (see [EG02], [BEG03], [FV02]). For any Coxeter group W and any multiplicity $m \in \mathcal{M}(W)$, $Q_m(W)$ is a free module over $\mathbb{C}[V]^W$ of rank |W|. Moreover, $Q_m(W)$ is a graded Gorenstein algebra with Gorenstein shift $a = \dim(V) - 2\sum_{\alpha \in \mathcal{A}} m_{\alpha}$.

Remark 2.4. For m=0 (i.e., for the polynomial ring $Q_0(W)=\mathbb{C}[V]$), Theorem 2.3 is a well-known result due to C. Chevalley [Che55]. For $m\neq 0$, it was first proven in the case of dihedral groups (i.e. Coxeter groups of rank 2) in [FV02]. For arbitrary Coxeter W, Theorem 2.3 was proven (by different methods) in [EG02] and [BEG03]. It is worth mentioning that the classical arguments of [Che55] do not work for nonzero m's.

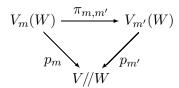
Remark 2.5. The first statement of Theorem 2.3 makes sense and holds true for an arbitrary finite pseudoreflection group W and for all multiplicities. In this generality, Theorem 2.3 was proven in [BC11] (see, loc. cit., Theorem 1.1). However, for W non-Coxeter, the module $Q_m(W)$ may not be Gorenstein even when it is an algebra.

2.2. Varieties of quasi-invariants. The algebraic properties of quasi-invariants can be recast geometrically. To this end, following [BEG03], we introduce the affine schemes $V_m(W) := \operatorname{Spec} Q_m(W)$ called the varieties of quasi-invariants of W. The schemes $V_m(W)$ come equipped with natural projections $p_m: V_m(W) \to V/\!/W$ and form a covariant diagram (tower) over the poset $\mathcal{M}(W)$:

(2.4)
$$V = V_0(W) \to \dots \to V_m(W) \xrightarrow{\pi_{m,m'}} V_{m'}(W) \to \dots$$

that is dual to (2.3). The following formal properties of (2.4) hold:

(1) Each $V_m(W)$ is a reduced irreducible scheme (of finite type over \mathbb{C}) equipped with an algebraic W-action, all morphisms in (2.4) being W-equivariant. The morphism $p_0: V_0(W) \to V//W$ coincides with the canonical projection $p: V \to V//W$, and the triangles



commute for all $m' \geq m$. Thus, (2.4) is a diagram of W-schemes over V//W.

¹Abusing notation, in the Coxeter case, we will often write $\alpha \in \mathcal{A}$ instead of $H \in \mathcal{A}$ for $H = \text{Ker}(\alpha)$.

(2) The diagram (2.4) 'converges' to V//W in the sense that the maps p_m induce

$$\operatorname{colim}_{\mathcal{M}^{\operatorname{alg}}(W)}[V_m(W)] \stackrel{\sim}{\to} V/\!/W$$
.

- (3) Each projection $p_m: V_m \to V/\!/W$ factors naturally (in m) through $V_m/\!/W$, inducing isomorphisms of schemes $V_m/\!/W \cong V/\!/W$ for all $m \in \mathcal{M}(W)$.
- (4) Each map $\pi_{m,m'}: V_m \to V_{m'}$ in (2.4) is a universal homeomorphism: i.e., a finite morphism of schemes that is surjective and set-theoretically injective on closed points.

Remark 2.6. The first three properties in the above list are formal consequences of Lemma 2.2. In contrast, Property (4) is a nontrivial geometric fact that does not follow immediately from definitions (see [BEG03, Lemma 7.3]). We recall that a morphism of schemes $f: S \to T$ is called a universal homeomorphism if for every morphism $T' \to T$ the pullback map $T' \times_T S \to T'$ is a homeomorphism in the category of schemes. For a map of algebraic varieties $f: S \to T$ defined over \mathbb{C} , this categorical property is known to be equivalent to the geometric property (4).

We will construct a topological analogue of the diagram (2.4), where the schemes $V_m(W)$ are replaced by topological spaces $X_m(G,T)$, with Properties (1)–(3) holding in a homotopy meaningful (i.e. homotopy invariant) way. The universal homeomorphisms in the category of schemes will be modeled homotopy theoretically by the classical fibre-cofibre construction.

2.3. **Borel Theorem.** Next, we recall a fundamental result of A. Borel on cohomology of classifying spaces of compact Lie groups [Bor53]. Let G be a compact connected Lie group. Fix a maximal torus $T \subseteq G$ and write $N = N_G(T)$ for its normalizer in G. Let W := N/T be the associated Weyl group. The W acts naturally on T by conjugation: $W \times T \to T$, $w \cdot t = ntn^{-1}$, and on the classifying space BT = EG/T via the right action of G on EG: $W \times BT \to BT$, $w \cdot [x]_T = [xn^{-1}]_T$, where $w = nT \in W$ and $[x]_T$ denotes the T-orbit of x in EG. Let $p:BT \to BG$ denote the natural fibration, i.e. the quitient map induced by the inclusion $T \hookrightarrow G$.

Theorem 2.7 (Borel). The map $p^*: H^*(BG, \mathbb{Q}) \to H^*(BT, \mathbb{Q})$ induced by p on rational cohomology is an injective ring homomorphism whose image is precisely the subring of W invariants in $H^*(BT, \mathbb{Q})$:

$$(2.5) H^*(BG, \mathbb{Q}) \cong H^*(BT, \mathbb{Q})^W.$$

In fact, more is true. Let $V := \pi_1(T) \otimes \mathbb{Q}$, which is \mathbb{Q} -vector space of dimension $n = \operatorname{rank}(G)$. The natural action of W on T induces a group homomorphism $W \to \operatorname{Aut}[\pi_1(T)]$ that extends by linearity to a group homomorphism

(2.6)
$$\varrho: W \to \mathrm{GL}_{\mathbb{Q}}(V) .$$

The latter is known to be faithful, with image being a reflection subgroup of $GL_{\mathbb{Q}}(V)$ (see, e.g., [DW98, Theorem 5.16]). Furthermore, since T is a connected topological group, there is a natural isomorphism $\pi_1(T) \cong \pi_2(BT)$ induced by the homotopy equivalence $T \stackrel{\sim}{\to} \Omega BT$; combining this with the rational Hurewicz isomorphism $\pi_2(BT) \otimes \mathbb{Q} \cong H_2(BT, \mathbb{Q})$, we get a natural isomorphism of \mathbb{Q} -vector spaces

$$(2.7) V \cong H_2(BT, \mathbb{Q})$$

which shows that $H_2(BT, \mathbb{Q})$ carries a reflection representation of W as a Coxeter group. Dualizing (2.7) gives an isomorphism

$$(2.8) H^2(BT, \mathbb{Q}) \cong V^*$$

which extends to an isomorphism of graded Q-algebras

$$(2.9) H^*(BT, \mathbb{Q}) \cong \operatorname{Sym}_{\mathbb{Q}}(V^*) = \mathbb{Q}[V]$$

where the linear forms on V (covectors in V^*) are given cohomological degree 2 (in agreement with (2.8)). Borel's Theorem 2.7 thus identifies $H^*(BG,\mathbb{Q})$ with the ring $\mathbb{Q}[V]^W$ of polynomial invariants on the (rational) reflection representation of W.

We are now in a position to state our main problem — the realization problem for algebras of quasi-invariants of Weyl groups — in an axiomatic way.

2.4. **Realization problem.** Given a compact connected Lie group G with maximal torus $T \subseteq G$ and associated Weyl group $W = W_G(T)$, construct a diagram of spaces $X_m(G,T)$ over the poset $\mathcal{M}(W)$:

$$(2.10) BT = X_0(G,T) \to \dots \to X_m(G,T) \xrightarrow{\pi_{m,m'}} X_{m'}(G,T) \to \dots$$

together with natural maps $p_m: X_m(G,T) \to BG$, one for each $m \in \mathcal{M}(W)$, such that

(QI₁) Each $X_m(G,T)$ is a W-space (i.e., a CW complex equipped with an action of W), and all maps are W-equivariant. The map $p_0: X_0(G,T) \to BG$ coincides with the canonical map $p: BT \to BG$, and for all $m' \ge m$, the following diagrams commute up to homotopy:

$$X_m(G,T) \xrightarrow{\pi_{m,m'}} X_{m'}(G,T)$$

$$p_m \xrightarrow{p_{m'}} p_{m'}$$

Thus, (2.10) is a diagram of W-spaces over BG.

(QI₂) The diagram (2.10) 'converges' to BG in the sense that the maps p_m induce a weak homotopy equivalence of spaces:

$$\operatorname{hocolim}_{\mathcal{M}(W)}[X_m(G,T)] \xrightarrow{\sim} BG$$
.

(QI₃) Each map $p_m: X_m(G,T) \to BG$ factors naturally (in m) through the fibre inclusion into the space $X_m(G,T)_{hW}$ of homotopy orbits of the action of W on $X_m(G,T)$:

$$X_m(G,T) \xrightarrow{p_m} BG$$

$$i_m \qquad \bar{p}_m$$

$$X_m(G,T)_{hW}$$

the induced map $\bar{p}_m: X_m(G,T)_{hW} \to BG$ being a cohomology isomorphism: thus, for all $m \in \mathcal{M}(W)$, we have algebra isomorphisms

$$H_W^*(X_m, \mathbb{Q}) \cong H^*(BG, \mathbb{Q})$$

(QI₄) Each map $\pi_{m,m'}$ in (2.10) induces an injective homomorphism on cohomology so that the Borel homomorphism p^* factors into a $\mathcal{M}(W)^{\text{op}}$ -diagram of algebra maps

$$H^*(BG,\mathbb{Q}) \hookrightarrow \ldots \hookrightarrow H^*(X_{m'},\mathbb{Q}) \stackrel{\pi^*_{m,m'}}{\hookrightarrow} H^*(X_m,\mathbb{Q}) \hookrightarrow \ldots \hookrightarrow H^*(BT,\mathbb{Q})$$

(QI₅) With natural identification $H^*(BT, \mathbb{Q}) = \mathbb{Q}[V]$ (see (2.9)), the maps $\pi_{0,m}^* : H^*(X_m, \mathbb{Q}) \to H^*(BT, \mathbb{Q})$ in (QI₄) induce isomorphisms

$$H^*(X_m,\mathbb{Q})\otimes\mathbb{C}\cong Q_m(W)$$

where $Q_m(W)$ are the subalgebras of quasi-invariants in $\mathbb{C}[V]$.

Remark 2.8. The first three properties of the spaces $X_m(G,T)$ are homotopy-theoretic analogues of the corresponding geometric properties of the varieties $V_m(W)$ listed in Section 2.2. Properties (QI_4) and (QI_5) reflect the fact that the diagram (2.10) is a topological realization of the diagram of algebras (2.3): in particular, the maps $\pi_{m,m'}^*$ in (QI_4) induced by the cohomology functor correspond to the natural inclusions (2.3) of algebras $Q_m(W)$ determined by their multiplicities.

Remark 2.9. The spaces $X_m(G,T)$ will arise naturally as homotopy G-orbit spaces

$$X_m(G,T) = EG \times_G F_m(G,T)$$
,

where $F_m(G,T)$ are the homotopy fibres of the maps $p_m: X_m(G,T) \to BG$ (see Theorem 3.9). These homotopy fibres form a diagram of G-spaces

$$G/T = F_0(G,T) \to \ldots \to F_m(G,T) \to F_{m'}(G,T) \to \ldots$$

that induces the diagram (2.10). We will call $F_m(G,T)$ the quasi-flag manifolds of the group G.

3. Spaces of quasi-invariants

In this section, we give a solution of our Realization Problem (see Section 2.4) in the rank one case. Our main observation (see Proposition 3.7 and Theorem 3.9) is that, for G = SU(2), the diagram of spaces (2.10) satisfying all five axioms (QI₁)-(QI₅) can be obtained inductively, using the so-called 'fibre-cofibre construction' introduced in homotopy theory by T. Ganea [Gan65].

3.1. **Ganea construction.** First, we recall some basic definitions from topology. If $f: X \to Y$ is a map of (well) pointed spaces, its *homotopy fibre* is defined by

$$hofib_*(f) := X \times_Y P_*(Y) = \{(x, \gamma) : \gamma(0) = *, \gamma(1) = f(x)\},\$$

where $P_*(Y) := \operatorname{Map}_*(I,Y) = \{ \gamma : I \to Y , \ \gamma(0) = * \}$ is the (based) path space over Y. Any map $f: X \to Y$ can be replaced by a fibration in the sense that it admits a factorization $X \overset{\sim}{\to} X' \overset{p}{\to} Y$ in Top_* , where the first arrow is a weak homotopy equivalence and the second is a (Serre) fibration. The homotopy fibre is a homotopy invariant in Top_* so that the pullback along a weak equivalence $X \overset{\sim}{\to} X'$ induces a weak equivalence: $\operatorname{hofib}_*(f) \overset{\sim}{\to} \operatorname{hofib}_*(p)$. On the other hand, for any fibration $p: X' \to Y$, the natural inclusion map

$$p^{-1}(*) \stackrel{\sim}{\to} \mathrm{hofib}_*(p) , \quad x \mapsto (x,*)$$

is a (based) homotopy equivalence. Thus, the homotopy fibres of fibrations can be represented in $Ho(Top_*)$ by usual (set-theoretic) fibres.

Dually, the homotopy cofibre of a map $f: X \to Y$ is defined by

$$\operatorname{hocof}_*(f) := Y \cup_X C_*(X),$$

where $C_*(X) := (X \times I)/(\{*\} \times I) \cup (X \times \{1\})$ is the reduced cone on X. Any map $f: X \to Y$ can be replaced by a cofibration in the sense that it admits a factorization $X \stackrel{j}{\hookrightarrow} Y' \stackrel{\sim}{\to} Y$ in Top_* , where the first arrow is a cofibration (i.e., an injective map) in Top_* and the second is (weak) homotopy equivalence. The homotopy cofibre is a homotopy invariant so that the pushout along the homotopy equivalence $Y' \stackrel{\sim}{\to} Y$ induces an equivalence: $\mathsf{hocof}_*(j) \stackrel{\sim}{\to} \mathsf{hocof}_*(f)$. On the other

hand, for a cofibration $j: X \hookrightarrow Y'$, the homotopy cofibre $hocof_*(j)$ is simply obtained by erecting the cone $C_*(\operatorname{Im} j)$ on the image of j in Y'. The natural map collapsing this cone to the basepoint gives then a natural map

$$\operatorname{hocof}_*(j) \cong Y' \cup C_*(\operatorname{Im} j) \xrightarrow{\sim} Y'/X$$

which is a (based) homotopy equivalence. Thus, the homotopy cofibres of cofibrations can be represented in $Ho(Top_*)$ by usual (set-theoretic) cofibres.

Formally, $hofib_*(f)$ and $hocof_*(f)$ can be defined in $Ho(Top_*)$ by the following homotopy limit and homotopy colimit:

$$(3.1) \qquad \operatorname{hofib}_*(f) = \operatorname{holim}\{* \to Y \xleftarrow{f} X\} \quad , \quad \operatorname{hocof}_*(f) = \operatorname{hocolim}\{* \leftarrow X \xrightarrow{f} Y\}$$

The advantage of these formal definitions is that they make sense in any homotopical context: in particular, in an arbitrary pointed model category or ∞ -category.

Now, the Ganea construction starts with a homotopy fibration sequence with a well-pointed base

$$(3.2) F \xrightarrow{j} X \xrightarrow{p} B$$

and produces another homotopy fibration sequence on the same base:

$$(3.3) F_1 \xrightarrow{j_1} X_1 \xrightarrow{p_1} B$$

The space X_1 in (3.3) is defined to be the homotopy cofibre of the fibre inclusion in (3.2): $X_1 := \text{hocof}_*(j)$. The map p_1 — called the (first) whisker map — is obtained by extending $p: X \to B$ to $X_1 = X \cup C_*(F)$ so that $C_*(F)$ maps to the basepoint of B. The F_1 is then defined to be the homotopy fibre of $p_1: F_1 := \text{hofib}_*(p_1)$.

The above construction can be iterated ad infinitum; as a result, one gets a tower of fibration sequences over B:

where X_m and F_m are defined by

(3.5)
$$X_m := \text{hocof}_*(j_{m-1})$$
, $F_m := \text{hofib}_*(p_m)$, $\forall m \ge 1$.

Note that the horizontal arrows p_m in (3.4) are whisker maps making each row $F_m \xrightarrow{j_m} X_m \xrightarrow{p_m} B$ of the above diagram a homotopy fibration sequence. On the other hand, the vertical arrows π_m are canonical maps making each triple $F_m \xrightarrow{j_m} X_m \xrightarrow{\pi_m} X_{m+1}$ a homotopy cofibration sequence. The main observation of [Gan65] is that the homotopy fibres in (3.4) can be described explicitly in terms of iterated joins² of based loop spaces ΩB . More precisely, we have

²We review the definition and basic topological properties of joins in Appendix A.

Theorem 3.1 (Ganea). (1) For all $m \ge 1$, there are natural homotopy equivalences

$$F_m \simeq F * \Omega B * \ldots * \Omega B \pmod{\text{m-fold join}}$$

compatible with the fibre inclusions $F_m \to F_{m+1}$ in (3.4).

(2) The whisker maps $p_m: X_m \to B$ induce a weak homotopy equivalence

$$\operatorname{hocolim} \left\{ X \xrightarrow{\pi_0} X_1 \xrightarrow{\pi_1} X_2 \to \ldots \to X_m \to \ldots \right\} \xrightarrow{\sim} B$$

where the homotopy colimit is taken over the telescope diagram in the middle of (3.4).

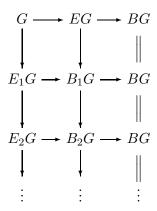
Note that the second claim of Theorem 3.1 follows from the first by Milnor's Lemma (see A.2).

Example 3.2 (LS-categories). Recall that the *LS-category* of a topological space B is defined to be $\operatorname{cat}(B) := n-1$, where n is the least cardinality of an open cover $\{U_1, \ldots, U_n\}$ of B such that each U_i is contractible as a subspace in B. Given a pointed connected space B, one applies the fibre-cofibre construction to the canonical path fibration $\Omega B \to P_* B \xrightarrow{p} B$. The result is the sequence of spaces

$$P_*B \xrightarrow{\pi_0} (P_*B)_1 \xrightarrow{\pi_1} (P_*B)_2 \xrightarrow{\pi_2} (P_*B)_3 \to \dots$$

called the *Ganea tower* of the space B. The main theorem of [Gan67] asserts that if B is a normal space, its LS category $cat(B) \leq m$ if and only if the m-th whisker map $p_m : (P_*B)_m \to B$ associated to the above tower splits (i.e., admits a section). Most applications of Ganea's construction in topology are related to or inspired by this observation (see, e.g., [CLOT03]).

Example 3.3 (Milnor bundles). If G is a topological group, we can apply the Ganea construction to the universal principal G-fibration $G \to EG \to BG$. In this case, the diagram (3.4) reads



where $E_nG := G^{*(n+1)}$ is the join of (n+1) copies of the group G. The group G acts freely on E_nG and hence $B_nG \simeq E_nG/G$. The induced fibration $\Omega BG \to E_nG \to B_nG$ associated to the Ganea fibration at the n-th step of the above tower is thus equivalent to Milnor's n-universal principal G-bundle $G \to E_nG \to B_nG$. We review the properties of such bundles in Appendix A Note that this example can be viewed as a special case of Example 3.2 if we take B = BG.

3.2. **Derived schemes of quasi-invariants.** The fibre-cofibre construction is essentially formal: it can be performed in an arbitrary (pointed) model category or ∞ -category. To see why this construction is relevant to our problem we will apply it first in a simple algebraic model category: the category $\mathtt{dAff}_{k,*}$ of pointed derived affine schemes over a field k of characteristic 0. As a model for $\mathtt{dAff}_{k,*}$, we take the category $(\mathtt{DGCA}_k \downarrow k)^{\mathrm{op}}$ dual to the category of non-negatively graded commutative DG k-algebras k equipped with augmentation map k and k be k becomes the standard algebro-geometric notation, we write $\mathrm{Spec}(k)$ for the object (affine DG scheme)

in \mathtt{dAff}_k corresponding to the DG algebra A in \mathtt{DGCA}_k . Since we assume that $\mathtt{char}(k) = 0$, the category \mathtt{DGCA}_k carries a natural (projective) model structure, where weak equivalences are the quasi-isomorphisms of DG algebras and fibrations are the DG algebra maps which are surjective in positive (homological) degrees (see, e.g., [BKR13, Appendix B]). The category $\mathtt{dAff}_k = \mathtt{DGCA}_k^{\mathrm{op}}$ is equipped with the dual (injective) model structure. The homotopy (co)fibres of morphisms in \mathtt{dAff}_k are defined in terms of homotopy (co)limits, using formulas (3.1). Explicitly, given a morphism of pointed affine DG schemes $f: \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ corresponding to a DG algebra homomorphism $f^*: B \to A$, its homotopy fibre and homotopy cofibre are given by

(3.6)
$$\operatorname{hofib}_*(f) \cong \operatorname{Spec}(A \otimes_B^{\mathbf{L}} k)$$
, $\operatorname{hocof}_*(f) \cong \operatorname{Spec}(B \times_A^{\mathbf{R}} k)$

where \otimes_B^L and \times_A^R denote the derived tensor product (homotopy pushout) and the derived direct product (homotopy pullback) in the model category DGCA_k.

We apply the fibre-cofibre construction in the category $\mathtt{dAff}_{k,*}$ to the canonical (algebrogeometric) quotient map $p: V \twoheadrightarrow V/\!/W$ in the situation of the following simple example.

Example 3.4. Let $W = \mathbb{Z}/2\mathbb{Z}$, acting in its one-dimensional reflection representation V. Choosing a basis vector in V, we can identify $V = \mathbb{C}$ and k[V] = k[x], with W acting on k[x] by the rule s(p)(x) = p(-x). In this case, $A = \{0\}$ and m is a non-negative integer. Condition (1.2) says that p(x) is a quasi-invariant of multiplicity m iff p(x) - p(-x) is divisible by x^{2m} . Hence $Q_m(W)$ is spanned by the monomials $\{x^{2i} : i \geq 0\}$ and $\{x^{2i+1} : i \geq m\}$, or equivalently

$$Q_m(W) = k[x^2] \oplus x^{2m+1}k[x^2] = k[x^2, x^{2m+1}]$$
.

Thus, we take V to be the affine line acted upon by $W = \mathbb{Z}/2\mathbb{Z}$ via the reflection at 0. Regarding $V \cong \operatorname{Spec} k[x]$ and $V//W \cong \operatorname{Spec} k[x^2]$ as affine (DG) schemes pointed at 0, we can compute the homotopy fibre $F := \operatorname{hofib}_*(p)$ in $\operatorname{\mathsf{dAff}}_{k,*}$, using formula (3.6):

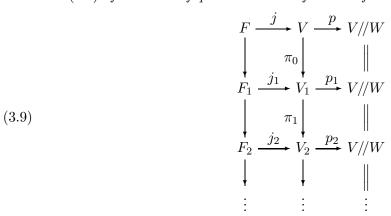
(3.7)
$$F \cong \operatorname{Spec}\left(k[x] \otimes_{k[x^2]}^{\mathbf{L}} k\right) \cong \operatorname{Spec}\left(k[x] \otimes_{k[x^2]} k\right) \cong \operatorname{Spec}(k[x]/x^2) .$$

Note that the second isomorphism in (3.7) is due to the fact that k[x] is a free module (and hence, a flat algebra) over $k[x^2]$. Thus, in $dAff_{k,*}$, we have the fibration sequence

$$(3.8) F \xrightarrow{j} V \xrightarrow{p} V//W$$

where F is given by (3.7). The following simple observation, which was the starting point of the present paper, provides a motivation for our topological results in the next section.

Proposition 3.5. The fibre-cofibre construction in $dAff_{k,*}$ applied to the fibration (3.8) produces the tower (2.4) of varieties of quasi-invariants for the reflection representation of $W = \mathbb{Z}/2\mathbb{Z}$:



Thus, for all $m \geq 0$, we have

(3.10)
$$V_m \cong \operatorname{Spec}(Q_m), \quad F_m \cong \operatorname{Spec}[Q_m/(x^2)],$$

where $Q_m = k[x^2, x^{2m+1}]$ and the maps π_m , p_m and j_m in (3.9) correspond to the natural inclusions $Q_{m+1} \hookrightarrow Q_m$, $k[x^2] \hookrightarrow Q_m$, and the projection $Q_m \twoheadrightarrow Q_m/(x^2)$, respectively.

Proof. The proof is an easy induction in m. For m=0, we have already shown in (3.7) that $F=F_0$, with (3.8) corresponding (i.e. dual) to the natural algebra maps $k[x^2] \hookrightarrow k[x] \twoheadrightarrow k[x]/(x^2)$. Now, assuming that V_m is given by (3.10) together with $p_m: V_m \twoheadrightarrow V//W$ corresponding to the inclusion $k[x^2] \hookrightarrow Q_m$, we compute the fibre F_m in the same way as in (3.7), using formula (3.6):

$$F_m := \mathrm{hofib}_*(p_m) \cong \mathrm{Spec}\left(Q_m \otimes_{k[x^2]}^{\mathbf{L}} k\right) \cong \mathrm{Spec}\left(Q_m \otimes_{k[x^2]} k\right) \cong \mathrm{Spec}\left[Q_m/(x^2)\right]$$

Again, crucial here is the fact that Q_m is a free module (and hence, a flat algebra) over $k[x]^W$, which is a general property of quasi-invariants (see Theorem 2.3). Next, we have

$$(3.11) \quad V_{m+1} := \operatorname{hocof}_*(j_m) \cong \operatorname{Spec}\left(Q_m \times_{Q_m/(x^2)}^{\mathbf{R}} k\right) \cong \operatorname{Spec}\left(Q_m \times_{Q_m/(x^2)} k\right) \cong \operatorname{Spec}(Q_{m+1})$$

The first isomorphism in (3.11) is the result of formula (3.6) for homotopy cofibres in $\mathtt{dAff}_{k,*}$. The second isomorphism is due to the fact that the canonical map $Q_m \to Q_m/(x^2)$ is surjective, and hence a fibration in the standard model structure on \mathtt{DGCA}_k (this implies that $\mathtt{hocof}_*(j_m)$ coincides with the usual cofibre of j_m in the category of affine k-schemes). Finally, the last isomorphism in (3.11) is given by the composition of canonical algebra maps

$$(3.12) Q_m \times_{Q_m/(x^2)} k \hookrightarrow Q_m \times k \twoheadrightarrow Q_m$$

It is easy to see that the map (3.12) is injective, and its image is precisely $Q_{m+1} = k[x^2, x^{2m+3}]$. This gives an identification $Q_m \times_{Q_m/(x^2)} k \cong Q_{m+1}$ together with the inclusion $Q_{m+1} \hookrightarrow Q_m$ that defines the morphism of schemes $\pi_m : V_m \to V_{m+1}$.

Remark 3.6. Proposition 3.5 does not extend directly to higher rank groups: the standard fibre-cofibre construction in $\mathtt{dAff}_{k,*}$ does not produce the tower of varieties of quasi-invariants, (2.4), for an arbitrary Coxeter group W (cf. Proposition 3.7 below).

3.3. Spaces of quasi-invariants of SU(2). Let G be a compact connected Lie group with a fixed maximal torus T and Weyl group $W = W_G(T)$. Associated to (G, T) there is a natural fibration sequence³

$$(3.13) G/T \xrightarrow{j} BT \xrightarrow{p} BG ,$$

where p is the map induced by the inclusion $T \hookrightarrow G$ and j is the classifying map for the principal T-bundle $G \to G/T$.

³If we choose a model for the universal G-bundle EG (for example, the Milnor model described in Section A) and let BG = EG/G and BT = EG/T, then (3.13) is represented by a canonical locally trivial fibre bundle $G/T \to EG/T \to EG/G$ (see, e.g., [Hus75], Chap 4, Sect. 7).

Proposition 3.7. Assume that W is simply-laced (i.e., of ADE type). Then the fibre-cofibre construction applied to (3.13) produces a tower of fibrations

$$G/T \xrightarrow{j} BT \xrightarrow{p} BG$$

$$\downarrow \qquad \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \parallel$$

$$F_{1}(G,T) \xrightarrow{j_{1}} X_{1}(G,T) \xrightarrow{p_{1}} BG$$

$$\downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \parallel$$

$$F_{2}(G,T) \xrightarrow{j_{2}} X_{2}(G,T) \xrightarrow{p_{2}} BG$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \parallel$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

where the diagram of spaces

$$(3.15) BT \xrightarrow{\pi_0} X_1(G,T) \xrightarrow{\pi_1} X_2(G,T) \xrightarrow{\pi_2} \dots \to X_m(G,T) \xrightarrow{\pi_m} \dots$$

together with maps $p_m: X_m(G,T) \to BG$ satisfy the first three properties (QI₁), (QI₂) and (QI₃) of Section 2.4.

Proof. If W is simply-laced, all reflection hyperplanes of W are in the same orbit, and the poset $\mathcal{M}(W)$ consists only of constant multiplicities which we identify with \mathbb{Z}_+ . By Ganea's Theorem 3.1, the homotopy fibre $F_m = F_m(G,T)$ at stage m in (3.14) can be represented by the iterated join

$$(3.16) F_m = G/T * \Omega BG * * \Omega BG \simeq G/T * G * * G = G/T * E_{m-1}G ,$$

where $E_{m-1}G$ is Milnor's model for the (m-1)-universal G-bundle (see Section A). The fibre (3.16) carries a natural left (holonomy) action $\Omega BG \times F_m \to F_m$ that under the identification (3.16), corresponds to the diagonal action of G:

$$(3.17) G \times F_m \to F_m , g \cdot (t_0(g_0T) + t_1g_1 + \ldots + t_mg_m) = t_0(gg_0T) + t_1gg_1 + \ldots + t_mgg_m$$

where $g, g_0, g_1, \ldots, g_m \in G$ and $(t_0, \ldots, t_m) \in \Delta^m$, see (A.2). The space $X_m = X_m(G, T)$ can then be represented as the homotopy quotient

$$(3.18) X_m = (F_m)_{hG} = EG \times_G (G/T * E_{m-1}G)$$

and the fibration $F_m \to X_m \to BG$ in (3.14) is identified with the Borel fibration

$$(3.19) F_m \to (F_m)_{hG} \to BG$$

Now, the Weyl group $W = N_G(T)/T$ acts on G/T by $w \cdot (gT) = gn^{-1}T$, where $w = nT \in W$. With identification (3.16), this action naturally induces a W-action on $F_m = G/T * E_{m-1}G$. The latter commutes with the G-action (3.17), and hence extends to the space X_m of homotopy G-orbits in F_m . Explicitly, with identification (3.18), the action of W on $X_m = EG \times_G (G/T * E_{m-1}G)$ is given by

$$(3.20) w \cdot (x, t_0(g_0T) + t_1g_1 + \ldots + t_mg_m) = (x, t_0(g_0n^{-1}T) + t_1g_1 + \ldots + t_mg_m)$$

where $x \in EG$ and $w = nT \in W$. The inclusions $F_m \hookrightarrow F_{m+1}$ defined by

$$t_0(g_0T) + t_1g_1 + \ldots + t_mg_m \mapsto t_0(g_0T) + t_1g_1 + \ldots + t_mg_m + 0e^{-t_0}$$

are obviously $(G \times W)$ -equivariant, hence induce W-equivariant maps on homotopy G-quotients: $\pi_m: X_m \to X_{m+1}$. The whisker maps $p_m: X_m \to BG$ are induced by the trivial maps $F_m \to \operatorname{pt}$

and hence are W-invariant. Thus, we have established property (P1) for the tower (3.15). Property (P2) follows directly from part (2) of Theorem 3.1.

For (P3), it suffices to show that

$$(3.21) H_W^*(F_m, \mathbb{Q}) \cong \mathbb{Q}$$

Indeed, since the actions of G and W on F_m commute, we have

$$(X_m)_{hW} = EW \times_W (EG \times_G F_m) \simeq EG \times_G (EW \times_W F_m) = EG \times_G (F)_{hW}$$

Whence

$$(3.22) H_W^*(X_m, \mathbb{Q}) \cong H_G^*((F_m)_{hW}, \mathbb{Q})$$

On the other hand, if (3.21) holds, the Serre spectral sequence of the Borel fibration

$$(F_m)_{hW} \to EG \times_G (F_m)_{hW} \to BG$$

degenerates, giving an isomorphism $H^*_G((F_m)_{hW}, \mathbb{Q}) \cong H^*(BG, \mathbb{Q})$. Combining this last isomorphism with (3.22) yields $H^*_W(X_m, \mathbb{Q}) \cong H^*(BG, \mathbb{Q})$, as required by (QI₃).

Now, since F_m is connected, (3.21) is equivalent to vanishing of higher cohomology:

$$(3.23) H_W^n(F_m, \mathbb{Q}) = 0 \quad \forall n > 0.$$

We prove (3.23) by induction on m. For m=0, we have $F_0=G/T$ and $(G/T)_{hW}\simeq (G/T)/W\cong G/N$, since the action of W on G/T is free. It follows that $H^n_W(F_0,\mathbb{Q})\cong H^n(G/N,\mathbb{Q})=0$ for all n>0 as it is well known that the space G/N is rationally acyclic for any compact connected Lie group (see [Bor67, Theorem 20.3]).

Now, assume that (3.23) holds for some $m \geq 0$ and consider $(F_{m+1})_{hW} = (F_m * G)_{hW}$. Representing this space by homotopy colimits (see (A.3)) and using the fact that the homotopy colimits commute, we have

$$\begin{array}{lcl} (F_{m+1})_{hW} & \simeq & \operatorname{hocolim}_{W} \operatorname{hocolim} \left[\, F_{m} \leftarrow F_{m} \times G \rightarrow G \, \right] \\ \\ & \simeq & \operatorname{hocolim} \operatorname{hocolim}_{W} \left[\, F_{m} \leftarrow F_{m} \times G \rightarrow G \, \right] \\ \\ & \simeq & \operatorname{hocolim} \left[\, (F_{m})_{hW} \leftarrow (F_{m} \times G)_{hW} \rightarrow (G)_{hW} \, \right] \\ \\ & \simeq & \operatorname{hocolim} \left[\, (F_{m})_{hW} \leftarrow (F_{m})_{hW} \times G \rightarrow BW \times G \, \right] \end{array}$$

This homotopy decomposition implies that the cohomology groups of $(F_{m+1})_{hW}$ and $(F_m)_{hW}$ are related by the following Mayer-Vietoris type long exact sequence:

$$H^{n-1}[(F_m)_{hW} \times G] \to H^n[(F_{m+1})_{hW}] \to H^n[(F_m)_{hW}] \oplus H^n[BW \times G] \to H^n[(F_m)_{hW} \times G]$$

Since W is a finite, its rational cohomology vanishes in positive degrees. Hence, by Künneth Theorem, we have $H^*(BW \times G, \mathbb{Q}) \cong H^*(G, \mathbb{Q})$. Furthermore, our induction assumption (3.23) implies that $H^*((F_m)_{hW} \times G, \mathbb{Q}) \cong H^*(G, \mathbb{Q})$ and for each $n \geq 1$, the last map in the above exact sequence is an isomorphism. Thus, for $n \geq 2$, the above sequence breaks up into short exact sequences

$$0 \to H^n((F_{m+1})_{hW}, \mathbb{Q}) \to H^n(G, \mathbb{Q}) \xrightarrow{\sim} H^n(G, \mathbb{Q}) \to 0$$

which show that $H_W^n(F_{m+1},\mathbb{Q}) = 0$ for all $n \geq 2$. On the other hand, in dimension 0 and 1, the above long exact sequence reads

$$H^0((F_m)_{hW}, \mathbb{Q}) \oplus H^0(G, \mathbb{Q}) \twoheadrightarrow H^0(G, \mathbb{Q}) \to H^1((F_{m+1})_{hW}, \mathbb{Q}) \to H^1(G, \mathbb{Q}) \xrightarrow{\sim} H^1(G, \mathbb{Q})$$

where the first arrow is surjective and the last is an isomorphism. This shows that $H_W^1(F_{m+1}, \mathbb{Q})$ also vanishes, thus finishing the induction and the proof of (QI₃).

Example 3.8. Let us describe the cohomology $H^*(X_1, \mathbb{Q})$ of the first space $X_1 = X_1(G, T)$ in the diagram (3.15) explicitly. By general properties of the Ganea construction (see Section 3.1), this space fits in the homotopy cofibration sequence

$$(3.24) G/T \xrightarrow{j} BT \xrightarrow{\pi_0} X_1$$

Since both BT and G/T have no cohomology classes in odd dimensions and the natural map $j^*: H^*(BT, \mathbb{Q}) \to H^*(G/T, \mathbb{Q})$ is surjective, the long cohomology sequence associated to (3.24) reduces to the short exact sequence

$$(3.25) 0 \to \tilde{H}^*(X_1, \mathbb{Q}) \xrightarrow{\pi_0^*} \tilde{H}^*(BT, \mathbb{Q}) \xrightarrow{j^*} \tilde{H}^*(G/T, \mathbb{Q}) \to 0$$

where \tilde{H}^* stands for the reduced cohomology. Since X_1 is connected, (3.25) shows that the algebra map $\pi_0^*: H^*(X_1, \mathbb{Q}) \to H^*(BT, \mathbb{Q})$ is injective, and with identification $H^*(BT, \mathbb{Q}) \cong \mathbb{Q}[V]$ (as in (2.9)), its image being

$$(3.26) H^*(X_1, \mathbb{Q}) \cong \mathbb{Q} + \langle \mathbb{Q}[V]_+^W \rangle \subset \mathbb{Q}[V],$$

where $\langle \mathbb{Q}[V]_+^W \rangle$ is the ideal in $\mathbb{Q}[V]$ generated by the W-invariant polynomials of positive degrees. Formula (3.26) shows that X_1 has no odd cohomology; moreover, the map $p_1^*: H^*(BG, \mathbb{Q}) \to H^*(X_1, \mathbb{Q})$ induced by the first whisker map in (3.14) is injective, and $H^*(X_1, \mathbb{Q})$ is a finite module over $H^*(BG, \mathbb{Q}) \cong \mathbb{Q}[V]^W$ via p_1^* . By Hilbert-Noether Theorem, this implies that $H^*(X_1, \mathbb{Q})$ is a finitely generated graded \mathbb{Q} -algebra, however it is not Cohen-Macaulay (and hence not Gorenstein) when $\dim_{\mathbb{Q}}(V) \geq 2$. To see this we set $R := H^*(X_1, \mathbb{Q}), \ S := H^*(BT, \mathbb{Q})$ and $S^W = H^*(BG, \mathbb{Q})$ to simplify the notation. Since S is a free S^W -module, the long exact sequence obtained by dualizing the short exact sequence $0 \to R \to S \to S/R \to 0$ over S^W yields

$$\operatorname{Ext}_{S^W}^i(R, S^W) \cong \operatorname{Ext}_{S^W}^{i+1}(R/S, S^W), \quad \forall i \ge 1$$

Since $R/S \cong \tilde{H}^*(G/T,\mathbb{Q})$ by (3.25), $\dim_{\mathbb{Q}}(R/S) = |W| - 1 < \infty$. Hence $\operatorname{Ext}^n_{SW}(R/S,S^W) \neq 0$ and therefore $\operatorname{Ext}^{n-1}_{SW}(R,S^W) \neq 0$, where $n := \dim_{\mathbb{Q}}(V)$. It follows that when n > 1, R is not free as a graded module over S^W , and hence not Cohen-Macaulay as a graded algebra (see, e.g., [Smo72, Prop. 6.8]).

Example 3.8 shows that, unfortunately, the tower of spaces (3.15) constructed in Proposition 3.7 cannot satisfy all five axioms of our realization problem for an arbitrary compact Lie group. Indeed, if $\mathrm{rk}(G) = n \geq 2$, then (QI₅) already fails for $H^*(X_1(G,T),\mathbb{Q})$, since $H^*(X_1(G,T),\mathbb{Q})$ is not a Gorenstein algebra, while $Q_1(W)$ is (see Theorem 2.3). Note, however, that in the rank one case, for G = SU(2), we still have $H^*(X_1(G,T),\mathbb{Q}) \cong Q_1(\mathbb{Z}/2\mathbb{Z})$ by formula (3.26). The next theorem shows that this is not a coincidence.

Theorem 3.9. Assume that G = SU(2) and $W = \mathbb{Z}/2\mathbb{Z}$. Then the diagram of spaces (3.15) together with whisker maps p_m produced by the fibre-cofibre construction satisfies all five properties $(QI_1)-(QI_5)$ of Section 2.4. In particular, for all $m \geq 0$, there are isomorphisms of graded commutative algebras

$$(3.27) H^*(X_m(G,T),\mathbb{Q}) \cong Q_m(W) ,$$

where $Q_m(W)$ is the subring of W-quasi-invariants of multiplicity m in $\mathbb{Q}[V]$. Moreover, $X_m(G,T)$ are unique, up to rational homotopy equivalence, topological spaces realizing the algebras $Q_m(W)$.

Proof. Properties (QI₁)-(QI₃) have already been established for arbitrary G in Proposition 3.7; we need only to check (QI₄) and (QI₅). As a topological space, SU(2) is homeomorphic to \mathbb{S}^3 and

 $G/T = \mathbb{CP}^1 \cong \mathbb{S}^2$. Hence, applying a well-known formula for the join of spheres, we can identify the fibre (3.16):

(3.28)
$$F_m = G/T * G^{*m} \cong \mathbb{S}^2 * (\mathbb{S}^3)^{*m} \cong \mathbb{S}^{4m+2}.$$

Thus, for G = SU(2), (3.19) is equivalent to the sphere fibration: $\mathbb{S}^{4m+2} \to X_m \to B\mathbb{S}^3$. We will look at the Serre spectral sequence of this fibration and apply the Leray-Hirsch Theorem. Since both the basespace and the fibre of (3.19) have no cohomology in odd dimensions, the Serre spectral sequence collapses, giving an isomorphism of graded vector spaces (see, e.g., [MT91, Lemma III.4.5(1)])

$$H^*(X_m,\mathbb{Q}) \cong H^*(BG,\mathbb{Q}) \otimes H^*(F_m,\mathbb{Q})$$

Then, the Leray-Hirsch Theorem (see, e.g., [MT91, Theorem III.4.2]) implies that $H^*(X_m, \mathbb{Q})$ is a free graded module over the algebra $H^*(BG, \mathbb{Q}) = H^*(BSU(2), \mathbb{Q})$, which is the rational polynomial algebra $\mathbb{Q}[c_2]$ generated by the second Chern class $c_2 \in H^4(BSU(2), \mathbb{Q})$. This graded module has rank two, with $H^*(BG, \mathbb{Q})$ identified with a direct summand in $H^*(X_m, \mathbb{Q})$ under the whisker map $p_m^*: H^*(BG, \mathbb{Q}) \hookrightarrow H^*(X_m, \mathbb{Q})$. The complement of $H^*(BG, \mathbb{Q})$ in $H^*(X_m, \mathbb{Q})$ is generated by a cohomology class ξ of dimension 4m+2 whose image under the projection $j_m^*: H^*(X_m, \mathbb{Q}) \to H^*(F_m, \mathbb{Q}) \cong H^*(\mathbb{S}^{4m+2}, \mathbb{Q})$ is the fundamental cohomology class of \mathbb{S}^{4m+2} . Thus, we have

$$(3.29) H^*(X_m, \mathbb{Q}) \cong \mathbb{Q}[c_2] \oplus \mathbb{Q}[c_2]\xi$$

where $|c_2| = 4$ and $|\xi| = 4m + 2$. Next, we look at the homotopy cofibration sequence in (3.14)

$$(3.30) F_m \xrightarrow{j_m} X_m \xrightarrow{\pi_m} X_{m+1}$$

arising from the Ganea construction. This gives a long exact sequence on (reduced) cohomology:

$$(3.31) \qquad \dots \to \tilde{H}^{n-1}(F_m, \mathbb{Q}) \to \tilde{H}^n(X_{m+1}, \mathbb{Q}) \xrightarrow{\pi_m^*} \tilde{H}^n(X_m, \mathbb{Q}) \xrightarrow{j_m^*} \tilde{H}^n(F_m, \mathbb{Q}) \to \dots$$

Since neither F_m nor X_m (by (3.29)) have odd cohomology, we see immediately from (3.31) that all algebra maps π_m^* must be injective, i.e. property (QI₄) holds for (3.15). For each $m \geq 0$, the composition of these maps then gives an embedding

$$(3.32) \pi_0^* \pi_1^* \dots \pi_{m-1}^* : H^*(X_m, \mathbb{Q}) \hookrightarrow H^*(X_{m-1}, \mathbb{Q}) \hookrightarrow \dots \hookrightarrow H^*(BT, \mathbb{Q})$$

If we identify $H^*(BT,\mathbb{Q})=\mathbb{Q}[x]$ by choosing $x\in H^2(BT,\mathbb{Q})=H^2(B\mathbb{S}^1,\mathbb{Q})$ to be the universal Euler class, which is the image of the canonical generator of $H^2(B\mathbb{S}^1,\mathbb{Z})=H^2(K(\mathbb{Z},2),\mathbb{Z})$, then the Chern class $c_2\in H^4(BG,\mathbb{Q})$ maps by (3.32) to $x^2\in H^*(BT,\mathbb{Q})$. Then, for degree reasons, the generator $\xi\in H^{4m+2}(X_m,\mathbb{Q})$ in (3.29) should map to (a scalar multiple of) $x^{2m+1}\in\mathbb{Q}[x]$. Thus the algebra homomorphism (3.32) identifies $H^*(X_m,\mathbb{Q})\cong\mathbb{Q}[x^2,x^{2m+1}]$, which is precisely the subring Q_m of W-quasi-invariants in $H^*(BT,\mathbb{Q})=\mathbb{Q}[x]$. This gives property (QI₅) and completes the proof of the first part of the theorem.

The last claim of the theorem follows from Sullivan's formality theorem [Sul77]. Indeed, the algebras $Q_m(W)$ have the presentation $\mathbb{Q}[\xi,\eta]/(\xi^2-\eta^{2m+1})$, where $|\eta|=4$ and $|\xi|=4m+2$ (see Example 3.4). Hence, by [Sul77, Remark (v), p. 317], they are *intrinsically* formal. This means that, for each $m \geq 0$, there is only one rational homotopy type that realizes Q_m .

From now on, we will assume that G = SU(2) and T = U(1) embedded in SU(2) in the standard way as a maximal torus.

Definition 3.10. We call the G-space $F_m(G,T) := G/T * E_{m-1}G$ the m-quasi-flag manifold and the associated homotopy quotient

$$X_m(G,T) := F_m(G,T)_{hG} = EG \times_G (G/T * E_{m-1}G)$$

the space of m-quasi-invariants for G = SU(2). These spaces fit in the Borel fibration sequence

$$(3.33) F_m(G,T) \xrightarrow{j_m} X_m(G,T) \xrightarrow{p_m} BG$$

that generalizes the fundamental sequence (3.13).

Remark 3.11. By definition, $H^*(X_m(G,T),\mathbb{Q}) = H_G^*(F_m(G,T),\mathbb{Q})$ for all $m \geq 0$. With this identification, the algebra homomorphisms $H^*(X_m,\mathbb{Q}) \to H^*(BT,\mathbb{Q})$ constructed in Theorem 3.9 (see (3.32)) are induced (on G-equivariant cohomology) by the natural inclusion maps

$$(3.34) i_0: G/T \hookrightarrow F_m(G,T), \quad gT \mapsto 1 \cdot (gT) + 0 \cdot x,$$

where $x \in E_{m-1}G$. Note that the maps (3.34) are null-homotopic in the category Top of ordinary spaces, the null homotopy being $i_t: gT \mapsto (1-t)\cdot (gT)+t\cdot x$; however, they are *not* null-homotopic in the category of G-spaces and G-equivariant maps. In fact, the proof of Theorem 3.9 shows that the maps induced by (3.34) on G-equivariant cohomology are injective and hence nontrivial.

3.4. T-equivariant cohomology. Our next goal is to compute the T-equivariant cohomology of the G-spaces $F_m(G,T)$ by restricting the G-action to the maximal torus $T \subset G$. The computation is based on the following simple observations.

Lemma 3.12. For all $m \geq 0$, there is a natural T-equivariant homeomorphism

$$(3.35) F_m(G,T) \cong \Sigma E_{2m}(T) ,$$

where Σ stands for the unreduced suspension in Top.

Proof. First, note that G is T-equivariantly homeomorphic to the (unreduced) join of two copies of T: the required homeomorphism

$$(3.36) T * T \cong G$$

can be explicitly written as $t\lambda + (1-t)\mu \mapsto t^{1/2}\lambda + (1-t)^{1/2}\mu j$, where G = SU(2) is identified with the group of unit quaternions in $\mathbb{H} = \mathbb{C} \oplus \mathbb{C} j$ and T = U(1) with unit complex numbers. Similarly, we can define a T-equivariant homeomorphism

$$(3.37) (G/T)^T * T \cong G/T$$

where $(G/T)^T$ denotes the set of T-fixed points in G/T. Combining (3.36) and (3.37) with natural associativity isomorphisms for joins, we get

(3.38)
$$F_m(G,T) = (G/T) * G^{*m} \cong (G/T)^T * T^{*(2m+1)} = \mathbb{S}^0 * E_{2m}(T)$$

which is equivalent to formula (3.35).

Lemma 3.13. For all $n \geq 0$, there are natural algebra isomorphisms

$$(3.39) H_T^*(\Sigma E_n(T), \mathbb{Q}) \cong \mathbb{Q}[x] \times_{\mathbb{Q}[x]/(x^{n+1})} \mathbb{Q}[x].$$

Proof. We compute

$$[\Sigma E_n(T)]_{hT} \simeq [\operatorname{hocolim}(\operatorname{pt} \leftarrow E_n(T) \to \operatorname{pt})]_{hT}$$

$$\simeq \operatorname{hocolim}(BT \leftarrow E_n(T)_{hT} \to BT)$$

$$\simeq \operatorname{hocolim}(BT \leftarrow B_n(T) \to BT)$$

where the last equivalence follows from the fact that $E_n(T)$ is an n-universal T-bundle, so that the T-action on $E_n(T)$ is free and hence $E_n(T)_{hT} \simeq E_n(T)/T = B_n(T)$ (see Section A). To complete the proof it remains to note that $BT \simeq \mathbb{CP}^{\infty}$ and $B_n(T) \cong \mathbb{CP}^n$ for T = U(1), with natural map $B_nT \to BT$ represented by the inclusion $\mathbb{CP}^{2m} \to \mathbb{CP}^{\infty}$ (see, e.g., [Sel97, Example 9.2.3]). Hence,

(3.40) shows that $[\Sigma E_n(T)]_{hT} \simeq \mathbb{CP}^{\infty} \bigvee_{\mathbb{CP}^n} \mathbb{CP}^{\infty}$, which, by Mayer-Vietoris sequence, yields the isomorphism (3.39).

As a consequence of Lemma 3.12 and Lemma 3.13, we get

Proposition 3.14. For all multiplicities $m \geq 0$, there are natural algebra isomorphisms

$$(3.41) H_T^*(F_m(G,T), \mathbb{Q}) \cong \mathbb{Q}[x] \times_{\mathbb{Q}[x]/(x^{2m+1})} \mathbb{Q}[x],$$

where $x \in H^2(BT, \mathbb{Q})$ is the universal (rational) Euler class.

Remark 3.15. For m=0, formula (3.41) is well known: it follows, for example, from a general combinatorial description of T-equivariant cohomology of equivariantly formal spaces in terms of moment graphs (see [GKM98]). In our subsequent paper, we will generalize the main localization theorem of [GKM98] to moment graphs with multiplicities, and as an application, extend the result of Proposition 3.14 to quasi-flag manifolds for an arbitrary compact connected Lie group.

Next, we recall the modules of $\mathbb{C}W$ -valued quasi-invariants, $\mathbf{Q}_k(W)$, introduced in [BC11]. In [BC11, Section 3.2], these modules are considered only for integral multiplicities $k \in \mathbb{Z}_+$; however, their definition makes sense — in the Coxeter case — for all $k \in \frac{1}{2}\mathbb{Z}_+$ (cf. [BC11, (3.8)]). We provide a natural topological interpretation of these modules.

Corollary 3.16. For all $n \geq 0$, there are natural isomorphisms of $\mathbb{Q}[x] \times W$ -modules

$$(3.42) H_T^*(\Sigma E_n(T), \mathbb{C}) \cong \mathbf{Q}_{\frac{n+1}{2}}(W).$$

In particular, $H_T^*(F_m(G,T),\mathbb{C}) \cong \mathbf{Q}_{m+\frac{1}{2}}(W)$ for all $m \geq 0$.

Proof. Under the isomorphism (3.39), the geometric action of $W = \mathbb{Z}/2\mathbb{Z}$ on $H_T^*(\Sigma E_n(T), \mathbb{Q})$ corresponds to the action $(p, q) \mapsto (s(q), s(p))$ on the fiber product. Relative to this action, we can then define the W-equivariant map

$$f: \mathbb{Q}[x] \times_{\mathbb{Q}[x]/(x^{n+1})} \mathbb{Q}[x] \to \mathbb{Q}[x] \otimes \mathbb{Q}W$$
, $(p, q) \mapsto \frac{1}{2}(p+qs)$

This map is obviously injective, and it is easy to see that its image is $\mathbb{Q}[x]e_0 + \mathbb{Q}[x]x^{n+1}e_1$, where $e_0 = (1+s)/2$ and $e_1 = (1-s)/2$ are the idempotents in $\mathbb{Q}W$ corresponding to the trivial and sign representations of W. Example 3.9 of [BC11] shows that $\mathrm{Im}(f)$ is precisely $\mathbb{Q}_{\frac{n+1}{2}}(W)$; thus, combining f with the isomorphism of Lemma 3.13 gives the required isomorphism (3.42). The last statement then follows from Proposition 3.14.

Remark 3.17. Recall that, for any compact connected Lie group G, there is a natural isomorphism

$$(3.43) H_G^*(X,\mathbb{Q}) \cong H_T^*(X,\mathbb{Q})^W$$

that extends the result of Borel's Theorem 2.7 to an arbitrary G-space X (see, e.g., [Hsi75, Chap III, Prop. 1]). For $X = F_m(G, T)$, it follows from Corollary 3.16 that

$$H_T^*(F_m(G,T), \mathbb{C})^W \cong e_0 \mathbf{Q}_{m+\frac{1}{2}}(W) \cong Q_m(W).$$

Thus the isomorphism (3.27) of Theorem 3.9 can be deduced from (3.41) by (3.43).

3.5. **Divided difference operators.** As an application of Theorem 3.9, we give a topological construction of generalized divided difference operators associated with quasi-invariants. Recall that the classical divided difference operators $\Delta_{\alpha}: \mathbb{Q}[V] \to \mathbb{Q}[V]$ are attached to reflections $s_{\alpha} \in W$ of a Coxeter group W by the rule (cf. [Dem73, Dem74]):

$$(3.44) (1 - s_{\alpha})p = \Delta_{\alpha}(p) \cdot \alpha_H$$

where $\alpha_H \subset V^*$ is a linear form vanishing on the reflection hyperplane $H = H_{\alpha}$. Note that (3.44) defines Δ_{α} uniquely up to a nonzero constant factor. The definition of quasi-invariants of Coxeter groups suggests the following natural generalization of (3.44):

$$(3.45) (1 - s_{\alpha})p = \Delta_{\alpha}^{(m_{\alpha})}(p) \cdot \alpha_{H}^{2m_{\alpha}+1}$$

To be precise, given a W-invariant multiplicity function $m: A \to \mathbb{Z}_+, \ \alpha \mapsto m_{\alpha}$, formulas (3.45) define unique (up to nonzero constants) linear maps

$$\Delta_{\alpha}^{(m_{\alpha})}: Q_m(W) \to Q_0(W)$$

one for each reflection $s_{\alpha} \in W$. Note that $Q_0(W) = \mathbb{Q}[V]$, and for m = 0, the maps (3.46) coincide with the classical divided difference operators: $\Delta_{\alpha}^{(0)} = \Delta_{\alpha}$.

Definition 3.18. We call (3.46) the divided difference operators of W of multiplicity m.

When W has rank one, i.e. W is generated by a single reflection s, the corresponding map $\Delta_s^{(m)}$ takes values in $\mathbb{Q}[V]^W$ thus defining a linear operator on W-quasi-invariants:

$$\Delta_{\mathbf{s}}^{(m)}: Q_m(W) \to Q_m(W) .$$

The operator (3.47) has a natural topological interpretation in terms of our spaces of quasiinvariants. The proof of Theorem 3.9 shows that the basic fibration (3.33) is equivalent to a sphere fibration with fibre $F_m \simeq \mathbb{S}^{4m+2}$. Hence, associated to (3.33) there is a Gysin long exact sequence of the form (see, e.g., [McC01, Example II.5.C]):

$$(3.48) \qquad \dots \to H^n(BG,\mathbb{Q}) \xrightarrow{p_m^*} H^n(X_m,\mathbb{Q}) \xrightarrow{(p_m)_*} H^{n-4m-2}(BG,\mathbb{Q}) \to H^{n+1}(BG,\mathbb{Q}) \to \dots$$

where p_m^* is the natural pullback map induced on cohomology by the m-th whisker map $p_m: X_m \to BG$ and $(p_m)_*$ is a 'wrong way' pushforward map called the Gysin homomorphism. Combining these last two maps, we get the graded linear endomorphism on $H^*(X_m, \mathbb{Q})$ of degree -(4m+2):

$$(3.49) p_m^* \circ (p_m)_* : H^*(X_m, \mathbb{Q}) \to H^*(X_m, \mathbb{Q})$$

The next proposition generalizes a well-known formula for the classical divided difference operators Δ_{α} (proven, for example, in [BE90]).

Proposition 3.19. Under the isomorphism of Theorem 3.9, the operator (3.49) coincides with the divided difference operator (3.47) of multiplicity m: i.e.,

$$\Delta_s^{(m)} = p_m^* \circ (p_m)_*$$

Proof. Since the algebra homomorphism $p_m^*: H^*(BG, \mathbb{Q}) \to H^*(X_m, \mathbb{Q})$ is injective (for all m), the Gysin sequence (3.48) breaks up into short exact sequences

$$(3.51) 0 \to H^*(BG, \mathbb{Q}) \xrightarrow{p_m^*} H^*(X_m, \mathbb{Q}) \xrightarrow{(p_m)_*} H^{*-4m-2}(BG, \mathbb{Q}) \to 0$$

Now, if we identify $H^*(BG,\mathbb{Q})=\mathbb{Q}[c_2]$ and $H^*(X_m,\mathbb{Q})=\mathbb{Q}[x^2,x^{2m+1}]$ as in (the proof of) Theorem 3.9, the map p_m^* takes c_2 to x^2 and hence c_2^k to x^{2k} for all $k\geq 0$. By exactness of (3.51),

we then conclude that $(p_m)_*(x^{2k}) = 0$, while $(p_m)_*(x^{2m+1}) = \kappa_m$, where $\kappa_m \in \mathbb{Q}^{\times}$ is a nonzero constant. Hence, $p_m^*(p_m)_*(x^{2k}) = 0$ for all $k \geq 0$; on the other hand, by projection formula,

$$p_m^*(p_m)_*(x^{2k+2m+1}) = p_m^*(p_m)_*(x^{2k} \cdot x^{2m+1})$$

$$= p_m^*(p_m)_*(p_m^*(c_2^k) \cdot x^{2m+1})$$

$$= p_m^*(c_2^k) \cdot (p_m)_*(x^{2m+1})$$

$$= \kappa_m x^{2k}$$

Thus, up to a nonzero constant factor, we have

$$p_m^*(p_m)_*(x^N) = \begin{cases} 0, & \text{if } N = 2k \\ x^{2k}, & \text{if } N = 2k + 2m + 1 \end{cases}$$

which agrees with the action of $\Delta_s^{(m)} = \frac{1}{x^{2m+1}}(1-s)$ on $Q_m(W) = \mathbb{Q}[x^2, x^{2m+1}]$.

4. 'FAKE' SPACES OF QUASI-INVARIANTS

By Theorem 3.9, the spaces $X_m(G,T)$ provide topological realizations for the algebras $Q_m(W)$ that are unique up to rational equivalence. This raises the question whether the $X_m(G,T)$'s are actually unique up to homotopy equivalence. In this section, we answer the above question in the negative by constructing a natural class of counterexamples related to finite loop spaces. These remarkable loop spaces — sometimes referred to as fake Lie groups — were originally constructed by D. L. Rector [Rec71a] as examples of nonstandard ('exotic') deloopings of \mathbb{S}^3 . We will show that the rational cohomology rings of the spaces of quasi-invariants associated to Rector's spaces are isomorphic to the 'genuine' spaces of quasi-invariants $X_m(G,T)$; however, the spaces themselves are not homotopy equivalent (in fact, as we will see in Section 5, they can be distinguished K-theoretically). Thus, we get many different topological realizations of $Q_m(W)$, but among these only the 'genuine' spaces of quasi-invariants $X_m(G,T)$ satisfy all properties (QI₁)-(QI₅).

4.1. **Finite loop spaces.** We recall the definition of a finite loop space which is a natural homotopy-theoretic generalization of a compact Lie group. An exposition of classical results as well as many interesting examples of finite loop spaces can be found in the monograph [Kan88]; for more recent developments, we refer to the survey papers [Not95], [Dwy98], and [Gro10].

Definition 4.1. A finite loop space is a pointed connected space B such that ΩB is homotopy equivalent to a finite CW-complex.

It is convenient to represent a finite loop space as a triple (X,B,e), where X is a finite CW-complex, B is a pointed connected space, and $e:X\stackrel{\sim}{\to}\Omega B$ is a homotopy equivalence. A prototypical example is (G,BG,e), where G is a compact Lie group, BG its classifying space, and $e:G\stackrel{\sim}{\to}\Omega BG$ is a canonical equivalence. In general, finite loop spaces have many properties in common with compact Lie groups; however, the class of such spaces is much larger. In fact, if G is a compact connected non-abelian Lie group, there exist uncountably many homotopically distinct spaces B such that $\Omega B \simeq G$; thus the underlying topological space of G carries uncountably many finite loop structures (see [MÖ2]). In the case G = SU(2), this striking phenomenon was originally discovered by Rector [Rec71a] (see Theorem 4.2 below).

4.2. Fake Lie groups of type SU(2). We will work with localizations of topological spaces in the sense of D. Sullivan. A modern exposition of this classical construction can be found in [MP12]. Given a space X and a prime number p, we denote the localization of X at p by $X_{(p)}$. Recall (cf. [MP12, 8.5.1]) that two (nilpotent, finite type) spaces X and Y are said to be in the same genus if $X_{(p)} \simeq Y_{(p)}$ for every prime p. We are interested in finite loop spaces B (see Definition 4.1) that are in the same genus as BG for some compact connected Lie group G. Such spaces (called fake Lie groups) have been studied extensively in the literature (see, e.g., [NS90]), since their original discovery in [Rec71a]. This last paper gave a complete homotopy classification of spaces in the genus of BG for G = SU(2), and proposed a simple criterion to distinguish the genuine BSU(2) among these spaces: more precisely,

Theorem 4.2 (Rector). Let G = SU(2), and let B be a space in the genus of BG. Then, for each prime p, there is a homotopy invariant $(B/p) \in \{\pm 1\}$ called the Rector invariant of B at p, such that

- (1) The set $\{(B/p)\}$, where p runs over all primes, is a complete set of invariants of B in the genus of BG.
- (2) Every combination of values of (B/p) can occur for some B. In particular, the genus of BG consists of uncountably many distinct homotopy types.
 - (3) The Rector invariant of B = BG equals 1 at all primes p.
 - (4) The space B admits a maximal torus⁴ if and only if B is homotopy equivalent to BG.

Remark 4.3. Each space B in the genus of BSU(2) defines a loop structure on \mathbb{S}^3 , i.e. $\Omega B \simeq \mathbb{S}^3$. Conversely, a uniqueness theorem of Dwyer, Miller and Wilkerson [DMW87] implies that every loop structure on \mathbb{S}^3 belongs to the genus of BSU(2). Thus, Theorem 4.2 combined with results of [DMW87] provides a complete classification of finite loop spaces of type SU(2).

Remark 4.4. It was a long-standing conjecture in homotopy theory (motivated in part by Theorem 4.2(4), cf. [Wil74]) that a finite loop space with a maximal torus is homotopy equivalent to the classifying space of a compact Lie group. This conjecture was eventually proved by Anderson and Grodal using the Classification Theorem of p-compact groups (see [AG09]). Thus, the existence of maximal tori provides a purely homotopy-theoretic characterization of compact Lie groups among finite loop spaces.

Even though the spaces $B \not\simeq BG$ do not admit maximal tori, this does not rule out the possibility that there could exist interesting maps $f:BT\to B$ whose homotopy fibres are *not* finite CW complexes. In his thesis (see [Yau04]), D. Yau refined Rector's classification by describing the spaces B in the genus of BSU(2) that can occur as targets of essential (i.e., non-nullhomotopic) maps from BT. Such spaces admit a beautiful arithmetic characterization:

Theorem 4.5 (Yau). Let G = SU(2), and let B be a space in the genus of BG. Then

- (1) B admits an essential map $f: BT \to B$ if and only if there is an integer $k \neq 0$ such that (B/p) = (k/p) for all but finitely many primes p, where (k/p) denotes the Legendre symbol⁵ of k.
- (2) If B satisfies condition (1), then there exists a unique (up to homotopy) map $p_B : BT \to B$ such that every essential map $f : BT \to B$ is homotopic to $g \circ p_B$ for some self-map g of B.
 - (3) For B = BG, the map $p_{BG}: BT \to BG$ is induced by the maximal torus inclusion.

⁴We say that a finite loop space B admits a maximal torus if there is a map $p: BT_n \to B$ from the classifying space of a finite-dimensional torus with homotopy fibre being a finite CW-complex (see [Rec71b]).

⁵Recall that, for a prime p, the Legendre symbol (k/p) of an integer k is defined whenever $p \nmid k$: for p odd, we have (k/p) = 1 (resp., -1) if k is a quadratic residue (resp., nonresidue) mod p, while for p = 2, (k/2) = 1 (resp., -1) if k is quadratic residue (resp. nonresidue) mod p.

4.3. 'Fake' spaces of quasi-invariants. Let B be a space in the genus of BG (for G = SU(2)) that admits an essential map from BT. Theorem 4.5 shows that, for such a space, there is a natural generalization of the maximal torus: namely, the 'maximal' essential map $p_B : BT \to B$. We let $F(\Omega B, T)$ denote the homotopy fibre of this map and apply the Ganea construction to the associated fibration sequence:

$$(4.1) \begin{array}{c|c} F(\Omega B,T) \xrightarrow{j_B} BT \xrightarrow{p_B} B \\ \downarrow & \pi_0 \downarrow & \parallel \\ F_1(\Omega B,T) \xrightarrow{j_{1,B}} X_1(\Omega B,T) \xrightarrow{p_{1,B}} B \\ \downarrow & \pi_1 \downarrow & \parallel \\ F_2(\Omega B,T) \xrightarrow{j_{2,B}} X_2(\Omega B,T) \xrightarrow{p_{2,B}} B \\ \downarrow & \downarrow & \parallel \\ \vdots & \vdots & \vdots \end{array}$$

As a result, we construct a tower of spaces $X_m(\Omega B, T)$ which we will refer to as the 'fake' spaces of quasi-invariants associated to the Rector space B. Note, if B = BG, then $\Omega B \simeq G$, and by Theorem 4.5(3), the map $p_B : BT \to BG$ is the maximal torus inclusion; hence, in this case, $X_m(\Omega B, T)$ are equivalent to the 'genuine' spaces $X_m(G, T)$ of quasi-invariants (see Definition 3.10).

To compute the cohomology of $X_m(\Omega B, T)$ we recall (cf. [Rec71a]) that any space B in the genus of BG can be represented as a (generalized) homotopy pullback:

$$(4.2) B = \operatorname{holim}_{\{p\}} \{ BG_{(p)} \xrightarrow{r_p} BG_{(0)} \xrightarrow{n_p} BG_{(0)} \} ,$$

where the indexing set $\{p\}$ runs over all primes, r_p denotes the natural map from the p-localization to the rationalization of BG, and the map n_p is induced by multiplication by an integer n_p which is relatively prime to p and such that $(n_p/p) = (B/p)$ for every p (for p = 2, one requires, in addition, that $n_p \equiv 1 \pmod{4}$).

Now, if a space B admits an essential map from BT, part (1) of Theorem 4.5 implies that the set of integers $\{n_p \in \mathbb{Z} : p \text{ prime}\}$ appearing in (4.2) can be chosen to be finite. Hence, for such spaces, we can define the natural number

$$(4.3) N_B := \min\{\operatorname{lcm}(n_p) \in \mathbb{N} : B = \operatorname{holim}_{\{p\}}(n_p \circ r_p)\},$$

which is clearly a homotopy invariant of B. Note that $N_B = 1$ iff B = BG; however, in general, N_B does not determine the homotopy type of B (see [Yau04, (1.8)] for a counterexample).

Lemma 4.6. For any space B in the genus of BG, $H^*(B,\mathbb{Z}) \cong \mathbb{Z}[u]$, where |u| = 4. If B admits an essential map from BT, then, with natural identification $H^*(BT,\mathbb{Z}) \cong \mathbb{Z}[x]$ as in Theorem 3.9, the map $p_B^*: H^*(B,\mathbb{Z}) \to H^*(BT,\mathbb{Z})$ is given by $p_B^*(u) = N_B x^2$, where N_B is defined by (4.3).

Proof. The first claim can be deduced easily from the fact that $\Omega B \simeq \mathbb{S}^3$ by looking at the Serre spectral sequence of the path fibration $\Omega B \to P_*B \to B$ (cf. [Rec71b, §4]). The second claim is a consequence of the last part of [Yau04, Theorem 1.7], which shows that (4.3) equals (up to sign) the degree of the map p_B^* on K-theory with coefficients in \mathbb{Z} and hence on cohomology.

Theorem 4.7. Let B be a space in the genus of BG admitting an essential map from BT.

- (i) All maps π_m in (4.1) are injective on rational cohomology. For each $m \geq 0$, the composite map $\tilde{\pi}_m = \pi_{m-1} \dots \pi_1 \pi_0$ induces an embedding $H^*(X_m(\Omega B, T), \mathbb{Q}) \hookrightarrow H^*(BT, \mathbb{Q}) = \mathbb{Q}[x]$ with image $Q_m(W) \subseteq \mathbb{Q}[x]$. Thus, $H^*(X_m(\Omega B, T), \mathbb{Q}) \cong Q_m(W)$ for all $m \geq 0$.
 - (ii) For each $m \geq 0$, there is an algebra isomorphism

$$H^*(X_m(\Omega B, T), \mathbb{Q}) \xrightarrow{\sim} H^*(X_m(G, T), \mathbb{Q})$$

making commutative the diagram

$$H^{*}(B,\mathbb{Q}) \xrightarrow{p_{m,B}^{*}} H^{*}(X_{m}(\Omega B,T),\mathbb{Q}) \xrightarrow{\tilde{\pi}_{m}^{*}} H^{*}(BT,\mathbb{Q})$$

$$(p_{BG}^{*})^{-1}p_{B}^{*} \downarrow \qquad \qquad \downarrow \qquad \qquad \parallel$$

$$H^{*}(BG,\mathbb{Q}) \xrightarrow{p_{m}^{*}} H^{*}(X_{m}(G,T),\mathbb{Q}) \xrightarrow{\tilde{\pi}_{m}^{*}} H^{*}(BT,\mathbb{Q})$$

where the map $(p_{BG}^*)^{-1}p_B^*$ is given explicitly by $u \mapsto N_B x^2$ (see Lemma 4.6).

Proof. We prove part (i) by induction on m. First, note that for m = 0, (i) as well as (ii) follow from Lemma 4.6. To perform the induction we define the subalgebras $Q'_m \subseteq \mathbb{Q}[x]$ for m > 0 by

$$Q'_0 := Q[x], \qquad Q'_m := \mathbb{Q} + N_B x^2 \cdot Q'_{m-1}, \ m > 0.$$

Clearly,

$$Q'_m = \mathbb{Q} + \mathbb{Q} \cdot N_B x^2 + \ldots + \mathbb{Q} \cdot (N_B x^2)^{m-1} + (N_B x^2)^m \mathbb{Q}[x] .$$

It follows that $Q'_m = Q_m$ as subrings of $\mathbb{Q}[x]$ for all m. Now assume that

$$H^*(X_m(\Omega B, T), \mathbb{Q}) \cong Q'_m,$$

and that $\tilde{\pi}_m^*$ is the inclusion $Q_m' \hookrightarrow \mathbb{Q}[x]$. To compute the cohomology of the fibre $F_m(\Omega B, T)$, we use the Eilenberg-Moore spectral sequence for the fibration sequence $F_m(\Omega B, T) \to X_m(\Omega B, T) \to B$, whose E_2 -term is

$$E_2^{*,*} \, = \, \operatorname{Tor}_{*,*}^{H^*(B)}(H^*(\mathrm{pt}),H^*(X_m(\Omega B,T))) \, \cong \, \operatorname{Tor}_{\mathbb{Q}[u]}^{*,*}(\mathbb{Q},Q_m')$$

By Lemma 4.6, $\operatorname{Tor}_{*,*}^{\mathbb{Q}[u]}(\mathbb{Q}, Q'_m)$ is the (co)homology of the complex

$$0 \longrightarrow Q'_m \xrightarrow{\cdot N_B x^2} Q'_m \longrightarrow 0 .$$

Since $Q'_m \subseteq \mathbb{Q}[x]$ is an integral domain, $\operatorname{Tor}_i^{\mathbb{Q}[x]}(\mathbb{Q}, Q'_m) = 0$ for i > 0. The Eilenberg-Moore spectral sequence for the fibration sequence $F_m(\Omega B, T) \to X_m(\Omega B, T) \to B$ therefore collapses to give

$$H^*(F_m(\Omega B, T), \mathbb{Q}) \cong Q'_m/(N_B x^2)$$
.

Further, since the Eilenberg-Moore spectral sequence is multiplicative, $j_{m,B}^*$ is the canonical quotient map. In particular, note that the cohomology of $F_m(\Omega B, T)$ is concentrated in even degree. The long exact sequence of cohomologies associated with the cofibration sequence $F_m(\Omega B, T) \to X_m(\Omega B, T) \to X_{m+1}(\Omega B, T)$ yields (for n even)

$$\tilde{H}^n(X_{m+1}[\Omega B,T)] \overset{\mathcal{T}^*_m}{\longleftrightarrow} \tilde{H}^n[X_m(\Omega B,T)] \overset{j^*_{m,B}}{\longleftrightarrow} \tilde{H}^n[F_m(\Omega B,T)] \overset{\partial}{\longrightarrow} \tilde{H}^{n+1}[X_{m+1}(\Omega B,T)]$$

Since $j_{m,B}^*$ is surjective, we have

$$\tilde{H}^{n+1}(X_{m+1}(\Omega B, T), \mathbb{Q}) = 0$$
 for n even.

Hence,

$$H^*(X_{m+1}(\Omega B, T), \mathbb{Q}) \cong \mathbb{Q} + \operatorname{Ker}(j_{m,B}^*) = \mathbb{Q} + (N_B x^2) \cdot Q_m' = Q_{m+1}',$$

with π_m^* being the inclusion $Q'_{m+1} \hookrightarrow Q'_m$. This completes the induction step, proving part (i). Part (ii) follows immediately from (i) combined with Lemma 4.6 (since $p_B = p_{m,B} \circ \tilde{\pi}_m$).

Corollary 4.8. For a fixed $m \geq 0$, all spaces $X_m(\Omega B, T)$ are rationally equivalent to $X_m(G, T)$ (and hence to each other).

Proof. This follows from Theorem 4.7 and the uniqueness part of Theorem 3.9. \Box

In Section 5.4 (see Corollary 5.12), we will show that $X_m(\Omega B, T) \not\simeq X_m(\Omega B', T)$ whenever $N_B \neq N_{B'}$. Thus Theorem 4.7 provides many different topological realizations⁶ for the algebras $Q_m(W)$. However, these do not give us different solutions to our realization problem (see Section 2.4), since none of the spaces B in the genus of BG (except for BG itself) admits a maximal torus and hence none carries a natural W-action. In addition, by Ganea's Theorem 3.1, hocolim $_m[X_m(\Omega B, T)] \simeq B$, which shows that property (QI₂) fails for $X_m(\Omega B, T)$ when $B \not\simeq BG$.

5. Equivariant K-theory

In this section, we compute the G-equivariant K-theory $K_G(F_m)$ of the m-quasi-flag manifold $F_m = F_m(G,T)$ associated to G = SU(2). We find that $K_G(F_m)$ is isomorphic to the ring $\mathcal{Q}_m(W)$ of exponential quasi-invariants of W. By the Atiyah-Segal Theorem, the (ordinary) K-theory of $X_m(G,T)$ is then isomorphic to the completion $\widehat{\mathcal{Q}}_m(W)$ of $\mathcal{Q}_m(W)$ with respect to the canonical augmentation ideal of R(G). For the 'fake' spaces of quasi-invariants, $X_m(\Omega B,T)$, associated to Rector spaces, the K-theory rings $K[X_m(\Omega B,T)]$ are new invariants that are not isomorphic to $\widehat{\mathcal{Q}}_m(W)$ in general and are strong enough to distinguish the $X_m(\Omega B,T)$ up to homotopy equivalence.

5.1. Equivariant K-theory. Recall that, for a compact Lie group G acting continuously on a compact topological space X, the $K_G(X)$ is defined to be the Grothendieck group of G-equivariant (complex topological) vector bundles on X. As shown in [Seg68], this construction extends to a $\mathbb{Z}/2$ -graded multiplicative generalized cohomology theory K_G^* on the category of (locally compact) G-spaces that is called the G-equivariant K-theory. We write $K_G^*(X) := K_G^0(X) \oplus K_G^1(X)$, with understanding that $K_G^0(X) \cong K_G^{2n}(X)$ and $K_G^1(X) \cong K_G^{2n+1}(X)$ for all $n \in \mathbb{Z}$. When G is trivial, $K_G^*(X)$ coincides with the ordinary complex K-theory $K^*(X)$, while for $X = \operatorname{pt}$, $K_G^*(\operatorname{pt})$ is the representation ring $K_G^*(G)$ of $K_G^*(G)$ in particular, we have $K_G^1(\operatorname{pt}) = 0$. In general, by functoriality of K_G^* , the trivial map $X \to \operatorname{pt}$ gives a canonical $K_G^*(G)$ -module structure on the ring $K_G^*(X)$ for any $K_G^*(X)$ has nice properties for which we refer the reader to [Seg68]. Here we only mention two technical results needed for our computations.

The first result is a well-known Künneth type formula for equivariant K-theory first studied by Hodgkin (see, e.g., [BZ00, Theorem 2.3]).

Theorem 5.1 (Hodgkin). Let G be a compact connected Lie group, such that $\pi_1(G)$ is torsion-free. Then, for any two G-spaces X and Y, there is a spectral sequence with E^2 -term

$$E_{*,*}^2 = \operatorname{Tor}_{*,*}^{R(G)}(K_G^*(X), K_G^*(Y))$$

that converges to $K_G^*(X \times Y)$, where $X \times Y$ is viewed as a G-space with the diagonal action.

The second result is the following Mayer-Vietoris type formula, which is also — in one form or another — well known to experts.

⁶It is tempting to conjecture that the (homotopy types of the) spaces $X_m(\Omega B, T)$ associated with the Rector spaces B admitting an essential map from BT constitute the set of all such realizations. Unfortunately, besides Theorem 3.9(2), we do not have much evidence for this conjecture.

Lemma 5.2. Let $f: U \to X$ and $g: U \to Y$ be proper equivariant maps of G-spaces. Let $Z = \text{hocolim}(X \xleftarrow{f} U \xrightarrow{g} Y)$, where 'hocolim' is taken in the category of G-spaces. Then, the abelian groups $K_G^*(X)$, $K_G^*(Y)$ and $K_G^*(Z)$ are related by the six-term exact sequence

$$K_G^0(Z) \xrightarrow{} K_G^0(X) \oplus K_G^0(Y) \xrightarrow{f^* - g^*} K_G^0(U)$$

$$\partial \downarrow \qquad \qquad \downarrow \partial$$

$$K_G^1(U) \xrightarrow{f^* - g^*} K_G^1(X) \oplus K_G^1(Y) \xrightarrow{} K_G^1(Z)$$

The proof of Lemma 5.2 can be found, for example, in [JO99].

5.2. K-theory of quasi-flag manifolds. We first introduce rings $\mathcal{Q}_m(W)$ of exponential quasi-invariants of a Weyl group W. Let \hat{G} be a compact connected Lie group with maximal torus T and associated Weyl group W. Let $\hat{T} := \operatorname{Hom}(T, U(1))$ denote the character lattice and R(T) the representation ring of T. It is well known that $R(T) \cong \mathbb{Z}[\hat{T}]$ via the canonical map induced by taking characters of representations, and $R(T)^W \cong R(G)$ via the restriction map $i^* : R(G) \to R(T)$ induced by the inclusion $i: T \hookrightarrow G$ (see, e.g., [Bou82, Chap. IX, Sect. 3]). Using the first isomorphism we identify $R(T) = \mathbb{Z}[\hat{T}]$ and write e^{λ} for the elements of R(T) corresponding to characters $\lambda \in \hat{T}$. Next, we let $\mathcal{R} \subseteq \hat{T}$ denote the root system of W determined by (G,T) and choose a subset $\mathcal{R}_+ \subset \mathcal{R}$ of positive roots in \mathcal{R} . If $s_\alpha \in W$ is the reflection in W corresponding to $\alpha \in \mathcal{R}_+$, then the difference $e^{\lambda} - e^{s_{\alpha}(\lambda)}$ in R(T) is uniquely divisible by $1 - e^{\alpha}$ for any $\lambda \in \hat{T}$. Following [Dem74], we define a linear endomorphism $\Lambda_\alpha : R(T) \to R(T)$ for each $\alpha \in \mathcal{R}_+$, such that

$$(5.1) (1 - s_{\alpha})f = \Lambda_{\alpha}(f) \cdot (1 - e^{\alpha}).$$

The operator Λ_{α} is an exponential analogue of the divided difference operator Δ_{α} introduced in Section 3.5(see (3.44)). Note that the conditions (1.2) defining the usual quasi-invariant polynomials can be written in terms of the divided difference operators as $\Delta_{\alpha}(p) \equiv 0 \mod(\alpha)^{2m_{\alpha}}$. This motivates the following definition of quasi-invariants in the exponential case.

Definition 5.3. An element $f \in R(T)$ is called an *exponential quasi-invariant of* W *of multiplicity* $m \in \mathcal{M}(W)$ if

(5.2)
$$\Lambda_{\alpha}(f) \equiv 0 \mod (1 - e^{\frac{\alpha}{2}})^{2m_{\alpha}}, \quad \forall \alpha \in \mathcal{R}_{+}.$$

Remark 5.4. In general, it may happen that $\frac{\alpha}{2} \notin \hat{T}$ for some $\alpha \in \mathcal{R}_+$, so that $e^{\frac{\alpha}{2}} \notin R(T)$. We view (5.2) as a congruence in the extended group ring $\mathbb{Z}[\frac{1}{2}\hat{T}]$ that naturally contains R(T).

We write $\mathcal{Q}_m(W)$ for the set of all $f \in R(T)$ satisfying (5.3) for a fixed multiplicity m. This set is closed under addition and multiplication in R(T), i.e. $\mathcal{Q}_m(W)$ is a commutative subring of R(T). (The latter can be easily seen from the twisted derivation property of Demazure operators: $\Lambda_{\alpha}(f_1f_2) = \Lambda_{\alpha}(f_1) \cdot f_2 + s_{\alpha}(f_1) \cdot \Lambda_{\alpha}(f_2)$ that holds for all $\alpha \in \mathcal{R}$, see [Dem74, Sect. 5.5].)

Example 5.5. We describe $\mathcal{Q}_m(W)$ explicitly in the case of G = SU(2) and T = U(1) the diagonal torus. In this case \hat{T} coincides with the weight lattice $P(\mathcal{R})$ which is generated by the fundamental weight $\varpi: T \to U(1)$ defined by $\varpi\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = t$. The corresponding (simple) root is $\alpha = 2\varpi$, and the Weyl group $W = \langle s_{\alpha} \rangle \cong \mathbb{Z}/2\mathbb{Z}$ acts on \hat{T} by $s_{\alpha}(\varpi) = -\varpi$. We have (5.3) $R(T) \cong \mathbb{Z}[z, z^{-1}] , \quad R(G) = R(T)^W \cong \mathbb{Z}[z + z^{-1}]$

where $z=e^{\varpi}=e^{\frac{\alpha}{2}}$. Now, with these identifications, we claim that

$$(5.4) \ \mathcal{Q}_m(W) = \mathbb{Z} \oplus \mathbb{Z} \cdot (z^{1/2} - z^{-1/2})^2 \oplus \mathbb{Z} \cdot (z^{1/2} - z^{-1/2})^4 \oplus \ldots \oplus (z^{1/2} - z^{-1/2})^{2m} \cdot \mathbb{Z}[z, z^{-1}] \ .$$

Indeed, if $f \in \mathbb{Z}[z, z^{-1}]$ can be written in the form (5.4), then

$$f - s_{\alpha}(f) \in (z^{1/2} - z^{-1/2})^{2m} (1 - s_{\alpha}) \mathbb{Z}[z, z^{-1}] = (z^{1/2} - z^{-1/2})^{2m} (z - z^{-1}) \mathbb{Z}[z, z^{-1}],$$

which shows that $\Lambda_{\alpha}(f) = (1-z^2)^{-1}(f-s_{\alpha}f)$ is divisible by $(1-z)^{2m} = (1-e^{\frac{\alpha}{2}})^{2m}$ in $\mathbb{Z}[z,z^{-1}]$. Thus $f \in \mathcal{Q}_m$. To see the converse denote the right-hand side of (5.4) by $\tilde{\mathcal{Q}}_m$. Note that there is a natural $\mathbb{Q}[z+z^{-1}]$ -module decomposition

$$\mathbb{Q}[z, z^{-1}] \cong \mathbb{Q}[z + z^{-1}] \oplus \mathbb{Q}[z + z^{-1}] \cdot \delta,$$

where $\delta := z - z^{-1}$. Writing $f = p + q \cdot \delta$ with $p, q \in \mathbb{Q}[z + z^{-1}]$, we find that $f - s_{\alpha}(f) = 2q\delta$. Thus, if $f \in \mathcal{Q}_m$ then $f - s_{\alpha}(f) \in (z^{1/2} - z^{-1/2})^{2m} (z - z^{-1}) \mathbb{Z}[z, z^{-1}]$ and hence $q \in (z^{1/2} - z^{-1/2})^{2m} \mathbb{Q}[z, z^{-1}]$. It follows that $f \in \tilde{\mathcal{Q}}_m \otimes \mathbb{Q}$. On the other hand, $(\tilde{\mathcal{Q}}_m \otimes \mathbb{Q}) \cap \mathbb{Z}[z, z^{-1}] = \tilde{\mathcal{Q}}_m$ which implies that $\mathcal{Q}_m \subseteq \tilde{\mathcal{Q}}_m$.

Let $F_m = F_m(G,T)$ be the m-quasi-flag manifold of G = SU(2) introduced in Section 3.3 (see Definition 3.10). Recall that F_m is a G-space of homotopy type of a finite CW-complex. The next theorem computes the G-equivariant K-theory of F_m , which is the main result of this section.

Theorem 5.6. This is a natural isomorphism of $\mathbb{Z}/2$ -graded commutative rings

$$K_G^*(F_m) \cong \mathcal{Q}_m(W)$$

Thus $K_G^0(F_m) \cong \mathcal{Q}_m(W)$ and $K_G^1(F_m) = 0$ for all $m \in \mathbb{Z}_+$.

Proof. Recall that $K_G^*(\mathrm{pt}) = R(G) \cong \mathbb{Z}[t]$, where t corresponds to the 2-dimensional irreducible representation of G = SU(2). The natural map $K_G^*(\mathrm{pt}) \to K_G^*(G) \cong K^*(\mathrm{pt})$ is then identified with the projection $\mathbb{Z}[t] \to \mathbb{Z}$ taking $t \mapsto 2$. For m = 0, by definition, we have $F_0 = G/T$, and hence (cf. Example 5.5)

(5.5)
$$K_G^*(G/T) \cong K_T^*(\text{pt}) = R(T) \cong \mathbb{Z}[z, z^{-1}]$$
.

Thus $K_G^0(F_0) \cong \mathbb{Z}[z,z^{-1}] = \mathcal{Q}_0(W)$ and $K_G^1(F_0) = 0$ as is well known. Further, the map $R(G) \to R(T)$ induced on G-equivariant K-theory by $G/T \to \operatorname{pt}$ is identified with $\mathbb{Z}[t] \to \mathbb{Z}[z,z^{-1}]$, $t \mapsto z+z^{-1}$.

Now, recall that $F_{m+1} = F_m * G$, which means

(5.6)
$$F_{m+1} \simeq \operatorname{hocolim}[F_m \leftarrow F_m \times G \to G] .$$

There is a canonical G-equivariant map $F_m \to F_{m+1}$ which we denote by $i_{m,m+1}$, which is nontrivial (not null-homotopic) in the homotopy category of G-spaces (see Remark 3.11). Let $i_{m,n}: F_m \to F_n$ denote the composite map $i_{m,n}:=i_{n-1,n}\circ\ldots\circ i_{m,m+1}$ for n>m. We claim that the map $i_{0,m}^*: K_G^*(F_m) \to K_G^*(G/T)$ induced by $i_{0,m}: G/T \to F_m$ is injective, and under the isomorphism (5.5), it is identified with the inclusion of $\mathcal{Q}_m(W)$ in $\mathbb{Z}[z,z^{-1}]$. We prove our claim by induction on m. For m=0, this is just (5.5).

Assume, for some $m \geq 0$, that $K_G^*(F_m) \cong \mathcal{Q}_m(W)$ and that the map $i_{0,m}^* : K_G^*(F_m) \to K_G^*(G/T)$ is identified with the inclusion of $\mathcal{Q}_m(W)$ in $\mathbb{Z}[z,z^{-1}]$ as a subring. Then the image of $t \in K_G^*(\mathrm{pt})$ in $K_G^*(F_m) \cong \mathcal{Q}_m(W)$ is $z+z^{-1}$. Since $K_G^*(G) \cong \mathbb{Z}$ has the free $K_G^*(\mathrm{pt}) \cong \mathbb{Z}[t]$ -module resolution $0 \to \mathbb{Z}[t] \to \mathbb{Z}[t] \xrightarrow{(t-2)} \mathbb{Z} \to 0$, the Tor-group

$$\operatorname{Tor}_{*}^{R(G)}(K_{G}^{*}(F_{m}), K_{G}^{*}(G)) \cong \operatorname{Tor}_{*}^{\mathbb{Z}[t]}(\mathcal{Q}_{m}, \mathbb{Z})$$

is identified with the homology of the two-term complex $0 \to \mathcal{Q}_m(W) \xrightarrow{\cdot (z+z^{-1}-2)} \mathcal{Q}_m(W) \to 0$, whose first homology vanishes since $\mathcal{Q}_m(W)$ is an integral domain. It follows that Hodgkin's spectral sequence (see Theorem 5.1) that

$$K_G^*(F_m \times G) \cong \mathcal{Q}_m(W)/(z+z^{-1}-2)$$
,

and that the map $K_G^*(F_m) \to K_G^*(F_m \times G)$ induced by the projection $F_m \times G \to F_m$ is the canonical quotient map $\pi: \mathcal{Q}_m(W) \to \mathcal{Q}_m(W)/(z+z^{-1}-2)$. Next, applying Lemma 5.2 to the homotopy pushout (5.6), we obtain the four-term exact sequence

$$(5.7) \quad 0 \to K_G^0(F_{m+1}) \xrightarrow{(i_{m,m+1},f)^*} \mathcal{Q}_m(W) \oplus \mathbb{Z} \xrightarrow{i^* - \pi^*} \mathcal{Q}_m(W)/(z+z^{-1}-2) \xrightarrow{\partial} K_G^1(F_{m+1}) \to 0 ,$$

where $i: \mathbb{Z} \to \mathcal{Q}_m(W)$ is the structure map of the ring $\mathcal{Q}_m(W)$) and $f: G \to F_{m+1}$ is the natural map associated to (5.6). It follows from (5.7) that $K_G^1(F_{m+1}) = 0$, and

$$K_G^0(F_{m+1}) \cong \text{Ker}(i^* - \pi^*) = \mathbb{Z} + (z + z^{-1} - 2) \cdot \mathcal{Q}_m(W) = \mathcal{Q}_{m+1}(W)$$
.

Furthermore, the inclusion $\mathcal{Q}_{m+1}(W) \hookrightarrow \mathcal{Q}_m(W)$ is identified with the map $i_{m,m+1}^*$. This completes the induction step, completing the proof of the theorem.

5.3. The equivariant Chern character. Recall that the space $X_m = X_m(G,T)$ of m-quasi-invariants is defined as the homotopy G-quotient $X_m := EG \times_G F_m$. The Borel construction yields a natural map

$$(5.8) \alpha: K_G^*(F_m) \to K^*(X_m)$$

where $K^*(X) = K^0(X) \oplus K^1(X)$ is the (complex) topological K-theory defined by $K^0(X) = [X, BU]$ and $K^1(X) = [X, U]$. Theorem 5.6 shows that $K_G^*(F_m)$ is a finitely generated R(G)-module for all $m \in \mathbb{Z}_+$. Hence, by Atiyah-Segal Completion Theorem [AS69], the map (5.8) extends to an isomorphism

$$\widehat{K}_G^*(F_m)_{I_G} \cong K^*(X_m)$$

where $\widehat{K}_{G}^{*}(F)_{I_{G}}$ denotes the (adic) completion of $K_{G}^{*}(F)$ (as an R(G)-module) with respect to the augmentation ideal of R(G) defined as the kernel of the dimension function $I_{G} := \text{Ker}[\dim : R(G) \to \mathbb{Z}]$. If we identify $R(G) \cong \mathbb{Z}[z+z^{-1}]$ as the invariant subring of $R(T) \cong \mathbb{Z}[z,z^{-1}]$ as in the proof of Theorem 5.6, then $I_{G} = (z+z^{-1}-2)$. Thus, as a consequence of (5.9), we get

Corollary 5.7. For all $m \geq 0$, there is an isomorphism

$$K^*(X_m) \cong \widehat{\mathcal{Q}}_m(W)_I$$

where $(\widehat{\mathcal{Q}}_m)_I$ denotes the completion of (5.4) with respect to the ideal $I = (z+z^{-1}-2) \subset \mathbb{Z}[z+z^{-1}]$.

Next, we compute a Chern character map relating equivariant K-theory to equivariant cohomology. Recall that the Chern character of an equivariant vector bundle on a G-space F is defined as the (non-equivariant) Chern character of the associated vector bundle on $EG \times_G F$. This gives a natural map

where $\widehat{H}_{G}^{*}(F,\mathbb{Q}):=\prod_{k=0}^{\infty}H_{G}^{k}(F,\mathbb{Q})$. The following proposition describes the map (5.10) for $F=F_{m}(G,T)$ explicitly, using the identifications of Theorem 3.9 and Theorem 5.6.

Proposition 5.8. (1) The Chern character map $\operatorname{ch}_G(F_m): K_G^*(F_m) \to \widehat{H}^*(X_m, \mathbb{Q})$ is given by

(5.11)
$$\exp: \ \mathcal{Q}_m(W) \to \widehat{Q}_m(W) \ , \quad z \mapsto \sum_{n=0}^{\infty} \frac{x^n}{n!} \ ,$$

where $\widehat{Q}_m(W) := Q_m(W) \otimes_{\mathbb{Q}[x^2]} \mathbb{Q}[[x^2]]$ is the completed ring of quasi-invariants of $W = \mathbb{Z}/2\mathbb{Z}$. (2) The map $\operatorname{ch}_G(F_m)$ factors through (5.8) inducing an isomorphism on rational K-theory

$$K(X_m)_{\mathbb{Q}} \cong \widehat{H}^*(X_m, \mathbb{Q}) \cong \widehat{Q}_m(W)$$

Proof. For $F_0 = G/T$, we can identify $K_G^*(G/T) \cong R(T) \cong \mathbb{Z}[z,z^{-1}]$ and $\widehat{H}_G^*(G/T,\mathbb{Q}) = \widehat{H}^*(BT,\mathbb{Q}) \cong \mathbb{Q}[[x]]$ as in (the proofs of) Theorem 3.9 and Theorem 5.6. With these identifications, it is well known that the equivariant Chern character is given by exponentiation (see, e.g., [FRW21, Example A.5]):

(5.12)
$$\operatorname{ch}_G(G/T): K_G^*(G/T) \to \widehat{H}^*(BT, \mathbb{Q}), \qquad z \mapsto \exp(x) .$$

Now, by functoriality of the Chern character, the maps $G/T \xrightarrow{i_{0,m}} F_m \to \text{pt}$ give a commutative diagram of ring homomorphisms

where the vertical maps as well as the top and the bottom horizontal maps are injective. Hence, the map in the middle, $\operatorname{ch}_G(F_m): K_G^*(F_m) \to \widehat{H}^*(F_m, \mathbb{Q})$, is also injective, and it is given by restriction of the exponential map (5.12). This proves the first claim of the proposition. The second claim follows from the first and Corollary 5.7.

5.4. K-theory of 'fake' spaces of quasi-invariants. In this section, we compute the K-theory of 'fake' spaces of quasi-invariants $X_m(\Omega B, T)$ constructed in Section 4.3. We will keep the notation G = SU(2) and T = U(1) and use the identification $K^*(BT) \cong \mathbb{Z}[[t]]$ as in the previous section. Let B be a space in the genus of BG that admits an essential map from BT. By [Yau04, Proposition 2.1], there is an isomorphism of rings $K^*(B) \cong \mathbb{Z}[[u]]$, such that for any essential map $f: BT \to B$, the induced map $f^*: K^*(B) \to K^*(BT)$ is given by

$$f^*(u) = \deg(f)t^2 + \text{higher order terms in } t$$
,

where the integer $\deg(f)$ coincides (up to sign) with the degree of f in integral (co)homology in dimension 4 (cf. Lemma 4.6). In fact, by a general result of Notbohm and Smith (see [NS90, Theorem 5.2]), the assignment $f \mapsto f^*$ gives a bijection between the homotopy classes of maps from BT to B and the λ -ring homomorphisms from $K^*(B)$ to $K^*(BT)$:

$$[BT, B]_* \cong \operatorname{Hom}_{\lambda}(K^*(B), K^*(BT))$$
.

Next, recall that, by Theorem 4.5, among all essential maps $BT \to B$, there is a 'maximal' one $p_B : BT \to B$, for which $\deg(p_B) = N_B$, where N_B is the integer defined by (4.3): the

corresponding power series

(5.14)
$$p_B^*(u) = N_B t^2 + \text{higher order terms in } t.$$

is a useful K-theoretic invariant of B that depends on the Rector invariants (B/p) (see [Yau04]). Using (5.14), we define a sequence of subrings $\mathcal{Q}_m(B)$ in $\mathbb{Z}[[t]]$ inductively by the rule:

(5.15)
$$Q_0(B) := \mathbb{Z}[[t]], \qquad Q_m(B) := \mathbb{Z} + p_B^*(u)Q_{m-1}(B), \quad m \ge 1.$$

Note that there are natural inclusions

$$Q_0(B) \supseteq Q_1(B) \supseteq \ldots \supseteq Q_m(B) \supseteq Q_{m+1}(B) \supseteq \ldots$$

which are all ring homomorphisms.

Example 5.9. For B = BG, one can easily compute the power series $p_B(u)$ in an explicit form. Recall that the Atiyah-Segal completion theorem gives an isomorphism $K^*(BG) \cong \widehat{K}_G^*(\operatorname{pt})_I$, where $I = I_G$ is the ideal of virtual representations in $K_G^*(\operatorname{pt}) \cong R(G)$ of dimension 0. If we identify $K_G^*(\operatorname{pt}) \cong \mathbb{Z}[v]$, where v is the standard 2-dimensional representation of G, then I = (v-2), and $K^*(BG) \cong \mathbb{Z}[[u]]$, where u = v-2. Similarly, $K^*(BT) \cong \mathbb{Z}[[t]]$, where t = z-1, with z standing for the generating character of T. The naturality of (5.8) (with respect to the G-equivariant map $p: G/T \to \operatorname{pt}$) yields the commutative diagram

$$K_G^*(\mathrm{pt}) \cong \mathbb{Z}[u] \xrightarrow{\alpha} K^*(BG) \cong \mathbb{Z}[[u]]$$

$$p^* \downarrow \qquad p_B^* \downarrow \qquad .$$

$$K_G^*(G/T) \cong \mathbb{Z}[t] \xrightarrow{\alpha} K^*(BT) \cong \mathbb{Z}[[t]]$$

Since $p^*(v)$ is the restriction of v to T, we have $p^*(v) = z + z^{-1}$. Hence,

$$p_B^*(u) = p_B^*(v-2) = z + z^{-1} - 2 = (1+t) + \frac{1}{1+t} - 2 = \frac{t^2}{1+t}$$

It follows that $\mathcal{Q}_m(BG) \cong \widehat{\mathcal{Q}}_m(W)$, where the right-hand side is the completion of $\mathcal{Q}_m(W)$ with respect to the ideal generated by $z + z^{-1} - 2$ (cf. Corollary 5.7).

Now, we state the main result of this section.

Theorem 5.10. There are isomorphisms of rings

(5.16)
$$K^*[X_m(\Omega B, T)] \cong \mathcal{Q}_m(B) , \forall m \ge 0.$$

In particular, $K^1[X_m(\Omega B,T)]=0$ for all $m\geq 0$. The maps $\pi_m^*: K^*[X_{m+1}(\Omega B,T)]\to K^*[X_m(\Omega B,T)]$ induced by the Ganea maps π_m in (4.1) correspond under (5.16) to the natural inclusions $\mathcal{Q}_{m+1}(B)\hookrightarrow \mathcal{Q}_m(B)$, and hence are all injective.

To prove Theorem 5.10 we will use an Eilenberg-Moore spectral sequence for K-theory in the following form.

Lemma 5.11. Let $F \to E \to B$ be a (homotopy) fibration sequence over a base B such that $K^*(\Omega B)$ is an exterior algebra in a finite number of generators of odd degrees. Then there is a multiplicative spectral sequence with $E_2^{i,*} \cong \operatorname{Tor}_i^{K^*(B)}(\mathbb{Z}, K^*(E))$ that strongly converges to $K^*(F)$.

The proof of Lemma 5.11 can be found, for example, in [JO99] (see Main Theorem, Part 3).

Proof of Theorem 5.10. We further claim that the ring homomorphisms

$$j_{m,B}^*: K^*[X_m(\Omega B, T)] \to K^*[F_m(\Omega B, T)]$$

induced by the fibre maps $j_{B,m}$ in (4.1) are surjective, and with (5.16), they induce isomorphisms

$$K^*[F_m(\Omega B, T)] \cong \mathcal{Q}_m(B)/(p_B^*(u))$$

We prove these facts together with the claims of Theorem 5.10 by induction on m.

For m = 0, we need only to compute $K^*[F_0(\Omega B, T)]$. This can be done using Lemma 5.11. Note that $K(\operatorname{pt}) \cong \mathbb{Z}$ has the obvious free resolution over $K^*(B) \cong \mathbb{Z}[[u]]$:

$$(5.17) 0 \to \mathbb{Z}[[u]] \xrightarrow{\cdot u} \mathbb{Z}[[u]] \to \mathbb{Z} \to 0$$

Hence $\operatorname{Tor}_*^{K^*(B)}(\mathbb{Z}, K^*(BT))$ can be identified with homology of the two-term complex $0 \to \mathbb{Z}[[t]] \xrightarrow{p_B^*(u)} \mathbb{Z}[[t]] \to 0$, where the map in the middle is given by the power series (5.14). Since $\mathbb{Z}[[t]]$ is an integral domain, it follows that $\operatorname{Tor}_i^{K^*(B)}(\mathbb{Z}, K^*(BT)) = 0$ for i > 0. The Eilenburg-Moore spectral sequence of Lemma 5.11 therefore collapses, giving an isomorphism

$$K^*[F_0(\Omega B, T)] \cong \mathbb{Z}[[t]]/(p_B^*(u))$$

Next, assume that $K^*[X_m(\Omega B, T)] \cong \mathcal{Q}_m(B)$ and that $K^*[F_m(\Omega B, T)] \cong \mathcal{Q}_m(B)/(p_B^*(u))$, with $j_{m,B}^*$ being the canonical quotient map. Since

$$X_{m+1}(\Omega B, T) \simeq \operatorname{hocolim} \left[\operatorname{pt} \stackrel{i_m}{\longleftarrow} F_m(\Omega B, T) \xrightarrow{j_{m,B}} X_m(\Omega B, T)\right],$$

and since $K^1(\mathrm{pt})=K^1[F_m(\Omega B,T)]=K^1[X_m(\Omega B,T)]=0$, Lemma 5.2 (with G trivial group) yields the four-term exact sequence

$$0 \to K^0[X_{m+1}(\Omega B, T)] \xrightarrow{(i_m^*, \pi_m^*)} \mathbb{Z} \oplus \mathcal{Q}_m(B) \xrightarrow{p_m^* - j_{m,B}^*} \mathcal{Q}_m(B)/(p_B^*(u)) \xrightarrow{\partial} K^1(X_{m+1}(\Omega B, T)) \to 0.$$

Here p_m is the trivial map from $F_m(\Omega B, T)$ to the point. Since $j_{m,B}^*$ is surjective, $K^1[X_{m+1}(\Omega B, T)] = 0$. The above six-term exact sequence also shows that $K^0[X_{m+1}(\Omega B, T)] \cong \operatorname{Ker}(p_m^* - j_{m,B}^*) \subseteq \mathbb{Z} \oplus \mathcal{Q}_m(B)$ (with isomorphism given by the map (i_m^*, π_m^*)). Projection to $\mathcal{Q}_m(B)$ identifies this kernel with $\mathcal{Q}_{m+1}(B) = \mathbb{Z} + p_B^*(u)\mathcal{Q}_m(B) \subset \mathcal{Q}_m(B)$. It follows that $K^*[X_{m+1}(\Omega B, T)] \cong \mathcal{Q}_{m+1}(B)$, with π_m^* being the inclusion of $\mathcal{Q}_{m+1}(B)$ into $\mathcal{Q}_m(B)$. Finally, by taking the (completed) tensor product of the resolution (5.17) with $\mathcal{Q}_{m+1}(B)$, we see that $\operatorname{Tor}_i^{K^*(B)}(\mathbb{Z}, \mathcal{Q}_{m+1}(B))$ is the homology of the complex

$$0 \to \mathcal{Q}_{m+1}(B) \xrightarrow{p_B^*(u)} \mathcal{Q}_{m+1}(B) \to 0$$

where the map in the middle is given by multiplication by the formal power series (5.14). Since $Q_{m+1}(B) \subseteq \mathbb{Z}[[t]]$ is an integral domain, $\operatorname{Tor}_{i}^{K^{*}(B)}(\mathbb{Z}, K^{*}(X_{m+1})) = 0$ for i > 0. The spectral sequence of Lemma 5.11 associated with the fibration sequence $F_{m+1} \to X_{m+1} \to B$ therefore collapses, giving

$$K^*[F_{m+1}(\Omega B, T)] \cong \mathcal{Q}_{m+1}(B)/(p_B^*(u))$$
,

with $j_{m+1,B}^*$ being the canonical quotient map. This completes the induction step and thus finishes the proof of the theorem.

Theorem 5.10 allows one to distinguish spaces of quasi-invariants of the same multiplicity associated to homotopically distinct spaces in the genus of BG. First, we recall that the topological K-theory $K^*(X)$ of any space X of homotopy type of a CW complex carries a natural filtration

$$F^0K^*(X) \supseteq F^1K^*(X) \supseteq \ldots \supseteq F^nK^*(X) \supseteq F^{n+1}K^*(X) \supseteq \ldots$$

where $F^nK^*(X)$ is defined to be the kernel of the restriction map $K^*(X) \to K^*(X_{n-1})$ corresponding to the (n-1)-skeleton of X. This filtration is functorial: any map $f: X \to X'$ of spaces, each of which has homotopy type of a CW complex, induces a morphism of filtered rings $f^*: K^*(X') \to K^*(X)$. Moreover, by Cellular Approximation Theorem, it is independent of the CW structure in the sense that using a different CW structure on X will not change the isomorphism type of $K^*(X)$ as a filtered ring.

Corollary 5.12. Let B and B' be two spaces in the genus of BG admitting essential maps from BT. Assume that $N_B \neq N_{B'}$. Then $X_m(\Omega B, T) \not\simeq X_m(\Omega B', T)$ for any $m \geq 0$. In particular, $X_m(\Omega B, T)$ is not homotopy equivalent to $X_m(G, T)$ for any $B \not\simeq BG$.

Proof. Let $\tilde{\pi}_m : BT \to X_m(\Omega B, T)$ denote the composite map $\pi_{m-1} \circ \ldots \circ \pi_0$ in (4.1). By Theorem 5.10, this map induces an embedding

$$\tilde{\pi}_m^* : K^*[X_m(\Omega B, T)] \cong \mathcal{Q}_m(B) \hookrightarrow \mathbb{Z}[[t]] \cong K^*(BT)$$

which is a morphism of filtered rings. Now, recall that $BT \simeq \mathbb{CP}^{\infty}$; the generator t in $K^*(BT) \cong K^*(\mathbb{CP}^{\infty}) = \mathbb{Z}[[t]]$ can be taken in the form $t = b\xi$, where $\xi \in F^2K^2(BT)$ and $b \in K^{-2}(\operatorname{pt})$ is the Bott element (see [Yau04, Sect. 3]). Hence $t \in F^2K^0(BT)$, and therefore, by (5.14), we have

$$p_B^*(u) \equiv N_B t^2 \pmod{F^5 K^*(BT)}$$
 in $\mathbb{Z}[[t]]$

Now, by Theorem 5.10,

$$K^*[X_m(\Omega B, T)] \cong \mathcal{Q}_m(B) = \mathbb{Z} + \mathbb{Z} \cdot p_B^*(u) + \ldots + \mathbb{Z} \cdot p_B^*(u)^{m-1} + p_B^*(u) \cdot \mathbb{Z}[[t]].$$

Hence

$$K^*[X_m(\Omega B, T)]/F^5K^*[X_m(\Omega B, T)] \cong \mathbb{Z} + \mathbb{Z} \cdot N_B t^2$$
,

where the generator $N_B t^2$ is square zero. It follows that if p is a prime then

$$K^*[X_m(\Omega B, T)]/(p, F^5 K^*[X_m(\Omega B, T)]) \cong \begin{cases} (\mathbb{Z}/p\mathbb{Z}) + (\mathbb{Z}/p\mathbb{Z}) \cdot \bar{N}_B t^2 & \text{if } p \nmid N_B \\ (\mathbb{Z}/p\mathbb{Z}) & \text{if } p \mid N_B \end{cases}$$

where $(p, F^5K^*(X_m))$ denotes the ideal in $K^*(X_m)$ generated by $p \in \mathbb{Z}$ and $F^5K^*(X_m)$. This shows that $X_m(\Omega B, T)$ is not homotopy equivalent to $X_m(\Omega B', T)$ unless $N_B = N_{B'}$.

Remark 5.13. The converse of Corollary 5.12 also holds true in the following sense: if two spaces B and B' in the genus of BG have homotopy equivalent towers of spaces of quasi-invariants $\{X_m(\Omega B,T),\pi_m\}_{m\geq 0}$ and $\{X_m(\Omega B',T),\pi'_m\}_{m\geq 0}$, then $B\simeq B'$. This simply follows from the fact that

$$\operatorname{hocolim}_{m \in \mathbb{Z}_+} X_m(\Omega B, T) \simeq B$$
,

which is a consequence of Ganea's Theorem 3.1.

6. Elliptic cohomology

In this section, we compute complex analytic T- and G-equivariant elliptic cohomology of the quasi-flag manifolds $F_m(G,T)$. We express the result in two ways: geometrically (in terms of coherent sheaves on a given elliptic curve E) and analytically (in terms of Θ -functions and q-difference equations). We also compute the spaces of global sections of the elliptic cohomology sheaves of $F_m(G,T)$ with coefficients twisted by tensor powers of the Looijenga line bundle on E. This last computation provides a motivation for our definition of elliptic quasi-invariants of W.

6.1. **Equivariant elliptic cohomology.** Complex analytic elliptic cohomology was introduced by I. Grojnowski (see [Gro07]). We will follow the approach of [Gan14] that relies on earlier topological results of [And00] and [Ros01]. We begin by briefly recalling the main definitions.

Let E be an elliptic curve defined analytically over \mathbb{C} . Given a compact connected abelian Lie group T (i.e., $T \cong U(1)^n$), we write $\check{T} := \text{Hom}(U(1), T)$ for its cocharacter lattice and set

$$\mathcal{M}_T := \check{T} \otimes_{\mathbb{Z}} E$$
,

which is an abelian variety of rank n = rk(T) defined over \mathbb{C} . The T-equivariant elliptic cohomology is defined as a functor on the (homotopy) category of finite T-CW complexes with values in the category of (complex-analytic) coherent sheaves on \mathcal{M}_T :

(6.1)
$$\mathcal{E}ll_T^*: \operatorname{Ho}(\operatorname{Top}_T^{fin}) \to \operatorname{Coh}(\mathcal{M}_T).$$

This functor has the basic property that $\mathcal{E}ll_T^*(T/T') \cong \mathcal{O}_{\mathcal{M}_{T'}}$ for any closed subgroup $T' \subseteq T$, where $\mathcal{O}_{\mathcal{M}_{T'}} = i^*\mathcal{O}_{\mathcal{M}_T}$ is the restriction of the structure sheaf of \mathcal{M}_T to $\mathcal{M}_{T'}$ (see [Gan14, 2.1(1)]). In particular, we have

(6.2)
$$\mathcal{E}ll_T^*(\mathrm{pt}) \cong \mathcal{O}_{\mathcal{M}_T}$$

Now, if G is a compact connected Lie group with maximal torus T and Weyl group W, we define the G-equivariant elliptic cohomology of a G-space X by

(6.3)
$$\mathcal{E}ll_C^*(X) := \mathcal{E}ll_T^*(X)^W,$$

where X is viewed as a T-space by restricting G-action (see [Gan14, 3.4]). To compute the G-equivariant elliptic cohomology we thus need to compute the T-equivariant elliptic cohomology of a G-space X and take its W-invariant sections.

The coherent sheaves $\mathcal{E}ll_T^*(X)$ do not have usually many interesting global sections. To remedy this one considers a twisted version of elliptic cohomology, where the sheaves $\mathcal{E}ll_T^*(X)$ are tensored with a certain ample line bundle on \mathcal{M}_T . Recall that, if G is a simple, simply connected compact Lie group with a root system \mathcal{R} , there is a canonical W-equivariant line bundle \mathcal{L} on \mathcal{M}_T associated to the invariant symmetric bilinear form I on the coroot lattice of \mathcal{R} such that $I(\alpha,\alpha)=2$ for all roots of smallest length in \mathcal{R} ; this line bundle has I as its Chern class and has degree equal to the order of the center of G (see [Loo77]). Following [And00, Gan14], we will refer to \mathcal{L} as the Looijenga bundle on \mathcal{M}_T and define the T- and G-equivariant elliptic cohomology of X with coefficients twisted by \mathcal{L} by

(6.4)
$$\operatorname{Ell}_{T}^{*}(X,\mathcal{L}) := \bigoplus_{n=0}^{\infty} H_{\operatorname{an}}^{0}(\mathcal{M}_{T}, \mathcal{E}ll_{T}^{*}(X) \otimes \mathcal{L}^{n})$$

(6.5)
$$\operatorname{Ell}_{G}^{*}(X,\mathcal{L}) := \bigoplus_{n=0}^{\infty} H_{\operatorname{an}}^{0}(\mathcal{M}_{T}, \mathcal{E}ll_{T}^{*}(X) \otimes \mathcal{L}^{n})^{W}$$

where $H_{\rm an}^0$ stands for the global (holomorphic) sections of the coherent sheaf $\mathcal{E}ll_T^*(X)$ twisted by the tensor powers of \mathcal{L} . Note that (6.4) and (6.5) are naturally graded modules over the graded commutative rings

(6.6)
$$\operatorname{Ell}_{T}^{*}(\operatorname{pt},\mathcal{L}) = \bigoplus_{n=0}^{\infty} H_{\operatorname{an}}^{0}(\mathcal{M}_{T},\mathcal{L}^{n}) \quad \text{and} \quad \operatorname{Ell}_{G}^{*}(\operatorname{pt},\mathcal{L}) = \bigoplus_{n=0}^{\infty} H_{\operatorname{an}}^{0}(\mathcal{M}_{T},\mathcal{L}^{n})^{W}$$

which we denote by S(E) and $S(E)^W$, respectively. Following [Loo77], we also write $S(E)^{-W}$ for the subspace of S(E) consisting of all W-anti-invariant sections. The main theorem of [Loo77] asserts that $S(E)^W$ is a graded polynomial algebra generated freely by l+1 homogeneous elements,

while $S(E)^{-W}$ is a free module over $S(E)^{W}$ of rank one (see [Loo77, (3.4)]). The generators of $S(E)^{W}$ are called the *Looijenga theta functions* on \mathcal{M}_{T} .

6.2. Elliptic cohomology of quasi-flag manifolds. In the rank one case (T = U(1)), we can identify $\mathcal{M}_T = E$ and take for a model of E the Tate curve $E_q := \mathbb{C}^*/q^{\mathbb{Z}}$ with 0 < |q| < 1. The latter is defined as the quotient of the punctured line $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ (viewed as a complex analytic manifold) by the free action of the infinite cyclic group \mathbb{Z} :

(6.7)
$$\mathbb{C}^* \to \mathbb{C}^* , \ z \mapsto q^n z .$$

We write $A := \mathcal{O}_{\mathrm{an}}(\mathbb{C}^*)$ for the ring of global analytic functions on \mathbb{C}^* and define $\mathcal{A}_q := A \rtimes_q \mathbb{Z}$ to be the crossed product algebra for the action of \mathbb{Z} on A induced by (6.7). Letting ξ be the (multiplicative) generator of \mathbb{Z} , we can present \mathcal{A}_q as a skew-polynomial algebra $A[\xi, \xi^{-1}]$ with multiplication determined by the commutation relation $\xi \cdot a(z) = a(qz) \cdot \xi$ for $a(z) \in A$. The left modules over \mathcal{A}_q can be identified with spaces of global sections of \mathbb{Z} -equivariant quasi-coherent sheaves on \mathbb{C}^* . The natural projection $\pi : \mathbb{C}^* \to E_q$ induces then the additive functor

(6.8)
$$\operatorname{Coh}(E_q) \to \operatorname{Mod}_A^{\text{f.p.}}(\mathcal{A}_q) , \quad \mathcal{F} \mapsto \widetilde{\mathcal{F}} := H_{\text{an}}^0(\mathbb{C}^*, \pi^* \mathcal{F}) ,$$

that maps the coherent sheaves on the analytic curve E_q to left \mathcal{A}_q -modules admitting a finite presentation $A^{\oplus m} \to A^{\oplus n} \to M \to 0$ over the subalgebra $A \subset \mathcal{A}_q$. The following proposition is a well-known result that provides a convenient algebraic description of the category $\operatorname{Coh}(E_q)$; its proof can be found in various places (see, for example, [SV03, Thm 2.2] or [vdPR07, Sect. 2]).

Proposition 6.1. The functor (6.8) is an exact equivalence of abelian tensor categories.

We remark that the tensor structure on $\operatorname{Coh}(E_q)$ is the standard geometric one (defined by tensor product of sheaves of \mathcal{O}_{E_q} -modules), while the tensor structure on $\operatorname{Mod}_A^{f.p.}(\mathcal{A}_q)$ is defined by tensoring \mathcal{A}_q -modules over the subalgebra A with the action of \mathcal{A}_q on $M_1 \otimes_A M_2$ given by $\xi \cdot (m_1 \otimes m_2) = (\xi \cdot m_1) \otimes (\xi \cdot m_2)$. The vector bundles on E_q correspond under (6.8) to \mathcal{A}_q -modules that are free of finite rank over A; such modules form a full subcategory of $\operatorname{Mod}_A^{f.p.}(\mathcal{A}_q)$ closed under the tensor product. The cohomology of a coherent sheaf \mathcal{F} on E_q can be computed algebraically in terms of \mathcal{A}_q -modules as invariants and coinvariants of the induced action of \mathbb{Z} on the corresponding A-module $\widetilde{\mathcal{F}}$ (see [vdPR07, Lemma 2.1]):

(6.9)
$$H_{an}^0(E_q, \mathcal{F}) \cong \operatorname{Ker}(\xi - \operatorname{id} : \widetilde{\mathcal{F}}), \quad H_{an}^1(E_q, \mathcal{F}) \cong \operatorname{Coker}(\xi - \operatorname{id} : \widetilde{\mathcal{F}}).$$

where ξ is the multiplicative generator of the copy of $\mathbb Z$ in $\mathcal A_q$ acting on the $\mathcal A_q$ -module $\widetilde{\mathcal F}$.

Example 6.2. The structure sheaf \mathcal{O}_{E_q} of E_q corresponds under (6.8) to the cyclic module $\widetilde{\mathcal{O}}_{E_q} = \mathcal{A}_q/\mathcal{A}_q(\xi-1)$, which can be identified as $\widetilde{\mathcal{O}}_{E_q} \cong Ae$ with generator e satisfying the relation $\xi e = e$. The line bundle $\mathcal{O}_{E_q}([1])$ corresponds to $\mathcal{A}_q/\mathcal{A}_q(\xi+z) \cong Ae$, with e satisfying $\xi e = -ze$. More generally, any line bundle on E_q of degree d corresponds to a cyclic \mathcal{A}_q -module Ae, where the generator e satisfies the relation $\xi e = cz^d e$ for some $c \in \mathbb{C}^*$ (see [vdPR07, Example 2.2]).

We now proceed with computing elliptic cohomology of the spaces $F_m = F_m(G, T)$. For a fixed Tate curve E_q , we first describe the T-equivariant cohomology, presenting the answer in two ways: in terms of coherent sheaves on E_q and in terms of \mathcal{A}_q -modules via the equivalence (6.8).

Theorem 6.3. For all $m \geq 0$, there are natural isomorphisms of coherent sheaves in $Coh(E_q)$

(6.10)
$$\mathcal{E}ll_T^*(F_m) \cong \mathcal{O}_{E_q} \times_{\mathcal{O}_{E_q}/\mathcal{J}^{2m+1}} \mathcal{O}_{E_q} ,$$

where \mathcal{J} is the subsheaf of ideals in the structure sheaf \mathcal{O}_{E_q} vanishing at the identity of E_q . Under the equivalence (6.8), the coherent sheaf (6.10) corresponds to the \mathcal{A}_q -module

(6.11)
$$\widetilde{\mathcal{E}ll}_T^*(F_m) \cong A \times_{A/\langle \Theta \rangle^{2m+1}} A,$$

where the action of \mathcal{A}_q on the fibre product is induced by the natural action of \mathcal{A}_q on A and $\langle\Theta\rangle$ denotes the (principal) ideal in $A = \mathcal{O}_{an}(\mathbb{C}^*)$ generated by the classical theta function

(6.12)
$$\Theta(z) := (1-z) \prod_{n>0} (1-q^n z)(1-q^n z^{-1}) = \frac{1}{(q;q)_{\infty}} \sum_{n \in \mathbb{Z}} q^{\frac{n(n-1)}{2}} (-z)^n.$$

Proof. Recall that, by Lemma 3.12, there is a T-equivariant homeomorphism

$$F_m \cong \Sigma E_{2m}(T) = \text{hocolim} (\text{pt} \leftarrow E_{2m}(T) \rightarrow \text{pt}),$$

where $E_{2m}(T) = T^{*(2m+1)}$ is Milnor's 2*m*-universal *T*-bundle. As equivariant *K*-theory, the *T*-equivariant elliptic cohomology is known to satisfy the Mayer-Vietoris property (see, e.g., [Ros01, Theorem 3.8]). Hence, as in Lemma 5.2, there is a six term long exact sequence of sheaves on E_q :

$$\mathcal{E}ll_{T}^{0}(F_{m}) \longrightarrow \mathcal{E}ll_{T}^{0}(\mathrm{pt}) \times \mathcal{E}ll_{T}^{0}(\mathrm{pt}) \xrightarrow{p_{1}^{*} - p_{2}^{*}} \mathcal{E}ll_{T}^{0}(E_{2m}(T))$$

$$\downarrow \qquad \qquad \downarrow$$

$$\mathcal{E}ll_{T}^{1}(E_{2m}T) \longleftarrow \mathcal{E}ll_{T}^{1}(\mathrm{pt}) \times \mathcal{E}ll_{T}^{1}(\mathrm{pt}) \longleftarrow \mathcal{E}ll_{T}^{1}(F_{m})$$

where the arrow on top of the diagram is given on sections by $(x_1, x_2) \mapsto p_1^*(x_1) - p_2^*(x_2)$, with p_1 and p_2 representing two copies of the canonical map $E_{2m}(T) \to \text{pt.}$ By (6.2), we know that $\mathcal{E}ll_T^*(\text{pt}) \cong \mathcal{O}_{E_q}$; on the other hand, by Lemma 6.4 (see below),

$$\mathcal{E}ll_T^*(E_{2m}T) \cong \mathcal{O}_{E_q}/\mathcal{J}^{2m+1},$$

where $\mathcal{J} \subset \mathcal{O}_{E_q}$ is the subsheaf of sections vanishing at $1 \in E_q$. Hence, by exactness, the above commutative diagram shows that $\mathcal{E}ll_T^1(F_m(G,T)) = 0$ and

$$\mathcal{E}ll_T^0(F_m) = \operatorname{Ker}(p_1^* - p_2^*) \cong \mathcal{O}_{E_q} \times_{\mathcal{O}_{E_q}/\mathcal{J}^{2m+1}} \mathcal{O}_{E_q}.$$

This proves the first claim of the theorem.

Now, to prove the second claim we observe that the skyscraper sheaf $\mathcal{F}:=i_{1,*}\mathbb{C}$ on E_q (with stalk \mathbb{C} supported at $\{1\}$) corresponds under (6.8) to the quotient module $\widetilde{\mathcal{F}}\cong A/\langle\Theta\rangle$, where the action of \mathcal{A}_q is induced by the natural action of \mathcal{A}_q on A. Indeed, $\widetilde{\mathcal{F}}$ is isomorphic to the cokernel of the map $\mathcal{O}_{E_q}\to\mathcal{O}_{E_q}([1])$, which is given (with identifications of Example 6.2) by $e\mapsto\Theta e$. This follows from the fact that as a global analytic function on \mathbb{C}^* , $\Theta=\Theta(z)$ has simple zeroes exactly at the points $z=q^n$ $(n\in\mathbb{Z})$. Hence the ideal sheaf $\mathcal{J}\subset\mathcal{O}_{E_q}$ corresponds to the ideal $\langle\Theta\rangle=A\Theta$ in A, and more generally, since (6.8) is a tensor functor, \mathcal{J}^{2m+1} corresponds to $\langle\Theta\rangle^{2m+1}=A\Theta^{2m+1}$ for all $m\geq 0$. Now, since (6.8) is an exact additive functor, it takes the fibre product $\mathcal{O}_{E_q}\times_{\mathcal{O}_{E_q}/\mathcal{J}^{2m+1}}\mathcal{O}_{E_q}$ in $\mathrm{Coh}(E_q)$ to the module $A\times_{A/\langle\Theta\rangle^{2m+1}}A$ in $\mathrm{Mod}_A^{\mathrm{f.p.}}(\mathcal{A}_q)$, thus completing the proof of the theorem. \square

Lemma 6.4. There are isomorphisms of sheaves $\mathcal{E}ll_T^*(E_nT) \cong \mathcal{O}_{E_q}/\mathcal{J}^{n+1}$ for all $n \geq 0$.

Proof. Note that T acts freely on $E_n(T) := T^{*(n+1)}$. Recall (see [Ros01, Sect. 3.2]) that if X is a finite T-space, the stalk at $a \in E$ of $\mathcal{E}ll_T^*(X)$ is isomorphic to $H_T^*(X^a; \mathbb{C}) \otimes_{\mathbb{C}[z]} \mathcal{O}_{\mathbb{C}^*,1}$, where $\mathcal{O}_{E_q,1}$ stands for the ring of germs of analytic functions at $1 \in E_q$. Here, X^a stands for the fixed point space X^{T_a} , where $T_a = \mathbb{Z}/k\mathbb{Z} \subset T$ if a is of finite order k in E, and $T_a = T$ if a is not of finite order in E. It follows that of T acts freely on X, the stalk $\mathcal{E}ll_T^*(X)_a$ of $\mathcal{E}ll_T^*(X)$ at a vanishes for

 $a \neq 1$. Hence, $\mathcal{E}ll_T(E_nT)_a = 0$ for $a \neq 1$, and for U a small neighborhood of 1 in E_q , $\mathcal{E}ll_T^*(X)|_U \cong H_T^*(E_nT;\mathbb{C}) \otimes_{\mathbb{C}[x]} \mathcal{O}_{E_q}|_U$, where $\mathcal{O}_{E_q}|_U$ acquires the structure of a sheaf of $\mathbb{C}[x]$ -modules via the map $\mathbb{C}[x] \to \mathcal{O}_{E_q}(U), x \mapsto \theta$, where θ is a generator of the maximal ideal of the local ring $\mathcal{O}_{\mathbb{C}^*,1}$. The desired lemma therefore, follows from the fact that $H_T^*(E_nT;\mathbb{C}) \cong H^*(B_nT;\mathbb{C}) \cong \mathbb{C}[x]/x^{n+1}$ (see the proof of Lemma 3.13 above).

To compute the G-equivariant elliptic cohomology of F_m we need to refine the result of Theorem 6.3 by taking into account the action of $W = \mathbb{Z}/2\mathbb{Z}$ on $\mathcal{E}ll_T^*(F_m)$. To this end we first refine the result of Proposition 6.1. Observe that the equivalence (6.8) extends to the category of W-equivariant coherent sheaves on E_g :

(6.13)
$$\operatorname{Coh}_{W}(E_{q}) \xrightarrow{\sim} \operatorname{Mod}_{A}^{f,p.}(\mathcal{A}_{q} \rtimes W) ,$$

where the category of \mathcal{A}_q -modules finitely presented over A is replaced by a similar category of modules over the crossed product algebra $\mathcal{A}_q \rtimes W$ associated to the geometric action of W on \mathbb{C}^* . The algebra $\mathcal{A}_q \rtimes W$ has the canonical presentation $A\langle \xi, \xi^{-1}, s \rangle$, where the generators ξ , s and $a(z) \in A$ are subject to the relations

$$s \cdot a(z) = a(z^{-1}) \cdot s$$
, $s \cdot \xi = \xi^{-1} \cdot s$, $\xi \cdot a(z) = a(qz) \cdot \xi$, $s^2 = 1$

We let $e_+ := (1+s)/2$ denote the symmetrizing idempotent in $\mathcal{A}_q \rtimes W$ and consider the subalgebra $e_+(\mathcal{A}_q \rtimes W)e_+$ of $\mathcal{A}_q \rtimes W$ (with identity element e_+). This subalgebra can be naturally identified with the invariant subalgebra \mathcal{A}_q^W of \mathcal{A}_q via the isomorphism: $\mathcal{A}_q^W \xrightarrow{\sim} e_+(\mathcal{A}_q \rtimes W)e_+$, $a \mapsto e_+a e_+$. With this identification, we can define the additive functor

(6.14)
$$\operatorname{Mod}(\mathcal{A}_q \rtimes W) \to \operatorname{Mod}(\mathcal{A}_q^W), \quad M \mapsto e_+ M,$$

that assigns to a W-equivariant \mathcal{A}_q -module its subspace of W-invariant elements viewed as a module over \mathcal{A}_q^W . The next result is well known for the algebra $\mathcal{A}_q^{\mathrm{alg}} := \mathcal{O}_{\mathrm{alg}}(\mathbb{C}^*) \rtimes_q \mathbb{Z}$ which is an algebraic (polynomial) analogue⁷ of $\mathcal{A}_q = \mathcal{O}_{\mathrm{an}}(\mathbb{C}^*) \rtimes_q \mathbb{Z}$. The analytic case easily reduces to the algebraic one as $\mathcal{A}_q^{\mathrm{alg}}$ is naturally a subalgebra of \mathcal{A}_q .

Lemma 6.5. The functor (6.14) is an equivalence of categories, its inverse being given by

$$\mathcal{A}_q \otimes_{\mathcal{A}_q^W} (-): \operatorname{Mod}(\mathcal{A}_q^W) \to \operatorname{Mod}(\mathcal{A}_q \rtimes W)$$

Proof. Lemma can be restated by saying that the algebra $\mathcal{A}_q \rtimes W$ is Morita equivalent to \mathcal{A}_q^W . To prove this, by standard Morita theory (see [MR01, 3.5.6]), it suffices to check that the idempotent e_+ generates the whole $\mathcal{A}_q \rtimes W$ as its two-sided ideal. This last condition holds for $\mathcal{A}_q^{\text{alg}} \rtimes W$, since $\mathcal{A}_q^{\text{alg}} \rtimes W$ is a simple algebra (has no proper two-sided ideals), if q is not a root of unity. But then it also holds for $\mathcal{A}_q \rtimes W$, since $\mathcal{A}_q^{\text{alg}} \rtimes W$ is a unital subalgebra of $\mathcal{A}_q \rtimes W$ containing e_+ . \square

Now, combining (6.13) with Morita equivalence (6.14), we get the equivalence

(6.15)
$$\operatorname{Coh}_{W}(E_{q}) \xrightarrow{\sim} \operatorname{Mod}_{A^{W}}^{\text{f.p.}}(\mathcal{A}_{q}^{W}) , \quad \mathcal{F} \mapsto H_{\operatorname{an}}^{0}(\mathbb{C}^{*}, \pi^{*}\mathcal{F})^{W} ,$$

that allows us to describe the W-equivariant coherent sheaves on E_q in terms of \mathcal{A}_q^W -modules. Recall that $\mathcal{E}ll_G^*(F_m)$ is defined to be the subsheaf of W-invariant sections of the coherent sheaf $\mathcal{E}ll_T^*(F_m)$ (see (6.3)). In the next theorem, we describe $\mathcal{E}ll_G^*(F_m)$ explicitly as an \mathcal{A}_q^W -submodule of A, where the action of \mathcal{A}_q^W on A is obtained by restricting the natural action of \mathcal{A}_q .

⁷The algebra $\mathcal{A}_q^{\mathrm{alg}}$ is usually referred to as a quantum Weyl algebra.

Theorem 6.6. Under the equivalence (6.15), the W-equivariant sheaf $\mathcal{E}ll_T^*(F_m)$ maps to the \mathcal{A}_q^W -module representing the G-equivariant elliptic cohomology of F_m :

(6.16)
$$\widetilde{\mathcal{E}ll}_G^*(F_m) \cong A^W + A^W(\Theta(z) - \Theta(z^{-1})) \vartheta(z)^{2m} \subseteq A,$$

where A^W is the subspace of W-invariant functions in $A = \mathcal{O}_{an}(\mathbb{C}^*)$ and $\vartheta(z) \in A[z^{\pm 1/2}]$ is the Jacobi theta function

(6.17)
$$\vartheta(z) := (z^{1/2} - z^{-1/2}) \prod_{n>0} (1 - q^n z)(1 - q^n z^{-1})$$

Proof. Observe that the T-space F_m comes together with a natural T-equivariant map

(6.18)
$$(G/T)^T \hookrightarrow (G/T)^T * E_{2m}(T) \cong F_m(G,T),$$

where $(G/T)^T \subset G/T$ is the set of T-fixed points in G/T (see (3.38)). On T-equivariant elliptic cohomology, the map (6.18) induces an injective map $\mathcal{E}ll_T^*(F_m) \hookrightarrow \mathcal{E}ll_T^*[(G/T)^T]$, which under the isomorphism (6.10) of Theorem 6.3, corresponds to the canonical inclusion

$$(6.19) \mathcal{O}_{E_q} \times_{\mathcal{O}_{E_q}/\mathcal{J}^{2m+1}} \mathcal{O}_{E_q} \hookrightarrow \mathcal{O}_{E_q} \times \mathcal{O}_{E_q}$$

Now, the map (6.18) is also equivariant under the action of W which is given on $(G/T)^T = \mathbb{S}^0$ simply by transposition of points. It follows that (6.19) is a morphism of W-equivariant sheaves on E_q that, under equivalence (6.13), corresponds to the W-equivariant inclusion $A \times_{A/\langle\Theta\rangle^{2m+1}} A \hookrightarrow A \times A$, where W acts on $A \times A$ by $s \cdot (f(z), g(z)) = (g(z^{-1}), f(z^{-1}))$. As a $(\mathcal{A}_q \rtimes W)$ -module, the product $A \times A$ is thus isomorphic to $A[W] := A \otimes \mathbb{C}W$, where the action of $\mathcal{A}_q \rtimes W$ is given by

$$(6.20) a \cdot (f(z) \otimes w) = a(z)f(z) \otimes w ,$$

$$\xi \cdot (f(z) \otimes w) = f(qz) \otimes w ,$$

$$s \cdot (f(z) \otimes w) = f(z^{-1}) \otimes sw .$$

Choosing a basis in $\mathbb{C}W$ consisting of the idempotents $\{e_+, e_-\}$, we can describe $\widetilde{\mathcal{E}ll}_T^*(F_m)$ as the $(\mathcal{A}_q \rtimes W)$ -submodule of A[W]

(6.21)
$$\widetilde{\mathcal{E}ll}_T^*(F_m) \cong A e_+ + A \Theta(z)^{2m+1} e_-,$$

where the isomorphism is explicitly given by $(f,g)\mapsto (f+g)e_+ + (f-g)e_-$. Now, applying to (6.21) the restriction functor (6.14) and using the (obvious) algebraic identities for theta functions $\vartheta(z)=-z^{-1/2}\Theta(z)$ and $\Theta(z^{-1})=-z^{-1}\Theta(z)$, we get

$$\widetilde{\mathcal{E}ll}_{T}^{*}(F_{m})^{W} \cong e_{+}A e_{+} + e_{+}A \Theta(z)^{2m+1} e_{-}
= e_{+}A^{W} + e_{+}A \Theta(z)\vartheta(z)^{2m} e_{-}
= e_{+}A^{W} + e_{+}A \Theta(z)e_{-}\vartheta(z)^{2m}
= e_{+}A^{W} + e_{+}A e_{+} (\Theta(z) - \Theta(z^{-1}))\vartheta(z)^{2m}
= e_{+} (A^{W} + A^{W} (\Theta(z) - \Theta(z^{-1}))\vartheta(z)^{2m}),$$

which, with our identifications $\widetilde{\mathcal{E}ll}_G^*(F_m) = \widetilde{\mathcal{E}ll}_T^*(F_m)^W$ (see (6.3)) and $e_+(\mathcal{A}_q \rtimes W)e_+ = \mathcal{A}_q^W$, is precisely the isomorphism (6.16).

6.3. Elliptic cohomology with twisted coefficients. The coherent sheaves $\mathcal{E}ll_T^*(F_m)$ (and a fortiori $\mathcal{E}ll_G^*(F_m)$) do not have nontrivial global sections. Indeed, by Theorem 6.3, $\mathcal{E}ll_T^*(F_m)$ fits in the short exact sequence in $Coh(E_g)$:

$$(6.22) 0 \to \mathcal{E}ll_T^*(F_m) \to \mathcal{O}_{E_q} \oplus \mathcal{O}_{E_q} \to \mathcal{O}_{E_q}/\mathcal{J}^{2m+1} \to 0$$

that shows at once that $H^0_{\mathrm{an}}(E_q,\mathcal{E}ll_T^*(F_m))\cong\mathbb{C}$ for all $m\geq 0$. With a little more work, using the long exact cohomology sequence associated to (6.22) we can also find that $H^1_{\mathrm{an}}(E_q,\mathcal{E}ll_T^*(F_m))\cong\mathbb{C}^{2m+2}$, which — as a W-module — admits decomposition

$$(6.23) H_{\mathrm{an}}^{1}(E_{q}, \mathcal{E}ll_{T}^{*}(F_{m})) \cong \mathbb{C}_{+}^{\oplus (m+1)} \oplus \mathbb{C}_{-}^{\oplus (m+1)}$$

where ${}^{\circ}\mathbb{C}_{+}{}^{\circ}$ and ${}^{\circ}\mathbb{C}_{-}{}^{\circ}$ denote the trivial and the sign representations of W, respectively.

A much richer picture emerges if we twist the elliptic cohomology sheaves $\mathcal{E}ll_T^*(F_m)$ with the Looijenga line bundle \mathcal{L} on E_q (see definitions (6.4) and (6.5)). Under the equivalence (6.8), this line bundle corresponds to the rank one free A-module $\tilde{\mathcal{L}} = Av$, where the action of \mathcal{A}_q and W are determined by the relations $\xi \cdot v = q z^2 v$ and $s \cdot v = v$ (cf. Example 6.2). Since (6.8) preserves tensor products, the tensor powers $\mathcal{L}^n = \mathcal{L}^{\otimes n}$ of \mathcal{L} in $Coh(E_q)$ correspond to the \mathcal{A}_q -modules $\tilde{\mathcal{L}}^n = Av_n$ with $\xi \cdot v_n = q^n z^{2n} v_n$ and $s \cdot v_n = v_n$. By (6.9), we can then identify the spaces of global sections of these line bundles as

(6.24)
$$H_{\rm an}^0(E_q, \mathcal{L}^n) \cong \{ f(z) \in A : f(qz) = q^{-n} z^{-2n} f(z) \}, \quad \forall n \ge 0.$$

Following [Loo77], we set

(6.25)
$$S(E) := \bigoplus_{n>0} H_{\mathrm{an}}^0(E_q, \mathcal{L}^n),$$

which, with identifications (6.24), is a graded subalgebra of A stable under the action of W. To describe this subalgebra we decompose it as the direct sum of W-invariants and anti-invariants:

$$(6.26) S(E) = S(E)^W \oplus S(E)^{-W}$$

Then, by Looijenga Theorem (see[Loo77, (3.4)]), we know that $S(E)^W$ is a free polynomial algebra on 2 generators, while $S(E)^{-W}$ is a free module over $S(E)^W$ of rank one. The generators of $S(E)^W$ and $S(E)^{-W}$ can be explicitly given in terms of the Jacobi theta function (6.17): namely, $S(E)^W$ is generated (as an algebra) by $\vartheta^2(z)$ and $\vartheta^2(-z)$, which are both invariant functions in S(E) of degree 1, while $S(E)^{-W}$ is generated (as a module) by the function $\vartheta(z^2)$ which is an anti-invariant in S(E) of degree 2.

Now to state our last result in this section we recall the definitions of equivariant elliptic cohomology with twisted coefficients: see formulas (6.4) and (6.5) (with $\mathcal{M}_T = E_q$). For X = G/T, it is well known that (see, e.g., [Gan14]):

(6.27)
$$\operatorname{Ell}_{G}^{*}(G/T,\mathcal{L}) \cong \operatorname{Ell}_{T}^{*}(\operatorname{pt},\mathcal{L}) = S(E)$$

We extend this result to the quasi-flag manifolds $F_m = F_m(G, T)$.

Theorem 6.7. The natural maps

$$G/T = F_0(G,T) \to F_1(G,T) \to \ldots \to F_{m-1}(G,T) \to F_m(G,T) \to \ldots$$

induce injective homomorphisms on twisted elliptic cohomology:

$$\ldots \hookrightarrow \operatorname{Ell}_{G}^{*}(F_{m}, \mathcal{L}) \hookrightarrow \operatorname{Ell}_{G}^{*}(F_{m-1}, \mathcal{L}) \hookrightarrow \ldots \hookrightarrow \operatorname{Ell}_{G}^{*}(G/T, \mathcal{L}).$$

Under the identification (6.27), the composite map $\mathrm{Ell}_G^*(F_m,\mathcal{L}) \hookrightarrow \mathrm{Ell}_G^*(G/T,\mathcal{L})$ corresponds to the inclusion $S(E)^W \oplus \vartheta^{2m}(z) S(E)^{-W} \hookrightarrow S(E)$, so that

(6.28)
$$\operatorname{Ell}_{G}^{*}(F_{m},\mathcal{L}) \cong S(E)^{W} \oplus \vartheta^{2m}(z) S(E)^{-W},$$

where
$$S(E)^W = \mathbb{C}[\vartheta^2(z), \vartheta^2(-z)]$$
 and $S(E)^{-W} = \mathbb{C}[\vartheta^2(z), \vartheta^2(-z)]\vartheta(z^2)$.

Proof. We use the description of $\widetilde{\mathcal{E}ll}_T^*(F_m)$ given in the proof of Theorem 6.6: namely, $\widetilde{\mathcal{E}ll}_T^*(F_m) = A \, e_+ + A \, \Theta^{2m+1} \, e_-$ as an $(\mathcal{A}_q \rtimes W)$ -submodule of $A[W] = A \, e_+ + A \, e_-$. Under the equivalence (6.8), the twisted sheaves $\mathcal{E}ll_T^*(F_m) \otimes \mathcal{L}^n$ can then be described by

(6.29)
$$\widetilde{\mathcal{E}ll}_T^*(F_m) \otimes_A \widetilde{\mathcal{L}}^n = A v_n \otimes e_+ \oplus A \Theta^{2m+1} v_n \otimes e_-$$

and we can compute their global sections using formula (6.9):

$$H_{\mathrm{an}}^{0}(E_{q}, \mathcal{E}ll_{T}^{*}(F_{m}) \otimes \mathcal{L}^{n}) \cong \operatorname{Ker}(\xi - \operatorname{id} : \widetilde{\mathcal{E}ll}_{T}^{*}(F_{m}) \otimes_{A} \widetilde{\mathcal{L}}^{n})$$

$$\cong \operatorname{Ker}(\xi - \operatorname{id} : Av_{n} \otimes e_{+}) \oplus \operatorname{Ker}(\xi - \operatorname{id} : A\Theta^{2m+1}v_{n} \otimes e_{-})$$

$$\cong H_{\mathrm{an}}^{0}(E_{q}, \mathcal{L}^{n}) e_{+} \oplus (H_{\mathrm{an}}^{0}(E_{q}, \mathcal{L}^{n}) | \Theta^{2m+1}) e_{-},$$

where $(H_{\rm an}^0(E_q,\mathcal{L}^n) | \Theta^{2m+1})$ denotes the subspace of all sections in $H_{\rm an}^0(E_q,\mathcal{L}^n)$ that are divisible by Θ^{2m+1} under the identification (6.24). Summing up over all $n \geq 0$, we find

$$\operatorname{Ell}_{T}^{*}(F_{m},\mathcal{L}) \cong S(E) e_{+} \oplus (S(E) | \Theta^{2m+1}) e_{-},$$

where S(E) is the Looijenga ring (6.25). To compute $(S(E) | \Theta^{2m+1})$ we observe that an element of S(E) is divisible by Θ^{2m+1} in A if and only if its invariant and anti-invariant parts in $S(E)^W$ and $S(E)^{-W}$ are both divisible by Θ^{2m+1} . Now, for $f(z) \in S(E)^W$, $\Theta^{2m+1}(z)$ divides f(z) if and only if $\vartheta^{2m+2}(z)$ divides f(z), while for $f(z) \in S(E)^{-W}$, $\Theta^{2m+1}(z)$ divides f(z) if and only if $\vartheta^{2m}(z)\vartheta(z^2)$ divides f(z). Thus

(6.30)
$$\operatorname{Ell}_{T}^{*}(F_{m},\mathcal{L}) \cong S(E) e_{+} \oplus (\vartheta^{2m+2}(z) S(E)^{W} + \vartheta^{2m}(z) S(E)^{-W}) e_{-}$$

Now, applying to (6.30) the restriction functor (6.14), we get

$$\operatorname{Ell}_{T}^{*}(F_{m},\mathcal{L})^{W} \cong e_{+} S(E) e_{+} \oplus e_{+} \left(\vartheta^{2m+2}(z) S(E)^{W} + \vartheta^{2m}(z) S(E)^{-W} \right) e_{-}$$

$$\cong S(E)^{W} \oplus \vartheta^{2m}(z) S(E)^{-W}$$

which gives (6.28) since $\mathrm{Ell}_T^*(F_m,\mathcal{L})^W=\mathrm{Ell}_G^*(F_m,\mathcal{L})$. To complete the proof it suffices to note that the map of spaces $G/T\to F_m$ induces the natural inclusion

$$S(E) e_{+} \oplus (\vartheta^{2m+2}(z) S(E)^{W} + \vartheta^{2m}(z) S(E)^{-W}) e_{-} \hookrightarrow S(E) e_{+} \oplus (\vartheta^{2}(z) S(E)^{W} + S(E)^{-W}) e_{-}$$

as a map representing $\mathrm{Ell}_T^*(F_m,\mathcal{L}) \to \mathrm{Ell}_T^*(G/T,\mathcal{L})$ under the isomorphism (6.30). When restricted to W-invariants this yields the inclusion

$$S(E)^W \oplus \vartheta^{2m}(z) S(E)^{-W} \hookrightarrow S(E)^W \oplus S(E)^{-W} = S(E)$$

that represents
$$\mathrm{Ell}_G^*(F_m,\mathcal{L}) \hookrightarrow \mathrm{Ell}_G^*(G/T,\mathcal{L})$$
.

Remark 6.8. The above calculation of elliptic cohomology suggests a natural algebraic definition of quasi-invariants in the elliptic case (cf. (6.28)). This differs, however, from the definition of elliptic quasi-invariants that has already been used in the literature (see, e.g., the beautiful work of O. Chalykh on Macdonald's conjectures [Cha02]). The difference seems to be an instance of 'elliptic' duality studied in the theory of integrable systems (see, e.g., [KS19]).

7. Topological Gorenstein duality

The realization of algebras of quasi-invariants raises natural questions about homotopy-theoretic analogues (refinements) of basic properties and structures associated with these algebras. In this section, we make first steps in this direction by showing that the spaces of quasi-invariants $X_m(G,T)$ satisfy Gorenstein duality in the sense of stable homotopy theory. Our main result — Theorem 7.1 — should be viewed as a topological analogue of Theorem 2.3 on Gorensteinness of rings of quasi-invariants. For reader's convenience, we collect basic definitions from stable homotopy theory concerning duality and regularity properties of commutative ring spectra, in Appendix B. We refer to Appendix B for all unexplained notation used in this section.

7.1. Gorenstein duality of spaces of quasi-invariants. It is well known that, if X is a pointed connected topological space, the singular cochain complex $C^*(X,\mathbb{Q})$, computing cohomology of X with coefficients in \mathbb{Q} , admits a commutative DG algebra model⁸. When \mathbb{Q} is replaced by an arbitrary field k, this last fact is no longer true: in general, the cochain complex $C^*(X,k)$ is not quasi-isomorphic to any commutative DG algebra over k if $\operatorname{char}(k) \neq 0$. A natural way to remedy this problem is to use commutative ring spectra – instead of DGAs – as models for $C^*(X,k)$. Specifically, for any commutative ring k, the cochain spectrum of the space X with coefficients in k is defined by $(cf. [\operatorname{Man}01])$

(7.1)
$$C^*(X,k) := \operatorname{Map}_{\mathbb{S}}(\Sigma^{\infty} X_+, Hk)$$

where $\Sigma^{\infty}X_{+}$ is the suspension spectrum associated to X, Hk is the Eilenberg-MacLane spectrum of k, and Map_S denotes the mapping spectrum in the category of (symmetric) spectra. By definition, (7.1) is a commutative ring spectrum with multiplication induced by the multiplication map on Hk and the diagonal map on X. In addition, following [DGI06], we introduce the *chain spectrum* of X:

(7.2)
$$C_*(\Omega X, k) := Hk \wedge \Sigma^{\infty}(\Omega X)_+$$

which is a noncommutative ring spectrum that models the singular chain complex of the based loop space of X. Both $C^*(X,k)$ and $C_*(\Omega X,k)$ are augmented k-algebras, with augmentation on $C^*(X,k)$ induced by the basepoint inclusion pt $\to X$ and on $C_*(\Omega X,k)$ by the trivial map $\Omega \to \operatorname{pt}$. For all $i \in \mathbb{Z}$, there are natural isomorphisms

(7.3)
$$\pi_i [C^*(X,k)] \cong H^{-i}(X,k) , \quad \pi_i [C_*(\Omega X,k)] \cong H_i(\Omega X,k)$$

which show that $C^*(X,k)$ and $C_*(\Omega X,k)$ are coconnective and connective spectra, respectively.

We are now in position to state and prove the main theorem of this section.

Theorem 7.1. Let $X_m = X_m(G,T)$ be the space of m-quasi-invariants associated to G = SU(2). Let $R_m := C^*(X_m, k)$ and $E_m := C_*(\Omega X_m, k)$ denote the cochain and chain spectra of X_m with coefficients in an arbitrary field k. Then, for any $m \ge 0$,

(1) R_m and E_m are proxy-regular (Definition B.1) and dc-complete (Definition B.4) with

$$\operatorname{Map}_{R_m}(k,k) \simeq E_m$$
 and $\operatorname{Map}_{E_m}(k,k) \simeq R_m$

- (2) R_m is orientable Gorenstein of shift a = 1 4m (Definition B.2)
- (3) R_m satisfies Gorenstein duality of shift a = 1 4m (Definition B.3)

⁸Such a model can be constructed in a functorial way, using, for example, piecewise polynomial differential forms on X defined over \mathbb{Q} (see [BG76]).

Proof. (1) We start with Borel fibration sequence that comes from the Ganea construction of spaces of quasi-invariants (see (3.33)):

$$(7.4) F_m(G,T) \to X_m(G,T) \xrightarrow{p_m} BG$$

To simplify the notation we set

$$Q_m := C^*(F_m, k) , \quad R_m := C^*(X_m, k) , \quad S := C^*(BG, k) .$$

Since F_m is a finite connected complex (see (3.28)), by [DGI06, Prop. 5.3], the augmentation morphism $Q_m \to k$ is cosmall (see Definition B.1). Since G is connected, the classifying space BG is simply-connected; moreover, the cohomology of BG is free, of finite type over \mathbb{Z} , and hence, a fortiori, over any field k (see, e.g., [MT91, III.3.17]). Therefore, in terminology of [DGI06, Sect. 4.22], the pair (BG, k) is of Eilenberg-Moore type. Since $H_*(\Omega BG, k) \cong H_*(G, k)$ is finite-dimensional over k, it follows from [DGI06, 5.5(3)] that $S \to k$ is a regular morphism, i.e. k is small as an S-module. Next, since (BG, k) is of Eilenberg-Moore type, the fibration sequence (7.4) gives an equivalence of cochain spectra (see, e.g., [BCHV21, Lemma 3.7])

$$(7.5) Q_m \simeq k \wedge_S R_m$$

Now, by [DGI06, Prop. 4.18(1)], we conclude from (7.5) together with our earlier observations that $S \to k$ is regular and $Q_m \to k$ is cosmall that $R_m \to k$ is proxy-regular. To complete the proof of part (1) it remains to note that the pair (X_m, k) is of Eilenberg-Moore type for any field k. Indeed, from the fibration sequence (7.4) it follows that X_m is simply-connected (since so are F_m and BG); on the other hand, from the homotopy cofibration sequences (see (3.30))

$$F_m \to X_m \xrightarrow{\pi_m} X_{m+1}$$

it follows (by induction) that X_m is of finite type over k for any $m \geq 0$. By construction of the Eilenberg-Moore spectral sequence, for $E_m = C_*(\Omega X_m, k)$, we have $E_m \simeq \operatorname{Map}_{R_m}(k, k)$, while the equivalence $R_m \simeq \operatorname{Map}_{E_m}(k, k)$ holds in general (see remarks in [DGI06, Sect. 4.22]). It follows that the augmented k-algebras R_m and E_m are both dc-complete, and then, by [DGI06, Prop. 4.17], E_m is proxy-regular (since so is R_m).

(2) By the proof of Theorem 3.9, we know that (7.4) is a sphere fibration with $F_m \simeq \mathbb{S}^{4m+2}$. Hence, F_m is a Poincaré duality space of dimension 4m+2, then its cochain spectrum $Q_m = C^*(F_m, k)$ satisfies Poincaré duality of dimension a = -4m-2 (in the sense of [DGI06, 8.11]). Since Q_m is cosmall, by [DGI06, Prop. 8.12], we conclude that Q_m is Gorenstein of shift a = -4m-2. Further, by [DGI06, 10.2], we also know that $S = C^*(BG, k)$ is Gorenstein of dimension $a = \dim(G) = 3$.

Now, consider the morphism of cochain spectra $p_m^*: S \to R_m$ induced by the whisker map $p_m: X_m \to BG$ in (7.4). We claim that R_m is finitely built from S via p_m^* . To see this denote by $\mathcal{E} := C_*(\Omega BG, k) \cong C_*(G, k)$ the chain spectrum of BG. Since G is simply-connected, \mathcal{E} is a connective k-algebra with $\pi_0(\mathcal{E}) \cong k[\pi_1(G)] = k$ (see (7.3)). Since S is of Eilenberg-Moore type, there is an equivalence $S \simeq \operatorname{Map}_{\mathcal{E}}(k,k)$. Furthermore, if we set $M_m := C_*(F_m,k)$, the action of G on F_m induces a left \mathcal{E} -module structure on M_m , and by a standard Eilenberg-Moore spectral sequence argument there is an equivalence $R_m \simeq \operatorname{Map}_{\mathcal{E}}(M_m,k)$. Since $\pi_*(M_m) \cong H_*(F_m,k)$ is finite-dimensional over k, the \mathcal{E} -module M_m is finitely built from k. Now, Proposition 3.18 of [DGI06] implies that $R_m \simeq \operatorname{Map}_{\mathcal{E}}(M_m,k)$ is finitely built from $S \simeq \operatorname{Map}_{\mathcal{E}}(k,k)$ as we claimed. Since R_m is proxy-regular and both S and Q_m are Gorenstein, it follows from [DGI06, Prop. 8.10] that R_m is Gorenstein as well. The Gorenstein shift of R_m can be computed from the following

equivalence of k-modules induced by (7.5) (see [DGI06, Prop. 8.6]):

$$\operatorname{Map}_{R_m}(k, R_m) \simeq \operatorname{Map}_{Q_m}(k, \operatorname{Map}_S(k, S) \wedge_k Q_m)$$

$$\simeq \operatorname{Map}_{Q_m}(k, (\Sigma^3 k) \wedge_k Q_m)$$

$$\simeq \Sigma^3 \operatorname{Map}_{Q_m}(k, Q_m)$$

$$\simeq \Sigma^3 (\Sigma^{-4m-2} k)$$

$$\sim \Sigma^{1-4m} k$$

To complete part (2) it remains to note that, for a simply-connected space X of finite type over k, the cochain spectrum $C^*(X, k)$ is automatically orientable Gorenstein whenever it is Gorenstein. This follows from the fact that, under the above assumptions, k carries a unique action of $E = \operatorname{Map}_{C^*(X,k)}(k,k) \simeq C_*(\Omega X,k)$ (see [Gre18, Sect. 18.3] and also the proof of [BCHV21, Lemma 3.8]).

- (3) follows from (2) by a standard argument. If an augmented k-algebra R is orientable Gorenstein of shift a, then
- (7.6) $\operatorname{Cell}_k(R) \simeq \operatorname{Map}_R(k,R) \wedge_E k \simeq \Sigma^a \operatorname{Map}_R(k,\operatorname{Map}_k(R,k)) \wedge_E k \simeq \Sigma^a \operatorname{Cell}_k[\operatorname{Map}_k(R,k)]$ where the first and the last equivalences are given by (B.2) and the one in the middle is induced by (B.5). For $R = C^*(X,k)$ with $\pi_0(R) \cong H^0(X,k) \cong k$, we have $\pi_i \operatorname{Map}_k(R,k) = 0$ for $i \ll 0$. By [DGI06, Remark 3.17], the R-module $\operatorname{Map}_k(R,k)$ is then built from k and therefore k-cellular in Mod_R . Condition B.8 thus follows from (7.6). This completes the proof of the theorem.
- 7.2. Generalized spaces of quasi-invariants. It is natural to ask whether the result of Theorem 7.1, i.e. the topological Gorenstein property, holds for generalized ('fake') spaces of quasi-invariants introduced in Section 4. In view of Corollary 4.8, the answer is obviously affirmative when k is a field of characteristic 0. The next theorem shows that this is also true when $k = \mathbb{F}_p$. We keep the notation G = SU(2) and T = U(1); however, as in Section 4, we do not identify T as a maximal torus in G.

Theorem 7.2. Let B be a space in the genus of BG that admits an essential map from BT, and let $X_m = X_m(\Omega B, T)$ be the space of m-quasi-invariants associated to B. Then, for any prime p, the morphism $C^*(X_m, \mathbb{F}_p) \to \mathbb{F}_p$ is Gorenstein of shift a = 1 - 4m.

Proof. We give the part of the proof that differs from that of Theorem 7.1. First, observe that, for any space B in the genus of BG, we have equivalences of cochain spectra

$$C^*(B, \mathbb{F}_p) \simeq C^*(B_p^{\wedge}, \mathbb{F}_p) \simeq C^*((BG)_p^{\wedge}, \mathbb{F}_p) \simeq C^*(BG, \mathbb{F}_p)$$

where $(-)_p^{\wedge}$ denotes the \mathbb{F}_p -completion functor on pointed spaces. This follows from the fact that both B and BG are \mathbb{F}_p -good spaces (in the sense of [Bou75]) and $B_p^{\wedge} \simeq (BG)_p^{\wedge}$ for any prime p. The above equivalences are compatible with augmentation; hence, by [DGI06, 10.2], we conclude that $C^*(B, \mathbb{F}_p) \to \mathbb{F}_p$ is a regular map, Gorenstein of shift $\dim(G) = 3$.

Now, assume that B satisfies the conditions of Theorem 4.5. Let $F = F(\Omega B, T)$ denote the homotopy fibre of the maximal essential map $p_B : BT \to B$. Recall that this last space is not equivalent to a finite CW complex (unless $B \simeq BG$), and hence its cochain spectrum $C^*(F, \mathbb{F}_p)$ need not be cosmall (as in the case of BG). Nevertheless, we claim that $C^*(F, \mathbb{F}_p) \to \mathbb{F}_p$ is always proxy-regular and satisfies the Gorenstein property of shift (-2). To see this consider the homotopy fibration sequence $\Omega B \to F \to BT$ associated to the map $p_B : BT \to B$. Since $BT \simeq \mathbb{CP}^{\infty}$ is of Eilenberg-Moore type (see [DGI06, 4.22]), we have

$$C^*(\Omega B, \mathbb{F}_p) \simeq C^*(F, \mathbb{F}_p) \wedge_{C^*(BT, \mathbb{F}_p)} \mathbb{F}_p$$

In view of the fact that $\Omega B \simeq \mathbb{S}^3$, the map $C^*(\Omega B, \mathbb{F}_p) \to \mathbb{F}_p$ is cosmall, and hence, by [DGI06, Prop. 4.18], $C^*(F, \mathbb{F}_p) \to \mathbb{F}_p$ is proxy-regular. Furthermore, since \mathbb{F}_p is small over $C^*(BT) = C^*(BT, \mathbb{F}_p)$, we have a natural equivalence of $C^*(BT)$ -modules

$$\operatorname{Map}_{C^*(BT)}(\mathbb{F}_p, C^*(BT)) \wedge_{C^*(BT)} C^*(F) \xrightarrow{\sim} \operatorname{Map}_{C^*(BT)}(\mathbb{F}_p, C^*(F)),$$

which, by the proof of [DGI06, Prop. 8.6], implies that $C^*(F, \mathbb{F}_p) \to \mathbb{F}_p$ is Gorenstein of shift a = 1 + (-3) = -2.

The rest of the proof is parallel to that of Theorem 7.1. In brief, by Theorem 3.1, the fibre of the m-th Ganea fibration $F_m \to X_m \to B$ defining the space $X_m = X_m(\Omega B, T)$ has the homotopy type of $\Sigma^{4m}F$. Hence its cochain spectrum $C^*(F_m, \mathbb{F}_p)$ is Gorenstain of shift a = -2 - 4m. By induction, each space X_m is of finite type over \mathbb{F}_p . Since $C^*(B, \mathbb{F}_p) \to \mathbb{F}_p$ is a regular Gorenstein map of shift 3, it follows from the above fibration sequence that $C^*(X_m, \mathbb{F}_p) \to \mathbb{F}_p$ is Gorenstein of shift a = -2 - 4m + 3 = 1 - 4m.

Remark 7.3. We point out that the topological Gorenstein shifts a of Theorem 7.1 and Theorem 7.2 agree with the algebraic one of Theorem 2.3: to see this it suffices to change the standard polynomial grading on $Q_m(W)$ to the cohomological one (by 'doubling' degrees of the generators).

APPENDIX A. MILNOR BUNDLES

Recall that, if G is a topological group, its classifying space BG is defined to be the basespace of a principal G-bundle $EG \to BG$ that is universal among all (numerable) principal G-bundles over pointed spaces. This universal property determines the space BG uniquely up to homotopy, i.e. as a unique (up to unique isomorphism) object in the homotopy category $Ho(Top_*)$ of pointed spaces. For a general G there are two classical models for the classifying space: the Milgram-Segal model [Mil67, Seg68] that defines G as the geometric realization G of a simplicial space G (topological bar construction) and the Milnor model [Mil56] that represents G as a quotient of an infinite join of spaces homeomorphic to G. The Milnor model will play a key role in our construction of spaces of quasi-invariants; we therefore review it in some detail beginning with the classical topological operation of a join.

Recall that the join X * Y of two spaces is defined to be the space of all line segments joining points in X to points in Y: i.e., X * Y is the quotient space of $X \times I \times Y$ under the identifications $(x,0,y) \sim (x',0,y)$ and $(x,1,y) \sim (x,1,y')$ for all $x,x' \in X$ and $y,y' \in Y$. If X and Y are both (well) pointed, it is convenient to work with a reduced version of the join obtained by collapsing to a point the line segment joining the basepoints in X and Y (i.e., by imposing on X * Y the extra identification $(*,t,*) \sim (*,t',*)$ for all $t,t' \in I$). Note that inside X * Y, there are two cones CX and CY embedded via the canonical maps $CX \hookrightarrow X * Y$, $(x,t) \mapsto (x,t,*)$, and $CY \hookrightarrow X * Y$, $(y,t) \mapsto (*,1-t,y)$. Collapsing these cones converts X * Y into the suspension of the smash product of spaces: $\Sigma(X \wedge Y) = (X * Y)/(CX \vee CY)$. Since CX and CY are both contractible in X * Y, the quotient map $X * Y \to \Sigma(X \wedge Y)$ is a homotopy equivalence. Thus, in the homotopy category $\operatorname{Ho}(\operatorname{Top}_*)$ of pointed spaces, we have natural isomorphisms

$$(A.1) X * Y \cong \Sigma(X \wedge Y) \cong (\Sigma X) \wedge Y \cong X \wedge (\Sigma Y)$$

These are useful in practice for computing the homotopy types of joins.

Using standard notation, we will write the points of X*Y as formal linear combinations t_0x+t_1y , where $x \in X$, $y \in Y$ and $(t_0, t_1) \in \Delta^1 := \{(t_0, t_1) \in \mathbb{R}^2 : t_0 + t_1 = 1, t_0, t_1 \geq 0\}$. The identification

 $^{^9}$ For special groups (for example, classical Lie groups), there are also nice geometric models representing BG as infinite-dimensional manifolds (Grassmannians).

with topological presentation is given by (x, t, y) = tx + (1 - t)y. The advantage of this notation is that it naturally extends to 'higher dimensions': the *iterated joins* of spaces

(A.2)
$$X_0 * X_1 * \dots * X_n = \{t_0 x_0 + t_1 x_1 + \dots + t_n x_n : (t_0, \dots, t_n) \in \Delta^n, x_i \in X_i\} / \sim$$

where the equivalence relation is defined by $\sum_{i=0}^{n} t_i x_i \sim \sum_{i=0}^{n} t_i' x_i'$ if and only if $t_i = t_i'$ (for all i) and $x_i = x_i'$ whenever $t_i = t_i' > 0$. Note that, under this equivalence relation, if $t_i = 0$ for some i, the point x_i in $t_0 x_0 + \ldots + 0 x_i + \ldots + t_n x_n \in X_0 * \ldots * X_n$ can be chosen arbitrarily (or simply omitted).

There is also a convenient way to represent joins by homotopy colimits. For example, it is well-known that the join of two spaces is represented by the homotopy pushout

$$(A.3) X * Y = \text{hocolim}[X \leftarrow X \times Y \rightarrow Y]$$

where the maps are canonical projections and the "hocolim" is taken either in the category of pointed or unpointed spaces depending on whether we consider reduced or unreduced joins. Formula (A.3) generalizes to iterated joins (see, e.g. [WZv99, Prop. 5.1])

(A.4)
$$X_0 * X_1 * \dots * X_n = \operatorname{hocolim}_{\mathcal{P}(\Delta^n)}(F_X)$$

where $\mathcal{P}(\Delta^n)$ is the poset of all non-empty faces of the n-simplex Δ^n (ordered by reversed inclusions) and the diagram $F_X: \mathcal{P}(\Delta^n) \to \text{Top}$ is defined by assigning to a face $\Delta_I \in \mathcal{P}(\Delta^n)$ the product of spaces $\prod_{i \in I} X_i$ (with indices corresponding to the vertices of Δ_I) and to an inclusion of faces $\Delta_I \subset \Delta_I$ the canonical projection $\prod_{i \in I} X_i \to \prod_{j \in J} X_j$. It is easy to see that formula (A.4) boils down to (A.3) in case of two spaces.

Now, we can describe the Milnor model. For integer $n \ge 0$, we define a sequence of spaces E_nG by taking the (unreduced) iterated joins of copies of G:

$$(A.5) E_n G := G * G * \dots * G (n+1 \text{ times}).$$

Each space E_nG carries natural (diagonal) left and right G-actions each of which is free. We will use the right G-action $E_nG \times G \to E_nG$ that can be written explicitly (with notation (A.2)) as

(A.6)
$$(t_0g_0 + t_1g_1 + \ldots + t_ng_n) \cdot g = t_0g_0g + t_1g_1g + \ldots + t_ng_ng$$

where $g_0, \ldots, g_n, g \in G$. Moreover, there are natural G-equivariant maps $E_nG \hookrightarrow E_{n+1}G$:

$$t_0g_0 + \ldots + t_ng_n \mapsto t_0g_0 + \ldots + t_ng_n + 0 \cdot e$$

making $\{E_nG\}_{n\geqslant 0}$ into a direct system of (right) G-spaces. We set $B_nG=E_nG/G$ and define

(A.7)
$$EG := \lim_{n \to \infty} E_n G \quad \text{and} \quad BG := \lim_{n \to \infty} B_n G.$$

By construction, the spaces EG and BG come equipped with canonical filtrations

$$(A.8) E_0G \hookrightarrow \ldots \hookrightarrow E_nG \hookrightarrow E_{n+1}G \hookrightarrow \ldots \hookrightarrow EG$$

$$(A.9) B_0G \hookrightarrow \ldots \hookrightarrow B_nG \hookrightarrow B_{n+1}G \hookrightarrow \ldots \hookrightarrow BG$$

with consecutive terms (at each level n) forming the principal G-bundles

(A.10)
$$G \to E_n G \to B_n G$$
.

The main observation of [Mil56] (see *loc. cit.*, Theorem 3.1) is that the principal G-bundle (A.10) is n-universal in the sense that its total space is (n-1)-connected (i.e., $\pi_i(E_nG) = 0$ for all i < n). In the inductive limit, this gives

Theorem A.1 (Milnor). For any topological group G the natural (quotient) map $EG \to BG$ is a numerable principal G-bundle, which is universal among all such G-bundles.

A detailed proof of Theorem A.1 can be found in [Hus75] (see Chap. 4, Theorem 11.2). We only recall one basic topological fact behind this proof that we will use repeatedly in this paper.

Lemma A.2 ([Mil56], Lemma 2.3). If each space X_i in the iterated join (A.4) is (c_i-1) -connected, then the space $X_0 * X_1 * ... * X_n$ is $(\sum c_i + n - 1)$ -connected.

APPENDIX B. DUALITY OF COMMUTATIVE RING SPECTRA

In this Appendix, we collect basic definitions from stable homotopy theory concerning duality and regularity properties of commutative ring spectra. Our main references are the paper [DGI06] by Dwyer, Greenlees and Iyengar, where many concepts that we need were originally introduced, and the lecture notes of Greenlees [Gre18] that supplement [DGI06] with motivation and examples. As in [DGI06], we will work in the (stable model) category of symmetric spectra, which can be succinctly described as the category Mod_S of modules 10 over the symmetric sphere spectrum $\mathbb{S} = ((S^1)^{\wedge n})_{n \geq 0}$ (see [HSS00]). The category $\mathrm{Mod}_{\mathbb{S}}$ is equipped with a symmetric monoidal product which is denoted as a smash $A \wedge B$ or tensor product $A \otimes_{\mathbb{S}} B$ (depending on the context). A ring spectrum is then, by definition, an S-algebra, i.e. an S-module R given with two structure maps $\mathbb{S} \to R$ and $R \wedge R \to R$ satisfying the usual unitality and associativity properties. We denote the category of ring spectra by $Alg_{\mathbb{S}}$. There is a natural (Eilenberg-MacLane) functor $H:Alg_{\mathbb{Z}}\to$ $Alg_{\mathbb{S}}, k \mapsto Hk$ that embeds the category $Alg_{\mathbb{Z}}$ of usual (discrete) associative rings into S-algebras by identifying a ring k with its symmetric Eilenberg-MacLane spectrum $Hk = (K(k,n))_{n>0}$ (see [HSS00, 1.2.5]). The category Alg_S can be thought of as a homotopical refinement ('thickening') of $Alg_{\mathbb{Z}}$ in the same way as the category $Mod_{\mathbb{S}}$ is a homotopical refinement of the category $Mod_{\mathbb{Z}}$ of (discrete) abelian groups.

For a ring spectrum $R \in \mathrm{Alg}_{\mathbb{S}}$, we let Mod_R denote the category of left module spectra over R. This is a stable model category enriched over $\mathrm{Mod}_{\mathbb{S}}$. The latter means that, for two R-modules A and B, there is a mapping spectrum of R-module maps $A \to B$ that we denote $\mathrm{Map}_R(A,B)$. Moreover, if A is a right R-module and B is a left R-module, there is an associated smash product $A \wedge_R B$ defined as the (homotopy) coequalizer $A \wedge R \wedge B \rightrightarrows A \wedge B$ of structure maps $A \wedge R \to A$ and $R \wedge B \to B$ in $\mathrm{Mod}_{\mathbb{S}}$. Note that both $\mathrm{Map}_R(A,B)$ and $A \wedge_R B$ are understood as 'derived' objects in the sense that their first arguments are (replaced by) cofibrant objects in Mod_R . In particular, if A and B are usual (discrete) modules over a usual (discrete) ring R, viewed as symmetric spectra via the Eilenberg-MacLane functor, then $\pi_i \, \mathrm{Map}_R(A,B) \cong \mathrm{Ext}_R^{-i}(A,B)$ and $\pi_i(A \wedge_R B) \cong \mathrm{Tor}_i^R(A,B)$, where π_i stand for the (stable) homotopy groups of spectra. If R is a commutative ring spectrum, then both $\mathrm{Map}_R(A,B)$ and $A \wedge_R B$ are naturally R-modules, i.e. objects in Mod_R .

Next, we recall that a subcategory of a (stable) model category \mathcal{M} is called *thick* if it is closed under weak equivalences, cofibration sequences (distinguished triangles) and retracts in \mathcal{M} . Further, a subcategory of \mathcal{M} is called *localizing* if it is thick and, in addition, closed under arbitrary coproducts (and hence homotopy colimits) in \mathcal{M} . Given two objects A and B in \mathcal{M} , we say that B is *built* from A if B belongs to the localizing subcategory of \mathcal{M} generated by A, and B is *finitely built* from A if it belongs to the thick subcategory generated by A ([DGI06, 3.15]). Now, if $\mathcal{M} = \text{Mod}_R$, an R-module A is called *small* if it is finitely built from R in Mod_R . This agrees with the usual definition of small (compact) objects in Mod_R : an R-module A is small iff $\text{Map}_R(A, -)$ commutes with arbitrary coproducts.

¹⁰Unfortunately, the term 'S-module' in reference to spectra is very ambiguous: apart from symmetric, other popular types of spectra (e.g., orthogonal and EKMM ones) are also S-modules. A nice recent survey comparing properties and applications of different types of spectra can be found in [Dug22].

The notion of a localizing subcategory is closely related to that of cellularization. For a fixed object $A \in \text{Mod}_R$, we say that a morphism $f: M \to N$ in Mod_R is an A-cellular equivalence if f induces a (weak) equivalence on mapping spectra:

$$f_*: \operatorname{Map}_R(A, M) \xrightarrow{\sim} \operatorname{Map}_R(A, N)$$

Note that every equivalence in Mod_R is automatically an A-cellular equivalence, but the converse, in general, is not true. Now, an R-module B is called A-cellular if any A-cellular equivalence $f:M\to N$ induces an equivalence $\operatorname{Map}_R(B,M)\overset{\sim}{\to}\operatorname{Map}_R(B,N)$. This terminology is motivated by the fact that the A-cellular modules are precisely those objects of Mod_R that are built from A (see [Hir03, 5.1.15]). Moreover, for any R-module B, there is a A-cellular module $\operatorname{Cell}_A^R(B)$ together with a A-equivalence in Mod_R :

$$\operatorname{Cell}_{A}^{R}(B) \to B$$

called an A-cellular approximation¹¹ of B. Such an approximation is determined by B uniquely up to canonical equivalence; we will use the simpler notation $\operatorname{Cell}_A(B)$ for $\operatorname{Cell}_A^R(B)$ when the ring spectrum R is understood.

The above categorical notions can be used to impose some finiteness and regularity conditions on commutative ring spectra. First, we say that a morphism of commutative ring spectra $R \to k$ is called regular if k is small as an R-module. This definition is motivated by the fact that, in classical commutative algebra, a local Noetherian ring R with residue field $k = R/\mathfrak{m}$ is regular iff k has a finite length resolution by f.g. free R-modules (see [Ser00]); for the associated Eilenberg-MacLane spectra, the latter means that Hk is finitely built from HR. A more flexible and technically useful condition is obtained by weakening the regularity assumption on $R \to k$ in the following way.

Definition B.1 ([DGI06], 4.6). A morphism of commutative ring spectra $R \to k$ is called *proxy-regular* if k is a *proxy-small* R-module via $R \to k$ in the sense that there is a small R-module K that builds k and is finitely built from k in Mod_R . Note that if K = k, then $R \to k$ is regular. On the other extreme, if K = R then $R \to k$ is called *cosmall*.

Let $E := \operatorname{Map}_R(k, k)$ denote the endomorphism ring spectrum of k viewed as a left R-module via the morphism $R \to k$. There is a standard Quillen adjunction relating right E-modules to left R-modules:

(B.1)
$$(-) \wedge_E k : \operatorname{Mod}_{E^{\operatorname{op}}} \leftrightarrows \operatorname{Mod}_R : \operatorname{Map}_R(k, -)$$

If $R \to k$ is regular, the functors (B.1) induce an equivalence between $Ho(Mod_{E^{op}})$ and the full subcategory of $Ho(Mod_R)$ consisting of k-cellular R-modules (see [Gre18, Theorem 6.1]).

If $R \to k$ is proxy-regular, (B.1) does not induce an equivalence in general, but the counit of this adjunction still provides a k-cellular approximation for modules in Mod_R (see [Gre18, Lemma 6.3]):

(B.2)
$$\operatorname{Cell}_k(M) \simeq \operatorname{Map}_R(k, M) \wedge_E k$$

Moreover, for all R-modules M, there is a natural equivalence (see [Gre18, Lemma 6.6])

(B.3)
$$\operatorname{Cell}_k(M) \simeq \operatorname{Cell}_k(R) \wedge_R M$$

Formula (B.2) shows that when $R \to k$ is proxy-regular, the k-cellular approximation $\operatorname{Cell}_k(M)$ is functorial and effectively constructible in Mod_R (cf. [DGI06, Definition 4.3]).

 $^{^{11}}$ Cellularization is an example of a general model-categorical construction called right Bousfield localization (colocalization) with respect to an object A. In this language, A-cellular equivalences are called A-colocal equivalences, A-cellular objects are A-colocal objects, and A-cellular approximations are functorial cofibrant replacements in the A-colocal model structure on Mod_R (see [Hir03, 3.1.19]).

Now, we come to the key definition of a Gorenstein ring spectrum that we state under the regularity assumptions of Definition B.1 (which is a slightly less general form than in [DGI06]):

Definition B.2 (cf. [DGI06], 8.1 and 8.4). A morphism of commutative ring spectra $R \to k$ is called Gorenstein of shift $a \in \mathbb{Z}$, if $R \to k$ is proxy-regular and there is an equivalence of k-modules

(B.4)
$$\operatorname{Map}_{R}(k,R) \simeq \Sigma^{a}k$$

where Σ denotes the suspension functor on Mod_k .

We will be mostly interested in ring spectra R that are augmented k-algebras over a field k. For such algebras, we will always assume that $R \to k$ is the given augmentation morphism on R, and we will simply say that R is Gorenstein if so is $R \to k$. The Gorenstein condition (B.4) can be slightly refined in this case. Note that, if R is a k-algebra, using the k-module structure on R, we can rewrite (B.4) in the form

(B.5)
$$\operatorname{Map}_{R}(k,R) \simeq \Sigma^{a} \operatorname{Map}_{R}(k,\operatorname{Map}_{k}(R,k))$$

Both sides of (B.5) have natural right module structures over the endomorphism ring $E = \operatorname{Map}_{R}(k,k)$ but, in general, these module structures need not to agree under the equivalence (B.5). Following [DGI06] (see also [Gre18, Section 18.2])), we say that an augmented k-algebra R is orientable Gorenstein if (B.5) is an equivalence of right E-modules.

If R is a local Noetherian ring of Krull dimension d with residue field $k = R/\mathfrak{m}$, then R is Gorenstein (in the sense of commutative algebra) iff

(B.6)
$$\operatorname{Ext}_{R}^{i}(k,R) \cong \left\{ \begin{array}{ll} k & i=d \\ 0 & \text{otherwise} \end{array} \right.$$

The isomorphism (B.6) can be written as an equivalence $RHom_R(k,R) \simeq \Sigma^d k$ in the derived category $\mathcal{D}(R)$ and thus corresponds to the Gorenstein condition (B.4) of Definition B.2. In classical commutative algebra, there is another well-known characterization of Gorenstein rings in terms of local cohomology:

(B.7)
$$H_{\mathfrak{m}}^{i}(R) \cong \left\{ \begin{array}{cc} \operatorname{Hom}_{k}(R,k) & i = d \\ 0 & \text{otherwise} \end{array} \right.$$

which can be viewed as a special case of Grothendieck's local duality theorem. The following definition is a topological analogue of (B.7).

Definition B.3. An augmented k-algebra R satisfies Gorenstein duality of shift a if there is an equivalence of R-modules

(B.8)
$$\operatorname{Cell}_k(R) \simeq \Sigma^a \operatorname{Map}_k(R, k)$$

While the algebraic conditions (B.6) and (B.7) are known to be equivalent, their topological analogues (B.4) and (B.8) are, in general, not (see, e.g., [BCHV21, Remark 2.11] for a counterexample). This necessitates two separate definitions for Gorensteinness of commutative ring spectra.

The last property of ring spectra that we want to review is concerned with double centralizers. Recall, for a morphism $R \to k$, the double centralizer of R is defined to be $\hat{R} := \operatorname{Map}_E(k, k)$, where $E = \operatorname{Map}_R(k, k)$ is the endomorphism spectrum of k in Mod_R . The left multiplication on k gives a morphism of ring spectra $R \to \hat{R}$, and following [DGI06], we say

Definition B.4 ([DGI06], 4.16). $R \to k$ is dc-complete if $R \xrightarrow{\sim} \hat{R}$ is an equivalence in $Alg_{\mathbb{S}}$.

Note that, in algebra, a surjective homomorphism $R \to k$ from a Noetherian commutative ring R to a field k is dc-complete iff $R \cong \hat{R}_I$, where $\hat{R}_I := \varprojlim R/I^n$ is the I-adic completion of R with respect to the ideal $I = \operatorname{Ker}(R \to k)$. This motivates the above terminology. One can show that if $R \to k$ is dc-complete, the regularity properties of the ring spectra R and E are strongly connected (see, e.g., [DGI06, Proposition 4.17]).

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