1 Introduction

The mathematical formulation of curvature was not developed until the late 17th century, however it was calculated, although implicitly, by mathematicians in ancient Greece. As a topic, the study of curvature is referred to as differential geometry and relies heavily on the ideas of calculus, as such it might not be a surprise to hear that the person to introduce curvature as we might think of it today is Newton, who originated calculus along with Leibniz. Leibniz himself also dabbled in curvature with an idea he called the Osculating circle(Essentially the circle that best fits a curve at a point). The way curvature was originally found without all of our notions of modern day calculus was through geometry of infinitesimals(Stillwell). While not a technically accurate approach, it is more illuminating than just repeated manipulation of equations. The goal of this exploration is to define curvature on plane curves and surfaces and find formulas and interesting results for them.

2 Plane Curves

Plane curves are curves in a flat plane, such as $y = x^2$ or $(\cos(x), \sin(x))$ and are a special case of curvature since there's only one interpretation possible. This will become more clear once we talk about curvature of surfaces later. However first we need to define what curvature means. Especially in higher dimensions, a suitable definition for curvature becomes different.

2.1 Defining Curvature

Imagine we are on a roller coaster with a bunch of curves, and we are coming across a particularly sharp one, we get thrown into the side of the coaster. Next, we come across a much less curved one and we are thrown much more lightly. As a naive definition we can say that the curvature is the force felt when thrown into the side of the coaster, which makes even more sense if we consider a roller coaster going on a circular track. The curvature is constant and the force we feel pressed to the side is constant. Now, let's go back to our first two examples. We can say that they are of different curvatures but in more precise terms, what is the differentiating factor? Well, if we imagine ourselves in the roller coaster again, it's the rate of turning as the curve goes on but how can we define that precisely?

First, let's hammer out what we mean by ”as the curve goes on”. We are trying to find a definition for a curvature at a point and ”as the curve goes on” makes sense if we consider a roller coaster which has a direction but does not if we consider something like $y = x^3$. So what we really mean is we imagine we have some starting point and then we consider some small change in the arclength of the curve. Now we can rewrite our definition of curvature at a point as the rate of turning with respect to some small change of arclength.

Now what about rate of turning? Rate of turning seems to imply there’s soem angle to change, but which angle and where do we find it? Well, let’s imagine that we are once again on the roller coaster, at each point we have a velocity vector(Essentially an that tells us where we’re headed at that point in time) and we can look at the change in the angles of the velocity vectors as we change the arclength a bit. Now, let’s remove the physical aspect of this. What we are really doing here is that we are looking at the tangent lines of the curve and how their angles change for some small change in arclength.

Now our definition is nearly complete, the curvature at a point is the change in the angles of the tangent line with respect to a small change of arclength. However, there’s still a small detail here that we need to clear up. How small of a change? Well, here we can take a hint from calculus and say that we look at the instantaneous change in angle, in other words we take the limit of the change in angles of the tangenet lines with respect to a small change in arclength. However, all of these words make it a bit clunky and a picture(Figure 1) can really help to clear it up. Note that ds represents a small change in arclength.

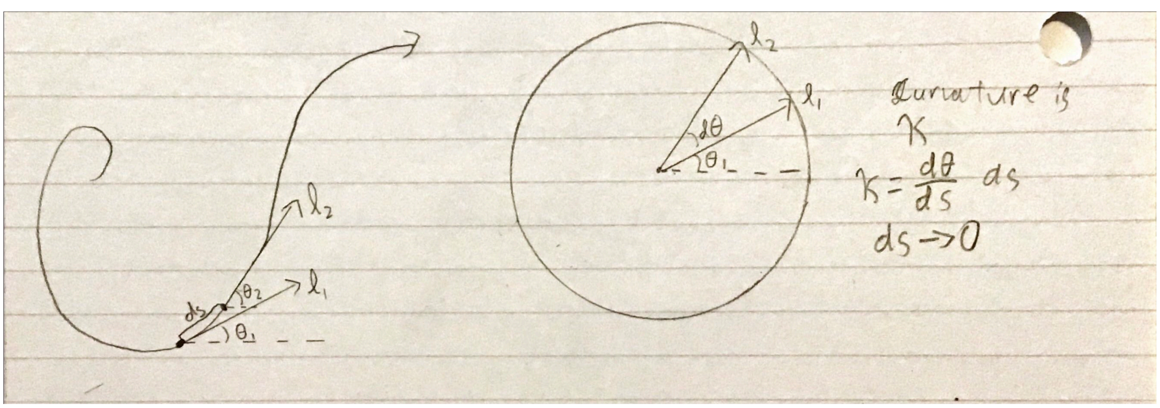

Figure 1: Definition of Curvature

Now that we have a definition(Given by Needham), we can start working with it. We will be using parametric plots since it allows the most freedom in defining curves, and we will be letting κ stand for curvature. We will be writing parametric plots as $(x(t), y(t))$. Here we only have one variable and so ideally, what we are looking for is some equation for κ in terms of x, y and t.

2.2 Deriving a Curvature Formula

Notice that we can rewrite $\frac{d\theta}{ds}$ as $\frac{d\theta/dt}{ds/dt}$ and now instead of trying to express $\frac{d\theta}{ds}$ in terms of x, y, t, we can look for an expression of $d\theta/dt$ and ds/dt individually. This can be more friendly because there's actually a standard way of finding $\frac{ds}{dt}$ which we will show below.

Let's now try to find the formula for $\frac{ds}{dt}$ where we interpret ds as a small change in arclength and dt as a small change in time. We can think of this nearly purely geometrically and use Pythagorean's Theorem. We can imagine taking a curve and zooming in to a very small part of it, so close that it's nearly completely straight and using Pythagorean's theorem as shown below in Figure 2. It's not completely accurate but it is a good approximation and becomes accurate once we introduce limits to it.

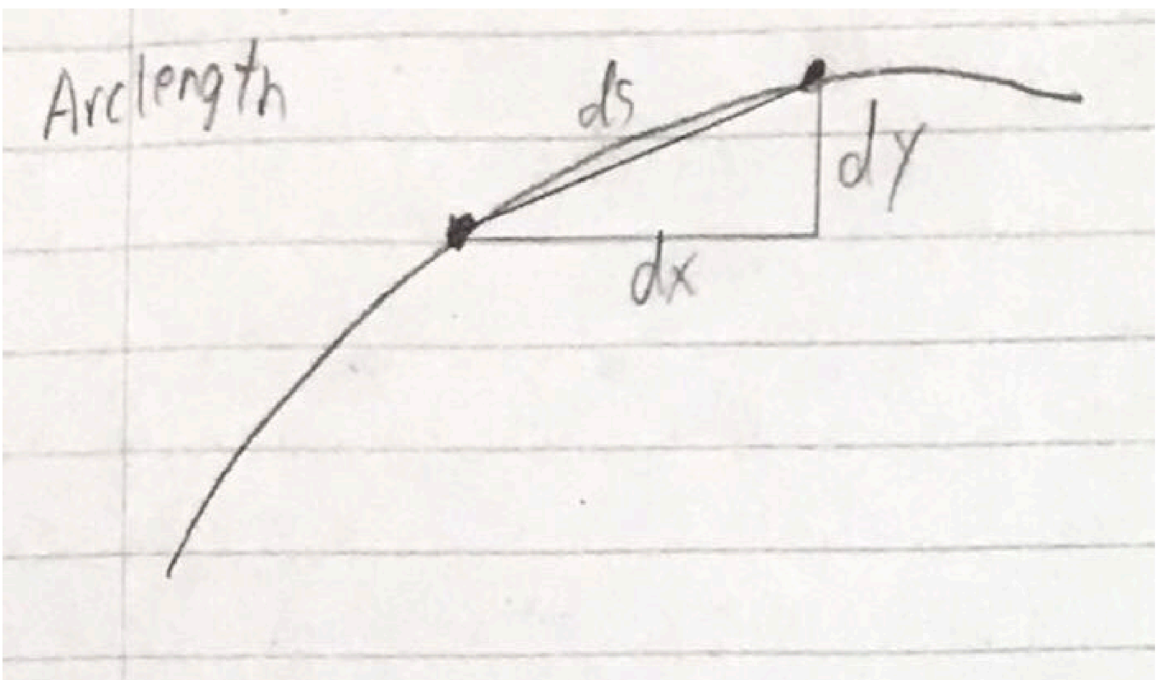

Figure 2: $ds^2 = dx^2 + dy^2$

Here we use Pythagorean's theorem to get $ds^2 = dx^2 + dy^2$, which we can rearrange to

$ds = \sqrt{dx^2 + dy^2}$, and if we divide by dt we get

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \tag{1}$$

As ds and dt approach 0, $\frac{dx}{dt}$ approaches the instantaneous rate of change of x with respect to t, in other words the derivative of x with respect to t, similarly $\frac{dy}{dt}$ also becomes a derivative.

It's a bit tougher to find an expression for $\frac{d\theta}{dt}$. However, we can apply the methodology used to find an expression for $\frac{ds}{dt}$. First, we find a useful identity. Since we are dealing with angles, we will likely come across trig at some point so we know where to look. It often helps to draw a diagram to try and visualize the situation. So let's imagine we have some curve and at some point we have some angle θ and some small increments dx and dy in the x and y directions respectively as shown in Figure 3.

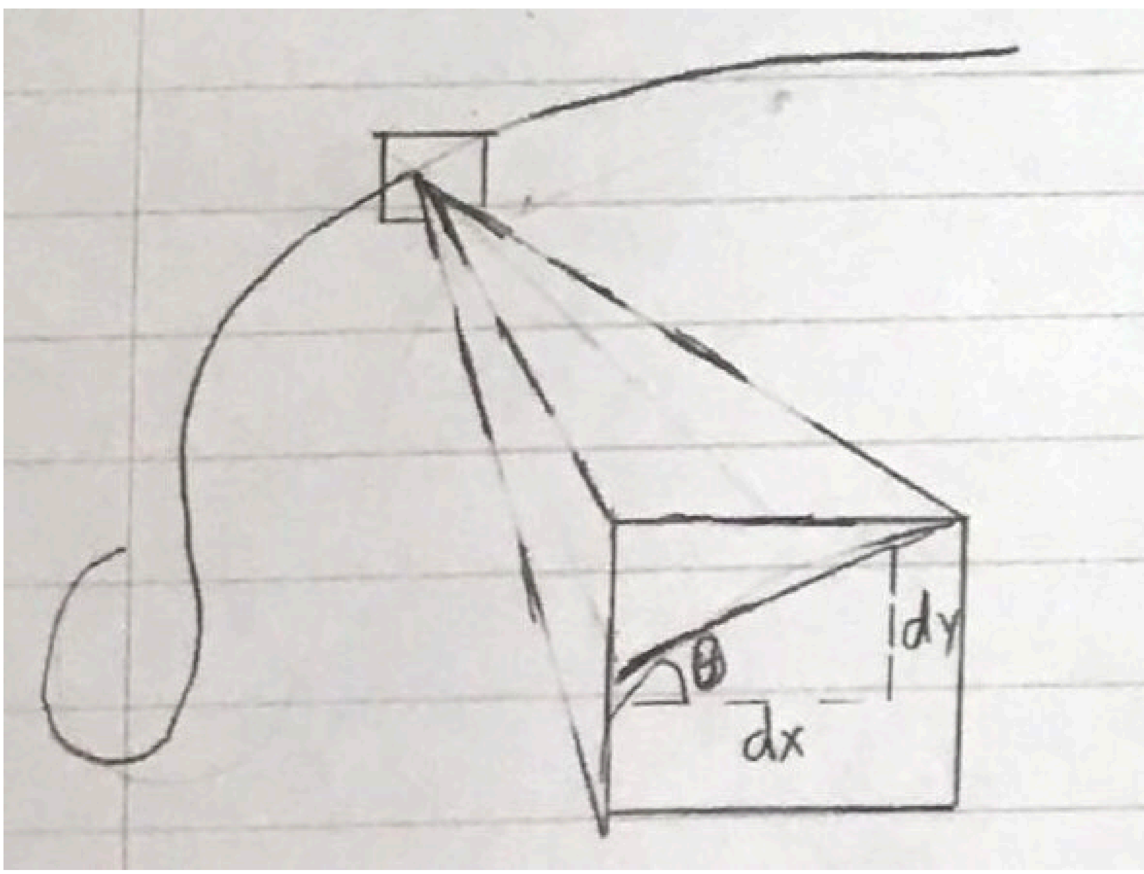

Figure 3: $\tan(\theta) = \frac{dy}{dx}$

If we apply a bit of trigonometry we can get $\tan(\theta) = \frac{opposite}{adjacent} = \frac{dy}{dx}$ which we can rewrite as $\frac{dy/dt}{dx/dt}$. And as dx,dy,dt approach 0, it all becomes derivatives, which we will then shorten to $\frac{y'}{x'}$. So now we have our identity

$$\tan(\theta) = \frac{y'}{x'} \tag{2}$$

If you'll remember from chain rule, differentiating $f(g(x))$ gives $f'(g(x))g'(x)$, and in this case our g is θ. The idea here is that we can take the derivative of $\tan(\theta)$ using chain rule and quotient rule/ and hope that it all works out when we isolate for θ'

$$\frac{d}{dt}\tan(\theta) = \sec^2(\theta)\frac{d\theta}{dt} = \frac{d}{dt}\left(\frac{x'}{y'}\right) = \frac{x'y" - y'x"}{x'^2}$$

And now we isolate for $\frac{d\theta}{dt}$

$$\begin{aligned}\frac{d\theta}{dt} &= \left(\frac{1}{\sec^2\theta}\right)\left(\frac{x'y" - y'x"}{x'^2}\right)\\ &= \left(\frac{1}{1+\tan^2\theta}\right)\left(\frac{x'y" - y'x"}{x'^2}\right)\\ &= \left(\frac{1}{1+\left(\frac{y'}{x'}\right)^2}\right)\left(\frac{x'y" - y'x"}{x'^2}\right)\\ &= \frac{x'y" - y'x"}{x'^2 + y'^2}\end{aligned}$$

Then,

$$\begin{aligned}\frac{d\theta}{ds} &= \frac{\frac{d\theta}{dt}}{\frac{ds}{dt}}\\ &= \frac{\frac{x'y" - y'x"}{x'^2+y'^2}}{\sqrt{x'^2+y'^2}}\\ &= \left(\frac{x'y" - y'x"}{x'^2 + y'^2}\right)\left(\frac{1}{(x'^2+y'^2)^{\frac{1}{2}}}\right)\\ &= \frac{x'y" - y'x"}{(x'^2+y'^2)^{\frac{3}{2}}}\end{aligned}$$

In other words for a curve given by $(x(t), y(t))$,

$$\kappa = \frac{x'y" - y'x"}{(x'^2+y'^2)^{\frac{3}{2}}} \tag{3}$$

(Derivation due to Weisstein) Now what about for a plot $(x, f(x))$? Well we want $x(t)$ to be the independent variable, $x(t) = t$. Now we just need to compute $x', x", y', y"$. First of all since $x = t$, we have $x' = 1$ and $x" = 0$ and $y' = \frac{df}{dt}$ and so on. Since $x = t$ we can also write $y' = \frac{df}{dx}$ and $y" = \frac{d^2f}{dx^2}$ and if we substitute this into the formula we get

$$\kappa = \frac{1 \cdot \frac{d^2f}{dx^2} - \frac{df}{dx} \cdot 0}{1^2 + \left(\frac{df}{dx}\right)^2}$$

which we can then simplify to

$$\kappa = \frac{\frac{d^2f}{dx^2}}{1^2 + \left(\frac{df}{dx}\right)^2} \tag{4}$$

which is the formula for curvature originally discovered by Newton(Stillwell). If you are familiar with physics, another interpretation of curvature is the centripetal acceleration at each point in time along a path. This can be used to find the g-force a person experiences on a roller coaster at each point along the path of the ride. Similarly it tells us about the force experienced by someone/something as you go around a corner, for example the force experienced by a train going through a bend in the tracks. This can be used in engineering when designing tracks, whether fro trains or roller coasters and even for ramps on and off highways.

3 Curvature of Surfaces

There are two ways to go about the curvature of surfaces, the first is the historical approach which favours more geometric reasoning and the second is the modern approach which favours using clean analytical arguments. The historical approach refers to the sequence in which the major results were first discovered however it is a bit of a misnomer since the approach favoured for teaching it in the modern day was developed in parallel to the historical approach. The timeline is a bit scattered, but one way to view it is that the modern approach wasn't tied together in the way it is now until later. Our goal here will be to define curvature on surfaces and explore an application to Earth's curvature in particular.

3.1 Geodesics

The earth is not flat, and yet to us on the surface, it has straight lines. How can that be? This is an example of a geodesic, a straight line on earth is called a great circle because if you keep going it will wrap you all the way around back to your starting point as shown in Figure 4. Some examples of geodesics are given in Figure 5

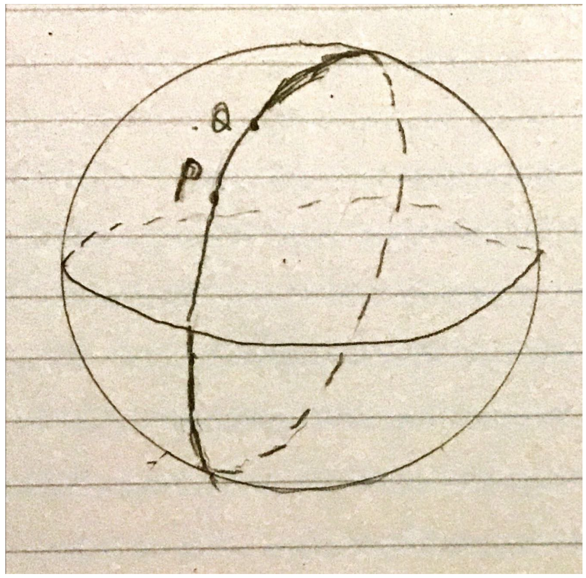

Figure 4: Great Circle

It's curved if we take an outside observer's point of view, it's curved in *space*. However, it is straight in the surface. We still haven't answered how we know if a line is straight(or rather a geodesic) for a general surface. The idea here is that we can define something called geodesic curvature for each point in the curve. The idea behind geodesic curvature is that we approach it like normal curvature and we assign some value to each point in the curve. The only tiny hiccup here however is that the surface isn't flat and if we could make it flat, it would be really easy to compute the curvature.

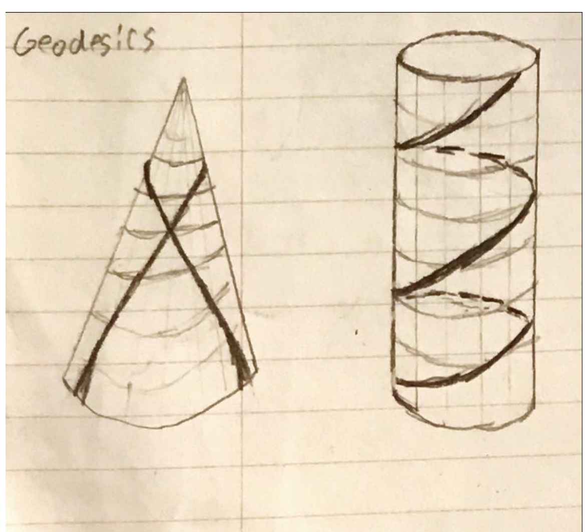

Figure 5: Some Examples of Geodesics

To remedy this, let's say we wanted to find the geodesic curvature at a point p on the curve. Then we project it onto the tangent plane at p. What this does is it says that for a region close to p, this is what it would look like if it was a curve on a plane. It takes a segment of the curve and flattens it so you can find the curvature at one specific point as shown below.

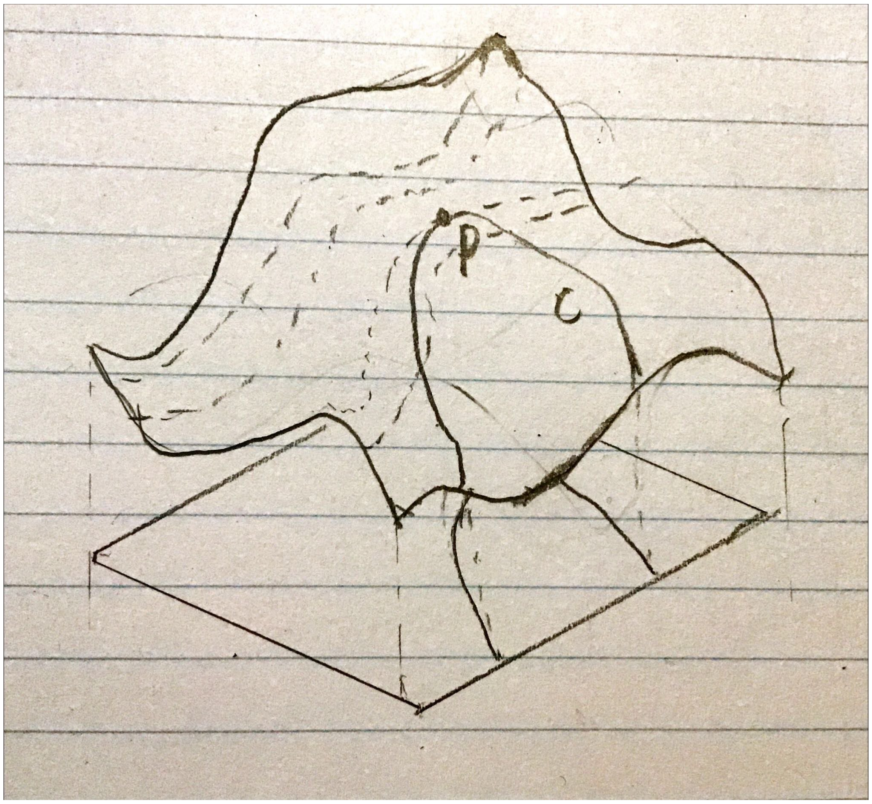

Figure 6: Projection of Curve to Tangent Plane at p

The geodesic curvature of a curve at a point p is then the plane curvature of the projection of the curve at p. This projection is shown in Figure 6. The idea is if you took a really thin strip of paper and wrapped it around, say a pear, you would find a geodesic. So then, you should be able to take a geodesic and flatten it into a straight line where you started flattening it. Geodesic curvature tests how much it deviates from a straight line, or in other words, how much it deviates from a geodesic. It then follows that a geodesic should have geodesic curvature 0 everywhere since, being a geodesic itself, it doesn't deviate from a geodesic. Geodesics and geodesic curvature might seem innocuous enough but they're actually vitally important in aspects of physics like Relativity, which makes plenty use of differential geometry, which includes the study of geodesics.

Geodesic curvature is mostly a tangent, it can help to clarify what exactly geodesics are(straight lines in surfaces), which will be used plenty in the next subsection

3.2 Angular Excess

What is a triangle? Well a triangle is a 2d figure that has 3 angles and 3 sides, all of which are straight. Now what about a triangle on, for example, a sphere? Well we can still have 3 straight sides and angles, the sides are geodesics and the angles are, well, angles. Now a triangle on a plane has a sum of angles of π. What about a triangle on a sphere? Well let's give a simple example. Imagine we are at the equator, we stand 10 metres from each other and we walk all the way up to the north pole. We'd meet each other there. We have just formed a triangle, since it has 3 angles and 3 sides. However, if we look at the two angles formed on the equator, they are each $\pi/2$ since we've walked straight up from the equator, however there's still some angle θ they make once they've reached the equator as shown in Figure 7. So what we have is an angle sum of $\pi + \theta$ and an angular excess of θ as shown below.

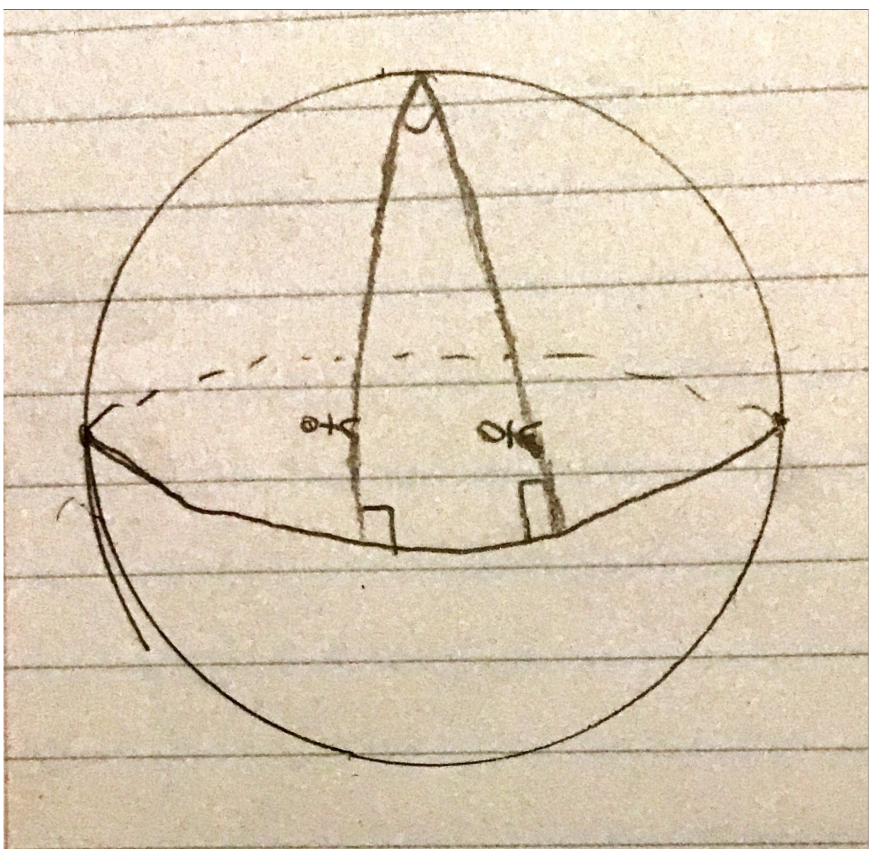

Figure 7: Angular Excess of a Triangle on a Sphere

We will write the angular excess of a triangle Δ as $\epsilon(\Delta)$, and now a question appears. Can we calculate how much angular excess there is for a given triangle on a sphere? Well first, let's draw some arbitrary triangle on a sphere with angles α, β, and γ as shown in Figure 8. Notice that each great circle splits the sphere in half.

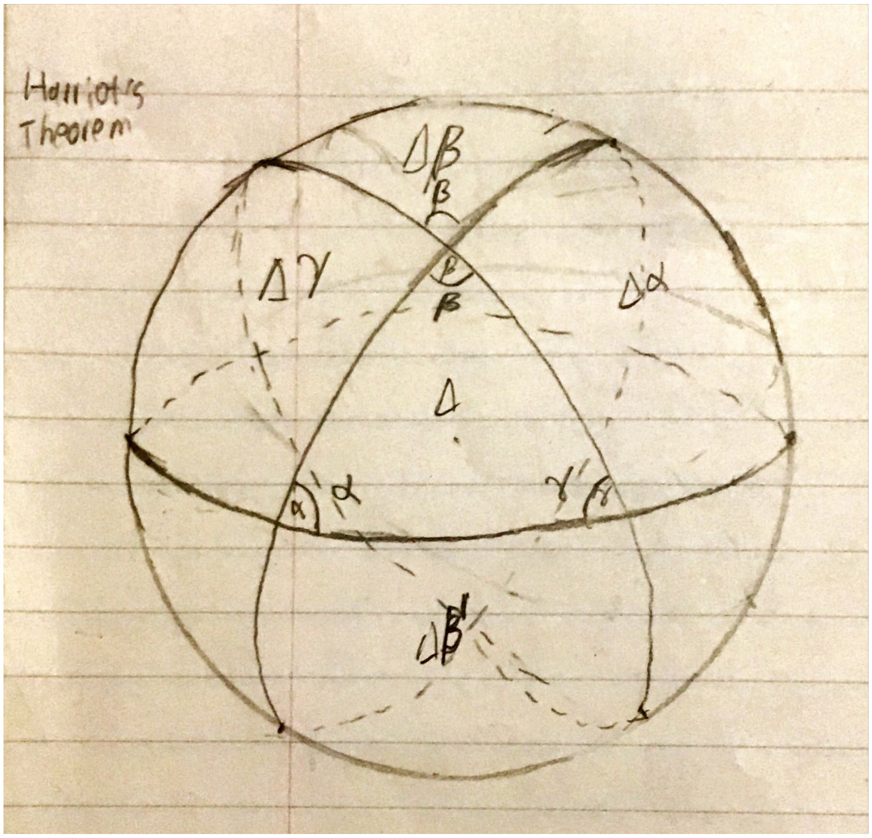

Figure 8: A Geodesic Triangle on a Sphere

Now we extend the lines of the triangle. If we count the total number of triangles we see, there are 8 of them. We know that the surface area of a sphere is $4\pi R^2$ where R is the radius of the sphere. so then if we take all the triangles in a hemisphere($\Delta, \Delta_\alpha, \Delta_\beta, \Delta_\gamma$) and add up their areas it gives us

$$A(\Delta) + A(\Delta_\alpha) + A(\Delta_\beta) + A(\Delta_\gamma) = 2\pi R^2 \tag{5}$$

Notice that $A(\Delta_\beta) = A(\Delta_{\beta'})$ and that $A(\Delta) + A(\Delta_{\beta'}) = \frac{\beta}{\pi}(2\pi R^2) = 2\beta R^2$ and so $A(\Delta) + A(\Delta_\beta) = 2\beta R^2$. Similarly, $A(\Delta) + A(\Delta_\alpha) = 2\alpha R^2$ and $A(\Delta) + A(\Delta_\gamma) = 2\gamma R^2$. And if we added all of this together we get

$$3A(\Delta) + A(\Delta_\alpha) + A(\Delta_\beta) + A(\Delta_\gamma) = 2(\alpha + \beta + \gamma)R^2 \tag{6}$$

If we now subtract (5) from (6) we get

$$2A(\Delta) = 2(\alpha + \beta + \gamma - \pi)R^2$$
$$= 2\epsilon(\Delta)R^2$$

And so the equation we get is

$$\frac{A(\Delta)}{R^2} = \epsilon(\Delta) \tag{7}$$

(Proof due to Needham)What this tells us is that for larger and larger spheres, the less the angular excess if $A(\Delta)$ is conserved. This motivates our next definition. The curvature at a point is the angular excess per unit area at that point(Needham). A rough sketch of this idea is shown in Figure 9. In other words, the curvature K of a point p is the limit of the angular excess divided by the area a triangle shrinking to p. Another way to write this is

$$\mathrm{K} = \lim_{\Delta \to p} \frac{\epsilon(\Delta)}{A(\Delta)} \tag{8}$$

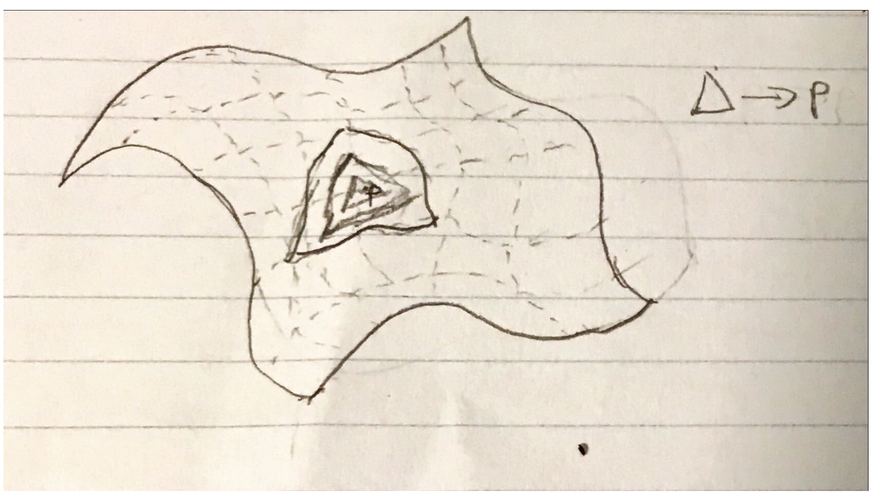

Figure 9: A Geodesic Triangle Shrinking to a Point

Now let's compute the curvature for a sphere. We can rearrange equation (7) as $\frac{\epsilon(\Delta)}{A(\Delta)} = \frac{1}{R^2}$. This then implies that

$$\mathrm{K} = \lim_{\Delta \to p} \frac{\epsilon(\Delta)}{A(\Delta)}$$
$$= \frac{1}{R^2}$$

Which would make sense considering that from our intuitive notion of curvature, for a sphere it decreases as radius increases and is constant everywhere if we fix the radius.

3.3 Approximating the Curvature of the Earth

Let's give a bit of an example now. We can now find approximate the curvature of the earth using the circumference of a small circle on it. However, before we talk about that we have to introduce polynomial approximations of functions. The ones we will be using are called Taylor Series. Essentially it's a sum of polynomial functions where adding on each new part helps approximate the function better and better. Let's call f_n the nth term added to the sum and f the function we are approximating, then we can say that as $n \to \infty$, $\left(\sum_{n \in \mathbb{N}} f_n\right) \to f$.

Another important part about Taylor series is that they have a sort of centre of approximation where they approximate really well close to that region, and less well the farther you get. Unless of course you are adding on more terms. The relevant Taylor series for us is $\sin(\phi) \approx \phi - \frac{\phi^3}{3!}$ (for small ϕ).

The last point to view before we hop in is some review.

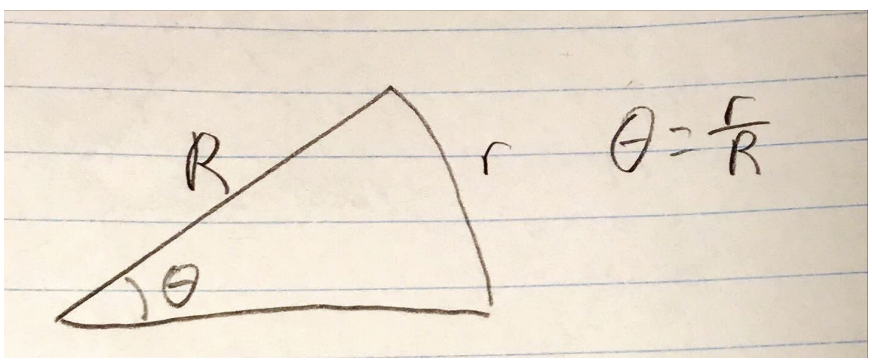

Figure 10: Angles/Radians Definition

For our purposes the definition of radians/angles is very important. As shown in Figure 10, an angle θ in a circle is given the arclength r divided by the radius R.
So now let's take a look at a diagram of a sphere with, for example, radius R, as shown below in Figure 11.

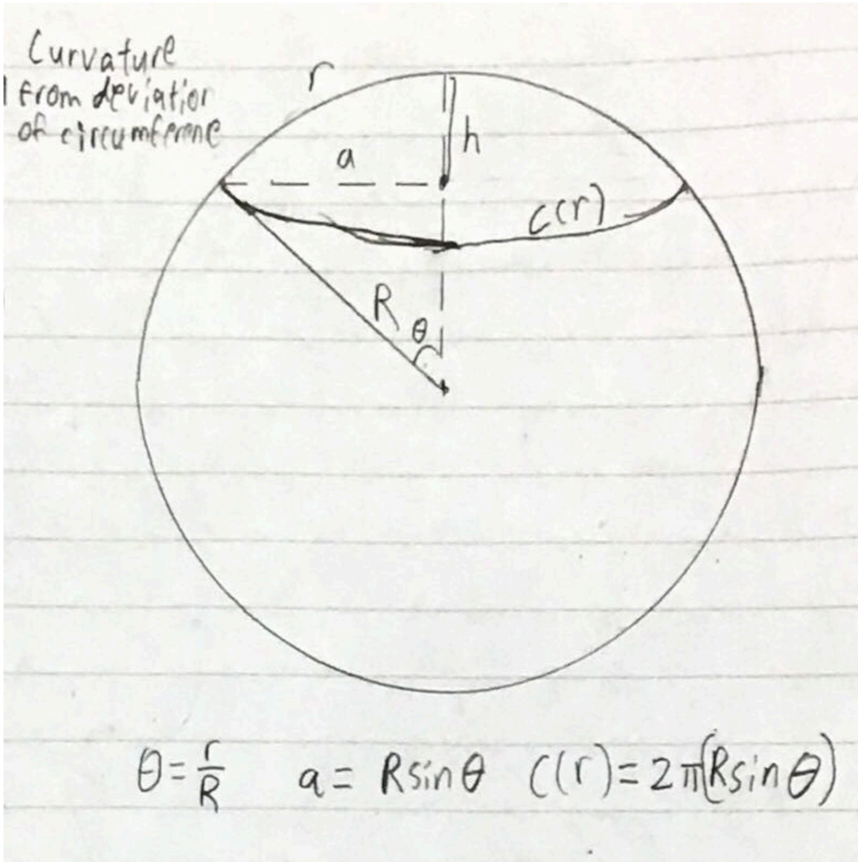

Figure 11: Circle on a Sphere

Let's calculate the circumference $C(r)$, and to calculate the circumference we first need to find a. Notice that, we have $\sin(\theta) = \frac{a}{R}$, as shown in Figure 11, and so we can rearrange this to get $a = R\sin(\theta)$. We can use the definition of radians/angles to get $\theta = \frac{r}{R}$, and so we can write

$$C(r) = 2\pi a = 2\pi R \sin\left(\frac{r}{R}\right) \tag{9}$$

Now let's temporarily put this on the backburner and turn our eye to our sine approximation. We can rearrange it to get

$$\theta - \sin(\theta) \approx \frac{1}{3!}\theta^3 = \frac{1}{6}\theta^3 \tag{10}$$

Now we can write, using (9) and (10)

$$\begin{aligned} 2\pi r - C(r) &= 2\pi R\left[\frac{r}{R} - \sin\left(\frac{r}{R}\right)\right] \\ &\approx 2\pi R\left(\frac{1}{6}\left(\frac{r}{R}\right)^3\right) \\ &= \frac{2\pi R r^3}{6R^3} \\ &= \frac{\pi r^3}{3R^2} \end{aligned}$$

Recall that the curvature K of a sphere is given by $\frac{1}{R^2}$, and so we end up with

$$2\pi r - C(r) \approx \left(\frac{\pi r^3}{3}\right)\mathrm{K} \tag{11}$$

and if we isolate for K here, we get

$$\mathrm{K} \approx \frac{3}{\pi}\left[\frac{2\pi r - C(r)}{r^3}\right] \tag{12}$$

And now we have our approximation(Proof and approximation due to Needham). Recall that our approximation for sine only works for very small ϕ, and in this case, our ϕ is equal to $\frac{r}{R}$. What this tells us is that since R is constant, we need very small r for the approximation to be more accurate. So our approximation only works for very small r. So hypothetically if we could make a really good circle and measure it extremely accurately, we could find the curvature of the earth purely from measuring the circumference of that circle.

4 Conclusion

In this article, we've first defined curvature on both plane curves and surfaces, found a formula for defining plane curves and rederived Newton's original formula, as well as found a way to approximate the curvature of the earth. That last example in particular has very deep implications since it tells us that the curvature we've defined can be locally determined. We just need a very accurate measurement of a very round and small circle. Curvature is not necessarily determined from space, and in fact this begs the question whether anyone has ever found the curvature of the Earth purely by looking at the terrain of the earth and the answer is yes. Gauss, in his study of Geodesy(surveying and mapping), noticed the curvature of the earth through many expeditions(Needham).

Works Cited

Needham, Tristan. *Visual Differential Geometry and Forms*. Princeton P, 2021.

Stillwell, John. *Mathematics and Its History*. Springer, 2010. Undergraduate Texts In Mathematics.

Weisstein, Eric. “Curvature,” mathworld.wolfram.com/Curvature.html.