Riemann Integration

Brian Wu

January 2, 2025

You can use anything for a Vector Space. You can make a Vector Space with cows as long as you can define what a negative cow would be.

Some person's dad

Remark 1. Many of the ideas of Riemann Integration extend equally well into functions that map into more general spaces than just $\mathbb{R}$ or $\mathbb{C}$

Here we will show a generalization of the concept of Riemann Integration to a "vector-valued function". More specifically, a function that maps into a vector space that is complete(every cauchy sequence converges). The definition for vector space is long an easily googleable so it will not be provided here.

A vector space is a set where that you can define multiplication(scalar multiplication) by an algebraically "nice" subset of the complex numbers(A subfield of $\mathbb{C}$), adding elements of the set(vector addition), an identity element(a zero element), inverse elements(negative elements), and is associative and closed under the operations(you cannot make an element not in the set from the set).

TLDR. A vector space is where you can add things, multiply some things, and are nice enough that we can do simple algebra on it.

Definition 2 *Norm*. Let $(V, \mathbb{F})$ be a vector space. A function $\nu : V \to \mathbb{R}$ is a seminorm if for all $k \in \mathbb{F}$, $v, w \in V$

(a) $\nu(kv) = |k|\nu(v)$

(b) $\nu(v) \geq 0$

(c) $\nu(v + w) \leq \nu(v) + \nu(w)$

If $\nu(v) = 0 \iff v = 0$ then we call ν a norm.

Definition 3 *Normed Linear Spaces*. A normed linear space is a vector space $\mathcal{X}$ over a field $\mathbb{F}$ with a norm $||\cdot||$ on $\mathcal{X}$.

A normed linear space generalizes the idea of "size" on a set where things can be added and stretched(think $\mathbb{R}^2$).

Definition 4 ***Banach Space.*** The *metric space induced by the norm* of a Normed Linear Space(NLS) $(\mathcal{X}, ||\cdot||)$ is the metric space (X, d) where $d(x,y) = ||x-y||$. If this induced metric space is complete, we call it a *Banach space*.

Definition 5 ***Riemann Sum.*** Let $a < b$ be in $\mathbb{R}$, let $\mathcal{X}$ be a Banach space and $f : [a,b] \to \mathcal{X}$ be a function. A *Partition* of $[a,b]$ is a finite set $\{a = x_0 < x_1 < \cdots < x_N = b\}$ where $N \in \mathbb{N}$. The set of all partitions of $[a,b]$ is denoted $\mathcal{P}[a,b]$.

A finite set $P^* = \{p_i | 1 \leq i \leq N, x_{i-1} \leq p_i \leq x_i, x_i \in P\}$ is called a set of test values for the partition P.
We define the Riemann sum

$$S(f, P, P^*) = \sum_{i=1}^{N} f(p_i)(x_i - x_{i-1})$$

Definition 6 ***Refinement.*** A refinement of a partition $P \in \mathcal{P}[a,b]$ is a set $Q \in \mathcal{P}[a,b]$ such that $P \subset Q$.

The Riemann Sum is defined in an analogous way to how it is defined for functions from $\mathbb{R}$ to $\mathbb{R}$. In fact, if you set $(\mathcal{X}, ||\cdot||) = (\mathbb{R}, |\cdot|)$ then you have exactly the Riemann Sum studied in standard first year calculus.

Definition 7. A function $f : [a,b] \to \mathcal{X}$ is *Riemann Integrable* if there exists $x_0 \in \mathcal{X}$ for all $\epsilon > 0$ such that there exists a partition P of $[a,b]$ such that for all refinements Q of P and test values Q^* of Q,

$$||x_0 - S(f, Q, Q^*)|| < \epsilon$$

We will then say

$$x_0 = \int_a^b f = \int_a^b f(x)\mathrm{d}x$$

Remark 8. Here, we can see why at least a Normed Linear Space is required. The norm is used in the definition of Riemann Integrability to measure the "error" which is needed to ensure that the partitions are close enough eventually.

Proposition 9 ***Uniqueness of the Riemann Integral.*** Suppose $f : [a,b] \to \mathcal{X}$ is Riemann integrable and integral is x_0. This integral is unique.

Proof. Concepts of a plan will be provided here instead.

Suppose for contradiction it is not unique and there exists y_0 that satisfies these conditions. Let $\epsilon = ||y_0 - x_0||/2$ and choose P_1 to be a partition of $[a, b]$ for ϵ and x_0 and P_2 for y_0 respectively. Take Q to be a refinement of $P_1 \cup P_2$ and observe that Q is a refinement of P_1 and P_2. Notice

$$\begin{aligned} 2\epsilon &= ||y_0 - x_0|| \\ &\leq ||y_0 - S(f, Q, Q^*)|| + ||S(f, Q, Q^*)|| \\ &< \epsilon + \epsilon \\ &= 2\epsilon \end{aligned}$$

which is a contradiction.

The definition of Riemann Integrability requires an x_0. In other words, it requires knowing the integral before being able to check if a function if Riemann Integrable. It would be nice if we could check if a fucntion is Riemann Integrable without having to know the intgeral beforehand. Here is where the "completeness" requiremenet comes in handy.

Theorem 10 ***Cauchy Criterion for Integrability.*** Let $\mathcal{X}$ be a Banach space, $a < b$ be real numbers and $f : [a, b] \to \mathcal{X}$ be a function. The following are equivalent.

(a) f is Riemann Integrable

(b) For all $\epsilon > 0$, there exists a partition R of $[a, b]$ such that if P, Q are refinements of R and P^*, Q^* are test values of P, Q respectively, then

$$||S(f, P, P^*) - S(f, Q, Q^*)|| < \epsilon$$

Proof. $(a) \Rightarrow (b)$ by a standard argument

$(b) \Rightarrow (a)$ is analogous to the case of real-valued functions and is left as an exercise.

Theorem 11 ***Continuity implies Integrability.*** Let $\mathcal{X}$ be a Banach space, $a < b$ be real numbers and $f : [a, b] \to \mathcal{X}$ be a continuous function. f is Riemann-Integrable.

Proof. The proof is an adaptation of the proof for $\mathbb{R}$ and is left as an exercise to the reader's ChatGPT/Copilot/Gemini/Generative AI.

Corollary 12. Convex functions over finite intervals are integrable.