

1 Yoneda Lemma

Definiton 1.1. For any category \mathcal{A} , it's opposite category, \mathcal{A}^{op} is the category having the objects of \mathcal{A} . And for objects $A, B \in \mathcal{A}$, a morphism $f \in \mathcal{A}^{op}(A, B)$ only if there is a morphism $g \in \mathcal{A}(B, A)$.

Proposition 1.1. For a locally small category \mathcal{A} , fixing an object $A \in \mathcal{A}$ gives a functor, $H_A : \mathcal{A}^{op} \rightarrow Set$ defined as:

- (i) For any object $B \in \mathcal{A}$, $H_A(B) := Hom_{Set}(B, A)$
- (ii) For any morphism, $g : X \rightarrow Y$ in \mathcal{A} ,

$$H_A(g) : \mathcal{A}(Y, A) \rightarrow \mathcal{A}(X, A) \text{ is given by } p \mapsto p \circ g$$

Definiton 1.2. For a locally small category \mathcal{A} , the category of presheaves on \mathcal{A} , denoted by $[\mathcal{A}^{op}, Set]$ is defined to have functors from \mathcal{A}^{op} to Set as objects, and natural transformations between them as morphisms.

Lemma 1.2.1. Lemma 1.3.11

Lemma 1.2.2. Exccercise 1.3.29

Theorem 1.1. Yoneda Lemma If \mathcal{A} is a locally small category then, for any object $A \in \mathcal{A}$ and $X \in [\mathcal{A}^{op}, Set]$, there exists an isomorphism,

$$[\mathcal{A}^{op}, Set](H_A, X) \cong X(A) \text{ which is natural in } A \text{ and } X.$$

Notation:

- We denote the category of presheaves on \mathcal{A} by \mathcal{C} .
- For the map $\hat{}$, instead of writing $\hat{}(a)$, we use $\hat{a} = b$ to denote $a \xrightarrow{\hat{}} b$.
- For the map $\tilde{}$, instead of writing $\tilde{}(a)$, we use $\tilde{a} = b$ to denote $a \xrightarrow{\tilde{}} b$.

- $[\mathcal{A}^{op}, Set](H_A, X)$ denotes the collection of morphisms $\alpha : \mathcal{A}^{op} \begin{matrix} \xrightarrow{H_A} \\ \Downarrow \alpha \\ \xrightarrow{X} \end{matrix} Set$.

To prove the theorem, first, we show that $[\mathcal{A}^{op}, Set](H_A, X)$ is isomorphic to $X(A)$ as set, and then that this isomorphism is natural in X and A .

Proof. Let \mathcal{A} be a locally small category. Fix an object $A \in \mathcal{A}$ and a presheaf X on \mathcal{A} .

I Showing isomorphism between $[\mathcal{A}^{op}, Set](H_A, X)$ and $X(A)$

Define $\hat{} : \mathcal{C}(H_A, X) \rightarrow X(A)$ for any $\alpha : H_A \rightarrow X$, as $\hat{\alpha} := \alpha_A(1_A)$. As $1_A \in Hom_{Set}(A, A) = H_A(A)$, definition of α_A gives that $\alpha_A(1_A) \in X(A)$

Define $\tilde{} : X(A) \rightarrow [\mathcal{A}^{op}, Set](H_A, X)$ for any $x \in X(A)$ as the natural transformation $\tilde{x} : H_A \rightarrow X$ whose K-component is the function mapping each morphism $p \in \mathcal{A}(K, A)$ to $(X(p))(x)$. That is, $\tilde{x}_K(p) := (X(p))(x)$.

We are going to show that \tilde{x} is a natural transformation. Fix objects $K, L \in \mathcal{A}$ and morphism $q \in \mathcal{A}^{op}(K, L)$.

$$\text{Need to show that the square } \begin{array}{ccc} H_A(K) & \xrightarrow{H_A(q)} & H_A(L) \\ \tilde{x}_K \downarrow & & \downarrow \tilde{x}_L \\ X(K) & \xrightarrow{X(q)} & X(L) \end{array} \text{ , that is } \begin{array}{ccc} \mathcal{A}(K, A) & \xrightarrow{- \circ q} & \mathcal{A}(L, A) \\ \tilde{x}_K \downarrow & & \downarrow \tilde{x}_L \\ X(K) & \xrightarrow{X(q)} & X(L) \end{array} \text{ commutes .}$$

So, for any $f : K \rightarrow A$, need that $\tilde{x}_L(f \circ q) = X(q) \circ \tilde{x}_K(f)$. Using the definition of \tilde{x} gives the following.

$$\begin{aligned} LHS &= \tilde{x}_L(f \circ q) = (X(f \circ q))(x) \\ RHS &= X(q) \circ \tilde{x}_K(f) = (X(q))(X(f)(x)) = (X(q) \circ X(f))(x) \end{aligned}$$

And as X is a contravariant functor, $X(f \circ q) = X(q) \circ X(f)$, giving that LHS=RHS. Now going to show that $\hat{}$ and $\tilde{}$ define an isomorphism. Need to show that $\hat{}$ and $\tilde{}$ are mutually inverse.

- (i) For any $x \in X(A)$, $\hat{x} = \tilde{x}_A(1_A) = (X(1_A))(x) = 1_{X(A)}(x) = x$.
- (ii) For any $\alpha \in \mathcal{C}(H_A, X)$, need to show that $\hat{\alpha} = \alpha$. So, it's required that each of their component are equal. As both $\hat{\alpha}$ and α are natural transformations between functors that go to the category *Set*, each of the components is a function. So, need to show that for any $f \in \mathcal{A}(K, A) = H_A(K)$, $(\hat{\alpha})_K(f) = \alpha_K(f)$. Using first the definition of \sim and then that of $\hat{\alpha}$ gives:

$$LHS = \hat{\alpha}_B(f) = (X(f))(\hat{\alpha}) = (X(f))(\alpha_A(1_A)) \quad (1)$$

And as $f \in \mathcal{A}(K, A)$, we also have the following.

$$RHS = \alpha_K(f) = \alpha_K(1_A \circ f) \quad (2)$$

Because α is a natural transformation, the square following square commutes for 1_A .

$$\begin{array}{ccc} \mathcal{A}(A, A) & \xrightarrow{- \circ f} & \mathcal{A}(K, A) \\ \alpha_A \downarrow & & \downarrow \alpha_K \\ X(A) & \xrightarrow{X(f)} & X(K) \end{array}$$

This gives that $\alpha_K(1_A \circ f) = (X(f))(\alpha_A(1_A))$. Hence, we have from (1) and (2), we get that $RHS = LHS$.

II Showing naturality of this isomorphism

By Using Lemma 1.2.1, it's enough to show that $\hat{\cdot}$ is natural in A for any choice of X , and that it is natural in X for any A .

- (i) We are going to show naturality in X . Fix any $A \in \mathcal{A}$. Need that for presheaves $X, Y \in \mathcal{C}$ and natural transformation $\beta \in \mathcal{C}(X, Y)$, the following square commutes.

$$\begin{array}{ccc} \mathcal{C}(H_A, X) & \xrightarrow{\beta \circ -} & \mathcal{C}(H_A, Y) \\ \hat{\cdot} \downarrow & & \downarrow \hat{\cdot} \\ X(A) & \xrightarrow{\beta_A} & Y(A) \end{array}$$

So, for any $\alpha : H_A \rightarrow X$, we need that $(\hat{\cdot} \circ H_\beta)(\alpha) = (\beta_A \circ \hat{\cdot})(\alpha)$. Using definition of H_β and $\hat{\cdot}$ gives:

$$LHS = (\hat{\cdot} \circ H_\beta)(\alpha) = \widehat{(H_\beta(\alpha))} = \widehat{(\beta \circ \alpha)} = (\beta \circ \alpha)_A(1_A) \quad (3)$$

$$RHS = (\beta_A \circ \hat{\cdot})(\alpha) = \beta_A(\hat{\alpha}) = (\beta_A \circ \alpha_A)(1_A) \quad (4)$$

As $\alpha \in \mathcal{C}(H_A, X)$ and $\beta \in \mathcal{C}(X, Y)$ are morphisms in \mathcal{C} , composition in \mathcal{C} gives $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$. From (3) and (4), we directly get that $RHS = LHS$.

- (ii) We are going to show naturality in A . Fix any $X \in \mathcal{C}$. Need that for objects $A, B \in \mathcal{A}$ and morphism $f \in \mathcal{A}^{op}(A, B)$, the following square commutes.

$$\begin{array}{ccc} \mathcal{C}(H_A, X) & \xrightarrow{- \circ H_f} & \mathcal{C}(H_B, Y) \\ \hat{\cdot} \downarrow & & \downarrow \hat{\cdot} \\ X(A) & \xrightarrow{X(f)} & X(B) \end{array}$$

So, for any $\alpha : H_A \rightarrow X$, we need that $(\hat{\cdot} \circ H_f)(\alpha) = ((X(f)) \circ \hat{\cdot})(\alpha)$. Using definition of H_f and $\hat{\cdot}$, we get:

$$LHS = (\hat{\cdot} \circ H_f)(\alpha) = \widehat{\alpha \circ H_f} = (\alpha \circ H_f)_B(1_B) = \alpha_B(f \circ 1_B) = \alpha_B(1_A \circ f) \quad (5)$$

$$RHS = ((X(f)) \circ \hat{\cdot})(\alpha) = (X(f))(\hat{\alpha}) = (X(f))(\alpha_A(1_A)) \quad (6)$$

The last equality in (5) is justified because the morphism f goes from B to A in \mathcal{A} . By using equality of (1) and (2) from I(i), for $f \in \mathcal{A}(B, A)$, we get that $(X(f))(\alpha_A(1_A)) = \alpha_B(1_A \circ f)$. Hence, $RHS = LHS$.

□

1.1 Cayley's Theorem

In group theory, Cayley's theorem is a result that similarly allows us to study any group by instead studying a subgroup of some symmetric group.

Definiton 1.3 (Symmetric group on a set).

Theorem 1.2. Cayley's Theorem Every group, (G, \cdot) is isomorphic to a subgroup of symmetric group on G .

1.2 Embedding of a category in Presheaf category

Definiton 1.4 (Embedding of a category). A category, \mathcal{A} is said to be embedded in a category, \mathcal{B} if there exists a functor $F : \mathcal{A} \rightarrow \mathcal{B}$ such that F is full, faithful and injective (on objects).

2 Prorelations

Definiton 2.1. A prorelation is a partially ordered, down-directed, up-set of relations $X \rightarrow Y$.

That is, $P \subseteq \mathcal{P}(X \times Y)$ is a prorelation if it satisfies the following conditions:

- (i) Partial Order: Containment of relations defines a partial order. That is, $r \subseteq s$ meaning that for any $(x, y) \in X \times Y$, if $(x, y) \in r$ then $(x, y) \in s$.
- (ii) Down-directed: For any $r, s \in P$, there exists $t \in P$ such that $t \subseteq r$ and $t \subseteq s$.
- (iii) Up-set: For any relation $u : X \rightarrow Y$, if there exists $p \in P$ such that $p \subseteq u$ then $u \in P$.

Definiton 2.2. A prorelation $P : X \rightarrow Y$ can be composed to a prorelation $Q : Y \rightarrow Z$ by taking composition of the relations belonging to them. Then, the set $Q.P$ is defined as $Q.P = \{q \circ p : p \in P \text{ and } q \in Q\}$.

Lemma 2.2.1. Composition of two prorelations is a prorelation.

Proof. For prorelations $P : X \rightarrow Y$ and $Q : Y \rightarrow Z$, need to show that $Q.P$ is a prorelation.

- (i) (Partial Order) Inclusion of relations gives a partial order.
- (ii) (Down-Directed) If $k, k' \in Q.P$, then $k = q \circ p$ and $k' = q' \circ p'$ for some $q, q' \in Q$ and $p, p' \in P$. Because Q and P are prorelations, and hence down-directed sets there exists, $a \in Q$ such that $a \subseteq q, q'$ and $b \in P$ such that $b \subseteq p, p'$. Thus, giving an element, $a \circ b$ of $Q.P$ such that $a \circ b \subseteq k, k'$.
- (iii) (Up-Set) Let $l : X \rightarrow Z$ be a relation, and $k \in Q.P$ such that $l \supseteq k$. Define relations $p : X \rightarrow Y$ and $q : Y \rightarrow Z$ as, $p = \{(x, y) : x \in \text{domain}(l) \text{ and } y \in Y\}$ and $q = \{(y, z) : y \in Y \text{ and } z \in \text{range}(l)\}$. Because $k \in Q.P$, there exist $q' \in Q$ and $p' \in P$ such that $k = q' \circ p'$. Thus by definition of p and q , we get that $p \supseteq p'$ and $q \supseteq q'$. Hence $p \in P$ and $q \in Q$ because P and Q are up-sets, which gives us that $q \circ p \in Q.P$. For any $(x, z) \in l$, by definition of p and q , we get that for every $y \in Y$, $(x, y) \in p$ and $(y, z) \in q$. By definition of composition, this gives that $(x, z) \in q \circ p$, giving that $l \subseteq q \circ p$. And, by definition of $q \circ p$ we get that $l \supseteq q \circ p$. Finally giving that $l = q \circ p \in Q.P$.

□

Definiton 2.3. For prorelations $P, Q : X \rightarrow Y$, if every relation in Q is contained inside some relation in P , then P is said to be contained in Q . That is $P \leq Q$ only if $\forall q \in Q, \exists p \in P$ such that $p \subseteq q$.

Definiton 2.4. For a relation $r : X \rightarrow Y$, it's opposite relation $r^o : Y \rightarrow X$ is defined as

$$\forall (x, y) \in X \times Y, (y, x) \in r^o \text{ only if } (x, y) \in r.$$

Lemma 2.4.1. For any function $f : X \rightarrow Y$, Δ_X is contained in the composition $f^o \circ f$.

Proof. As f is a function, it must be defined on every element of it's domain. Thus, for every $x \in X$, there exists some (x, y) in f . By definition of f^o , (y, x) is in f^o . Hence, by definition of composition, (x, x) is in $f^o \circ f$. □

Lemma 2.4.2. For any relation $r : X \rightarrow Y$, the composition $r \circ r^o$ contains $\subseteq \Delta_Y$.

Proof. Suppose there exist $x \in X$ and $y \in Y$ such that $x r y$. By definition of r^o , this gives us that $y r^o x$. Using definition of composition, $y r^o x r y$ gives that $y (r \circ r^o) y$. □

Lemma 2.4.3. For relations $r, s : X \rightarrow Y$ and $t : Y \rightarrow Z$, if $r \subseteq s$ then $(t \circ r) \subseteq (t \circ s)$.

Proof. Suppose relations r, s and t are as given above, and let $x(tr)z$. By definition of composition, there exists, $y \in Y$ such that xry and ytz . Using the hypothesis, as $r \subseteq s$, xry gives xsy . And via composition of xsy with ytz , we get $x(tz)z$. We started with any element of $(t \circ r)$ and showed that it must also be in $t \circ s$ and thus have that $(t \circ r) \subseteq (t \circ s)$. \square

Lemma 2.4.4. For relations $r : X \rightarrow Y$ and $s, t : Y \rightarrow Z$, if $s \subseteq t$ then $(s \circ r) \subseteq (t \circ r)$.

Proof. Suppose relations r, s and t are as given above, and let $x(sr)z$. By definition of composition of relations, we get that there exists some $y \in Y$ such that xry and ysz . Because $s \subseteq t$, ysz implies that ytz . Taking the composition, $xrysz$ yields $x(tr)z$. We started with any element of $(s \circ r)$ and showed that it must also be in $t \circ r$ and thus have that $(s \circ r) \subseteq (t \circ r)$. \square

3 Propositions

Definiton 3.1. A function, $f : (X, A) \rightarrow (Y, B)$ is said to be uniformly continuous if $f.A \leq B.f$. That is, for each

$$b \in B, \text{ there exists } a \in A \text{ such that } f \circ a \subseteq b \circ f. \text{ Meaning that } \begin{array}{ccc} X & \xrightarrow{f} & Y \\ A \downarrow & \leq & \downarrow B \\ X & \xrightarrow{f} & Y \end{array}$$

Definiton 3.2. A prorelation on a set, $P : X \rightarrow X$ is a quasi-uniformity if it satisfies the following conditions:

- (i) Every relation in P is reflexive. That is, for each $p \in P$, if $x \in X$ then $(x, x) \in p$.
- (ii) For each p in P , there exists p' in P such that $p' \circ p' \subseteq p$.

Definiton 3.3. If X is a set, and A is a quasi-uniformity on X , then (X, A) is a quasi-uniform space.

Lemma 3.3.1. If A is a quasi-uniformity on a set X , then $A.A = A$

Proof. Fix any $a \in A$, as A is a quasi-uniformity, $\exists b \in A : bb \subseteq a$, we get that $A.A \leq A$. And as A is a prorelation, and is hence down-directed, $\exists c \in A : a.a \supseteq c$, giving that $A.A \geq A$. \square

Proposition 3.1. QUnif is a category defined as having quasi-uniform spaces as objects, and uniformly continuous maps between them as morphisms, with composition defined as that of functions.

Proof. (i) (Associativity) The composition of functions is associative by definition.

- (ii) (Identity) For each object (X, A) , define it's identity to be the identity function $\Delta_X : (X, A) \rightarrow (X, A)$. This function is uniformly continuous as $\Delta_X.A = A \leq A = A.\Delta_X$. \square

Definiton 3.4. A prorelation, $\phi : X \multimap Y$ is called a promodule $\phi : (X, A) \multimap (Y, B)$ if it satisfies:

$$\phi.A \leq \phi \text{ and } B.\phi \leq \phi$$

Definiton 3.5. Containment of promodules is defined as that of prorelations.

Definiton 3.6. Promodules are composed as prorelations.

Proposition 3.2. ProMod is a 2-category defined as having quasi-uniform spaces as its 0-cells, promodules as 1-cells and containment of promodules as 2-cells.

Proof. In order to show that ProMod is a 2-category, need the following:

- (a) (1-Identities) For each quasi-uniform space (X, A) , define $A : (X, A) \multimap (X, A)$ to be the identity 1-cell for (X, A) . A is a promodule because $A.A = A$ (Lemma 2.2.1)

- (b) (1-Composition) Need promodules to be closed under composition.

Let $\phi : (X, A) \multimap (Y, B)$ and $\psi : (Y, B) \multimap (Z, C)$ be promodules. To show that $\psi.\phi : (X, A) \multimap (Z, C)$ is a promodule, need it to be a prorelation that satisfies the two conditions required to be a promodule:

- (i) By Lemma 1.2.1, prorelations are closed under composition. Hence, $\psi.\phi$ is a prorelation
- (ii) Need to show that $\psi.\phi.A \leq \psi.\phi$. So, Fix $p \in \psi$ and $q \in \phi$. As ϕ is a promodule, $\phi.A \leq \phi$ gives that there exists $q' \in \phi$ and $a \in A$ such that $q'a \subseteq q$. Thus, $pq'a \subseteq pq$.

- (iii) Need to show that $C.\psi.\phi \leq \psi.\phi$. Fix $p \in \psi$ and $q \in \phi$. Because ψ is a promodule, $C.\psi \leq \psi$ gives that there exists $c \in C$ and $p' \in \psi$ such that $c p' \subseteq p$. Thus, $c p' q \subseteq p q$
- (c) (2-Identities) As every promodule is contained in itself, always have $\psi \leq \psi$. Define this comparison to be the identity 2-cell for ψ and denote it by \leq_ψ
- (d) (Vertical 2-composition) For promodules $\psi, \phi, \delta : (X, A) \multimap (Y, B)$, if there is a 2-cell from ψ to ϕ and another one from ϕ to δ i.e. $\psi \leq \phi \leq \delta$, then by transitivity of the partial order, $\psi \leq \delta$ i.e. there's a 2-cell from ψ to δ .
- (e) (Horizontal 2-composition) If there are promodules $\psi, \psi' : (X, A) \multimap (Y, B)$ and $\phi, \phi' : (Y, B) \multimap (Z, C)$ such that $\psi \leq \psi'$ and $\phi \leq \phi'$, need to show that $\psi.\phi \leq \psi'.\phi'$. Fix $p' \in \psi'$ and $q' \in \phi'$. As $\psi \leq \psi'$, $\exists p \in \psi : p \subseteq p'$ and as $\psi \leq \psi'$, $\exists q \in \phi : q \subseteq q'$. Thus, $p q \subseteq p' q'$
- (f) (1-Identity) Need to show that for any promodule $\phi : (X, A) \multimap (Y, B)$, $\phi.A = \phi = B.\phi$. By quasi-uniformity of A , every $a \in A$, is reflexive. Thus, for any $p \in \phi$ and $a \in A$, $p = p.\Delta_X \subseteq p a$ giving that $\phi \leq \phi.A$. And as ϕ is a promodule, $\phi \geq \phi.A$. Hence, by anti-symmetry of the partial order, $\phi = \phi.A$.
Similarly, By quasi-uniformity of B , every $b \in B$, is reflexive. Thus, for any $p \in \phi$ and $b \in B$, $p = \Delta_Y.p \subseteq b p$ giving that $\phi \leq B.\phi$. And as ϕ is a promodule, $\phi \geq B.\phi$. Hence, $\phi = B.\phi$.
- (g) (1-Associativity) As composition of relations is associative, so too is the composition of prorelations directly giving that composition of promodules i.e. 1-cells is associative.
- (h) (Vertical 2-Identity) Let $\leq : \psi \rightarrow \phi$ be a 2-cell i.e. $\psi \leq \phi$. By our definition of identity 2-cell, $\leq_\psi . \leq_1$ means precisely that $\psi \leq \psi \leq \phi$, and by transitivity, this is equivalent to $\psi \leq \phi$. Similarly, $\leq_1 . \leq_\phi$ means exactly that $\psi \leq \phi \leq \phi$, and this is equivalent to $\psi \leq \phi$.
- (i) (Vertical 2-Associativity) Associativity of the partial order on promodules directly gives the associativity of composition of 2-cells in ProMod.
- (j) (Horizontal 2-Identity) Let $\psi, \phi : (X, A) \multimap (Y, B)$ be promodules. For any 2-cell $\leq : \psi \rightarrow \phi$, need to show that the 2-cell given by the horizontal composition, $\leq * \leq_A$ is equal to \leq , as well as equal to $\leq_B * \leq$. So, it's required that $\psi.A \leq \phi.A \iff \psi \leq \phi \iff B.\psi \leq B.\phi$. And this holds as a direct consequence of (f).
- (k) (Horizontal 2-Associativity) As there's a unique 2-cell between any two promodules, and composition of promodules is associative, horizontal composition of 2-cells is associative.
- (l) (2-Identity) For promodules $\psi : (X, A) \multimap (Y, B)$ and $\phi : (Y, B) \multimap (Z, C)$ need $(\leq_\psi * \leq_\phi) = \leq_{\psi.\phi}$. Both sides of the required equality are 2-cells $\leq : \psi.\phi \rightarrow \psi.\phi$. Thus, they are equal by the uniqueness of 2-cells between any two 1-cells.
- (m) (2-Interchange) Let $\psi, \phi, \delta : (X, A) \multimap (Y, B)$ and $\psi', \phi', \delta' : (Y, B) \multimap (Z, C)$ be promodules. For 2-cells $\leq_1 : \psi \rightarrow \phi, \leq_2 : \phi \rightarrow \delta, \leq_a : \psi' \rightarrow \phi'$ and $\leq_b : \phi' \rightarrow \delta'$, need to show $(\leq_b . \leq_a) * (\leq_2 . \leq_1) = (\leq_b * \leq_2) . (\leq_a * \leq_1)$. Both RHS and LHS are 2-cells from $\psi.\psi'$ to $\delta.\delta'$ and are hence equal.

□

Proposition 3.3. Functor, $(-)_* : \text{QUnif} \rightarrow \text{ProMod}$ is defined as fixing quasi-uniform spaces (objects) and taking uniformly continuous functions (morphisms) to the it's pre-composition with it's domain space's identity (promodule) i.e.

- (a) for $(X, A) \in \text{QUnif}$, $(X, A)_* := (X, A) \in \text{ProMod}$
- (b) for $f : (X, A) \rightarrow (Y, B)$ in QUnif , $f_* := B.f$

Proof. We will first show that $B.f = b \circ f : b \in B$ is a promodule, and then that $(-)_*$ defines a functor.

- (i) (Partial-Order) Inclusion of relations acts as the partial order.
- (ii) (Down-Directed) Fix any k, k' belonging to $B.f$. Thus, there exist b, b' in B such that $k = b f$ and $k' = b' f$. Using down-directedness of B , there exists a $c \in B$ such that $c \subseteq b, b'$. Hence, by Lemma 2.4.4, $c f \subseteq k, k'$.
- (iii) (Up-set) Let k belong to $B.f$ and $l : (X, A) \rightarrow (Y, B)$ be a uniformly continuous function such that $l \supseteq k$. Define a relation $b' := \{(f(d), l(d)) : d \in \text{domain}(l)\}$. By definition, for any $x \in X$ and $z \in Y$ such that $(x, z) \in l$, we get that $(f(x), z) \in b'$. And $l \supseteq k = b f$ implies $\text{domain}(l) \supseteq \text{domain}(f)$ giving $(x, f(x)) \in f$. Thus, by definition of composition, $(x, z) \in b'.f$. Conversely, suppose $(x, z) \in b'.f$. By definition of composition, there exists $f(x) \in Y$ such that $(f(x), z) \in b'$. Again using the definition of b' , we get that $z = l(x)$ i.e. $(x, z) \in l$. Hence, $l = b'.f$. Now we will show that $b' \supseteq b$. Because $b' f = l \supseteq k = b f$, for any $x \in X$ we have that $b'(f(x)) \supseteq b(f(x))$. Thus, $b'|_{f(x)} \supseteq b|_{f(x)}$. By down-directedness of B , the restriction $b|_{f(x)} \subseteq b$ implies $b(x)|_{f(x)} \in B$. Finally, $b' \supseteq b'|_{f(x)} \supseteq b|_{f(x)}$ gives $b' \in B$. Hence, $b'.f \in B.f$.

- (iv) Need to show that $(B.f).A \leq B.f$. So, fix any $b \in B$, we will find $b' \in B$ and $a \in A$ such that $b' f a \subseteq b f$. By quasi-uniformity of B , there exists $b' \in B$ such that $b' b' \subseteq b$. Using Lemma 2.4.3, we get that $b' b' f \subseteq b f$. As f is uniformly continuous, $f.A \leq B.f$ gives that there is some $a \in A$ such that $f a \subseteq b' f$. Using this in the previous inequality, we get $b' f a \subseteq b' b' f \subseteq b f$.
- (v) Need to show that $B.B.f \leq B.f$. Fix any $b \in B$, we will find $b' \in B$ such that $b' b' f \subseteq b f$. By quasi-uniformity of B , there exists $b \in B$ such that $b' b' \subseteq b$. Using Lemma 2.4.4, we get $b' b' f \subseteq b f$.

Thus, $B.f$ is a promodule. Going to show that $(-)_*$ defines a functor:

- (i) (Composition) Need to show that $(g \circ f)_* = g_* f_*$ i.e. $C.g.f = C.g.B.f$.

In order to show $C.g.f \leq C.g.B.f$, fix any $b \in B, c \in C$. We will show that $c g f \subseteq c g b f$. As f is uniformly continuous, $f.A \leq B.f$ gives that there exists $a \in A$ such that $f a \subseteq b f$. Using Lemma 2.4.3, we get $(c g) f a \subseteq (c g) b f$. Now, using reflexivity of a , we get $c g f \subseteq c g b f$.

Now, to show that $C.g.f \geq C.g.B.f$. Fix any $c \in X$, we will find $c' \in C$ and $b \in B$ such that $c g f \supseteq c g b f$. By quasi-uniformity of C , there exists $c' \in C$ such that $c \subseteq c' c'$. Using Lemma 2.4.4 gives that $c(g f) \supseteq c' c'(g f)$. Because g is uniformly continuous, $C.g \geq g.B$ gives us $b \in B$ such that $g c' \supseteq b g$. Using this in the previous inequality gives that $c g f \supseteq c' g b f$.

- (ii) (Identity) let (X, A) be in object of QUnif and $1_{(X,A)} : (X, A) \rightarrow (X, A)$ be the identity function on (X, A) . That is, $1_{(X,A)}$ is defined as $x \mapsto x$. Need to show that $(1_{(X,A)})_* = 1_{(X,A)_*}$. Using functor's definition, $LHS = (1_{(X,A)})_* = A.(1_{(X,A)}) = A.1_{(X,A)} = A$ and $RHS = 1_{(X,A)_*} = 1_{(X,A)}$ Using Proposition 3.2(f), we get that $A = 1_{(X,A)} = RHS$.

□

Proposition 3.4. Functor, $(-)^* : \text{QUnif}^{\text{op}} \rightarrow \text{ProMod}$ is defined as fixing quasi-uniform spaces and taking uniformly continuous functions to the composition of their opposite relation with it's domain space's identity i.e.

- (a) for $(X, A) \in \text{QUnif}^{\text{op}}$, $(X, A)^* := (X, A) \in \text{ProMod}$
- (b) for $f : (X, A) \rightarrow (Y, B)$ in QUnif, $f^* := f^o.B$

Proof. Showing that $f^o.B : (Y, B) \rightrightarrows (X, A)$ is a promodule.

So, need to show $f^o.B$ a prorelation $Y \rightarrow X$ and that $(f^o.B).B \subseteq f^o.B$ and $A.(f^o.B) \subseteq f^o.B$

To show prorelation,

- (i) (Partial-order) Inclusion of relations i.e. for $k = f^o \circ b$ and $k' = f^o \circ b'$ in $f^o.B$, $k \subseteq k' \iff b \subseteq b'$
- (ii) (Down directed) for $k, k' \in f^o.B$, need that $\exists l \in f^o.B$ such that $l \subseteq k, k'$
Fix $k, k' \in f^o.B \implies \exists b, b' \in B : k = f^o \circ b$ and $k' = f^o \circ b'$
By down-directedness of B , there exists $c \in B$ such that $c \subseteq b, b'$, define $l = f^o \circ c$. Now, using Lemma 2.4.3 gives $l = f^o \circ c \subseteq k, k'$.
- (iii) (Up-set) for a relation $l : Y \rightarrow X$ and $k \in f^o.B$ such that $l \supseteq k$, need $l \in f^o.B$
Let $b \in B$ be such that $k = f^o \circ b$ and define $b' := \{(y, y') : y \in \text{domain}(l) \text{ and } y' \in (f^o)^{-1}(l(y))\}$
As $l \supseteq k = f^o \circ b$, $\text{domain}(b') = \text{domain}(l) \supseteq \text{domain}(b)$
and $\text{range}(l) \supseteq \text{range}(f^o \circ b) \implies \forall y \in \text{domain}(b), \text{range}(b') = (f^o)^{-1}(l(y)) \supseteq (f^o)^{-1}(f^o \circ b) = \text{range}(b)$
Now, by definition of b' , $f^o \circ b' \supseteq l$. To show $f^o \circ b' \subseteq l$,
 $(x, y) \in f^o \circ b' \implies \exists z \in Y : (x, z) \in b' \text{ and } (z, y) \in f^o \implies x \in \text{domain}(l) \text{ and } z \in l(x) \text{ i.e. } (x, z) \in l$

To show $(f^o.B).B \leq f^o.B$, need that $\forall b \in B, \exists b' \in B : f^o \circ b' \circ b' \subseteq f^o \circ b$,

Fix any $b \in B$, as B is a quasi-uniformity, $\exists b' \in B : b' \circ b' \subseteq b \implies f^o \circ b' \circ b' \subseteq f^o \circ b$

To show $A.(f^o.B) \leq f^o.B$, need that $\forall b \in B, \exists b' \in B, a \in A : a \circ f^o \circ b' \subseteq f^o \circ b$,

As f is uniformly continuous, $f.A \leq B.f$ i.e. $\forall b \in B, \exists a \in A : f \circ a \subseteq b \circ f \implies a = f^o \circ f \circ a \subseteq f^o \circ b \circ f$

Fix any $b \in B$, so, $\exists b' \in B : b' b' \subseteq b$ And, for this $b', \exists a : a \subseteq f^o b' f \implies a f^o b' \subseteq f^o b' f f^o b' \subseteq f^o b' b' \subseteq f^o b \implies a f^o b' \subseteq f^o b$

Now, need to show that $(-)^*$ respects composition and identity.

- (i) (Composition) let f, g be uniformly continuous, $(X, A) \xrightarrow{f} (Y, B) \xrightarrow{g} (Z, C)$ need that $(g \circ f)^* = f^*.g^*$
LHS = $(g \circ f)^* = (g \circ f^o).C = (f^o \circ g^o).C$ and RHS = $f^*.g^* = (f^o.B).(g^o.C)$
For equality, showing that LHS \geq RHS and LHS \leq RHS:

To show $(f^\circ \circ g^\circ).C \geq (f^\circ.B).(g^\circ.C)$, need that $\forall c \in C, \exists b \in B, c' \in C : f^\circ g^\circ c \supseteq f^\circ b g c'$
 Fix any $c \in C$, so, $\exists c' \in C : c' \circ c' \subseteq c \implies f^\circ g^\circ c \supseteq f^\circ g^\circ (c' c') = f^\circ g^\circ (c' \Delta_Z c') \supseteq f^\circ g^\circ c' (g g^\circ) c'$
 By uniform continuity of g , for $c' \in C, \exists b \in B : g b \subseteq c' g$
 Thus, $f^\circ g^\circ c \supseteq f^\circ g^\circ (c' g) g^\circ c' \supseteq f^\circ (g^\circ g) b g^\circ c' = f^\circ b g^\circ c'$.

To show $(f^\circ \circ g^\circ).C \leq (f^\circ.B).(g^\circ.C)$, need that $\forall b \in B, c \in C, \exists c' \in C : f^\circ g^\circ c \subseteq f^\circ b g^\circ c'$

Fix any $c \in C, b \in B$ will show that $c' := c$ works:

As B is a quasi-uniformity, $\Delta_Y \subseteq b \implies f^\circ \Delta_Y g^\circ c = f^\circ g^\circ c \subseteq f^\circ b g^\circ c = f^\circ b g^\circ c'$

- (ii) (Identity) let $(X, A) \in \text{QUnif}^{\text{op}}$, and $1_{(X,A)} : (X, A) \rightarrow (X, A)$ as $x \mapsto x$ need that $(1_{(X,A)})^* = 1_{(X,A)^*}$
 LHS = $(1_{(X,A)})^* = (1_{(X,A)})^\circ.A = 1_{(X,A)}.A = A$.
 And as $RHS = 1_{(X,A)^*} = 1_{(X,A)}$ Using Proposition 3.2(f), we get that $A = 1_{(X,A)} = RHS$.

□

Definiton 3.7. Let $f : (X, A) \rightarrow (Y, B)$ be a uniformly continuous function.

I f is said to be fully faithful if $f^*.f_* = A$

II f is said to be fully dense if $f_*.f^* = B$

Proposition 3.5. Fix a uniformly continuous map, $f : (X, A) \rightarrow (Y, B)$

- (a) f is fully faithful if and only if $A \geq f^\circ.B.f$
- (b) f is fully dense if and only if for any $b \in B, \exists b' \in B$ such that $b' \subseteq b f f^\circ b$
- (c) f is topologically dense if and only if for any $b \in B, b f f^\circ b$ is reflexive
- (d) f is fully dense if and only if f is topologically dense

Proof.

- (a) (i) (\implies) Let f be fully faithful i.e. $f^*.f_* = A \implies f^\circ.B.B.f = A$
 Need to show that $A = f^\circ.B.f$ i.e. $A \leq f^\circ.B.f$ and $A \geq f^\circ.B.f$
 By hypothesis and quasi-uniformity of B , $A \geq f^\circ.B.B.f \geq f^\circ.B.f$
 To show $A \leq f^\circ.B.f$, need that $\forall b \in B, \exists a \in A : a \subseteq f^\circ b f$
 Fix $b \in B$, hypothesis gives that $f^\circ.B.B.f \leq A$ so,
 $\exists a \in A : a \subseteq f^\circ b b f$ and also, by quasi-uniformity of B , for $b, \exists b' \in B : b' b' \subseteq b \implies f^\circ b' b' f \subseteq f^\circ b f$
 Combining the above two inequalities, $a \subseteq f^\circ b b f \subseteq f^\circ b f$
- (ii) (\impliedby) Let $A = f^\circ.B.f$ need to show $A = f^\circ.B.B.f$ i.e. $A \geq f^\circ.B.B.f$ and $A \leq f^\circ.B.B.f$
 To show $A \geq f^\circ.B.B.f$, need to show that $\forall a \in A, \exists b, b' \in B : a \supseteq f^\circ b b' f$
 Have that $A \geq f^\circ.B.f$ and $B.B \leq B$
 So, fix $a \in A$, now $\exists b \in B : a \subseteq f^\circ b f$ and for this $b, \exists b' \in B : b' b' \subseteq b$. Therefore, $a \supseteq f^\circ b f \supseteq f^\circ b' b' f$
 To show $A \leq f^\circ.B.B.f$, need $\forall b, b' \in B, \exists a \in A : a \subseteq f^\circ b b' f$
 Before that, uniform continuity of f along with Lemma 2.1.1 gives that
 $f.A \leq B.f \implies A = f^\circ f.A \leq f^\circ.B.f$
 So, fix $b, b' \in B$, now, as ,
 $A \leq f^\circ.B.f$ giving
 $\exists a \in A : a \subseteq f^\circ b f$ and $\exists a' \in A : a' \subseteq f^\circ b' f \implies \Delta_X \subseteq f^\circ b' f$.
 Therefore $a = a \Delta_X \subseteq (f^\circ b f)(f^\circ b' f) \subseteq f^\circ b b' f$
- (b) (i) (\implies) Let f be fully dense i.e. $B = f_*.f^* = B.f.f^\circ.B$. showing that $\forall b \in B, \exists b' \in B : b' \subseteq b f f^\circ b$:
 So, fix $b \in B$, as $B \leq B.f.f^\circ.B$, there exists $b' \in B$ such that $b' \subseteq b f f^\circ b$.
- (ii) (\impliedby) Suppose $\forall b \in B, \exists b' \in B : b' \subseteq b f f^\circ b$. This gives $B \leq B.f.f^\circ.B$, in order to show equality, also need $B \geq B.f.f^\circ.B$. By quasi-uniformity of B , for any $b \in B, \exists b' \in B : b' b' \subseteq b$. Now, by Lemma 2.4.2,

$$f f^\circ \subseteq \Delta_Y \implies b' f f^\circ b' \subseteq b' \Delta_Y b' = b' b' \subseteq b$$

- (c) (i) (\implies) Let f be topologically dense. We will show that for any $b \in B, y \in Y, (y, y) \in b f f^\circ b$. Fix any $b \in B$ and $y \in Y$. As f is topologically dense, $\overline{f(X)} = Y$, implying that $y \in \overline{f(X)}$, by definition giving that

$$\exists x \in X \text{ such that } (f(x), y) \in b \text{ and } (y, f(x)) \in b$$

Re-writing the above statement in terms of relations, and considering f as a relation:

$$(f(x), y) \in b \text{ gives } x(b \circ f)y \text{ i.e. } y \in (b \circ f)(x) \quad (7)$$

$$(y, f(x)) \in b \text{ gives } f(x) \subseteq b(y) \quad (8)$$

Repeatedly applying Lemma 2.4.3 to (2),

$$f(x) \subseteq b(y) \implies (f \circ f^o)(f(x)) \subseteq (f \circ f^o)b(y) \implies (f \circ f^o \circ f)(x) \subseteq (f \circ f^o \circ b)(y)$$

Applying Lemma 2.4.1 to the final inequality in the above statement gives that

$$f(x) = (f \circ \Delta_X)(x) \subseteq (f \circ f^o \circ f)(x) \subseteq (f \circ f^o \circ b)(y)$$

Applying Lemma 2.4.3 and then using (1) on the above inequality completes the result:

$$f(x) \subseteq (ff^ob)(y) \implies (b \circ f)(x) \subseteq (bff^ob)(y) \implies y \in (bff^ob)(y) \text{ i.e. } y(bff^ob)y$$

- (ii) (\Leftarrow) Fix any $y \in Y$ and $b \in B$. Also, suppose that $\Delta_Y \leq bff^ob$. As f is a function with domain as X , $f^o : Y \rightarrow X$, $\phi \neq (f^o \circ b)(y) \subseteq X$. So, fix $x \in (f^o \circ b)(y)$, going to show that $(f(x), y) \in b$ and $(y, f(x)) \in b$. Again, while viewing f as a relation.

$$\Delta_Y \leq bff^ob \implies \Delta_Y(y) \subseteq bff^ob(y) = (bf)(f^ob(y))$$

Last inequality of the above statement gives $y \in (bf)(x)$ i.e. $(f(x), y) \in b$.

Applying Lemma 2.4.2 to f , and then using Lemma 2.4.4,

$$ff^o \subseteq \Delta_Y \implies ff^ob \subseteq \Delta_Y b = b$$

Thus $ff^ob(y) \subseteq b(y)$ and hence $f(x) \subseteq b(y) \implies (y, f(x)) \in b$

- (d) (i) (\implies) Let f be topologically dense. As B is a quasi-uniformity, for any $b \in B$,

$$\exists b' \in B : b'b' \subseteq b \text{ and } \Delta_Y \subseteq b' \implies b' = b'\Delta_Y \subseteq b'b' \subseteq b \quad (9)$$

By the characterisation of topologically dense in (c), have that $\Delta_Y \subseteq b'ff^ob'$. Now, using the (3) and Lemma 2.4.3,

$$\Delta_Y \subseteq b'ff^ob' \implies b' = b'\Delta_Y \subseteq b'b'ff^ob' \subseteq bff^ob' \subseteq bff^ob$$

Hence, we have $b' \in B : b' \subseteq bff^ob$ giving us that f is fully dense (from (b)).

- (ii) (\Leftarrow) From (b), we have for $b \in B$, the existstence of $b' \in B$ such that $b' \subseteq bff^ob$. As B is a quasi-uniformity, $\Delta_Y \subseteq b'$. So, $\Delta_Y \subseteq bff^ob$, and from (c), this gives us that f is topologically dense. \square

Definiton 3.8. The set PX is defined to be the collection of all promodules from the quasi-uniform space (X, A) to the quasi-uniform space 1.

$$PX := \{\psi : \psi : (X, A) \multimap 1 \text{ is a promodule}\}$$

Proposition 3.6. For any $a \in A$, \tilde{a} is defined to be a relation $PX \rightarrow PX$ as

$$\text{for } \phi, \psi \in PX, \phi \tilde{a} \psi \text{ only if } \phi \leq \psi.a$$

The set, $\tilde{A} := \{\tilde{a} : a \in A\}$ defines a quasi-uniformity on PX .

Proof. First need to show that \tilde{A} is a prorelation,

- (i) (Partial order) For any two relations $\tilde{a}, \tilde{b} : PX \rightarrow PX$, define $\tilde{a} \leq \tilde{b}$ to be true only if $a \subseteq b$.
- (ii) (Down-Directed) Need for any $\tilde{a}, \tilde{b} \in \tilde{A}$, the existstence of some $\tilde{c} \in \tilde{A}$ such that $c \subseteq a, b$. If $\tilde{a}, \tilde{b} \in \tilde{A}$ then there exist $a, b \in A$. By down-directedness of A , there exists a $c \in A$ such that $c \subseteq a, b$. Now the definition of \tilde{A} gives that $\tilde{c} \in \tilde{A}$. And the definition of the partial order on \tilde{A} ensures $\tilde{c} \leq \tilde{a}, \tilde{b}$.
- (iii) (Upset) For any relation $l : PX \rightarrow PX$, need that if \tilde{k} belongs to \tilde{A} such that $l \geq \tilde{k}$, then $l \in \tilde{A}$. Fix any $k : PX \rightarrow PX$, and $\tilde{k} \in \tilde{A}$ such that $l \geq \tilde{k}$. As k is a relation between promodules $X \multimap 1$, it can be thought of as a relation a on X , defined as:

$$a := \{(x, y) : x \in \text{domain}(\psi) \text{ and } y \in \text{domain}(\phi) \text{ whenever } \exists \psi, \phi \in PX : \psi l \phi\}$$

So, $l \geq \tilde{k}$ gives that $\tilde{a} \geq \tilde{k}$ i.e. $a \supseteq k$. And as A is an upper-set, we get $a \in A$. Now, by definition of \tilde{A} , $l \in \tilde{A}$.

Secondly, need show that the other two conditions hold for \tilde{A} ,

- (i) For all $\tilde{a} \in \tilde{A}$, need \tilde{a} to be reflexive i.e if $\psi \in PX$ then $\psi \tilde{a} \psi$. By definition of \tilde{a} , need to show that $\psi \leq \psi.a$. So, fix a $p \in \psi$, we will show that $p \subseteq p.a$. Quasi-uniformity of A gives that $\Delta_X \subseteq a$. Hence, by Lemma 2.4.3, $p = p\Delta_X \subseteq p.a$.

- (ii) For all $\tilde{a} \in \tilde{A}$, need to find $\tilde{b} \in \tilde{A}$ such that $\tilde{b}\tilde{b} \leq \tilde{a}$

Before showing the result, proving that for any $x, y \in A$, $\tilde{x}\tilde{y} \leq \widetilde{xy}$ i.e. $\forall \psi, \phi \in PX$, $\psi(\tilde{x}\tilde{y})\phi \implies \psi\widetilde{xy}\phi$. If $\psi_1(\tilde{a}\tilde{b})\psi_3$, then, the definition of composition gives that $\exists \psi_2$ such that $\psi_1\tilde{b}\psi_2\tilde{a}\psi_3$. Now, the definition of \tilde{b} gives $\psi_1 \leq \psi_2\tilde{b}$ and that of \tilde{a} gives $\psi_2 \leq \psi_3\tilde{a}$. Combining these inequalities, $\psi_1 \leq \psi_2\tilde{b} \leq \psi_3\tilde{a}\tilde{b}$. Hence, by definition of \tilde{ab} , $\psi_1(\tilde{ab})\psi_3$. Now, to show the result, fix any $\tilde{a} \in \tilde{A}$. Therefore, $a \in A$, and by quasi-uniformity of A , $\exists b \in A : b \circ b \subseteq a$. Thus, by the partial-order defined on \tilde{A} , $\tilde{b}\tilde{b} \leq \tilde{a}$. Now, transitivity of the partial order gives us the required result, $\tilde{b}\tilde{b} \leq \tilde{a}$.

□

Proposition 3.7 (Yoneda Embedding).

For a quasi-uniform space (X, A) , function $y_X : X \rightarrow PX$ is defined by $x \mapsto x^*$ for $x \in X$.

- (a) $y_X : (X, A) \rightarrow (PX, \tilde{A})$ is a uniformly continuous map
(b) $y_X : (X, A) \rightarrow (PX, \tilde{A})$ is fully faithful

Proof.

- (a) In order to show y_X is uniformly continuous, need to show that $y_X.A \leq \tilde{A}.y_X$. By definition of \leq , need $\forall a \in A, \exists \tilde{b} \in \tilde{A} : y_X \circ b \subseteq \tilde{a} \circ y_X$. Applying the relations to some element, x of the set X :

$$(y_X \circ b)(x) \subseteq (\tilde{a} \circ y_X)(x) \implies y_X(b(x)) \subseteq \tilde{a}(x^*) \quad (10)$$

So, for the condition given by (4) to hold, if $y \in b(x)$, then it's required that $y^* = y_X(y) \in \tilde{a}(x^*)$ i.e. $x^*\tilde{a}y^*$. Using the definition of x^*, y^* and \tilde{a} ,

$$x^*\tilde{a}y^* \iff x^o.A \leq y^o.A.a \iff \forall a' \in A, \exists a'' \in A : x^o a'' \subseteq y^o a' a \quad (11)$$

Now, fix any $a \in A, x \in X$. Thus, quasi-uniformity of A , gives $a'' \in A$ such that $a''a'' \subseteq a$. Also, choose some $y \in a''(x)$. Hence, in order to show that the condition from (5) holds, need that $\forall b \in A, x^o a'' \subseteq y^o b a$, and by applying the relations to an element z gives the following condition

$$\forall b \in B, \forall x \in X, (x^o a'')(z) \subseteq (y^o b a)(z) \quad (12)$$

Examining the left side of (6),

$$(x^o a'')(z) = x^o(a''(z)) = \begin{cases} \phi & \text{if } x \notin a''(z) \\ \star & \text{if } x \in a''(z) \end{cases}$$

Thus, to show that (6) holds, need to show that (for any $b \in A$ and $z \in X$):

$$x \in a''(z) \implies z(y^o b a) \star \text{ i.e. } y \in (ba)(z) \quad (13)$$

To show that (7) holds, fix any $z \in X : x \in a''(z)$. Also, by our choice of y , have that $y \in a''(x)$. And as $b \in A$, it's reflexive, giving that $y \in b(y)$. So, by composition of relations, we get:

$$za''x, xa''y \text{ and } yby \implies z(a''a''b)y \implies z(ab)y \text{ i.e. } y \in (ba)(z)$$

- (b) By using Proposition 2.3(a), need to show that $A \geq y_X^o.\tilde{A}.y_X$ i.e. $\forall a \in A, \exists \tilde{b} \in \tilde{A} : a \geq y_X^o \tilde{b} y_X$. Applying to an element, $x \in X$ gives the condition

$$(y_X^o \tilde{b} y_X)(x) \subseteq a(x) \implies (y_X^o \tilde{b})(x^*) = y_X^o(\tilde{b}(x^*)) \subseteq a(x) \quad (14)$$

Thus, if $y^* \in PX$ such that $x^*\tilde{b}y^*$, then $y \in y_X^o(\tilde{b}(x^*))$. Now, for (8) to hold, $y \in a(x)$ i.e. xay . Thus, need only to show that for any $a \in A, \exists b \in A$ such that $\forall x, y \in X, x^*\tilde{b}y^* \implies xay$. So, fix $a \in A$, and take $b \in A : bb \subseteq a$. Now, let $x^*\tilde{b}y^*$ i.e. $x^o.A \leq y^o.A.b$. Hence, $\exists c \in A : x^o c \subseteq y^o bb$. And as c is reflexive,

$$xcx \implies x(cx^o)\star \implies x(bby^o)\star \implies x(bb)y \implies xay$$

□

Theorem 3.1 (Yoneda Lemma). For every $\psi \in PX$, in the following diagram,

$$\begin{array}{ccc} X & \xrightarrow{(y_X)^*} & PX \\ & \searrow \psi & \downarrow \psi^* \\ & & 1 \end{array}$$

$$(a) \psi \geq \psi^*.(y_X)^*$$

$$(b) \psi \in \overline{y_X(X)} \implies \psi \leq \psi^*.(y_X)_*$$

Proof. (a) By definition, $(y_X)_* = \tilde{A}.y_X$, and $\psi^* = \psi^o.\tilde{A}$. Need that $\psi \geq (y_X)_*.\psi^* = \psi^o.\tilde{A}.\tilde{A}.y_X$. And applying Lemma 2.2.1 to \tilde{A} , the required condition becomes $\psi \geq \psi^o.\tilde{A}.y_X$. Fix $p \in \psi$, we will find $a \in A : p \supseteq \psi^o a y_X$. Examining the right side of the condition, (for any $a \in A, x \in X$)

$$\left(\psi^o.\tilde{a}.y_X \right)(x) = \psi^o.\tilde{a}(x^*) = \psi^o(\tilde{a}(x^*)) = \begin{cases} \phi & \text{if } \psi \notin \tilde{a}(x^*) \\ \star & \text{if } \psi \in \tilde{a}(x^*) \end{cases} \quad (15)$$

In case $\psi \notin \tilde{a}(x^*)$, the condition holds trivially. As ψ is a promodule, $\psi.A \leq \psi$ gives $\exists q \in \psi, a \in A : qa \subseteq p$. Thus, fix $x \in X$ and $\psi \in PX$ such that $x^* \tilde{a} \psi$. We will now show that $x p \star$. Using the definition of \tilde{a} ,

$$x^* \tilde{a} \psi \implies x^o.A \leq \psi.a \implies \exists b \in A : x^o b \subseteq qa \implies \forall z \in X, (x^o b)(z) \subseteq (qa)(z) \quad (16)$$

Thus, in particular for $z = x$, as b is reflexive, xbx , which gives:

$$(x^o b)(x) \subseteq (qa)(x) \implies x^o x \subseteq (qa)(x) \implies \star \in (qa)(x) \quad (17)$$

But, as $qa \subseteq p$, (11) gives that $x p \star$.

(b) Suppose $\psi \in \overline{y_X(X)}$, need to show $\psi \leq \psi^*.(y_X)_* = \psi^o.\tilde{A}.y_X$ i.e. for $a \in A, \exists p \in \psi : p \subseteq \psi^o.\tilde{a}.y_X$. For any $x \in \text{domain}(p)$, the condition requires:

$$p(x) \subseteq \psi^o.\tilde{a}.y_X(x) = \psi^o(\tilde{a}(x^*)) \quad (18)$$

By definition of p , for (12) to hold, need that $x p \star \implies \psi \in \tilde{a}(x^*)$. Fix any $a \in A$, we will find $p \in \psi$ such that (12) holds. By quasi-uniformity of A , $\exists b \in A : bb \subseteq a$. From Proposition 2.5(a), y_X is uniformly continuous, $y_X.A \leq \tilde{A}.y_X$ giving that $\exists c \in A : y_x c \subseteq \tilde{b} y_X$. Thus, for any $z, w \in X$ such that $z c w$,

$$(y_X c)(z) \subseteq (\tilde{b} y_X)(z) \implies y_X(c(z)) \subseteq \tilde{b}(z^*) \implies w^* \in \tilde{b}(z^*) \text{ i.e. } z^* \tilde{b} w^* \quad (19)$$

As A is a quasi-uniformity, $\exists d \in A : dd \subseteq c$. Also, because A is a down-directed set, $\exists a' \in A : a' \subseteq b, d$. This along with (13) gives that for any $x, y \in X$

$$x(a'a')y \implies x(dd)y \implies xcy \implies x^* \tilde{b} y^* \quad (20)$$

Now, because $\psi \in \overline{y_X(X)}$, we get $\exists x^* \in y_X(X)$ such that $\psi \tilde{a}' x^*$ and $x^* \tilde{a}' \psi$. By definition of \tilde{a} , $\psi \tilde{a}' x^*$ gives

$$\psi \leq x^o.A.a' \implies \exists p \in \psi : p \subseteq x^o a' a' \quad (21)$$

Fix any $z \in X : z p \star$, using (15) and (14) gives:

$$z p \star \xrightarrow{\approx} (x^o a' a') \star \xrightarrow{(15)} z(a' a') x \xrightarrow{(14)} z^* \tilde{b} x^* \quad (22)$$

Finally, by definition of the partial order on $\tilde{A}, a' \subseteq b \implies \tilde{a}' \subseteq \tilde{b}$. Therefore, $x^* \tilde{a}' \psi \implies x^* \tilde{b} \psi$. Now, using (16), $z^* \tilde{b} x^*$ and $x^* \tilde{b} \psi$ gives the desired result $z^* \tilde{b} x^*$. \square

Corollary 3.1. For $\psi \in PX$, $\psi \in \overline{y_X(X)}$ if and only if ψ is a right-adjoint.

Proof. Fix any $\psi \in PX$.

(i) (\implies)

(ii) (\impliedby) Suppose ψ is a right adjoint. Need to show that for any $a \in A, \exists x^* \in y_X(X)$ such that $\psi \tilde{a} x^* \tilde{a} \psi$. Fix $a \in A$. Because ψ is a right-adjoint, there exists a promodule $\phi : 1 \dashv \multimap X$ such that $\phi.\psi \leq A$ and $1 \leq \psi.\phi$. From $\phi.\psi \leq A$, we get that

$$\exists p \in \phi, q \in \psi \text{ such that } a \supseteq p.q \quad (1)$$

Because ϕ and ψ are promodules,

$$A.\phi \leq \phi \text{ gives the existence of } p' \in \phi \text{ such that } p \supseteq a' p' \quad (2)$$

$$A.\psi \leq \psi \text{ gives the existence of } q' \in \psi \text{ and } a'' \in A \text{ such that } q \supseteq a'' q' \quad (3)$$

Now, from $1 \leq \psi.\phi$, we get that $q'p'$ is reflexive i.e. $\star(q'p')\star$. By the definition of composition we get the existence of an $x \in X$ such that $\star p'xq'\star$. Now, considering x as a map, $x : 1 \rightarrow X$ defined as $\star \mapsto x$,

$$xq'\star \text{ i.e. } \star \in q'(x) \text{ gives that } q' \supseteq x^o \quad (4)$$

$$\star p'x \text{ i.e. } x \in p'(\star) \text{ gives that } p' \supseteq x \quad (5)$$

Thus, by using inequalities (1),(2) and (3), we get that

$$a \supseteq pq \supseteq a'p'q'a'' \quad (6)$$

By definition of \tilde{a} , to show $\psi \tilde{a}x^*$, we need that $\psi \leq x^*a = x^o.A.a$. Showing for any $b \in A$, $x^ob a \supseteq q'$:

$$x^ob a \supseteq x^ob a'p'q' \supseteq x^ob a'xq' \supseteq x^oxq' \subseteq q'$$

Where the first inequality comes from (6) by using reflexivity of a'' and then left-multiplying by x^o . The second inequality comes from (5), third one from reflexivity of b and a' , and the last one is given by Lemma 2.4.1.

In order to show $x^*\tilde{a}\psi$, by definition of \tilde{a} , need that $x^o.A = x^* \leq \psi a$. Fix $k \in \psi$. We will show $ka \supseteq x^oa''$.

$$a \supseteq a'p'q'a'' \supseteq p'q'a'' \supseteq p'x^oa'' \quad (7)$$

Where the first inequality is given by (6), second one is due to reflexivity of a' and the third inequality comes by using (4). Left-multiplying (7) with k gives

$$ka \supseteq kp'x^oa'' \quad (8)$$

FINAL STEP LEFT !!

□