

# 1 Categories

**Definiton 1.1** (Category). A category,  $\mathcal{A}$  is defined to have each of the following,

- (i) A collection of objects, denoted by  $\text{ob}(\mathcal{A})$  and written  $A, B, C \in \mathcal{A}$ .  
Such that each object has an ‘identity’,  $1_A \in \mathcal{A}(A, A), 1_B \in \mathcal{A}(B, B), 1_C \in \mathcal{A}(C, C)$
- (ii) For each pair of objects, a collection of ‘links’/morphisms between them, denoted by  $\mathcal{A}(A, B)$  and written as  $f \in \mathcal{A}(A, B) \ g \in \mathcal{A}(B, C)$ . Such that,
  - (a) morphisms with matching domain,co-domain can be ‘chained’/composed  $(g, f) = g \circ f$
  - (b) with this composition being associative,  $(h \circ g) \circ f = h \circ (g \circ f)$
  - (c) and they are ‘fixed’ by the identity  $f \circ 1_A = f = 1_B \circ f$

**Example 1.1. Non-trivial Identity** Consider the objects to be groups, and morphisms to be direct product between them:

- i  $\text{ob}(\mathcal{A}) = \{G \mid G \text{ is a group}\}$
- ii  $\mathcal{A}(A, B) := A \times B$
- iii  $\mathcal{A}(B, C) \circ \mathcal{A}(A, B) \rightarrow \mathcal{A}(A, C)$

So, there’s a unique morphism between any two objects i.e groups. And the identity morphism,

$\forall A, B \in \mathcal{A}$ , if  $f \in \mathcal{A}(A, B)$ , then  $f \circ 1_A \in \mathcal{A}(A, B) \times \mathcal{A}(A, A) \rightarrow \mathcal{A}(A, B)$  and  $1_B \circ f \in \mathcal{A}(B, B) \times \mathcal{A}(A, B) \rightarrow \mathcal{A}(A, B)$

Thus,  $\text{ob}(\mathcal{A})$  along with  $\circ$  is actually a group. And hence has a unique inverse. [But how exactly?](#)

**Example 1.2. Set** The objects are defined to be sets, and morphisms are the functions between them, with the usual composition law:

- i  $\text{ob}(\mathcal{A}) = \{S \mid S \text{ is a set}\}$
- ii  $(f : A \rightarrow B) \in \mathcal{A}(A, B)$
- iii  $(g \in \mathcal{A}(B, C)) \circ (f \in \mathcal{A}(A, B)) \rightarrow g(f) \in \mathcal{A}(A, C)$

**Example 1.3. Pre-ordered Set** A pre-ordered, can be made into a category via the binary operation, so that the morphism  $a \rightarrow b$  is defined iff  $a \leq b$  where  $\leq$  is the preorder. The interesting part about this category is that there’s at most one morphism between any two objects.

**Example 1.4. Grp** Objects are groups, with homomorphisms between them being the morphisms, and composition being as usual:

- i  $\text{ob}(\mathcal{A}) = \{G \mid G \text{ is a group}\}$
- ii  $\mathcal{A}(A, B) = \text{Hom}(A, B)$  i.e. all  $f$  such that  $\forall x, y \in A \ f((x) \cdot_A (y)) = (f(x)) \cdot_B (f(y))$
- iii composition is defined as that between two group homomorphisms

In this example, the set of all morphisms along with composition forms a group.

**Example 1.5. Ring** Objects are rings, and arrows are ring homomorphisms between them.

- i  $\text{ob}(\mathcal{A}) = \{G \mid G \text{ is a ring}\}$
- ii  $\mathcal{A}(A, B) = \text{Hom}(A, B)$
- iii composition is defined as that between two ring homomorphisms

**Definiton 1.2** (Dual Category). Given a category  $\mathcal{A}$ , it’s opposite/dual,  $\mathcal{A}^{op}$  is a category with the same objects, but reversed arrows, while keeping the composition :

$$\text{ob}(\mathcal{A}^{op}) = \text{ob}(\mathcal{A}) \text{ and } \forall A, B \in \text{ob}(\mathcal{A}), \mathcal{A}^{op}(A, B) = \mathcal{A}(B, A)$$

**Example 1.6.  $\mathbf{Vect}_k$**  Objects are vector spaces *over field  $k$* , and the morphisms between them are linear transformations

- i  $ob(\mathcal{A}) = \{A | A \text{ is a vector space}\}$
- ii  $\mathcal{A}(A, B) = \mathcal{L}(A, B)$
- iii composition is defined as that of linear transformations

**Definiton 1.3 (Isomorphism).** An isomorphism, between objects, is a morphism between them such that it's 'inverse' is also a morphism. So,

$$f : A \rightarrow B \text{ is an isomorphism} \iff \exists g \in \mathcal{A}(B, A) : gf = 1_A \text{ and } fg = 1_B$$

**Definiton 1.4 (Product Category).** Somewhat like a cartesian product of categories. Given categories  $\mathcal{A}$  and  $\mathcal{B}$ ,  $\mathcal{A} \times \mathcal{B}$  is defined as:

- i  $ob(\mathcal{A} \times \mathcal{B}) := ob(\mathcal{A}) \times ob(\mathcal{B})$
- ii  $(\mathcal{A} \times \mathcal{B})((A, B), (A', B')) := \mathcal{A}(A, A') \times \mathcal{B}(B, B')$
- iii  $(f, g) \in \mathcal{A} \times \mathcal{B}((A, B), (C, D))$ ,  $(a, b) \in \mathcal{A} \times \mathcal{B}((C, D), (E, F)) \implies (a, b) \circ (f, g) := (a \circ f, b \circ g)$
- iv  $\forall (A, B) \in ob(\mathcal{A} \times \mathcal{B})$ ,  $1_{(A, B)} := (1_A, 1_B)$

**Example 1.7 (CAT).** The category of all categories with morphisms being functors.

- i  $ob(\mathcal{A}) = \{A | A \text{ is a category}\}$
- ii  $\mathcal{A}(A, B) = F(A, B)$
- iii  $F : \mathcal{A} \rightarrow \mathcal{B}$ ,  $G : \mathcal{B} \rightarrow \mathcal{C} \implies G \circ F := H : \mathcal{A} \rightarrow \mathcal{C}$

And thus, the identity of  $\mathcal{A}$  is the functor,  $1_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}$

**Example 1.8. Functor Category** Fix categories  $\mathcal{A}$  and  $\mathcal{B}$ . Take objects to be the functors  $F : \mathcal{A} \rightarrow \mathcal{B}$  and morphisms as the natural transformations between the objects.

This **Functor category** is written as  $[\mathcal{A}, \mathcal{B}]$  and  $\mathcal{B}^{\mathcal{A}}$

**Example 1.9. Top**

- i objects are topological spaces
- ii morphisms are continous functions

## 2 Functors

**Definiton 2.1 ((Covariant)Functor).** A functor is a map between categories, written  $F : \mathcal{A} \rightarrow \mathcal{B}$ , consists :

- (i) function taking objects of  $\mathcal{A}$  to those of  $\mathcal{B}$  i.e.  $ob(\mathcal{A}) \rightarrow ob(\mathcal{B})$ . Written as  $A \rightarrow F(A)$ .
- (ii) associative, identity-preserving function taking links between objects of  $\mathcal{A}$  to those for  $\mathcal{B}$ ,  $f \mapsto F(f)$ , i.e.

$$\forall A, B \in \mathcal{A}, \mathcal{A}(A, B) \mapsto \mathcal{B}(F(A), F(B)) \text{ such that } (a) f : A \rightarrow B, g : B \rightarrow C \implies F(g \circ f) = F(g) \circ F(f) = F(g \circ f)$$

$$(b) F(1_A) = 1_{F_A}$$

**Example 2.1. Forgetful Functors** They essentially ignore some of the structure of the 'domain'.

- (a)  $U : Grp \rightarrow Set$  takes groups to their underlying set, and homomorphisms to maps between the sets. Similarly,  $Ring \rightarrow Set$  and  $Vect_k \rightarrow Set$
- (b) Let  $Ab$  be the category of abelian groups, then,  $U : Ring \rightarrow Ab$  takes rings to their additive group, 'forgetting' the multiplicative group. And if  $Mon$  is the category of monoids,  $U : Ring \rightarrow Mon$  'forgets' the additive group.
- (c)  $U : Ab \rightarrow Grp$  just takes each abelian group to itself, and does the same for (homo)morphisms.

### Example 2.2. Free Functors

- (a) let  $F(S)$  denote the free group on a set  $S$ . Then,  $F : Set \rightarrow Grp$  is a ‘free’ functor taking sets to their free group, and thus the maps between them become homomorphisms between their free groups. As,

$$f \in Set(S, S') \mapsto F(f) \in Grp(F(S), F(S')) \text{ i.e. } f : s \rightarrow s' \text{ goes to } F(f) \text{ defined as } g := F(s) \mapsto f(g)$$

- (b) Similarly, there’s a ‘free’ functor  $F : Set \rightarrow CRing$  to the category of commutative rings. Defined as taking sets to polynomial rings having each element as a commuting variable, and coefficients from  $\mathbb{Z}$ .
- (c) Fix any field  $\mathbb{F}$ , and define  $F(S)$  to be a vector space over it with (Schauder) basis  $S$ . As basis completely determines a vector space,

$$F(S) := \{L : S \rightarrow \mathbb{F} \mid L \text{ takes only finitely many } s \in S \text{ to a non-zero } k \in \mathbb{F}\} \text{ i.e. } F(S) \mapsto \sum_{s \in S} k_s s$$

$$\text{and } f \in Set(S, S') \text{ goes to } F(f) : L(F(S), F(S'))$$

**Example 2.3.** Let  $\mathcal{G}, \mathcal{H}$  be the one object categories of monoids  $G, H$  respectively. Then, due to composition being associative and identity preserving, possible functors are precisely the homomorphisms.

**Example 2.4.** Let monoid  $G$  be regarded as a one-object category,  $\mathcal{G}$ . Then, functor  $F : \mathcal{G} \rightarrow Set$  has one object, a set  $S$ . And,  $\forall g \in G, F(g) : S \rightarrow S$  is defined as  $(F(g))(s) = g * s$  where  $*$  is an associative identity-preserving function. Thus,  $(g, s) \mapsto g.s$  i.e.  $S$  is a left  $G$ -set.

**Definiton 2.2** (Contravariant Functor). For categories  $\mathcal{A}$  and  $\mathcal{B}$ ,  $\mathcal{A}^{op} \mapsto \mathcal{B}$  is a contravariant functor from  $\mathcal{A}$  to  $\mathcal{B}$ .

**Example 2.5.** Let  $k$  be a field and  $V, V', W$  be vector spaces over it. Then fixing  $W$ ,

$$\forall f \in Hom(V, V'), \exists f^* : Hom(V', W) \rightarrow Hom(V, W) \text{ as } g \in Hom(V', W) \implies V \xrightarrow{f} V' \xrightarrow{g} W$$

**recheck the following argument** So, for each  $V \in ob(Vect_k)$ ,  $Hom(V, W)$  defines a (contravariant) functor on  $Vect_k$ , as, fixing  $W=V$ , the above argument can be restated as

$$f \in Vect_k^{op}(V', V) = Vect_k(V, V') \mapsto g \in Vect_k(V', V)$$

**Definiton 2.3** (Faithful Functor). A functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  is faithful iff the map  $\mathcal{A}(A, A') \mapsto \mathcal{B}(F(A), F(A'))$  is injective for any  $A, A' \in \mathcal{A}$  i.e. each arrow between  $A, A'$  goes to at most one arrow between  $F(A), F(A')$

**Definiton 2.4** (Full Functor). A functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  is full iff the map  $\mathcal{A}(A, A') \mapsto \mathcal{B}(F(A), F(A'))$  is surjective for any  $A, A' \in \mathcal{A}$  i.e. each arrow between  $A, A'$  goes to at least one arrow between  $F(A), F(A')$

**Definiton 2.5** (Subcategory). A subcategory of  $\mathcal{A}$  is a category with objects from  $\mathcal{A}$ , but not necessarily all of them. Similarly for the morphisms.

**Definiton 2.6** (Full Subcategory). A full subcategory of  $\mathcal{A}$  that retains as many morphisms of  $\mathcal{A}$  as possible.

## 3 Natural Transformation

**Definiton 3.1** (Natural Transformation). Let  $\mathcal{A}$  and  $\mathcal{B}$  be categories and functors,  $F, G : \mathcal{A} \rightarrow \mathcal{B}$ . Then, a natural transformation,  $\alpha : F \rightarrow G$  is a family of arrows in  $\mathcal{B}$ ,  $(F(A) \xrightarrow{\alpha_A} G(A))_{A \in \mathcal{A}}$  such that

$$\text{(Naturality Axiom)} \quad \forall f \in \mathcal{A}(A, A'), \text{ the square } \begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(A') \\ \alpha_A \downarrow & & \downarrow \alpha_{A'} \\ G(A) & \xrightarrow{G(f)} & G(A') \end{array} \text{ commutes}$$

This is written as  $\mathcal{A} \begin{array}{c} \curvearrowright \\ F \\ \Downarrow \alpha \\ G \\ \curvearrowleft \end{array} \mathcal{B}$ . And  $\alpha_A$  are called the components of  $\alpha$ .

**Lemma 3.1** (Unique factorization through components). For any  $A, B \in \mathcal{A}$

$$\forall f \in \mathcal{A}(A, B), \exists! f' \in \mathcal{B}(F(A), G(B))$$

*Proof.* Because of the naturality axiom, there's at least one such map,  $f' = G(f) \circ \alpha_A$ . And if there exist two such maps, say  $a, b$  then  $\square$

**Example 3.1. From a discrete category** The natural transformation has one component for every object,  $A \in \mathcal{A}$ , that takes  $1_{F(A)} \mapsto 1_{G(A)}$ .

**Example 3.2. Determinant (of an  $n \times n$  matrix)** Let  $R$  be a commutative ring with unity. So, the matrices on it form a monoid under matrix multiplication. Also, a ring homomorphism,  $f : R \rightarrow S$  would induce a monoid homomorphism,  $g : M_n(R) \rightarrow M_n(S)$  as

$$f(rr') = f(r)f(r') \implies g(MM') = g(M)g(M')$$

Now, this defines a functor,  $M_n : CRing \rightarrow Mon$  which takes each ring to monoid of matrices with entries from it (And each ring homomorphism,  $h$  to a map that applies  $h$  pointwise to the matrices). Also, there's a forgetful functor,  $F : CRing \rightarrow Mon$  that retains only multiplication. Every  $n \times n$  matrix over  $X$  over  $R$  has a determinant in  $R$  which, due to linearity, is a monoid homomorphism,  $det_R : M_n(R) \rightarrow F(R)$ . In order to show that  $det_R$  is a natural transformation,

$$\forall h \in Cring(R, S), \text{ the square } \begin{array}{ccc} M_n(R) & \xrightarrow{M_n(h)} & M_n(S) \\ det_R \downarrow & & \downarrow det_S \\ F(R) & \xrightarrow{F(h)} & F(S) \end{array} \text{ must commute}$$

So, given any matrix  $M$  over  $R$ , and  $A := M_n(h)$  ;  $B := F(h)$ , need to show that  $B(|M|_R) = |A(M)|_S$ . I.e. that taking the determinant, and then applying only the multiplicative part of  $h$  to it is equivalent to first applying, pointwise to the entries of  $M$ , the homomorphism  $h$ , and then taking the determinant.

**Construction 3.1** (Composition of Natural Transforms). Given  $\mathcal{A} \begin{array}{c} \xrightarrow{F} \\ \Downarrow \alpha \\ \xrightarrow{G} \end{array} \mathcal{B}$  and  $\mathcal{A} \begin{array}{c} \xrightarrow{G} \\ \Downarrow \beta \\ \xrightarrow{H} \end{array} \mathcal{B}$ ,

define their composition,  $\beta \circ \alpha$  as

$$\forall A \in \mathcal{A}, (\beta \circ \alpha)_A = \beta_A \circ \alpha_A \text{ i.e. } \left( F(A) \xrightarrow{\alpha_A} G(A) \xrightarrow{\beta_A} H(A) \right)_{A \in \mathcal{A}} \xrightarrow{(\beta \circ \alpha)_A}$$

**Example 3.3.**  $[2, \mathcal{B}]$  Let  $2$  be the discrete category with two objects. So, a functor,  $F : 2 \rightarrow \mathcal{B}$  is a pair of objects in  $\mathcal{B}$  and a natural transformation is a pair of maps in  $\mathcal{B}$ . Thus, the functor category  $[2, \mathcal{B}]$  a.k.a.  $\mathcal{B}^2$  is isomorphic to the product category  $\mathcal{B} \times \mathcal{B}$ .

**Definiton 3.2** (Natural Isomorphism). Let  $\mathcal{A}$  and  $\mathcal{B}$  be categories, a natural isomorphism between functors from  $\mathcal{A}$  to  $\mathcal{B}$  is an isomorphism in  $[\mathcal{A}, \mathcal{B}]$ . I.e. a natural transformation such that it's 'inverse' is also a natural transformation between some functors in  $[\mathcal{A}, \mathcal{B}]$ .

**Lemma 3.2** (Alternate Defintion of Natural Isomorphism). Given a natural transformation,  $\mathcal{A} \begin{array}{c} \xrightarrow{F} \\ \Downarrow \alpha \\ \xrightarrow{G} \end{array} \mathcal{B}$ . It is

a natural isomorphism iff  $\forall A \in \mathcal{A}, \alpha_A : F(A) \rightarrow G(A)$  is an isomorphism.

*Proof.* soon  $\square$

**Definiton 3.3** (Isomorphy of functors). For functors  $\mathcal{A} \xrightleftharpoons[G]{F} \mathcal{B}$ , it's said that  $F(A) \cong G(A)$  naturally in  $A$  iff  $F$  and  $G$  are naturally isomorphic.

It gives not only that  $\forall A \in \mathcal{A}, F(A) \cong G(A)$  but that there's a family of *isomorphisms*,  $\left( F(A) \xleftarrow{\alpha_A} G(A) \right)_{A \in \mathcal{A}}$  in  $\mathcal{B}$  that satisfies the naturality axiom.

**Definiton 3.4** (Equivalent categories). Categories  $\mathcal{A}$  and  $\mathcal{B}$  are said to be equivalent iff there's an *equivalence* between them. An equivalence is a pair of functors  $F, G$  along with a natural isomorphisms  $\alpha, \beta$  such that:

$$\alpha : 1_{\mathcal{A}} \rightrightarrows G \circ F \text{ and } \beta : F \circ G \rightrightarrows 1_{\mathcal{B}}$$

And it's writtten  $\mathcal{A} \simeq \mathcal{B}$

**Definiton 3.5** (Essentially Surjective on objects). A functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  is said to be **essentially surjective** on objects iff  $\forall B \in \mathcal{B}, \exists A \in \mathcal{A}$  such that  $F(A) \cong B$ .

**Lemma 3.3.** A functor ?? is an equivalence iff it is full, faithful and essentially surjective on objects.

*Proof.*

□

## 4 Representables

Hereon, only regarding locally small categories.

**Definiton 4.1** (Functor  $H^A$  aka  $\mathcal{A}(f, -)$ ). For a fixed  $A \in \mathcal{A}$ , functor  $H^A : \mathcal{A} \rightarrow \mathbf{Set}$  is defined:

- i on objects  $B \in \mathcal{A}, H^A(B) := \mathcal{A}(A, B)$
- ii for morphisms  $f \in \mathcal{A}(X, Y), H^A(g) : \mathcal{A}(A, X) \rightarrow \mathcal{A}(A, Y)$  mapping each arrow,  $p : A \rightarrow X$  as  $p \mapsto f \circ p$

**Definiton 4.2** (Representable functor). Functor  $F : \mathcal{A} \rightarrow \mathbf{Set}$  is said to be representable iff it's isomorphic to  $H^A$  for some  $A \in \mathcal{A}$ . And in that case, the object  $A$  along with the isomorphism are called a arepresentation of  $F$ .

**Definiton 4.3** ( $H^f$  aka  $\mathcal{A}(f, -)$ ). Any morphism in  $\mathcal{A}, f : X \rightarrow Y$  induces a natural transformation  $H^Y \Rightarrow H^X$ :

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{H^Y} & \mathbf{Set} \\ & \Downarrow H^f & \\ \mathcal{A} & \xrightarrow{H^X} & \mathbf{Set} \end{array}$$

At  $B \in \mathcal{A}$  for  $p \in \text{Hom}(Y, B)$  i.e.  $p : Y \rightarrow B$  as  $p \mapsto p \circ f$

**Definiton 4.4** ( $H^\bullet$ ). A functor,  $H^\bullet : \mathcal{A}^{op} \rightarrow [\mathcal{A}, \mathbf{Set}]$  defined on

- i objects  $A \in \mathcal{A}$  as  $H^\bullet(A) = H^A$
- ii morphisms  $f : X \rightarrow Y$  as  $H^\bullet(f) = H^f$

**Definiton 4.5** ( $H_A$  or  $\mathcal{A}(-, A)$  i.e. dual of  $H^A$ ). A functor,  $H_A : \mathcal{A}^{op} \rightarrow \mathbf{Set}$  defined on:

- i objects  $B \in \mathcal{A}$  as  $\text{Hom}(B, A)$
- ii on a morphism,  $f : X \rightarrow Y$  in  $\mathcal{A}$ ,  $H_A(g) : \mathcal{A}(Y, A) \rightarrow \mathcal{A}(X, A)$  as, for each  $p \in \mathcal{A}(Y, A)$  as  $p \mapsto p \circ f$

**Definiton 4.6** (contravariant representables). Functor  $X : \mathcal{A}^{op} \rightarrow \mathbf{Set}$  is representable iff there is some object  $A \in \mathcal{A}$  such that  $X \cong H_A$ . And that choice of object and isomorphism is called a representation.

**Definiton 4.7** ( $H_f$ ). Any map,  $f : X \rightarrow Y$  in  $\mathcal{A}$  induces a natural transformation  $H_f : \mathcal{A}^{op} \rightarrow \mathbf{Set}$ :

$$\begin{array}{ccc} \mathcal{A}^{op} & \xrightarrow{H_X} & \mathbf{Set} \\ & \Downarrow H_f & \\ \mathcal{A}^{op} & \xrightarrow{H_Y} & \mathbf{Set} \end{array}$$

with component for  $B \in \mathcal{A}$  being,  $p \in \mathcal{A}(B, X) \mapsto f \circ p \in \mathcal{A}(B, Y)$

**Definiton 4.8** (Yoneda embedding). A functor,  $H_\bullet : \mathcal{A} \rightarrow [\mathcal{A}^{op}, \mathbf{Set}]$  defined on

- i objects  $A \in \mathcal{A}$  as  $H_\bullet(A) := H^A$
- ii morphisms  $f : X \rightarrow Y$  as  $H_\bullet(f) = H_f$

**Definiton 4.9** (Generalized Element). A generalized element, of an object  $A$  (of some category) is a map with co-domain  $A$ . The domain of that map is called the shape. So, a generalized element of  $A$ , of shape  $S$  is a map  $S \rightarrow A$ .

**Definiton 4.10** (Presheaf). A presheaf on a category  $\mathcal{A}$  is a functor  $\mathcal{A}^{op} \rightarrow \mathbf{Set}$ .

To be continued.