1 Definitions

Definiton 1.1 (Prorelation)**.** A partial-ordered set of relations $X \to Y$, which is down-directed and an upper set. i.e A set, $P \subseteq \mathcal{P}(X \times Y)$ such that

(i) A partial-order defined to be containment as relations, $r \subseteq s$ only if $\forall (x, y) \in X \times Y$, $(x, y) \in r \implies (x, y) \in s$

(ii) (Down-directed), $\forall r, s \in P, \exists t \in P$ such that $t \subseteq r$ and $t \subseteq s$

(iii) (Up-set) for any relation $u : X \to Y$, if $\exists p \in P$ such that $p \leq u$ then $u \in P$

Definiton 1.2 (Composition of prorelations)**.** Prorelations can be composed by taking all compositions of their elements as relations: for prorelations $P : X \to Y$ and $Q : Y \to Z$,

$$Q.P := \{q \circ p : p \in P \text{ and } q \in Q\}$$

Definiton 1.3 (Comparison of Prorelations)**.** Two prorelations with same domain, co-domain are comparable as

$$\text{for } P, Q : X \to Y \ , \ P \leq Q \text{ if } \forall q \in Q, \exists p \in P \text{ such that } p \subseteq q$$

Definiton 1.4 (Quasi-uniformity)**.** A prorelation on a set X, $P : X \to X$ is a quasi-uniformity if it follows :

i $\forall p \in P$, for any $x \in X$, $(x, x) \in p$ i.e. xpx

ii $\forall p \in P, \exists p' \in P$ such that $p' \circ p' \subseteq p$

And in this case, (X, A) is called a *quasi-uniform space.*

Definiton 1.5 (Uniformly Continuous function)**.** A function, $f : X \to Y$ is called a uniformly continuous function, $f : (X, A) \to (Y, B)$ if, $\forall b \in B, \exists a \in A$ such that $f \circ a \subseteq b \circ f$. meaning that $f.A \leq B.f$ or $\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "A"'] \arrow[dr, phantom, "\leq"] & Y \arrow[d, "B"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$.

Definiton 1.6 (Promodule)**.** A prorelation, $\phi : X \to\!\!\!\!\circ\ Y$ is called a promodule $\phi : (X, A) \to\!\!\!\!\circ\ (Y, B)$ if it obeys: $\phi.A \leq \phi$ and $B.\phi \leq \phi$ where . denotes composition as prorelations.

Definiton 1.7 (Comparison of Promodules)**.** Promodules with same domain and co-domain are compared as prorelations, for $\phi, \psi : (X, A) \to\!\!\!\!\circ\ (Y, B)$, $\phi \sqsubseteq \psi$, only if $\phi \leq \psi$.

Definiton 1.8 (Composition of Promodules)**.** Promodules are composed as prorelations.
For promodules $\phi : (X, A) \to\!\!\!\!\circ\ (Y, B)$ and $\psi : (Y, B) \to\!\!\!\!\circ\ (Z, C)$, $\psi\phi := \psi.\phi = \{q \circ p : p \in \phi \text{ and } q \in \psi\}$

Definiton 1.9 (Opposite relation)**.** For relation $r : X \to Y$, r^o is defined to be a relation $r^o : Y \to X$ as

$$\forall (x, y) \in X \times Y, (x, y) \in r \iff (y, x) \in r^o$$

Lemma 1.9.1. For any relation $r : X \to Y$, $r^o \circ r = \Delta_X$

Lemma 1.9.2. For any relation $r : X \to Y$, $r \circ r^o \subseteq \Delta_Y$

Definiton 1.10 ($(-)_*$)**.**

Definiton 1.11 ($(-)^*$)**.**

Definiton 1.12 (Fully Faithful)**.**

Definiton 1.13 (Fully Dense)**.**

Definiton 1.14 (Topologically Dense)**.**

2 Propositions

Definiton 2.1 (QUnif)**.** QUnif is defined to be the category having quasi-uniform spaces as objects, and uniformly continous maps between them as morphisms.

Lemma 2.1.1. QUnif does define a category, as

i Composition

ii Identity

Definiton 2.2 (ProMod).

Lemma 2.2.1. ProMod does define a category, as

i Composition

ii Identity

Proposition 2.1 ($(-)_*$:QUnif$\to$ $ProMod$ is a Functor).

Proposition 2.2 ($(-)^*$:QUnif^{op} $\to$ $ProMod$ is a Functor). Defined as fixing objects and taking morphisms to their image under $(-)^*$

(a) for $(X, A) \in$ QUnif^{op}, $(X, A)^* := (X, A) \in$ ProMod

(b) for $f : (X, A) \to (Y, B)$ in QUnif, $f^* := f^o.B$

Proof.

Showing that $f^o.B : (Y, B) \nrightarrow (X, A)$ is a promodule
So, need to show $f^o.B$ a prorelation $Y \to X$ and that $(f^o.B).B \sqsubseteq f^o.B$ and $A.(f^o.B) \sqsubseteq f^o.B$
To show prorelation,

(i) (Partial-order) Inclusion of relations i.e. for $k = f^o \circ b$ and $k' = f^o \circ b'$ in $f^o.B$, $k \subseteq k' \iff b \subseteq b'$

(ii) (Down directed) for $k, k' \in f^o.B$, need that $\exists l \in f^o.B$ such that $l \subseteq k, k'$
Fix $k, k' \in f^o.B \implies \exists b, b' \in B : k = f^o \circ b$ and $k' = f^o \circ b'$
And as B is a quasi-uniformity, it's down directed so, $\exists c \in B : c \subseteq b, b' \implies l := f^o \circ c \subseteq k, k'$

(iii) (Up-set) for a relation $l : Y \to X$ and $k \in f^o.B$ such that $l \supseteq k$, need $l \in f^o.B$
Let $b \in B$ be such that $k = f^o \circ b$ and define $b' := \{(y, y') : y \in domain(l)$ and $y' \in (f^o)^{-1}(l(y))$
As $l \supseteq k = f^o \circ b$, $domain(b') = domain(l) \supseteq domain(b)$
and $range(l) \supseteq range(f^o \circ b) \implies \forall y \in domain(b), range(b') = (f^o)^{-1}(l(y)) \supseteq (f^o)^{-1}(f^o \circ b) = range(b)$
Now, by definition of b', $f^o \circ b' \supseteq l$. To show $f^o \circ b \subseteq l$,
$(x, y) \in f^o \circ b' \implies \exists z \in Y : (x, z) \in b'$ and $(z, y) \in f^o \implies x \in domain(l)$ and $z \in l(x)$ i.e. $(x, z) \in l$

To show $(f^o.B).B \leq f^o.B$, need that $\forall b \in B, \exists b' \in B : f^o \circ b' \circ b' \subseteq f^o \circ b$,
Fix any $b \in B$ as B is a quasi-uniformity, $\exists b' \in B : b' \circ b' \subseteq b \implies f^o \circ b' \circ b' \subseteq f^o \circ b$
To show $A.(f^o.B) \leq f^o.B$, need that $\forall b \in B, \exists b' \in B, a \in A : a \circ f^o \circ b' \subseteq f^o \circ b$,
As f is uniformly continuous, $f.A \leq B.f$ i.e. $\forall b \in B, \exists a \in A : f \circ a \subseteq b \circ f \implies a = f^o \circ f \circ a \subseteq f^o \circ b \circ f$
Fix any $b \in B$, so, $\exists b' \in B : b'b' \subseteq b$ (for brevity,omitting $\circ$ to explicitly denote composition)
And, for this $b', \exists a : a \subseteq f^o b' f \implies a f^o b' \subseteq f^o b' f f^o b' \subseteq f^o b' b' \subseteq f^o b \implies a f^o b' \subseteq f^o b$

Now, need to show that $(-)^*$ respects composition and identity.

(i) (Composition) let f, g be uniformly continuous, $(X, A) \xrightarrow{f} (Y, B) \xrightarrow{g} (Z, C)$ need that $(g \circ f)^* = f^*.g^*$
LHS=$(g \circ f)^* = (g \circ f)^o.C = (f^o \circ g^o).C$ and RHS=$f^*.g^* = (f^o.B).(g^o.C)$
For equality, showing that LHS$\geq$RHS and LHS$\leq$RHS:

To show $(f^o \circ g^o).C \geq (f^o.B).(g^o.C)$, need that $\forall c \in C, \exists b \in B, c' \in C : f^o g^o c \supseteq f^o b g c'$
Fix any $c \in C$, so, $\exists c' \in C : c' \circ c' \subseteq c \implies f^o g^o c \supseteq f^o g^o(c'c') = f^o g^o(c' \Delta_Z c') \supseteq f^o g^o c'(g g^o)c'$
By uniform conntinuity of g, for $c' \in C, \exists b \in B : gb \subseteq c'g$
Thus, $f^o g^o c \supseteq f^o g^o(c'g)g^o c' \supseteq f^o(g^o g)b g^o c' = f^o b g^o c'$.

To show $(f^o \circ g^o).C \leq (f^o.B).(g^o.C)$, need that $\forall b \in B, c \in C, \exists c' \in C : f^o g^o c \subseteq f^o b g^o c$
Fix any $c \in C, b \in B$ will show that $c' := c$ works:
As B is a quasi-uniformity, $\Delta_Y \subseteq b \implies f^o \Delta_Y g^o c = f^o g^o c \subseteq f^o b g^o c = f^o b g^o c'$

(ii) (Identity) let $(X, A) \in$ QUnif ^{op}, and $1_{(X,A)} : (X, A) \to (X, A)$ as $x \mapsto x$ need that $(1_{(X,A)})^* = 1_{(X,A)^*}$
LHS=$(1_{(X,A)})^* = (1_{(X,A)})^o.A = 1_{(X,A)}.A = A$.
Now, it's required that A is the identity of (X, A) in ProMod.
So, fix $\phi : (X, A) \nrightarrow (Y, B)$, need to show $\phi.A = \phi$
As ϕ is a promodule, $\phi.A \leq \phi$ and as A is quasi-uniformity on X,
$\forall a \in A, \Delta_X \subseteq a \implies \forall a \in A, \forall p \in \phi, p = p\Delta_X \subseteq pa \implies \phi \leq \phi.A$
Also, fix $\psi : (Y, B) \nrightarrow (X, A)$, need to show $A.\psi = \psi$
As ψ is a promodule, $A.\psi \leq \psi$ and as A is quasi-uniformity on X,
$\forall a \in A, \Delta_X \subseteq a \implies \forall a \in A, \forall q \in \psi, q = \Delta_X q \subseteq aq \implies \psi \leq \psi.A$

$\square$

Proposition 2.3 (Proposition 1)**.** Fix a uniformly continuous map, $f : (X, A) \to (Y, B)$

(a) f is fully faithful $\iff A = f^o.B.f$

(b) f is fully dense $\iff \forall b \in B, \exists b' \in B$ such that

Definiton 2.3 (PX)**.** PX:=$\{\psi : \psi : (X, A) \looparrowright 1$ is a promodule$\}$

Proposition 2.4 (Prorelation $\tilde{A}$)**.** for an $\tilde{A} := \{\tilde{a} : a \in A\}$

Proposition 2.5 (Yoneda Embedding)**.**

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