Question 4 (b) $f(x,y) := \frac{2xy}{\sqrt{x^2+y^2}}$
For partial derivative along x at $(\delta, 0)$,

$$\frac{\partial f}{\partial x}(\delta,0) = \lim_{h\to 0} \frac{f(\delta+h,0)-f(\delta,0)}{h}$$
$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{2(\delta+h)\times 0}{\sqrt{(\delta+h)^2+0}} - \frac{2\delta\times 0}{\sqrt{\delta^2+0}}\right) = 0$$

In particular, if $\delta = 0$ then the second term is zero as $f(0,0) := 0$. And, due to symmetry, $f_y(0,\delta) = 0$. As :

$$\frac{\partial f}{\partial y}(0,\delta) = \lim_{h\to 0} \frac{f(0,\delta+h)-f(0,\delta)}{h}$$
$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{2\times 0\times(\delta+h)}{\sqrt{0+(\delta+h)^2}} - \frac{2\times 0\times\delta}{\sqrt{0+\delta^2}}\right) = 0$$

So, not only are the partial derivatives f_x and f_y 0 at (0,0), but also along x and y axes respectively.
To show the existence of second derivatives ,

$$f_{xx}(0,0) = \lim_{h\to 0}\frac{f_x(h,0)-f_x(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$$

Similarly, $f_{yy}(0,0) = 0$

Now, if the derivative of f existed at (0,0), then, the following limit must be 0

$$\lim_{h^2+k^2\to 0} \frac{f(0+h,0+k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}}$$
$$= \lim_{h^2+k^2\to 0}\frac{2hk}{h^2+k^2}$$

But this limit is equal to 1 along the line of slope 1, and thus f isn't differentiable.

$$\text{(a) } f(x,y) := \frac{(x^2y + xy^2)sin(x-y)}{x^2+y^2}$$

For partial derivative along x at $(\delta, 0)$,

$$\frac{\partial f}{\partial x}(\delta,0) = \lim_{h\to 0}\frac{f(\delta+h,0) - f(\delta,0)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\Big(\frac{\big((\delta+h)^2\times 0 + (\delta+h)\times 0\big)sin((\delta+h))}{(\delta+h)^2+0} - 0\Big) = 0$$

In particular, if $\delta = 0$ then the second term is still zero as $f(0,0) := 0$. And, due to symmetry, $f_y(0,\delta) = 0$.

So, not only are the partial derivatives f_x and f_y 0 at (0,0), but also along x and y axes respectively.

To show the existence of second derivatives ,

$$f_{xx}(0,0) = \lim_{h\to 0}\frac{f_x(h,0) - f_x(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$$

Similarly, $f_{yy}(0,0) = 0$

Now, to show the existence of derivative of f at (0,0), then, the following limit must be 0

$$\lim_{h^2+k^2\to 0}\frac{f(0+h,0+k) - f(0,0) - hf_x(0,0) - kf_y(0,0)}{\sqrt{h^2+k^2}}$$

$$= \lim_{h^2+k^2\to 0}\frac{(h^2k+hk^2)(sin(h-k))}{(h^2+k^2)\sqrt{h^2+k^2)}}$$

The above limit goes to zero iff it's modulus goes to zero:

$$\lim_{h^2+k^2\to 0}\Big|\frac{(h^2k+hk^2)(sin(h-k))}{(h^2+k^2)\sqrt{h^2+k^2)}}\Big| = 0$$

Seperating the numerator and dividing by them , this limit becomes:

$$\lim_{h^2+k^2\to 0} |(sin(h-k))\frac{(h^2k)}{(h^2+k^2)\sqrt{h^2+k^2)}} + \frac{(hk^2)}{(h^2+k^2)\sqrt{h^2+k^2)}}|$$
$$= \lim_{h^2+k^2\to 0} |\frac{(sin(h-k))(k)}{(1+(\frac{k}{h})^2)\sqrt{h^2+k^2)}} + \frac{(sin(h-k))(h)}{((\frac{h}{k})^2+1)\sqrt{h^2+k^2)}}|$$
$$= \lim_{h^2+k^2\to 0} |\frac{(sin(h-k))}{(1+(\frac{k}{h})^2)\sqrt{(\frac{h}{k})^2+1)}} + \frac{(sin(h-k))}{((\frac{h}{k})^2+1)\sqrt{1+(\frac{k}{h})^2)}}|$$

Now, for any given values of h, k, by trichotomy, either

$$\frac{h}{k} \leq 1 \text{ and } \frac{k}{h} \geq 1 \textbf{ or } \frac{h}{k} \geq 1 \text{ and } \frac{k}{h} \leq 1$$

Thus, the denominators of

$$= \lim_{h^2+k^2\to 0} |\frac{(sin(h-k))}{(1+(\frac{k}{h})^2)\sqrt{(\frac{h}{k})^2+1)}} + |\frac{(sin(h-k))}{((\frac{h}{k})^2+1)\sqrt{1+(\frac{k}{h})^2)}}$$

are always greater than 1. But the numerators tend to 0. Thus the differential exists at the origin. And must be equal to the gradient, i.e. a matrix containing the partial derivatives at the origin. Hence, the derivative is (0,0).

Now, as we want to evaluate differentiablity at (0,0), it's enough to consider all the lines of the form $y = mx$ i.e. (i, mi) with $i \in \mathbb{R}$. Also, for points along the x-axis, the limit is zero as,

$$\lim_{h^2+k^2\to 0} \frac{(0 \times k + 0 \times k^2)(sin(0-k))}{(0+k^2)\sqrt{0+k^2)}} = 0$$

By symmetry, the limit is zero along y-axis as well, so, let m be some finite, non-zero number, and consider the line (i, mi),

$$\begin{aligned}
&\lim_{i\to 0} \frac{(i^2 \times mi + i \times m^2 i^2)(sin(i - mi))}{(i^2 + m^2 i^2)\sqrt{i^2 + m^2 i^2)}} \\
&= \lim_{i\to 0} \frac{i^3(m+m^2)(sin(i(1-m))}{i^2(1+m^2)|i|\sqrt{1+m^2}} \\
&= \lim_{i\to 0} \frac{m+m^2}{(1+m^2)\sqrt{1+m^2}} \times sgn(i) \times sin(i(1-m))
\end{aligned}$$

For any particular line, the first term is a constant, second one is ± 1, while the third one goes to 0 as $i \to 0$. Thus, the limit is 0 for every line going through the origin.

Thus, the differential exists at the origin.

Question 5 (Marked)

$$f(x,y) = xy(1-x^2-y^2)$$

For each partial derivative to be zero,

$$f_x = y - 3x^2y - y^3 = 0 \text{ and } f_y = x - x^3 - 3xy^2 = 0$$
$$\implies f_x = y(1-3x^2-y^2) = 0 \text{ and } f_y = x(1-x^2-3y^2) = 0$$

So, a critical point is $(0,0)$.

And,
If $x = 0, y \neq 0$ then $f_x = y(1-y^2) = 0 \implies y = \pm 1$
gives two critical points: $(0, \pm 1)$

If $y = 0, x \neq 0$ then $0 = f_y = y(1-x^2) \implies x = \pm 1$
gives two critical points: $(\pm 1, 0)$

If $x \neq 0 \neq y$, then

$$\begin{aligned} f_x = y - 3x^2y - y^3 = 0 &\implies y(1-3x^2) = y^3 \\ &\implies 1 - 3x^2 = y^2 \qquad (equation A) \end{aligned}$$

And, also,

$$\begin{aligned} f_y = x - 3y^2x - x^3 = 0 &\implies x(1-3y^2) = x^3 \\ &\implies 1 - 3y^2 = x^2 \qquad (equation B) \end{aligned}$$

Thus,

Substituting B into A, $1 - 3 + 9y^2 = y^2 \implies y = \pm 0.5$
And, substituting A into B, $1 - 3 + 9x^2 = x^2 \implies x = \pm 0.5$

Thus, the four such possible points are also critical:

$$(0.5, 0.5), (0.5, -0.5), (-0.5, 0.5), (-0.5, -0.5)$$

Now, to classify these critical points, looking at $rt - s^2$

$$r = f_{xx} = -6xy = f_{yy} = t \text{ and } s = f_{xy} = 1 - 3x^2 - 3y^2$$

$$\text{So, } rt = 36x^2y^2 \text{ and } s^2 = (1 - 3(x^2 + y^2))^2$$

At $(0, 0), rt = 0$ and $s = 1$. Thus, $rt - s^2 = 0 - 1 < 0$
Thus, $(0, 0)$ is a saddle point.

At $(\pm 1, 0)$ and $(0, \pm 1)$,
$rt = 36 \times 1 \times 0 = 36 \times 0 \times 1 = 0$ and $s^2 = (1 - 3)^2 = 4$.
Thus, $rt - s^2 = 0 - 4 < 0$
Thus, $(\pm 1, 0), (0, \pm 1)$ are saddle points.

At $(0.5, 0.5), (0.5, -0.5), (-0.5, 0.5), (-0.5, -0.5)$,
$rt = 36x^2y^2 = 36 \times 0.25 \times 0.25 = \frac{9}{4} = 2.25$
and $s^2 = (1 - 3(0.25 + 0.25))^2 = (-0.5)^2 = 0.25$
Hence, $rt - s^2 = 2.25 - 0.25 = 2 > 0$
Thus, $(0.5, -0.5), (-0.5, 0.5)$ are minima(as $r = 1.5$),
while $(0.5, 0.5), (-0.5, -0.5)$ are maxima(as $r = 1.5$).

Question 3 (Marked) **(a)** $f(x,y) := \frac{x^2y^2}{x^2y^2+(x-y)^2} = \frac{1}{1+(\frac{x-y}{xy})^2} = \frac{1}{1+(\frac{1}{y}-\frac{1}{x})^2}$

$$\lim_{x\to 0} f(x,y) = \lim_{x\to 0} \frac{1}{1+(\frac{1}{y}-\frac{1}{x})^2} = 0 \implies \lim_{y\to 0}\lim_{x\to 0} f(x,y) = 0$$

And as the expression is symmetric in x and y,

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{x\to 0}\lim_{y\to 0} f(x,y)$$

But the simultaneous limit at$(0,0)$ along $T(t) := (t,t)$ is

$$\lim_{t\to 0} f(T(t)) = \lim_{t\to 0} \frac{1}{1+(\frac{1}{t}-\frac{1}{t})^2} = 1$$

If the simultaneous limit existed, all the iterated limits would be equal to it. So, there is a curve, $S(t) := (\frac{1}{t}, \frac{1}{t+1})$ with

$$\lim_{t\to \infty} f(S(t)) = \lim_{t\to \infty} \frac{1}{1+(t+1-t)^2} = \frac{1}{2}$$

Thus, f is discontinous at $(0,0)$

(b) $f(x,y) := \frac{e^{\frac{-1}{x^2}}y}{e^{\frac{-1}{x^2}}+y^2} = \frac{ye^{\frac{1}{x^2}}}{1+(ye^{\frac{1}{x^2}})^2} = \frac{1}{\frac{1}{ye^{1/x^2}}+ye^{1/x^2}}$

$$\lim_{y\to 0}\lim_{x\to 0} \frac{1}{\frac{1}{ye^{1/x^2}}+ye^{1/x^2}} = \lim_{y\to 0} 0 = 0$$

$$\lim_{x\to 0}\lim_{y\to 0} \frac{ye^{\frac{1}{x^2}}}{1+(ye^{\frac{1}{x^2}})^2} = \lim_{x\to 0} 0 = 0$$

To show the non-existence of simulatenous limit at $(0,0)$, consider the curve $T(t) := (t, e^{-1/t^2})$

$$\lim_{t\to 0} f(T(t)) = \frac{e^{-1/t^2}\times e^{1/t^2}}{1+(e^{-1/t^2}\times e^{1/t^2})^2} = \frac{1}{2}$$