Question 1

$$C \subseteq D \subseteq \mathbb{R}; \qquad \big(f_n\big)_{n\in\mathbb{N}} \text{ is uniformly convergent on } C$$
$$\text{and } \forall i \in \mathbb{N},\ f_i : D \to \mathbb{R} \text{ is continous}$$

Need to show that, there's a continous function,

$$f \text{ such that } f_n \rightrightarrows f \text{ on } closure(C) \cap D$$

Proof. As the sequence uniformly converges on C, there's a function, f such that for $\epsilon = 1$,

$$\exists \delta : n \geq \delta \implies \forall x \in C\ ,\ |f_n(x) - f(x)| \leq 1$$

So, just need to show that $f_n \xrightarrow{C'} f$ uniformly i.e.

$$\exists K : n \geq K \implies \forall x \in C',\ |f_n(x) - f(x)| \leq 1$$

Now, fix any $x \in C'$, so, there's a sequence in C, $\big(x_n\big)$ that converges to x. Thus, $\exists N : n \geq N \implies |x_n - x| \leq 1$. And, as each f_i is continous on D, $\lim_{a\to x} f_n(a) = f_n(x)$ i.e.

$$\exists \delta' : |a - x| \leq \delta' \implies \forall i \in \mathbb{N},\ |f_i(a) - f_i(x)| \leq 1$$

Hence, if $k \in C'$

$\square$

I $\liminf a_n \leq \liminf \sigma_n$
If a_n is unbounded below, then $\liminf a_n = -\infty \leq \liminf \sigma_n$ So, let a_n be bounded below, thus LHS is a real number,

$$m := \liminf a_n$$

Now, if a_n is constant, and equal to a,

$$\forall n \in \mathbb{N}\ ,\ \inf\{a_i | i \geq n\} = a = \frac{na}{n} = \sigma_n$$

Otherwise, if a_n is not constant, then,

$$\exists i, j \in \mathbb{N} \text{ such that(wlog) } a_i < a_j$$

Suppose if possible, $\liminf a_n = m >\ \liminf \sigma_n$. But, $\forall n \in \mathbb{N}$ such that $n \geq i, j$,

$$\sigma_n = \frac{a_1 + ... + a_i + ... + a_j + ... + a_n}{n}$$

$$\implies \sigma_n \geq \frac{(n-2)m + a_i + a_j}{n} \geq \frac{(n-1)m + a_j}{n} > m$$

$[\ \because m \leq a_i < a_j\ ]$

$$\text{Hence, } \inf \sigma_n \geq \frac{(n-1)m + a_j}{n}$$

$$\implies \liminf \sigma_n \geq \lim_{n\to\infty} \frac{(n-1)m + a_j}{n} = m$$

But this contradicts the initial assumption.

II $\limsup a_n \geq \limsup \sigma_n$ If a_n is unbounded above, then $\limsup a_n = \infty \geq \limsup \sigma_n$.

So, let a_n be bounded above, thus LHS is a real number,

$$M := \text{lim sup } a_n$$

Now, if a_n is constant, and equal to a,

$$\forall n \in \mathbb{N} \text{ , } \sup\{a_i | i \geq n\} = a = \frac{na}{n} = \sigma_n$$

Otherwise, if a_n is not constant, then,

$$\exists i, j \in \mathbb{N} \text{ such that(wlog) } a_i < a_j$$

Suppose if possible, lim sup $a_n = M <$ lim sup σ_n
But,

$$\forall n \in \mathbb{N} \text{ such that } n \geq i, j,$$

$$\begin{aligned} \sigma_n = \frac{a_1 + ... + a_i + ... + a_j + ... + a_n}{n} &\leq \frac{(n-2)M + a_i + a}{n} \\ &\leq \frac{(n-1)M + a_j}{n} \\ &< M \; [\because M \geq a_j > a \end{aligned}$$

$$\text{Hence, inf } \sigma_n \leq \frac{(n-1)M + a_j}{n}$$

$$\implies \text{lim inf } \sigma_n \leq \lim_{n \to \infty} \frac{(n-1)M + a_j}{n} = M$$

But this contradicts the initial assumption.

Question 2

$$\text{lim inf } \frac{a_{n+1}}{a_n} \leq \text{lim inf } (a_n)^{\frac{1}{n}} \leq \text{lim sup } (a_n)^{\frac{1}{n}} \leq \text{lim sup } \frac{a_{n+1}}{a_n}$$

I Showing that lim inf $\frac{a_{n+1}}{a_n} \leq$ lim inf $(a_n)^{\frac{1}{n}}$
Case 1: lim inf $\frac{a_{n+1}}{a_n} = 0$

$$\forall n \in \mathbb{N}, a_n > 0 \implies (a_n)^{\frac{1}{n}} > 0$$

$$\text{Thus , lim inf } (a_n)^{\frac{1}{n}} \geq 0 = \text{lim inf } \frac{a_{n+1}}{a_n}$$

Case 2: lim inf $\frac{a_{n+1}}{a_n} = \infty$
So, for any $a \in \mathbb{N}$,

$$\exists M_a \in \mathbb{N} : n \geq M_a \implies \frac{a_{n+1}}{a_n} > a$$

Fix any a, and choose $n > M_a$. For any such n,

$$\frac{a_{n+1}}{a_n} > a \implies a_{n+1} > aa_n$$

$$\implies a_n > aa_{n-1} > a^2 a_{n-2} > ... > a^{n-M} a_M$$

$$\implies (a_n)^{\frac{1}{n}} > a(\frac{a_M}{a^M})^{\frac{1}{n}}$$

Now as for fixed a, $\frac{a_M}{a^M}$ is constant,

$$\lim_{n\to\infty} (\frac{a_M}{a^M})^{\frac{1}{n}} = 1 \implies \lim_{n\to\infty} a(\frac{a_M}{a^M})^{\frac{1}{n}} = a$$

Thus,

$$\forall \epsilon > 0, \exists K \in \mathbb{N} : n > K \implies a(\frac{a_M}{a^M})^{\frac{1}{n}} > a - \epsilon$$

And hence, in particular, $a(\frac{a_M}{a^M})^{\frac{1}{n}} > a - 1$ for any natural number a.

$$(a_n)^{\frac{1}{n}} > a(\frac{a_M}{a^M})^{\frac{1}{n}} > a - 1 \implies \lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \infty$$

$$\therefore \text{lim inf } \frac{a_{n+1}}{a_n} = \infty = \text{lim inf } (a_n)^{\frac{1}{n}}$$

Case 3: lim inf $\frac{a_{n+1}}{a_n} = a \in \mathbb{R}$
So, $\forall \epsilon > 0, \exists M \in \mathbb{N} : n > M \implies \frac{a_{n+1}}{a_n} > a - \epsilon$

$$a_n > (a-\epsilon)a_{n-1} > (a-\epsilon)^2 a_{n-2} > ... > (a-\epsilon)^{n-M} a_M$$

$$\implies (a_n)^{\frac{1}{n}} > (a-\epsilon)(\frac{a_M}{a^M})^{\frac{1}{n}}$$

But, as $\lim_{n\to\infty}(\frac{a_M}{a^M})^{\frac{1}{n}} = 1$,

$$(a_n)^{\frac{1}{n}} > (a-\epsilon)(\frac{a_M}{a^M})^{\frac{1}{n}} > (a-\epsilon)(1-\epsilon) = a-(1+a)\epsilon+\epsilon^2$$

$$\implies (a_n)^{\frac{1}{n}} > a - (1+a)\epsilon$$

But as this holds for every $\epsilon > 0$,

$$\text{lim inf } (a_n)^{\frac{1}{n}} \geq a - 0 = \text{lim inf } \frac{a_{n+1}}{a_n}$$

II $\limsup (a_n)^{\frac{1}{n}} \leq \limsup \frac{a_{n+1}}{a_n}$

Case 1: $\limsup \ \frac{a_{n+1}}{a_n} = \infty$

As $\limsup (a_n)^{\frac{1}{n}} \leq \infty = \limsup \frac{a_{n+1}}{a_n}$,done

Case 2: $\limsup \ \frac{a_{n+1}}{a_n} = -\infty$

$$\limsup \frac{a_{n+1}}{a_n} = -\infty \implies \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = -\infty$$

But, as all a_n are positive, so is their raito, and hence it cant be unbounded below.

Case 3: $\limsup \ \frac{a_{n+1}}{a_n} = a \in \mathbb{R}$

So, $\forall \epsilon > 0, \exists M \in \mathbb{N} : n > M \implies \frac{a_{n+1}}{a_n} < a + \epsilon$

$a_n < (a+\epsilon)a_{n-1} < (a+\epsilon)^2 a_{n-2} < ... < (a+\epsilon)^{n-M} a_M$

Now, as $(\frac{a_M}{a^M})^{\frac{1}{n}}$ is constant,

$$\exists K \in \mathbb{N} : n \geq K \implies (\frac{a_M}{a^M})^{\frac{1}{n}} < 1 + \epsilon$$

$$\implies (a_n)^{\frac{1}{n}} < (a+\epsilon)(\frac{a_M}{a^M})^{\frac{1}{n}} < (a+\epsilon)(1+\epsilon) = a+(a+1)\epsilon+\epsilon^2$$

Thus,

$\limsup (a_n)^{\frac{1}{n}} \leq a+(a+1)\epsilon+\epsilon^2 < a+(a+1)(1+\epsilon)\epsilon,$

but as this holds for every $\epsilon > 0$,

$$\liminf (a_n)^{\frac{1}{n}} \leq a = \liminf \frac{a_{n+1}}{a_n}$$

Question 3

a_n & b_n are bounded, non-negative sequences; $a_n \to a > 0$

As a_n, b_n are bounded sequences, so is $a_n b_n$.
Hence, L:=lim sup $a_n b_n$ and R:=lim sup b_n are real numbers