1 Definitions

Definiton 1.1 (Prorelation)**.** A partial-ordered set of relations $X \to Y$, which is down-directed and an upper set. i.e A set, $P \subseteq \mathcal{P}(X \times Y)$ such that

(i) A partial-order defined to be containment as relations, $r \subseteq s$ only if $\forall (x,y) \in X \times Y$, $(x,y) \in r \implies (x,y) \in s$

(ii) (Down-directed), $\forall r, s \in P, \exists t \in P$ such that $t \subseteq r$ and $t \subseteq s$

(iii) (Up-set) for any relation $u : X \to Y$, if $\exists p \in P$ such that $p \leq u$ then $u \in P$

Definiton 1.2 (Composition of prorelations)**.** Prorelations can be composed by taking all compositions of their elements as relations: for prorelations $P : X \to Y$ and $Q : Y \to Z$,

$$Q.P := \{q \circ p : p \in P \text{ and } q \in Q\}$$

Definiton 1.3 (Comparison of Prorelations)**.** Two prorelations with same domain, co-domain are comparable as

$$\text{for } P, Q : X \to Y \ ,\ P \leq Q \text{ if } \forall q \in Q, \exists p \in P \text{ such that } p \subseteq q$$

Definiton 1.4 (Quasi-uniformity)**.** A prorelation on a set X, $P : X \to X$ is a quasi-uniformity if it follows :

i $\forall p \in P$, for any $x \in X$, $(x,x) \in p$ i.e. xpx

ii $\forall p \in P, \exists p' \in P$ such that $p' \circ p' \subseteq p$

And in this case, (X, A) is called a *quasi-uniform space.*

Definiton 1.5 (Uniformly Continuous function)**.** A function, $f : X \to Y$ is called a uniformly continuous function, $f : (X, A) \to (Y, B)$ if, $\forall b \in B, \exists a \in A$ such that $f \circ a \subseteq b \circ f$. meaning that $f.A \leq B.f$ or $\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "A"'] \arrow[dr, phantom, "\leq"] & Y \arrow[d, "B"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$.

Definiton 1.6 (Promodule)**.** A prorelation, $\phi : X \to\!\!\!\circ\; Y$ is called a promodule $\phi : (X, A) \to\!\!\!\circ\; (Y, B)$ if it obeys: $\phi.A \leq \phi$ and $B.\phi \leq \phi$ where . denotes composition as prorelations.

Definiton 1.7 (Comparison of Promodules)**.** Promodules with same domain and co-domain are compared as prorelations, for $\phi, \psi : (X, A) \to\!\!\!\circ\; (Y, B)$, $\phi \sqsubseteq \psi$, only if $\phi \leq \psi$.

Definiton 1.8 (Composition of Promodules)**.** Promodules are composed as prorelations.
For promodules $\phi : (X, A) \to\!\!\!\circ\; (Y, B)$ and $\psi : (Y, B) \to\!\!\!\circ\; (Z, C)$, $\psi\phi := \psi.\phi = \{q \circ p : p \in \phi \text{ and } q \in \psi\}$

Definiton 1.9 (Opposite relation)**.** For relation $r : X \to Y$, r^o is defined to be a relation $r^o : Y \to X$ as

$$\forall (x,y) \in X \times Y, (x,y) \in r \iff (y,x) \in r^o$$

Lemma 1.9.1. For any relation $r : X \to Y$, $r^o \circ r = \Delta_X$

Lemma 1.9.2. For any relation $r : X \to Y$, $r \circ r^o \subseteq \Delta_Y$

Lemma 1.9.3. For relations $r, s : X \to Y$ and $t : Y \to Z$, for any $x, x' \in X$, $r(x) \subseteq s(x') \implies (t \circ r)(x) \subseteq (t \circ s)(x)$

Lemma 1.9.4. For relations $r : X \to Y$ and $s, t : Y \to Z$, $s \subseteq t \implies (s \circ r) \subseteq (t \circ r)(x)$

Definiton 1.10 ($(-)_*$)**.**

Definiton 1.11 ($(-)^*$)**.**

Definiton 1.12 (Fully Faithful)**.**

Definiton 1.13 (Fully Dense)**.**

Definiton 1.14 (Topologically Dense)**.**

2 Propositions

Definiton 2.1 (QUnif)**.** QUnif is defined to be the category having quasi-uniform spaces as objects, and uniformly continous maps between them as morphisms.

Lemma 2.1.1. QUnif does define a category, as

i Composition

ii Identity

Definiton 2.2 (ProMod)**.**

Lemma 2.2.1. ProMod does define a category, as

i Composition

ii Identity

Proposition 2.1 ($(-)_*$:QUnif$\to$ $ProMod$ is a Functor)**.**

Proof. □

Proposition 2.2 ($(-)^*$:QUnif^{op} $\to$ $ProMod$ is a Functor)**.** Defined as fixing objects and taking morphisms to their image under $(-)^*$

(a) for $(X, A) \in$ QUnif^{op}, $(X, A)^* := (X, A) \in$ ProMod

(b) for $f : (X, A) \to (Y, B)$ in QUnif, $f^* := f^o.B$

Proof.

Showing that $f^o.B : (Y, B) \looparrowright (X, A)$ **is a promodule**
So, need to show $f^o.B$ a prorelation $Y \to X$ and that $(f^o.B).B \sqsubseteq f^o.B$ and $A.(f^o.B) \sqsubseteq f^o.B$
To show prorelation,

(i) (Partial-order) Inclusion of relations i.e. for $k = f^o \circ b$ and $k' = f^o \circ b'$ in $f^o.B$, $k \subseteq k' \iff b \subseteq b'$

(ii) (Down directed) for $k, k' \in f^o.B$, need that $\exists l \in f^o.B$ such that $l \subseteq k, k'$
Fix $k, k' \in f^o.B \implies \exists b, b' \in B : k = f^o \circ b$ and $k' = f^o \circ b'$
And as B is a quasi-uniformity, it's down directed so, $\exists c \in B : c \subseteq b, b' \implies l := f^o \circ c \subseteq k, k'$

(iii) (Up-set) for a relation $l : Y \to X$ and $k \in f^o.B$ such that $l \supseteq k$, need $l \in f^o.B$
Let $b \in B$ be such that $k = f^o \circ b$ and define $b' := \{(y, y') : y \in domain(l)$ and $y' \in (f^o)^{-1}(l(y))$
As $l \supseteq k = f^o \circ b$, $domain(b') = domain(l) \supseteq domain(b)$
and $range(l) \supseteq range(f^o \circ b) \implies \forall y \in domain(b), range(b') = (f^o)^{-1}(l(y)) \supseteq (f^o)^{-1}(f^o \circ b) = range(b)$
Now, by definition of b', $f^o \circ b' \supseteq l$. To show $f^o \circ b \subseteq l$,
$(x, y) \in f^o \circ b' \implies \exists z \in Y : (x, z) \in b'$ and $(z, y) \in f^o \implies x \in domain(l)$ and $z \in l(x)$ i.e. $(x, z) \in l$

To show $(f^o.B).B \leq f^o.B$, need that $\forall b \in B, \exists b' \in B : f^o \circ b' \circ b' \subseteq f^o \circ b$,
Fix any $b \in B$ as B is a quasi-uniformity, $\exists b' \in B : b' \circ b' \subseteq b \implies f^o \circ b' \circ b' \subseteq f^o \circ b$
To show $A.(f^o.B) \leq f^o.B$, need that $\forall b \in B, \exists b' \in B, a \in A : a \circ f^o \circ b' \subseteq f^o \circ b$,
As f is uniformly continuous, $f.A \leq B.f$ i.e. $\forall b \in B, \exists a \in A : f \circ a \subseteq b \circ f \implies a = f^o \circ f \circ a \subseteq f^o \circ b \circ f$
Fix any $b \in B$, so, $\exists b' \in B : b'b' \subseteq b$ (for brevity, omitting $\circ$ to explicitly denote composition)
And, for this $b', \exists a : a \subseteq f^o b' f \implies a f^o b' \subseteq f^o b' f f^o b' \subseteq f^o b' b' \subseteq f^o b \implies a f^o b' \subseteq f^o b$

Now, need to show that $(-)^*$ respects composition and identity.

(i) (Composition) let f, g be uniformly continuous, $(X, A) \xrightarrow{f} (Y, B) \xrightarrow{g} (Z, C)$ need that $(g \circ f)^* = f^*.g^*$
LHS$=(g \circ f)^* = (g \circ f)^o.C = (f^o \circ g^o).C$ and RHS$=f^*.g^* = (f^o.B).(g^o.C)$
For equality, showing that LHS$\geq$RHS and LHS$\leq$RHS:
To show $(f^o \circ g^o).C \geq (f^o.B).(g^o.C)$, need that $\forall c \in C, \exists b \in B, c' \in C : f^o g^o c \supseteq f^o b g c'$
Fix any $c \in C$, so, $\exists c' \in C : c' \circ c' \subseteq c \implies f^o g^o c \supseteq f^o g^o (c'c') = f^o g^o (c' \Delta_Z c') \supseteq f^o g^o c' (g g^o) c'$
By uniform conntinuity of g, for $c' \in C, \exists b \in B : gb \subseteq c' g$
Thus, $f^o g^o c \supseteq f^o g^o (c' g) g^o c' \supseteq f^o (g^o g) b g^o c' = f^o b g^o c'$.
To show $(f^o \circ g^o).C \leq (f^o.B).(g^o.C)$, need that $\forall b \in B, c \in C, \exists c' \in C : f^o g^o c \subseteq f^o b g^o c$
Fix any $c \in C, b \in B$ will show that $c' := c$ works:
As B is a quasi-uniformity, $\Delta_Y \subseteq b \implies f^o \Delta_Y g^o c = f^o g^o c \subseteq f^o b g^o c = f^o b g^o c'$

(ii) (Identity) let $(X,A) \in$ QUnif ^{op}, and $1_{(X,A)} : (X,A) \to (X,A)$ as $x \mapsto x$ need that $(1_{(X,A)})^* = 1_{(X,A)^*}$
LHS=$(1_{(X,A)})^* = (1_{(X,A)})^o.A = 1_{(X,A)}.A = A$.
Now, it's required that A is the identity of (X,A) in ProMod.
So, fix $\phi : (X,A) \to\!\!\!\!\circ\!\!\!\to (Y,B)$, need to show $\phi.A = \phi$
As ϕ is a promodule, $\phi.A \leq \phi$ and as A is quasi-uniformity on X,
$\forall a \in A, \Delta_X \subseteq a \implies \forall a \in A, \forall p \in \phi, p = p\Delta_X \subseteq pa \implies \phi \leq \phi.A$
Also, fix $\psi : (Y,B) \to\!\!\!\!\circ\!\!\!\to (X,A)$, need to show $A.\psi = \psi$
As ψ is a promodule, $A.\psi \leq \psi$ and as A is quasi-uniformity on X,
$\forall a \in A, \Delta_X \subseteq a \implies \forall a \in A, \forall q \in \psi, q = \Delta_X q \subseteq aq \implies \psi \leq \psi.A$

□

Proposition 2.3 (Proposition 1)**.** Fix a uniformly continuous map, $f : (X,A) \to (Y,B)$

(a) f is fully faithful $\iff A \geq f^o.B.f$

(b) f is fully dense $\iff \forall b \in B, \exists b' \in B$ such that $b' \subseteq bff^ob$

(c) f is topologically dense $\iff \forall b \in B, \Delta_Y \subseteq b \circ f \circ f^o \circ b$

(d) f is fully dense $\iff$ f is topologically dense

Proof.

(a) (i) ($\implies$) Let f be fully faithful i.e. $f^*.f_* = A \implies f^o.B.B.f = A$
Need to show that $A = f^o.B.f$ i.e. $A \leq f^o.B.f$ and $A \geq f^o.B.f$
By hypothesis and quasi-uniformity of B, $A \geq f^o.B.B.f \geq f^oB.f$
To show $A \leq f^o.B.f$, need that $\forall b \in B, \exists a \in A : a \subseteq f^obf$
Fix $b \in B$, hypothesis gives that $f^o.B.B.f \leq A$ so,
$\exists a \in A : a \subseteq f^obbf$ and also, by quasi-uniformity of B, for $b, \exists b' \in B : b'b' \subseteq b \implies f^ob'b'f \subseteq f^obf$
Combining the above two inequalities, $a \subseteq f^obbf \subseteq f^obf$

(ii) ($\impliedby$) Let $A = f^o.B.f$ need to show $A = f^o.B.B.f$ i.e. $A \geq f^oB.B.f$ and $A \leq f^oB.B.f$
To show $A \geq f^o.B.B.f$, need to show that $\forall a \in A, \exists b, b' \in B : a \supseteq f^obb'f$
Have that $A \geq f^o.B.f$ and $B.B \leq B$
So, fix $a \in A$, now $\exists b \in B : a \subseteq f^obf$ and for this b, $\exists b' \in B : b'b' \subseteq b$. Therefore, $a \supseteq f^obf \supseteq f^ob'b'f$
To show $A \leq f^o.B.B.f$, need $\forall b, b' \in B, \exists a \in A : a \subseteq f^obb'f$
Before that, uniform continuity of f along with Lemma 2.1.1 gives that
$f.A \leq B.f \implies A = f^of.A \leq f^o.B.f$
So, fix $b, b' \in B$, now, as ,
$A \leq f^o.B.f$ giving
$\exists a \in A : a \subseteq f^obf$ and $\exists a' \in A : a' \subseteq f^ob'f \implies \Delta_X \subseteq f^ob'f.$
Therefore $a = a\Delta_X \subseteq (f^obf)(f^ob'f) \subseteq f^obb'f$

(b) (i) ($\implies$) Let f be fully dense i.e. $B = f_*f^* = B.f.f^o.B$. showing that $\forall b \in B, \exists b' \in B : b' \subseteq bff^ob$:
So, fix $b \in B$, as $B \leq B.f.f^o.B$, there exists $b' \in B$ such that $b' \subseteq bff^ob$.

(ii) ($\impliedby$) Suppose $\forall b \in B, \exists b' \in B : b' \subseteq bff^ob$. This gives $B \leq B.f.f^o.B$, in order to show equality, also need $B \geq B.f.f^o.B$. By quasi-uniformity of B, for any $b \in B$, $\exists b' \in B : b'b' \subseteq b$. Now, by Lemma 1.9.2,

$$ff^o \subseteq \Delta_Y \implies b'ff^ob' \subseteq b'\Delta_Y b' = b'b' \subseteq b$$

(c) (i) ($\implies$) Let f be topologically dense, going to show that $\forall b \in B$, $(y,y) \in bff^ob$. So, fix any $b \in B$ and $y \in Y$. Now, by definition of $\overline{f(X)} = Y$, we get

$$\exists x \in X \text{ such that } (f(x), y) \in b \text{ and } (y, f(x)) \in b$$

Re-writing the above statement in terms of relations, and considering f as a relation:

$$(f(x), y) \in b \text{ gives } x(b \circ f)y \text{ i.e. } y \in (b \circ f)(x) \quad (1)$$
$$(y, f(x)) \in b \text{ gives } f(x) \subseteq b(y) \quad (2)$$

Repeatedly applying Lemma 1.9.3 to (2),

$$f(x) \subseteq b(y) \implies \big(f \circ f^o\big)(f(x) \subseteq \big(f \circ f^o\big)b(y) \implies \big(f \circ f^o \circ f\big)(x) \subseteq \big(f \circ f^o \circ b\big)(y)$$

Applying Lemma 1.9.1 to the above statement gives that

$$f(x) = \big(f \circ f^o \circ f\big)(x) \subseteq \big(f \circ f^o \circ b\big)(y)$$

Applying Lemma 1.9.3 and then using (1) to this inequality completes the result:

$$f(x) \subseteq \big(ff^o b\big)(y) \implies \big(b \circ f\big)(x) \subseteq \big(bff^o b\big)(y) \implies y \in \big(bff^o b\big)(y) \text{ i.e. } y\big(bff^o b\big)y$$

(ii) ($\Longleftarrow$) Fix any $y \in Y$ and $b \in B$. Also, suppose that $\Delta_Y \leq bff^o b$. As f is a function with domain as X, $f^o : Y \to X$, $\phi \neq (f^o \circ b)(y) \subseteq X$. So, fix $x \in (f^o \circ b)(y)$, going to show that $(f(x), y) \in b$ and $(y, f(x)) \in b$. Again, while viewing f as a relation.

$$\Delta_Y \leq bff^o b \implies \Delta_Y(y) \subseteq bff^o b(y) = \big(bf\big)(f^o b(y))$$

Using Lemma 1.9.3 on the above statement, gives $y \in (bf)(x)$ i.e. $(f(x), y) \in b$.
Applying Lemma 1.9.3 to f, and then using Lemma 1.9.4,

$$ff^o \subseteq \Delta_Y \implies ff^o b \subseteq \Delta_Y b = b$$

Thus $ff^o b(y) \subseteq b(y)$ and hence $f(x) \subseteq b(y) \implies (y, f(x)) \in b$

(d) (i) ($\Longrightarrow$) Let f be topologically dense. As B is a quasi-uniformity, for any $b \in B$,

$$\exists b' \in B : b'b' \subseteq b \text{ and } \Delta_Y \subseteq b' \implies b' = b'\Delta_Y \subseteq b'b' \subseteq b \tag{3}$$

By the characterisation of topologically dense in (c), have that $\Delta_Y \subseteq b'ff^o b'$. Now, using the (3) and Lemma 1.9.2,

$$\Delta_Y \subseteq b'ff^o b' \implies b' = b'\Delta_Y \subseteq b'b'ff^o b' \subseteq bff^o b' \subseteq bff^o b$$

Hence, we have $b' \in B : b' \subseteq bff^o b$ giving us that f is fully dense (from (b)).

(ii) ($\Longleftarrow$) From (b), we have for $b \in B$, the existstence of $b' \in B$ such that $b' \subseteq bff^o b$. As B is a quasi-uniformity, $\Delta_Y \subseteq b'$. So, $\Delta_Y \subseteq bff^o b$, and from (c), this gives us that f is topologically dense.

□

Definiton 2.3 (PX)**.** $PX := \{\psi : \psi : (X, A) \nrightarrow 1 \text{ is a promodule}\}$

Definiton 2.4 $(\tilde{a})$**.** for any $a \in A$, $\tilde{a}$ is defined to be a relation $PX \to PX$ as

$$\text{for } \phi, \psi \in PX, \phi\tilde{a}\psi \text{ only if } \phi \leq \psi.a$$

Proposition 2.4 (Prorelation $\tilde{A}$)**.** The set, $\tilde{A} := \{\tilde{a} : a \in A\}$ defines a quasi-uniformity on PX.

Proof. First, need to show that $\tilde{A}$ is a prorelation,

(i) (Partial order) Define, for any two relations $\tilde{a}, \tilde{b} : PX \to PX$, that $\tilde{a} \leq \tilde{b}$ only if $a \subseteq b$

(ii) (Down-Directed) Need that $\forall \tilde{a}, \tilde{b} \in \tilde{A}, \exists \tilde{c} \in A : c \subseteq a, b$
$\tilde{a}, \tilde{b} \in A \implies a, b \in A \implies \exists c \in A : c \subseteq a, b \implies \tilde{c} \leq \tilde{a}, \tilde{b}$

(iii) (Upset) Need that, for any relation $l : PX \to PX$, if $\exists \tilde{k} \in \tilde{A}$ such that $l \geq \tilde{k}$, then $l \in \tilde{A}$
Fix any $k : PX \to PX$, and $\tilde{k} \in \tilde{A}$ such that $l \geq \tilde{k}$
Now, k is a relation between promodules $X \nrightarrow 1$. Thus, it can be thought of as a relation on X,
$a := \{(x, y) : x \in domain(\psi) and y \in domain(\phi) \text{ whenever } \exists \psi, \phi \in PX : \psi l \phi\}$
So, $l = \tilde{a}$ and thus, $\tilde{a} \geq \tilde{k} \implies a \supseteq k \implies a \in A \implies l \in \tilde{A}$

Now to show that the other two conditions hold,

(i) need that $\forall \tilde{a} \in \tilde{A}, \forall \psi \in PX, \psi\tilde{a}\psi$
So, need to show that $\psi \leq \psi.a$ i.e. $\forall p \in \psi, \exists q \in \psi : q \subseteq p.a$. Take $q := p$, and as A is a quasi-uniformity, $\Delta_X \subseteq a \implies p = p.\Delta_X \subseteq p.a$

(ii) Need that $\forall \tilde{a} \in \tilde{A}, \exists \tilde{b} \in \tilde{A} : \tilde{b}\tilde{b} \leq \tilde{a}$
Before that, showing , for any $x, y \in A, \tilde{x}\tilde{y} \leq \widetilde{xy}$ i.e. $\forall \psi, \phi \in PX$, $\psi(\tilde{x}.\tilde{y})\phi \implies \psi\widetilde{xy}\phi$
Let $\psi_1(\tilde{a}.\tilde{b})\psi_3 \implies \exists \psi_2 : \psi_1\tilde{b}\psi_2\tilde{a}\psi_3 \implies \psi_1 \leq \psi_2.b$ and $\psi_2 \leq \psi_3.a \implies \psi_1 \leq \psi_2.b \leq \psi_3.ab \implies \psi_1(\widetilde{ab})\psi_3$
Fix any $\tilde{a} \in \tilde{A} \implies a \in A \implies \exists b \in A : b \circ b \subseteq a \implies \widetilde{bb} \leq \tilde{a} \implies \tilde{b}\tilde{b} \leq \widetilde{bb} \leq \tilde{a}$

□

Proposition 2.5 (Yoneda Embedding)**.**

For a quasi-uniform space (X, A), function $y_X : X \to PX$ is defined by $x \mapsto x^*$ for $x \in X$.

(a) $y_X : (X, A) \to (PX, \tilde{A})$ is a uniformly continuous map

(b) $y_X : (X, A) \to (PX, \tilde{A})$ is fully faithful

Proof.

(a) In order to show y_X is uniformly continuous, need to show that $y_X.A \leq \tilde{A}.y_X$. By definition of $\leq$, need $\forall a \in A, \exists b \in A : y_X \circ b \subseteq \tilde{a} \circ y_X$. Applying the relations to some element, x of the set X:

$$\big(y_X \circ b\big)(x) \subseteq \big(\tilde{a} \circ y_X\big)(x) \implies y_X(b(x)) \subseteq \tilde{a}(x^*) \tag{4}$$

So, for the condition given by (4) to hold, if $y \in b(x)$, then it's required that $y^* = y_X(y) \in \tilde{a}(x^*)$ i.e. $x^*\tilde{a}y^*$. Using the definition of x^*, y^* and $\tilde{a}$,

$$x^*\tilde{a}y^* \iff x^o.A \leq y^o.A.a \iff \forall a' \in A, \exists a'' \in A : x^o a'' \subseteq y^o a' a \tag{5}$$

Now, fix any $a \in A$, $x \in X$. Thus, quasi-uniformity of A, gives $a'' \in A$ such that $a''a'' \subseteq a$.Also, choose some $y \in a''(x)$. Hence, in order to show that the condition from (5) holds, need that $\forall b \in A, x^o a'' \subseteq y^o ba$, and by applying the relations to an element z gives the following condition

$$\forall b \in B, \forall x \in X \ , \ \big(x^o a''\big)(z) \subseteq \big(y^o ba\big)(z) \tag{6}$$

Examining the left side of (6),

$$\big(x^o a''\big)(z) = x^o(a''(z)) = \begin{cases} \phi & \text{if } x \notin a''(z) \\ \star & \text{if } \in a''(z) \end{cases}$$

Thus, to show that (6) holds, need to show that (for any $b \in A$ and $z \in X$):

$$x \in a''(z) \implies z(y^o ba)\star \text{ i.e. } y \in (ba)(z) \tag{7}$$

To show that (7) holds, fix any $z \in X : x \in a''(z)$. Also, by our choice of y, have that $y \in a''(x)$. And as $b \in A$, it's reflexive, giving that $y \in b(y)$. So, by composition of relations, we get:

$$za''x \ , \ xa''y \text{ and } yby \implies z(a''a''b)y \implies z(ab)y \text{ i.e. } y \in (ba)(z)$$

(b) By using Proposition2.3(a), need to show that $A \geq y_X^o.\tilde{A}.y_X$ i.e. $\forall a \in A, \exists \tilde{b} \in \tilde{A} : a \supseteq y_X^o \ \tilde{b} \ y_X$. Applying to an element, $x \in X$ gives the condition

$$\Big(y_X^o \ \tilde{b} \ y_X\Big)(x) \subseteq a(x) \implies \Big(y_X^o \ \tilde{b}\Big)(x^*) = y_x^o\Big(\tilde{b}(x^*)\Big) \subseteq a(x) \tag{8}$$

Thus, if $y^* \in PX$ such that $x^*\tilde{b}y^*$, then $y \in y_x^o\Big(\tilde{b}(x^*)\Big)$. Now, for (8) to hold, $y \in a(x)$ i.e. xay. Thus, need only to show that for any $a \in A, \exists b \in A$ such that $\forall x, y \in X, x^*\tilde{b}y^* \implies xay$. So, fix $a \in A$, and take $b \in A : bb \subseteq a$. Now, let $x^*\tilde{b}y^*$ i.e. $x^o.A \leq y^o.A.b$. Hence, $\exists c \in A : x^o c \subseteq y^o bb$. And as c is reflexive,

$$xcx \implies x(cx^o)\star \implies x(bby^o)\star \implies x(bb)y \implies xay$$

□

Lemma 2.4.1. If A is a quasi-uniformity on a set X, then $A.A = A$

Proof. Fix any $a \in A$, as A is a quasi-uniformity, $\exists b \in A : bb \subseteq a$, we get that $A.A \leq A$. And as A is a prorelation, and is hence down-directed, $\exists c \in A : a.a \supseteq c$, giving that $A.A \geq A$ □

Theorem 2.1 (Yoneda Lemma). *For every $\psi \in PX$, in the following digram,*

$$\begin{tikzcd} X \arrow[r, "(y_X)_*"] \arrow[dr, "\psi"'] & PX \arrow[d, "\psi^*"] \\ & 1 \end{tikzcd}$$

(a) $\psi \geq \psi^*.(y_X)^*$

(b) $\psi \in \overline{y_X(X)} \implies \psi \leq \psi^*.(y_X)_*$

Proof. (a) By definition, $(y_X)_* = \tilde{A}.y_X$, and $\psi^* = \psi^o.\tilde{A}$. Need that $\psi \geq (y_X)_*.\psi^* = \psi^o.\tilde{A}.\tilde{A}.y_X$. And applying Lemma 2.4.1 to $\tilde{A}$, the required condition becomes $\psi \geq \psi^o.\tilde{A}.y_X$ Fix $p \in \psi$, we will find $a \in A : p \supseteq \psi^o a y_X$. Examining the right side of the condition, (for any $a \in A$, $x \in X$)

$$\Big(\psi^o.\tilde{a}.y_X\Big)(x) = \psi^o.\tilde{a}(x^*) = \psi^o\big(\tilde{a}(x^*)\big) = \begin{cases} \phi & \text{if } \psi \notin \tilde{a}(x^*) \\ \star & \text{if } \psi \in \tilde{a}(x^*) \end{cases} \tag{9}$$

In case $\psi \notin \tilde{a}(x^*)$, the condition holds trivially. As ψ is a promodule, $\psi.A \leq \psi$ gives $\exists q \in \psi, a \in A : qa \subseteq p$. Thus, fix $x \in X$ and $\psi \in PX$ such that $x^*\tilde{a}\psi$. We will now show that $xp\star$. Using the definition of $\tilde{a}$,

$$x^*\tilde{a}\psi \implies x^o.A \leq \psi.a \implies \exists b \in A : x^o b \subseteq qa \implies \forall z \in X, \big(x^o b\big)(z) \subseteq (qa)(z) \tag{10}$$

Thus, in particular for $z = x$, as b is reflexive, xbx, which gives:

$$\big(x^o b\big)(x) \subseteq (qa)(x) \implies x^o x \subseteq (qa)(x) \implies \star \in (qa)(x) \tag{11}$$

But, as $qa \subseteq p$, (11) gives that $xp\star$.

(b) Suppose $\psi \in \overline{y_X(X)}$, need to show $\psi \leq \psi^*.(y_X)_* = \psi^o.\tilde{A}.y_X$ i.e. for $a \in A$, $\exists p \in \psi : p \subseteq \psi^o.\tilde{a}.y_X$. For any $x \in domain(p)$, the condition requires:

$$p(x) \subseteq \psi^o.\tilde{a}.y_X(x) = \psi^o\big(\tilde{a}(x^*)\big) \tag{12}$$

By definition of p, for (12) to hold, need that $xp\star \implies \psi \in \tilde{a}(x^*)$. Fix any $a \in A$, we will find $p \in \psi$ such that (12) holds. By quasi-uniformity of A, $\exists b \in A : bb \subseteq a$. From Proposition 2.5(a), y_X is uniformly continuous, $y_X.A \leq \tilde{A}.y_X$ giving that $\exists c \in A : y_x c \subseteq \tilde{b}y_X$. Thus, for any $z, w \in X$ such that zcw,

$$\big(y_X c\big)(z) \subseteq \big(\tilde{b}y_X\big)(z) \implies y_X(c(z)) \subseteq \tilde{b}(z^*) \implies w^* \in \tilde{b}(z^*) \text{ i.e. } z^*\tilde{b}w^* \tag{13}$$

As A is a quasi-uniformity, $\exists d \in A : dd \subseteq c$. Also, because A is a down- directed set, $\exists a' \in A : a' \subseteq b, d$. This along with (13) gives that for any $x, y \in X$

$$x(a'a')y \implies x(dd)y \implies xcy \implies x^*\tilde{b}y^* \tag{14}$$

Now, because $\psi \in \overline{y_X(X)}$, we get $\exists x^* \in y_X(X)$ such that $\psi\tilde{a'}x^*$ and $x^*\tilde{a'}\psi$. By definition of $\tilde{a}$, $\psi\tilde{a'}x^*$ gives

$$\psi \leq x^o.A.a' \implies \exists p \in \psi : p \subseteq x^o a'a' \tag{15}$$

Fix any $z \in X : zp\star$, using (15) and (14) gives:

$$zp\star \overset{z}{\Rightarrow} (x^o a'a')\star \overset{(15)}{\Longrightarrow} z(a'a')x \overset{(14)}{\Longrightarrow} z^*\tilde{b}x^* \tag{16}$$

Finally, by definition of the partial order on $\tilde{A}$,$a' \subseteq b \implies \tilde{a'} \subseteq \tilde{b}$. Therefore, $x^*\tilde{a'}\psi \implies x^*\tilde{b}\psi$. Now, using (16), $z^*\tilde{b}x^*$ and $x^*\tilde{b}\psi$ gives the desired result $z^*\tilde{b}x^*$.

□

Corollary 2.1. *For $\psi \in PX$, $\psi \in \overline{y_X(X)} \iff \psi$ is a right-adjoint.*