

1 Definitions

Definiton 1.1 (Prorelation). A partial-ordered set of relations $X \rightarrow Y$, which is down-directed and an upper set. i.e A set, $P \subseteq \mathcal{P}(X \times Y)$ such that

- (i) A partial-order defined to be containment as relations, $r \subseteq s$ only if $\forall (x, y) \in X \times Y, (x, y) \in r \implies (x, y) \in s$
- (ii) (Down-directed), $\forall r, s \in P, \exists t \in P$ such that $t \subseteq r$ and $t \subseteq s$
- (iii) (Up-set) for any relation $u : X \rightarrow Y$, if $\exists p \in P$ such that $p \leq u$ then $u \in P$

Definiton 1.2 (Composition of prorelations). Prorelations can be composed by taking all compositions of their elements as relations: for prorelations $P : X \rightarrow Y$ and $Q : Y \rightarrow Z$,

$$Q.P := \{q \circ p : p \in P \text{ and } q \in Q\}$$

Definiton 1.3 (Comparison of Prorelations). Two prorelations with same domain, co-domain are comparable as

$$\text{for } P, Q : X \rightarrow Y, P \leq Q \text{ if } \forall q \in Q, \exists p \in P \text{ such that } p \subseteq q$$

Definiton 1.4 (Quasi-uniformity). A prorelation on a set X , $P : X \rightarrow X$ is a quasi-uniformity if it follows :

- i $\forall p \in P$, for any $x \in X$, $(x, x) \in p$ i.e. xpx
- ii $\forall p \in P, \exists p' \in P$ such that $p' \circ p' \subseteq p$

And in this case, (X, A) is called a *quasi-uniform space*.

Definiton 1.5 (Uniformly Continuous function). A function, $f : X \rightarrow Y$ is called a uniformly continuous function,

$$f : (X, A) \rightarrow (Y, B) \text{ if, } \forall b \in B, \exists a \in A \text{ such that } f \circ a \subseteq b \circ f. \text{ meaning that } f.A \leq B.f \text{ or } \begin{array}{ccc} X & \xrightarrow{f} & Y \\ A \downarrow & \leq & \downarrow B \\ X & \xrightarrow{f} & Y \end{array}$$

Definiton 1.6 (Promodule). A prorelation, $\phi : X \multimap Y$ is called a promodule $\phi : (X, A) \multimap (Y, B)$ if it obeys: $\phi.A \leq \phi$ and $B.\phi \leq \phi$ where $.$ denotes composition as prorelations.

Definiton 1.7 (Comparison of Promodules). Promodules with same domain and co-domain are compared as prorelations, for $\phi, \psi : (X, A) \multimap (Y, B)$, $\phi \subseteq \psi$, only if $\phi \leq \psi$.

Definiton 1.8 (Composition of Promodules). Promodules are composed as prorelations.

For promodules $\phi : (X, A) \multimap (Y, B)$ and $\psi : (Y, B) \multimap (Z, C)$, $\psi\phi := \psi.\phi = \{q \circ p : p \in \phi \text{ and } q \in \psi\}$

Definiton 1.9 (Opposite relation). For relation $r : X \rightarrow Y$, r^o is defined to be a relation $r^o : Y \rightarrow X$ as

$$\forall (x, y) \in X \times Y, (x, y) \in r \iff (y, x) \in r^o$$

Lemma 1.9.1. For any relation $r : X \rightarrow Y$, $r^o \circ r = \Delta_X$

Lemma 1.9.2. For any relation $r : X \rightarrow Y$, $r \circ r^o \subseteq \Delta_Y$

Lemma 1.9.3. For relations $r, s : X \rightarrow Y$ and $t : Y \rightarrow Z$, for any $x, x' \in X$, $r(x) \subseteq s(x') \implies (t \circ r)(x) \subseteq (t \circ s)(x')$

Lemma 1.9.4. For relations $r : X \rightarrow Y$ and $s, t : Y \rightarrow Z$, $s \subseteq t \implies (s \circ r) \subseteq (t \circ r)(x)$

Definiton 1.10 $((-)_*)$.

Definiton 1.11 $((-)^*)$.

Definiton 1.12 (Fully Faithful).

Definiton 1.13 (Fully Dense).

Definiton 1.14 (Topologically Dense).

2 Propositions

Definiton 2.1 (QUnif). QUnif is defined to be the category having quasi-uniform spaces as objects, and uniformly continous maps between them as morphisms.

Lemma 2.1.1. QUnif does define a category, as

- i Composition
- ii Identity

Definiton 2.2 (ProMod).

Lemma 2.2.1. ProMod does define a category, as

- i Composition
- ii Identity

Proposition 2.1 $((-)_* : \text{QUnif} \rightarrow \text{ProMod} \text{ is a Functor})$.

Proof. □

Proposition 2.2 $((-)^* : \text{QUnif}^{\text{op}} \rightarrow \text{ProMod} \text{ is a Functor})$. Defined as fixing objects and taking morphisms to their image under $(-)^*$

- (a) for $(X, A) \in \text{QUnif}^{\text{op}}$, $(X, A)^* := (X, A) \in \text{ProMod}$
- (b) for $f : (X, A) \rightarrow (Y, B)$ in QUnif, $f^* := f^o.B$

Proof.

Showing that $f^o.B : (Y, B) \rightrightarrows (X, A)$ is a promodule

So, need to show $f^o.B$ a prorelation $Y \rightarrow X$ and that $(f^o.B).B \sqsubseteq f^o.B$ and $A.(f^o.B) \sqsubseteq f^o.B$

To show prorelation,

- (i) (Partial-order) Inclusion of relations i.e. for $k = f^o \circ b$ and $k' = f^o \circ b'$ in $f^o.B$, $k \subseteq k' \iff b \subseteq b'$
- (ii) (Down directed) for $k, k' \in f^o.B$, need that $\exists l \in f^o.B$ such that $l \subseteq k, k'$
 Fix $k, k' \in f^o.B \implies \exists b, b' \in B : k = f^o \circ b$ and $k' = f^o \circ b'$
 And as B is a quasi-uniformity, it's down directed so, $\exists c \in B : c \subseteq b, b' \implies l := f^o \circ c \subseteq k, k'$
- (iii) (Up-set) for a relation $l : Y \rightarrow X$ and $k \in f^o.B$ such that $l \supseteq k$, need $l \in f^o.B$
 Let $b \in B$ be such that $k = f^o \circ b$ and define $b' := \{(y, y') : y \in \text{domain}(l) \text{ and } y' \in (f^o)^{-1}(l(y))\}$
 As $l \supseteq k = f^o \circ b$, $\text{domain}(b') = \text{domain}(l) \supseteq \text{domain}(b)$
 and $\text{range}(l) \supseteq \text{range}(f^o \circ b) \implies \forall y \in \text{domain}(b), \text{range}(b') = (f^o)^{-1}(l(y)) \supseteq (f^o)^{-1}(f^o \circ b) = \text{range}(b)$
 Now, by definition of b' , $f^o \circ b' \supseteq l$. To show $f^o \circ b \subseteq l$,
 $(x, y) \in f^o \circ b \implies \exists z \in Y : (x, z) \in b' \text{ and } (z, y) \in f^o \implies x \in \text{domain}(l) \text{ and } z \in l(x) \text{ i.e. } (x, z) \in l$

To show $(f^o.B).B \leq f^o.B$, need that $\forall b \in B, \exists b' \in B : f^o \circ b' \circ b' \subseteq f^o \circ b$,

Fix any $b \in B$ as B is a quasi-uniformity, $\exists b' \in B : b' \circ b' \subseteq b \implies f^o \circ b' \circ b' \subseteq f^o \circ b$

To show $A.(f^o.B) \leq f^o.B$, need that $\forall b \in B, \exists b' \in B, a \in A : a \circ f^o \circ b' \subseteq f^o \circ b$,

As f is uniformly continuous, $f.A \leq B.f$ i.e. $\forall b \in B, \exists a \in A : f \circ a \subseteq b \circ f \implies a = f^o \circ f \circ a \subseteq f^o \circ b \circ f$

Fix any $b \in B$, so, $\exists b' \in B : b' \circ b' \subseteq b$ (for brevity, omitting \circ to explicitly denote composition)

And, for this b' , $\exists a : a \subseteq f^o b' f \implies a f^o b' \subseteq f^o b' f f^o b' \subseteq f^o b' b' \subseteq f^o b \implies a f^o b' \subseteq f^o b$

Now, need to show that $(-)^*$ respects composition and identity.

- (i) (Composition) let f, g be uniformly continuous, $(X, A) \xrightarrow{f} (Y, B) \xrightarrow{g} (Z, C)$ need that $(g \circ f)^* = f^*.g^*$

LHS $= (g \circ f)^* = (g \circ f)^o.C = (f^o \circ g^o).C$ and RHS $= f^*.g^* = (f^o.B).(g^o.C)$

For equality, showing that LHS \geq RHS and LHS \leq RHS:

To show $(f^o \circ g^o).C \geq (f^o.B).(g^o.C)$, need that $\forall c \in C, \exists b \in B, c' \in C : f^o g^o c \supseteq f^o b g^o c'$

Fix any $c \in C$, so, $\exists c' \in C : c' \circ c' \subseteq c \implies f^o g^o c \supseteq f^o g^o (c' c') = f^o g^o (c' \Delta_Z c') \supseteq f^o g^o c' (g g^o) c'$

By uniform continuity of g , for $c' \in C, \exists b \in B : g b \subseteq c' g$

Thus, $f^o g^o c \supseteq f^o g^o (c' g) g^o c' \supseteq f^o (g^o g) b g^o c' = f^o b g^o c'$.

To show $(f^o \circ g^o).C \leq (f^o.B).(g^o.C)$, need that $\forall b \in B, c \in C, \exists c' \in C : f^o g^o c \subseteq f^o b g^o c'$

Fix any $c \in C, b \in B$ will show that $c' := c$ works:

As B is a quasi-uniformity, $\Delta_Y \subseteq b \implies f^o \Delta_Y g^o c = f^o g^o c \subseteq f^o b g^o c = f^o b g^o c'$

- (ii) (Identity) let $(X, A) \in \mathbf{QUnif}^{op}$, and $1_{(X,A)} : (X, A) \rightarrow (X, A)$ as $x \mapsto x$ need that $(1_{(X,A)})^* = 1_{(X,A)^*}$
 $\text{LHS} = (1_{(X,A)})^* = (1_{(X,A)})^o.A = 1_{(X,A)}.A = A$.
 Now, it's required that A is the identity of (X, A) in \mathbf{ProMod} .
 So, fix $\phi : (X, A) \rightarrow (Y, B)$, need to show $\phi.A = \phi$
 As ϕ is a promodule, $\phi.A \leq \phi$ and as A is quasi-uniformity on X ,
 $\forall a \in A, \Delta_X \subseteq a \implies \forall a \in A, \forall p \in \phi, p = p\Delta_X \subseteq pa \implies \phi \leq \phi.A$
 Also, fix $\psi : (Y, B) \rightarrow (X, A)$, need to show $A.\psi = \psi$
 As ψ is a promodule, $A.\psi \leq \psi$ and as A is quasi-uniformity on X ,
 $\forall a \in A, \Delta_X \subseteq a \implies \forall a \in A, \forall q \in \psi, q = \Delta_X q \subseteq aq \implies \psi \leq \psi.A$

□

Proposition 2.3 (Proposition 1). Fix a uniformly continuous map, $f : (X, A) \rightarrow (Y, B)$

- (a) f is fully faithful $\iff A \geq f^o.B.f$
 (b) f is fully dense $\iff \forall b \in B, \exists b' \in B$ such that $b' \subseteq bff^ob$
 (c) f is topologically dense $\iff \forall b \in B, \Delta_Y \subseteq b \circ f \circ f^o \circ b$
 (d) f is fully dense $\iff f$ is topologically dense

Proof.

- (a) (i) (\implies) Let f be fully faithful i.e. $f^*.f_* = A \implies f^o.B.B.f = A$
 Need to show that $A = f^o.B.f$ i.e. $A \leq f^o.B.f$ and $A \geq f^o.B.f$
 By hypothesis and quasi-uniformity of B , $A \geq f^o.B.B.f \geq f^o.B.f$
 To show $A \leq f^o.B.f$, need that $\forall b \in B, \exists a \in A : a \subseteq f^obf$
 Fix $b \in B$, hypothesis gives that $f^o.B.B.f \leq A$ so,
 $\exists a \in A : a \subseteq f^obb f$ and also, by quasi-uniformity of B , for $b, \exists b' \in B : b'b' \subseteq b \implies f^ob'b'f \subseteq f^obf$
 Combining the above two inequalities, $a \subseteq f^obb f \subseteq f^obf$
 (ii) (\impliedby) Let $A = f^o.B.f$ need to show $A = f^o.B.B.f$ i.e. $A \geq f^o.B.B.f$ and $A \leq f^o.B.B.f$
 To show $A \geq f^o.B.B.f$, need to show that $\forall a \in A, \exists b, b' \in B : a \supseteq f^obb'f$
 Have that $A \geq f^o.B.f$ and $B.B \leq B$
 So, fix $a \in A$, now $\exists b \in B : a \subseteq f^obf$ and for this $b, \exists b' \in B : b'b' \subseteq b$. Therefore, $a \supseteq f^obf \supseteq f^ob'b'f$
 To show $A \leq f^o.B.B.f$, need $\forall b, b' \in B, \exists a \in A : a \subseteq f^obb'f$
 So, fix $b, b' \in B$, now, by hypothesis,
 NOPE, USE UNIFORM CTY OF f HERE $A \leq f^o.B.f$ giving
 $\exists a \in A : a \subseteq f^obf$ and $\exists a' \in A : a' \subseteq f^ob'f \implies \Delta_X \subseteq f^ob'f$.
 Therefore $a = a\Delta_X \subseteq (f^obf)(f^ob'f) \subseteq f^obb'f$
 (b) (i) (\implies) Let f be fully dense i.e. $B = f_*f^* = B.f.f^o.B$. showing that $\forall b \in B, \exists b' \in B : b' \subseteq bff^ob$:
 So, fix $b \in B$, as $B \leq B.f.f^o.B$, there exists $b' \in B$ such that $b' \subseteq bff^ob$.
 (ii) (\impliedby) Suppose $\forall b \in B, \exists b' \in B : b' \subseteq bff^ob$. This gives $B \leq B.f.f^o.B$, in order to show equality, also
 need $B \geq B.f.f^o.B$. By quasi-uniformity of B , for any $b \in B, \exists b' \in B : b'b' \subseteq b$. Now, by Lemma 1.9.2,

$$ff^o \subseteq \Delta_Y \implies b'ff^ob' \subseteq b'\Delta_Y b' = b'b' \subseteq b$$

- (c) (i) (\implies) Let f be topologically dense, going to show that $\forall b \in B, (y, y) \in bff^ob$. So, fix any $b \in B$ and
 $y \in Y$. Now, by definition of $\overline{f(X)} = Y$, we get

$$\exists x \in X \text{ such that } (f(x), y) \in b \text{ and } (y, f(x)) \in b$$

Re-writing the above statement in terms of relations, and considering f as a relation:

$$(f(x), y) \in b \text{ gives } x(b \circ f)y \text{ i.e. } y \in (b \circ f)(x) \quad (1)$$

$$(y, f(x)) \in b \text{ gives } f(x) \subseteq b(y) \quad (2)$$

Repeatedly applying Lemma 1.9.3 to (2),

$$f(x) \subseteq b(y) \implies (f \circ f^o)(f(x)) \subseteq (f \circ f^o)b(y) \implies (f \circ f^o \circ f)(x) \subseteq (f \circ f^o \circ b)(y)$$

Applying Lemma 1.9.1 to the above statement gives that

$$f(x) = (f \circ f^o \circ f)(x) \subseteq (f \circ f^o \circ b)(y)$$

Applying Lemma 1.9.3 and then using (1) to this inequality completes the result:

$$f(x) \subseteq (ff^ob)(y) \implies (b \circ f)(x) \subseteq (bff^ob)(y) \implies y \in (bff^ob)(y) \text{ i.e. } y(bff^ob)y$$

- (ii) (\Leftarrow) Fix any $y \in Y$ and $b \in B$. Also, suppose that $\Delta_Y \leq bff^ob$. As f is a function with domain as X , $f^o : Y \rightarrow X$, $\phi \neq (f^o \circ b)(y) \subseteq X$. So, fix $x \in (f^o \circ b)(y)$, going to show that $(f(x), y) \in b$ and $(y, f(x)) \in b$. Again, while viewing f as a relation.

$$\Delta_Y \leq bff^ob \implies \Delta_Y(y) \subseteq bff^ob(y) = (bf)(f^ob(y))$$

Using Lemma 1.9.3 on the above statement, gives $y \in (bf)(x)$ i.e. $(f(x), y) \in b$.
Applying Lemma 1.9.3 to f , and then using Lemma 1.9.4,

$$ff^o \subseteq \Delta_Y \implies ff^ob \subseteq \Delta_Y b = b$$

Thus $ff^ob(y) \subseteq b(y)$ and hence $f(x) \subseteq b(y) \implies (y, f(x)) \in b$

- (d) (i) (\implies) Let f be topologically dense. As B is a quasi-uniformity, for any $b \in B$,

$$\exists b' \in B : b'b' \subseteq b \text{ and } \Delta_Y \subseteq b' \implies b' = b'\Delta_Y \subseteq b'b' \subseteq b \quad (3)$$

By the characterisation of topologically dense in (c), have that $\Delta_Y \subseteq b'ff^ob'$. Now, using the (3) and Lemma 1.9.2,

$$\Delta_Y \subseteq b'ff^ob' \implies b' = b'\Delta_Y \subseteq b'b'ff^ob' \subseteq bff^ob' \subseteq bff^ob$$

Hence, we have $b' \in B : b' \subseteq bff^ob$ giving us that f is fully dense (from (b)).

- (ii) (\Leftarrow) From (b), we have for $b \in B$, the existstence of $b' \in B$ such that $b' \subseteq bff^ob$. As B is a quasi-uniformity, $\Delta_Y \subseteq b'$. So, $\Delta_Y \subseteq bff^ob$, and from (c), this gives us that f is topologically dense. \square

Definiton 2.3 (PX). $PX := \{\psi : \psi : (X, A) \rightarrow 1 \text{ is a promodule}\}$

Definiton 2.4 (\tilde{a}). for any $a \in A$, \tilde{a} is defined to be a relation $PX \rightarrow PX$ as

$$\text{for } \phi, \psi \in PX, \phi \tilde{a} \psi \text{ only if } \phi \leq \psi.a$$

Proposition 2.4 (Prorelation \tilde{A}). The set, $\tilde{A} := \{\tilde{a} : a \in A\}$ defines a quasi-uniformity on PX .

Proof. First, need to show that \tilde{A} is a prorelation,

- (i) (Partial order) Define, for any two relations $\tilde{a}, \tilde{b} : PX \rightarrow PX$, that $\tilde{a} \leq \tilde{b}$ only if $a \subseteq b$
- (ii) (Down-Directed) Need that $\forall \tilde{a}, \tilde{b} \in \tilde{A}, \exists \tilde{c} \in \tilde{A} : \tilde{c} \subseteq \tilde{a}, \tilde{c} \subseteq \tilde{b}$
 $\tilde{a}, \tilde{b} \in \tilde{A} \implies a, b \in A \implies \exists c \in A : c \subseteq a, c \subseteq b \implies \tilde{c} \leq \tilde{a}, \tilde{c} \leq \tilde{b}$
- (iii) (Upset) Need that, for any relation $l : PX \rightarrow PX$, if $\exists \tilde{k} \in \tilde{A}$ such that $l \geq \tilde{k}$, then $l \in \tilde{A}$
Fix any $k : PX \rightarrow PX$, and $\tilde{k} \in \tilde{A}$ such that $l \geq \tilde{k}$
Now, k is a relation between promodules $X \rightarrow 1$. Thus, it can be thought of as a relation on X ,
 $a := \{(x, y) : x \in \text{domain}(\psi) \text{ and } y \in \text{domain}(\phi) \text{ whenever } \exists \psi, \phi \in PX : \psi l \phi\}$
So, $l = \tilde{a}$ and thus, $\tilde{a} \geq \tilde{k} \implies a \supseteq k \implies a \in A \implies l \in \tilde{A}$

Now to show that the other two conditions hold,

- (i) need that $\forall \tilde{a} \in \tilde{A}, \forall \psi \in PX, \psi \tilde{a} \psi$
So, need to show that $\psi \leq \psi.a$ i.e. $\forall p \in \psi, \exists q \in \psi : q \subseteq p.a$. Take $q := p$, and as A is a quasi-uniformity, $\Delta_X \subseteq a \implies p = p.\Delta_X \subseteq p.a$
- (ii) Need that $\forall \tilde{a} \in \tilde{A}, \exists \tilde{b} \in \tilde{A} : \tilde{b} \leq \tilde{a}$
Before that, showing, for any $x, y \in A, \tilde{x} \tilde{y} \leq \tilde{x} \tilde{y}$ i.e. $\forall \psi, \phi \in PX, \psi(\tilde{x}.\tilde{y})\phi \implies \psi \tilde{x} \tilde{y} \phi$
Let $\psi_1(\tilde{a}.\tilde{b})\psi_3 \implies \exists \psi_2 : \psi_1 \tilde{b} \psi_2 \tilde{a} \psi_3 \implies \psi_1 \leq \psi_2.b$ and $\psi_2 \leq \psi_3.a \implies \psi_1 \leq \psi_2.b \leq \psi_3.ab \implies \psi_1(\tilde{a}b)\psi_3$
Fix any $\tilde{a} \in \tilde{A} \implies a \in A \implies \exists b \in A : b \circ b \subseteq a \implies \tilde{b} \tilde{b} \leq \tilde{a} \implies \tilde{b} \tilde{b} \leq \tilde{b} \tilde{b} \leq \tilde{a}$

\square

Proposition 2.5 (Yoneda Embedding).

For a quasi-uniform space (X, A) , function $y_X : X \rightarrow PX$ is defined by $x \mapsto x^*$ for $x \in X$.

- (a) $y_X : (X, A) \rightarrow (PX, \tilde{A})$ is a uniformly continuous map
- (b) $y_X : (X, A) \rightarrow (PX, \tilde{A})$ is fully faithful

Proof.

- (a) In order to show y_X is uniformly continuous, need to show that $y_X.A \leq \tilde{A}.y_X$. By definition of \leq , need $\forall a \in A, \exists b \in A : y_X \circ b \subseteq \tilde{a} \circ y_X$. Applying the relations to some element, x of the set X :

$$(y_X \circ b)(x) \subseteq (\tilde{a} \circ y_X)(x) \implies y_X(b(x)) \subseteq \tilde{a}(x^*) \quad (4)$$

So, for the condition given by (4) to hold, if $y \in b(x)$, then it's required that $y^* = y_X(y) \in \tilde{a}(x^*)$ i.e. $x^* \tilde{a} y^*$. Using the definition of x^*, y^* and \tilde{a} ,

$$x^* \tilde{a} y^* \iff x^o.A \leq y^o.A.a \iff \forall a' \in A, \exists a'' \in A : x^o a'' \subseteq y^o a' a \quad (5)$$

Now, fix any $a \in A, x \in X$. Thus, quasi-uniformity of A , gives $a'' \in A$ such that $a'' a'' \subseteq a$. Also, choose some $y \in a''(x)$. Hence, in order to show that the condition from (5) holds, need that $\forall b \in A, x^o a'' \subseteq y^o b a$, and by applying the relations to an element z gives the following condition

$$\forall b \in B, \forall x \in X, (x^o a'')(z) \subseteq (y^o b a)(z) \quad (6)$$

Examining the left side of (6),

$$(x^o a'')(z) = x^o(a''(z)) = \begin{cases} \phi & \text{if } a''(z) \neq x \\ \star & \text{if } a''(z) = x \end{cases}$$

Thus, to show that (6) holds, need to show that (for any $b \in A$ and $z \in X$):

$$a''(z) = x \implies z(y^o b a) \star \text{ i.e. } y \in (b a)(z) \quad (7)$$

To show that (7) holds, fix any $z \in X : a''(z) = x$. Also, by our choice of y , have that $y \in a''(x)$. And as $b \in A$, it's reflexive, giving that $y \in b(y)$. So, by composition of relations, we get:

$$z a'' x, x a'' y \text{ and } y b y \implies z(a'' a'' b) y \implies z(a b) y \text{ i.e. } y \in (b a)(z)$$

- (b) By using Proposition 2.3(a), need to show that $A \geq y_X^o.\tilde{A}.y_X$ i.e. $\forall a \in A, \exists \tilde{b} \in \tilde{A} : a \supseteq y_X^o \tilde{b} y_X$. Applying to an element, $x \in X$ gives the condition

$$(y_X^o \tilde{b} y_X)(x) \subseteq a(x) \implies (y_X^o \tilde{b})(x^*) = y_X^o(\tilde{b}(x^*)) \subseteq a(x) \quad (8)$$

Thus, if $y^* \in PX$ such that $x^* \tilde{b} y^*$, then $y \in y_X^o(\tilde{b}(x^*))$. Now, for (8) to hold, $y \in a(x)$ i.e. $x a y$. Thus, need only to show that for any $a \in A, \exists b \in A$ such that $\forall x, y \in X, x^* \tilde{b} y^* \implies x a y$. So, fix $a \in A$, and take $b \in A : b b \subseteq a$. Now, let $x^* \tilde{b} y^*$ i.e. $x^o.A \leq y^o.A.b$. Hence, $\exists c \in A : x^o c \subseteq y^o b b$. And as c is reflexive,

$$x c x \implies x(c x^o) \star \implies x(b b y^o) \star \implies x(b b) y \implies x a y$$

□

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