

Final Exam

Question 1

Given that $c \in (a, b)$, with

(a) $f \in R[a, c]$ and

(b) $f \in R[b, c]$

Need that $f \in R[a, b]$

Proof. Need to show that,

$$\forall \epsilon > 0, \exists P \in \mathbb{P}([a, b]) : U(P, f) - L(P, f) < \epsilon$$

Fix any $\epsilon > 0$.

By (a), have that

$$\exists P \in \mathbb{P}([a, c]) : U(P, f) - L(P, f) < \frac{\epsilon}{2}$$

By (b), have that

$$\exists Q \in \mathbb{P}([c, b]) : U(Q, f) - L(Q, f) < \frac{\epsilon}{2}$$

Consider

$$U(P, f) + U(Q, f) = \sum_{i=1}^p M_i \delta t_i + \sum_{i=1}^q M'_i \delta t'_i$$

where M_i and t_i correspond to P, while where M'_i and t'_i correspond to Q.

Taking $S := P \cup Q$ so that $S \in \mathbb{P}([a, b])$,
giving $S = \{a < t_1 < t_2 < \dots < c < t'_1 < t'_2 < \dots < b\}$
where $t_i \in P$ and $t'_i \in Q$, so that,

$$\begin{aligned} U(S, f) &= \sum_{i=1}^{p+q} M''_i \delta t_i = \sum_{i=1}^p M_i \delta t_i + \sum_{i=p+1}^q M'_{i-p} \delta t'_{i-p} \\ &= U(P, f) + U(Q, f) \end{aligned}$$

Repeating the same argument for lower sums,

$$L(P, f) + L(Q, f) = \sum_{i=1}^p m_i \delta t_i + \sum_{i=1}^q m'_i \delta t'_i$$

where m_i and t_i correspond to P , while where m'_i and t'_i correspond to Q . Taking $S := P \cup Q$ so that $S \in \mathbb{P}([a, b])$,
giving $S = \{a < t_1 < t_2 < \dots < c < t'_1 < t'_2 < \dots < b\}$
where $t_i \in P$ and $t'_i \in Q$, so that,

$$\begin{aligned} L(S, f) &= \sum_{i=1}^{p+q} m''_i \delta t_i = \sum_{i=1}^p m_i \delta t_i + \sum_{i=p+1}^q m'_{i-p} \delta t'_{i-p} \\ &= L(P, f) + L(Q, f) \end{aligned}$$

Hence,

$$\begin{aligned} U(S, f) - L(S, f) &= (U(P, f) + U(Q, f)) - (L(P, f) + L(Q, f)) \\ &= U(P, f) - L(P, f) + U(Q, f) - L(Q, f) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

□

Question 2

Suppose $P \in \mathbb{P}([a, b])$ such that $U(P, f) = L(P, f)$.
Prove that f is a constant function.

Proof. As $U(P, f) = L(P, f)$,

$$\sum_{i=1}^p M_i \delta t_i = \sum_{i=1}^p m_i \delta t_i \implies \sum_{i=1}^p (M_i - m_i) \delta t_i = 0$$

But, as each $\delta t_i > 0$, it must be that each $M_i - m_i = 0$.

Hence, f is constant in every $\Delta t_i = [t_{i-1}, t_i]$.

So, suppose(if possible) f is not constant over $[a, b]$,

Then, there must be some $\alpha \in \mathbb{N}$ and $\Delta t_k, \Delta t_{k+\alpha}$ such that f takes distinct values in them.

Then, as f is constant on them, $f(t_k) \neq f(t_{k+\alpha})$

But, f is constant on $\Delta t_k = [t_{k-1}, t_k]$ and $\Delta t_{k+1} = [t_k, t_{k+1}]$
and as, t_k , is in both, f attains same value on both.

Now, Δt_{k+1} and Δt_{k+2} share t_{k+1} thus fixing the value of f to be the same over both of them. And, this is the same value as that on Δt_k , i.e. $f(t_k)$.

Repeating the above argument $\alpha - 2$ more times, the value on $\Delta t_{k+\alpha}$ also becomes $f(t_k)$, i.e. $f(t_k) = f(t_{k+\alpha})$

Hence, our supposition was incorrect, and f is constant over $[a, b]$. \square

Question 4

Determine whether $x(1 + \frac{1}{n})$ converges uniformly on \mathbb{R} .

Proof. To show pointwise convergence,

$$\lim_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} \left(x + \frac{x}{n}\right) = x$$

To show that it converges non uniformly,
need an ϵ for which every δ fails for some $x \in \mathbb{R}$

$$\exists \epsilon : \forall \delta, \exists x \text{ such that } \left(\exists n, m > \delta \text{ with } |f_n(x) - f_m(x)| \geq \epsilon \right)$$

Take $\epsilon = 0.5$, and fix any $\delta > 0$.

Now, take $n = \delta + 1$ and $m = \delta + 2$. Take $x = \frac{1}{(\frac{1}{n} - \frac{1}{m})}$

So,

$$\begin{aligned} |f_n(x) - f_m(x)| &= \left| x\left(1 + \frac{1}{n}\right) - x\left(1 + \frac{1}{m}\right) \right| \\ &= \left| x \left(\left(1 + \frac{1}{n}\right) - \left(1 + \frac{1}{m}\right) \right) \right| \\ &= \left| \frac{1}{\left(\frac{1}{n} - \frac{1}{m}\right)} \left(\frac{1}{n} - \frac{1}{m} \right) \right| = 1 > 0.5 = \epsilon \end{aligned}$$

□