Yoneda Lemma and Quasi-Uniform Spaces

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Abstract

We work out the details of the proof for Yoneda Lemma using the text from [3]. Roughly speaking, Yoneda Lemma allows us to embed locally small categories into *Set* via representable functors. We then give two consequences of the Lemma: first is to show that Cayley's theorem from group theory is a particular case of Yoneda Lemma, and second is to derive Yoneda Embedding, a fully faithful functor from locally small categories to their presheaf category. Further, we discuss quasi-uniform spaces from the paper [1]. Here we discuss categories of quasi-uniform spaces and Promodules. We define the Yoneda embedding and prove a (weak) Yoneda Lemma for quasi-uniform spaces. We stop our work here; though the paper goes on a step further to discuss the Cauchy completion monad for quasi-uniform spaces.

Representables

Definition

For any category $\mathcal{A}$, it's opposite category, $\mathcal{A}^{op}$ is the category having the objects of $\mathcal{A}$. And for objects $A, B \in \mathcal{A}$, a morphism $f \in \mathcal{A}^{op}(A, B)$ if and only if there is a morphism $g \in \mathcal{A}(B, A)$.

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Proposition

For a locally small category $\mathcal{A}$, fixing an object $A \in \mathcal{A}$ gives a functor, $H_A : \mathcal{A}^{op} \to Set$ defined as:

1. *For any object $B \in \mathcal{A}$, $H_A(B) := \mathcal{A}(B, A)$.*
2. *For any morphism, $g : X \to Y$ in $\mathcal{A}$,*

$$H_A(g) : \mathcal{A}(Y, A) \to \mathcal{A}(X, A) \text{ is given by } p \mapsto p \circ g.$$

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A functor $\mathcal{A}^{op} \to Set$ that is isomorphic to H_A is called a representable.

Required Results

If a transformation is natural in two individual variables simultaneously, then it is natural their pair.

Lemma

Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be categories. Suppose there are functors $F, G : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$.
For every $A \in \mathcal{A}$, there are functors, $F^A, G^A : \mathcal{B} \to \mathcal{C}$ defined as taking $B \in \mathcal{B}$ to $F(A, B)$, $G(A, B)$ and morphism f to $F((1_A, f))$, $G((1_A, f))$. And, for every $B \in \mathcal{B}$, there are functors $F_B, G_B : \mathcal{A} \to \mathcal{C}$ defined similarly.
A family of maps, $\left(\alpha_{A,B} : F(A, B) \to G(A, B)\right)_{A \in \mathcal{A}, B \in \mathcal{B}}$ is a natural transformation $F \to G$ if the following conditions are satisfied:

1. *For each $A \in \mathcal{A}$, the family $\left(\alpha_{A,B} : F^A(B) \to G^A(B)\right)_{B \in \mathcal{B}}$ is a natural transformation $F^A \to G^A$;*
2. *For each $B \in \mathcal{B}$, the family $\left(\alpha_{A,B} : F_B(A) \to G_B(A)\right)_{A \in \mathcal{A}}$ is a natural transformation $F_B \to G_B$.*

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Following is an equivalent definition of natural isomorphism.

Lemma

Let $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B} \end{tikzcd}$ *be a natural transformation. If for every $A \in \mathcal{A}$, $\alpha_A : F(A) \to G(A)$ is an isomorphism then α is a natural isomorphism.*

Yoneda Lemma

Theorem

If $\mathcal{A}$ is a locally small category then, for any object $A \in \mathcal{A}$ and $X \in [\mathcal{A}^{op}, Set]$, there exists an isomorphism,

$$[\mathcal{A}^{op}, Set](H_A, X) \cong X(A) \text{ which is natural in } A \text{ and } X. \tag{1}$$

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Outline of the proof:

- Define a map from $[\mathcal{A}^{op}, Set](H_A, X)$ to $X(A)$.
- Define a map from $X(A)$ to $[\mathcal{A}^{op}, Set](H_A, X)$.

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- Define a map from $[\mathcal{A}^{op}, Set](H_A, X)$ to $X(A)$.
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- Show the maps to be mutually inverse.

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Cayley's Theorem

Theorem

Cayley's Theorem *Every group, G is isomorphic to a subgroup of symmetric group on G.*

Outline of the proof: Take a group G.

- Define category $\mathcal{A}$ to have a single object, $\star$. With morphisms corresponding to elements of G. With their product also being as elements of G. So that $\mathcal{A}(\star, \star)$ is isomorphic to group G.

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- Prove that each member of the collection $[\mathcal{A}^{op}, Set](H_\star, H_\star)$ can be considered as a bijection on G.

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- Show that the above collection of bijections is actually a group.
- Use Yoneda Lemma to get a set isomorphism between $[\mathcal{A}^{op}, Set](H_\star, H_\star)$ and $\mathcal{A}(\star, \star)$.

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Yoneda Embedding

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The proof proceeds by showing functor $H_\bullet$ to be full and faithful.

Prorelation

Definition

A prorelation is a partially ordered, down-directed, up-set of relations $X \to Y$. That is, $P \subseteq \mathcal{P}(X \times Y)$ is a prorelation if it satisfies the following conditions:

1. Partial Order: Containment of relations defines a partial order. That is, $r \subseteq s$ meaning that for any $(x, y) \in X \times Y$, if $(x, y) \in r$ then $(x, y) \in s$.
2. Down-directed: For any $r, s \in P$, there exists $t \in P$ such that $t \subseteq r$ and $t \subseteq s$.
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Example

For any positive real number ϵ, define a relation on $\mathbb{R}$ as $A_\epsilon = \{(x, y)| \, |x - y| < \epsilon\}$. The collection of all relations on $\mathbb{R}$ that contains some A_ϵ will be a prorelation, K on $\mathbb{R}$. That is, $K = \{a : \mathbb{R} \to \mathbb{R} \,|\, a \supseteq A_\epsilon$ for some $\epsilon > 0 \}$ forms a prorelation. If $k, l \in K$, then there exist δ, $\epsilon > 0$ such that $k \supseteq A_\delta$ and $l \supseteq A_\epsilon$. Thus, the relation $A_{\frac{\delta+\epsilon}{2}}$ is in both k and l. Moreover, K is an up-set by definition.

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Lemma

Composition of two prorelations is a prorelation.

Quasi-Uniform Space

Quasi-uniformity is a particular kind of prorelation.

Definition

A prorelation P on a set X is said to be a quasi-uniformity if it satisfies the following conditions:

1. Every relation in P is reflexive. That is, for each $p \in P$, if $x \in X$ then $(x, x) \in p$.
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Definition

If X is a set, and A is a quasi-uniformity on X, then (X,A) is a quasi-uniform space.

The following gives us a partial order on prorelations:

Definition

For prorelations $P, Q : X \to Y$, if for each $q \in Q$, there exists $p \in P$ such that $p \subseteq q$, then we write $P \leq Q$.

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We will require the following definition in the next slide.

Definition

A function, $f : (X, A) \to (Y, B)$ is said to be uniformly continuous if and only if $f.A \leq B.f$. That is, for each $b \in B$, there exists $a \in A$ such that $f \circ a \subseteq b \circ f$.

Category definition

On this slide, we define two categories.

Definition

The collection of quasi-uniform spaces can be given a categorical structure by considering the uniformly continuous maps between two spaces as the morphisms between them. We call this category QUnif. Composition is as that of functions, and identity morphisms are the identity functions.

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Definition

Now, we define a 2-category called ProMod as having quasi-uniform spaces as 0-cells and the promodules between them being 1-cells. The promodule A will work as the identity of (X, A).
Let promodules $P, Q : (X, A) \to (Y, B)$. Then, there is a 2-cell from P to Q if and only if $P \leq Q$ as prorelations.

Functors from QUnif to ProMod

We have a covariant functor from QUnif to ProMod:

Proposition

Functor $(\_)_* : QUnif \to ProMod$ defined as:

1. *for $(X, A) \in QUnif$, $(X, A)_* := (X, A) \in ProMod$,*
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as well as a contravariant functor from QUnif to ProMod,

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Functor $(\_)^* : QUnif^{op} \to ProMod$ defined as as:

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Topological definitions

Definition

For any quasi-uniform space (X, A), an element $x \in X$ is said to belong in the topological closure of set $M \subseteq X$ if and only if for each $a \in A$, there exists $y \in M$ such that $x\,a\,y$ and $y\,a\,x$.

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Definition

Let $f : (X, A) \to (Y, B)$ be a uniformly continuous function.

1. f is said to be fully faithful if and only if $f^*.f_* = A$.
2. f is said to be fully dense if and only if $f_*.f^* = B$.
3. f is said to be topologically dense of and only if $\overline{f(X)} = Y$.

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The following proposition gives us easier to apply versions of previous definitions.

Proposition

Let $f : (X, A) \to (Y, B)$ be a uniformly continuous map.

1. *f is fully faithful if and only if $A = f^o.B.f$, that is $A \geq f^o.B.f$.*
2. *f is fully dense if and only if for any $b \in B$, $\exists b' \in B$ such that $b' \subseteq b\,f\,f^o\,b$.*
3. *f is topologically dense if and only if for any $b \in B$, $b\,f\,f^o\,b$ is reflexive.*
4. *f is fully dense if and only if f is topologically dense.*

Quasi-uniform space of promodules

Definition

The set PX is defined to be the collection of all promodules from the quasi-uniform space (X, A) to the quasi-uniform space 1.

$$PX := \{\psi : (X, A) \to 1 | \psi \text{ is a promodule}\}$$

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On this set, we can define a quasi-uniformity.

Proposition

For any $a \in A$, $\tilde{a}$ is defined to be a relation on PX as:

$$\text{for } \phi, \psi \in PX, \ \phi \, \tilde{a} \, \psi \text{ if and only if } \phi \leq \psi.a \ .$$

The set, $\tilde{A} := \{\tilde{a} : a \in A\}$ defines a quasi-uniformity on PX.

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Just like we could embed any category into its presheaf category via $H_\bullet$, we can embed any quasi-uniform space into its 'quasi-uniform space of promodules':

Proposition

For a quasi-uniform space (X, A), function $y_X : X \to PX$ is defined by $x \mapsto x^*$ for $x \in X$.

- $y_X : (X, A) \to (PX, \tilde{A})$ *is a uniformly continuous map.*
- $y_X : (X, A) \to (PX, \tilde{A})$ *is fully faithful.*

Yoneda Lemma for Quasi-Uniform Spaces

To think of Yoneda Lemma in quasi-uniform spaces,

- consider the promodule $(y_X)_*$ to be the representable H_A in our initial statement of Yoneda Lemma,
- consider the promodule ψ^* to be the presheaf X in our initial statement of Yoneda Lemma.

This gives us a weak version of Yoneda Lemma that holds in quasi-uniform space:

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Theorem

The following statements hold for any $\psi \in PX$:

1. $\psi \geq \psi^*.(y_X)_*$,
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And the full strength Yoneda Lemma holds only for elements of PX that are also in the topological closure of $y_X(X)$:

Corollary

For $\psi \in PX$, $\psi \in \overline{y_X(X)}$ if and only if ψ is a right-adjoint.

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Showing $\psi \geq \psi^*.(y_X)^*$

- By definition, $(y_X)_* = \tilde{A}.y_X$, and $\psi^* = \psi^o.\tilde{A}$,
  so, we need to show that $\psi \geq (y_X)_*.\psi^* = \psi^o.\tilde{A}.\tilde{A}.y_X$.
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- Examining the right side of the condition, for any $a \in A$, $x \in X$

$$\left(\psi^o.\tilde{a}.y_X\right)(x) == \psi^o\left(\tilde{a}(x^*)\right) = \begin{cases} \phi & \text{if } \psi \notin \tilde{a}(x^*) \\ \star & \text{if } \psi \in \tilde{a}(x^*) \end{cases}.$$

  So, if $\psi \notin \tilde{a}(x^*)$, then, as $p \supseteq \phi$, we're done. Thus, let $\psi \in \tilde{a}(x^*)$, that is, $x^* \tilde{a} \psi$.
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- As ψ is a promodule, $\psi.A \leq \psi$ gives us the the existence of $q \in \psi$ and $a \in A$ such that $qa \subseteq p$. Using the definition of $\tilde{a}$, $x^* \tilde{a} \psi$ gives:

$$x^o.A \leq \psi.a \implies \exists b \in A : x^o b \subseteq qa \implies \forall z \in X, \left(x^o b\right)(z) \subseteq (qa)(z).$$

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- By quasi-uniformity of A, b is reflexive, giving that in particular, for xbx,

$$\left(x^o b\right)(x) \subseteq (qa)(x) \implies x^o x \subseteq (qa)(x) \implies \star \in (qa)(x) \implies \star \in p(x)$$

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8. Finally, as $\tilde{a}' \subseteq \tilde{b}$, From (5), we have that $z^* \tilde{b} x^*$ and $x^* \tilde{b} \psi$. Using composition, $z^* \tilde{b} x^*$.

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7. Thus, to show that (1) holds, need to show that, for any $b \in A$ and $z \in X$:

$$x \in a''(z) \implies z(y^o ba) \star \text{ i.e. } y \in (ba)(z)$$

8. Fix any $z \in X : x \in a''(z)$. Also, by our choice of y, have that $y \in a''(x)$. Now, going to show that the above condition holds: As A is a quasi-uniformity, b is reflexive, giving that $y \in b(y)$.
9. Hence, by composition of relations, we get:

$$za''x \text{ , } xa''y \text{ and } yby \implies z(a''a''b)y \implies z(ab)y \text{ i.e. } y \in (ba)(z).$$

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- Applying the relations to an element, $x \in X$ gives the condition:

$$\left(y_X^o \ \tilde{b} \ y_X\right)(x) \subseteq a(x) \implies \left(y_X^o \ \tilde{b}\right)(x^*) = y_x^o\left(\tilde{b}(x^*)\right) \subseteq a(x). \tag{3}$$

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- Thus, if $y^* \in PX$ such that $x^*\tilde{b}y^*$, then $y \in y_x^o\left(\tilde{b}(x^*)\right)$. For the above condtition to hold, need that $y \in a(x)$, that is, xay.

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- So, fix $a \in A$, and take $b \in A : bb \subseteq a$. Now, let $x^*\tilde{b}y^*$ i.e. $x^o.A \leq y^o.A.b$.

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- Thus, if $y^* \in PX$ such that $x^* \tilde{b} y^*$, then $y \in y_x^o\left(\tilde{b}(x^*)\right)$. For the above condtition to hold, need that $y \in a(x)$, that is, xay.
- Thus, need only to show that for any $a \in A$, there exists $b \in A$ such that for any $x, y \in X$, $x^* \tilde{b} y^*$ implies xay.
- So, fix $a \in A$, and take $b \in A : bb \subseteq a$. Now, let $x^* \tilde{b} y^*$ i.e. $x^o.A \leq y^o.A.b$.
- Hence, $\exists c \in A : x^o c \subseteq y^o bb$. And as c is reflexive,

$$xcx \implies x(cx^o)\star \implies x(bby^o)\star \implies x(bb)y \implies xay\ .$$

Proof: Yoneda Corollary I

Fix any $\psi \in PX$.

1. ($\implies$) Let $\psi \in \overline{y_X(X)}$, from Theorem 5.4, we get that $\psi = \psi^*.(y_X)_*$. In order to show ψ is a right-adjoint, we will show that ψ^* is a right adjoint and that $(y_X)_*$ is an equivalence.
   1. In order to show that $(y_X)_*$ is an equivalence, we need that $A = (y_X)^*.(y_X)_*$ and $\tilde{A} = (y_X)_*.(y_X)^*$.
      From proposition 5.3 (b), we have that y_X is fully faithful, and by Proposition 4.14 (a), this gives us that $A = (y_X)^*.(y_X)_*$.
      - We are now going to show that $\tilde{A} \leq (y_X)_*.(y_X)^*$. Fix any $a, b \in A$, we need to find $c \in A$ such that $\tilde{c} \subseteq \tilde{a}\, y_X\, y_X^o\, \tilde{b}$.
        $$(\tilde{a}\, y_X . y_X^o\, \tilde{b})(\psi) = (\tilde{a}\, \tilde{b})(\psi) \supseteq \tilde{c}\tilde{c}(\psi) \supseteq \tilde{c}(\psi)$$
        In the above equation, the equality holds because $\psi \in \overline{y_X(X)}$, gives the existence of $x^* = \tilde{b}(\psi)$. And the first inequality is given by down-directedness of $\tilde{A}$, whereas the second one holds because $\tilde{c}$ is reflexive, as $\tilde{A}$ is a quasi-uniformity.
      - To show that $\tilde{A} \geq (y_X)_*.(y_X)^*$, fix any $a \in A$. By quasi-uniformity of $\tilde{A}$, there exists $\tilde{b} \in \tilde{A}$ such that $\tilde{b}\,\tilde{b} \subseteq a$. We will show that $\tilde{a} \supseteq \tilde{b}\, y_X\, y_X^o\, \tilde{b}$:
        $$\psi(\tilde{b}\, y_X\, y_X^o\, \tilde{b})\phi \implies \psi(\tilde{b}\tilde{b})\phi \implies \psi\tilde{a}\phi\ .$$
   2. In order to show that ψ^* is a right adjoint to ψ_*, due to the 2-categorical structure of ProMod, we need to show that $\tilde{A} \geq \psi_\star.\psi^\star$ and $\psi_\star.\psi^\star \geq 1$.

Proof: Yoneda Corollary II

- To show that $\tilde{A} \geq \psi_*.\psi^* = \psi_*.\psi^o.\tilde{A}$, fix any $a \in A$. We will show that $\psi_*.\psi^o.\tilde{a} \subseteq \tilde{a}$. Using definition of ψ_*, for any $\phi \in \overline{y_X(X)}$, we get:

$$(\psi_*.\psi^o.\tilde{a})(\phi) = \psi_*.\psi^o(\tilde{a}(\phi)) = \begin{cases} \phi & \text{if } \tilde{a}(\phi) \neq \psi \\ \psi = \psi_*.\psi^o(\psi) & \text{if } \tilde{a}(\phi) = \psi \end{cases} .$$

  The above equation gives that $\phi(\psi_*.\psi^o.\tilde{a})\psi$ implies $\phi\tilde{a}\psi$. Hence, we have that $\tilde{a} \supseteq \psi_*.\psi^o.\tilde{a}$.
- We will show that $\psi_\star.\psi^\star \geq 1$, that is $\star(\psi^o.\tilde{a}.\psi_*)\star$. Using definition of ψ_*,

$$(\psi^o.\tilde{a}.\psi_*)(\star) = (\psi^o.\tilde{a})(\psi_*(\star)) = (\psi^o.\tilde{a})(\psi) = \psi^o(\tilde{a}(\psi)).$$

  By the quasi-uniformity of $\tilde{A}$, we get that $\tilde{a}$ is reflexive, and hence, $\psi\tilde{a}\psi$. So, from the above equation, we have that $\star \in \psi^o(\psi) \subseteq (\psi^o.\tilde{a}.\psi_*)(\star)$.

2. ($\Longleftarrow$) Suppose ψ is a right adjoint. Need to show that for any $a \in A$, $\exists x^* \in y_X(X)$ such that $\psi\, \tilde{a}\, x^*\, \tilde{a}\, \psi$. Fix $a \in A$. Because ψ is a right-adjoint, there exists a promodule $\phi : 1 \to X$ such that $\phi.\psi \leq A$ and $1 \leq \psi.\phi$. From $\phi.\psi \leq A$, we get that:

$$\exists p \in \phi, q \in \psi \text{ such that } a \supseteq p.q \ . \tag{1}$$

Because ϕ and ψ are promodules,

$$A.\phi \leq \phi \text{ gives the existence of } p' \in \phi \text{ such that } p \supseteq a'p' \ , \tag{2}$$

$$A.\psi \leq \psi \text{ gives the existence of } q' \in \psi \text{ and } a'' \in A \text{ such that } q \supseteq a''q' \ . \tag{3}$$

Proof: Yoneda Corollary III

Now, from $1 \leq \psi.\phi$, we get that $q'\, p'$ is reflexive i.e. $\star(q'\, p')\star$. By the definition of composition we get the existence of an $x \in X$ such that $\star\, p'\, x\, q'\, \star$. Now, considering x as a map, $x : 1 \to X$ defined as $\star \mapsto x$,

$$x\, q'\, \star \text{ i.e. } \star \in q'(x) \text{ gives that } q' \supseteq x^o \,, \tag{4}$$

$$\star\, p'\, x \text{ i.e. } x \in p'(\star) \text{ gives that } p' \supseteq x \,. \tag{5}$$

Thus, by using inequalities (1),(2) and (3), we get:

$$a \supseteq p\, q \supseteq a'\, p'\, q'\, a'' \,. \tag{6}$$

By definition of $\tilde{a}$, to show $\psi\, \tilde{a}\, x^*$, we need that $\psi \leq x^*\, a = x^o.\, A.\, a$. We are now going to show that for any $b \in A$, $x^o\, b\, a \supseteq q'$:

$$x^o\, b\, a \supseteq x^o\, b\, a'\, p'\, q' \supseteq x^o\, b\, a'\, x\, q' \supseteq x^o\, x\, q' = q' \,.$$

Where the first inequality comes from (6) by using reflexiveness of a'' and then left-multiplying by x^o. The second inequality comes from (5), third one from reflexiveness of b and a', and the last one is given by Lemma 3.7.
In order to show $x^*\, \tilde{a}\, \psi$, by definition of $\tilde{a}$, need that $x^o.A = x^* \leq \psi\, a$. Fix $k \in \psi$. We will now show $k\, a \supseteq x^o\, a''$:

$$a \supseteq a'\, p'\, q'\, a'' \supseteq p'\, q'\, a'' \supseteq p'\, x^o\, a'' \,. \tag{7}$$

Proof: Yoneda Corollary IV

Where the first inequality is given by (6), second one is due to reflexiveness of a' and the third inequality comes by using (4). Left-multiplying (7) with k gives the following:

$$ka \supseteq k\, p'\, x^o\, a'' \text{ that is, for any } z \in X, \;\; z(k\, a)\star \implies z(k\, p'\, x^o\, a'')\star\; . \qquad (8)$$

As ψ is a right adjoint to ϕ, we have $1 \leq \psi.\phi$, giving that $\star(k\, p')\star$. So, using the implication in(8), we get that $z(k\, a)\star$ implies $z(x^o\, a'')\star(k\, p')\star$, which in turn gives that $z(x^o\, a'')\star$. Hence, we get that $ka \supseteq x^o\, a''$

Showing isomorphism between $[\mathcal{A}^{op}, Set](H_A, X)$ and $X(A)$

Let $\mathcal{A}$ be a locally small category. Fix an object $A \in \mathcal{A}$ and a presheaf X on $\mathcal{A}$.

- Define $\hat{} : \mathcal{C}(H_A, X) \to X(A)$ for any $\alpha : H_A \to X$, as $\hat{\alpha} := \alpha_A(1_A)$. As $1_A \in Set(A, A) = H_A(A)$, definition of α_A gives that $\alpha_A(1_A) \in X(A)$.

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- Define $\tilde{} : X(A) \to \mathcal{C}(H_A, X)$ for any $x \in X(A)$ as the natural transformation $\tilde{x} : H_A \to X$ whose K-component is the function mapping each morphism $p \in \mathcal{A}(K, A)$ to $\Big(X(p)\Big)(x)$. That is, $\tilde{x}_K(p) := \Big(X(p)\Big)(x)$.

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- We are going to show that $\tilde{x}$ is a natural transformation.
- Fix objects $K, L \in \mathcal{A}$ and morphism $q \in \mathcal{A}^{op}(K, L)$.

Need to show that the square

$$\begin{tikzcd} \mathcal{A}(K, A) \arrow[r, "- \circ q"] \arrow[d, "\tilde{x}_K"'] & \mathcal{A}(L, A) \arrow[d, "\tilde{x}_L"] \\ X(K) \arrow[r, "X(q)"'] & X(L) \end{tikzcd}$$

commutes .

So, for any $f : K \to A$, need that $\tilde{x}_L(f \circ q) = X(q) \circ \tilde{x}_K(f)$. Using the definition of $\tilde{x}$ gives the following.

$$LHS = \tilde{x}_L(f \circ q) = \Big(X(f \circ q)\Big)(x)$$

$$RHS = X(q) \circ \tilde{x}_K(f) = \Big(X(q)\Big)\big(X(f)(x)\big) = \Big(X(q) \circ X(f)\Big)(x)$$

And as X is a contravariant functor, $X(f \circ q) = X(q) \circ X(f)$, giving that LHS=RHS.

Showing isomorphism between $[\mathcal{A}^{op}, Set](H_A, X)$ and $X(A)$

- Need to show that ^ and ~ are mutually inverse.

Showing isomorphism between $[\mathcal{A}^{op}, Set](H_A, X)$ and $X(A)$

- Need to show that ˆ and ˜ are mutually inverse.
- For any $x \in X(A)$, $\hat{\tilde{x}} = \tilde{x}_A(1_A) = \Big(X(1_A)\Big)(x) = 1_{X(A)}(x) = x$. For any $\alpha \in \mathcal{C}(H_A, X)$, need to show that $\tilde{\hat{\alpha}} = \alpha$. So, it's required that each of their component are equal.

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- As both $\tilde{\hat{\alpha}}$ and α are natural transformations between functors that go to the category *Set*, each of their components is a function. So, need to show that for any $f \in \mathcal{A}(K, A) = H_A(K)$, $\left(\tilde{\hat{\alpha}}\right)_K(f) {=} \alpha_K(f)$.

Showing isomorphism between $[\mathcal{A}^{op}, Set](H_A, X)$ and $X(A)$

- Need to show that $\hat{}$ and $\tilde{}$ are mutually inverse.
- For any $x \in X(A)$, $\hat{\tilde{x}} = \tilde{x}_A(1_A) = \Big(X(1_A)\Big)(x) = 1_{X(A)}(x) = x$. For any $\alpha \in \mathcal{C}(H_A, X)$, need to show that $\tilde{\hat{\alpha}} = \alpha$. So, it's required that each of their component are equal.
- As both $\tilde{\hat{\alpha}}$ and α are natural transformations between functors that go to the category *Set*, each of their components is a function. So, need to show that for any $f \in \mathcal{A}(K, A) = H_A(K)$, $\left(\tilde{\hat{\alpha}}\right)_K(f)$=$\alpha_K(f)$.
- Using first the definition of $\tilde{}$ and then that of $\hat{\alpha}$ gives:

$$LHS = \tilde{\hat{\alpha}}_K(f) = \Big(X(f)\Big)(\hat{\alpha}) = \Big(X(f)\Big)(\alpha_A(1_A)) \tag{1}$$

And as $f \in \mathcal{A}(K, A)$, we also have the following.

$$RHS = \alpha_K(f) = \alpha_K(1_A \circ f) \tag{2}$$

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- Because α is a natural transformation, the following square commutes for 1_A:

$$\begin{tikzcd}
\mathcal{A}(A, A) \arrow[r, "- \circ f"] \arrow[d, "\alpha_A"'] & \mathcal{A}(K, A) \arrow[d, "\alpha_K"] \\
X(A) \arrow[r, "X(f)"'] & X(K)
\end{tikzcd},$$

which gives that $\alpha_K(1_A \circ f) = \Big(X(f)\Big)(\alpha_A(1_A))$. Hence, we get that $RHS = LHS$, giving us that $\hat{}$ and $\tilde{}$ define a a set isomorphism, as α_K being a function, RHS is a set.

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1. By using the very first two Lemmas, it's enough to show that $\hat{}$ is natural in X and natural in A.

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2. Fix any $A \in \mathcal{A}$. Need, for presheaves $X, Y \in \mathcal{C}$ and natural transformation $\beta \in \mathcal{C}(X, Y)$, the following square to commute:

$$\begin{tikzcd} \mathcal{C}(H_A, X) \arrow[r, "\beta \circ -"] \arrow[d, "\hat{}"'] & \mathcal{C}(H_A, Y) \arrow[d, "\hat{}"] \\ X(A) \arrow[r, "\beta_A"'] & Y(A) \end{tikzcd} .$$

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3. So, for any $\alpha : H_A \to X$, we need that $\left(\hat{} \circ (\beta \circ \_)\right)(\alpha) = \left(\beta_A \circ \hat{}\right)(\alpha)$.

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3. So, for any $\alpha : H_A \to X$, we need that $(\hat{} \circ (\beta \circ \_))(\alpha) = (\beta_A \circ \hat{})(\alpha)$.
4. Using the definition of $(\beta \circ \_)$ and $\hat{}$ gives:

$$LHS = (\hat{} \circ (\beta \circ \_))(\alpha) = (\widehat{(\beta \circ \_)(\alpha)}) = (\widehat{\beta \circ \alpha}) = (\beta \circ \alpha)_A(1_A)$$
$$RHS = (\beta_A \circ \hat{})(\alpha) = \beta_A(\widehat{\alpha}) = (\beta_A \circ \alpha_A)(1_A)$$

As $\alpha \in \mathcal{C}(H_A, X)$ and $\beta \in \mathcal{C}(X, Y)$ are morphisms in $\mathcal{C}$, composition in $\mathcal{C}$ gives $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$.

Showing natural in A

1. Fix any $X \in \mathcal{C}$ Need that for objects $A, B \in \mathcal{A}$ and morphism $f \in \mathcal{A}^{op}(A, B)$, the following square commutes:

$$\begin{tikzcd}
\mathcal{C}(H_A, X) \arrow[r, "- \circ H_f"] \arrow[d, "\hat{}"'] & \mathcal{C}(H_B, Y) \arrow[d, "\hat{}"] \\
X(A) \arrow[r, "X(f)"'] & X(B)
\end{tikzcd},$$

Where H_f denotes $(f \circ \_)$. So, for any $\alpha : H_A \to X$, we need that $(\hat{} \circ H_f)(\alpha) = \Big((X(f)) \circ \hat{}\Big)(\alpha)$.

Showing natural in A

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   Where H_f denotes $(f \circ \_)$. So, for any $\alpha : H_A \to X$, we need that $(\hat{} \circ H_f)(\alpha) = \Big((X(f)) \circ \hat{}\Big)(\alpha)$.

2. Using definition of H_f and $\hat{}$, we get:

$$LHS = (\hat{} \circ H_f)(\alpha) = \widehat{\alpha \circ H_f} = (\alpha \circ H_f)_B(1_B) = \alpha_B(f \circ 1_B) = \alpha_B(1_A \circ f)$$

$$RHS = \Big((X(f)) \circ \hat{}\Big)(\alpha) = (X(f))(\hat{\alpha}) = \Big(X(f)\Big)(\alpha_A(1_A))$$

Showing natural in A

1. Fix any $X \in \mathcal{C}$ Need that for objects $A, B \in \mathcal{A}$ and morphism $f \in \mathcal{A}^{op}(A, B)$, the following square commutes:

$$\begin{tikzcd} \mathcal{C}(H_A, X) \arrow[r, "- \circ H_f"] \arrow[d, "\hat{}"'] & \mathcal{C}(H_B, Y) \arrow[d, "\hat{}"] \\ X(A) \arrow[r, "X(f)"'] & X(B) \end{tikzcd},$$

   Where H_f denotes $(f \circ \_)$. So, for any $\alpha : H_A \to X$, we need that $(\hat{} \circ H_f)(\alpha) = \Big((X(f)) \circ \hat{}\Big)(\alpha)$.

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3. By using equality of equations (1) and (2), for $f \in \mathcal{A}(B, A)$, we get that $\Big(X(f)\Big)(\alpha_A(1_A)) = \alpha_B(1_A \circ f)$. Hence, $RHS = LHS$.

Natural transformations from $H_\star$ to $H_\star$ are bijections on G.

- Let G be a group. Define category $\mathcal{A}$ with a single object $\star$. And let the morphisms of $\mathcal{A}$ be elements of G. Then, G and $\mathcal{A}(\star, \star)$ have the same elements and rule of composition, so there exists a group isomorphism $\psi : \mathcal{A}(\star, \star) \to G$.

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- As $\mathcal{A}^{op}$ is a category with a single object, each natural transformation

$$\alpha : \begin{tikzcd} \mathcal{A}^{op} \arrow[r, bend left, "H_\star"] \arrow[r, bend right, "H_\star"'] \arrow[r, phantom, "\Downarrow\alpha"] & Set \end{tikzcd}$$

has only one component, that is $\alpha_\star$. Therefore, we can identify α with $\alpha_\star$.

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- Using naturality of α, we get that

the square $\begin{tikzcd} \mathcal{A}(\star,\star) \arrow[r, "\_\circ f"] \arrow[d, "\alpha_\star"'] & \mathcal{A}(\star,\star) \arrow[d, "\alpha_\star"] \\ \mathcal{A}(\star,\star) \arrow[r, "\_\circ f"'] & \mathcal{A}(\star,\star) \end{tikzcd}$ commutes for any $f \in \mathcal{A}(\star,\star)$.

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- Applying the identity of $\star$ in $\mathcal{A}$ in above square gives us the following equation:

$$((\_\circ f)\circ\alpha_\star)(1_\star) = (\alpha_\star\circ(\_\circ f))(1_\star) \implies \alpha_\star(f) = \alpha_\star(1_\star)\circ f \implies \alpha_\star(f) = \alpha_\star(1_\star).f$$

Thus, every natural transformation α is defined in terms of its value at $1_\star$. This can be considered as left multiplication by $\alpha_\star(1_\star)$ in G, which we know is a bijection on G.

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- Let G be a group. Define category $\mathcal{A}$ with a single object $\star$. And let the morphisms of $\mathcal{A}$ be elements of G. Then, G and $\mathcal{A}(\star,\star)$ have the same elements and rule of composition, so there exists a group isomorphism $\psi : \mathcal{A}(\star,\star) \to G$.
- As $\mathcal{A}^{op}$ is a category with a single object, each natural transformation

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Thus, every natural transformation α is defined in terms of its value at $1_\star$. This can be considered as left multiplication by $\alpha_\star(1_\star)$ in G, which we know is a bijection on G.
- So far we have shown that the collection $[\mathcal{A}^{op}, Set](H_\star, H_\star)$ of all $\alpha : H_\star \to H_\star$ is a collection of bijections on G.

The collection $[\mathcal{A}^{op}, Set](H_\star, H_\star)$ is a group.

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- As $[\mathcal{A}^{op}, Set]$ is a category, we have that the composition is associative. Also, because this collection contains morphisms with the same source and destination, it is closed under composition. Identity of $[\mathcal{A}^{op}, Set](H_\star, H_\star)$ will act as the identity for its group structure.

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- We know that any natural transformation α is defined in terms of $\alpha_\star(1_\star) \in \mathcal{A}(\star, \star)$. Thus, we define $\delta : H_\star \to H_\star$ with $\delta_\star(1_\star) = a$. Giving us that $h^{-1} = \psi\big(\delta_\star(1_\star)\big)$. And as ψ is a group isomorphism,

$$1_\star = \psi^{-1}(h.h^{-1}) = \psi^{-1}(h)\,.\psi^{-1}(h^{-1}) = \big(\gamma_\star(1_\star)\big).\big(\delta_\star(1_\star)\big).$$

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Applying Yoneda Lemma, and showing group isomorphism

- As the collection of elements of G form a set, $\mathcal{A}(\star, \star)$ is also a set. Hence, $\mathcal{A}$ is a locally small category. Because $\mathcal{A}^{op}$ has the same number of morphisms as $\mathcal{A}$, it is also a locally small category, and we may apply Yoneda Lemma to it.

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$$[\mathcal{A}^{op}, Set](H_\star, H_\star) \overset{\wedge}{\cong} H_\star(\star),$$

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- From the proof of Yoneda Lemma, we know that the map $\hat{}$ acts as $\alpha \mapsto \alpha_\star(1_\star)$. Hence, for any $\alpha, \beta : H_\star \to H_\star$,

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- Using I and II, we get that $[\mathcal{A}^{op}, Set](H_\star, H_\star)$ is a group with all of it's elements being bijections on G. Thus, it is a subgroup of the symmetric group on G.

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- Using I and II, we get that $[\mathcal{A}^{op}, Set](H_\star, H_\star)$ is a group with all of it's elements being bijections on G. Thus, it is a subgroup of the symmetric group on G.
- Finally, the isomorphism $\hat{}$ is between groups, with the *LHS* being the above mentioned subgroup. And *RHS* being $\mathcal{A}(\star, \star)$, which is further isomorphic to group G:

$$G \overset{\psi}{\cong} \mathcal{A}(\star, \star) \overset{\wedge}{\cong} [\mathcal{A}^{op}, Set](H_\star, H_\star) \leq Sym(G).$$

This is precisely the statement of Cayley's theorem.

Functor $H_\bullet$ is full

We will show that the functor $H_\bullet$ is full and faithful. Fix any objects X, Y in a locally small category $\mathcal{A}$.

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- Thus, we need to show that their K-components are equal for every $K \in \mathcal{A}$. Using the definition of $H_\bullet(f)$, this amounts to showing that

$$\text{for any morphism } k \in H_X(K), \Big(H_\bullet(f)\Big)_K(k) = \alpha_K(k), \text{ that is } f \circ k = \alpha_K(k).$$

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commutes.
- Thus, for the identity morphism $1_X \in \mathcal{A}(X, X)$, we get the following:

$$\Big(H_Y(k) \circ \alpha_X\Big)(1_X) = \Big(\alpha_K \circ H_X(k)\Big)(1_X) \implies \alpha_K(1_X) \circ k = \alpha_K(k).$$

- Thus, we have that $H_\bullet$ is a full functor.

Functor $H_\bullet$ is faithful

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- Using the definition of $H_\bullet$, we get that $f \circ 1_X = g \circ 1_X$. And as both g and f are morphisms from X, we get that $f = g$.

Proof: Promod is a 2-category I

In order to show that ProMod is a 2-category, need the following:

1. (1-Identities) For each quasi-uniform space (X, A), $A : (X, A) \to (X, A)$ a promodule because $A.A = A$ by Lemma 4.5.
2. (1-Composition) Need composition of promodules to be a promodule. Let $\phi : (X, A) \to (Y, B)$ and $\psi : (Y, B) \to (Z, C)$ be promodules. To show that $\psi.\phi : (X, A) \to (Z, C)$ is a promodule, need it to be a prorelation that satisfies the two conditions required to be a promodule:
    1. By Lemma 3.4, prorelations are closed under composition. Hence, $\psi.\phi$ is a prorelation
    2. Need to show that $\psi.\phi.A \leq \psi.\phi$. So, Fix $p \in \psi$ and $q \in \phi$. As ϕ is a promodule, $\phi.A \leq \phi$ gives that there exists $q' \in \phi$ and $a \in A$ such that $q'\,a \subseteq q$. Thus, $p\,q'\,a \subseteq p\,q$.
    3. Need to show that $C.\psi.\phi \leq \psi.\phi$.Fix $p \in \psi$ and $q \in \phi$. Because ψ is a promodule, $C.\psi \leq \psi$ gives that there exists $c \in C$ and $p' \in \psi$ such that $c\,p' \subseteq p$. Thus, $c\,p'\,q \subseteq p\,q$
3. (2-Identities) As every promodule is contained in itself, always have $\psi \leq \psi$. Define this comparison to be the identity 2-cell for ψ and denote it by $\leq_\psi$
4. (Vertical 2-composition) For promodules $\psi, \phi, \delta : (X, A) \to (Y, B)$, if there is is a 2-cell from ψ to ϕ and another one from ϕ to δ i.e. $\psi \leq \phi \leq \delta$, then by transitivity of the partial order, $\psi \leq \delta$ i.e. there's a 2-cell from ψ to δ.
5. (Horizontal 2-composition) If there are promodules $\psi, \psi' : (X, A) \to (Y, B)$ and $\phi, \phi' : (Y, B) \to (Z, C)$ such that $\psi \leq \psi'$ and $\phi \leq \phi'$, need to show that $\psi.\phi \leq \psi'.\phi'$. Fix $p' \in \psi'$ and $q' \in \phi'$. As $\psi \leq \psi'$, $\exists p \in \psi : p \subseteq p'$ and as $\psi \leq \psi'$, $\exists q \in \phi : q \subseteq q'$. Thus, $p\,q \subseteq p'\,q'$

Proof: Promod is a 2-category II

6. (1-Identity) Need to show that for any promodule $\phi : (X, A) \to (Y, B)$, $\phi.A = \phi = B.\phi$. By quasi-uniformity of A, every $a \in A$, is reflexive. Thus, for any $p \in \phi$ and $a \in A$, $p = p.\Delta_X \subseteq p\, a$ giving that $\phi \leq \phi.A$. And as ϕ is a promodule, $\phi \geq \phi.A$. Hence, by anti-symmetry of the partial order, $\phi = \phi.A$.
   Similarly, By quasi-uniformity of B, every $b \in B$, is reflexive. Thus, for any $p \in \phi$ and $b \in B$, $p = \Delta_Y.p \subseteq b\, p$ giving that $\phi \leq B.\phi$. And as ϕ is a promodule, $\phi \geq B.\phi$. Hence, $\phi = B.\phi$.
7. (1-Associativity) As composition of relations is associative, so too is the composition of prorelations directly giving that composition of promodules i.e. 1-cells is associative.
8. (Vertical 2-Identity) Let $\leq : \psi \to \phi$ be a 2-cell i.e. $\psi \leq \phi$. By our definition of identity 2-cell, $\leq_\psi \,.\, \leq_1$ means precisely that $\psi \leq \psi \leq \phi$, and by transitivity, this is equivalent to $\psi \leq \phi$. Similarly, $\leq_1 \,.\, \leq_\phi$ means exactly that $\psi \leq \phi \leq \phi$, and this is equivalent to $\psi \leq \phi$.
9. (Vertical 2-Associativity) Associativity of the partial order on promodules directly gives the associativity of composition of 2-cells in ProMod.
10. (Horizontal 2-Identity) Let $\psi, \phi : (X, A) \to (Y, B)$ be promodules. For any 2-cell $\leq : \psi \to \phi$, need to show that the 2-cell given by the horizontal composition, $\leq * \leq_A$ is equal to $\leq$, as well as equal to $\leq_B * \leq$. So, it's required that $\psi.A \leq \phi.A \iff \psi \leq \phi \iff B.\psi \leq B.\phi$. And this holds as a direct consequence of (f).

Proof: Promod is a 2-category III

11. (Horizontal 2-Associativity) As there's a unique 2-cell between any two promodules, and composition of promodules is associative, horizontal composition of 2-cells is associative.
12. (2-Identity) For promodules $\psi : (X, A) \to (Y, B)$ and $: \phi(Y, B) \to (Z, C)$ need $(\leq_\psi * \leq_\phi) = \leq_{\psi.\phi}$. Both sides of the required equality are 2-cells $\leq: \psi.\phi \to \psi.\phi$. Thus, they are equal by the uniqueness of 2-cells between any two 1-cells.
13. (2-Interchange) Let $\psi, \phi, \delta : (X, A) \to (Y, B)$ and $\psi', \phi', \delta' : (Y, B) \to (Z, C)$ be promodules. For 2-cells $\leq_1: \psi \to \phi$,$\leq_2: \phi \to \delta$, $\leq_a: \psi' \to \phi'$ and $\leq_b: \phi' \to \delta'$,need to show $(\leq_b . \leq_a) * (\leq_2 . \leq_1) = (\leq_b * \leq_2).(\leq_a * \leq_1)$. Both RHS and LHS are 2-cells from $\psi.\psi'$ to $\delta.\delta'$ and are hence equal.

Proof: QUnif is a category I

1. (Associativity) The composition of functions is associative by definition.
2. (Identity) For each object (X, A), the identity function $\Delta_X : (X, A) \to (X, A)$ is uniformly continuous as $\Delta_X.A = A \leq A = A.\Delta_X$.

Proof: Covariant Functor I

1. (Partial-Order) Inclusion of relations acts as the partial order.
2. (Down-Directed) Fix any k, k' belonging to $B.f$. Thus, there exist b, b' in B such that $k = b\,f$ and $k' = b\,f$. Using down-directedness of B, there exists $c \in B$ such that $c \subseteq b, b'$. Hence, by Lemma 3.10, $c\,f \subseteq k, k'$.
3. (Up-set) Let k belong to $B.f$ and $l : (X, A) \to (Y, B)$ be a uniformly continuous function such that $l \supseteq k$. Define a relation $b' = \{(f(d), l(d)) : d \in Dom(l)\}$. By definition, for any $x \in X$ and $z \in Y$ such that $(x, z) \in l$, we get that $(f(x), z) \in b'$. And $l \supseteq k = b\,f$ implies $Dom(l) \supseteq Dom(f)$, giving $(x, f(x)) \in f$. Therefore, by definition of composition, $(x, z) \in b'.f$. Conversely, suppose $(x, z) \in b'.f$. By definition of composition, there exists $f(x) \in Y$ such that $(f(x), z) \in b'$. Again using the definition of b', we get that $z = l(x)$ i.e. $(x, z) \in l$. Hence, $l = b'\,f$. Now we will show that $b' \supseteq b$. Because $b'\,f = l \supseteq k = b\,f$, for any $x \in X$ we have that $b'(f(x)) \supseteq b(f(x))$. Thus, $b'|_{f(x)} \supseteq b|_{f(x)}$. By down-directedness of B, $b|_{f(x)} \subset b$ implies $b(x)|_{f(x)} \in B$. Finally, $b' \supseteq b'|_{f(x)} \supseteq b|_{f(x)}$ gives $b' \in B$. Hence, $b'.f \in B.f$.
4. Need to show that $(B.f).A \leq B.f$. So, fix any $b \in B$, we will find $b' \in B$ and $a \in A$ such that $b'\,f\,a \subseteq bf$. By quasi-uniformity of B, there exists $b' \in B$ such that $b'\,b' \subseteq b$. Using Lemma 3.10, we get that $b'\,b'\,f \subseteq b\,f$. As f is uniformly continuous, $f.A \leq B.f$ gives that there is some $a \in A$ such that $f\,a \subseteq b'\,f$. Using this in the previous inequality, we get $b'\,f\,a \subseteq b'\,b'\,f \subseteq b\,f$.

Proof: Covariant Functor II

5. Need to show that $B.B.f \leq B.f$. Fix any $b \in B$, we will find $b' \in B$ such that $b'\, b'\, f \subseteq b\, f$. By quasi-uniformity of B, there exists $b \in B$ such that $b'\, b' \subseteq b$. Using Lemma 3.10, we get $b'\, b'\, f \subseteq bf$.

Thus, $B.f$ is a promodule. We now proceed to show that $(\_)_*$ defines a functor.

1. (Composition) Need to show that $(g \circ f)_* = g_* f_*$ i.e. $C.g.f = C.g.B.f$.
   In order to show $C.g.f \leq C.g.B.f$, fix any $b \in B, c \in C$. We will show that $c\, g\, f \subseteq c\, g\, b\, f$. As f is uniformly continuous, $f.A \leq B.f$ gives that there exists $a \in A$ such that $f\, a \subseteq b\, f$. Using Lemma 3.9, we get $(c\, g) f\, a \subseteq (c\, g) b\, f$. Now, using reflexiveness of a, we get $c\, g\, f \subseteq c\, g\, b\, f$.
   Now, to show that $C.g.f \geq C.g.B.f$. Fix any $c \in C$, we will find $c' \in C$ and $b \in B$ such that $c\, g\, f\ \supseteq c\, g\, b\, f$. By quasi-uniformity of C, there exists $c' \in C$ such that $c \supseteq c'\, c'$. Using Lemma 3.10 gives that $c\, (g\, f) \supseteq c'\, c'\, (g\, f)$. Because g is uniformly continuous, $C.g \geq g.B$ gives us $b \in B$ such that $c'\, g \supseteq b\, g$. Using this in the previous inequality gives that $c\, g\, f \supseteq c'\, g\, b\, f$.
2. (Identity) Let (X, A) be an object of QUnif and $1_{(X,A)} : (X, A) \to (X, A)$ be the identity of (X, A). That is, $1_{(X,A)}$ is defined as $x \mapsto x$. Need to show that $(1_{(X,A)})_* = 1_{(X,A)_*}$. Using functor's definition,
   $LHS = (1_{(X,A)})_* = A.(1_{(X,A)}) = A.1_{(X,A)} = A$ and $RHS = 1_{(X,A)_*} = 1_{(X,A)}$
   Using Proposition 4.8 (f), we get that $A = 1_{(X,A)} = RHS$.

Proof: Contravariant Functor I

Showing that $f^o.B : (Y, B) \to (X, A)$ is a promodule.
So, need to show $f^o.B$ a prorelation $Y \to X$ and that $(f^o.B).B \leq f^o.B$ and $A.(f^o.B) \leq f^o.B$
To show prorelation,

1. (Partial-order) Inclusion of relations is the partial order.
2. (Down directed) for $k, k' \in f^o.B$, need that $\exists l \in f^o.B$ such that $l \subseteq k, k'$
   Fix $k, k' \in f^o.B \implies \exists b, b' \in B : k = f^o \circ b$ and $k' = f^o \circ b'$
   By down-directedness of B, there exists $c \in B$ such that $c \subseteq b, b'$, define $l = f^o \circ c$. Now, using Lemma 3.9 gives $l = f^o \circ c \subseteq k, k'$.
3. (Up-set) for a relation $l : Y \to X$ and $k \in f^o.B$ such that $l \supseteq k$, need $l \in f^o.B$
   Let $b \in B$ be such that $k = f^o \circ b$ and define
   $b' := \{(y, y') : y \in Dom(l) \text{ and } y' \in (f^o)^{-1}(l(y))\}$
   As $l \supseteq k = f^o \circ b$, $Dom(b') = Dom(l) \supseteq Dom(b)$
   and $Ran(l) \supseteq Ran(f^o \circ b) \implies \forall y \in Dom(b), Ran(b') = (f^o)^{-1}(l(y)) \supseteq (f^o)^{-1}(f^o \circ b) = Ran(b)$.
   Now, by definition of b', $f^o \circ b' \supseteq l$. To show $f^o \circ b \subseteq l$,
   $(x, y) \in f^o \circ b' \implies \exists z \in Y : (x, z) \in b'$ and $(z, y) \in f^o \implies x \in Dom(l)$ and $z \in l(x)$ i.e. $(x, z) \in l$.

Proof: Contravariant Functor II

4. To show $(f^o.B).B \leq f^o.B$, need that $\forall b \in B, \exists b' \in B : f^o \circ b' \circ b' \subseteq f^o \circ b$,
   Fix any $b \in B$, as B is a quasi-uniformity,
   $\exists b' \in B : b' \circ b' \subseteq b \implies f^o \circ b' \circ b' \subseteq f^o \circ b$.
   To show $A.(f^o.B) \leq f^o.B$, need that $\forall b \in B$,
   $\exists b' \in B, a \in A : a \circ f^o \circ b' \subseteq f^o \circ b$.
   As f is uniformly continuous, $f.A \leq B.f$ i.e.
   $\forall b \in B, \exists a \in A : f \circ a \subseteq b \circ f \implies a = f^o \circ f \circ a \subseteq f^o \circ b \circ f$.
   Fix any $b \in B$, so, $\exists b' \in B : b'b' \subseteq b$. And, for this
   $b', \exists a : a \subseteq f^o b' f \implies a f^o b' \subseteq f^o b' f f^o b' \subseteq f^o b' b' \subseteq f^o b \implies a f^o b' \subseteq f^o b$.

Now, need to show that $(\_)^*$ respects composition and identity.

1. (Composition) let f, g be uniformly continuous, $(X, A) \xrightarrow{f} (Y, B) \xrightarrow{g} (Z, C)$ need that $(g \circ f)^* = f^*.g^*$
   LHS=$(g \circ f)^* = (g \circ f)^o.C = (f^o \circ g^o).C$ and RHS=$f^*.g^* = (f^o.B).(g^o.C)$
   For equality, showing that LHS$\geq$RHS and LHS$\leq$RHS:
   To show $(f^o \circ g^o).C \geq (f^o.B).(g^o.C)$, need that
   $\forall c \in C, \exists b \in B, c' \in C : f^o g^o c \supseteq f^o b g c'$
   Fix any $c \in C$, so, $\exists c' \in C : c' \circ c' \subseteq c \implies f^o g^o c \supseteq f^o g^o (c'c') =$
   $f^o g^o (c' \Delta_Z c') \supseteq f^o g^o c' (g g^o) c'$
   By uniform continuity of g, for $c' \in C, \exists b \in B : gb \subseteq c'g$
   Thus, $f^o g^o c \supseteq f^o g^o (c'g) g^o c' \supseteq f^o (g^o g) b g^o c' = f^o b g^o c'$.
   To show $(f^o \circ g^o).C \leq (f^o.B).(g^o.C)$, need that
   $\forall b \in B, c \in C, \exists c' \in C : f^o g^o c \subseteq f^o b g^o c$

Proof: Contravariant Functor III

Fix any $c \in C, b \in B$ will show that $c' := c$ works:
As B is a quasi-uniformity,
$\Delta_Y \subseteq b \implies f^o \Delta_Y g^o c = f^o g^o c \subseteq f^o b g^o c = f^o b g^o c'$

2. (Identity) let $(X, A) \in$ QUnif ^{op}, and $1_{(X,A)} : (X, A) \to (X, A)$ as $x \mapsto x$ need that $(1_{(X,A)})^* = 1_{(X,A)^*}$ LHS$=(1_{(X,A)})^* = (1_{(X,A)})^o.A = 1_{(X,A)}.A = A$.
And as $RHS = 1_{(X,A)^*} = 1_{(X,A)}$ Using Proposition 3.2(f), we get that $A = 1_{(X,A)} = RHS$.

Proof: Composition of Prorelations is a prorelation I

For prorelations $P : X \to Y$ and $Q : Y \to Z$, need to show that Q.P is a prorelation.

1. (Partial Order) Inclusion of relations gives a partial order.
2. (Down-Directed) If $k, k' \in Q.P$, then $k = q\,p$ and $k' = q'\,p'$ for some $q, q' \in Q$ and $p, p' \in P$. Because Q and P are prorelations, and hence down-directed sets there exists, $a \in Q$ such that $a \subseteq q, q'$ and $b \in P$ such that $b \subseteq p, p'$. Thus, giving an element, $a \circ b$ of $Q.P$ such that $a \circ b \subseteq k, k'$.
3. (Up-Set) Let $l : X \to Z$ be a relation, and $k \in Q.P$ such that $l \supseteq k$. Define relations $p : X \to Y$ and $q : Y \to Z$ as, $p = \{(x, y) : x \in Dom(l) \text{ and } y \in Y\}$ and $q = \{(y, z) : y \in Y \text{ and } z \in Ran(l)\}$. Because $k \in Q.P$, there exist $q' \in Q$ and $p' \in P$ such that $k = q' \circ p'$. Thus by definition of p and q, we get that $p \supseteq p'$ and $q \supseteq q'$. Hence $p \in P$ and $q \in Q$ because P and Q are up-sets, which gives us that $q \circ p \in Q.P$. For any $(x, z) \in l$, by definition of p and q, we get that for every $y \in Y$, $(x, y) \in p$ and $(y, z) \in q$. By definition of composition, this gives that $(x, z) \in q \circ p$, giving that $l \subseteq q \circ p$. And, by definition of $q \circ p$ we get that $l \supseteq q \circ p$. Finally giving that $l = q \circ p \in Q.P$.

Topology induced by a quasi-uniformity

A subfamily $\mathbb{B}$ of quasi-uniformity A is called a base for A if each relation in A contains a relation in $\mathbb{B}$.

Proposition

Let $\mathbb{B}$ be the base for quasi-uniformity A on X. For $x \in X$, define $\mathbb{B}(x) = \{B(x) | B \in \mathbb{B}\}$. Then there is a unique topology on X such that for each $x \in X$, $\mathbb{B}(x)$ is a base for the neighborhood of x in this topology.

We skip the proof as we have no requirement of it. But refer the interested reader to [2] for similar results.