

Question 1

$$C \subseteq D \subseteq \mathbb{R};$$

$(f_n)_{n \in \mathbb{N}}$ is uniformly convergent on C ;

$\forall i \in \mathbb{N}, f_i : D \rightarrow \mathbb{R}$ is continuous

Show $\exists f$ such that $f_n \xrightarrow[\text{uniformly}]{\overline{C} \cap D} f$ and f is continuous.

Proof. Fix an $\epsilon > 0$ so, by uniform continuity of f ,

$$\exists M : p, m \geq M \implies \forall x \in C, |f_p(x) - f_m(x)| < \frac{\epsilon}{3}$$

Fix $p, m \geq M$. Now, fix a $k \in \overline{C} \cap D$ thus, $\exists (x_n) \subseteq C$ such that $x_n \rightarrow k$. As each f_i is continuous ,

$$\exists \delta(i) \text{ s.t. } |x - y| \leq \delta \implies |f_i(x) - f_i(y)| < \frac{\epsilon}{3}$$

So, there's K such that

$$i \geq K \implies |f_p(x_i) - f_p(k)| < \frac{\epsilon}{3} \text{ and } |f_m(x_i) - f_m(k)| < \frac{\epsilon}{3}$$

Thus, by triangle inequality,

$$\begin{aligned} 2 \times \frac{\epsilon}{3} &> |f_p(x_i) - f_p(k)| + |f_m(x_i) - f_m(k)| \\ &\geq |f_p(x_i) - f_m(x_i) + f_p(k) - f_m(k)| \\ &\geq ||f_p(x_i) - f_m(x_i)| - |f_m(k) - f_p(k)|| \end{aligned}$$

Now, as $|f_p(x_i) - f_m(x_i)| < \frac{\epsilon}{3}$, $-|f_p(x_i) - f_m(x_i)| > \frac{-\epsilon}{3}$,

$$\begin{aligned} 2 \times \frac{\epsilon}{3} &\geq ||f_p(x_i) - f_m(x_i)| - |f_m(k) - f_p(k)|| \\ &\geq |f_p(x_i) - f_m(x_i)| - |f_m(k) - f_p(k)| \\ &\geq |f_p(x_i) - f_m(x_i)| - \frac{\epsilon}{3} \end{aligned}$$

Therefore, $|f_p(x_i) - f_m(x_i)| \leq \epsilon$

□

There's a much better way of doing this though,

Proof. Fix an $\epsilon > 0$ so, by uniform continuity of f ,

$$\exists M : p, m \geq M \implies \forall x \in C, |f_p(x) - f_m(x)| < \frac{\epsilon}{4}$$

Fix $p, m \geq M$

Now, fix a $k \in \overline{C} \cap D$. Going to show that (f_n) uniformly converges at k .

As f_p and f_m are continuous, in particular at k ,

$$\exists \delta_1 \text{ s.t. } |x - k| \leq \delta_1 \implies |f_p(x) - f_p(k)| < \frac{\epsilon}{4}$$

$$\exists \delta_2 \text{ s.t. } |x - k| \leq \delta_2 \implies |f_m(x) - f_m(k)| < \frac{\epsilon}{4}$$

Take $\delta := \min\{\delta_1, \delta_2\}$

Now, if $k \in C$, then by hypothesis, the given sequence uniformly converges at k . Else, k is a limit point of C . Thus,

$$\exists c \in C \text{ such that } |c - k| < \delta$$

Fix this c and consider the following inequality

$$|f_p(k) - f_m(k)| \leq |f_p(k) - f_p(c)| + |f_p(c) - f_m(c)| + |f_m(c) - f_m(k)|$$

Each of the right hand side term is $< \frac{\epsilon}{4}$ (1st, 3rd term due to continuity, 2nd term due to uniform convergence on C), forcing the left side to be $< \epsilon$. \square

Question 2 Prove that $\sum x^n(1-x)$ converges pointwise on $[0, 1]$ but not uniformly. While $\sum (-1)^n x^n(1-x)$ converges uniformly on $[0, 1]$.

Proof. As $x^n(1-x) = x^n - x^{n+1}$, the first sum telescopes:

$$\sum_{i=1}^k x^n(1-x) = (x-x^2) + (x^2-x^3) + \dots + (x^k-x^{k+1}) = x-x^{k+1}$$

So, for $x = 1$, every partial sum is 0, and for $0 \leq x < 1$,

$$\lim_{k \rightarrow \infty} \sum_{i=1}^k x^i(1-x) = \lim_{k \rightarrow \infty} (x - x^{k+1}) = x$$

Thus, the series converges pointwise on $[0, 1]$. Suppose it also converges uniformly to f . Then, as the k^{th} partial sum is $x - x^{k+1}$, a polynomial, and hence continuous on $[0, 1]$, it's limit function, f must be continuous on $[0, 1]$. But, f is discontinuous at 1 as

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} x = 1 \neq 0 = f(1)$$

The partial sums for the second series of functions,

$$\sum_{i=1}^k (-x)^n(1-x) = -x + 2[(-x)^2 + (-x)^3 + \dots + (-x)^k] + (-x)^{k+1}$$

So, for $x = 1$, every partial sum is 0, and for $0 \leq x < 1$,

$$\begin{aligned}\lim_{k \rightarrow \infty} \sum_{i=1}^k (-x)^i (1-x) &= x + 2 \lim_{k \rightarrow \infty} ((-x)^{k+1} + \sum_{i=1}^k (-x)^i) \\ &= x + 2 \lim_{k \rightarrow \infty} \frac{-x(1 - (-x)^k)}{1+x} \\ &= x + \frac{-2x}{1+x}\end{aligned}$$

To show uniform convergence, going to use the Dirichlet test:

I Take $b_n(x) := \frac{x^n}{2} = (\frac{x}{\sqrt[2]{2}})^n$. Going to show that

(i) $\forall x \in [0,1], b_n(x) \geq b_{n+1}(x) :$

$$\text{As } b_{n+1}(x) - b_n(x) = \frac{x^{n+1}}{n+1\sqrt[2]{2}} - \frac{x^n}{\sqrt[2]{2}} = \frac{x^n}{\sqrt[2]{2}} (\frac{x}{\sqrt[2]{2}} - 1) \leq 0$$

(ii) $b_n \Rightarrow 0(C)$

$$\begin{aligned}\text{Take } \delta > \frac{\ln 2}{\ln(\epsilon + 1)} &\implies \ln(\epsilon + 1) > \frac{1}{\delta} \ln(2) \\ &\implies \epsilon > \sqrt[\delta]{2}\end{aligned}$$

$$\begin{aligned}|b_n(x) - b_m(x)| &= \left| \frac{x}{\sqrt[2]{2}} - \frac{x}{\sqrt[2]{2}} \right| = x \left(\frac{1}{\sqrt[2]{2}} - \frac{1}{\sqrt[2]{2}} \right) \\ &\leq \left(\frac{1}{\sqrt[2]{2}} - \frac{1}{\sqrt[2]{2}} \right) = \frac{\sqrt[2]{2} - \sqrt[2]{2}}{\sqrt[2]{2} \sqrt[2]{2}} \leq \sqrt[2]{2} - \sqrt[2]{2} \\ &\leq \sqrt[\delta]{2} - 1 \leq \epsilon\end{aligned}$$

II Take $a_n(x) := 2(-1)^n(1-x)$. So,

$$|(A_n(x))| = \left| \sum_{i=1}^n a_i(x) \right| = 2(1-x) \left| \sum_{i=1}^n (-1)^i \right|$$

$$= \begin{cases} 0 & n \text{ is even} \\ 2(1-x) & n \text{ is odd} \end{cases}$$

\therefore for any n, x $|A_n(x)| \leq 2$, a_n is uniformly bounded.

III So, Dirichlet's test is applicable and

$$\begin{aligned} \sum b_n a_n &= \sum \frac{x^n}{2} \times 2(-1)^n(1-x) \\ &= \sum (-1)^n x^n (1-x) \\ &\text{uniformly converges on } [0,1] \end{aligned}$$

□

Question 3

- A is closed and bounded ;
 (f_n) is a sequence of continuous functions on A ;
 $(f_n) \xrightarrow{p.w.} f$, with f continuous on A ;
 $\forall x \in A, f_n(x) \geq f_{n+1}(x)$, with $n \in \mathbb{N}$;

Prove that $f_n \rightrightarrows f(A)$

Question 4 Construct a sequence of functions, (f_n) on $[0,1]$ such that

- (a) each f_i is discontinuous at every point of $[0,1]$;
and
(b) $\exists f$, a continuous function on $[0,1]$ such that $f_n \rightrightarrows f$

Proof. Define $f_n(x) = \begin{cases} \frac{1}{n} & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

- (a) This sequence is discontinuous on $[0,1]$ as :
(i) For $q \in \mathbb{Q}$, take any sequence of irrational numbers, $(p_n) \rightarrow q$. So,

$$\lim_{k \rightarrow \infty} f_n(p_k) = 0 \neq \frac{1}{n} = f_n(q)$$

- (ii) For $p \notin \mathbb{Q}$, take any sequence of rational numbers, $(q_n) \rightarrow p$. So,

$$\lim_{k \rightarrow \infty} f_n(q_k) = \frac{1}{n} \neq 0 = f_n(q)$$

- (b) The defined sequence uniformly converges to 0 as:
 Fix $\epsilon = \frac{1}{n}$, and choose $\delta > n$,

$$m \geq \delta \implies |f_m(x)| = \frac{1}{m} \leq \frac{1}{\delta} < \frac{1}{n} = \epsilon$$

□

Question 3

A is closed and bounded ;

(f_n) is a sequence of continous functions on A ;

$(f_n) \xrightarrow{p.w.} f$, with f continous on A ;

$\forall x \in A, f_n(x) \geq f_{n+1}(x)$, with $n \in \mathbb{N}$;

Prove that $f_n \rightrightarrows f(A)$

Question 5 Prove If $\sum a_n$ is absolutely convergent then $\sum \frac{a_n x^n}{1+x^{2n}}$ converges uniformly on \mathbb{R} .