

Question 1

$\langle a_n \rangle$ is a real sequence; $\sigma_n := \frac{a_1 + a_2 + \dots + a_n}{n}$

Going to show that

I $\liminf a_n \leq \liminf \sigma_n$

If a_n is unbounded below, then $\liminf a_n = -\infty \leq \liminf \sigma_n$. So, let a_n be bounded below, thus LHS is a real number,

$$m := \liminf a_n$$

Now, if a_n is constant, and equal to a ,

$$\forall n \in \mathbb{N}, \inf\{a_i | i \geq n\} = a = \frac{na}{n} = \sigma_n$$

Otherwise, if a_n is not constant, then,

$$\exists i, j \in \mathbb{N} \text{ such that (wlog) } a_i < a_j$$

Suppose if possible, $\liminf a_n = m > \liminf \sigma_n$

But,

$$\forall n \in \mathbb{N} \text{ such that } n \geq i, j,$$

$$\begin{aligned} \sigma_n &= \frac{a_1 + \dots + a_i + a_j + \dots + a_n}{n} \geq \frac{(n-2)m + a_i + a_j}{n} \\ &\geq \frac{(n-1)m + a_j}{n} \\ &> m \quad [\because m \leq a_i < a_j] \end{aligned}$$

$$\begin{aligned} \text{Hence, } \inf \sigma_n &\geq \frac{(n-1)m + a_j}{n} \\ \implies \liminf \sigma_n &\geq \lim_{n \rightarrow \infty} \frac{(n-1)m + a_j}{n} = m \end{aligned}$$

But this contradicts the initial assumption.

II $\limsup a_n \geq \limsup \sigma_n$ If a_n is unbounded above, then $\limsup a_n = \infty \geq \limsup \sigma_n$.

So, let a_n be bounded above, thus LHS is a real number,

$$M := \limsup a_n$$

Now, if a_n is constant, and equal to a ,

$$\forall n \in \mathbb{N}, \sup\{a_i | i \geq n\} = a = \frac{na}{n} = \sigma_n$$

Otherwise, if a_n is not constant, then,

$$\exists i, j \in \mathbb{N} \text{ such that (wlog) } a_i < a_j$$

Suppose if possible, $\limsup a_n = M < \limsup \sigma_n$

But,

$$\begin{aligned} \forall n \in \mathbb{N} \text{ such that } n \geq i, j, \\ \sigma_n = \frac{a_1 + \dots + a_i + a_j + \dots + a_n}{n} &\leq \frac{(n-2)M + a_i + a_j}{n} \\ &\leq \frac{(n-1)M + a_j}{n} \\ &< M \quad [\because M \geq a_j > a_i] \end{aligned}$$

$$\begin{aligned} \text{Hence, } \inf \sigma_n &\leq \frac{(n-1)M + a_j}{n} \\ \implies \liminf \sigma_n &\leq \lim_{n \rightarrow \infty} \frac{(n-1)M + a_j}{n} = M \end{aligned}$$

But this contradicts the initial assumption.

Question 2

$$\liminf \frac{a_{n+1}}{a_n} \leq \liminf (a_n)^{\frac{1}{n}} \leq \limsup (a_n)^{\frac{1}{n}} \leq \limsup \frac{a_{n+1}}{a_n}$$

I We have that a_n is bounded below by 0.

$$\text{II } \liminf \frac{a_{n+1}}{a_n} \leq \liminf (a_n)^{\frac{1}{n}}$$

$$\text{III } \limsup (a_n)^{\frac{1}{n}} \leq \limsup \frac{a_{n+1}}{a_n}$$

Question 4

1.

2.

3.

4. Some convergent sequences are:

$$(a) \ a_n := 1$$

5. Some divergent sequences are:

(a) $a_n := n$

$$\frac{a_{n+1}}{a_n}$$

Appendix

1. also, make sure to show $\liminf \leq \limsup$