Question 1

$$\begin{aligned} C \subseteq D \subseteq \mathbb{R}; \\ \big(f_n\big)_{n\in\mathbb{N}} \text{ is uniformly convergent on } C; \\ \forall i \in \mathbb{N},\ f_i : D \to \mathbb{R} \text{ is continous} \end{aligned}$$

Show $\exists f$ such that $f_n \xrightarrow[\text{uniformly}]{\overline{C}\cap D} f$ and f is continous.

Proof. Fix any $\epsilon > 0$. Need to show that

$\exists K > 0$ s.t. $k \geq K \implies \forall x \in \overline{C} \cap D,\ |f_k(x) - f(x)| < \epsilon$

So, fix any $c \in \overline{C} \cap D$.

As each f_i is given continous on D, $\exists \delta(i, c) > 0$ such that

$$\forall y \in D, |c - y| \leq \delta \implies |f_i(c) - f_i(y)| < \epsilon/2$$

So, in particular, for any sequence in C, $(c_n) \to c$,

$$\begin{aligned} \exists N((c_n), \delta) \text{ such that } n \geq N &\implies |c_n - c| < \delta \\ &\implies |f_i(c_n) - f_i(c)| < \epsilon/2 \end{aligned}$$

Now, as $f_i \overset{C}{\rightrightarrows} f$, $\exists \beta$ such that for any $c \in C$,

$$k \geq \beta \implies |f_k(c) - f(c)| < \epsilon/2$$

By triangle inequality,

$|f_i(c) - f_j(c)| \leq |f_i(c) - f_i(c_k)| + |f_i(c_k) - f_j(c)| + |f_i(c_k) - f_j(c)|$

Thus,for $L > max\{\beta, N\}$, both (1) and (2) will hold:

$$\begin{aligned} i \geq L \implies \epsilon/2 + \epsilon/2 &> |f_i(c_i) - f_i(c)| + |f_i(c_i) - f(c)| \\ &\geq |f_i(c) - f(c)| \end{aligned}$$

Hence, the sequence uniformly converges to f on $\overline{C} \cap D$. And as $(f_i)_{i \in \mathbb{N}}$ is a sequence of continous functions on D, that uniformly converges to f, f is continous on $\overline{C} \cap D$. $\square$

Note: In case D were bounded, $\overline{C} \cap D$ would be compact. Then, Heine-Cantor Theorem would give the existence of f , and Uniform Limit Theorem would give it's continuity.

Question 2 Prove that $\Sigma x^n(1-x)$ converges pointwise on $[0, 1]$ but not uniformly. While $\Sigma(-1)^n x^n(1-x)$ converges uniformly on $[0, 1]$.

Proof. As $x^n(1-x) = x^n - x^{n+1}$, the first sum telescopes:

$$\sum_{i=1}^{k} x^n(1-x) = (x-x^2)+(x^2-x^3)+...+(x^k-x^{k+1}) = x-x^{k+1}$$

So, for $x = 1$, every partial sum is 0, and for $0 \leq x < 1$,

$$\lim_{k\to\infty} \sum_{i=1}^{k} x^i(1-x) = \lim_{k\to\infty} (x - x^{k+1}) = x$$

Thus, the series converges pointwise on $[0, 1]$. Suppose it also converges uniformly to f . Then, as the k^{th} partial sum is $x - x^{k+1}$, a polynomial, and hence continous on $[0, 1]$, it's limit function, f must be continous on $[0, 1]$. But, f is discontinous at 1 as

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} x = 1 \neq 0 = f(1)$$

The partial sums for the second series of functions,

$$\sum_{i=1}^{k}(-x)^n(1-x) = -x+2[(-x)^2+(-x)^3+...+(-x)^k]+(-x)^{k+1})$$

So, for $x = 1$, every partial sum is 0, and for $0 \leq x < 1$,

$$\begin{aligned}
\lim_{k\to\infty}\sum_{i=1}^{k}(-x)^i(1-x) &= x + 2\lim_{k\to\infty}((-x)^{k+1} + \sum_{i=1}^{k}(-x)^i) \\
&= x + 2\lim_{k\to\infty}\frac{-x(1-(-x)^k)}{1+x} \\
&= x + \frac{-2x}{1+x}
\end{aligned}$$

To show uniform convergence, try drichilet-test?

$\square$

Question 3

a_n & b_n are bounded, non-negative sequences; $a_n \to a > 0$

As a_n, b_n are bounded sequences, so is $a_n b_n$.
Hence, L:=lim sup $a_n b_n$ and R:=lim sup b_n are real numbers