1 Yoneda

2 Prorelations

Definiton 2.1. A prorelation is a partially ordered, down-directed, upper set of relations $X \to Y$, i.e a set, $P \subseteq \mathcal{P}(X \times Y)$ such that

(i) (Partial Order)Containment of relations, $r \subseteq s$ only if for any $(x, y) \in X \times Y,\ (x, y) \in r \implies (x, y) \in s$

(ii) (Down-directed), For any $r, s \in P, \exists t \in P$ such that $t \subseteq r$ and $t \subseteq s$

(iii) (Up-set) for any relation $u : X \to Y$, if $\exists p \in P$ such that $p \leq u$ then $u \in P$

Definiton 2.2. Prorelations are composed by taking compositions of the relations belonging to them.

$$\text{For prorelations } P : X \to Y \text{ and } Q : Y \to Z,\ \ Q.P := \{q \circ p : p \in P \text{ and } q \in Q\}$$

Lemma 2.2.1. Prorelations are closed under composition.

Proof. For prorelations $P : X \to Y$ and $Q : Y \to Z$, need to show that Q.P is a prorelation.

(i) (Partial Order) Inclusion of relations gives a partial order.

(ii) (Down-Directed) If $k, k' \in Q.P$ then, $k = q\,p$ and $k' = q'\,p'$ for some $q, q' \in Q$ and $p, p' \in P$. Because Q and P are prorelations, and hence down-directed sets, there exists $a \in Q$ such that $a \subseteq q, q'$ and $b \in P$ such that $b \subseteq p, p'$. Thus, $a \circ b \subseteq k, k'$.

(iii) (Up-Set) Let $l : X \to Z$ be a relation, and $k \in Q.P$ such that $l \supseteq k$. Define relations $a : X \to Y$ and $b : Y \to Z$ as $a := \{(x, y) : x \in domain(l) \text{ and } y \in Y\}$ and $b := \{(y, z) : z \in range(l) \text{ and } y \in Y\}$. Because there exist $a' \in A$ and $b' \in B$ such that $l \supseteq k = b' \circ a'$, by definition of a and b, we get that $b \supseteq b'$ and $a \supseteq a'$. Because Q and P are up-sets, this gives that$b \in B$ and $a \in A$, and hence $b \circ a \in Q.P$. Lastly for $(x, z) \in X \times Z$, if $(x, z) \in l$, then $\forall y \in Y$, $(x, y) \in a$ and $(y, z) \in b$ gives that $(x, z) \in b \circ a$, giving that $l = b \circ a$.

□

Definiton 2.3. Prorelations with same domain and co-domain are said to be comparable when one of them contains all the relations of the other one.

$$\text{For prorelations } P, Q : X \to Y,\ \ P \leq Q \ \text{ if } \ \forall q \in Q, \exists p \in P \text{ such that } p \subseteq q$$

Definiton 2.4. For a relation $r : X \to Y$, it's opposite relation $r^o : Y \to X$ is defined as

$$\forall (x, y) \in X \times Y,\ \ (x, y) \in r \text{ if and only if } (y, x) \in r^o$$

Lemma 2.4.1. For any function $f : X \to Y,\ \ f^o \circ f \supseteq \Delta_X$

Proof. As f is a function, it must be defined on every element of it's domain. So, for every $x \in X$, there exists some $(x, y) \in r$, which, by definition of f^o also gives that $(y, x) \in f^o$. Hence, by definition of composition, for every $x \in X$, $(x, x) \in f^o\, f$ giving that $\Delta_X \subseteq f^o\, f$. □

Lemma 2.4.2. For any relation $r : X \to Y,\ \ r \circ r^o \subseteq \Delta_Y$

Proof. Suppose there exist $x \in X$ and $y \in Y$ such that $x\, r\, y$. By definition of r^o, this gives us that $y\, r^o\, x$. Now, using the definition of composition, $y\, r^o\, x\, r\, y$ gives that $y\,(r \circ r^o)\, y$. □

Lemma 2.4.3. For relations $r, s : X \to Y$ and $t : Y \to Z,\ \ r \subseteq s \implies (t \circ r) \subseteq (t \circ s)$

Proof. Suppose relations r, s and t are as given above and let $x \in X, z \in Z$ such that $x\,(t\,r)\,z$. By definition of composition, there exists some element, y of Y such that $x\,r\,y$ and $y\,t\,z$. By hypothesis, $x\,r\,y$ gives $x\,s\,y$, and composition with $y\,t\,z$ gives $x\,(t\,s)\,z$. We started with any element of $(t \circ r)$ and showed that it must also be in $t \circ s$. Thus, $(t \circ r) \subseteq (t \circ s)$. □

Lemma 2.4.4. For relations $r : X \to Y$ and $s, t : Y \to Z$, $s \subseteq t \implies (s \circ r) \subseteq (t \circ r)$

Proof. Suppose relations r, s and t are as given above and let $x \in X, z \in Z$ such that $x\,(s\,r)\,z$. By definition of composition of relations, we get that there exists some $y \in Y$ such that $x\,r\,y$ and $y\,s\,z$. Because $s \subseteq t$, $y\,s\,z$ implies that $y\,t\,z$. Again, by composition, we get that $x(t\,r)z$. □

3 Propositions

Definiton 3.1. A function, $f : (X, A) \to (Y, B)$ is said to be uniformly continuous if $f.A \leq B.f$ i.e.

$$\forall b \in B, \exists a \in A \text{ such that } f \circ a \subseteq b \circ f \quad \text{i.e.} \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "A"'] \arrow[dr, phantom, "\leq"] & Y \arrow[d, "B"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Definiton 3.2. A prorelation on a set, $P : X \to X$ is a quasi-uniformity if it satisfies :

(i) $\forall p \in P$, p is reflexive i.e. for any $x \in X$, $(x, x) \in p$

(ii) $\forall p \in P, \exists p' \in P$ such that $p' \circ p' \subseteq p$

And in that case, (X, A) is called a *quasi-uniform space.*

Lemma 3.2.1. If A is a quasi-uniformity on a set X, then $A.A = A$

Proof. Fix any $a \in A$, as A is a quasi-uniformity, $\exists b \in A : bb \subseteq a$, we get that $A.A \leq A$. And as A is a prorelation, and is hence down-directed, $\exists c \in A : a.a \supseteq c$, giving that $A.A \geq A$ □

Proposition 3.1. QUnif is a category defined as having quasi-uniform spaces as objects, and uniformly continuous maps between them as morphisms, with composition defined as that of functions.

Proof. (i) (Associativity) The composition of functions is associative by definition.

(ii) (Identity) For each object (X, A), define it's identity to be the identity function $\Delta_X : (X, A) \to (X, A)$. This function is uniformly continuous as $\Delta_X.A = A \leq A = A.\Delta_X$.

□

Definiton 3.3. A prorelation, $\phi : X \nrightarrow Y$ is called a promodule $\phi : (X, A) \nrightarrow (Y, B)$ if it satisfies:

$$\phi.A \leq \phi \text{ and } B.\phi \leq \phi$$

Definiton 3.4. Comparison of promodules is defined as that of prorelations.

Definiton 3.5. Promodules are composed as prorelations.

Proposition 3.2. ProMod is a 2-category defined as having quasi-uniform spaces as its 0-cells, promodules as 1-cells and containment of promodules as 2-cells.

Proof. In order to show that ProMod is a 2-category, need the following:

(a) (1-Identities) For each quasi-uniform space (X, A), define $A : (X, A) \nrightarrow (X, A)$ to be the identity 1-cell for (X, A). A is a promodule because $A.A = A$ (Lemma 2.2.1)

(b) (1-Composition) Need promodules to be closed under composition.
Let $\phi : (X, A) \nrightarrow (Y, B)$ and $\psi : (Y, B) \nrightarrow (Z, C)$ be promodules. To show that $\psi.\phi : (X, A) \nrightarrow (Z, C)$ is a promodule, need it to be a prorelation that satisfies the two conditions required to be a promodule:

   (i) By Lemma 1.2.1, prorelations are closed under composition. Hence, $\psi.\phi$ is a prorelation

   (ii) Need to show that $\psi.\phi.A \leq \psi.\phi$. So, Fix $p \in \psi$ and $q \in \phi$. As ϕ is a promodule, $\phi.A \leq \phi$ gives that there exists $q' \in \phi$ and $a \in A$ such that $q'\,a \subseteq q$. Thus, $p\,q'\,a \subseteq p\,q$.

   (iii) Need to show that $C.\psi.\phi \leq \psi.\phi$.Fix $p \in \psi$ and $q \in \phi$. Because ψ is a promodule, $C.\psi \leq \psi$ gives that there exists $c \in C$ and $p' \in \psi$ such that $c\,p' \subseteq p$. Thus, $c\,p'\,q \subseteq p\,q$

(c) (2-Identities) As every promodule is contained in itself, always have $\psi \leq \psi$. Define this comparison to be the identity 2-cell for ψ and denote it by $\leq_\psi$

(d) (Vertical 2-composition) For promodules $\psi, \phi, \delta : (X, A) \nrightarrow (Y, B)$, if there is is a 2-cell from ψ to ϕ and another one from ϕ to δ i.e. $\psi \leq \phi \leq \delta$, then by transitivity of the partial order, $\psi \leq \delta$ i.e. there's a 2-cell from ψ to δ.

(e) (Horizontal 2-composition) If there are promodules $\psi, \psi' : (X, A) \nrightarrow (Y, B)$ and $\phi, \phi' : (Y, B) \nrightarrow (Z, C)$ such that $\psi \leq \psi'$ and $\phi \leq \phi'$, need to show that $\psi.\phi \leq \psi'.\phi'$. Fix $p' \in \psi'$ and $q' \in \phi'$. As $\psi \leq \psi'$, $\exists p \in \psi : p \subseteq p'$ and as $\psi \leq \psi'$, $\exists q \in \phi : q \subseteq q'$. Thus, $p\,q \subseteq p'\,q'$

(f) (1-Identity) Need to show that for any promodule $\phi : (X, A) \nrightarrow (Y, B)$, $\phi.A = \phi = B.\phi$. By quasi-uniformity of A, every $a \in A$, is reflexive. Thus, for any $p \in \phi$ and $a \in A$, $p = p.\Delta_X \subseteq p\, a$ giving that $\phi \leq \phi.A$. And as ϕ is a promodule, $\phi \geq \phi.A$. Hence, by anti-symmetry of the partial order, $\phi = \phi.A$.

Similarly, By quasi-uniformity of B, every $b \in B$, is reflexive. Thus, for any $p \in \phi$ and $b \in B$, $p = \Delta_Y .p \subseteq b\,p$ giving that $\phi \leq B.\phi$. And as ϕ is a promodule, $\phi \geq B.\phi$. Hence, $\phi = B.\phi$.

(g) (1-Associativity) As composition of relations is associative, so too is the composition of prorelations directly giving that composition of promodules i.e. 1-cells is associative.

(h) (Vertical 2-Identity) Let $\leq : \psi \to \phi$ be a 2-cell i.e. $\psi \leq \phi$. By our definition of identity 2-cell, $\leq_\psi . \leq_1$ means precisely that $\psi \leq \psi \leq \phi$, and by transitivity, this is equivalent to $\psi \leq \phi$. Similarly, $\leq_1 . \leq_\phi$ means exactly that $\psi \leq \phi \leq \phi$, and this is equivalent to $\psi \leq \phi$.

(i) (Vertical 2-Associativity) Associativity of the partial order on promodules directly gives the associativity of composition of 2-cells in ProMod.

(j) (Horizontal 2-Identity) Let $\psi, \phi : (X, A) \nrightarrow (Y, B)$ be promodules. For any 2-cell $\leq : \psi \to \phi$, need to show that the 2-cell given by the horizontal composition, $\leq * \leq_A$ is equal to $\leq$, as well as equal to $\leq_B * \leq$. So, it's required that $\psi.A \leq \phi.A \iff \psi \leq \phi \iff B.\psi \leq B.\phi$. And this holds as a direct consequence of (f).

(k) (Horizontal 2-Associativity) As there's a unique 2-cell between any two promodules, and composition of promodules is associative, horizontal composition of 2-cells is associative.

(l) (2-Identity) For promodules $\psi : (X, A) \nrightarrow (Y, B)$ and $: \phi(Y, B) \nrightarrow (Z, C)$ need $(\leq_\psi * \leq_\phi) = \leq_{\psi.\phi}$. Both sides of the required equality are 2-cells $\leq : \psi.\phi \to \psi.\phi$. Thus, they are equal by the uniqueness of 2-cells between any two 1-cells.

(m) (2-Interchange) Let $\psi, \phi, \delta : (X, A) \nrightarrow (Y, B)$ and $\psi', \phi', \delta' : (Y, B) \nrightarrow (Z, C)$ be promodules. For 2-cells $\leq_1 : \psi \to \phi$,$\leq_2 : \phi \to \delta$, $\leq_a : \psi' \to \phi'$ and $\leq_b : \phi' \to \delta'$,need to show $(\leq_b . \leq_a) * (\leq_2 . \leq_1) = (\leq_b * \leq_2).(\leq_a * \leq_1)$. Both RHS and LHS are 2-cells from $\psi.\psi'$ to $\delta.\delta'$ and are hence equal.

□

Proposition 3.3. Functor, $(-)_* : \mathrm{QUnif} \to \mathrm{ProMod}$ is defined as fixing objects and taking morphisms to their image under $(-)_*$

(a) for $(X, A) \in \mathrm{QUnif}$, $(X, A)_* := (X, A) \in \mathrm{ProMod}$

(b) for $f : (X, A) \to (Y, B)$ in QUnif, $f_* := f^o.B$

Proof. □

Proposition 3.4. Functor, $(-)^* : \mathrm{QUnif}^{op} \to \mathrm{ProMod}$ is defined as fixing objects and taking morphisms to their image under $(-)^*$

(a) for $(X, A) \in \mathrm{QUnif}^{op}$, $(X, A)^* := (X, A) \in \mathrm{ProMod}$

(b) for $f : (X, A) \to (Y, B)$ in QUnif, $f^* := f^o.B$

Proof.

Showing that $f^o.B : (Y, B) \nrightarrow (X, A)$ is a promodule
So, need to show $f^o.B$ a prorelation $Y \to X$ and that $(f^o.B).B \sqsubseteq f^o.B$ and $A.(f^o.B) \sqsubseteq f^o.B$
To show prorelation,

(i) (Partial-order) Inclusion of relations i.e. for $k = f^o \circ b$ and $k' = f^o \circ b'$ in $f^o.B$, $k \subseteq k' \iff b \subseteq b'$

(ii) (Down directed) for $k, k' \in f^o.B$, need that $\exists l \in f^o.B$ such that $l \subseteq k, k'$

Fix $k, k' \in f^o.B \implies \exists b, b' \in B : k = f^o \circ b$ and $k' = f^o \circ b'$

And as B is a quasi-uniformity, it's down directed so, $\exists c \in B : c \subseteq b, b' \implies l := f^o \circ c \subseteq k, k'$

(iii) (Up-set) for a relation $l : Y \to X$ and $k \in f^o.B$ such that $l \supseteq k$, need $l \in f^o.B$

Let $b \in B$ be such that $k = f^o \circ b$ and define $b' := \{(y, y') : y \in \mathit{domain}(l) \text{ and } y' \in (f^o)^{-1}(l(y))\}$
As $l \supseteq k = f^o \circ b$, $\mathit{domain}(b') = \mathit{domain}(l) \supseteq \mathit{domain}(b)$
and $\mathit{range}(l) \supseteq \mathit{range}(f^o \circ b) \implies \forall y \in \mathit{domain}(b), \mathit{range}(b') = (f^o)^{-1}(l(y)) \supseteq (f^o)^{-1}(f^o \circ b) = \mathit{range}(b)$
Now, by definition of b', $f^o \circ b' \supseteq l$. To show $f^o \circ b \subseteq l$,
$(x, y) \in f^o \circ b' \implies \exists z \in Y : (x, z) \in b'$ and $(z, y) \in f^o \implies x \in \mathit{domain}(l)$ and $z \in l(x)$ i.e. $(x, z) \in l$

To show $(f^o.B).B \leq f^o.B$, need that $\forall b \in B, \exists b' \in B : f^o \circ b' \circ b' \subseteq f^o \circ b$,
Fix any $b \in B$, as B is a quasi-uniformity, $\exists b' \in B : b' \circ b' \subseteq b \implies f^o \circ b' \circ b' \subseteq f^o \circ b$

To show $A.(f^o.B) \leq f^o.B$, need that $\forall b \in B, \exists b' \in B, a \in A : a \circ f^o \circ b' \subseteq f^o \circ b$,
As f is uniformly continuous, $f.A \leq B.f$ i.e. $\forall b \in B, \exists a \in A : f \circ a \subseteq b \circ f \implies a = f^o \circ f \circ a \subseteq f^o \circ b \circ f$
Fix any $b \in B$, so, $\exists b' \in B : b'b' \subseteq b$ (for brevity, omitting $\circ$ to explicitly denote composition)
And, for this $b', \exists a : a \subseteq f^o b' f \implies a f^o b' \subseteq f^o b' f f^o b' \subseteq f^o b' b' \subseteq f^o b \implies a f^o b' \subseteq f^o b$

Now, need to show that $(-)^*$ respects composition and identity.

(i) (Composition) let f, g be uniformly continuous, $(X, A) \xrightarrow{f} (Y, B) \xrightarrow{g} (Z, C)$ need that $(g \circ f)^* = f^*.g^*$
LHS=$(g \circ f)^* = (g \circ f)^o.C = (f^o \circ g^o).C$ and RHS=$f^*.g^* = (f^o.B).(g^o.C)$
For equality, showing that LHS$\geq$RHS and LHS$\leq$RHS:

To show $(f^o \circ g^o).C \geq (f^o.B).(g^o.C)$, need that $\forall c \in C, \exists b \in B, c' \in C : f^o g^o c \supseteq f^o b g c'$
Fix any $c \in C$, so, $\exists c' \in C : c' \circ c' \subseteq c \implies f^o g^o c \supseteq f^o g^o (c'c') = f^o g^o (c' \Delta_Z c') \supseteq f^o g^o c' (g g^o) c'$
By uniform conntinuity of g, for $c' \in C, \exists b \in B : gb \subseteq c' g$
Thus, $f^o g^o c \supseteq f^o g^o (c' g) g^o c' \supseteq f^o (g^o g) b g^o c' = f^o b g^o c'$.

To show $(f^o \circ g^o).C \leq (f^o.B).(g^o.C)$, need that $\forall b \in B, c \in C, \exists c' \in C : f^o g^o c \subseteq f^o b g^o c$
Fix any $c \in C, b \in B$ will show that $c' := c$ works:
As B is a quasi-uniformity, $\Delta_Y \subseteq b \implies f^o \Delta_Y g^o c = f^o g^o c \subseteq f^o b g^o c = f^o b g^o c'$

(ii) (Identity) let $(X, A) \in$ QUnif ^{op}, and $1_{(X,A)} : (X, A) \to (X, A)$ as $x \mapsto x$ need that $(1_{(X,A)})^* = 1_{(X,A)^*}$
LHS=$(1_{(X,A)})^* = (1_{(X,A)})^o.A = 1_{(X,A)}.A = A$.
Now, it's required that A is the identity of (X, A) in ProMod.
So, fix $\phi : (X, A) \multimap (Y, B)$, need to show $\phi.A = \phi$
As ϕ is a promodule, $\phi.A \leq \phi$ and as A is quasi-uniformity on X,
$\forall a \in A, \Delta_X \subseteq a \implies \forall a \in A, \forall p \in \phi, p = p\Delta_X \subseteq pa \implies \phi \leq \phi.A$
Also, fix $\psi : (Y, B) \multimap (X, A)$, need to show $A.\psi = \psi$
As ψ is a promodule, $A.\psi \leq \psi$ and as A is quasi-uniformity on X,
$\forall a \in A, \Delta_X \subseteq a \implies \forall a \in A, \forall q \in \psi, q = \Delta_X q \subseteq aq \implies \psi \leq \psi.A$

□

Definiton 3.6. Let $f : (X, A) \to (Y, B)$ be a uniformly continuous function.

I f is said to be fully faithful if $f^*.f_* = A$

II f is said to be fully dense if $f_*.f^* = B$

Proposition 3.5. Fix a uniformly continuous map, $f : (X, A) \to (Y, B)$

(a) f is fully faithful if and only if $A \geq f^o.B.f$

(b) f is fully dense if and only if for any $b \in B$, $\exists b' \in B$ such that $b' \subseteq b\, f\, f^o\, b$

(c) f is topologically dense if and only if for any $b \in B$, $b\, f\, f^o\, b$ is reflexive

(d) f is fully dense if and only if f is topologically dense

Proof.

(a) (i) ($\implies$) Let f be fully faithful i.e. $f^*.f_* = A \implies f^o.B.B.f = A$
Need to show that $A = f^o.B.f$ i.e. $A \leq f^o.B.f$ and $A \geq f^o.B.f$
By hypothesis and quasi-uniformity of B, $A \geq f^o.B.B.f \geq f^o B.f$
To show $A \leq f^o.B.f$, need that $\forall b \in B, \exists a \in A : a \subseteq f^o b f$
Fix $b \in B$, hypothesis gives that $f^o.B.B.f \leq A$ so,
$\exists a \in A : a \subseteq f^o b b f$ and also, by quasi-uniformity of B, for $b, \exists b' \in B : b'b' \subseteq b \implies f^o b' b' f \subseteq f^o b f$
Combining the above two inequalities, $a \subseteq f^o b b f \subseteq f^o b f$

(ii) ($\impliedby$) Let $A = f^o.B.f$ need to show $A = f^o.B.B.f$ i.e. $A \geq f^o B.B.f$ and $A \leq f^o B.B.f$
To show $A \geq f^o.B.B.f$, need to show that $\forall a \in A, \exists b, b' \in B : a \supseteq f^o b b' f$
Have that $A \geq f^o.B.f$ and $B.B \leq B$
So, fix $a \in A$, now $\exists b \in B : a \subseteq f^o b f$ and for this b, $\exists b' \in B : b'b' \subseteq b$. Therefore, $a \supseteq f^o b f \supseteq f^o b' b' f$
To show $A \leq f^o.B.B.f$, need $\forall b, b' \in B, \exists a \in A : a \subseteq f^o b b' f$
Before that, uniform continuity of f along with Lemma 2.1.1 gives that

$f.A \leq B.f \implies A = f^o f.A \leq f^o.B.f$
So, fix $b, b' \in B$, now, as ,
$A \leq f^o.B.f$ giving
$\exists a \in A : a \subseteq f^o bf$ and $\exists a' \in A : a' \subseteq f^o b' f \implies \Delta_X \subseteq f^o b' f$.
Therefore $a = a\Delta_X \subseteq (f^o bf)(f^o b' f) \subseteq f^o bb' f$

(b) (i) ($\implies$) Let f be fully dense i.e. $B = f_* f^* = B.f.f^o.B$. showing that $\forall b \in B, \exists b' \in B : b' \subseteq bff^o b$:
So, fix $b \in B$, as $B \leq B.f.f^o.B$, there exists $b' \in B$ such that $b' \subseteq bff^o b$.

(ii) ($\impliedby$) Suppose $\forall b \in B, \exists b' \in B : b' \subseteq bff^o b$. This gives $B \leq B.f.f^o.B$, in order to show equality, also need $B \geq B.f.f^o.B$. By quasi-uniformity of B, for any $b \in B$, $\exists b' \in B : b'b' \subseteq b$. Now, by Lemma 2.4.2,

$$ff^o \subseteq \Delta_Y \implies b' ff^o b' \subseteq b' \Delta_Y b' = b'b' \subseteq b$$

(c) (i) ($\implies$) Let f be topologically dense, going to show that $\forall b \in B$, $(y, y) \in bff^o b$. So, fix any $b \in B$ and $y \in Y$. Now, by definition of $\overline{f(X)} = Y$, we get

$$\exists x \in X \text{ such that } (f(x), y) \in b \text{ and } (y, f(x)) \in b$$

Re-writing the above statement in terms of relations, and considering f as a relation:

$$(f(x), y) \in b \text{ gives } x(b \circ f)y \text{ i.e. } y \in (b \circ f)(x) \tag{1}$$
$$(y, f(x)) \in b \text{ gives } f(x) \subseteq b(y) \tag{2}$$

Repeatedly applying Lemma 2.4.3 to (2),

$$f(x) \subseteq b(y) \implies \big(f \circ f^o\big)(f(x) \subseteq \big(f \circ f^o\big)b(y) \implies \big(f \circ f^o \circ f\big)(x) \subseteq \big(f \circ f^o \circ b\big)(y)$$

Applying Lemma 2.4.1 to the final inequality in the above statement gives that

$$f(x) = (f \circ \Delta_X)(x) \subseteq \big(f \circ f^o \circ f\big)(x) \subseteq \big(f \circ f^o \circ b\big)(y)$$

Applying Lemma 2.4.3 and then using (1) on the above inequality completes the result:

$$f(x) \subseteq \big(ff^o b\big)(y) \implies \big(b \circ f\big)(x) \subseteq \big(bff^o b\big)(y) \implies y \in \big(bff^o b\big)(y) \text{ i.e. } y\big(bff^o b\big)y$$

(ii) ($\impliedby$) Fix any $y \in Y$ and $b \in B$. Also, suppose that $\Delta_Y \leq bff^o b$. As f is a function with domain as X, $f^o : Y \to X$, $\phi \neq (f^o \circ b)(y) \subseteq X$. So, fix $x \in (f^o \circ b)(y)$, going to show that $(f(x), y) \in b$ and $(y, f(x)) \in b$. Again, while viewing f as a relation.

$$\Delta_Y \leq bff^o b \implies \Delta_Y(y) \subseteq bff^o b(y) = \big(bf\big)(f^o b(y))$$

Last inequality of the above statement gives $y \in (bf)(x)$ i.e. $(f(x), y) \in b$.
Applying Lemma 2.4.2 to f, and then using Lemma 2.4.4,

$$ff^o \subseteq \Delta_Y \implies ff^o b \subseteq \Delta_Y b = b$$

Thus $ff^o b(y) \subseteq b(y)$ and hence $f(x) \subseteq b(y) \implies (y, f(x)) \in b$

(d) (i) ($\implies$) Let f be topologically dense. As B is a quasi-uniformity, for any $b \in B$,

$$\exists b' \in B : b'b' \subseteq b \text{ and } \Delta_Y \subseteq b' \implies b' = b'\Delta_Y \subseteq b'b' \subseteq b \tag{3}$$

By the characterisation of topologically dense in (c), have that $\Delta_Y \subseteq b' ff^o b'$. Now, using the (3) and Lemma 2.4.3,

$$\Delta_Y \subseteq b' ff^o b' \implies b' = b'\Delta_Y \subseteq b'b' ff^o b' \subseteq bff^o b' \subseteq bff^o b$$

Hence, we have $b' \in B : b' \subseteq bff^o b$ giving us that f is fully dense (from (b)).

(ii) ($\impliedby$) From (b), we have for $b \in B$, the existstence of $b' \in B$ such that $b' \subseteq bff^o b$. As B is a quasi-uniformity, $\Delta_Y \subseteq b'$. So, $\Delta_Y \subseteq bff^o b$, and from (c), this gives us that f is topologically dense.

□

Definiton 3.7. The set PX is defined to be the collection of all promodules from the quasi-uniform space(X,A) to the quasi-uniform space 1.

$$PX := \{\psi : \psi : (X, A) \to\!\!\!\!\!\circ\!\!\to 1 \text{ is a promodule}\}$$

Proposition 3.6. For any $a \in A$, $\tilde{a}$ is defined to be a relation $PX \to PX$ as

$$\text{for } \phi, \psi \in PX,\ \phi\, \tilde{a}\, \psi \ \text{ only if } \ \phi \leq \psi.a$$

The set, $\tilde{A} := \{\tilde{a} : a \in A\}$ defines a quasi-uniformity on PX.

Proof. First need to show that $\tilde{A}$ is a prorelation,

(i) (Partial order) For any two relations $\tilde{a}, \tilde{b} : PX \to PX$, define $\tilde{a} \leq \tilde{b}$ to be true only if $a \subseteq b$.

(ii) (Down-Directed) Need for any $\tilde{a}, \tilde{b} \in \tilde{A}$, the existstence of some $\tilde{c} \in A$ such that $c \subseteq a, b$
If $\tilde{a}, \tilde{b} \in A$ then there exist $a, b \in A$. By down-directedness of A, there exists a $c \in A$ such that $c \subseteq a, b$. Now the definition of $\tilde{A}$ gives that $\tilde{c} \in \tilde{A}$. And the definition of the partial order on $\tilde{A}$ ensures $\tilde{c} \leq \tilde{a}, \tilde{b}$.

(iii) (Upset) For any relation $l : PX \to PX$, need that if $\tilde{k}$ belongs to $\tilde{A}$ such that $l \geq \tilde{k}$, then $l \in \tilde{A}$.
Fix any $k : PX \to PX$, and $\tilde{k} \in \tilde{A}$ such that $l \geq \tilde{k}$. As k is a relation between promodules $X \twoheadrightarrow 1$, it can be thought of as a relation a on X, defined as:

$$a := \{(x, y) : x \in domain(\psi) \text{ and } y \in domain(\phi) \text{ whenever } \exists \psi, \phi \in PX : \psi l \phi\}$$

So, $l \geq \tilde{k}$ gives that $\tilde{a} \geq \tilde{k}$ i.e. $a \supseteq k$. And as A is an upper-set, we get $a \in A$. Now, by definition of $\tilde{A}$, $l \in \tilde{A}$.

Secondly, need show that the other two conditions hold for $\tilde{A}$,

(i) For all $\tilde{a} \in \tilde{A}$, need $\tilde{a}$ to be reflexive i.e if $\psi \in PX$ then $\psi\, \tilde{a}\, \psi$.
By definition of $\tilde{a}$, need to show that $\psi \leq \psi.a$. So, fix a $p \in \psi$, we will show that $p \subseteq p.a$. Quasi-uniformity of A gives that $\Delta_X \subseteq a$. Hence, by Lemma 2.4.3, $p = p\, \Delta_X \subseteq p\, a$.

(ii) For all $\tilde{a} \in \tilde{A}$, need to find $\tilde{b} \in \tilde{A}$ such that $\tilde{b}\tilde{b} \leq \tilde{a}$
Before showing the result, proving that for any $x, y \in A$, $\tilde{x}\, \tilde{y} \leq \widetilde{xy}$ i.e. $\forall \psi, \phi \in PX$, $\psi(\tilde{x}\, \tilde{y})\phi \implies \psi\, \widetilde{xy}\, \phi$. If $\psi_1(\tilde{a}.\tilde{b})\psi_3$, then, the definition of composition gives that $\exists \psi_2$ such that $\psi_1\, \tilde{b}\, \psi_2\, \tilde{a}\, \psi_3$. Now, the definition of $\tilde{b}$ gives $\psi_1 \leq \psi_2\, b$ and that of $\tilde{a}$ gives $\psi_2 \leq \psi_3\, a$. Combining these inequalities, $\psi_1 \leq \psi_2.b \leq \psi_3.ab$. Hence, by definition of $\widetilde{ab}$, $\psi_1\, (\widetilde{ab})\, \psi_3$. Now, to show the result, fix any $\tilde{a} \in \tilde{A}$. Therefore,$a \in A$, and by quasi-uniformity of A, $\exists b \in A : b \circ b \subseteq a$. Thus, by the partial-order defined on $\tilde{A}$, $\widetilde{bb} \leq \tilde{a}$. Now, transitivity of the partial order gives us the required result, $\tilde{b}\, \tilde{b} \leq \widetilde{bb} \leq \tilde{a}$.

□

Proposition 3.7 (Yoneda Embedding)**.**
For a quasi-uniform space (X, A), function $y_X : X \to PX$ is defined by $x \mapsto x^*$ for $x \in X$.

(a) $y_X : (X, A) \to (PX, \tilde{A})$ is a uniformly continuous map

(b) $y_X : (X, A) \to (PX, \tilde{A})$ is fully faithful

Proof.

(a) In order to show y_X is uniformly continuous, need to show that $y_X.A \leq \tilde{A}.y_X$. By definition of $\leq$, need $\forall a \in A, \exists b \in A : y_X \circ b \subseteq \tilde{a} \circ y_X$. Applying the relations to some element, x of the set X:

$$\big(y_X \circ b\big)(x) \subseteq \big(\tilde{a} \circ y_X\big)(x) \implies y_X(b(x)) \subseteq \tilde{a}(x^*) \tag{4}$$

So, for the condition given by (4) to hold, if $y \in b(x)$, then it's required that $y^* = y_X(y) \in \tilde{a}(x^*)$ i.e. $x^* \tilde{a} y^*$. Using the definition of x^*, y^* and $\tilde{a}$,

$$x^* \tilde{a} y^* \iff x^o.A \leq y^o.A.a \iff \forall a' \in A, \exists a'' \in A : x^o a'' \subseteq y^o a' a \tag{5}$$

Now, fix any $a \in A$, $x \in X$. Thus, quasi-uniformity of A, gives $a'' \in A$ such that $a''a'' \subseteq a$.Also, choose some $y \in a''(x)$. Hence, in order to show that the condition from (5) holds, need that $\forall b \in A, x^o a'' \subseteq y^o ba$, and by applying the relations to an element z gives the following condition

$$\forall b \in B, \forall x \in X\ ,\ \big(x^o a''\big)(z) \subseteq \big(y^o ba\big)(z) \tag{6}$$

Examining the left side of (6),

$$\big(x^o a''\big)(z) = x^o(a''(z)) = \begin{cases} \phi & \text{if } x \notin a''(z) \\ \star & \text{if } \in a''(z) \end{cases}$$

Thus, to show that (6) holds, need to show that (for any $b \in A$ and $z \in X$):

$$x \in a''(z) \implies z(y^o ba)\star \text{ i.e. } y \in (ba)(z) \tag{7}$$

To show that (7) holds, fix any $z \in X : x \in a''(z)$. Also, by our choice of y, have that $y \in a''(x)$. And as $b \in A$, it's reflexive, giving that $y \in b(y)$. So, by composition of relations, we get:

$$za''x \text{ , } xa''y \text{ and } yby \implies z(a''a''b)y \implies z(ab)y \text{ i.e. } y \in (ba)(z)$$

(b) By using Proposition2.3(a), need to show that $A \geq y_X^o.\tilde{A}.y_X$ i.e. $\forall a \in A, \exists \tilde{b} \in \tilde{A} : a \supseteq y_X^o \; \tilde{b} \; y_X$. Applying to an element, $x \in X$ gives the condition

$$\Big(y_X^o \; \tilde{b} \; y_X\Big)(x) \subseteq a(x) \implies \Big(y_X^o \; \tilde{b}\Big)(x^*) = y_x^o\Big(\tilde{b}(x^*)\Big) \subseteq a(x) \tag{8}$$

Thus, if $y^* \in PX$ such that $x^*\tilde{b}y^*$, then $y \in y_x^o\Big(\tilde{b}(x^*)\Big)$. Now, for (8) to hold, $y \in a(x)$ i.e. xay. Thus, need only to show that for any $a \in A, \exists b \in A$ such that $\forall x, y \in X, x^*\tilde{b}y^* \implies xay$. So, fix $a \in A$, and take $b \in A : bb \subseteq a$. Now, let $x^*\tilde{b}y^*$ i.e. $x^o.A \leq y^o.A.b$. Hence, $\exists c \in A : x^o c \subseteq y^o bb$. And as c is reflexive,

$$xcx \implies x(cx^o)\star \implies x(bby^o)\star \implies x(bb)y \implies xay$$

□

Theorem 3.1 (Yoneda Lemma). *For every $\psi \in PX$, in the following digram,*

$$\begin{tikzcd} X \arrow[r, "(y_X)_*"] \arrow[dr, "\psi"'] & PX \arrow[d, "\psi^*"] \\ & 1 \end{tikzcd}$$

(a) $\psi \geq \psi^*.(y_X)^*$

(b) $\psi \in \overline{y_X(X)} \implies \psi \leq \psi^*.(y_X)_*$

Proof. (a) By definition, $(y_X)_* = \tilde{A}.y_X$, and $\psi^* = \psi^o.\tilde{A}$. Need that $\psi \geq (y_X)_*.\psi^* = \psi^o.\tilde{A}.\tilde{A}.y_X$. And applying Lemma 2.2.1 to $\tilde{A}$, the required condition becomes $\psi \geq \psi^o.\tilde{A}.y_X$ Fix $p \in \psi$, we will find $a \in A : p \supseteq \psi^o a y_X$. Examining the right side of the condition, (for any $a \in A$, $x \in X$)

$$\Big(\psi^o.\tilde{a}.y_X\Big)(x) = \psi^o.\tilde{a}(x^*) = \psi^o\big(\tilde{a}(x^*)\big) = \begin{cases} \phi & \text{if } \psi \notin \tilde{a}(x^*) \\ \star & \text{if } \psi \in \tilde{a}(x^*) \end{cases} \tag{9}$$

In case $\psi \notin \tilde{a}(x^*)$, the condition holds trivially. As ψ is a promodule, $\psi.A \leq \psi$ gives $\exists q \in \psi, a \in A : qa \subseteq p$. Thus, fix $x \in X$ and $\psi \in PX$ such that $x^*\tilde{a}\psi$. We will now show that $xp\star$. Using the definition of $\tilde{a}$,

$$x^*\tilde{a}\psi \implies x^o.A \leq \psi.a \implies \exists b \in A : x^o b \subseteq qa \implies \forall z \in X, \big(x^o b\big)(z) \subseteq (qa)(z) \tag{10}$$

Thus, in particular for $z = x$, as b is reflexive, xbx, which gives:

$$\big(x^o b\big)(x) \subseteq (qa)(x) \implies x^o x \subseteq (qa)(x) \implies \star \in (qa)(x) \tag{11}$$

But, as $qa \subseteq p$, (11) gives that $xp\star$.

(b) Suppose $\psi \in \overline{y_X(X)}$, need to show $\psi \leq \psi^*.(y_X)_* = \psi^o.\tilde{A}.y_X$ i.e. for $a \in A$, $\exists p \in \psi : p \subseteq \psi^o.\tilde{a}.y_X$. For any $x \in domain(p)$, the condition requires:

$$p(x) \subseteq \psi^o.\tilde{a}.y_X(x) = \psi^o\big(\tilde{a}(x^*)\big) \tag{12}$$

By definition of p, for (12) to hold, need that $xp\star \implies \psi \in \tilde{a}(x^*)$. Fix any $a \in A$, we will find $p \in \psi$ such that (12) holds. By quasi-uniformity of A, $\exists b \in A : bb \subseteq a$. From Proposition 2.5(a), y_X is uniformly continuous, $y_X.A \leq \tilde{A}.y_X$ giving that $\exists c \in A : y_x c \subseteq \tilde{b} y_X$. Thus, for any $z, w \in X$ such that zcw,

$$\big(y_X c\big)(z) \subseteq \big(\tilde{b} y_X\big)(z) \implies y_X(c(z)) \subseteq \tilde{b}(z^*) \implies w^* \in \tilde{b}(z^*) \text{ i.e. } z^*\tilde{b}w^* \tag{13}$$

As A is a quasi-uniformity, $\exists d \in A : dd \subseteq c$. Also, because A is a down- directed set, $\exists a' \in A : a' \subseteq b, d$. This along with (13) gives that for any $x, y \in X$

$$x(a'a')y \implies x(dd)y \implies xcy \implies x^*\tilde{b}y^* \tag{14}$$

Now, because $\psi \in \overline{y_X(X)}$, we get $\exists x^* \in y_X(X)$ such that $\psi\tilde{a'}x^*$ and $x^*\tilde{a'}\psi$. By definition of $\tilde{a}$, $\psi\tilde{a'}x^*$ gives

$$\psi \leq x^o.A.a' \implies \exists p \in \psi : p \subseteq x^o a' a' \tag{15}$$

Fix any $z \in X : zp\star$, using (15) and (14) gives:

$$zp\star \overset{z}{\Rightarrow} (x^o a' a')\star \overset{(15)}{\Longrightarrow} z(a'a')x \overset{(14)}{\Longrightarrow} z^*\tilde{b}x^* \tag{16}$$

Finally, by definition of the partial order on $\tilde{A}$,$a' \subseteq b \implies \tilde{a'} \subseteq \tilde{b}$. Therefore, $x^*\tilde{a'}\psi \implies x^*\tilde{b}\psi$. Now, using (16), $z^*\tilde{b}x^*$ and $x^*\tilde{b}\psi$ gives the desired result $z^*\tilde{b}x^*$.

□

Corollary 3.1. *For $\psi \in PX$, $\psi \in \overline{y_X(X)}$ if and only if ψ is a right-adjoint.*

Proof. (i) ($\implies$)

(ii) ($\impliedby$) Suppose ψ is a right adjoint. Need to show that for any $a \in A$, $\exists x^* \in y_X(X)$ such that $\psi\, \tilde{a}\, x^* \tilde{a}\, \psi$. Fix $a \in A$. Because ψ is a right-adjoint, there exists a promodule $\phi : 1 \to\!\!\!\!\circ\!\!\to X$ such that $\phi.\psi \leq A$ and $1 \leq \psi.\phi$. From $\phi.\psi \leq A$, we get that

$$\exists p \in \phi, q \in \psi \text{ such that } a \supseteq p.q \tag{1}$$

Because ϕ and ψ are promodules,

$$A.\phi \leq \phi \text{ gives the existence of } p' \in \phi \text{ such that } p \supseteq a'p' \tag{2}$$
$$A.\psi \leq \psi \text{ gives the existence of } q' \in \psi \text{ and } a'' \in A \text{ such that } q \supseteq a''q' \tag{3}$$

Now, from $1 \leq \psi.\phi$, we get that $q'\, p'$ is reflexive i.e. $\star(q'\, p')\star$. By the definition of composition we get the existence of an $x \in X$ such that $\star\, p'\, x\, q'\, \star$. Now, considering x as a map, $x : 1 \to X$ defined as $\star \mapsto x$,

$$x\, q'\, \star \text{ i.e. } \star \in q'(x) \text{ gives that } q' \supseteq x^o \tag{4}$$
$$\star\, p'\, x \text{ i.e. } x \in p'(\star) \text{ gives that } p' \supseteq x \tag{5}$$

Thus, by using inequalities (1),(2) and (3), we get that

$$a \supseteq p\, q \supseteq a'\, p'\, q'\, a'' \tag{6}$$

By definition of $\tilde{a}$, to show $\psi\, \tilde{a}\, x^*$, we need that $\psi \leq x^*\, a = x^o.\, A.\, a$. Showing for any $b \in A$, $x^o\, b\, a \supseteq q'$:

$$x^o\, b\, a \supseteq x^o\, b\, a'\, p'\, q' \supseteq x^o\, b\, a'\, x\, q' \supseteq x^o\, x\, q' \subseteq q'$$

Where the first inequality comes from (6) by using reflexiviness of a'' and then left-multiplying by x^o. The second inequality comes from (5), third one from reflexiviness of b and a', and the last one is given by Lemma 2.4.1.

In order to show $x^*\, \tilde{a}\, \psi$, by definition of $\tilde{a}$, need that $x^o.A = x^* \leq \psi\, a$. Fix $k \in \psi$. We will show $k\, a \supseteq x^o\, a''$.

$$a \supseteq a'\, p'\, q'\, a'' \supseteq p'\, q'\, a'' \supseteq p'\, x^o\, a'' \tag{7}$$

Where the first inequality is given by (6), second one is due to reflexiviness of a' and the third inequality comes by using (4). Left-multiplying (7) with k gives

$$ka \supseteq k\, p'\, x^o\, a'' \tag{8}$$

FINAL STEP LEFT !!

□