

# 1 Definitions

**Definiton 1.1** (Prorelation). A partial-ordered set of relations  $X \rightarrow Y$ , which is down-directed and an upper set. i.e A set,  $P \subseteq \mathcal{P}(X \times Y)$  such that

- (i) A partial-order defined to be containment as relations,  $r \subseteq s$  only if  $\forall (x, y) \in X \times Y, (x, y) \in r \implies (x, y) \in s$
- (ii) (Down-directed),  $\forall r, s \in P, \exists t \in P$  such that  $t \subseteq r$  and  $t \subseteq s$
- (iii) (Up-set) for any relation  $u : X \rightarrow Y$ , if  $\exists p \in P$  such that  $p \leq u$  then  $u \in P$

**Definiton 1.2** (Composition of prorelations). Prorelations can be composed by taking all compositions of their elements as relations: for prorelations  $P : X \rightarrow Y$  and  $Q : Y \rightarrow Z$ ,

$$Q.P := \{q \circ p : p \in P \text{ and } q \in Q\}$$

**Definiton 1.3** (Comparison of Prorelations). Two prorelations with same domain, co-domain are comparable as

$$\text{for } P, Q : X \rightarrow Y, P \leq Q \text{ if } \forall q \in Q, \exists p \in P \text{ such that } p \subseteq q$$

**Definiton 1.4** (Quasi-uniformity). A prorelation on a set  $X$ ,  $P : X \rightarrow X$  is a quasi-uniformity if it follows :

- i  $\forall p \in P$ , for any  $x \in X$ ,  $(x, x) \in p$  i.e.  $xpx$
- ii  $\forall p \in P, \exists p' \in P$  such that  $p' \circ p' \subseteq p$

And in this case,  $(X, A)$  is called a *quasi-uniform space*.

**Definiton 1.5** (Uniformly Continuous function ). A function,  $f : X \rightarrow Y$  is called a uniformly continuous function,

$$f : (X, A) \rightarrow (Y, B) \text{ if, } \forall b \in B, \exists a \in A \text{ such that } f \circ a \subseteq b \circ f. \text{ meaning that } f.A \leq B.f \text{ or } \begin{array}{ccc} X & \xrightarrow{f} & Y \\ A \downarrow & \leq & \downarrow B \\ X & \xrightarrow{f} & Y \end{array}$$

**Definiton 1.6** (Promodule). A prorelation,  $\phi : X \multimap Y$  is called a promodule  $\phi : (X, A) \multimap (Y, B)$  if it obeys:  $\phi.A \leq \phi$  and  $B.\phi \leq \phi$  where  $.$  denotes composition as prorelations.

**Definiton 1.7** (Comparison of Promodules). Promodules with same domain and co-domain are compared as prorelations, for  $\phi, \psi : (X, A) \multimap (Y, B)$ ,  $\phi \subseteq \psi$ , only if  $\phi \leq \psi$ .

**Definiton 1.8** (Composition of Promodules). Promodules are composed as prorelations.

For promodules  $\phi : (X, A) \multimap (Y, B)$  and  $\psi : (Y, B) \multimap (Z, C)$ ,  $\psi\phi := \psi.\phi = \{q \circ p : p \in \phi \text{ and } q \in \psi\}$

**Definiton 1.9** (Opposite relation). For relation  $r : X \rightarrow Y$ ,  $r^o$  is defined to be a relation  $r^o : Y \rightarrow X$  as

$$\forall (x, y) \in X \times Y, (x, y) \in r \iff (y, x) \in r^o$$

**Lemma 1.9.1.** For any relation  $r : X \rightarrow Y$ ,  $r^o \circ r = \Delta_X$

**Lemma 1.9.2.** For any relation  $r : X \rightarrow Y$ ,  $r \circ r^o \subseteq \Delta_Y$

**Lemma 1.9.3.** For relations  $r, s : X \rightarrow Y$  and  $t : Y \rightarrow Z$ , for any  $x, x' \in X$ ,  $r(x) \subseteq s(x') \implies (t \circ r)(x) \subseteq (t \circ s)(x')$

**Lemma 1.9.4.** For relations  $r : X \rightarrow Y$  and  $s, t : Y \rightarrow Z$ ,  $s \subseteq t \implies (s \circ r) \subseteq (t \circ r)(x)$

**Definiton 1.10**  $((-)_*)$ .

**Definiton 1.11**  $((-)^*)$ .

**Definiton 1.12** (Fully Faithful).

**Definiton 1.13** (Fully Dense).

**Definiton 1.14** (Topologically Dense).

## 2 Propositions

**Definiton 2.1** (QUnif). QUnif is defined to be the category having quasi-uniform spaces as objects, and uniformly continous maps between them as morphisms.

**Lemma 2.1.1.** QUnif does define a category, as

- i Composition
- ii Identity

**Definiton 2.2** (ProMod).

**Lemma 2.2.1.** ProMod does define a category, as

- i Composition
- ii Identity

**Proposition 2.1**  $((-)_* : \text{QUnif} \rightarrow \text{ProMod} \text{ is a Functor})$ .

*Proof.* □

**Proposition 2.2**  $((-)^* : \text{QUnif}^{\text{op}} \rightarrow \text{ProMod} \text{ is a Functor})$ . Defined as fixing objects and taking morphisms to their image under  $(-)^*$

- (a) for  $(X, A) \in \text{QUnif}^{\text{op}}$ ,  $(X, A)^* := (X, A) \in \text{ProMod}$
- (b) for  $f : (X, A) \rightarrow (Y, B)$  in QUnif,  $f^* := f^o.B$

*Proof.*

**Showing that  $f^o.B : (Y, B) \rightrightarrows (X, A)$  is a promodule**

So, need to show  $f^o.B$  a prorelation  $Y \rightarrow X$  and that  $(f^o.B).B \sqsubseteq f^o.B$  and  $A.(f^o.B) \sqsubseteq f^o.B$

To show prorelation,

- (i) (Partial-order) Inclusion of relations i.e. for  $k = f^o \circ b$  and  $k' = f^o \circ b'$  in  $f^o.B$ ,  $k \subseteq k' \iff b \subseteq b'$
- (ii) (Down directed) for  $k, k' \in f^o.B$ , need that  $\exists l \in f^o.B$  such that  $l \subseteq k, k'$   
 Fix  $k, k' \in f^o.B \implies \exists b, b' \in B : k = f^o \circ b$  and  $k' = f^o \circ b'$   
 And as  $B$  is a quasi-uniformity, it's down directed so,  $\exists c \in B : c \subseteq b, b' \implies l := f^o \circ c \subseteq k, k'$
- (iii) (Up-set) for a relation  $l : Y \rightarrow X$  and  $k \in f^o.B$  such that  $l \supseteq k$ , need  $l \in f^o.B$   
 Let  $b \in B$  be such that  $k = f^o \circ b$  and define  $b' := \{(y, y') : y \in \text{domain}(l) \text{ and } y' \in (f^o)^{-1}(l(y))\}$   
 As  $l \supseteq k = f^o \circ b$ ,  $\text{domain}(b') = \text{domain}(l) \supseteq \text{domain}(b)$   
 and  $\text{range}(l) \supseteq \text{range}(f^o \circ b) \implies \forall y \in \text{domain}(b), \text{range}(b') = (f^o)^{-1}(l(y)) \supseteq (f^o)^{-1}(f^o \circ b) = \text{range}(b)$   
 Now, by definition of  $b'$ ,  $f^o \circ b' \supseteq l$ . To show  $f^o \circ b \subseteq l$ ,  
 $(x, y) \in f^o \circ b \implies \exists z \in Y : (x, z) \in b' \text{ and } (z, y) \in f^o \implies x \in \text{domain}(l) \text{ and } z \in l(x) \text{ i.e. } (x, z) \in l$

To show  $(f^o.B).B \leq f^o.B$ , need that  $\forall b \in B, \exists b' \in B : f^o \circ b' \circ b' \subseteq f^o \circ b$ ,

Fix any  $b \in B$  as  $B$  is a quasi-uniformity,  $\exists b' \in B : b' \circ b' \subseteq b \implies f^o \circ b' \circ b' \subseteq f^o \circ b$

To show  $A.(f^o.B) \leq f^o.B$ , need that  $\forall b \in B, \exists b' \in B, a \in A : a \circ f^o \circ b' \subseteq f^o \circ b$ ,

As  $f$  is uniformly continuous,  $f.A \leq B.f$  i.e.  $\forall b \in B, \exists a \in A : f \circ a \subseteq b \circ f \implies a = f^o \circ f \circ a \subseteq f^o \circ b \circ f$

Fix any  $b \in B$ , so,  $\exists b' \in B : b' \circ b' \subseteq b$  (for brevity, omitting  $\circ$  to explicitly denote composition)

And, for this  $b'$ ,  $\exists a : a \subseteq f^o b' f \implies a f^o b' \subseteq f^o b' f f^o b' \subseteq f^o b' b' \subseteq f^o b \implies a f^o b' \subseteq f^o b$

Now, need to show that  $(-)^*$  respects composition and identity.

- (i) (Composition) let  $f, g$  be uniformly continuous,  $(X, A) \xrightarrow{f} (Y, B) \xrightarrow{g} (Z, C)$  need that  $(g \circ f)^* = f^*.g^*$

LHS  $= (g \circ f)^* = (g \circ f)^o.C = (f^o \circ g^o).C$  and RHS  $= f^*.g^* = (f^o.B).(g^o.C)$

For equality, showing that LHS  $\geq$  RHS and LHS  $\leq$  RHS:

To show  $(f^o \circ g^o).C \geq (f^o.B).(g^o.C)$ , need that  $\forall c \in C, \exists b \in B, c' \in C : f^o g^o c \supseteq f^o b g^o c'$

Fix any  $c \in C$ , so,  $\exists c' \in C : c' \circ c' \subseteq c \implies f^o g^o c \supseteq f^o g^o (c' c') = f^o g^o (c' \Delta_Z c') \supseteq f^o g^o c' (g g^o) c'$

By uniform continuity of  $g$ , for  $c' \in C, \exists b \in B : g b \subseteq c' g$

Thus,  $f^o g^o c \supseteq f^o g^o (c' g) g^o c' \supseteq f^o (g g^o) b g^o c' = f^o b g^o c'$ .

To show  $(f^o \circ g^o).C \leq (f^o.B).(g^o.C)$ , need that  $\forall b \in B, c \in C, \exists c' \in C : f^o g^o c \subseteq f^o b g^o c'$

Fix any  $c \in C, b \in B$  will show that  $c' := c$  works:

As  $B$  is a quasi-uniformity,  $\Delta_Y \subseteq b \implies f^o \Delta_Y g^o c = f^o g^o c \subseteq f^o b g^o c = f^o b g^o c'$

- (ii) (Identity) let  $(X, A) \in \mathbf{QUnif}^{op}$ , and  $1_{(X,A)} : (X, A) \rightarrow (X, A)$  as  $x \mapsto x$  need that  $(1_{(X,A)})^* = 1_{(X,A)^*}$   
 $\text{LHS} = (1_{(X,A)})^* = (1_{(X,A)})^o.A = 1_{(X,A)}.A = A$ .  
 Now, it's required that  $A$  is the identity of  $(X, A)$  in  $\mathbf{ProMod}$ .  
 So, fix  $\phi : (X, A) \rightarrow (Y, B)$ , need to show  $\phi.A = \phi$   
 As  $\phi$  is a promodule,  $\phi.A \leq \phi$  and as  $A$  is quasi-uniformity on  $X$ ,  
 $\forall a \in A, \Delta_X \subseteq a \implies \forall a \in A, \forall p \in \phi, p = p\Delta_X \subseteq pa \implies \phi \leq \phi.A$   
 Also, fix  $\psi : (Y, B) \rightarrow (X, A)$ , need to show  $A.\psi = \psi$   
 As  $\psi$  is a promodule,  $A.\psi \leq \psi$  and as  $A$  is quasi-uniformity on  $X$ ,  
 $\forall a \in A, \Delta_X \subseteq a \implies \forall a \in A, \forall q \in \psi, q = \Delta_X q \subseteq aq \implies \psi \leq \psi.A$

□

**Proposition 2.3** (Proposition 1). Fix a uniformly continuous map,  $f : (X, A) \rightarrow (Y, B)$

- (a)  $f$  is fully faithful  $\iff A = f^o.B.f$   
 (b)  $f$  is fully dense  $\iff \forall b \in B, \exists b' \in B$  such that  $b' \subseteq bff^ob$   
 (c)  $f$  is topologically dense  $\iff \forall b \in B, \Delta_Y \subseteq b \circ f \circ f^o \circ b$   
 (d)  $f$  is fully dense  $\iff f$  is topologically dense

*Proof.*

- (a) (i) ( $\implies$ ) Let  $f$  be fully faithful i.e.  $f^*.f_* = A \implies f^o.B.B.f = A$   
 Need to show that  $A = f^o.B.f$  i.e.  $A \leq f^o.B.f$  and  $A \geq f^o.B.f$   
 By hypothesis and quasi-uniformity of  $B$ ,  $A \geq f^o.B.B.f \geq f^o.B.f$   
 To show  $A \leq f^o.B.f$ , need that  $\forall b \in B, \exists a \in A : a \subseteq f^obf$   
 Fix  $b \in B$ , hypothesis gives that  $f^o.B.B.f \leq A$  so,  
 $\exists a \in A : a \subseteq f^obbff$  and also, by quasi-uniformity of  $B$ , for  $b, \exists b' \in B : b'b' \subseteq b \implies f^ob'b'f \subseteq f^obf$   
 Combining the above two inequalities,  $a \subseteq f^obbff \subseteq f^obf$   
 (ii) ( $\impliedby$ ) Let  $A = f^o.B.f$  need to show  $A = f^o.B.B.f$  i.e.  $A \geq f^o.B.B.f$  and  $A \leq f^o.B.B.f$   
 To show  $A \geq f^o.B.B.f$ , need to show that  $\forall a \in A, \exists b, b' \in B : a \supseteq f^obb'b'f$   
 Have that  $A \geq f^o.B.f$  and  $B.B \leq B$   
 So, fix  $a \in A$ , now  $\exists b \in B : a \subseteq f^obf$  and for this  $b$ ,  $\exists b' \in B : b'b' \subseteq b$ . Therefore,  $a \supseteq f^obf \supseteq f^ob'b'f$   
 To show  $A \leq f^o.B.B.f$ , need  $\forall b, b' \in B, \exists a \in A : a \subseteq f^obb'b'f$   
 So, fix  $b, b' \in B$ , now, by hypothesis,  $A \leq f^o.B.f$  giving  
 $\exists a \in A : a \subseteq f^obf$  and  $\exists a' \in A : a' \subseteq f^ob'b'f \implies \Delta_X \subseteq f^ob'b'f$ .  
 Therefore  $a = a\Delta_X \subseteq (f^obf)(f^ob'b'f) \subseteq f^obb'b'f$   
 (b) (i) ( $\implies$ ) Let  $f$  be fully dense i.e.  $B = f_*f^* = B.f.f^o.B$ . showing that  $\forall b \in B, \exists b' \in B : b' \subseteq bff^ob$ :  
 So, fix  $b \in B$ , as  $B \leq B.f.f^o.B$ , there exists  $b' \in B$  such that  $b' \subseteq bff^ob$ .  
 (ii) ( $\impliedby$ ) Suppose  $\forall b \in B, \exists b' \in B : b' \subseteq bff^ob$ . This gives  $B \leq B.f.f^o.B$ , in order to show equality, also  
 need  $B \geq B.f.f^o.B$ . By quasi-uniformity of  $B$ , for any  $b \in B, \exists b' \in B : b'b' \subseteq b$ . Now, by Lemma 1.9.2,

$$ff^o \subseteq \Delta_Y \implies b'ff^ob' \subseteq b'\Delta_Y b' = b'b' \subseteq b$$

- (c) (i) ( $\implies$ ) Let  $f$  be topologically dense, going to show that  $\forall b \in B, (y, y) \in bff^ob$ . So, fix any  $b \in B$  and  
 $y \in Y$ . Now, by definition of  $\overline{f(X)} = Y$ , we get

$$\exists x \in X \text{ such that } (f(x), y) \in b \text{ and } (y, f(x)) \in b$$

Re-writing the above statement in terms of relations, and considering  $f$  as a relation:

$$(f(x), y) \in b \text{ gives } x(b \circ f)y \text{ i.e. } y \in (b \circ f)(x) \quad (1)$$

$$(y, f(x)) \in b \text{ gives } f(x) \subseteq b(y) \quad (2)$$

Repeatedly applying Lemma 1.9.3 to (2),

$$f(x) \subseteq b(y) \implies (f \circ f^o)(f(x)) \subseteq (f \circ f^o)b(y) \implies (f \circ f^o \circ f)(x) \subseteq (f \circ f^o \circ b)(y)$$

Applying Lemma 1.9.1 to the above statement gives that

$$f(x) = (f \circ f^o \circ f)(x) \subseteq (f \circ f^o \circ b)(y)$$

Applying Lemma 1.9.3 and then using (1) to this inequality completes the result:

$$f(x) \subseteq (ff^ob)(y) \implies (b \circ f)(x) \subseteq (bff^ob)(y) \implies y \in (bff^ob)(y) \text{ i.e. } y(bff^ob)y$$

- (ii) (  $\Leftarrow$  ) Fix any  $y \in Y$  and  $b \in B$ . Also, suppose that  $\Delta_Y \leq bff^ob$ . As  $f$  is a function with domain as  $X$ ,  $f^o : Y \rightarrow X$ ,  $\phi \neq (f^o \circ b)(y) \subseteq X$ . So, fix  $x \in (f^o \circ b)(y)$ , going to show that  $(f(x), y) \in b$  and  $(y, f(x)) \in b$ . Again, while viewing  $f$  as a relation.

$$\Delta_Y \leq bff^ob \implies \Delta_Y(y) \subseteq bff^ob(y) = (bf)(f^ob(y))$$

Using Lemma 1.9.3 on the above statement, gives  $y \in (bf)(x)$  i.e.  $(f(x), y) \in b$ . Applying Lemma 1.9.3 to  $f$ , and then using Lemma 1.9.4,

$$ff^o \subseteq \Delta_Y \implies ff^ob \subseteq \Delta_Y b = b$$

Thus  $ff^ob(y) \subseteq b(y)$  and hence  $f(x) \subseteq b(y) \implies (y, f(x)) \in b$

- (d) (i) (  $\implies$  ) Let  $f$  be topologically dense. As  $B$  is a quasi-uniformity, for any  $b \in B$ ,

$$\exists b' \in B : b'b' \subseteq b \text{ and } \Delta_Y \subseteq b' \implies b' = b'\Delta_Y \subseteq b'b' \subseteq b \quad (3)$$

By the characterisation of topologically dense in (c), have that  $\Delta_Y \subseteq b'ff^ob'$ . Now, using the above statement and Lemma 1.9.2,

$$\Delta_Y \subseteq b'ff^ob' \implies b' = b'\Delta_Y \subseteq b'b'ff^ob' \subseteq bff^ob' \subseteq bff^ob$$

Hence, we have  $b' \in B : b' \subseteq bff^ob$  giving us that  $f$  is fully dense (from (b)).

- (ii) (  $\Leftarrow$  ) From (b), we have for  $b \in B$ , the existstence of  $b' \in B$  such that  $b' \subseteq bff^ob$ . As  $B$  is a quasi-uniformity,  $\Delta_Y \subseteq b'$ . So,  $\Delta_Y \subseteq bff^ob$ , and from (c), this gives us that  $f$  is topologically dense.  $\square$

**Definiton 2.3** ( $PX$ ).  $PX := \{\psi : \psi : (X, A) \multimap 1 \text{ is a promodule}\}$

**Definiton 2.4** ( $\tilde{a}$ ). for any  $a \in A$ ,  $\tilde{a}$  is defined to be a relation  $PX \rightarrow PX$  as

$$\text{for } \phi, \psi \in PX, \phi \tilde{a} \psi \text{ only if } \phi \leq \psi.a$$

**Proposition 2.4** (Prorelation  $\tilde{A}$ ). The set,  $\tilde{A} := \{\tilde{a} : a \in A\}$  defines a quasi-uniformity on  $PX$ .

*Proof.* First, need to show that  $\tilde{A}$  is a prorelation,

- (i) (Partial order) Define, for any two relations  $\tilde{a}, \tilde{b} : PX \rightarrow PX$ , that  $\tilde{a} \leq \tilde{b}$  only if  $a \subseteq b$
- (ii) (Down-Directed) Need that  $\forall \tilde{a}, \tilde{b} \in \tilde{A}, \exists \tilde{c} \in \tilde{A} : \tilde{c} \subseteq \tilde{a}, \tilde{b}$   
 $\tilde{a}, \tilde{b} \in \tilde{A} \implies a, b \in A \implies \exists c \in A : c \subseteq a, b \implies \tilde{c} \leq \tilde{a}, \tilde{b}$
- (iii) (Upset) Need that, for any relation  $l : PX \rightarrow PX$ , if  $\exists \tilde{k} \in \tilde{A}$  such that  $l \geq \tilde{k}$ , then  $l \in \tilde{A}$   
 Fix any  $k : PX \rightarrow PX$ , and  $\tilde{k} \in \tilde{A}$  such that  $l \geq \tilde{k}$   
 Now,  $k$  is a relation between promodules  $X \multimap 1$ . Thus, it can be thought of as a relation on  $X$ ,  
 $a := \{(x, y) : x \in \text{domain}(\psi) \text{ and } y \in \text{domain}(\phi) \text{ whenever } \exists \psi, \phi \in PX : \psi l \phi\}$   
 So,  $l = \tilde{a}$  and thus,  $\tilde{a} \geq \tilde{k} \implies a \supseteq k \implies a \in A \implies l \in \tilde{A}$

Now to show that the other two conditions hold,

- (i) need that  $\forall \tilde{a} \in \tilde{A}, \forall \psi \in PX, \psi \tilde{a} \psi$   
 So, need to show that  $\psi \leq \psi.a$  i.e.  $\forall p \in \psi, \exists q \in \psi : q \subseteq p.a$ . Take  $q := p$ , and as  $A$  is a quasi-uniformity,  
 $\Delta_X \subseteq a \implies p = p.\Delta_X \subseteq p.a$
- (ii) Need that  $\forall \tilde{a} \in \tilde{A}, \exists \tilde{b} \in \tilde{A} : \tilde{b} \tilde{b} \leq \tilde{a}$   
 Before that, showing, for any  $x, y \in A, \tilde{x} \tilde{y} \leq \tilde{x} \tilde{y}$  i.e.  $\forall \psi, \phi \in PX, \psi(\tilde{x}.\tilde{y})\phi \implies \psi \tilde{x} \tilde{y} \phi$   
 Let  $\psi_1(\tilde{a}.\tilde{b})\psi_3 \implies \exists \psi_2 : \psi_1 \tilde{b} \psi_2 \tilde{a} \psi_3 \implies \psi_1 \leq \psi_2.b$  and  $\psi_2 \leq \psi_3.a \implies \psi_1 \leq \psi_2.b \leq \psi_3.ab \implies \psi_1(\tilde{a}b)\psi_3$   
 Fix any  $\tilde{a} \in \tilde{A} \implies a \in A \implies \exists b \in A : b \circ b \subseteq a \implies \tilde{b} \tilde{b} \leq \tilde{a} \implies \tilde{b} \tilde{b} \leq \tilde{b} \tilde{b} \leq \tilde{a}$

□

**Proposition 2.5** (Yoneda Embedding).

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