

# 1 Yoneda Lemma

**Lemma 1.1** ( $H_A$  or  $\mathcal{A}(-, A)$ ). For any category  $\mathcal{A}$ , fixing an object,  $A \in \mathcal{A}$ , there's a functor,  $H_A : \mathcal{A}^{op} \rightarrow Set$  defined as:

i For object  $B \in \mathcal{A}$ ,  $F(B) := Hom(B, A)$

ii For any morphism in  $\mathcal{A}$ ,  $g : X \rightarrow Y$ ,

$$H_A(g) : \mathcal{A}(Y, A) \rightarrow \mathcal{A}(X, A), \text{ as, } \forall p \in \mathcal{A}(Y, A), p \mapsto p \circ g \text{ i.e. } (H_A(g))(p) := p \circ g$$

**Theorem 1.1. Yoneda** If  $\mathcal{A}$  is a locally small category, for any object  $A \in \mathcal{A}$  and  $X \in [\mathcal{A}^{op}, Set]$ , there's exists a natural isomorphism:

$$[\mathcal{A}^{op}, Set](H_A, X) \cong X(A) \text{ naturally in } A \in \mathcal{A}$$

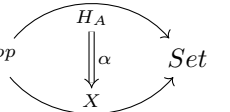
## Explanation:

First, fix any category,  $\mathcal{A}$ . Now, choose two things (independent of each other):

i an object,  $A$  from the category  $\mathcal{A} = \mathcal{A}^{op}$

ii an object,  $X \in [\mathcal{A}^{op}, Set]$ , the presheaf category  
i.e. a functor  $X : \mathcal{A}^{op} \rightarrow Set$

Here,  $[\mathcal{A}^{op}, Set](H_A, X)$  denotes morphisms  $H_A \rightarrow X$  in  $[\mathcal{A}^{op}, Set]$ , i.e. natural transformations,  $\alpha : H_A \rightarrow X$



Each of these natural transformations is a collection of, morphisms in  $Set$ , hence each of their components is exactly a function. i.e.  $\forall \alpha \in [\mathcal{A}^{op}, Set](H_A, X), \forall K \in \mathcal{A}, \alpha_K$  is a function  $: H_A(K) \rightarrow X(K)$

$X(A)$  is precisely a set, because  $X(A)$  is the image of (our chosen object,)  $A$ , under (our chosen functor,)  $X$ .

The key idea is that the choice of  $A$  and  $X$  completely determines all possible maps (i.e. natural transformations) from  $H_A$  to  $X$ .

Moreover, this isomorphism is *natural* in  $A$  and  $X$ .

Meaning that  $[\mathcal{A}^{op}, Set](H_A, X)$  and  $X(A)$  are *functorial* in both  $A$  and  $X$

## Notation:

- Denoting the category of all presheaves on  $\mathcal{A}$  by  $\mathcal{C}$ , i.e.  $\mathcal{C} := [\mathcal{A}^{op}, Set]$
- using  $\hat{\phantom{a}}$  as a map i.e.  $\hat{a} = b$  stands for  $a \xrightarrow{\hat{\phantom{a}}} b$
- using  $\tilde{\phantom{a}}$  as a map i.e.  $\tilde{a} = b$  stands for  $a \xrightarrow{\tilde{\phantom{a}}} b$

To prove the theorem, first, going to show that  $[\mathcal{A}^{op}, Set](H_A, X)$  is isomorphic to  $X(A)$ . And then that this isomorphism is natural.

*Proof.* Let a locally small category,  $\mathcal{A}$  be given.

Let  $A \in \mathcal{A}$  and  $X \in [\mathcal{A}^{op}, Set]$

**1. Defining  $\phi$  and  $\psi$**  Define  $\phi$  (on natural transformations) as the  $A$ -component (of that natural transformation) at the identity of  $A$ . i.e. for  $\alpha \in \mathcal{C}(H_A, X), \phi(\alpha) := \alpha_A(1_A)$

Define  $\psi$  on an object,  $x \in X(A)$ , by defining it's  $K$ -component for any  $K \in \mathcal{A}$ :

$$(\psi(x))_K : H_A(K) \rightarrow X(K) \text{ as, for each } p \in Hom_{\mathcal{A}}(K, A), p \mapsto (X(p))(x)$$

That is to say that the  $K$ -component maps any arrow  $p : K \rightarrow A$  to the image of  $x$  under the map  $X(p)$ .

□

## 2 Cayley's Theorem

## 3 Embedding of a category in Presheaf category

**Definiton 3.1** (Embedding of a category). A category,  $\mathcal{A}$  is said to be embedded in a category,  $\mathcal{B}$  if there exists a functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  such that  $F$  is full, faithful and injective (on objects).

*Proof.* Prove that  $H_A$  is indeed a functor

□

## 4 Quasi-Paper