

1 Yoneda Lemma

Lemma 1.1 (H_A or $\mathcal{A}(-, A)$). For any category \mathcal{A} , fixing an object, $A \in \mathcal{A}$, there's a functor, $H_A : \mathcal{A}^{op} \rightarrow Set$ defined as:

i For object $B \in \mathcal{A}$, $F(B) := Hom(B, A)$

ii For any morphism in \mathcal{A} , $g : X \rightarrow Y$,

$$H_A(g) : \mathcal{A}(Y, A) \rightarrow \mathcal{A}(X, A), \text{ as, } \forall p \in \mathcal{A}(Y, A), p \mapsto p \circ g \text{ i.e. } (H_A(g))(p) := p \circ g$$

Theorem 1.1. Yoneda If \mathcal{A} is a locally small category, for any object $A \in \mathcal{A}$ and $X \in [\mathcal{A}^{op}, Set]$, there's exists a natural isomorphism:

$$[\mathcal{A}^{op}, Set](H_A, X) \cong X(A) \text{ naturally in } A \in \mathcal{A}$$

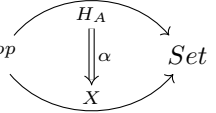
Explanation:

First, fix any category, \mathcal{A} . Now, choose two things (independent of each other):

i an object, A from the category $\mathcal{A} = \mathcal{A}^{op}$

ii an object, $X \in [\mathcal{A}^{op}, Set]$, the presheaf category
i.e. a functor $X : \mathcal{A}^{op} \rightarrow Set$

Here, $[\mathcal{A}^{op}, Set](H_A, X)$ denotes morphisms $H_A \rightarrow X$ in $[\mathcal{A}^{op}, Set]$, i.e. natural transformations, $\alpha : H_A \rightarrow X$



Each of these natural transformations is a collection of, morphisms in Set , hence each of their components is exactly a function. i.e. $\forall \alpha \in [\mathcal{A}^{op}, Set](H_A, X), \forall K \in \mathcal{A}, \alpha_K$ is a function $: H_A(K) \rightarrow X(K)$

$X(A)$ is precisely a set, because $X(A)$ is the image of (our chosen object,) A , under (our chosen functor,) X .

The key idea is that the choice of A and X completely determines all possible maps (i.e. natural transformations) from H_A to X .

Moreover, this isomorphism is *natural* in A and X .

Meaning that $[\mathcal{A}^{op}, Set](H_A, X)$ and $X(A)$ are *functorial* in both A and X

Notation:

- Denoting the category of all presheaves on \mathcal{A} by \mathcal{C} , i.e. $\mathcal{C} := [\mathcal{A}^{op}, Set]$
- using $\hat{}$ as a map i.e. $\hat{a} = b$ stands for $a \xrightarrow{\hat{}} b$
- using $\tilde{}$ as a map i.e. $\tilde{a} = b$ stands for $a \xrightarrow{\tilde{}} b$

To prove the theorem, first, going to show that $[\mathcal{A}^{op}, Set](H_A, X)$ is isomorphic to $X(A)$. And then that this isomorphism is natural.

Proof. Let a locally small category, \mathcal{A} be given.

Let $A \in \mathcal{A}$ and $X \in [\mathcal{A}^{op}, Set]$

Showing isomorphism, Define ϕ (on natural transformations) as the A -component (of that natural transformation) at the identity of A . i.e. for $\alpha \in \mathcal{C}(H_A, X), \phi(\alpha) := \alpha_A(1_A)$

Define ψ on an object, $x \in X(A)$, by defining it's K -component for any $K \in \mathcal{A}$:

$$(\psi(x))_K : H_A(K) \rightarrow X(K) \text{ as, for each } p \in Hom_{\mathcal{A}}(K, A), p \mapsto (X(p))(x)$$

That is to say that the K -component maps any arrow $p : K \rightarrow A$ to the image of x under the map $X(p)$.

□

2 Cayley's Theorem

3 Embedding of a category in Presheaf category

Definiton 3.1 (Embedding of a category). A category, \mathcal{A} is said to be embedded in a category, \mathcal{B} if there exists a functor $F : \mathcal{A} \rightarrow \mathcal{B}$ such that F is full, faithful and injective (on objects).

Proof. Prove that H_A is indeed a functor

□

4 Quasi-Paper