Question 5

$$f(x, y) = xy(1 - x^2 - y^2)$$

For each partial derivative to be zero,

$$f_x = y - 3x^2y - y^3 = 0 \text{ and } f_y = x - x^3 - 3xy^2 = 0$$
$$\implies f_x = y(1 - 3x^2 - y^2) = 0 \text{ and } f_y = x(1 - x^2 - 3y^2) = 0$$

So, a critical point is $(0, 0)$.

And,
If $x = 0, y \neq 0$ then $f_x = y(1 - y^2) = 0 \implies y = \pm 1$
gives two critical points: $(0, \pm 1)$

If $y = 0, x \neq 0$ then $0 = f_y = y(1 - x^2) \implies x = \pm 1$
gives two critical points: $(\pm 1, 0)$

If $x \neq 0 \neq y$, then

$$f_x = y - 3x^2y - y^3 = 0 \implies y(1 - 3x^2) = y^3$$
$$\implies 1 - 3x^2 = y^2 \qquad (equation A)$$

And, also,

$$f_y = x - 3y^2x - x^3 = 0 \implies x(1 - 3y^2) = x^3$$
$$\implies 1 - 3y^2 = x^2 \qquad (equation B)$$

Thus,

Substituting B into A, $1 - 3 + 9y^2 = y^2 \implies y = \pm 0.5$
And, substituting A into B, $1 - 3 + 9x^2 = x^2 \implies x = \pm 0.5$

Thus, the four such possible points are also critical:

$$(0.5, 0.5), (0.5, -0.5), (-0.5, 0.5), (-0.5, -0.5)$$

Now, to classify these critical points, looking at $rt - s^2$

$$r = f_{xx} = -6xy = f_{yy} = t \text{ and } s = f_{xy} = 1 - 3x^2 - 3y^2$$

$$\text{So, } rt = 36x^2y^2 \text{ and } s^2 = (1 - 3(x^2 + y^2))^2$$

At $(0, 0), rt = 0$ and $s = 1$. Thus, $rt - s^2 = 0 - 1 < 0$
Thus, $(0, 0)$ is a saddle point.

At $(\pm 1, 0)$ and $(0, \pm 1)$,
$rt = 36 \times 1 \times 0 = 36 \times 0 \times 1 = 0$ and $s^2 = (1 - 3)^2 = 4$.
Thus, $rt - s^2 = 0 - 4 < 0$
Thus, $(\pm 1, 0), (0, \pm 1)$ are saddle points.

At $(0.5, 0.5), (0.5, -0.5), (-0.5, 0.5), (-0.5, -0.5)$,
$rt = 36x^2y^2 = 36 \times 0.25 \times 0.25 = \frac{9}{4} = 2.25$
and $s^2 = (1 - 3(0.25 + 0.25))^2 = (-0.5)^2 = 0.25$
Hence, $rt - s^2 = 2.25 - 0.25 = 2 > 0$
Thus, $(0.5, -0.5), (-0.5, 0.5)$ are minima(as $r = 1.5$),
while $(0.5, 0.5), (-0.5, -0.5)$ are maxima(as $r = 1.5$).

Question 4 (b) $f(x,y) := \frac{2xy}{\sqrt{x^2+y^2}}$

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{1}{h}\Big(\frac{2h\times 0}{\sqrt{h^2+0}} - 0\Big) = 0$$

So, the partial derivative along x exists at $(0,0)$. And by symmetry, it also exists along y.
And at a point, (x,y) in a strict neighbourhood of $(0,0)$,

$$\frac{\partial f}{\partial y}(x,y) = \lim_{k\to 0}\frac{1}{k}\Big(\frac{2x(y+k)}{\sqrt{x^2+(y+k)^2}} - \frac{2xy}{\sqrt{x^2+y^2}}\Big)$$
$$= \lim_{k\to 0}\frac{1}{k}\Big(\frac{2x(y+k)\sqrt{x^2+y^2} - 2xy(\sqrt{x^2+(y+k)^2}}{\sqrt{x^2+y^2}\sqrt{x^2+(y+k)^2}}\Big)$$
$$= \lim_{k\to 0}\frac{2x}{k}\Big(\frac{(y+k)\sqrt{x^2+y^2} - y\sqrt{x^2+(y+k)^2}}{\sqrt{x^2+y^2}\sqrt{x^2+(y+k)^2}}\Big)$$

If the above derivative is continous,then $\lim_{(j,0)\to(0,0)} \frac{\partial f}{\partial x} = 0$

$$\frac{\partial f}{\partial y}(j,0) = \lim_{k\to 0}\frac{2j}{k}\Big(\frac{(0+k)\sqrt{j^2+0^2} - 0\sqrt{j^2+(0+k)^2}}{\sqrt{j^2+0^2}\sqrt{j^2+(0+k)^2}}\Big)$$
$$= \lim_{k\to 0}\frac{2j}{k}\Big(\frac{k|j|}{|j|\sqrt{j^2+k^2}}\Big) = \lim_{k\to 0}\frac{2j}{\sqrt{j^2+k^2}} = 2$$

Thus, $\frac{\partial f}{\partial y}$ is discontinous at (0,0) and by symmetry, so is $\frac{\partial f}{\partial x}$.

Hence $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial x^2}$ don't exist at $(0,0)$. For differentiability at (0,0), consider the following limit along line $i = j$,

$$\lim_{h^2+k^2\to 0} \frac{f(0+h,0+k) - f(0,0) - h\frac{\partial f}{\partial x}(0,0) - k\frac{\partial f}{\partial y}(0,0)}{\sqrt{h^2+k^2}}$$

$$= \lim_{h^2+k^2\to 0} \frac{\frac{2hk}{\sqrt{h^2+k^2}} - 0 - h\times 0 - k\times 0)}{\sqrt{h^2+k^2}} = \lim_{j=i\to 0} \frac{1}{\sqrt{2j^2}}\frac{2j^2}{\sqrt{2j^2}} = 1$$

Hence, as aforementioned limit is non-zero for some path through $(0,0)$, the function isn't differentiable at $(0,0)$.

(a) $f(x,y) := \frac{xy(x+y)sin(x-y)}{x^2+y^2}$

The partial derivative along x at $(0,0)$,

$$\frac{\partial f}{\partial x} = \lim_{h\to 0}\Big(\frac{f(h,0) - f(0,0)}{h}\Big)$$

$$= \lim_{h\to 0} \frac{(h\times 0(h+0)sin(h-0) - 0)}{h^2+0} = 0$$

And at a point, (a,b) in the neighbourhood of $(0,0)$ partial derivative along x is,

$$\lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} =$$

$$\lim_{h\to 0} \frac{1}{h}\Big(\frac{(x+h)y(x+h+y)sin(x+h-y)}{(x+h)^2+y^2} - \frac{xy(x+y)sin(x-}{x^2+y^2}$$

(WIP)

$$\frac{\partial f}{\partial x} = \frac{y\Big((-x^2y+2xy^2+y^3)sin(x-y) + (x^3+x^3y+x^2y^2+xy}{(x^2+y^2)^2}$$

Thus, for the second partial derivative along x,

$$\frac{\partial^2 f}{\partial x^2} = \lim_{h \to 0} \frac{1}{h}\Big(\frac{\partial f(h,0)}{\partial x} - \frac{\partial f(0,0)}{\partial x}\Big) = \lim_{h \to 0} \frac{(0 \times (...)) - 0}{h} = 0$$

Similarly, along the y-direction,

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

And,

$$\frac{\partial f}{\partial x} = \frac{-x}{(x^2+y^2)^2}[(-x^3 - 2x^2y + xy^2)sin(x-y) + ...]$$

the second order derivative is,

$$\frac{\partial^2 f}{\partial x^2} = \lim_{h \to 0} \frac{f_y(0,h) - f_y(0,0)}{h} = 0$$

Now, as the second partial derivative exists along x and y directions, the first partial derivatives are continous. Hence, f is differentiable at (0,0).

Question 3 (Marked) **(a)** $f(x,y) := \frac{x^2y^2}{x^2y^2+(x-y)^2} = \frac{1}{1+(\frac{x-y}{xy})^2} = \frac{1}{1+(\frac{1}{y}-\frac{1}{x})^2}$

$$\lim_{x\to 0} f(x,y) = \lim_{x\to 0} \frac{1}{1+(\frac{1}{y}-\frac{1}{x})^2} = 0 \implies \lim_{y\to 0}\lim_{x\to 0} f(x,y) = 0$$

And as the expression is symmetric in x and y,

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{x\to 0}\lim_{y\to 0} f(x,y)$$

But the simultaneous limit at$(0,0)$ along $T(t) := (t,t)$ is

$$\lim_{t\to 0} f(T(t)) = \lim_{t\to 0} \frac{1}{1+(\frac{1}{t}-\frac{1}{t})^2} = 1$$

If the simultaneous limit existed, all the iterated limits would be equal to it. So, there is a curve, $S(t) := (\frac{1}{t}, \frac{1}{t+1})$ with

$$\lim_{t\to\infty} f(S(t)) = \lim_{t\to\infty} \frac{1}{1+(t+1-t)^2} = \frac{1}{2}$$

Thus, f is discontinous at $(0,0)$

(b) $f(x,y) := \frac{e^{\frac{-1}{x^2}}y}{e^{\frac{-1}{x^2}}+y^2} = \frac{ye^{\frac{1}{x^2}}}{1+(ye^{\frac{1}{x^2}})^2} = \frac{1}{\frac{1}{ye^{1/x^2}}+ye^{1/x^2}}$

$$\lim_{y\to 0}\lim_{x\to 0} \frac{1}{\frac{1}{ye^{1/x^2}}+ye^{1/x^2}} = \lim_{y\to 0} 0 = 0$$

$$\lim_{x\to 0}\lim_{y\to 0} \frac{ye^{\frac{1}{x^2}}}{1+(ye^{\frac{1}{x^2}})^2} = \lim_{x\to 0} 0 = 0$$

To show the non-existence of simulatenous limit at $(0,0)$, consider the curve $T(t) := (t, e^{-1/t^2})$

$$\lim_{t\to 0} f(T(t)) = \frac{e^{-1/t^2}\times e^{1/t^2}}{1+(e^{-1/t^2}\times e^{1/t^2})^2} = \frac{1}{2}$$