Question 1

$$< a_n > \text{ is a real sequence; } \sigma_n := \frac{a_1 + a_2 + ... + a_n}{n}$$

Going to show that

I $\liminf a_n \leq \liminf \sigma_n$

If a_n is unbounded below, then $\liminf a_n = -\infty \leq \liminf \sigma_n$ So, let a_n be bounded below, thus LHS is a real number,

$$m := \liminf a_n$$

Now, if a_n is constant, and equal to a,

$$\forall n \in \mathbb{N} \text{ , } \inf\{a_i | i \geq n\} = a = \frac{na}{n} = \sigma_n$$

Otherwise, if a_n is not constant, then,

$$\exists i, j \in \mathbb{N} \text{ such that(wlog) } a_i < a_j$$

Suppose if possible, $\liminf a_n = m >$ $\liminf \sigma_n$ But,

$$\forall n \in \mathbb{N} \text{ such that } n \geq i, j,$$

$$\sigma_n = \frac{a_1 + ...a_i + a_j + ... + a_n}{n} \geq \frac{(n-2)m + a_i + a_j}{n}$$
$$\geq \frac{(n-1)m + a_j}{n}$$
$$> m \text{ [} \because m \leq a_i < a_j \text{]}$$

$$\text{Hence, } \inf \sigma_n \geq \frac{(n-1)m + a_j}{n}$$

$$\implies \liminf \sigma_n \geq \lim_{n\to\infty} \frac{(n-1)m + a_j}{n} = m$$

But this contradicts the initial assumption.

II $\limsup a_n \geq \limsup \sigma_n$ If a_n is unbounded above, then $\limsup a_n = \infty \geq \limsup \sigma_n$.

So, let a_n be bounded above, thus LHS is a real number,

$$M := \limsup a_n$$

Now, if a_n is constant, and equal to a,

$$\forall n \in \mathbb{N} \text{ , } \sup\{a_i | i \geq n\} = a = \frac{na}{n} = \sigma_n$$

Otherwise, if a_n is not constant, then,

$$\exists i, j \in \mathbb{N} \text{ such that(wlog) } a_i < a_j$$

Suppose if possible, $\limsup a_n = M <$ $\limsup \sigma_n$ But,

$$\forall n \in \mathbb{N} \text{ such that } n \geq i, j,$$

$$\sigma_n = \frac{a_1 + ...a_i + a_j + ... + a_n}{n} \leq \frac{(n-2)M + a_i + a_j}{n}$$

$$\leq \frac{(n-1)M + a_j}{n}$$

$$< M \;[\; \because M \geq a_j > a_i \;]$$

$$\text{Hence, } \inf \sigma_n \leq \frac{(n-1)M + a_j}{n}$$

$$\implies \liminf \sigma_n \leq \lim_{n\to\infty} \frac{(n-1)M + a_j}{n} = M$$

But this contradicts the initial assumption.

Question 2

$$\liminf \frac{a_{n+1}}{a_n} \leq \liminf\, (a_n)^{\frac{1}{n}} \leq \limsup\, (a_n)^{\frac{1}{n}} \leq \limsup \frac{a_{n+1}}{a_n}$$

I $\liminf \frac{a_{n+1}}{a_n} \leq \liminf\, (a_n)^{\frac{1}{n}}$

II $\limsup\, (a_n)^{\frac{1}{n}} \leq \limsup \frac{a_{n+1}}{a_n}$

Appendix

1. also, make sure to show liminf $\leq$ limsup