

# 1 Yoneda Lemma

**Theorem 1.1. Yoneda** If  $\mathcal{A}$  be a locally small category, then,

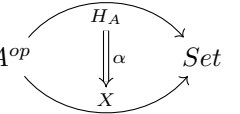
$$[\mathcal{A}^{op}, Set](H_A, X) \cong X(A) \text{ naturally in } A \in \mathcal{A} \text{ and } X \in [\mathcal{A}^{op}, Set]$$

**Explanation:**

First of all, we fix any category,  $\mathcal{A}$ . Now we choose two things (independent of each other):

- i an object,  $A$  from the category  $\mathcal{A} = \mathcal{A}^{op}$
- ii an object,  $X$  of the category  $[\mathcal{A}^{op}, Set]$  which is precisely a functor  $X : \mathcal{A}^{op} \rightarrow Set$

Here,  $[\mathcal{A}^{op}, Set](H_A, X)$  denotes arrows  $H_A \rightarrow X$  in  $[\mathcal{A}^{op}, Set]$  i.e. natural transformations,  $\alpha : \mathcal{A}^{op} \rightarrow Set$



Each of these natural transformations is a collection of maps in  $Set$ , hence each of their components is exactly a function. i.e.  $\forall \alpha \in [\mathcal{A}^{op}, Set](H_A, X), \forall K \in \mathcal{A}, \alpha_K$  is a function  $: H_A(K) \rightarrow X(K)$

$X(A)$  is precisely a set, because  $X(A)$  is the image of (our chosen object,)  $A$ , under (our chosen functor,)  $X$ .

So, the idea is that our choice of  $A$  and  $X$  completely determines all possible maps (i.e. natural transformations) from  $H_A$  to  $X$ . This answers our big question of "what are all the maps  $H_A \rightarrow X$ " or "how does  $H_A$  see other presheaves on  $\mathcal{A}$ ".

*The theorem says not just that the two are isomorphic, but that they're **naturally** isomorphic.*

This means that  $[\mathcal{A}^{op}, Set](H_A, X)$  and  $X(A)$  are *functorial* in both  $A$  and  $X$

Also, So, the aforementioned collection of natural transformations also must be a set: As  $\mathcal{A}$  is locally small, for each choice of  $K \in \mathcal{A}$ ,  $H_A(K) = Hom_{\mathcal{A}}(K, A)$  is a set. Thus, as  $\alpha_K$  is a function, and hence a relation, it's a subset of  $Hom_{\mathcal{A}}(K, A) \times X(A)$ , which is a set as it's a cartesian product of sets.

Thus,  $K$ -component of every natural transformation is a set. i.e.  $\forall \alpha \in [\mathcal{A}^{op}, Set](H_A, X), \forall K \in \mathcal{A}, \alpha_K$  is a set

Now, the question is whether the set of all natural transformations is a set

*Proof.* Let a locally small category,  $\mathcal{A}$  be given.

Denoting the category of all presheaves on  $\mathcal{A}$  by  $\mathcal{C}$ , i.e.  $\mathcal{C} := [\mathcal{A}^{op}, Set]$

Now, fix any object,  $A \in \mathcal{A}$ , and any object,  $X \in \mathcal{C}$ .

Need to show that  $\mathcal{C}(H_A, X)$  is naturally isomorphic to  $X(A)$ .

Thus, need two mutually inverse natural transformations, such that :  $\mathcal{C}(H_A, X) \xrightleftharpoons[\phi]{\psi} X(A)$

**Need to show that the RHS is a set, and then,** As both the RHS and LHS above are sets, the natural transformations between them are maps in  $Set$  i.e. functions.

Going to show the following in order,

1. Define  $\phi$  and  $\psi$
2. Show that  $\phi$  and  $\psi$  are mutually inverse
3. Show naturality of  $\phi$  in  $X$
4. Show naturality of  $\phi$  in  $A$
5. Show naturality of  $\psi$  in  $X$
6. Show naturality of  $\phi$  in  $A$

**1. Defining  $\phi$  and  $\psi$**  Define  $\phi$  (on natural transformations) as the  $A$ -component (of that natural transformation) at the identity of  $A$ . i.e. for  $\alpha \in \mathcal{C}(H_A, X), \phi(\alpha) := \alpha_A(1_A)$

Define  $\psi$  on an object,  $x \in X(A)$ , by defining it's  $K$ -component for any  $K \in \mathcal{A}$ :

$$(\psi(x))_K : H_A(K) \rightarrow X(K) \text{ as, for each } p \in Hom_{\mathcal{A}}(K, A), p \mapsto (X(p))(x)$$

That is to say that the  $K$ -component maps any arrow  $p : K \rightarrow A$  to the image of  $x$  under the map  $X(p)$ .

**2. Showing inverses** Firstly, to show  $\psi \circ \phi(\alpha) = \alpha$ , for any natural transformation  $\alpha : H_A \rightarrow X$

$$\psi \circ \phi(\alpha) = \psi\left(\alpha_A(1_A)\right) =$$

Secondly, for every  $x \in X(A)$ , need  $\psi \circ \phi(x) = x$

**3. Naturality of  $\phi$  in  $X$**  Need to show that the following square commutes  
i.e.

**4. Naturality of  $\phi$  in  $A$**  Need to show that the following square commutes  
i.e.

**5. Naturality of  $\psi$  in  $X$**  Need to show that the following square commutes  
i.e.

**6. Naturality of  $\psi$  in  $A$**  Need to show that the following square commutes  
i.e.

□

## 2 Embedding of a category in Presheaf category

**Definiton 2.1** (Embedding of a category). A category,  $\mathcal{A}$  is said to be embedded in another  $\mathcal{B}$

**3 THE PAPER**

## 4 Scratchpad