

Question 4 (b) $f(x, y) := \frac{2xy}{\sqrt{x^2+y^2}}$

For partial derivative along x at $(\delta, 0)$,

$$\begin{aligned}\frac{\partial f}{\partial x}(\delta, 0) &= \lim_{h \rightarrow 0} \frac{f(\delta + h, 0) - f(\delta, 0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{2(\delta + h) \times 0}{\sqrt{(\delta + h)^2 + 0}} - \frac{2\delta \times 0}{\sqrt{\delta^2 + 0}} \right) = 0\end{aligned}$$

In particular, if $\delta = 0$ then the second term is zero as $f(0, 0) := 0$. And, due to symmetry, $f_y(0, \delta) = 0$. As :

$$\begin{aligned}\frac{\partial f}{\partial y}(0, \delta) &= \lim_{h \rightarrow 0} \frac{f(0, \delta + h) - f(0, \delta)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{2 \times 0 \times (\delta + h)}{\sqrt{0 + (\delta + h)^2}} - \frac{2 \times 0 \times \delta}{\sqrt{0 + \delta^2}} \right) = 0\end{aligned}$$

So, not only are the partial derivatives f_x and f_y 0 at $(0,0)$, but also along x and y axes respectively.

To show the existence of second derivatives ,

$$f_{xx}(0, 0) = \lim_{h \rightarrow 0} \frac{f_x(h, 0) - f_x(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0$$

Similarly, $f_{yy}(0, 0) = 0$

Now, if the derivative of f existed at $(0,0)$, then, the following limit must be 0

$$\begin{aligned}\lim_{h^2+k^2 \rightarrow 0} \frac{f(0 + h, 0 + k) - f(0, 0) - hf_x(0, 0) - kf_y(0, 0)}{\sqrt{h^2 + k^2}} \\ = \lim_{h^2+k^2 \rightarrow 0} \frac{2hk}{h^2 + k^2}\end{aligned}$$

But this limit is equal to 1 along the line of slope 1, and thus f isn't differentiable.

$$(a) \quad f(x, y) := \frac{(x^2y + xy^2)\sin(x - y)}{x^2 + y^2}$$

For partial derivative along x at $(\delta, 0)$,

$$\begin{aligned} \frac{\partial f}{\partial x}(\delta, 0) &= \lim_{h \rightarrow 0} \frac{f(\delta + h, 0) - f(\delta, 0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{((\delta + h)^2 \times 0 + (\delta + h) \times 0)\sin((\delta + h))}{(\delta + h)^2 + 0} - 0 \right) = 0 \end{aligned}$$

In particular, if $\delta = 0$ then the second term is still zero as $f(0, 0) := 0$. And, due to symmetry, $f_y(0, \delta) = 0$.

So, not only are the partial derivatives f_x and f_y 0 at $(0, 0)$, but also along x and y axes respectively.

To show the existence of second derivatives ,

$$f_{xx}(0, 0) = \lim_{h \rightarrow 0} \frac{f_x(h, 0) - f_x(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0$$

Similarly, $f_{yy}(0, 0) = 0$

Now, if the derivative of f existed at $(0, 0)$, then, the following limit must be 0

$$\begin{aligned} \lim_{h^2+k^2 \rightarrow 0} \frac{f(0 + h, 0 + k) - f(0, 0) - hf_x(0, 0) - kf_y(0, 0)}{\sqrt{h^2 + k^2}} \\ = \lim_{h^2+k^2 \rightarrow 0} \frac{2hk}{h^2 + k^2} \end{aligned}$$

But this limit is equal to 1 along the line of slope 1, and thus f isn't differentiable.

Question 5 (Marked)

$$f(x, y) = xy(1 - x^2 - y^2)$$

For each partial derivative to be zero,

$$\begin{aligned} f_x &= y - 3x^2y - y^3 = 0 \text{ and } f_y = x - x^3 - 3xy^2 = 0 \\ \implies f_x &= y(1 - 3x^2 - y^2) = 0 \text{ and } f_y = x(1 - x^2 - 3y^2) = 0 \end{aligned}$$

So, a critical point is $(0, 0)$.

And,

If $x = 0, y \neq 0$ then $f_x = y(1 - y^2) = 0 \implies y = \pm 1$
gives two critical points: $(0, \pm 1)$

If $y = 0, x \neq 0$ then $0 = f_y = x(1 - x^2) \implies x = \pm 1$
gives two critical points: $(\pm 1, 0)$

If $x \neq 0 \neq y$, then

$$\begin{aligned} f_x &= y - 3x^2y - y^3 = 0 \implies y(1 - 3x^2) = y^3 \\ &\implies 1 - 3x^2 = y^2 \quad (\text{equation A}) \end{aligned}$$

And, also,

$$\begin{aligned} f_y &= x - 3y^2x - x^3 = 0 \implies x(1 - 3y^2) = x^3 \\ &\implies 1 - 3y^2 = x^2 \quad (\text{equation B}) \end{aligned}$$

Thus,

$$\text{Substituting B into A, } 1 - 3 + 9y^2 = y^2 \implies y = \pm 0.5$$

$$\text{And, substituting A into B, } 1 - 3 + 9x^2 = x^2 \implies x = \pm 0.5$$

Thus, the four such possible points are also critical:

$$(0.5, 0.5), (0.5, -0.5), (-0.5, 0.5), (-0.5, -0.5)$$

Now, to classify these critical points, looking at $rt - s^2$

$$r = f_{xx} = -6xy = f_{yy} = t \text{ and } s = f_{xy} = 1 - 3x^2 - 3y^2$$

$$\text{So, } rt = 36x^2y^2 \text{ and } s^2 = (1 - 3(x^2 + y^2))^2$$

At $(0, 0)$, $rt = 0$ and $s = 1$. Thus, $rt - s^2 = 0 - 1 < 0$
Thus, $(0, 0)$ is a saddle point.

At $(\pm 1, 0)$ and $(0, \pm 1)$,
 $rt = 36 \times 1 \times 0 = 36 \times 0 \times 1 = 0$ and $s^2 = (1 - 3)^2 = 4$.
Thus, $rt - s^2 = 0 - 4 < 0$
Thus, $(\pm 1, 0), (0, \pm 1)$ are saddle points.

At $(0.5, 0.5), (0.5, -0.5), (-0.5, 0.5), (-0.5, -0.5)$,
 $rt = 36x^2y^2 = 36 \times 0.25 \times 0.25 = \frac{9}{4} = 2.25$
and $s^2 = (1 - 3(0.25 + 0.25))^2 = (-0.5)^2 = 0.25$
Hence, $rt - s^2 = 2.25 - 0.25 = 2 > 0$
Thus, $(0.5, -0.5), (-0.5, 0.5)$ are minima(as $r = 1.5$),
while $(0.5, 0.5), (-0.5, -0.5)$ are maxima(as $r = 1.5$).

Question 3 (Marked) (a) $f(x, y) := \frac{x^2 y^2}{x^2 y^2 + (x-y)^2} = \frac{1}{1 + (\frac{x-y}{xy})^2} = \frac{1}{1 + (\frac{1}{y} - \frac{1}{x})^2}$

$$\lim_{x \rightarrow 0} f(x, y) = \lim_{x \rightarrow 0} \frac{1}{1 + (\frac{1}{y} - \frac{1}{x})^2} = 0 \implies \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x, y) = 0$$

And as the expression is symmetric in x and y ,

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x, y) = \lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x, y)$$

But the simultaneous limit at $(0, 0)$ along $T(t) := (t, t)$ is

$$\lim_{t \rightarrow 0} f(T(t)) = \lim_{t \rightarrow 0} \frac{1}{1 + (\frac{1}{t} - \frac{1}{t})^2} = 1$$

If the simultaneous limit existed, all the iterated limits would be equal to it. So, there is a curve, $S(t) := (\frac{1}{t}, \frac{1}{t+1})$ with

$$\lim_{t \rightarrow \infty} f(S(t)) = \lim_{t \rightarrow \infty} \frac{1}{1 + (t + 1 - t)^2} = \frac{1}{2}$$

Thus, f is discontinuous at $(0, 0)$

(b) $f(x, y) := \frac{\frac{-1}{e^{x^2}} y}{\frac{-1}{e^{x^2}} + y^2} = \frac{y e^{\frac{1}{x^2}}}{1 + (y e^{\frac{1}{x^2}})^2} = \frac{1}{\frac{1}{y e^{1/x^2}} + y e^{1/x^2}}$

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} \frac{1}{\frac{1}{y e^{1/x^2}} + y e^{1/x^2}} = \lim_{y \rightarrow 0} 0 = 0$$

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} \frac{y e^{\frac{1}{x^2}}}{1 + (y e^{\frac{1}{x^2}})^2} = \lim_{x \rightarrow 0} 0 = 0$$

To show the non-existence of simultaneous limit at $(0, 0)$, consider the curve $T(t) := (t, e^{-1/t^2})$

$$\lim_{t \rightarrow 0} f(T(t)) = \frac{e^{-1/t^2} \times e^{1/t^2}}{1 + (e^{-1/t^2} \times e^{1/t^2})^2} = \frac{1}{2}$$