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Theorem 1.1. Yoneda If $\mathcal{A}$ be a locally small category, then,

$$[\mathcal{A}^{op}, Set](H_A, X) \cong X(A) \text{ naturally in } A \in \mathcal{A} \text{ and } X \in [\mathcal{A}^{op}, Set]$$

Explaination:

First of all, we fix any category, $\mathcal{A}$. Now we choose two things (independent of each other):

i an object, A from the category $\mathcal{A} = \mathcal{A}^{op}$

ii an object, X of the category $[\mathcal{A}^{op}, Set]$ which is precisely a functor $X : A^{op} \to Set$

Here, $[\mathcal{A}^{op}, Set](H_A, X)$ denotes arrows $H_A \to X$ in $[\mathcal{A}^{op}, Set]$ i.e. natural transformations, $\alpha :$

$$\begin{tikzcd} A^{op} \arrow[r, bend left, "H_A"] \arrow[r, bend right, "X"'] \arrow[r, phantom, "\Downarrow\alpha"] & Set \end{tikzcd}$$

$X(A)$ is precisely a set. As $X(A)$ is the image of (our chosen object,) A, under (our chosen functor,) X.

So, the idea is that our choice of A and X completely determines all possible maps (i.e. natural transformations) from H_A to X. This answers out big question of "what are all the maps $H_A \to X$ " or "how does H_A see other presheaves on $\mathcal{A}$ ".

*More than the above must be true, as it's not just that the two are isomorphic, but they're **naturally** isomorphic.*
This more is supposed to be that both $[\mathcal{A}^{op}, Set]$ and $X(A)$ are *functorial* in *both* A and X