Question 3 **(a)** $f(x,y) := \frac{x^2y^2}{x^2y^2+(x-y)^2} = \frac{1}{1+(\frac{x-y}{xy})^2} = \frac{1}{1+(\frac{1}{y}-\frac{1}{x})^2}$

$$\lim_{x\to 0} f(x,y) = \lim_{x\to 0} \frac{1}{1+(\frac{1}{y}-\frac{1}{x})^2} = 0 \implies \lim_{y\to 0}\lim_{x\to 0} f(x,y) = 0$$

And as the expression is symmetric in x and y,

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{x\to 0}\lim_{y\to 0} f(x,y)$$

But the simultaneous limit at$(0,0)$ along $T(t) := (t,t)$ is

$$\lim_{t\to 0} f(T(t)) = \lim_{t\to 0} \frac{1}{1+(\frac{1}{t}-\frac{1}{t})^2} = 1$$

If the simultaneous limit existed, all the iterated limits would be equal to it. So, there is a curve, $S(t) := (\frac{1}{t}, \frac{1}{t+1})$ with

$$\lim_{t\to\infty} f(S(t)) = \lim_{t\to\infty} \frac{1}{1+(t+1-t)^2} = \frac{1}{2}$$

Thus, f is discontinous at $(0,0)$

(b) $f(x,y) := \frac{e^{\frac{-1}{x^2}}y}{e^{\frac{-1}{x^2}}+y^2} = \frac{ye^{\frac{1}{x^2}}}{1+(ye^{\frac{1}{x^2}})^2} = \frac{1}{\frac{1}{ye^{1/x^2}}+ye^{1/x^2}}$

$$\lim_{y\to 0}\lim_{x\to 0} \frac{1}{\frac{1}{ye^{1/x^2}}+ye^{1/x^2}} = \lim_{y\to 0} 0 = 0$$

$$\lim_{x\to 0}\lim_{y\to 0} \frac{ye^{\frac{1}{x^2}}}{1+(ye^{\frac{1}{x^2}})^2} = \lim_{x\to 0} 0 = 0$$

To show the non-existence of simulatenous limit at $(0,0)$, consider the curve $T(t) := (t, e^{-1/t^2})$

$$\lim_{t\to 0} f(T(t)) = \frac{e^{-1/t^2}\times e^{1/t^2}}{1+(e^{-1/t^2}\times e^{1/t^2})^2} = \frac{1}{2}$$

Question 4 (b) $f(x,y) := \frac{2xy}{\sqrt{x^2+y^2}}$

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{1}{h}\Big(\frac{2h\times 0}{\sqrt{h^2+0}} - 0\Big) = 0$$

So, the partial derivative along x exists at $(0,0)$. And by symmetry, it also exists along y. And in a neighbourhood of $(0,0)$, $\frac{\partial f}{\partial x} = \frac{2y\sqrt{x^2+y^2} - 2xy(\frac{2x}{2\sqrt{x^2+y^2}})}{x^2+y^2} = \frac{y\Big(2\sqrt{x^2+y^2} - 2x(\frac{2x}{2\sqrt{x^2+y^2}})\Big)}{x^2+y^2}$.

$$\frac{\partial^2 f}{\partial x^2} = \lim_{h\to 0} \frac{f_x(h,0)-f_x(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{0} = 0$$

Again, by symmetry, $\frac{\partial^2 f}{\partial y^2}(0,0) = 0$. Thus, as second partial derivative exists along x and y, the first derivatives are continous at $(0,0)$, and hence, f is differentiable at $(0,0)$.

(a) $f(x,y) := \frac{xy(x+y)sin(x-y)}{x^2+y^2}$

The partial derivative along x,

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \Big(\frac{f(h,0)-f(0,0)}{h}\Big)$$

$$= \lim_{h\to 0} \frac{(h\times 0(h+0)sin(h-0)-0)}{h^2+0} = 0$$

And in the neighbourhood of $(0,0)$,

$$\frac{\partial f}{\partial x} = \frac{y\big((-x^2y+2xy^2+y^3)sin(x-y)+(x^3+x^3y+x^2y^2+xy^3}{(x^2+y^2)^2}$$

Thus, for the second partial derivative along x,

$$\frac{\partial^2 f}{\partial x^2} = \lim_{h\to 0} \frac{1}{h}\Big(\frac{\partial f(h,0)}{\partial x} - \frac{\partial f(0,0)}{\partial x}\Big) = \lim_{h\to 0} \frac{(0\times(...))-0}{h} = 0$$

Similarly, along the y-direction,

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

And,

$$\frac{\partial f}{\partial x} = \frac{-x}{(x^2+y^2)^2}[(-x^3 - 2x^2y + xy^2)sin(x-y) + ...]$$

the second order derivative is,

$$\frac{\partial^2 f}{\partial x^2} = \lim_{h \to 0} \frac{f_y(0,h) - f_y(0,0)}{h} = 0$$

Now, as the second partial derivative exists along x and y directions, the first partial derivatives are continous. Hence, f is differentiable at (0,0).

Question 5

$$f(x, y) = xy(1 - x^2 - y^2)$$

$$f_x = y - 3x^2y - y^3 \text{ and } f_y = x - x^3 - 3xy^2$$

So, a critical point is $(0, 0)$. And,
If $x = 0, y \neq 0$ then $0 = f_x = y(1 - y^2) \implies y = \pm 1$ gives a critical point $(0, \pm 1)$
If $y = 0, x \neq 0$ then $0 = f_y = y(1 - x^2) \implies x = \pm 1$ gives a critical point $(\pm 1, 0)$

If $x \neq 0 \neq y$, then $f_y = y - 3x^2y - y^3 = 0 = x - x^3 - 3xy^2 = f_y$ gives $(\pm 0.5, \pm 0.5)$
Now, to classify these critical points, looking at $rt - s^2$

$$r = f_{xx} = -6xy = f_{yy} = t \text{ and } s = f_{xy} = 1 - 3x^2 - 3y^2$$

At $(0, 0), r = 0 = t$ and $s = 1$. Thus, $rt - s^2 = 0 - 1 < 0$
Thus, $(0, 0)$ is a maxima.

At $(\pm 1, 0)$ and $(0, \pm 1)$,
$rt = 36x^2y^2 = 0$ and $s = 1 - 3$. Thus, $rt - s^2 = 0 - 4 < 0$
Thus, $(\pm 1, 0), (0, \pm 1)$ are maxima.

At $(\pm 0.5, \pm 0.5)$ and $(-0.5, -0.5)$,
$rt = 36x^2y^2 = \frac{9}{4}$ and $s = 1 - 3(0.25 + 0.25) = -0.5$
Hence, $rt - s^2 = 2.25 - 0.25 = 2 > 0$
Thus, all four points, $(\pm 0.5, \pm 0.5)$ are minima.