Question 1

$$< a_n > \text{ is a real sequence; } \sigma_n := \frac{a_1 + a_2 + \dots + a_n}{n}$$

Going to show that

I $\lim \inf a_n \leq \lim \inf \sigma_n$

If a_n is unbounded below, then $\lim \inf a_n = -\infty \leq \lim \inf \sigma_n$ So, let a_n be bounded below, thus LHS is a real number,

$$m := \lim \inf a_n$$

Now, if a_n is constant, and equal to a,

$$\forall n \in \mathbb{N} \text{ , } \inf\{a_i | i \geq n\} = a = \frac{na}{n} = \sigma_n$$

Otherwise, if a_n is not constant, then,

$$\exists i, j \in \mathbb{N} \text{ such that(wlog) } a_i < a_j$$

Suppose if possible, $\lim \inf a_n = m > \lim \inf \sigma_n$
But,
$$\forall n \in \mathbb{N} \text{ such that } n \geq i, j,$$
$$\sigma_n = \frac{a_1 + \dots a_i + a_j + \dots + a_n}{n} \geq \frac{(n-2)m + a_i + a_j}{n}$$
$$\geq \frac{(n-1)m + a_j}{n}$$
$$> m \; [\because m \leq a_i < a_j]$$

$$\text{Hence, if } \sigma_n \geq \frac{(n-1)m + a_j}{n}$$

$$\implies \liminf \sigma_n \geq \lim_{n \to \infty} \frac{(n-1)m + a_j}{n} = m$$

But this contradicts the initial assumption.

II $\limsup a_n \geq \limsup \sigma_n$ If a_n is unbounded above, then $\limsup a_n = \infty \geq \limsup \sigma_n$ So, let a_n be bounded above, thus LHS is a real number,

$$M := \limsup a_n$$

Now, if a_n is constant, and equal to a,

$$\forall n \in \mathbb{N} \text{ , } \sup\{a_i | i \geq n\} = a = \frac{na}{n} = \sigma_n$$

Otherwise, if a_n is not constant, then,

$$\exists i, j \in \mathbb{N} \text{ such that(wlog) } a_i < a_j$$

Suppose if possible, $\liminf a_n = m > \liminf \sigma_n$
But,

$$\forall n \in \mathbb{N} \text{ such that } n \geq i, j,$$

$$\sigma_n = \frac{a_1 + ...a_i + a_j + ... + a_n}{n} \geq \frac{(n-2)m + a_i + a_j}{n}$$

$$\geq \frac{(n-1)m + a_j}{n}$$

$$> m \ [\because m \leq a_i < a_j\ ]$$

2

Hence, if $\sigma_n \geq \dfrac{(n-1)m + a_j}{n}$

$$\implies \liminf \sigma_n \geq \lim_{n \to \infty} \frac{(n-1)m + a_j}{n} = m$$

But this contradicts the initial assumption.