1 Yoneda Lemma

In this section, we set forth some basic definitions, state and prove Yoneda lemma, and then show Cayley's Theorem and Yoneda Embedding as it's consequences.

For any category $\mathcal{A}$, and it's objects $X, Y \in \mathcal{A}$, we denote $Hom_{\mathcal{A}}(X, Y)$ with $\mathcal{A}(X, Y)$.

Definiton 1.1. For any category $\mathcal{A}$, it's opposite category, $\mathcal{A}^{op}$ is the category having the objects of $\mathcal{A}$. And for objects $A, B \in \mathcal{A}$, a morphism $f \in \mathcal{A}^{op}(A, B)$ only if there is a morphism $g \in \mathcal{A}(B, A)$.

We denote composition in categories with the usual symbol $\circ$. And whenever we mention composition of morphism in any opposite category $\mathcal{A}^{op}$, we adjust it to become composition in $\mathcal{A}$ instead. Similarly, whenever we talk about contravariant functors, we adjust composition to proceed in the covariant version without changing implications.

Proposition 1.2. For a locally small category $\mathcal{A}$, fixing an object $A \in \mathcal{A}$ gives a functor, $H_A : \mathcal{A}^{op} \to Set$ defined as:

(i) For any object $B \in \mathcal{A}$, $H_A(B) := \mathcal{A}(B, A)$.

(ii) For any morphism, $g : X \to Y$ in $\mathcal{A}$,

$$H_A(g) : \mathcal{A}(Y, A) \to \mathcal{A}(X, A) \text{ is given by } p \mapsto p \circ g.$$

Proof. Fix any objects $K, L, M \in \mathcal{A}$.

I **(Composition)** As H_A is a contravariant functor, for any morphisms $f \in \mathcal{A}(K, L)$ and $g \in \mathcal{A}(L, M)$, we need to show that $H_A(g \circ f) = H_A(f) \circ H_A(g)$. Hence, using the definition of H_A gives us that for any $k \in H_A(M)$, we must have

$$\begin{aligned} LHS &= \Big(H_A(g \circ f)\Big)(k) = k \circ g \circ f \\ \text{and } RHS &= \Big(H_A(f) \circ H_A(g)\Big)(k) = \Big(H_A(f)\Big)(k \circ g) = (k \circ g) \circ f. \end{aligned}$$

II **(Identity)** We will show that for any $k \in \mathcal{A}(K, L)$, H_A respects the identities of K and L in $\mathcal{A}$ (as they're equal to the identities in $\mathcal{A}^{op}$). Using the definition of H_A, for any object $L \in \mathcal{A}$ and morphism $p \in H_A(L)$:

$$\begin{aligned} \text{Right Identity: } & \Big(\big(H_A(1_K)\big) \circ \big(H_A(k)\big)\Big)(p) = \big(H_A(1_K)\big)(p \circ k) = p \circ k \circ 1_K = p \circ k = \big(H_A(k)\big)(p) \\ \text{Left Identity: } & \Big(\big(H_A(k)\big) \circ \big(H_A(1_L)\big)\Big)(p) = \big(H_A(k)\big)(p \circ 1_L) = \big(H_A(k)\big)(p) \end{aligned}$$

Hence, H_A is indeed a functor. □

Definiton 1.3. For a locally small category $\mathcal{A}$, the category of presheaves on $\mathcal{A}$, denoted by $[\mathcal{A}^{op}, Set]$ is defined to have functors from $\mathcal{A}^{op}$ to Set as objects, and natural transformations between them as morphisms.

Lemma 1.4. Lemma 1.3.11

Lemma 1.5. Excercise 1.3.29

Theorem 1.6. ***Yoneda Lemma*** *If $\mathcal{A}$ is a locally small category then, for any object $A \in \mathcal{A}$ and $X \in [\mathcal{A}^{op}, Set]$, there exists an isomorphism,*

$$[\mathcal{A}^{op}, Set](H_A, X) \cong X(A) \text{ which is natural in } A \text{ and } X. \tag{1}$$

Notation:

- We denote the category of presheaves on $\mathcal{A}$ by $\mathcal{C}$.
- For the map $\hat{}$, instead of writing $\hat{}(a) = b$, we use $\hat{a} = b$ to denote $a \mapsto b$.
- For the map $\tilde{}$, instead of writing $\tilde{}(a) = b$, we use $\tilde{a} = b$ to denote $a \mapsto b$.
- $[\mathcal{A}^{op}, Set](H_A, X)$ denotes the collection of morphisms α : $\begin{tikzcd} \mathcal{A}^{op} \arrow[r, bend left, "H_A"] \arrow[r, bend right, "X"'] \arrow[r, phantom, "\Downarrow\alpha"] & Set \end{tikzcd}$.

To prove the theorem, first, we show that $[\mathcal{A}^{op}, Set](H_A, X)$ is isomorphic to $X(A)$ as set, and then that this isomorphism is natural in X and A.

Proof. Let $\mathcal{A}$ be a locally small category. Fix an object $A \in \mathcal{A}$ and a presheaf X on $\mathcal{A}$.

I **Showing isomorphism between $[\mathcal{A}^{op}, Set](H_A, X)$ and $X(A)$**
Define $\hat{}: \mathcal{C}(H_A, X) \to X(A)$ for any $\alpha : H_A \to X$, as $\hat{\alpha} := \alpha_A(1_A)$. As $1_A \in Set(A, A) = H_A(A)$, definition of α_A gives that $\alpha_A(1_A) \in X(A)$.

Define $\tilde{}: X(A) \to [\mathcal{A}^{op}, Set](H_A, X)$ for any $x \in X(A)$ as the natural transformation $\tilde{x} : H_A \to X$ whose K-component is the function mapping each morphsim $p \in \mathcal{A}(K, A)$ to $\Big(X(p)\Big)(x)$. That is, $\tilde{x}_K(p) := \Big(X(p)\Big)(x)$.

We are going to show that $\tilde{x}$ is a natural transformation. Fix objects $K, L \in \mathcal{A}$ and morphism $q \in \mathcal{A}^{op}(K, L)$.

Need to show that the square
$$\begin{tikzcd} H_A(K) \arrow[r, "H_A(q)"] \arrow[d, "\tilde{x}_K"'] & H_A(L) \arrow[d, "\tilde{x}_L"] \\ X(K) \arrow[r, "X(q)"'] & X(L) \end{tikzcd}$$
, that is
$$\begin{tikzcd} \mathcal{A}(K, A) \arrow[r, "- \circ q"] \arrow[d, "\tilde{x}_K"'] & \mathcal{A}(L, A) \arrow[d, "\tilde{x}_L"] \\ X(K) \arrow[r, "X(q)"'] & X(L) \end{tikzcd}$$
commutes .

So, for any $f : K \to A$, need that $\tilde{x}_L(f \circ q) = X(q) \circ \tilde{x}_K(f)$. Using the definition of $\tilde{x}$ gives the following.

$$LHS = \tilde{x}_L(f \circ q) = \Big(X(f \circ q)\Big)(x)$$
$$RHS = X(q) \circ \tilde{x}_K(f) = \Big(X(q)\Big)\big(X(f)(x)\big) = \Big(X(q) \circ X(f)\Big)(x)$$

And as X is a contravariant functor, $X(f \circ q) = X(q) \circ X(f)$, giving that LHS=RHS. Now going to show that $\hat{}$ and $\tilde{}$ define an isomorphism. Need to show that $\hat{}$ and $\tilde{}$ are mutually inverse.

(i) For any $x \in X(A)$, $\hat{\tilde{x}} = \tilde{x}_A(1_A) = \Big(X(1_A)\Big)(x) = 1_{X(A)}(x) = x$.

(ii) For any $\alpha \in \mathcal{C}(H_A, X)$, need to show that $\tilde{\hat{\alpha}} = \alpha$. So, it's required that each of their component are equal. As both $\tilde{\hat{\alpha}}$ and α are natural transformations between functors that go to the category *Set*, each of the components is a function. So, need to show that for any $f \in \mathcal{A}(K, A) = H_A(K)$, $\Big(\tilde{\hat{\alpha}}\Big)_K(f)$=$\alpha_K(f)$. Using first the definition of $\tilde{}$ and then that of $\hat{\alpha}$ gives:

$$LHS = \tilde{\hat{\alpha}}_B(f) = \Big(X(f)\Big)(\hat{\alpha}) = \Big(X(f)\Big)(\alpha_A(1_A)) \tag{1}$$

And as $f \in \mathcal{A}(K, A)$, we also have the following.

$$RHS = \alpha_K(f) = \alpha_K(1_A \circ f) \tag{2}$$

Because α is a natural transformation, the square following square commutes for 1_A.

$$\begin{tikzcd} \mathcal{A}(A, A) \arrow[r, "- \circ f"] \arrow[d, "\alpha_A"'] & \mathcal{A}(K, A) \arrow[d, "\alpha_K"] \\ X(A) \arrow[r, "X(f)"'] & X(K) \end{tikzcd}$$

This gives that $\alpha_K(1_A \circ f) = \Big(X(f)\Big)(\alpha_A(1_A))$. Hence, we have from (2) and (3), we get that $RHS = LHS$.

II **Showing naturality of this isomorphism**
By Using Lemma 1.4 and 1.5, it's enough to show that $\hat{}$ is natural in X and natural in A.

(i) We are going to show naturality in X. Fix any $A \in \mathcal{A}$. Need that for presheaves $X, Y \in \mathcal{C}$ and natural transformation $\beta \in \mathcal{C}(X, Y)$, the following square commutes.

$$\begin{tikzcd} \mathcal{C}(H_A, X) \arrow[r, "\beta \circ -"] \arrow[d, "\hat{}"'] & \mathcal{C}(H_A, Y) \arrow[d, "\hat{}"] \\ X(A) \arrow[r, "\beta_A"'] & Y(A) \end{tikzcd}$$

So, for any $\alpha : H_A \to X$, we need that $\big(\hat{} \circ H_\beta\big)(\alpha) = \big(\beta_A \circ \hat{}\big)(\alpha)$. Using definition of H_β and $\hat{}$ gives:

$$LHS = \big(\hat{} \circ H_\beta\big)(\alpha) = \big(\widehat{H_\beta(\alpha)}\big) = \big(\widehat{\beta \circ \alpha}\big) = \big(\beta \circ \alpha\big)_A(1_A) \tag{3}$$

$$RHS = \big(\beta_A \circ \hat{}\big)(\alpha) = \beta_A(\widehat{\alpha}) = \big(\beta_A \circ \alpha_A\big)(1_A) \tag{4}$$

As $\alpha \in \mathcal{C}(H_A, X)$ and $\beta \in \mathcal{C}(X, Y)$ are morphisms in $\mathcal{C}$, composition in $\mathcal{C}$ gives $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$. From (4) and (5), we directly get that $RHS = LHS$.

(ii) We are going to show naturality in A. Fix any $X \in \mathcal{C}$ Need that for objects $A, B \in \mathcal{A}$ and morphism $f \in \mathcal{A}^{op}(A, B)$, the following square commutes.

$$\begin{tikzcd}
\mathcal{C}(H_A, X) \arrow[r, "-\circ H_f"] \arrow[d, "\hat{}"'] & \mathcal{C}(H_B, Y) \arrow[d, "\hat{}"] \\
X(A) \arrow[r, "X(f)"'] & X(B)
\end{tikzcd}$$

So, for any $\alpha : H_A \to X$, we need that $\big(\hat{} \circ H_f\big)(\alpha) = \Big((X(f)) \circ \hat{}\Big)(\alpha)$. Using definition of H_f and $\hat{}$, we get:

$$LHS = \big(\hat{} \circ H_f\big)(\alpha) = \widehat{\alpha \circ H_f} = (\alpha \circ H_f)_B(1_B) = \alpha_B(f \circ 1_B) = \alpha_B(1_A \circ f) \tag{5}$$

$$RHS = \Big((X(f)) \circ \hat{}\Big)(\alpha) = (X(f))(\hat{\alpha}) = \Big(X(f)\Big)\big(\alpha_A(1_A)\big) \tag{6}$$

The last equality in (6) is justified as f goes from B to A in $\mathcal{A}$. By using equality of (2) and (3) from I(i), for $f \in \mathcal{A}(B, A)$, we get that $\Big(X(f)\Big)\big(\alpha_A(1_A)\big) = \alpha_B(1_A \circ f)$. Hence, $RHS = LHS$. □

1.1 Cayley's Theorem

Informally, given a locally small category$\mathcal{A}$, we can fix a presheaf X on $\mathcal{A}$, and for any object $A \in \mathcal{A}$, study the set $X(A)$ and gain information about all possible natural transformations between H_A and X. Moreover, by part I(ii) of the proof of Yoneda Lemma, each of the natural transformations is determined by its action on the identity morphisms in $\mathcal{A}$. Thus, no matter how complicated $\mathcal{A}$ is, if we choose X carefully, we can hope to understand the structure of $\mathcal{A}$ by looking at how $X(A)$ changes as we vary the chosen presheaf and object.

In group theory, Cayley's theorem says every group G is isomorphic to a subgroup of the symmetric group on G. Thus, instead of having to study a complicated group directly, we can study a subgroup of the symmetric group on it.

Cayley's theorem and Yoneda Lemma are similar in the sense that both allow us to change the environment that we study in by putting few restrictions on what we are allowed to study. Cayley allows us to change setting for groups, and Yoneda does that for locally small categories.

Also, as groups themselves can be considered as small categories, we can apply Yoneda Lemma to any group. In fact, we can get Cayley's theorem as a consequence of Yoneda Lemma by a suitable choice of X and A.

Definiton 1.7. Symmetric group on a set X is the set of all bijections on X, with the binary operation defined as composition of bijections.

Theorem 1.8. ***Cayley's Theorem*** *Every group, (G,.) is isomorphic to a subgroup of symmetric group on G.*

Proof. Let a group $(G, .)$ be given. Define category $\mathcal{A}$ as having: $\star$ as it's only object, one morphism for each element of G and composition being as that of elements of G. That is, for morphisms f and g in $\mathcal{A}$, the morphism $f \circ g$ is defined to be $f.g$. Thus, by definition of $\mathcal{A}$, G and $\mathcal{A}(\star, \star)$ are isomorphic as groups, becuase they have the same elements and rule of composition. Using naturality of α, we get that

$$\text{the square } \begin{tikzcd}
H_\star(\star) \arrow[r, "H_\star(f)"] \arrow[d, "\alpha_\star"'] & H_\star(\star) \arrow[d, "\alpha_\star"] \\
H_\star(\star) \arrow[r, "H_\star(f)"'] & H_\star(\star)
\end{tikzcd}, \text{ that is } \begin{tikzcd}
\mathcal{A}(\star, \star) \arrow[r, "_\circ f"] \arrow[d, "\alpha_\star"'] & \mathcal{A}(\star, \star) \arrow[d, "\alpha_\star"] \\
\mathcal{A}(\star, \star) \arrow[r, "_\circ f"'] & \mathcal{A}(\star, \star)
\end{tikzcd} \text{ commutes for any } f \in \mathcal{A}(\star, \star). \tag{1}$$

Using (1) for the identity $1_\star$ of $\star$ in $\mathcal{A}$, gives us the following equation.

$$\big((\_ \circ f) \circ \alpha_\star\big)(1_\star) = \big(\alpha_\star \circ (\_ \circ f)\big)(1_\star) \implies \alpha_\star(1_\star) \circ f = \alpha_\star(f) \tag{2}$$

As $\mathcal{A}^{op}$ has only one object, each natural transformation $\alpha : H_\star \to H_\star$ has only one component, that is $\alpha_\star$. which is a funtion from $\mathcal{A}(\star, \star)$ to $\mathcal{A}(\star, \star)$. Thus, from (2), we get that every natural transformation, α is determined by it's action on the identity of $\star$. And as $\mathcal{A}(\star,\star)$ is isomorphic to the group G, we can write the final equation from (2) with the product being in G as, $\alpha_\star(1_\star).f = \alpha_\star(f)$. Because multiplying every element of a group by a fixed element gives us a bijection on the group, each $\alpha_\star$, and hence each α can be thought of as a bijection on G.

Now we will show that the collection, $[\mathcal{A}^{op}, Set](H_\star, H_\star)$ is a group. As $[\mathcal{A}^{op}, Set]$ is a category, we get that the composition of natural transformations α is associative, and has an identity. Also, because the aforementioned collection contains morphisms with the same source and destination, that is $\star$, we get that it is closed under composition. In order to show inverses, we use the group isomorphism between $\mathcal{A}(\star,\star)$ and $\alpha : H_\star \to H_\star$ defined as mapping each element $g \in G$ to the natural transformation α such that $\alpha_\star(1_\star) = g \in \mathcal{A}$. Giving us that for $\beta : H_\star \to H_\star$ corresponding to $j \in G$ has, as it's inverse, $\gamma : H_\star \to H_\star$ corresponding to $j^{-1} \in G$. Thus, the collection $[\mathcal{A}^{op}, Set](H_\star, H_\star)$ is a group, and each of it's members is a bijection on G, giving us that it's a subgroup of the symmetric group on the underlying set of G.

Also, G being a set gives us that the collection of morphisms of$\mathcal{A}$ is a set, that is, $\mathcal{A}$ is a locally small category. Because $\mathcal{A}^{op}$ has the same number of morphism as $\mathcal{A}$, it is also a locally small category, and we may apply Yoneda Lemma. Taking $A = \star \in \mathcal{A}^{op}$ and $X = H_\star$ in Thm 1.6 (1), we get:

$$[\mathcal{A}^{op}, Set](H_\star, H_\star) \cong H_\star(\star) = \mathcal{A}(\star,\star) \cong G. \tag{3}$$

where the first isomorphism above is between sets. We will show that this isomorphism is also a homomorphism. The map $\hat{}$, defined in Theorem 1.1 will act here as $\alpha \mapsto \alpha_\star(1_\star)$. Hence, for any $\alpha, \beta : H_\star \to H_\star$,

$$\widehat{\alpha \circ \beta} = (\alpha \circ \beta)_\star(1_\star) = (\alpha)_\star\Big((\beta)_\star(1_\star)\Big) = \Big((\alpha)_\star(1_\star)\Big).\Big((\beta)_\star(1_\star)\Big). \tag{4}$$

Where the last equality is due to (2) being applicable as $\Big((\beta)_\star(1_\star)\Big)$ is an element of $\mathcal{A}(\star,\star)$, and hence corresponds to some element of G. Finally, we have shown that in statement (3), the isomorphism is between groups, with the left most expression being a subgroup of symmetric group on G. This is precisely the statement of Cayley's theorem.

$$Sym(G) \geq [\mathcal{A}^{op}, Set](H_\star, H_\star) \cong G$$

□

1.2 Yoneda Embedding

Definiton 1.9. A category, $\mathcal{A}$ is said to be embedded in a category, $\mathcal{B}$ if there exists a functor $F : \mathcal{A} \to \mathcal{B}$ such that F is full and faithful.

We define a functor from locally small category $\mathcal{A}$ to the presheaf category on $\mathcal{A}$ as taking any object $A \in \mathcal{A}$ to the functor H_A. And for any $X, Y, K \in \mathcal{A}$, taking morphism $f \in \mathcal{A}(X, Y)$ to the natural transformation f^*, whose K-component is defined as taking any $k \in H_X(K)$ to $f \circ k \in \mathcal{A}(K, Y)$.

Proposition 1.10. The map defined above is a functor from $\mathcal{A}$ to $[\mathcal{A}^{op}, Set]$. Henceforth, we denote it by $H_\bullet$.

Proof. Fix any objects $K, L, M \in \mathcal{A}$.

I **(Composition)** Let $f \in \mathcal{A}(K, L)$ and $g \in \mathcal{A}(L, M)$. As $(g \circ f)^*$ and $g^* \circ f^*$ are natural transformations from H_K to H_M, need to show that for any $X \in \mathcal{A}^{op}$, X-components of $(g \circ f)^*$ and $g^* \circ f^*$ are equal. So, fix $k \in H_K(X)$, and using the definition of $(\_)^*$, we get that

$$LHS = \Big((g \circ f)^*\Big)(k) = g \circ f \circ k$$
$$\text{and } RHS = \Big(f^* \circ g^*\Big)(k) = g^*\big(f \circ k\big) = g \circ f \circ k.$$

II **(Identity)** We will show that for any $g \in \mathcal{A}(K, L)$, $H_\bullet$ respects the identities of K and L in $\mathcal{A}$. Thus, for any object $X \in \mathcal{A}$, we need to show that $\Big(H_\bullet(g) \circ H_\bullet(1_K)\Big)_X = \Big(H_\bullet(g)\Big)_X = \Big(H_\bullet(g) \circ H_\bullet(1_K)\Big)_X$. So, fix any morphism $p \in H_A(L)$. Using the definition of $(\_)^*$, we get the following equations.

$$\text{Right Identity: } \Big(\big(H_\bullet(g)\big) \circ \big(H_\bullet(1_K)\big)\Big)(p) = \big(H_A(g)\big)(1_K \circ p) = \big(H_A(g)\big)(p)$$
$$\text{Left Identity: } \Big(\big(H_\bullet(1_L)\big) \circ \big(H_\bullet(g)\big)\Big)(p) = \big(H_\bullet(1_L)\big)(g \circ p) = g \circ p \circ 1_L = g \circ p = \big(H_\bullet(g)\big)(p)$$

Hence, $H_\bullet$ is indeed a functor. □

Theorem 1.11. *Any locally small category $\mathcal{A}$ can be embedded in the presheaf category on $\mathcal{A}$.*

Proof. We will show that the functor from Proposition 1.10 is full and faithful. Fix any objects $X, Y \in \mathcal{A}$.

I To show that $H_\bullet$ is a full, we need to show that for every $\alpha \in [\mathcal{A}^{op}, Set](H_X, H_Y)$, there exists a morphism $f \in \mathcal{A}(X,Y)$ such that $H_\bullet(f) = \alpha$. Thus, we need to show that their K-components are equal for every $K \in \mathcal{A}$. Using the definition of $H_\bullet(f)$, this amounts to showing that

$$\text{for any morphism } k \in H_X(K), \Big(H_\bullet(f)\Big)_K(k) = \alpha_K(k), \text{ that is } f \circ k = \alpha_K(k). \tag{1}$$

Because α_X goes from $H_X(X)$ to $H_Y(X)$, $\alpha_X(1_X)$ is a morphism in $\mathcal{A}(X,Y)$. We will show that choosing this morphism to be f will give us the required result, that is $\big(\alpha_X(1_X)\big) \circ k = \alpha_K(k)$. Using the naturality of α,

$$\text{we get that } \begin{array}{ccc} H_X(X) & \xrightarrow{H_X(k)} & H_X(K) \\ {\scriptstyle\alpha_X}\big\downarrow & & \big\downarrow{\scriptstyle\alpha_K} \\ H_Y(X) & \xrightarrow[H_Y(k)]{} & H_Y(K) \end{array}, \text{ that is } \begin{array}{ccc} \mathcal{A}(X,X) & \xrightarrow{\_\circ k} & \mathcal{A}(K,X) \\ {\scriptstyle\alpha_X}\big\downarrow & & \big\downarrow{\scriptstyle\alpha_K} \\ \mathcal{A}(X,Y) & \xrightarrow[\_\circ k]{} & \mathcal{A}(K,Y) \end{array} \text{ commutes.}$$

Thus, for the identity morphism $1_X \in \mathcal{A}(X,X)$, we get the following

$$\Big(H_Y(k) \circ \alpha_X\Big)(1_X) = \Big(\alpha_K \circ H_X(k)\Big)(1_X) \implies \alpha_X(1_X) \circ k \implies \alpha_K(1_X) \circ k = \alpha_K(k)$$

Thus, we have that $H_\bullet$ is a full functor.

II Fix any morphisms f, g in $\mathcal{A}(X,Y)$ and suppose $H_\bullet(f) = H_\bullet(g)$. In order to show $H_\bullet$ is faithfull, we need to show that $f = g$. As $H_\bullet(f)$ and $H_\bullet(g)$ are equal natural transformations, we have that the action of their X-components is equal. Thus, in particular, for the identity of X, $\big(H_\bullet(f)\big)_X(1_X) = \big(H_\bullet(g)\big)_X(1_X)$. Using the difinition of $H_\bullet$, we get that $f \circ 1_X = g \circ 1_X$. And as both g and f are morphisms from X, we get that $f = g$.

□

2 Prorelations

Definiton 2.1. A prorelation is a partially ordered, down-directed, up-set of relations $X \to Y$.
That is, $P \subseteq \mathcal{P}(X \times Y)$ is a prorelation if it satisfies the following conditions:

(i) Partial Order: Containment of relations defines a partial order. That is, $r \subseteq s$ meaning that for any $(x,y) \in X \times Y$, if $(x,y) \in r$ then $(x,y) \in s$.

(ii) Down-directed: For any $r, s \in P$, there exists $t \in P$ such that $t \subseteq r$ and $t \subseteq s$.

(iii) Up-set: For any relation $u : X \to Y$, if there exists $p \in P$ such that $p \subseteq u$ then $u \in P$.

Definiton 2.2. A prorelation $P : X \to Y$ can be composed to a prorelation $Q : Y \to Z$ by taking composition of the relations belonging to them. Then, the set $Q.P$ is defined as $Q.P = \{q \circ p : p \in P \text{ and } q \in Q\}$.

Lemma 2.3. Composition of two prorelations is a prorelation.

Proof. For prorelations $P : X \to Y$ and $Q : Y \to Z$, need to show that Q.P is a prorelation.

(i) (Partial Order) Inclusion of relations gives a partial order.

(ii) (Down-Directed) If $k, k' \in Q.P$, then $k = q\,p$ and $k' = q'\,p'$ for some $q, q' \in Q$ and $p, p' \in P$. Because Q and P are prorelations, and hence down-directed sets there exists, $a \in Q$ such that $a \subseteq q, q'$ and $b \in P$ such that $b \subseteq p, p'$. Thus, giving an element, $a \circ b$ of $Q.P$ such that $a \circ b \subseteq k, k'$.

(iii) (Up-Set) Let $l : X \to Z$ be a relation, and $k \in Q.P$ such that $l \supseteq k$. Define relations $p : X \to Y$ and $q : Y \to Z$ as, $p = \{(x,y) : x \in \mathit{domain}(l) \text{ and } y \in Y\}$ and $q = \{(y,z) : y \in Y \text{ and } z \in \mathit{range}(l)\}$. Because $k \in Q.P$, there exist $q' \in Q$ and $p' \in P$ such that $k = q' \circ p'$. Thus by definition of p and q, we get that $p \supseteq p'$ and $q \supseteq q'$. Hence $p \in P$ and $q \in Q$ because P and Q are up-sets, which gives us that $q \circ p \in Q.P$. For any $(x,z) \in l$, by definition of p and q, we get that for every $y \in Y$, $(x,y) \in p$ and $(y,z) \in q$. By definition of composition, this gives that $(x,z) \in q \circ p$, giving that $l \subseteq q \circ p$. And, by definition of $q \circ p$ we get that $l \supseteq q \circ p$. Finally giving that $l = q \circ p \in Q.P$. □

Definiton 2.4. For prorelations $P, Q : X \to Y$, if every relation in Q is contained inside some relation in P, then P is said to be contained in Q. That is $P \leq Q$ only if $\forall q \in Q, \exists p \in P$ such that $p \subseteq q$.

Definiton 2.5. For a relation $r : X \to Y$, it's opposite relation $r^o : Y \to X$ is defined as

$$(y,x) \in r^o \text{ if and only if } (x,y) \in r \text{ for } x \in X \text{ and } y \in Y.$$

Lemma 2.6. For any function $f : X \to Y$, Δ_X is contained in the composition $f^o \circ f$.

Proof. As f is a function, it must be defined on every element of it's domain. Thus, for every $x \in X$, there exists some (x, y) in f. By definition of f^o, (y, x) is in f^o. Hence, by definition of composition, (x, x) is in $f^o\, f$. □

Lemma 2.7. For any relation $r : X \to Y$, the composition $r \circ r^o$ contains $\subseteq \Delta_Y$.

Proof. Suppose there exist $x \in X$ and $y \in Y$ such that $x\, r\, y$. By definition of r^o, this gives us that $y\, r^o\, x$. Using definition of composition, $y\, r^o\, x\, r\, y$ gives that $y\, (r \circ r^o)\, y$. □

Lemma 2.8. For relations $r, s : X \to Y$ and $t : Y \to Z$, if $r \subseteq s$ then $(t \circ r) \subseteq (t \circ s)$.

Proof. Suppose relations r, s and t are as given above, and let $x\, (t\, r)\, z$. By definition of composition, there exists, $y \in Y$ such that $x\, r\, y$ and $y\, t\, z$. Using the hypothesis, as $r \subseteq s$, $x\, r\, y$ gives $x\, s\, y$. And via composition of $x\, s\, y$ with $y\, t\, z$, we get $x\, (t\, s)\, z$. We started with any element of $(t \circ r)$ and showed that it must also be in $t \circ s$ and thus have that $(t \circ r) \subseteq (t \circ s)$. □

Lemma 2.9. For relations $r : X \to Y$ and $s, t : Y \to Z$, if $s \subseteq t$ then $(s \circ r) \subseteq (t \circ r)$.

Proof. Suppose relations r, s and t are as given above, and let $x\, (s\, r)\, z$. By definition of composition of relations, we get that there exists some $y \in Y$ such that $x\, r\, y$ and $y\, s\, z$. Because $s \subseteq t$, $y\, s\, z$ implies that $y\, t\, z$. Taking the composition, $x\, r\, y\, s\, z$ yields $x(t\, r)z$. We started with any element of $(s \circ r)$ and showed that it must also be in $t \circ r$ and thus have that $(s \circ r) \subseteq (t \circ r)$. □

3 Quasi-Uniform Spaces

Definiton 3.1. A prorelation on a set, $P : X \to X$ is said to be a quasi-uniformity if it satisfies the following conditions:

(i) Every relation in P is reflexive. That is, for each $p \in P$, if $x \in X$ then $(x, x) \in p$.

(ii) For each p in P, there exists p' in P such that $p' \circ p' \subseteq p$.

Definiton 3.2. If X is a set, and A is a quasi-uniformity on X, then (X,A) is a quasi-uniform space.

Definiton 3.3. A function, $f : (X, A) \to (Y, B)$ is said to be uniformly continuous if $f.A \leq B.f$. That is, for each $b \in B$, there exists $a \in A$ such that $f \circ a \subseteq b \circ f$. Meaning that
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "A"'] \arrow[dr, phantom, "\leq"] & Y \arrow[d, "B"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}.$$

Lemma 3.4. If A is a quasi-uniformity on a set X, then $A.A = A$

Proof. Fix any $a \in A$, as A is a quasi-uniformity, $\exists b \in A : bb \subseteq a$, we get that $A.A \leq A$. And as A is a prorelation, and is hence down-directed, $\exists c \in A : a.a \supseteq c$, giving that $A.A \geq A$ □

Proposition 3.5. QUnif is a category defined as having quasi-uniform spaces as objects, and uniformly continuous maps between them as morphisms, with composition defined as that of functions.

Proof. (i) (Associativity) The composition of functions is associative by definition.

(ii) (Identity) For each object (X, A), define it's identity to be the identity function $\Delta_X : (X, A) \to (X, A)$. This function is uniformly continuous as $\Delta_X.A = A \leq A = A.\Delta_X$. □

Definiton 3.6. A prorelation, $\phi : X \twoheadrightarrow Y$ is called a promodule $\phi : (X, A) \twoheadrightarrow (Y, B)$ if it satisfies:

$$\phi.A \leq \phi \text{ and } B.\phi \leq \phi$$

Definiton 3.7. Containment of promodules is defined as that of prorelations.

Definiton 3.8. Promodules are composed as prorelations.

Proposition 3.9. ProMod is a 2-category defined as having quasi-uniform spaces as its 0-cells, promodules as 1-cells and containment of promodules as 2-cells.

Proof. In order to show that ProMod is a 2-category, need the following:

(a) (1-Identities) For each quasi-uniform space (X, A), define $A : (X, A) \to (X, A)$ to be the identity 1-cell for (X, A). A is a promodule because $A.A = A$ (Lemma 2.2.1)

(b) (1-Composition) Need promodules to be closed under composition.
Let $\phi : (X, A) \to (Y, B)$ and $\psi : (Y, B) \to (Z, C)$ be promodules. To show that $\psi.\phi : (X, A) \to (Z, C)$ is a promodule, need it to be a prorelation that satisfies the two conditions required to be a promodule:

(i) By Lemma 1.2.1, prorelations are closed under composition. Hence, $\psi.\phi$ is a prorelation

(ii) Need to show that $\psi.\phi.A \leq \psi.\phi$. So, Fix $p \in \psi$ and $q \in \phi$. As ϕ is a promodule, $\phi.A \leq \phi$ gives that there exists $q' \in \phi$ and $a \in A$ such that $q'\,a \subseteq q$. Thus, $p\,q'\,a \subseteq p\,q$.

(iii) Need to show that $C.\psi.\phi \leq \psi.\phi$.Fix $p \in \psi$ and $q \in \phi$. Because ψ is a promodule, $C.\psi \leq \psi$ gives that there exists $c \in C$ and $p' \in \psi$ such that $c\,p' \subseteq p$. Thus, $c\,p'\,q \subseteq p\,q$

(c) (2-Identities) As every promodule is contained in itself, always have $\psi \leq \psi$. Define this comparison to be the identity 2-cell for ψ and denote it by $\leq_\psi$

(d) (Vertical 2-composition) For promodules $\psi, \phi, \delta : (X, A) \to (Y, B)$, if there is is a 2-cell from ψ to ϕ and another one from ϕ to δ i.e. $\psi \leq \phi \leq \delta$, then by transitivity of the partial order, $\psi \leq \delta$ i.e. there's a 2-cell from ψ to δ.

(e) (Horizontal 2-composition) If there are promodules $\psi, \psi' : (X, A) \to (Y, B)$ and $\phi, \phi' : (Y, B) \to (Z, C)$ such that $\psi \leq \psi'$ and $\phi \leq \phi'$, need to show that $\psi.\phi \leq \psi'.\phi'$. Fix $p' \in \psi'$ and $q' \in \phi'$. As $\psi \leq \psi'$, $\exists p \in \psi : p \subseteq p'$ and as $\psi \leq \psi'$, $\exists q \in \phi : q \subseteq q'$. Thus, $p\,q \subseteq p'\,q'$

(f) (1-Identity) Need to show that for any promodule $\phi : (X, A) \to (Y, B)$, $\phi.A = \phi = B.\phi$. By quasi-uniformity of A, every $a \in A$, is reflexive. Thus, for any $p \in \phi$ and $a \in A$, $p = p.\Delta_X \subseteq p\,a$ giving that $\phi \leq \phi.A$. And as ϕ is a promodule, $\phi \geq \phi.A$. Hence, by anti-symmetry of the partial order, $\phi = \phi.A$.

Similarly, By quasi-uniformity of B, every $b \in B$, is reflexive. Thus, for any $p \in \phi$ and $b \in B$, $p = \Delta_Y.p \subseteq b\,p$ giving that $\phi \leq B.\phi$. And as ϕ is a promodule, $\phi \geq B.\phi$. Hence, $\phi = B.\phi$.

(g) (1-Associativity) As composition of relations is associative, so too is the composition of prorelations directly giving that composition of promodules i.e. 1-cells is associative.

(h) (Vertical 2-Identity) Let $\leq : \psi \to \phi$ be a 2-cell i.e. $\psi \leq \phi$. By our definition of identity 2-cell, $\leq_\psi . \leq_1$ means precisely that $\psi \leq \psi \leq \phi$, and by transitivity, this is equivalent to $\psi \leq \phi$. Similarly, $\leq_1 . \leq_\phi$ means exactly that $\psi \leq \phi \leq \phi$, and this is equivalent to $\psi \leq \phi$.

(i) (Vertical 2-Associativity) Associativity of the partial order on promodules directly gives the associativity of composition of 2-cells in ProMod.

(j) (Horizontal 2-Identity) Let $\psi, \phi : (X, A) \to (Y, B)$ be promodules. For any 2-cell $\leq : \psi \to \phi$, need to show that the 2-cell given by the horizontal composition, $\leq * \leq_A$ is equal to $\leq$, as well as equal to $\leq_B * \leq$. So, it's required that $\psi.A \leq \phi.A \iff \psi \leq \phi \iff B.\psi \leq B.\phi$. And this holds as a direct consequence of (f).

(k) (Horizontal 2-Associativity) As there's a unique 2-cell between any two promodules, and composition of promodules is associative, horizontal composition of 2-cells is associative.

(l) (2-Identity) For promodules $\psi : (X, A) \to (Y, B)$ and $: \phi(Y, B) \to (Z, C)$ need $(\leq_\psi * \leq_\phi) = \leq_{\psi.\phi}$. Both sides of the required equality are 2-cells $\leq : \psi.\phi \to \psi.\phi$. Thus, they are equal by the uniqueness of 2-cells between any two 1-cells.

(m) (2-Interchange) Let $\psi, \phi, \delta : (X, A) \to (Y, B)$ and $\psi', \phi', \delta' : (Y, B) \to (Z, C)$ be promodules. For 2-cells $\leq_1 : \psi \to \phi$,$\leq_2 : \phi \to \delta$, $\leq_a : \psi' \to \phi'$ and $\leq_b : \phi' \to \delta'$,need to show $(\leq_b . \leq_a) * (\leq_2 . \leq_1) = (\leq_b * \leq_2).(\leq_a * \leq_1)$. Both RHS and LHS are 2-cells from $\psi.\psi'$ to $\delta.\delta'$ and are hence equal. $\square$

Proposition 3.10. $(\_)_* :$ QUnif $\to$ ProMod defined above is indeed a functor.

Proof. We will first show that $B.f = b \circ f : b \in B$ is a promodule, and then that $(-)_*$ defines a functor.

(i) (Partial-Order) Inclusion of relations acts as the partial order.

(ii) (Down-Directed) Fix any k, k' belonging to $B.f$. Thus, there exist b, b' in B such that $k = b\,f$ and $k' = b\,f$. Using down-directedness of B, there exists a $c \in B$ such that $c \subseteq b, b'$. Hence, by Lemma 2.4.4, $c\,f \subseteq k, k'$.

(iii) (Up-set) Let k belong to $B.f$ and $l : (X, A) \to (Y, B)$ be a uniformly continous function such that $l \supseteq k$. Define a relation $b' := \{(f(d), l(d)) : d \in domain(l)\}$. By definition, for any $x \in X$ and $z \in Y$ such that $(x, z) \in l$, we get that $(f(x), z) \in b'$. And $l \supseteq k = b\, f$ implies $domain(l) \supseteq domain(f)$ giving $(x, f(x)) \in f$. Thus, by defniition of composition, $(x, z) \in b'.f$. Conversely, suppose $(x, z) \in b'.f$. By definition of composition, there exists $f(x) \in Y$ such that $(f(x), z) \in b'$. Again using the definition of b', we get that $z = l(x)$ i.e. $(x, z) \in l$. Hence, $l = b'\, f$. Now we will show that $b' \supseteq b$. Because $b'\, f = l \supseteq k = b\, f$, for any $x \in X$ we have that $b'\big(f(x)\big) \supseteq b\big(f(x)\big)$. Thus, $b'|_{f(x)} \supseteq b|_{f(x)}$. By down-directedness of B, the restriction $b|_{f(x)} \subset b$ implies $b(x)|_{f(x)} \in B$. Finally, $b' \supseteq b'|_{f(x)} \supseteq b|_{f(x)}$ gives $b' \in B$. Hence, $b'.f \in B.f$.

(iv) Need to show that $(B.f).A \leq B.f$. So, fix any $b \in B$, we will find $b' \in B$ and $a \in A$ such that $b'\, f\, a \subseteq bf$. By quasi-uniformity of B, there exists $b' \in B$ such that $b'\, b' \subseteq b$. Using Lemma 2.4.3, we get that $b'\, b'\, f \subseteq b\, f$. As f is uniformly continous, $f.A \leq B.f$ gives that there is some $a \in A$ such that $f\, a \subseteq b'\, f$. Using this in the previous inequality, we get $b'\, f\, a \subseteq b'\, b'\, f \subseteq b\, f$.

(v) Need to show that $B.B.f \leq B.f$. Fix any $b \in B$, we will find $b' \in B$ such that $b'\, b'\, f \subseteq b\, f$. By quasi-uniformity of B, there exists $b \in B$ such that $b'\, b' \subseteq b$. Using Lemma 2.4.4, we get $b'\, b'\, f \subseteq bf$.

Thus, $B.f$ is a promodule. We now proceed to show that $(\_)_*$ defines a functor.

(i) (Composition) Need to show that $(g \circ f)_* = g_* f_*$ i.e. $C.g.f = C.g.B.f$.

In order to show $C.g.f \leq C.g.B.f$, fix any $b \in B, c \in C$. We will show that $c\, g\, f \subseteq c\, g\, b\, f$. As f is uniformly continuous, $f.A \leq B.f$ gives that there exists $a \in A$ such that $f\, a \subseteq b\, f$. Using Lemma 2.4.3, we get $(c\, g) f\, a \subseteq (c\, g) b\, f$. Now, using reflexiviness of a, we get $c\, g\, f \subseteq c\, g\, b\, f$.

Now, to show that $C.g.f \geq C.g.B.f$. Fix any $c \in X$, we will find $c' \in C$ and $b \in B$ such that $c\, g\, f\ \supseteq c\, g\, b\, f$. By quasi-uniformity of C, there exists $c' \in C$ such that $c \subseteq c'\, c'$. Using Lemma 2.4.4 gives that $c\,(g\, f) \supseteq c'\, c'\,(g\, f)$. Because g is uniformly continuous, $C.g \geq g.B$ gives us $b \in B$ such that $g\, c' \supseteq b\, g$. Using this in the previous inequality gives that $c\, g\, f \supseteq c'\, g\, b\, f$.

(ii) (Identity) let (X, A) be in object of QUnif and $1_{(X,A)} : (X, A) \to (X, A)$ be the identity funtion on (X, A). That is, $1_{(X,A)}$ is defined as $x \mapsto x$. Need to show that $(1_{(X,A)})_* = 1_{(X,A)_*}$. Using functor's definition, $LHS = (1_{(X,A)})_* = A.(1_{(X,A)}) = A.1_{(X,A)} = A$ and $RHS = 1_{(X,A)_*} = 1_{(X,A)}$ Using Proposition 3.2(f), we get that $A = 1_{(X,A)} = RHS$. ☐

Proposition 3.11. Functor, $(-)^* :$ QUnif^{op} $\to$ ProMod is defined as fixing quasi-uniform spaces and taking uniformly continuous functions to the composition of their opposite relation with it's domain space's identity i.e.

(a) for $(X, A) \in$ QUnif^{op}, $(X, A)^* := (X, A) \in$ ProMod

(b) for $f : (X, A) \to (Y, B)$ in QUnif, $f^* := f^o.B$

Proof. Showing that $f^o.B : (Y, B) \looparrowright (X, A)$ is a promodule.
So, need to show $f^o.B$ a prorelation $Y \to X$ and that $(f^o.B).B \sqsubseteq f^o.B$ and $A.(f^o.B) \sqsubseteq f^o.B$
To show prorelation,

(i) (Partial-order) Inclusion of relations i.e. for $k = f^o \circ b$ and $k' = f^o \circ b'$ in $f^o.B$, $k \subseteq k' \iff b \subseteq b'$

(ii) (Down directed) for $k, k' \in f^o.B$, need that $\exists l \in f^o.B$ such that $l \subseteq k, k'$

Fix $k, k' \in f^o.B \implies \exists b, b' \in B : k = f^o \circ b$ and $k' = f^o \circ b'$

By down-directedness of B, there exists $c \in B$ such that $c \subseteq b, b'$, define $l = f^o \circ c$. Now, using Lemma 2.4.3 gives $l = f^o \circ c \subseteq k, k'$.

(iii) (Up-set) for a relation $l : Y \to X$ and $k \in f^o.B$ such that $l \supseteq k$, need $l \in f^o.B$

Let $b \in B$ be such that $k = f^o \circ b$ and define $b' := \{(y, y') : y \in domain(l)$ and $y' \in (f^o)^{-1}(l(y))\}$
As $l \supseteq k = f^o \circ b$, $domain(b') = domain(l) \supseteq domain(b)$
and $range(l) \supseteq range(f^o \circ b) \implies \forall y \in domain(b), range(b') = (f^o)^{-1}(l(y)) \supseteq (f^o)^{-1}(f^o \circ b) = range(b)$
Now, by definition of b', $f^o \circ b' \supseteq l$. To show $f^o \circ b \subseteq l$,
$(x, y) \in f^o \circ b' \implies \exists z \in Y : (x, z) \in b'$ and $(z, y) \in f^o \implies x \in domain(l)$ and $z \in l(x)$ i.e. $(x, z) \in l$

To show $(f^o.B).B \leq f^o.B$, need that $\forall b \in B, \exists b' \in B : f^o \circ b' \circ b' \subseteq f^o \circ b$,
Fix any $b \in B$, as B is a quasi-uniformity, $\exists b' \in B : b' \circ b' \subseteq b \implies f^o \circ b' \circ b' \subseteq f^o \circ b$
To show $A.(f^o.B) \leq f^o.B$, need that $\forall b \in B, \exists b' \in B, a \in A : a \circ f^o \circ b' \subseteq f^o \circ b$,
As f is uniformly continuous, $f.A \leq B.f$ i.e. $\forall b \in B, \exists a \in A : f \circ a \subseteq b \circ f \implies a = f^o \circ f \circ a \subseteq f^o \circ b \circ f$
Fix any $b \in B$, so, $\exists b' \in B : b'b' \subseteq b$ And, for this b', $\exists a : a \subseteq f^o b' f \implies a f^o b' \subseteq f^o b' f f^o b' \subseteq f^o b' b' \subseteq f^o b \implies a f^o b' \subseteq f^o b$

Now, need to show that $(-)^*$ respects composition and identity.

(i) (Composition) let f, g be uniformly continuous, $(X,A) \xrightarrow{f} (Y,B) \xrightarrow{g} (Z,C)$ need that $(g\circ f)^* = f^*.g^*$
LHS=$(g\circ f)^* = (g\circ f)^o.C = (f^o \circ g^o).C$ and RHS=$f^*.g^* = (f^o.B).(g^o.C)$
For equality, showing that LHS≥RHS and LHS≤RHS:

To show $(f^o \circ g^o).C \geq (f^o.B).(g^o.C)$, need that $\forall c \in C, \exists b \in B, c' \in C : f^o g^o c \supseteq f^o b g c'$
Fix any $c \in C$, so, $\exists c' \in C : c' \circ c' \subseteq c \implies f^o g^o c \supseteq f^o g^o (c'c') = f^o g^o (c' \Delta_Z c') \supseteq f^o g^o c' (g g^o) c'$
By uniform conntinuity of g, for $c' \in C, \exists b \in B : gb \subseteq c'g$
Thus, $f^o g^o c \supseteq f^o g^o (c'g) g^o c' \supseteq f^o (g^o g) b g^o c' = f^o b g^o c'$.

To show $(f^o \circ g^o).C \leq (f^o.B).(g^o.C)$, need that $\forall b \in B, c \in C, \exists c' \in C : f^o g^o c \subseteq f^o b g^o c$
Fix any $c \in C, b \in B$ will show that $c' := c$ works:
As B is a quasi-uniformity, $\Delta_Y \subseteq b \implies f^o \Delta_Y g^o c = f^o g^o c \subseteq f^o b g^o c = f^o b g^o c'$

(ii) (Identity) let $(X,A) \in$ QUnif ^{op}, and $1_{(X,A)} : (X,A) \to (X,A)$ as $x \mapsto x$ need that $(1_{(X,A)})^* = 1_{(X,A)^*}$
LHS=$(1_{(X,A)})^* = (1_{(X,A)})^o.A = 1_{(X,A)}.A = A$.
And as $RHS = 1_{(X,A)^*} = 1_{(X,A)}$ Using Proposition 3.2(f), we get that $A = 1_{(X,A)} = RHS$. ☐

Definiton 3.12. Let $f : (X,A) \to (Y,B)$ be a uniformly continuous function.

I f is said to be fully faithful if $f^*.f_* = A$

II f is said to be fully dense if $f_*.f^* = B$

Proposition 3.13. Fix a uniformly continuous map, $f : (X,A) \to (Y,B)$

(a) f is fully faithful if and only if $A \geq f^o.B.f$

(b) f is fully dense if and only if for any $b \in B$, $\exists b' \in B$ such that $b' \subseteq b\, f\, f^o\, b$

(c) f is topologically dense if and only if for any $b \in B$, $b\, f\, f^o\, b$ is reflexive

(d) f is fully dense if and only if f is topologically dense

Proof.

(a) (i) ($\implies$) Let f be fully faithful i.e. $f^*.f_* = A \implies f^o.B.B.f = A$
Need to show that $A = f^o.B.f$ i.e. $A \leq f^o.B.f$ and $A \geq f^o.B.f$
By hypothesis and quasi-uniformity of B, $A \geq f^o.B.B.f \geq f^o B.f$
To show $A \leq f^o.B.f$, need that $\forall b \in B, \exists a \in A : a \subseteq f^o b f$
Fix $b \in B$, hypothesis gives that $f^o.B.B.f \leq A$ so,
$\exists a \in A : a \subseteq f^o b b f$ and also, by quasi-uniformity of B, for $b, \exists b' \in B : b'b' \subseteq b \implies f^o b' b' f \subseteq f^o b f$
Combining the above two inequalities, $a \subseteq f^o b b f \subseteq f^o b f$

(ii) ($\impliedby$) Let $A = f^o.B.f$ need to show $A = f^o.B.B.f$ i.e. $A \geq f^o B.B.f$ and $A \leq f^o B.B.f$
To show $A \geq f^o.B.B.f$, need to show that $\forall a \in A, \exists b, b' \in B : a \supseteq f^o b b' f$
Have that $A \geq f^o.B.f$ and $B.B \leq B$
So, fix $a \in A$, now $\exists b \in B : a \subseteq f^o b f$ and for this b, $\exists b' \in B : b'b' \subseteq b$. Therefore, $a \supseteq f^o b f \supseteq f^o b' b' f$
To show $A \leq f^o.B.B.f$, need $\forall b, b' \in B, \exists a \in A : a \subseteq f^o b b' f$
Before that, uniform continuity of f along with Lemma 2.1.1 gives that
$f.A \leq B.f \implies A = f^o f.A \leq f^o.B.f$
So, fix $b, b' \in B$, now, as ,
$A \leq f^o.B.f$ giving
$\exists a \in A : a \subseteq f^o b f$ and $\exists a' \in A : a' \subseteq f^o b' f \implies \Delta_X \subseteq f^o b' f$.
Therefore $a = a\Delta_X \subseteq (f^o b f)(f^o b' f) \subseteq f^o b b' f$

(b) (i) ($\implies$) Let f be fully dense i.e. $B = f_* f^* = B.f.f^o.B$. showing that $\forall b \in B, \exists b' \in B : b' \subseteq b f f^o b$:
So, fix $b \in B$, as $B \leq B.f.f^o.B$, there exists $b' \in B$ such that $b' \subseteq b f f^o b$.

(ii) ($\impliedby$) Suppose $\forall b \in B, \exists b' \in B : b' \subseteq b f f^o b$. This gives $B \leq B.f.f^o.B$, in order to show equality, also need $B \geq B.f.f^o.B$. By quasi-uniformity of B, for any $b \in B$, $\exists b' \in B : b'b' \subseteq b$. Now, by Lemma 2.4.2,

$$f f^o \subseteq \Delta_Y \implies b' f f^o b' \subseteq b' \Delta_Y b' = b'b' \subseteq b$$

(c) (i) ($\Longrightarrow$) Let f be topologically dense. We will show that for any $b \in B, y \in Y, \ \underline{(y,y)} \in bff^ob$. Fix any $b \in B$ and $y \in Y$. As f is topologically dense, $\overline{f(X)} = Y$, implying that $y \in \overline{f(X)}$, by definition giving that

$$\exists x \in X \text{ such that } (f(x), y) \in b \text{ and } (y, f(x)) \in b$$

Re-writing the above statement in terms of relations, and considering f as a relation:

$$(f(x), y) \in b \text{ gives } x(b \circ f)y \text{ i.e. } y \in (b \circ f)(x) \tag{1}$$
$$(y, f(x)) \in b \text{ gives } f(x) \subseteq b(y) \tag{2}$$

Repeatedly applying Lemma 2.4.3 to (2),

$$f(x) \subseteq b(y) \implies \big(f \circ f^o\big)(f(x) \subseteq \big(f \circ f^o\big)b(y) \implies \big(f \circ f^o \circ f\big)(x) \subseteq \big(f \circ f^o \circ b\big)(y)$$

Applying Lemma 2.4.1 to the final inequality in the above statement gives that

$$f(x) = (f \circ \Delta_X)(x) \subseteq \big(f \circ f^o \circ f\big)(x) \subseteq \big(f \circ f^o \circ b\big)(y)$$

Applying Lemma 2.4.3 and then using (1) on the above inequality completes the result:

$$f(x) \subseteq \big(ff^ob\big)(y) \implies \big(b \circ f\big)(x) \subseteq \big(bff^ob\big)(y) \implies y \in \big(bff^ob\big)(y) \text{ i.e. } y\big(bff^ob\big)y$$

(ii) ($\Longleftarrow$) Fix any $y \in Y$ and $b \in B$. Also, suppose that $\Delta_Y \leq bff^ob$. As f is a function with domain as X, $f^o : Y \to X$, $\phi \neq (f^o \circ b)(y) \subseteq X$. So, fix $x \in (f^o \circ b)(y)$, going to show that $(f(x), y) \in b$ and $(y, f(x)) \in b$. Again, while viewing f as a relation.

$$\Delta_Y \leq bff^ob \implies \Delta_Y(y) \subseteq bff^ob(y) = \big(bf\big)\big(f^ob(y)\big)$$

Last inequality of the above statement gives $y \in (bf)(x)$ i.e. $(f(x), y) \in b$.
Applying Lemma 2.4.2 to f, and then using Lemma 2.4.4,

$$ff^o \subseteq \Delta_Y \implies ff^ob \subseteq \Delta_Y b = b$$

Thus $ff^ob(y) \subseteq b(y)$ and hence $f(x) \subseteq b(y) \implies (y, f(x)) \in b$

(d) (i) ($\Longrightarrow$) Let f be topologically dense. As B is a quasi-uniformity, for any $b \in B$,

$$\exists b' \in B : b'b' \subseteq b \text{ and } \Delta_Y \subseteq b' \implies b' = b'\Delta_Y \subseteq b'b' \subseteq b \tag{3}$$

By the characterisation of topologically dense in (c), have that $\Delta_Y \subseteq b'ff^ob'$. Now, using the (3) and Lemma 2.4.3,

$$\Delta_Y \subseteq b'ff^ob' \implies b' = b'\Delta_Y \subseteq b'b'ff^ob' \subseteq bff^ob' \subseteq bff^ob$$

Hence, we have $b' \in B : b' \subseteq bff^ob$ giving us that f is fully dense (from (b)).

(ii) ($\Longleftarrow$) From (b), we have for $b \in B$, the existstence of $b' \in B$ such that $b' \subseteq bff^ob$. As B is a quasi-uniformity, $\Delta_Y \subseteq b'$. So, $\Delta_Y \subseteq bff^ob$, and from (c), this gives us that f is topologically dense. □

4 Yoneda Lemma in Quasi-Uniform Spaces

Definiton 4.1. The set PX is defined to be the collection of all promodules from the quasi-uniform space(X,A) to the quasi-uniform space 1.

$$PX := \{\psi : (X, A) \multimap 1 \text{ is a promodule}\}$$

Proposition 4.2. For any $a \in A$, $\tilde{a}$ is defined to be a relation $PX \to PX$ as

$$\text{for } \phi, \psi \in PX, \ \phi\, \tilde{a}\, \psi \ \text{ only if } \ \phi \leq \psi.a$$

The set, $\tilde{A} := \{\tilde{a} : a \in A\}$ defines a quasi-uniformity on PX.

Proof. First need to show that $\tilde{A}$ is a prorelation,

(i) (Partial order) For any two relations $\tilde{a}, \tilde{b} : PX \to PX$, define $\tilde{a} \leq \tilde{b}$ to be true only if $a \subseteq b$.

(ii) (Down-Directed) Need for any $\tilde{a}, \tilde{b} \in \tilde{A}$, the existstence of some $\tilde{c} \in A$ such that $c \subseteq a, b$
If $\tilde{a}, \tilde{b} \in A$ then there exist $a, b \in A$. By down-directedness of A, there exists a $c \in A$ such that $c \subseteq a, b$. Now the definition of $\tilde{A}$ gives that $\tilde{c} \in \tilde{A}$. And the definition of the partial order on $\tilde{A}$ ensures $\tilde{c} \leq \tilde{a}, \tilde{b}$.

(iii) (Upset) For any relation $l : PX \to PX$, need that if $\tilde{k}$ belongs to $\tilde{A}$ such that $l \geq \tilde{k}$, then $l \in \tilde{A}$.
Fix any $k : PX \to PX$, and $\tilde{k} \in \tilde{A}$ such that $l \geq \tilde{k}$. As k is a relation between promodules $X \to\!\!\!\!\circ\; 1$, it can be thought of as a relation a on X, defined as:

$$a := \{(x, y) : x \in domain(\psi) \text{ and } y \in domain(\phi) \text{ whenever } \exists \psi, \phi \in PX : \psi l \phi\}$$

So, $l \geq \tilde{k}$ gives that $\tilde{a} \geq \tilde{k}$ i.e. $a \supseteq k$. And as A is an upper-set, we get $a \in A$. Now, by definition of $\tilde{A}$, $l \in \tilde{A}$.

Secondly, need show that the other two conditions hold for $\tilde{A}$,

(i) For all $\tilde{a} \in \tilde{A}$, need $\tilde{a}$ to be reflexive i.e if $\psi \in PX$ then $\psi\, \tilde{a}\, \psi$.
By definition of $\tilde{a}$, need to show that $\psi \leq \psi.a$. So, fix a $p \in \psi$, we will show that $p \subseteq p.a$. Quasi-uniformity of A gives that $\Delta_X \subseteq a$. Hence, by Lemma 2.4.3, $p = p\,\Delta_X \subseteq p\,a$.

(ii) For all $\tilde{a} \in \tilde{A}$, need to find $\tilde{b} \in \tilde{A}$ such that $\tilde{b}\tilde{b} \leq \tilde{a}$
Before showing the result, proving that for any $x, y \in A$, $\tilde{x}\,\tilde{y} \leq \widetilde{xy}$ i.e. $\forall \psi, \phi \in PX$, $\psi(\tilde{x}\,\tilde{y})\phi \implies \psi\,\widetilde{xy}\,\phi$. If $\psi_1(\tilde{a}.\tilde{b})\psi_3$, then, the definition of composition gives that $\exists\psi_2$ such that $\psi_1\,\tilde{b}\,\psi_2\,\tilde{a}\,\psi_3$. Now, the definition of $\tilde{b}$ gives $\psi_1 \leq \psi_2\, b$ and that of $\tilde{a}$ gives $\psi_2 \leq \psi_3\, a$. Combining these inequalities, $\psi_1 \leq \psi_2.b \leq \psi_3.ab$. Hence, by definition of $\widetilde{ab}$, $\psi_1\,(\widetilde{ab})\,\psi_3$. Now, to show the result, fix any $\tilde{a} \in \tilde{A}$. Therefore,$a \in A$, and by quasi-uniformity of A, $\exists b \in A : b \circ b \subseteq a$. Thus, by the partial-order defined on $\tilde{A}$, $\widetilde{bb} \leq \tilde{a}$. Now, transitivity of the partial order gives us the required result, $\tilde{b}\,\tilde{b} \leq \widetilde{bb} \leq \tilde{a}$. □

Proposition 4.3 (Yoneda Embedding)**.**

For a quasi-uniform space (X, A), function $y_X : X \to PX$ is defined by $x \mapsto x^*$ for $x \in X$.

(a) $y_X : (X, A) \to (PX, \tilde{A})$ is a uniformly continuous map.

(b) $y_X : (X, A) \to (PX, \tilde{A})$ is fully faithful.

Proof.

(a) In order to show y_X is uniformly continuous, need to show that $y_X.A \leq \tilde{A}.y_X$. By definition of $\leq$, need $\forall a \in A, \exists b \in A : y_X \circ b \subseteq \tilde{a} \circ y_X$. Applying the relations to some element, x of the set X:

$$\big(y_X \circ b\big)(x) \subseteq \big(\tilde{a} \circ y_X\big)(x) \implies y_X(b(x)) \subseteq \tilde{a}(x^*) \tag{1}$$

So, for the condition given by (4) to hold, if $y \in b(x)$, then it's required that $y^* = y_X(y) \in \tilde{a}(x^*)$ i.e. $x^*\tilde{a}y^*$. Using the definition of x^*, y^* and $\tilde{a}$,

$$x^*\tilde{a}y^* \iff x^o.A \leq y^o.A.a \iff \forall a' \in A, \exists a'' \in A : x^o a'' \subseteq y^o a' a \tag{2}$$

Now, fix any $a \in A$, $x \in X$. Thus, quasi-uniformity of A, gives $a'' \in A$ such that $a''a'' \subseteq a$.Also, choose some $y \in a''(x)$. Hence, in order to show that the condition from (5) holds, need that $\forall b \in A, x^o a'' \subseteq y^o ba$, and by applying the relations to an element z gives the following condition

$$\forall b \in B, \forall x \in X \ , \ \big(x^o a''\big)(z) \subseteq \big(y^o ba\big)(z) \tag{3}$$

Examining the left side of (6),

$$\big(x^o a''\big)(z) = x^o(a''(z)) = \begin{cases} \phi & \text{if } x \notin a''(z) \\ \star & \text{if } \in a''(z) \end{cases}$$

Thus, to show that (6) holds, need to show that (for any $b \in A$ and $z \in X$):

$$x \in a''(z) \implies z(y^o ba)\star \text{ i.e. } y \in (ba)(z) \tag{4}$$

To show that (7) holds, fix any $z \in X : x \in a''(z)$. Also, by our choice of y, have that $y \in a''(x)$. And as $b \in A$, it's reflexive, giving that $y \in b(y)$. So, by composition of relations, we get:

$$za''x \ , \ xa''y \text{ and } yby \implies z(a''a''b)y \implies z(ab)y \text{ i.e. } y \in (ba)(z)$$

(b) By using Proposition2.3 (a), need to show that $A \geq y_X^o.\tilde{A}.y_X$ i.e. $\forall a \in A, \exists \tilde{b} \in \tilde{A} : a \supseteq y_X^o\ \tilde{b}\ y_X$. Applying to an element, $x \in X$ gives the condition

$$\Big(y_X^o\ \tilde{b}\ y_X\Big)(x) \subseteq a(x) \implies \Big(y_X^o\ \tilde{b}\Big)(x^*) = y_x^o\Big(\tilde{b}(x^*)\Big) \subseteq a(x) \tag{5}$$

Thus, if $y^* \in PX$ such that $x^*\tilde{b}y^*$, then $y \in y_x^o\Big(\tilde{b}(x^*)\Big)$. Now, for (8) to hold, $y \in a(x)$ i.e. xay. Thus, need only to show that for any $a \in A, \exists b \in A$ such that $\forall x, y \in X, x^*\tilde{b}y^* \implies xay$. So, fix $a \in A$, and take $b \in A : bb \subseteq a$. Now, let $x^*\tilde{b}y^*$ i.e. $x^o.A \leq y^o.A.b$. Hence, $\exists c \in A : x^o c \subseteq y^o bb$. And as c is reflexive,

$$xcx \implies x(cx^o)\star \implies x(bby^o)\star \implies x(bb)y \implies xay$$ □

Theorem 4.4 (Yoneda Lemma). *For every $\psi \in PX$, in the following digram,*

(a) $\psi \geq \psi^*.(y_X)^*$

(b) $\psi \in \overline{y_X(X)} \implies \psi \leq \psi^*.(y_X)_*$

Proof. (a) By definition, $(y_X)_* = \tilde{A}.y_X$, and $\psi^* = \psi^o.\tilde{A}$. Need that $\psi \geq (y_X)_*.\psi^* = \psi^o.\tilde{A}.\tilde{A}.y_X$. And applying Lemma 2.2.1 to $\tilde{A}$, the required condition becomes $\psi \geq \psi^o.\tilde{A}.y_X$ Fix $p \in \psi$, we will find $a \in A : p \supseteq \psi^o a y_X$. Examining the right side of the condition, (for any $a \in A$, $x \in X$)

$$\Big(\psi^o.\tilde{a}.y_X\Big)(x) = \psi^o.\tilde{a}(x^*) = \psi^o\big(\tilde{a}(x^*)\big) = \begin{cases} \phi & \text{if } \psi \notin \tilde{a}(x^*) \\ \star & \text{if } \psi \in \tilde{a}(x^*) \end{cases} \tag{1}$$

In case $\psi \notin \tilde{a}(x^*)$, the condition holds trivially. As ψ is a promodule, $\psi.A \leq \psi$ gives $\exists q \in \psi, a \in A : qa \subseteq p$. Thus, fix $x \in X$ and $\psi \in PX$ such that $x^*\tilde{a}\psi$. We will now show that $xp\star$. Using the definition of $\tilde{a}$,

$$x^*\tilde{a}\psi \implies x^o.A \leq \psi.a \implies \exists b \in A : x^o b \subseteq qa \implies \forall z \in X, \big(x^o b\big)(z) \subseteq (qa)(z) \tag{2}$$

Thus, in particular for $z = x$, as b is reflexive, xbx, which gives:

$$\big(x^o b\big)(x) \subseteq (qa)(x) \implies x^o x \subseteq (qa)(x) \implies \star \in (qa)(x) \tag{3}$$

But, as $qa \subseteq p$, (11) gives that $xp\star$.

(b) Suppose $\psi \in \overline{y_X(X)}$, need to show $\psi \leq \psi^*.(y_X)_* = \psi^o.\tilde{A}.y_X$ i.e. for $a \in A$, $\exists p \in \psi : p \subseteq \psi^o.\tilde{a}.y_X$. For any $x \in domain(p)$, the condition requires:

$$p(x) \subseteq \psi^o.\tilde{a}.y_X(x) = \psi^o\big(\tilde{a}(x^*)\big) \tag{4}$$

By definition of p, for (12) to hold, need that $xp\star \implies \psi \in \tilde{a}(x^*)$. Fix any $a \in A$, we will find $p \in \psi$ such that (12) holds. By quasi-uniformity of A, $\exists b \in A : bb \subseteq a$. From Proposition 2.5(a), y_X is uniformly continuous, $y_X.A \leq \tilde{A}.y_X$ giving that $\exists c \in A : y_x c \subseteq \tilde{b} y_X$. Thus, for any $z, w \in X$ such that zcw,

$$\big(y_X c\big)(z) \subseteq \big(\tilde{b} y_X\big)(z) \implies y_X(c(z)) \subseteq \tilde{b}(z^*) \implies w^* \in \tilde{b}(z^*) \text{ i.e. } z^*\tilde{b}w^* \tag{5}$$

As A is a quasi-uniformity, $\exists d \in A : dd \subseteq c$. Also, because A is a down- directed set, $\exists a' \in A : a' \subseteq b, d$. This along with (13) gives that for any $x, y \in X$

$$x(a'a')y \implies x(dd)y \implies xcy \implies x^*\tilde{b}y^* \tag{6}$$

Now, because $\psi \in \overline{y_X(X)}$, we get $\exists x^* \in y_X(X)$ such that $\psi\tilde{a'}x^*$ and $x^*\tilde{a'}\psi$. By definition of $\tilde{a}$, $\psi\tilde{a'}x^*$ gives

$$\psi \leq x^o.A.a' \implies \exists p \in \psi : p \subseteq x^o a'a' \tag{7}$$

Fix any $z \in X : zp\star$, using (15) and (14) gives:

$$zp\star \overset{z}{\Rightarrow} (x^o a'a')\star \overset{(15)}{\implies} z(a'a')x \overset{(14)}{\implies} z^*\tilde{b}x^* \tag{8}$$

Finally, by definition of the partial order on $\tilde{A}$,$a' \subseteq b \implies \tilde{a'} \subseteq \tilde{b}$. Therefore, $x^*\tilde{a'}\psi \implies x^*\tilde{b}\psi$. Now, using (16), $z^*\tilde{b}x^*$ and $x^*\tilde{b}\psi$ gives the desired result $z^*\tilde{b}x^*$. □

Corollary 4.5. *For $\psi \in PX$, $\psi \in \overline{y_X(X)}$ if and only if ψ is a right-adjoint.*

Proof. Fix any $\psi \in PX$.

(i) ($\implies$)

(ii) ($\impliedby$) Suppose ψ is a right adjoint. Need to show that for any $a \in A$, $\exists x^* \in y_X(X)$ such that $\psi\,\tilde{a}\,x^*\tilde{a}\,\psi$. Fix $a \in A$. Because ψ is a right-adjoint, there exists a promodule $\phi : 1 \relbar\joinrel\mapsto X$ such that $\phi.\psi \leq A$ and $1 \leq \psi.\phi$. From $\phi.\psi \leq A$, we get that

$$\exists p \in \phi, q \in \psi \text{ such that } a \supseteq p.q \tag{1}$$

Because ϕ and ψ are promodules,

$$A.\phi \leq \phi \text{ gives the existence of } p' \in \phi \text{ such that } p \supseteq a'p' \tag{2}$$

$$A.\psi \leq \psi \text{ gives the existence of } q' \in \psi \text{ and } a'' \in A \text{ such that } q \supseteq a''q' \tag{3}$$

Now, from $1 \leq \psi.\phi$, we get that $q'\,p'$ is reflexive i.e. $\star(q'\,p')\star$. By the definition of composition we get the existence of an $x \in X$ such that $\star\, p'\, x\, q'\, \star$. Now, considering x as a map, $x : 1 \to X$ defined as $\star \mapsto x$,

$$x\, q'\, \star \;\text{ i.e. } \star \in q'(x) \text{ gives that } q' \supseteq x^o \tag{4}$$

$$\star\, p'\, x \text{ i.e. } x \in p'(\star) \text{ gives that } p' \supseteq x \tag{5}$$

Thus, by using inequalities (1),(2) and (3), we get that

$$a \supseteq p\, q \supseteq a'\, p'\, q'\, a'' \tag{6}$$

By definition of $\tilde{a}$, to show $\psi\, \tilde{a}\, x^*$, we need that $\psi \leq x^*\, a = x^o.\, A.\, a$. Showing for any $b \in A$, $x^o\, b\, a \supseteq q'$:

$$x^o\, b\, a \supseteq x^o\, b\, a'\, p'\, q' \supseteq x^o\, b\, a'\, x\, q' \supseteq x^o\, x\, q' \subseteq q'$$

Where the first inequality comes from (6) by using reflexiviness of a'' and then left-multiplying by x^o. The second inequality comes from (5), third one from reflexiviness of b and a', and the last one is given by Lemma 2.4.1.

In order to show $x^*\, \tilde{a}\, \psi$, by definition of $\tilde{a}$, need that $x^o.A = x^* \leq \psi\, a$. Fix $k \in \psi$. We will show $k\, a \supseteq x^o\, a''$.

$$a \supseteq a'\, p'\, q'\, a'' \supseteq p'\, q'\, a'' \supseteq p'\, x^o\, a'' \tag{7}$$

Where the first inequality is given by (6), second one is due to reflexiviness of a' and the third inequality comes by using (4). Left-multiplying (7) with k gives

$$ka \supseteq k\, p'\, x^o\, a'' \tag{8}$$

FINAL STEP LEFT !! $\square$