Question 1

$< a_n >$ is a real sequence; $\sigma_n := \dfrac{a_1 + a_2 + ... + a_n}{n}$

Going to show that

I $\liminf a_n \leq \liminf \sigma_n$

If a_n is unbounded below, then $\liminf a_n = -\infty \leq \liminf \sigma_n$ So, let a_n be bounded below, thus LHS is a real number,

$$m := \liminf a_n$$

Now, if a_n is constant, and equal to a,

$$\forall n \in \mathbb{N} \text{ , } \inf\{a_i | i \geq n\} = a = \frac{na}{n} = \sigma_n$$

Otherwise, if a_n is not constant, then,

$$\exists i, j \in \mathbb{N} \text{ such that(wlog) } a_i < a_j$$

Suppose if possible, lim inf a_n=m>lim inf σ_n But,

$$\forall n \in \mathbb{N} \text{ such that } n \geq i, j,$$

$$\begin{aligned} \sigma_n = \frac{a_1 + ...a_i + a_j + ... + a_n}{n} &\geq \frac{(n-2)m + a_i + a_j}{n} \\ &\geq \frac{(n-1)m + a_j}{n} \\ &> m \end{aligned}$$

roac

II $\limsup a_n \geq \limsup \sigma_n$