Quiz-4

$D \subseteq \mathbb{R}$ such that $\exists a_D \in \mathbb{R}^+ : a_D > 0$ and $\forall d \in D, d \geq a_D$

$$f_n(x) := \frac{nx}{1+n^2x^2}$$

Need to show that this sequence converges uniformly on D, but not on $[0, \infty)$

Proof:

$$f_n(x) := \frac{nx}{1+n^2x^2} = \frac{\frac{1}{nx}}{\frac{1}{nx}+nx} < \frac{1}{nx}$$

$$[\because n, x > 0 \implies \frac{1}{nx} + nx > 1]$$

Showing uniform convergence to 0 on D

Fix any set D as stipulated, with $a := a_D$;
Also, let $x \in D \implies x \geq a_D > 0 \implies \frac{1}{x} \leq \frac{1}{a_D}$
And let $\epsilon = \frac{1}{n}$ where n $\in \mathbb{N}$. For this ϵ, define $\delta_\epsilon := \frac{2}{a\epsilon} = \frac{2n}{a}$.
So that,

$$k > \delta_\epsilon \implies |f_k(x) - 0| < \frac{1}{kx} \leq \frac{1}{ka} < \frac{1}{\delta_\epsilon a}$$

$$\text{Thus, } |f_k(x)| < \frac{a\epsilon}{2} . \frac{1}{a} = \epsilon/2 < \epsilon$$

Showing non-uniform pointwise convergence to 0 on $[0, \infty)$

let $x \in [0, \infty)$, and $\epsilon = \frac{1}{n}$

$$\text{if } x = 0, \text{ then } f_n(x) = f_n(0) = 0 < \frac{1}{n} = \epsilon$$

$$\text{else, if } x \in (0, \infty) \text{ ,then define } \delta_\epsilon = \frac{2n}{x}$$

$$k > \delta_\epsilon \implies |f_k(x) - 0| < \frac{1}{kx} < \frac{1}{\delta_\epsilon x} = \frac{x}{2nx} = \epsilon/2 < \epsilon$$

So, if the sequence of $f_n(x)$ uniformly converges on $[0, \infty)$, then it does so to 0. Suppose, if at all possible, that $f_n(x) \rightrightarrows 0$. So, for $\epsilon = 0.5$, $\exists \delta > 0$ such that

$$\forall x \in [0, \infty), n > \delta \implies |f_n(x)| < 0.5$$

But, taking $x = x_\delta := \frac{1}{2(\delta+1)}$ gives

$$|f_{\delta+1(x_\delta)}| = \frac{(\delta+1)x}{1 + (\delta+1)^2 x^2} > \frac{1}{1 + (\delta+1)^2 x^2} = \frac{1}{1 + (1/4)}$$

Thus, for any δ, x_δ is such that

$$|f_{\delta+1(x_\delta)}| > \frac{1}{1 + (1/4)} = 0.8 > 0.5 = \epsilon$$

Hence, the sequence of funtions can't uniformly converge on $[0, \infty)$.