1 Yoneda Lemma

Lemma 1.1 (H_A or $\mathcal{A}(\_, A)$)**.** For any category $\mathcal{A}$, fixing an object, $A \in \mathcal{A}$, there's a functor, $H_A : \mathcal{A}^{op} \to Set$ defined as:

i For object $B \in \mathcal{A}$, $F(B) := Hom(B, A)$

ii For any morphism in $\mathcal{A}$, $g : X \to Y$,

$$H_A(g) : \mathcal{A}(Y, A) \to \mathcal{A}(X, A) \text{ , as, } \forall p \in \mathcal{A}(Y, A) \text{ , } p \mapsto p \circ g \text{ i.e. } \Big(H_A(g)\Big)(p) := p \circ g$$

Proof. Prove that H_A is indeed a functor

□

Lemma 1.2. For a natural transformation α, it's a natural isomorphism iff each of its components is an isomorphism.

Lemma 1.3. naturality in two variables simultaneously is equivalent to naturalityin each variable independently(1.3.29 pg 39)

Theorem 1.1. ***Yoneda*** *If $\mathcal{A}$ is a locally small category, for any object $A \in \mathcal{A}$ and $X \in [\mathcal{A}^{op}, Set]$, there's exists a natural isomorphism:*

$$[\mathcal{A}^{op}, Set](H_A, X) \cong X(A) \textit{ naturally in } A \in \mathcal{A}$$

Explaination:

First , fix any category, $\mathcal{A}$. Now ,choose two things (independent of each other):

i an object, A from the category $\mathcal{A} = \mathcal{A}^{op}$

ii an object, $X \in [\mathcal{A}^{op}, Set]$, the presheaf category
i.e. a functor $X : A^{op} \to Set$

Here, $[\mathcal{A}^{op}, Set](H_A, X)$ denotes morphisms $H_A \to X$ in $[\mathcal{A}^{op}, Set]$, i.e. natural transformations, α : $\begin{tikzcd} \mathcal{A}^{op} \arrow[r, bend left, "H_A"] \arrow[r, bend right, "X"'] \arrow[r, phantom, "\Downarrow\alpha"] & Set \end{tikzcd}$

Each of these natural transformations is a collection of, morphisms in Set, hence each of their components is exactly a function. i.e. $\forall \alpha \in [\mathcal{A}^{op}, Set](H_A, X), \forall K \in \mathcal{A}, \alpha_K$ is a function :$H_A(K) \to X(K)$

$X(A)$ is precisely a set, because $X(A)$ is the image of (our chosen object,) A, under (our chosen functor,) X.

The key idea is that the choice of A and X completely determines all possible maps (i.e. natural transformations) from functor H_A to functor$X$.
Moreover, that this isomorphism is *natural* in A and X.
Meaning that $[\mathcal{A}^{op}, Set](H_A, X)$ and $X(A)$ are *functorial* in *both* A and X

Notation:

- Denoting the category of presheaves on $\mathcal{A}$ by $\mathcal{C}$, i.e. $\mathcal{C} := [\mathcal{A}^{op}, Set]$
- using ˆ as a map i.e. $\hat{a} = b$ stands for $a \xrightarrow{\hat{}} b$
- using ˜ as a map i.e. $\hat{a} = b$ stands for $a \xrightarrow{\tilde{}} b$

To prove the theorem, first, going to show that $[\mathcal{A}^{op}, Set](H_A, X)$ is isomorphic to $X(A)$. And then that this isomorphism is natural.

Proof. Let a locally small category, $\mathcal{A}$ be given.
Now, fix any object $A \in \mathcal{A}$ and a presheaf on $\mathcal{A}$, $X \in \mathcal{C}$

Showing isomorphism between $[\mathcal{A}^{op}, Set](H_A, X)$ and $X(A)$

Define $\hat{}: \mathcal{C}(H_A, X) \to X(A)$ as the input's A-component, evaluated at the identity of A(in $\mathcal{A}$). i.e.

$$\text{for natural transformation } \alpha : H_A \to X \text{ , define } \hat{\alpha} := \alpha_A(1_A)\text{, an element of X(A)}$$

Define $\tilde{}: X(A) \to [\mathcal{A}^{op}, Set](H_A, X)$ on element, $x \in X(A)$, by defining it's K-component for any $K \in \mathcal{A}$ as

$$\tilde{x}_K : H_A(K) \to X(K) \text{ as, for each } p \in H_A(K) = Hom_{\mathcal{A}^{op}}(A, K) \text{ , } \tilde{x}_K(p) := \Big(X(p)\Big)(x)$$

Meaning that the $\tilde{x}_K$ maps any arrow $p : K \to A$ to the image of x under the function $X(p) : X(A) \to X(K)$.

Now, to show that $\tilde{x} = (\tilde{x}_K)_{K \in \mathcal{A}}$ is a natural transformation,

for any $q \in \mathcal{A}^{op}(K, L)$, the square
$$\begin{tikzcd} H_A(K) \arrow[r, "H_A(q)"] \arrow[d, "\tilde{x}_K"'] & H_A(L) \arrow[d, "\tilde{x}_L"] \\ X(K) \arrow[r, "X(q)"'] & X(L) \end{tikzcd}$$
i.e.
$$\begin{tikzcd} \mathcal{A}(K, A) \arrow[r, "- \circ q"] \arrow[d, "\tilde{x}_K"'] & \mathcal{A}(L, A) \arrow[d, "\tilde{x}_L"] \\ X(K) \arrow[r, "X(q)"'] & X(L) \end{tikzcd}$$
must commute .

So, for any $f : K \to A$, need that $\tilde{x}_L(f \circ q) = X(q) \circ \tilde{x}_K(f)$

Now, LHS=$\tilde{x}_L(f \circ q) = \Big(X(f \circ q)\Big)(x)$ while RHS=$X(q) \circ \tilde{x}_K(f) = \Big(X(q)\Big)\big(X(f)(x)\big) = \Big(X(q) \circ X(f)\Big)(x)$
And as X is a contravariant functor, $X(f \circ q) = X(q) \circ X(f)$, giving that LHS=RHS.
Finally, to show isomorphism, need to show that $\hat{}$ and $\tilde{}$ are mutually inverse,

$$\text{for any } x \in X(A), \hat{\tilde{x}} = \tilde{x}_A(1_A) = \Big(X(1_A)\Big)(x) = 1_{X(A)}(x) = x$$

And, for any $\alpha \in [\mathcal{A}^{op}, Set](H_A, X)$, $\tilde{\hat{\alpha}} = \alpha$ i.e. that each of their components are equal. As both $\tilde{\hat{\alpha}}$ and α are natural transformations between functors that go to the category *Set*, each of the components is a function.

So, need to show that for any $f \in \mathcal{A}(K, A) = H_A(K)$, $\Big(\tilde{\hat{\alpha}}\Big)_K(f)$=$\alpha_K(f)$

LHS=$\tilde{\hat{\alpha}}_B(f) = \Big(X(f)\Big)(\hat{\alpha}) = \Big(X(f)\Big)(\alpha_A(1_A))$ and RHS=$\alpha_K(f) = \alpha_K(1_A \circ f)$

Now, as α is a natural transformation, the square
$$\begin{tikzcd} \mathcal{A}(A, A) \arrow[r, "- \circ f"] \arrow[d, "\alpha_A"'] & \mathcal{A}(K, A) \arrow[d, "\alpha_K"] \\ X(A) \arrow[r, "X(f)"'] & X(K) \end{tikzcd}$$
commutes, for 1_A, giving that

$\alpha_K(1_A \circ f) = \Big(X(f)\Big)\big(\alpha_A(1_A)\big)$ thus, RHS=LHS, and the isomorphism is shown.

Showing naturality of this isomorphism

□

2 Cayley's Theorem

Definiton 2.1 (Symmetric group on a set)**.**

Proof. □

Theorem 2.1. ***Cayley's Theorem*** *Every group, (G,.) is isomorphic to a subgroup of symmetric group on G.*

3 Embedding of a category in Presheaf category

Definiton 3.1 (Embedding of a category)**.** A category, $\mathcal{A}$ is said to be embedded in a category, $\mathcal{B}$ if there exists a functor $F : \mathcal{A} \to \mathcal{B}$ such that F is full, faithful and injective (on objects).