1 Definitions

Definiton 1.1 (Prorelation)**.** A partial-ordered set of relations $X \to Y$, which is down-directed and an upper set. i.e A set, $P \subseteq \mathcal{P}(X \times Y)$ such that

(i) A partial-order defined to be containment as relations, $r \subseteq s$ only if $\forall (x, y) \in X \times Y$, $(x, y) \in r \implies (x, y) \in s$

(ii) (Down-directed), $\forall r, s \in P, \exists t \in P$ such that $t \subseteq r$ and $t \subseteq s$

(iii) (Up-set) for any relation $u : X \to Y$, if $\exists p \in P$ such that $p \leq u$ then $u \in P$

Definiton 1.2 (Composition of prorelations)**.** Prorelations can be composed by taking all compositions of their elements as relations: for prorelations $P : X \to Y$ and $Q : Y \to Z$,

$$Q.P := \{q \circ p : p \in P \text{ and } q \in Q\}$$

Definiton 1.3 (Comparison of Prorelations)**.** Two prorelations with same domain, co-domain are comparable as

$$\text{for } P, Q : X \to Y \ , \ P \leq Q \text{ if } \forall q \in Q, \exists p \in P \text{ such that } p \subseteq q$$

Definiton 1.4 (Quasi-uniformity)**.** A prorelation on a set X, $P : X \to X$ is a quasi-uniformity if it follows :

i $\forall p \in P$, for any $x \in X$, $(x, x) \in p$ i.e. xpx

ii $\forall p \in P, \exists p' \in P$ such that $p' \circ p' \subseteq p$

And in this case, (X, A) is called a *quasi-uniform space*.

Definiton 1.5 (Uniformly Continuous function)**.** A function, $f : X \to Y$ is called a uniformly continuous function, $f : (X, A) \to (Y, B)$ if, $\forall b \in B, \exists a \in A$ such that $f \circ a \subseteq b \circ f$. meaning that $f.A \leq B.f$ or

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "A"'] \arrow[dr, phantom, "\leq"] & Y \arrow[d, "B"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}.$$

Definiton 1.6 (Promodule)**.** A prorelation, $\phi : X \mathrel{\circ\!\!\!\rightarrow} Y$ is called a promodule $\phi : (X, A) \mathrel{\circ\!\!\!\rightarrow} (Y, B)$ if it obeys: $\phi.A \leq \phi$ and $B.\phi \leq \phi$ where . denotes composition as prorelations.

Definiton 1.7 (Comparison of Promodules)**.** Promodules with same domain and co-domain are compared as prorelations, for $\phi, \psi : (X, A) \mathrel{\circ\!\!\!\rightarrow} (Y, B)$, $\phi \sqsubseteq \psi$, only if $\phi \leq \psi$.

Definiton 1.8 (Composition of Promodules)**.** Promodules are composed as prorelations.
For promodules $\phi : (X, A) \mathrel{\circ\!\!\!\rightarrow} (Y, B)$ and $\psi : (Y, B) \mathrel{\circ\!\!\!\rightarrow} (Z, C)$, $\psi\phi := \psi.\phi = \{q \circ p : p \in \phi \text{ and } q \in \psi\}$

Definiton 1.9 (Opposite relation)**.** For relation $r : X \to Y$, r^o is defined to be a relation $r^o : Y \to X$ as

$$\forall (x, y) \in X \times Y, (x, y) \in r \iff (y, x) \in r^o$$

Lemma 1.9.1. For any relation $r : X \to Y$, $r^o \circ r = \Delta_X$

Lemma 1.9.2. For any relation $r : X \to Y$, $r \circ r^o \subseteq \Delta_Y$

Definiton 1.10 ($(-)_*$)**.**

Definiton 1.11 ($(-)^*$)**.**

Definiton 1.12 (Fully Faithful)**.**

Definiton 1.13 (Fully Dense)**.**

Definiton 1.14 (Topologically Dense)**.**

2 Propositions

Definiton 2.1 (QUnif)**.** QUnif is defined to be the category having quasi-uniform spaces as objects, and uniformly continous maps between them as morphisms.

Lemma 2.1.1. QUnif does define a category, as

i Composition

ii Identity

Definiton 2.2 (ProMod)**.**

Lemma 2.2.1. ProMod does define a category, as

i Composition

ii Identity

Proposition 2.1 ($(-)_*$:QUnif$\rightarrow$ $ProMod$ is a Functor)**.**

Proof. $\square$

Proposition 2.2 ($(-)^*$:QUnif^{op} $\rightarrow$ $ProMod$ is a Functor)**.** Defined as fixing objects and taking morphisms to their image under $(-)^*$

(a) for $(X, A) \in$ QUnif^{op}, $(X, A)^* := (X, A) \in$ ProMod

(b) for $f : (X, A) \to (Y, B)$ in QUnif, $f^* := f^o.B$

Proof.

Showing that $f^o.B : (Y, B) \multimap (X, A)$ **is a promodule**
So, need to show $f^o.B$ a prorelation $Y \to X$ and that $(f^o.B).B \sqsubseteq f^o.B$ and $A.(f^o.B) \sqsubseteq f^o.B$
To show prorelation,

(i) (Partial-order) Inclusion of relations i.e. for $k = f^o \circ b$ and $k' = f^o \circ b'$ in $f^o.B$, $k \subseteq k' \iff b \subseteq b'$

(ii) (Down directed) for $k, k' \in f^o.B$, need that $\exists l \in f^o.B$ such that $l \subseteq k, k'$
Fix $k, k' \in f^o.B \implies \exists b, b' \in B : k = f^o \circ b$ and $k' = f^o \circ b'$
And as B is a quasi-uniformity, it's down directed so, $\exists c \in B : c \subseteq b, b' \implies l := f^o \circ c \subseteq k, k'$

(iii) (Up-set) for a relation $l : Y \to X$ and $k \in f^o.B$ such that $l \supseteq k$, need $l \in f^o.B$
Let $b \in B$ be such that $k = f^o \circ b$ and define $b' := \{(y, y') : y \in domain(l)$ and $y' \in (f^o)^{-1}(l(y))$
As $l \supseteq k = f^o \circ b$, $domain(b') = domain(l) \supseteq domain(b)$
and $range(l) \supseteq range(f^o \circ b) \implies \forall y \in domain(b), range(b') = (f^o)^{-1}(l(y)) \supseteq (f^o)^{-1}(f^o \circ b) = range(b)$
Now, by definition of b', $f^o \circ b' \supseteq l$. To show $f^o \circ b \subseteq l$,
$(x, y) \in f^o \circ b' \implies \exists z \in Y : (x, z) \in b'$ and $(z, y) \in f^o \implies x \in domain(l)$ and $z \in l(x)$ i.e. $(x, z) \in l$

To show $(f^o.B).B \leq f^o.B$, need that $\forall b \in B, \exists b' \in B : f^o \circ b' \circ b' \subseteq f^o \circ b$,
Fix any $b \in B$ as B is a quasi-uniformity, $\exists b' \in B : b' \circ b' \subseteq b \implies f^o \circ b' \circ b' \subseteq f^o \circ b$
To show $A.(f^o.B) \leq f^o.B$, need that $\forall b \in B$, $\exists b' \in B, a \in A : a \circ f^o \circ b' \subseteq f^o \circ b$,
As f is uniformly continuous, $f.A \leq B.f$ i.e. $\forall b \in B, \exists a \in A : f \circ a \subseteq b \circ f \implies a = f^o \circ f \circ a \subseteq f^o \circ b \circ f$
Fix any $b \in B$, so, $\exists b' \in B : b'b' \subseteq b$ (for brevity, omitting $\circ$ to explicitly denote composition)
And, for this $b', \exists a : a \subseteq f^o b' f \implies a f^o b' \subseteq f^o b' f f^o b' \subseteq f^o b' b' \subseteq f^o b \implies a f^o b' \subseteq f^o b$

Now, need to show that $(-)^*$ respects composition and identity.

(i) (Composition) let f, g be uniformly continuous, $(X, A) \xrightarrow{f} (Y, B) \xrightarrow{g} (Z, C)$ need that $(g \circ f)^* = f^*.g^*$
LHS=$(g \circ f)^* = (g \circ f)^o.C = (f^o \circ g^o).C$ and RHS=$f^*.g^* = (f^o.B).(g^o.C)$
For equality, showing that LHS$\geq$RHS and LHS$\leq$RHS:
To show $(f^o \circ g^o).C \geq (f^o.B).(g^o.C)$, need that $\forall c \in C, \exists b \in B, c' \in C : f^o g^o c \supseteq f^o b g c'$
Fix any $c \in C$, so, $\exists c' \in C : c' \circ c' \subseteq c \implies f^o g^o c \supseteq f^o g^o (c'c') = f^o g^o (c' \Delta_Z c') \supseteq f^o g^o c' (g g^o) c'$
By uniform conntinuity of g, for $c' \in C, \exists b \in B : gb \subseteq c'g$
Thus, $f^o g^o c \supseteq f^o g^o (c'g) g^o c' \supseteq f^o (g^o g) b g^o c' = f^o b g^o c'$.
To show $(f^o \circ g^o).C \leq (f^o.B).(g^o.C)$, need that $\forall b \in B, c \in C, \exists c' \in C : f^o g^o c \subseteq f^o b g^o c$
Fix any $c \in C, b \in B$ will show that $c' := c$ works:
As B is a quasi-uniformity, $\Delta_Y \subseteq b \implies f^o \Delta_Y g^o c = f^o g^o c \subseteq f^o b g^o c = f^o b g^o c'$

(ii) (Identity) let $(X, A) \in$ QUnif ^{op}, and $1_{(X,A)} : (X, A) \to (X, A)$ as $x \mapsto x$ need that $(1_{(X,A)})^* = 1_{(X,A)^*}$
LHS=$(1_{(X,A)})^* = (1_{(X,A)})^o.A = 1_{(X,A)}.A = A$.
Now, it's required that A is the identity of (X, A) in ProMod.
So, fix $\phi : (X, A) \multimap (Y, B)$, need to show $\phi.A = \phi$
As ϕ is a promodule, $\phi.A \leq \phi$ and as A is quasi-uniformity on X,
$\forall a \in A, \Delta_X \subseteq a \implies \forall a \in A, \forall p \in \phi, p = p\Delta_X \subseteq pa \implies \phi \leq \phi.A$
Also, fix $\psi : (Y, B) \multimap (X, A)$, need to show $A.\psi = \psi$
As ψ is a promodule, $A.\psi \leq \psi$ and as A is quasi-uniformity on X,
$\forall a \in A, \Delta_X \subseteq a \implies \forall a \in A, \forall q \in \psi, q = \Delta_X q \subseteq aq \implies \psi \leq \psi.A$

□

Proposition 2.3 (Proposition 1)**.** Fix a uniformly continuous map, $f : (X, A) \to (Y, B)$

(a) f is fully faithful $\iff A = f^o.B.f$

(b) f is fully dense $\iff \forall b \in B, \exists b' \in B$ such that

(c) f is topologically dense $\iff \forall b \in B, \Delta_Y \subseteq b \circ f \circ f^o \circ b$

(d) f is fully dense $\iff$ f is topologically dense

Proof.

(a) (i) ($\implies$) Let f be fully faithful i.e. $f^*.f_* = A \implies f^o.B.B.f = A$
Need to show that $A = f^o.B.f$ i.e. $A \leq f^o.B.f$ and $A \geq f^o.B.f$
By hypothesis and quasi-uniformity of B, $A \geq f^o.B.B.f \geq f^oB.f$
To show $A \leq f^o.B.f$, need that $\forall b \in B, \exists a \in A : a \subseteq f^obf$
Fix $b \in B$, hypothesis gives that $f^o.B.B.f \leq A$ so,
$\exists a \in A : a \subseteq f^obbf$ and also, by quasi-uniformity of B, for $b, \exists b' \in B : b'b' \subseteq b \implies f^ob'b'f \subseteq f^obf$
Combining the above two inequalities, $a \subseteq f^obbf \subseteq f^obf$

(ii) ($\impliedby$) Let $A = f^o.B.f$ need to show $A = f^o.B.B.f$ i.e. $A \geq f^oB.B.f$ and $A \leq f^oB.B.f$
To show $A \geq f^o.B.B.f$, need to show that $\forall a \in A, \exists b, b' \in B : a \supseteq f^obb'f$
Have that $A \geq f^o.B.f$ and $B.B \leq B$
So, fix $a \in A$, now $\exists b \in B : a \subseteq f^obf$ and for this b, $\exists b' \in B : b'b' \subseteq b$. Therefore, $a \supseteq f^obf \supseteq f^ob'b'f$
To show $A \leq f^o.B.B.f$, need $\forall b, b' \in B, \exists a \in A : a \subseteq f^obb'f$
So, fix $b, b' \in B$, now, by hypothesis, $A \leq f^o.B.f$ giving
$\exists a \in A : a \subseteq f^obf$ and $\exists a' \in A : a' \subseteq f^ob'f \implies \Delta_X \subseteq f^ob'f$.
Therefore $a = a\Delta_X \subseteq (f^obf)(f^ob'f) \subseteq f^obb'f$

(b) (i)

□

Definiton 2.3 (PX)**.** $PX := \{\psi : \psi : (X, A) \nrightarrow 1 \text{ is a promodule}\}$

Definiton 2.4 ($\tilde{a}$)**.** for any $a \in A$, $\tilde{a}$ is defined to be a relation $PX \to PX$ as

$$\text{for } \phi, \psi \in PX, \phi\tilde{a}\psi \text{ only if } \phi \le \psi.a$$

Proposition 2.4 (Prorelation $\tilde{A}$)**.** The set, $\tilde{A} := \{\tilde{a} : a \in A\}$ defines a quasi-uniformity on PX.

Proof. First, need to show that $\tilde{A}$ is a prorelation,

(i) (Partial order) Define, for any two relations $\tilde{a}, \tilde{b} : PX \to PX$, that $\tilde{a} \le \tilde{b}$ only if $a \subseteq b$

(ii) (Down-Directed) Need that $\forall \tilde{a}, \tilde{b} \in \tilde{A}, \exists \tilde{c} \in A : c \subseteq a, b$
$\tilde{a}, \tilde{b} \in A \implies a, b \in A \implies \exists c \in A : c \subseteq a, b \implies \tilde{c} \le \tilde{a}, \tilde{b}$

(iii) (Upset) Need that, for any relation $l : PX \to PX$, if $\exists \tilde{k} \in \tilde{A}$ such that $l \ge \tilde{k}$, then $l \in \tilde{A}$
Fix any $k : PX \to PX$, and $\tilde{k} \in \tilde{A}$ such that $l \ge \tilde{k}$
Now, k is a relation between promodules $X \nrightarrow 1$. Thus, it can be thought of as a relation on X,
$a := \{(x, y) : x \in domain(\psi) andy \in domain(\phi)$ whenever $\exists \psi, \phi \in PX : \psi l \phi\}$
So, $l = \tilde{a}$ and thus, $\tilde{a} \ge \tilde{k} \implies a \supseteq k \implies a \in A \implies l \in \tilde{A}$

Now to show that the other two conditions hold,

(i) need that $\forall \tilde{a} \in \tilde{A}, \forall \psi \in PX, \psi\tilde{a}\psi$
So, need to show that $\psi \le \psi.a$ i.e. $\forall p \in \psi, \exists q \in \psi : q \subseteq p.a$. Take $q := p$, and as A is a quasi-uniformity, $\Delta_X \subseteq a \implies p = p.\Delta_X \subseteq p.a$

(ii) Need that $\forall \tilde{a} \in \tilde{A}, \exists \tilde{b} \in \tilde{A} : \tilde{b}\tilde{b} \le \tilde{a}$
Before that, showing , for any $x, y \in A, \tilde{x}\tilde{y} \le \widetilde{xy}$ i.e. $\forall \psi, \phi \in PX$, $\psi(\tilde{x}.\tilde{y})\phi \implies \psi\widetilde{xy}\phi$
Let $\psi_1(\tilde{a}.\tilde{b})\psi_3 \implies \exists \psi_2 : \psi_1\tilde{b}\psi_2\tilde{a}\psi_3 \implies \psi_1 \le \psi_2.b$ and $\psi_2 \le \psi_3.a \implies \psi_1 \le \psi_2.b \le \psi_3.ab \implies \psi_1(\widetilde{ab})\psi_3$
Fix any $\tilde{a} \in \tilde{A} \implies a \in A \implies \exists b \in A : b \circ b \subseteq a \implies \widetilde{bb} \le \tilde{a} \implies \tilde{b}\tilde{b} \le \widetilde{bb} \le \tilde{a}$

□

Proposition 2.5 (Yoneda Embedding)**.**

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