A note on Free groups(borrowed from Hungerford Ch1)

Definiton 0.1 (Concrete category)**.** A category $\mathcal{A}$, along with a faithful functor to *Set*

Definiton 0.2 (Free object)**.** Let A be an object of concrete category $\mathcal{A}$, $X \neq \phi$ a set, and a map $i : X \to A$. Then, A is said to be free on the set X iff

$$B \in \mathcal{B} \text{ and } f : X \to B \implies \begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & A \arrow[d, "g"] \\ & B \end{tikzcd} \text{ commutes i.e. } \exists! g \in \mathcal{A}(A, B) \text{ such that } g \circ i = f$$

Theorem 0.1 (Free objects depend only on the cardinality of the set they're free on)**.** If F and F' are objects of a concrete category $\mathcal{A}$ such that they're free on X and X' respectively. Then

$$|X| = |X'| \implies F \cong F'$$

Proof. As $|X| = |X'|$, there's a bijection $b : X \longleftrightarrow X'$. And as F, F' are free on X,X' ,
there are maps $i : X \to F$ and $i' : X \to F'$. Also, as the maps obey composition,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

$\square$

Corollary 0.1.1. Two objects of a category are free on the same set only if they're isomorphic.