CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome 33, nº 1 (1992), p. 71-93
<http://www.numdam.org/item?id=CTGDC_1992__33_1_71_0>

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ET GÉOMÉTRIE DIFFÉRENTIELLE
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VOL. XXXIII-1 (1992)

EPIREFLECTIONS WHICH ARE COMPLETIONS

by G.C.L. BRUMMER, E. GIULI and H. HERRLICH

We dedicate this paper to the memory of Siegfried Grässer

Résumé. Nous axiomatisons la situation où tout objet d'une catégorie $\mathbf{X}$ a un complété et où tout plongement dense dans un objet complet quelconque est une réflexion dans la sous-catégorie pleine des objets complets. On dit alors que $\mathbf{X}$ admet une sous-catégorie $\mathcal{S}$-fermement $\mathcal{E}$-réflexive. Ici, $\mathcal{S}$ est une classe de morphismes de $\mathbf{X}$ ayant des propriétés analogues aux plongements, et la classe $\mathcal{E}$ représente la densité appropriée. Pour le cas $\mathcal{E} = \mathrm{Epi}\mathbf{X}$ nous relions cette notion avec celles de fermeture $\mathcal{S}$-absolue, de $\mathcal{S}$-saturation, et de $(\mathcal{E} \cap \mathcal{S})$-injectivité; nous en donnons plusieurs caractérisations, en particulier la préservation des $\mathcal{S}$-morphismes; et nous considérons beaucoup d'exemples topologiques et algébriques. Quand $\mathbf{X}$ est une catégorie topologique on a un contexte naturel pour lequel $\mathcal{E}$ est plus large que la classe des épomorphismes.

0. Introduction

Among the various kinds of extensions that an object can have, compactifications and completions of spaces exhibit two very different forms of behaviour. A Tychonoff space, say, can have many mutually inequivalent Hausdorff compactifications, among which there is the

AMS Subject Classification: Primary 18A40, 18A32; secondary 18B30, 54B30, 54E15, 54E52.
Key words: Firm epireflection, completion, weak injectivity, saturated object, absolutely closed object, complete object, factorization structure, topological category.
The first author acknowledges support from the foundation for Research Development and the hospitality of the University of L'Aquila while working on this paper. The second author acknowledges a grant form the Italian Ministry of Public Education.

Čech-Stone compactification which is the reflection to compact spaces. On the other hand, a metric space, say, admits, up to isometry, just one completion, and that completion si the reflection to complete spaces.

The latter form of behaviour is paradigmatic for the kind of completion of objects that we study in this paper. When it occurs in a category $\mathbf{X}$, we say that the "complete" objects form an $\mathcal{S}$-firm $\mathcal{E}$-reflective subcategory of $\mathbf{X}$. Here $\mathcal{S}$ is a class of morphisms in $\mathbf{X}$ of which we like to think as embeddings, and $\mathcal{E}$ represents the kind of density that an object should have in its completion.

Our chosen setting then is an $(\mathcal{E}, \mathcal{M})$-category $\mathbf{X}$ with a designated class $\mathcal{S}$ of morphisms. Mild assumptions on the interplay between the $(\mathcal{E}, \mathcal{M})$-factorization and the class $\mathcal{S}$ determine the results. These become trivial when $\mathcal{S}$ coincides with $\mathcal{M}$.

For the sake of clarity, the main body of the paper is developed for the case that $(\mathcal{E}, \mathcal{M}) =$ (epi, extremal mono). The category $\mathbf{X}$ with the given $\mathcal{S}$ admits at most one $\mathcal{S}$-firmly epireflective subcategory $\mathbf{R}$. In this case, $\mathbf{R}$ consists of those $\mathbf{X}$-objects which are injective with respect to epimorphisms in $\mathcal{S}$, and $\mathbf{R}$ also coincides with the subcategory of $\mathcal{S}$-saturated objects; under an additional assumption, $\mathbf{R}$ is the subcategory of absolutely $\mathcal{S}$-closed objects. It is clear that $\mathbf{X}$ admits an $\mathcal{S}$-firm epireflection if and only if every $\mathbf{X}$-object admits an epic $\mathcal{S}$-morphism into an object which is $(\mathcal{S} \cap \mathrm{Epi}\mathbf{X})$-injective. We show (Theorem 1.6) that this is equivalent to $\mathbf{X}$ being $\mathcal{S}$-cogenerated by a class of $(\mathcal{S} \cap \mathrm{Epi}\mathbf{X})$-injectives. The main result (Theorem 2.5) is that a given $\mathcal{S}$-epireflector in $\mathcal{S}$-firm if and only if it preserves $\mathcal{S}$-morphisms (equivalently: $\mathcal{S}$-sources) and reflects into the class of $\mathcal{S}$-saturated objects. We give examples in topology and algebra. The applications to topological categories also provide a context for firm $\mathcal{E}$-reflections with $\mathcal{E}$ other than epi.

Injectives with respect to epimorphic embeddings were studied by P.D. Bacsich [2], with results partly of the same intent as the present paper. R.-E. Hoffmann [38] defined and investigated firm reflections in topological categories. The latter paper was the first, to our knowledge, which proved the firmness of the sobrification epireflector in the

T_0-topological spaces. It may be observed that the precategorical paper [10] by G. Birkhoff can still serve as a source of ideas connected with the notion of completeness.

1. Firm epireflections

Throughout the paper we consider a category $\mathbf{X}$ which is complete and well-powered and a fixed class $\mathcal{S}$ of morphisms of $\mathbf{X}$ satisfying the following properties:

($\mathbf{s}_1$) $\mathrm{Iso}\mathbf{X} \subset \mathcal{S} \subset \mathrm{Mono}\mathbf{X}$;
($\mathbf{s}_2$) $\mathcal{S}$ is closed under composition;
($\mathbf{s}_3$) If $me \in \mathcal{S}$ and $e \in \mathrm{Epi}\mathbf{X}$, $m \in Ex\mathrm{Mono}\mathbf{X}$, then $e \in \mathcal{S}$.

$\mathrm{Mor}\mathbf{X}$, $\mathrm{Epi}\mathbf{X}$, $\mathrm{Mono}\mathbf{X}$, $\mathrm{ExMono}\mathbf{X}$, $\mathrm{RegMono}\mathbf{X}$, $\mathrm{Iso}\mathbf{X}$ denote the class of all morphisms, epimorphisms, monomorphisms, extremal monomorphisms, regular monomorphisms and isomorphisms of $\mathbf{X}$, respectively. If $\mathbf{X}$ is concrete then $\mathrm{Emb}\mathbf{X}$ denotes the class of embeddings of $\mathbf{X}$ (i.e. initial maps whose underlying maps are mono); it satisfies conditions ($\mathbf{s}_1$)-($\mathbf{s}_3$).

Note that every $\mathcal{M}$ which is part of an $(\mathcal{E}, \mathcal{M})$-factorization structure of $\mathbf{X}$, with $\mathrm{Epi}\mathbf{X} \subset \mathcal{E}$, has the properties ($\mathbf{s}_1$)-($\mathbf{s}_3$) and in fact also satisfies the additional assumptions ($\mathbf{s}_4$) and ($\mathbf{s}_5$) which we impose later.

All subcategories will be taken as full and isomorphism-closed.

Since $\mathbf{X}$ is complete and well-powered, then (cf. [36, 34A]):
($\mathbf{x}_1$) $\mathbf{X}$ is an ($\mathrm{Epi}\mathbf{X}$, $\mathrm{ExMono}\ \mathbf{X}$)-category.

Definitions 1.1. *Let X be an $\mathbf{X}$-object.*

(1) *X is said to be $\mathcal{S}$-injective if, for each $e : Y \to Z$ in $\mathcal{S}$ and each $\mathbf{X}$-morphism f: $Y \to X$, there is an $\mathbf{X}$-morphism $g : Z \to X$ such that $ge = f$. Then g is called (an) extension of f (to $\mathbf{Z}$).*
(2) *X is said to be weakly $\mathcal{S}$-injective if it is ($Epi\mathbf{X} \cap \mathcal{S}$)-injective. $Inj(\mathcal{S})$ ($WInj(\mathcal{S})$) denotes the class of all (weakly) $\mathcal{S}$-injective objects of $\mathbf{X}$.*
(3) *X is said to be $\mathcal{S}$-saturated if an X-morphism $f : X \to Y$ is an isomorphism whenever $f \in \mathrm{Epi}\mathbf{X} \cap \mathcal{S}$. $\mathrm{Sat}(\mathcal{S})$ denotes the*

class of all $\mathcal{S}$-saturated objects of **X**.

(4) *X is said to be absolutely $\mathcal{S}$-closed if an* **X**-*morphism $f : X \to Y$ is a regular monomorphism whenever $f \in \mathcal{S}$. $AC(\mathcal{S})$ denotes the class of all absolutely $\mathcal{S}$-closed objects of* **X**.

We refer to [36] and [32] for other categorical terms not defined here.

Weakly $\mathcal{S}$-injective implies $\mathcal{S}$-saturated and absolutely $\mathcal{S}$-closed implies $\mathcal{S}$-saturated (see Proposition 1.3 below).

For the case **X** a category of algebras (in particular of rings or semigroups) and $\mathcal{S}$ the class of all monomorphisms, $\mathcal{S}$-saturated and absolutely $\mathcal{S}$-closed objects were introduced and investigated by Isbell [42] (see also [39]). It is shown in [42] that an $\mathcal{S}$-saturated algebra need not be absolutely $\mathcal{S}$-closed.

For the case **X** an epireflective subcategory of **Top** and $\mathcal{S} = \mathrm{Emb}\mathbf{X}$, $\mathcal{S}$-saturated and absolutely $\mathcal{S}$-closed objects were studied by Dikranjan and Giuli in [16] and [19]. It was observed in [16] that absolute $\mathcal{S}$-closedness coincides with **X**-closedness whenever the **X**-epimorphisms coincide with the dense continuous maps. The latter notion was introduced by Alexandroff and Urysohn [1] for the category of Hausdorff spaces (H-closedness) and was investigated by many authors (see e.g. [9] and [51])). In such a case also absolutely $\mathcal{S}$-closed = $\mathcal{S}$-saturated. It was also proved in [19] that, for **X** the category of Urysohn spaces (or, more generally for **X** the category of $S(n)$-spaces), an $\mathcal{S}$-saturated space need not be absolutely $\mathcal{S}$-closed and absolutely $\mathcal{S}$-closed need not be **X**-closed.

Weak $\mathcal{S}$-injectivity vanishes in some contexts in which the $\mathcal{S}$-injectivity is consistent (e.g. in abelian categories **X** with $\mathcal{S}$ =Mono**X**). In $\mathbf{Unif_o}$, with $\mathcal{S} = \{\text{Embeddings}\}$, $\mathcal{S}$-injectivity is a much stronger property than weak $\mathcal{S}$-injectivity [41]. The latter notion is equivalent to completeness. Analogously in $\mathbf{Top_o}$, with $\mathcal{S} = \{\text{Embeddings}\}$, $\mathcal{S}$-injective space means retract of a product of Sierpinski spaces [49] while weakly $\mathcal{S}$-injective space means sober space [38].

$\mathcal{S}$-injectivity with no conditions on **X** and on $\mathcal{S}$, is investigated by Herrlich [35] (see also [34]).

Let X be a topological space and let $\mathbf{X}$ be the epireflective hull of X in **Top**. Sobral [50] showed that the Eilenberg-Moore factorization of the functor $\mathrm{Hom}(-,X)$:$\mathbf{Top}^{\mathbf{op}} \to \mathbf{Set}$ can be obtained via the corresponding factorization of its restriction to the subcategory $AC(\mathrm{Emb}\mathbf{X})$, provided $\mathbf{X}$ satisfies an injectivity condition. This injectivity condition is weaker than the $(\mathrm{Emb}\mathbf{X})$-injectivity notion but it is not comparable with our weak $(\mathrm{Emb}\mathbf{X})$-injectivity.

If $\mathbf{R}$ is an epireflective subcategory of $\mathbf{X}$, $R : \mathbf{X} \to \mathbf{X}$ denotes the reflection functor, and, for each $\mathbf{X}$-object X, $r_X : X \to R(X)$ denotes the $\mathbf{R}$-reflection morphism. For each $\mathbf{X}$-morphism $f : X \to Y$, $Y \in \mathrm{Ob}\mathbf{R}$, $f^* : R(X) \to Y$ denotes the unique $\mathbf{X}$-morphism such that $f^* r_X = f$.

Definitions 1.2. *Let* $\mathbf{R}$ *be a reflective subcategory of* $\mathbf{X}$.

(1) $\mathbf{R}$ *is said to be* $\mathcal{S}$*-epireflective if for each* $\mathbf{X}$*-object* X, $r_X : X \to R(X)$ *belongs to* $\mathrm{Epi}\mathbf{X} \cap \mathcal{S}$.
(2) $\mathbf{R}$ *is said to be an* $\mathcal{S}$*-firm epireflective subcategory if it is* $\mathcal{S}$*-epireflective and, for each* $f : X \to Y$, *with* $Y \in \mathbf{R}$, f^* *is an isomorphism whenever* $f \in \mathrm{Epi}\mathbf{X} \cap \mathcal{S}$.

Proposition 1.3. *Let* $\mathbf{R}$ *be a subcategory of* $\mathbf{X}$ *and for an* $\mathbf{X}$*-object* X *consider the following conditions:*

(i) X *is* $\mathcal{S}$*-saturated;*
(ii) X *belongs to* $\mathbf{R}$*;*
(iii) X *is weakly* $\mathcal{S}$*-injective;*
(iv) X *is absolutely* $\mathcal{S}$*-closed.*

Then the following hold:

(a) *Always* (iii)$\Rightarrow$(i) *and* (iv)$\Rightarrow$(i)*;*
(b) (i)$\Rightarrow$(iv) *whenever* $\mathrm{ExMono}\mathbf{X} = \mathrm{RegMono}\mathbf{X}$*;*
(c) (i)$\Rightarrow$(ii) *whenever* $\mathbf{R}$ *is* $\mathcal{S}$*-epireflective;*
(d) (ii)$\Rightarrow$(iii) *whenever* $\mathbf{R}$ *is* $\mathcal{S}$*-firmly epireflective in* $\mathbf{X}$.

Proof.
(a) (iii)$\Rightarrow$(i): Since $\mathbf{X}$ is weakly $\mathcal{S}$-injective, then for each $(e : X \to$

$Z) \in \mathrm{Epi}\mathbf{X} \cap \mathcal{S}$, the identity 1_X has an extension g, so $ge = 1_X$, consequently $e \in \mathrm{Iso}\mathbf{X}$. (iv)$\Rightarrow$(i): Every $\mathbf{X}$-morphism which is both epi and regular mono is an isomorphism.
(b) If an $\mathbf{X}$-morphism $f : X \to Y$ belongs to $\mathcal{S}$ then, by property $(\mathbf{s}_3)$, in the (epi, extremal mono)-factorization $me = f$, $e \in \mathcal{S}$. If $X \in \mathrm{Sat}(\mathcal{S})$ then e is an isomorphism, so f is an extremal mono, hence, by assumption, it is a regular mono, consequently $X \in AC(\mathcal{S})$.
(c) If $\mathbf{R}$ is $\mathcal{S}$-epireflective then every $\mathbf{R}$-reflection $r_X : X \to R(X)$ belongs to $\mathrm{Epi}\mathbf{X} \cap \mathcal{S}$. If in addition $X \in \mathrm{Sat}(\mathcal{S})$, then r_X must be an isomorphism, consequently $X \in \mathbf{R}$.
(d) Whenever $e : Y \to Z$ belongs to $\mathrm{Epi}X \cap \mathcal{S}$, since $\mathbf{R}$ is $\mathcal{S}$-epireflective and since $\mathcal{S}$ has property $(\mathbf{s}_2)$, then $r_Z e : Y \to Z \to R(Z)$ belongs to $\mathrm{Epi}\mathbf{X} \cap \mathcal{S}$. Since $\mathbf{R}$ is supposed to be $\mathcal{S}$-firm, then $(r_Z e)^*$ in the commutative diagram below is an isomorphism.

$$\begin{tikzcd}
Y \arrow[r, "e"] \arrow[d, "f"'] \arrow[dr, "r_Y"] & Z \arrow[r, "r_Z"] & R(Z) \\
X & & R(Y) \arrow[ll, "f^*"'] \arrow[u, "(r_Z e)^*"']
\end{tikzcd}$$

Thus, for every $\mathbf{X}$-morphism $f : Y \to X$, the $\mathbf{X}$-morphism $g = f^*((r_Z e)^*)^{-1} r_Z$ is the needed extension of f.

We refer to [2, (Theorem 2.2)] for other conditions under which $\mathrm{Sat}(\mathcal{S})$, $AC(\mathcal{S})$ and $WInj(\mathcal{S})$ coincide.

Corollary 1.4. $\mathbf{X}$ *admits at most one $\mathcal{S}$-firm epireflective subcategory* $\mathbf{R}$. *In such a case* $\mathbf{R} = \mathrm{Sat}(\mathcal{S}) = WInj(\mathcal{S})$.

Question A. *Prove or disprove that, for each (concrete) category* $\mathbf{X}$ *(and* $\mathcal{S} = \mathrm{Emb}\mathbf{X}$*),* $\mathbf{Sat}(\mathcal{S}) = AC(\mathcal{S})$ *holds whenever* $\mathbf{X}$ *admits an $\mathcal{S}$-firm epireflective subcategory.*

Question B. *Prove or disprove that, for each (concrete) category* $\mathbf{X}$ *(and* $\mathcal{S} = \mathrm{Emb}\mathbf{X}$*),* $\mathbf{WInj}(\mathcal{S}) \subset \mathbf{A}C(\mathcal{S})$.

Definition 1.5. *Let* $\mathbf{P}$ *be a class of* $\mathbf{X}$*-objects. We say that* $\mathbf{X}$ *is $\mathcal{S}$-cogenerated by* $\mathbf{P}$ *if every* $\mathbf{X}$*-object is an $\mathcal{S}$-subobject (X is $\mathcal{S}$-subobject of Y if there is $s : X \to Y$, with $s \in \mathcal{S}$) of a product of objects in* $\mathbf{P}$.

The category $\mathbf{X}$ admits an $\mathcal{S}$-firm epireflection if and only if each $\mathbf{X}$-object is the domain of some $\mathcal{S} \cap \mathrm{Epi}\mathbf{X}$-morphism into some object of $WInj(\mathcal{S})$. (This is immediately clear from Proposition 1.3 (d).)

$WInj(\mathcal{S})$ is closed for the taking of products in $\mathbf{X}$ (again clear, with no assumptions on $\mathbf{X}$ or $\mathcal{S}$ other than $\mathcal{S} \subset \mathrm{Mor}\mathbf{X}$).

$WInj(\mathcal{S})$ is closed for the taking of extremal subobjects in $\mathbf{X}$ (this immediately follows from merely the (Epi$\mathbf{X}$, ExMono$\mathbf{X}$)-diagonalization property).

If $\mathbf{X}$ is assumed to be complete, well-powered and co-(well-powered), it follows by [36, (Theorem 37.1)] that $WInj(\mathcal{S})$ is epireflective in $\mathbf{X}$. Alternatively, it was shown in [6, (Proposition 1 and pp. 157-158)] that $WInj(\mathcal{S})$ is epireflective in $\mathbf{X}$, provided $\mathbf{X}$ is an (Epi, ExMonoSource)-category (cf. [34, (p. 331)].

It is worth noting that the following result does not assume $\mathbf{X}$ to be co-(well-powered).

Theorem 1.6. *The category $\mathbf{X}$ admits an $\mathcal{S}$-firm epireflective subcategory $\mathbf{R}$ iff there exists a class $\mathbf{P} \subset WInj(\mathcal{S})$ which $\mathcal{S}$-cogenerates $\mathbf{X}$. In this case, $\mathbf{R}$ is the epireflective hull in $\mathbf{X}$ of $\mathbf{P}$.*

Proof. Suppose $r_X : X \to R(X)$ is the firm $\mathcal{S}$-epireflection of X in $\mathbf{X}$. By definition, $r_X \in \mathcal{S}$, and by Proposition 1.3, $R(X) \in WInj(\mathcal{S})$. Hence $WInj(\mathcal{S})$ $\mathcal{S}$-cogenerates $\mathbf{X}$. Conversely, suppose there is a class $\mathbf{P} \subset WInj(\mathcal{S})$ which $\mathcal{S}$-cogenerates $\mathbf{X}$. Let $\mathbf{R}$ be the class of all extremal subobjects of products of $\mathbf{P}$-objects. By the above remarks, $\mathbf{R} \subset WInj(\mathcal{S})$. consider any $\mathbf{X}$-object X. Since $\mathbf{X}$ is $\mathcal{S}$-cogenerated by $\mathbf{P}$, there exists an $\mathcal{S}$-morphism $s : X \to \Pi P_i$ for some set of $P_i \in \mathbf{P}$. Let $X \overset{e}{\to} M \overset{m}{\to} \Pi P_i$ be the (Epi$\mathbf{X}$, ExMono$\mathbf{X}$)-factorization of s. Then $e \in \mathrm{Epi}\mathcal{E} \cap \mathcal{S}$ by ($\mathbf{s}_3$), and $M \in \mathbf{R}$ by definition of $\mathbf{R}$. Consider any $f : X \to Y$ with $Y \in \mathbf{R}$. The weak $\mathcal{S}$-injectivity of Y then gives a (unique) $f^* : M \to Y$ with $f^*e = f$. Thus $e : X \to M$ is an $\mathbf{R}$-reflection of X; we choose an equivalent $r_X : X \to R(X)$, whence by ($\mathbf{s}_1$) and ($\mathbf{s}_2$) $r_X \in \mathrm{Epi}\mathbf{X} \cap \mathcal{S}$. To see that $\mathbf{R}$ is $\mathcal{S}$-firm, consider $g : X \to Z$ in $\mathrm{Epi}\mathbf{X} \cap \mathcal{S}$ with $Z \in \mathbf{R}$. We have $g^* : R(X) \to Z$ with $g^* r_X = g$; the weak $\mathcal{S}$-injectivity of $R(X)$ also gives $h : Z \to R(X)$ with $hg = r_X$, and clearly h is inverse to g^*. Finally it is clear that $\mathbf{R}$

is the epireflective hull of **P**, also in the sense that **R** is the smallest epireflective subcategory of **X** that contains **P**.

Corollary 1.7. *$WInj(\mathcal{S})$ is $\mathcal{S}$-firmly epireflective in* **X** *if and only if* **X** *is $\mathcal{S}$-cogenerated by $WInj(\mathcal{S})$.*

A result by Bacsich [2, (Theorem 3.1)] partly overlaps with the content of Theorem 1.6, but under different assumptions. The corresponding results mentioned by Kiss et al. [44, (pp. 94-95)] and given by Tholen [52, (esp. Lemma 7)] are essentially different.

Example 1.8. In all examples below we assume that $\mathcal{S}$ is the class of embeddings and we drop "$\mathcal{S}$" in all terms in which it previously appeared.

(1) Firmness becomes trivial in concrete categories **X** in which epimorphisms are onto maps (in particular in topological categories as well as in abelian categories). Indeed in such a case, $\mathbf{R} = \mathbf{X}$.

(2) No non-trivial ($\neq$ **Singl** $=$ {spaces with at most one point}) epireflective subcategory of **Top** consisting of Hausdorff spaces admits a firm epireflection. In fact none of the subcategories above admits a class of weakly injective cogenerators: if **X** is as above, then the two-point discrete space $D_2(= \{0,1\})$, the discrete space of natural numbers **N** and its Alexandrov compactification $\mathbf{N}^*$ belong to **X**. Let $f : \mathbf{N} \to D_2$ be the continuous map defined by $f(2n) = 0$ and $f(2n+1) = 1$, $n \in \mathbf{N}$. Then f cannot be extended to $\mathbf{N}^*$, while the inclusion $e : \mathbf{N} \to \mathbf{N}^*$ is a dense, hence **X**-epi, embedding. We conclude that no space with more than one point is weakly injective in **X**, so, by Theorem 1.6, **X** does not admit a firm epireflection. The same proof also establishes that if **Y** is any epireflective subcategory of **Haus** containing **N**, then **Y** has no firm epireflective subcategory.

(3) in $\mathbf{Top_o}$, the category of T_o topological space, the Sierpinski two-point space is a cogenerator of $\mathbf{Top_o}$ and it is weakly injective in $\mathbf{Top_o}$. The epireflective hull in $\mathbf{Top_o}$ of the Sierpinski space is the category **Sob** of sober spaces, so, in virtue of Theorem 1.6, **Sob** is the firm epireflective subcategory of $\mathbf{Top_o}$. This restates a result which Hoffmann [38] established by an internal argument. In virtue

of Proposition 1.3, sober, weakly injective, saturated and absolutely closed coincide in $\mathbf{Top_o}$.

(4) By $\mathbf{Unif_o}$ we mean the category of separated (i.e. T_o, hence Hausdorff) uniform spaces and uniformly continuous maps. It is well known [53] that the complete spaces in $\mathbf{Unif_o}$ form the firm epireflective subcategory of $\mathbf{Unif_o}$.

(5) $\mathbf{Prox_o}$ denotes the category of separated proximity spaces and proximity maps [41]. The complete (which are here the same as compact) spaces in $\mathbf{Prox_o}$ form the firm epireflective subcategory of $\mathbf{Prox_o}$. This is an instance of Theorem 1.6, the required weakly injunctive cogenerator being the compact unit interval.

(6) The complete metric spaces form a firm epireflective subcategory of the category of metric spaces and non-expansive maps. (An interesting extended setting for this classical result is given by Hoffmann [38, (p. 321 example 3.4)].)

(7) $\mathbf{Qun}$ will denote the category of quasi-uniform spaces and quasi-uniform maps (see e.g. [15] or [24]). Its subcategory $\mathbf{Qun_o}$ of separated objects consists of those quasi-uniform spaces for which the join of the two induced topologies is T_o (hence Tychonoff), or equivalently the first topology is T_o, or equivalently the second topology is T_o (by the join of two structures we shall always mean the least fine structure finer than both). Császár ([14], [15]) showed that $\mathbf{Qun_o}$ has a firm epireflective subcategory whose objects he named "doubly complete". These are the spaces whose uniform coreflection, formed by joining a quasi-uniformity with its inverse, is a complete uniform space. A convenient construction, and proof of the firmness, is given in [24], where the objects are called "bicomplete" (cf. [13]).

(8) $\mathbf{Qprox}$ denotes the category of quasi-proximity spaces, known to be isomorphic to the full subcategory of totally bounded spaces in $\mathbf{Qun}$ (cf. [47] or [24]). $\mathbf{Qprox_o}$ denotes the corresponding subcategory of separated objects. $\mathbf{Qprox_o}$ has a cogenerator I which is injective with respect to $\mathbf{Qprox_o}$-epi embeddings; I is the closed unit interval with the quasi-proximity relation δ given by $A\delta B$ iff, for each $a > 0$, there exist $x \in A$ and $y \in B$ with $y > x - a$. Thus by Theorem 1.6, $\mathbf{Qprox_o}$ has a firm epireflective subcategory. The spaces in

this subcategory are those for which the join of the two underlying topologies is compact; equivalently, they are the bicomplete (in the sense of (7) above) totally bounded quasi-uniform spaces.

(9) **2Top** denotes the category of bitopological spaces (more briefly, "bispaces") in the sense of [43]. Objects are triples of the form $X = (|X|, \mathbf{O}_1X, \mathbf{O}_2X)$ where $|X|$ is a set and $\mathbf{O}_1X$, $\mathbf{O}_2X$ are topologies on $|X|$; a morphism $f : X \to Y$ is a function with $f : (|X|, \mathbf{O}_iX) \to (|Y|, \mathbf{O}_iY)$ continuous for $i = 1, 2$. The full subcategory $\mathbf{2Top_o}$ consists of those X with $(|X|, \mathbf{O}_1X \vee \mathbf{O}_2X) \in \mathbf{Top_o}$. $\mathbf{2Top_o}$ has a cogenerator Q which is weakly injective in $\mathbf{2Top_o}$. This was shown in [28], where the smallest such object was named "the quad":

$$Q = (\{0,1,2,3\},\ \{\emptyset, \{0,1\},\ |Q|\},\ \{\emptyset, \{0,3\},\ |Q|\})\ .$$

Thus by Theorem 1.6 $\mathbf{2Top_o}$ has a firm epireflective subcategory $\mathbf{R}$ whose objects are precisely the absolutely closed objects and also precisely the weakly injective objects of $\mathbf{2Top_o}$. (In [28] the coincidence $AC(\mathcal{S}) = WInj(\mathcal{S})$ was proved though the firmness was not observed). It was also shown in [28] that $\mathbf{R}$ is contained in, but differs from, the subcategory of sober bispaces in $\mathbf{2Top_o}$. The sober bispaces are given by the largest duality between bispaces and biframes [4]; they are those bispaces for which the join of the two topologies is a sober topological space. It is noteworthy that in $\mathbf{Qun_o}$ and $\mathbf{QProx_o}$ the "complete" objects are precisely those whose symmetrization is "complete" in $\mathbf{Unif_o}$ and $\mathbf{Prox_o}$ respectively, a phenomenon which fails for $\mathbf{2Top_o}$ versus $\mathbf{Top_o}$.

(10) The category **RegNear** of regular nearness spaces has a firm epireflective subcategory [8], whose objects are called complete. In fact the completion functor in **RegNear** restricts to the one in $\mathbf{Unif_o}$. In the larger category **SepNear** of separated nearness spaces, completeness misbehaves in interesting ways [7], [8].

(11) **PTop** ($\mathbf{PTop_o}$) will denote the category of (T_o) pretopological spaces (= Čech spaces). It is shown in [20] that $\mathbf{PTop_o}$-epis are onto. So $\mathbf{PTop_o}$ trivially admits a firm epireflective subcategory.

If $\mathbf{X}$ is a non trivial epireflective subcategory of **PTop** consisting

of Hausdorff pretopological spaces, then the argument in (2) can be used to show that **X** does not admit a firm epireflection.

Notice that the class of saturated (= absolutely closed = compact) Hausdorff pretopological spaces does not even form (the object class of) a reflective subcategory of **PTop** (cf. [12]).

The negative results above remain valid in other categories of filter convergence spaces which properly contain **Top** (e.g. in the category **PsTop** of pseudotopological spaces and in the category **Lim** of limit spaces).

It is also easy to see that firmness becomes trivial in all epireflective subcategories of both **Born** (the category of bornological spaces [40]) and **Simp** (the category of abstract simplicial complexes [46]).

(12) The category of normed vector spaces over a fixed subfield $\mathbf{K}$ of $\mathbf{C}$, with non-decreasing $\mathbf{K}$-linear maps, has the Banach spaces over $\mathbf{K}$ as firm epireflective subcategory. This fact is placed in the setting of topological categories in [38].

(13) $\mathbf{TopGrp_o}$ denotes the category of topological groups with T_o-(hence Hausdorff) topology; the morphisms are continuous homomorphisms. Completion with respect to the two-sided uniformity would provide the firm epireflection if we knew that every epimorphic embedding $G \to H$ with H complete in the two-sided uniformity was dense. Since the epimorphism problem for $\mathbf{TopGrp_o}$ is unsolved, we only know that a full subcategory **X** of $\mathbf{TopGrp_o}$ admits a firm epireflection if the epis of the stated kind in **X** are dense. This is the case for the subcategory $\mathbf{TopAb_o}$ of abelian topological groups with T_o-topology [11].

(14) Let **D** be the category of bounded distributive lattices, with lattice homomorphisms preserving 0 and 1. Emb**D** =Mono**D**. The 2-chain is an injective cogenerator of **D** [5]. Hence by Theorem 1.6 **D** has a firm epireflective subcategory, the epireflective hull of $\{2\}$, consisting of the Boolean algebras. Thus (cf. [2]) the reflection embeds any object of **D** in its Boolean envelope.

(15) The category of cancellative abelian monoids, with homomorphisms preserving neutral element, has the category of abelian groups as firm epireflective subcategory, the reflection being the group of dif-

ferences [10], [2]. The corresponding fact is true for the category of integral domains with ring homomorphisms preserving 1, the reflection giving the field of fractions (that this category does not satisfy our initial assumption of completeness is immaterial for the present purpose). Similarly, the category of torsion-free abelian groups admits a firm epireflection, which gives the divisible hulls of these groups [2]. This is a particular case of the following example.

(16) Let **R-Mod** be the category of all left modules over a fixed unitary ring **R** and, for a given radical **r** in **R-Mod**, let $\mathcal{F}_{\mathbf{r}}$ be the corresponding torsion free class, i.e. $\mathcal{F}_{\mathbf{r}} = \{X \in \mathbf{R}-\mathbf{Mod} | \mathbf{r}(X) = 0\}$. $\mathcal{F}_{\mathbf{r}}$ is epireflective in **R-Mod** and every epireflective subcategory of **R-Mod** is of the form $\mathcal{F}_{\mathbf{r}}$ for a unique radical **r** (cf. [21]). Denote by $\mathcal{FJ}_{\mathbf{r}}$ the class of all **r**-torsion free modules which are **U**-injective in **R-Mod**, where **U** is the class of all **r**-dense ($= \mathcal{F}_{\mathbf{r}}$-dense) monomorphisms. It is shown in [21] that $\mathcal{FJ}_{\mathbf{r}}$ is firmly epireflective in $\mathcal{F}_{\mathbf{r}}$ if and only if **r** is a hereditary radical.

(17) No category of algebras (as defined by Isbell [42]), in particular semigroups or rings, admits a firm epireflective subcategory. In fact it is shown in [42] Example 3.2, that the saturation of an algebra is not unique (and there is not a universal saturation which maps onto all others; so the saturated algebras do not even form an epireflective subcategory).

(18) A *separated projection space* [23] is a pair $(X, (\alpha_n : n \in \mathbf{N}))$ consisting of a set X and a sequence of maps $\alpha_n : X \to X$, subject to the following conditions:

(pro) $$\alpha_n \circ \alpha_p = \alpha_{\mathrm{Min}(n,p)} \quad \text{for all } n, p \in \mathbf{N} ;$$

(sep) $$\text{if } \alpha_n(x) = \alpha_n(y) \quad \text{for each} \quad n \in \mathbf{N} , \quad \text{then } x = y .$$

Abbreviated notation for $(X, (\alpha_n : n \in \mathbf{N}))$ is (X, α_n).

A *projection morphism* $f : (X, \alpha_n) \to (Y, \beta_n)$ is a map $f : X \to Y$ satisfying the condition

(mor) $$\beta_n \circ f = f \circ \alpha_n \quad \text{for each } n \in \mathbf{N} .$$

$\underline{\mathrm{PRO}}_s$ will denote the (concrete) category of separated projection spaces and projection morphisms.

A sequence (x_m) in (X, α_n) is called a **Cauchy sequence** if $\alpha_n(x_{n+1})$ $= x_n$, for each $n \in \mathbf{N}$. A Cauchy sequence (x_m) converges to a point x if $\alpha_n(x) = x_n$, for each $n \in \mathbf{N}$. A separated projection space is called *complete* if each Cauchy sequence converges.

It is shown in [26] that the subcategory of all complete separated projection spaces is firmly epireflective in $\underline{\mathrm{PRO}}_s$.

2. Preservation of $\mathcal{S}$-sources

We now have to extend the notion of $\mathcal{S}$-morphism to sources. We emphasize that we consider sources indexed over arbitrary classes.

Definitions 2.1. (1) *A source $(f_i : X \to Y_i | i \in I)$ is called an $\mathcal{S}$-source if for some set $J \subset I$ the* $\mathbf{X}$*-morphism $\langle f_j \rangle : X \to \Pi(Yj | j \in J)$ belongs to $\mathcal{S}$.*
(2) *We say that a functor $F : \mathbf{X} \to \mathbf{X}$ preserves $\mathcal{S}$-morphisms if $F(s) : F(X) \to F(Y)$ belongs to $\mathcal{S}$ whenever $s : X \to Y$ is in $\mathcal{S}$. The definition for the preservation of $\mathcal{S}$-sources is analogous.*

We shall sometimes need the following additional conditions on $\mathcal{S}$:
($\mathbf{s}_4$) Whenever $f_i : X_i \to Y_i$ is in $\mathcal{S}$ for each $i \in I$, I any set, the product morphism $\Pi f_i : \Pi(X_i | i \in I) \to \Pi(Y_i | i \in I)$ is in $\mathcal{S}$;
($\mathbf{s}_5$) ExMono$\mathbf{X} \subset \mathcal{S}$.

Proposition 2.2. *Assume condition* ($\mathbf{s}_1$)-($\mathbf{s}_4$). *Let* $\mathbf{R}$ *be an $\mathcal{S}$-epireflective subcategory of* $\mathbf{X}$ *with $R : \mathbf{X} \to \mathbf{X}$ the reflector to* $\mathbf{R}$. *If R preserves $\mathcal{S}$-morphism, then R preserves $\mathcal{S}$-sources.*

Proof. Let $(f_i : X \to Y_i | i \in I)$ be an $\mathcal{S}$-source. Thus there is a set $J \subset I$ such that the morphism $\langle f_j \rangle : X \to \Pi(Y_j | j \in J)$ is in $\mathcal{S}$. Let $P = \Pi(R(Y_j) | j \in J)$ with projection morphisms p_j. By properties ($\mathbf{s}_2$) and ($\mathbf{s}_4$), the morphism $h = \langle r_{Yj} f_j \rangle : X \to P$ is in $\mathcal{S}$, since each r_{Yj} is in $\mathcal{S}$.

$$\begin{tikzcd}
X \arrow[rr, "r_X"] \arrow[dr, "h"] \arrow[dd, "f_j"'] & & R(X) \arrow[dl, "R(h)"'] \arrow[dd, "R(f_j)"] \\
& P \arrow[r, "r_P"] \arrow[dr, "p_j"] & R(P) \\
Y_j \arrow[rr, "r_{Y_j}"] & & R(Y_j)
\end{tikzcd}$$

Since $\mathbf{R}$ is epireflective $P \in \mathbf{R}$, so that r_P is an isomorphism. We have $R(f_j)r_X = r_{Y_j}f_j = p_j h = p_j r_P^{-1} R(h) r_X$, whence $R(f_j) = p_j r_P^{-1} R(h)$, in other words $r_P^{-1}R(h) = \langle R(f_j)\rangle$; but this morphism is in $\mathcal{S}$ since both r_P^{-1} and $R(h)$ are in $\mathcal{S}$. Thus $(R(f_i) : R(X) \to R(Y_i) | i \in I)$ is an $\mathcal{S}$-source, as required.

Proposition 2.3. *Assume conditions* $(\mathbf{s}_1)$-$(\mathbf{s}_3)$ *and* $(\mathbf{s}_5)$. *If* $\mathbf{R}$ *is an* $\mathcal{S}$*-firm epireflective subcategory of* $\mathbf{X}$, *the* $\mathbf{R}$*-reflection functor* R *preserves* $\mathcal{S}$*-morphisms.*

Proof. Let $f : X \to Y$ be an $\mathcal{S}$-morphism and let $X \xrightarrow{e} M \xrightarrow{m} R(Y)$ be an (epi, extremal mono)-factorization of $r_Y f : X \to Y \to R(Y)$ (see property (x_1) in §1). Then $e \in \mathrm{Epi}\mathbf{X} \cap \mathcal{S}$ by $(\mathbf{s}_3)$. Since M is an extremal subobject of $R(Y) \in \mathbf{R}$, $M \in \mathbf{R}$. Since $\mathbf{R}$ is $\mathcal{S}$-firm and $e \in \mathrm{Epi}\mathbf{X} \cap \mathcal{S}$ the morphism $e^* : R(X) \to M$ is an isomorphism, so it belongs to $\mathcal{S}$, by $(\mathbf{s}_1)$. We have $me^* r_X = me = r_Y f = R(f) r_X$, whence $me^* = R(f)$. Since $m \in \mathrm{ExMono}\mathbf{X}$ ($\subset \mathcal{S}$ by assumption) and $e^* \in \mathcal{S}$, $R(f) = me^* \in \mathcal{S}$ by $(\mathbf{s}_2)$ and the proof is complete.

Corollary 2.4. *Assume conditions* $(\mathbf{s}_1)$-$(\mathbf{s}_5)$. *If* $\mathbf{R}$ *is* $\mathcal{S}$*-firmly epireflective in* $\mathbf{X}$, *then the* $\mathbf{R}$*-reflection functor preserves* $\mathcal{S}$*-sources.*

Theorem 2.5. *Let* $\mathbf{X}$ *be a complete and well-powered category and* $\mathcal{S}$ *a class of* $\mathbf{X}$*-morphisms satisfying conditions* $(\mathbf{s}_1)$-$(\mathbf{s}_5)$. *Let* $\mathbf{R}$ *be an* $\mathcal{S}$*-epireflective subcategory of* $\mathbf{X}$ *with* $\mathbf{R}$*-reflector* R.

Then, the following conditions are equivalent:

(i) $\mathbf{R}$ *is* $\mathcal{S}$*-firmly epireflective in* $\mathbf{X}$;

(ii) *R preserves $\mathcal{S}$-morphisms and* $\mathbf{R} \subset \mathrm{Sat}(\mathcal{S})$*;*
(iii) *R preserves $\mathcal{S}$-sources and* $\mathbf{R} \subset \mathrm{Sat}(\mathcal{S})$*;*
(iv) $\mathbf{R} \subset WInj(\mathcal{S})$*;*
(v) *There is a class* $\mathbf{P} \subset WInj(\mathcal{S})$ *such that* $\mathbf{R}$ *is the epireflective hull of* $\mathbf{P}$ *in* $\mathbf{X}$*;*
(vi) *R preserves $\mathcal{S}$-morphisms and* $\mathcal{S} \cap \mathrm{Epi}\mathbf{R} = \mathrm{Iso}\mathbf{R}$.

When one of these conditions holds, then $\mathbf{R} = \mathrm{Sat}(\mathcal{S}) = WInj(\mathcal{S})$.

Proof. (i)⇒(ii) follows from Proposition 2.3 and from (ii)⇒(i) of Proposition 1.3.
(ii)⇒(iii) follows from Proposition 2.2.
(iii)⇒(i): If $f : X \to Y$ is in $\mathrm{Epi}\mathbf{X} \cap \mathcal{S}$ and $Y \in \mathbf{R}$ then $f^* \in \mathcal{S}$ and $f^* \in \mathrm{Epi}\mathbf{X}$, so $\mathbf{R} \subset \mathrm{Sat}(\mathcal{S})$ gives f^* iso.
(i)⇒(iv) is precisely (ii)⇒(iii) of Proposition 1.3.
(iv)⇒(i): In such case $\mathbf{X}$ is trivially cogenerated by a class of weak $\mathcal{S}$-injective objects so Theorem 1.6 applies.
(iv)⇒(v) is trivial and (v)⇒(i) follows from Theorem 1.6.
(vi)⇒(iv): Let $A \in \mathbf{R}$. To show $A \in WInj(\mathcal{S})$, consider $f : X \to A$ and $e : X \to Y$ with $e \in \mathrm{Epi}\mathbf{X} \cap \mathcal{S}$. since $(R(e))r_X = r_Y e$, $R(e)$ is an $\mathbf{X}$-epimorphism and hence an $\mathbf{R}$-epimorphism. By (vi), $R(e) \in \mathcal{S}$, hence $R(e) \in \mathrm{Epi}\mathbf{R} \cap \mathcal{S}$, so that $R(e) \in \mathrm{Iso}\mathbf{R}$. With $f = f^* r_X$ we then have $f = f^*(R(e))^{-1} r_Y e$, which proves that $A \in Winj(\mathcal{S})$.
(ii)⇒(vi): Observe that always $\mathrm{Epi}\mathbf{R} = \mathrm{Epi}\mathbf{X} \cap \mathrm{Mor}\mathbf{R}$, which follows at once from the reflection maps being monic (by $(\mathbf{s}_1)$). Consider $f : X \to Y$ in $\mathrm{Epi}\mathbf{R} \cap \mathcal{S}$. Then $f \in \mathrm{Epi}\mathbf{X} \cap \mathcal{S}$ with X, $Y \in \mathbf{R}$. Assuming (ii) we have $X \in \mathrm{Sat}(\mathcal{S})$, so that f is an isomorphism. Thus (vi) holds.

The following example shows that the condition $\mathbf{R} \subset \mathrm{Sat}(\mathcal{S})$ cannot be deleted form conditions (ii) and (iii) in Theorem 2.5.

Example 2.6. Let **Alex** be the category of Alexandroff spaces [31], [25]. These are the same as the zero-set spaced of [30]. It is shown in [25] that **Alex** admits no (Emb**Alex**)-firm epireflection, but that the realcompact reflection maps $X \to \nu X$ in **Alex** are essential embeddings. This implies that the realcompact epireflector ν preserves

embeddings, so that by Proposition 2.2 ν also preserves embeddings-sources (i.e. initial sources). Moreover ν preserves arbitrary products in **Alex** [30]. (Christopher Gilmour kindly reminded us of this example).

It is worth noting that **Alex** is isomorphic to the full subcategory **SMF** of separable metric-fine spaces in $\mathbf{Unif_o}$ [31]. The completion epireflector in $\mathbf{Unif_o}$ then restricts to a firm epireflector in **SMF** with respect to uniform embeddings (note that **SMF** is stable under completion). But, being isomorphic to **Alex**, **SMF** admits no firm epireflector with respect to the larger class of **SMF**-embeddings.

Similarly the category **Tych**, which by Example 1.8. (2) admits no firm epireflector with respect to embeddings, has many completion-stable full embeddings into $\mathbf{Unif_o}$. Each of the embedded subcategories of $\mathbf{Unif_o}$ then admits a firm epireflector with respect to uniform embeddings.

Proposition 2.7. *Let* $\mathbf{X}$ *admit an* $\mathcal{S}$*-firm epireflector* R*. If* $\mathrm{Epi}\mathbf{X} \cap \mathcal{S}$ *is closed under the formation of products, then* R *preserves products.*

Proof. By assumption $\Pi r_{X_j} \in \mathrm{Epi}\mathbf{X} \cap \mathcal{S}$. Also $R(X_i) \in \mathbf{R}$. Then by the definition of $\mathcal{S}$-firmness, the extension of Πr_{X_j} to $R(\Pi X_i)$ is an isomorphism.

Definition 2.8. ([3], [52]) *An* $\mathbf{X}$*-morphism* $f \in \mathcal{S}$ *is called* $\mathcal{S}$*-essential if* $g \in \mathcal{S}$ *whenever* $gf \in \mathcal{S}$*. If* $f : X \to Y$ *is* $\mathcal{S}$*-essential and* Y *is* $\mathcal{S}$*-injective, then* f *is called an* $\mathcal{S}$*-injective hull of* $\mathbf{X}$*.*

Remarks 2.9. (1) If $\mathcal{S}$ satisfies $(\mathbf{s}_5)$, then an $\mathbf{X}$-epimorphism is $\mathcal{S}$-essential if and only if it is $\mathrm{Epi}\mathbf{X} \cap \mathcal{S}$-essential.
(2) Let $r : 1 \to R$ be an $\mathcal{S}$-epireflection in $\mathbf{X}$.
If each r_X is $\mathcal{S}$-essential, then R preserves $\mathcal{S}$-morphisms. The converse implication holds under a slight strengthening of condition $(\mathbf{s}_3)$, e.g. that $gf \in \mathcal{S}$ implies $f \in \mathcal{S}$ which is satisfied by e.g. extremal monomorphisms, initial morphisms, and embeddings.

Proposition 2.10. *Let condition* $(\mathbf{s}_5)$ *hold and let* $\mathbf{X}$ *admit an* $\mathcal{S}$*-firm epireflection* $r : 1 \to R$*. Then, for each* $X \in \mathbf{X}$*,* $r_X : X \to R(X)$ *is an* $\mathrm{Epi}\mathbf{X} \cap \mathcal{S}$*-injective hull of* X*.*

Proof. To see that r_X is Epi$\mathbf{X} \cap \mathcal{S}$-essential, consider $f : R(X) \to Y$ such that $fr_X \in$ Epi$\mathbf{X} \cap \mathcal{S}$. Since $f \in$ Epi$X$, we only have to prove $f \in \mathcal{S}$. By Proposition 2.3 $R(fr_X) \in \mathcal{S}$. Since $R(fr_X) = r_Y f$, in fact $R(fr_X) \in$ Epi$\mathbf{X} \cap \mathcal{S}$. With $R(X)$ being $\mathcal{S}$-saturated, this implies that $R(fr_X)$ is an isomorphism. Thus f is a section, hence an isomorphism, so that $f \in \mathcal{S}$ (and r_X is maximally Epi$X \cap \mathcal{S}$-essential).

3. Applications to topological categories

We consider a topological category $\mathbf{A}$ over **Set** in the sense of [33] and [45], cf. [34]. An object A of $\mathbf{A}$ is a T_o-**object** in the sense of [45] iff each $\mathbf{A}$-morphism from the indiscrete two-point object of $\mathbf{A}$ into A is constant.

Let $T_o\mathbf{A}$ denote the category of T_o-objects of $\mathbf{A}$ (our examples 1.8 (3)(4)(5)(7)(8)(9)(11) contain instances of this notion).

Proposition 3.1. ([45]) *$T_o\mathbf{A}$ is extremal-epireflective in $\mathbf{A}$, and $T_o\mathbf{A}$ is the largest epireflective, non-bireflective subcategory of $\mathbf{A}$.*

Proposition 3.2. *If $f : X \to Y$ is an initial morphism in $\mathbf{A}$ and $X \in T_o\mathbf{A}$, then f is injective.*
However, if $X \in \mathbf{A}$ is such that each initial morphism with domain X is injective, then X need not be in $T_o\mathbf{A}$ [54], [55].

Proposition 3.3. *Let $\mathbf{X}$ be an epireflective subcategory of the topological category $\mathbf{A}$. Then $\mathbf{X}$ is a well-powered and complete (*Epi$\mathbf{X}$, ExMono$\mathbf{X}$*)-category. The class* Mor$\mathbf{X} \cap$ Emb$\mathbf{A}$ *satisfies conditions* $(\mathbf{s}_1)$-$(\mathbf{s}_5)$.

Proof. All properties, except maybe $(\mathbf{s}_5)$, are well known (see e.g. [36]). To prove $(\mathbf{s}_5)$ consider $f : X \to Y$ in ExMono$\mathbf{X}$. Let $X \xrightarrow{e} A \xrightarrow{m} Y$ be its (Epi$\mathbf{A}$, ExMono$\mathbf{A}$)-factorization. Then $A \in \mathbf{X}$ since $\mathbf{X}$ is epireflective in $\mathbf{A}$. Therefore $e \in$ Epi$\mathbf{X}$, and since $f \in$ ExMono$\mathbf{X}$, e is an isomorphism, whence $f \in$ ExMono$\mathbf{A}$. But ExMono$\mathbf{A}$ =Emb$\mathbf{A}$ since $\mathbf{A}$ is topological [33], [32].

Thus, the hypotheses of Theorem 2.5 are satisfied by any epireflective subcategory $\mathbf{X}$ of a topological category, with $\mathcal{S} =$ Emb$\mathbf{X}$. There

are examples of such $\mathbf{X}$ which are not co-(well-powered) ([48], [27], [17], [18], [22], [29]).

Question C. *Find an example of a category* $\mathbf{X}$ *which is not co-(well-powered) and yet admits a non-trivial firm epireflective subcategory.*

For a subcategory $\mathbf{X}$ of $\mathbf{A}$, morphisms or sources which are initial in $\mathbf{X}$ need not be so in $\mathbf{A}$; we have the following version of Theorem 2.5:

Proposition 3.4. *Let* $\mathbf{X}$ *be an epireflective subcategory of the topological category* $\mathbf{A}$, *and let* $\mathcal{S} = \mathrm{Mor}\mathbf{X} \cap \mathrm{Emb}\mathbf{A}$. *If* $\mathbf{X}$ *admits an* $\mathcal{S}$*-firm epireflector* R, *then* R *preserves* $\mathbf{A}$*-initial sources in* $\mathbf{X}$.

Proof. In case $\mathbf{X}$ is bireflective in $\mathbf{A}$, $\mathbf{X}$ is itself a topological category, $\mathbf{X}$-epimorphisms are onto, and R is the identity functor on $\mathbf{X}$. In case $\mathbf{X}$ is non-bireflective in $\mathbf{A}$, then by Propositions 3.1 and 3.2, every $\mathbf{A}$-initial source in $\mathbf{X}$ is an embedding-source, and the required result follows from Proposition 3.3 and Theorem 2.5.

There is an occasion to extend the theory of firm epireflections to firm $\mathcal{E}$-reflections in a category $\mathbf{X}$ which has a factorization structure $(\mathcal{E}, \mathcal{M})$ on $\mathrm{Mor}\mathbf{X}$, where $\mathcal{E}$ need not necessarily consist of epimorphisms.

Definition 3.5. *Let* $\mathrm{Mor}\mathbf{X}$ *have a factorization structure* $(\mathcal{E}, \mathcal{M})$ *and let* $\mathcal{S} \subset \mathrm{Mor}\mathbf{X}$. *Let* $\mathbf{R}$ *be a reflective subcategory of* $\mathbf{X}$ *with reflection* $r : 1_{\mathbf{X}} \to R$. *We say that* $\mathbf{R}$ *is* $\mathcal{S}$*-firmly* $\mathcal{E}$*-reflective in* $\mathbf{X}$ *iff*

(1) *Each* $r_X : X \to R(X)$ *is in* $\mathcal{E} \cap \mathcal{S}$;
(2) *Whenever* $f : X \to Y$ *is in* $\mathcal{E} \cap \mathcal{S}$ *and* $Y \in \mathbf{R}$, *the morphism* $f^* : X \to Y$ *for which* $f^* r_X = f$, *is an isomorphism.*

Most of the results in this paper can be restated in this context, with suitable proviso's. We have the following concrete occurrence of $\mathcal{S}$-firm $\mathcal{E}$-reflections.

Let $\mathbf{A}$ be a topological category which is universal in the sense of [37] and [45]; this means that $\mathbf{A}$ is the bireflective hull of $T_o\mathbf{A}$, equivalently that the $T_o\mathbf{A}$-reflection morphisms, say $t_A : A \to T(A)$,

are initial.
Examples of such **A** are **Top**, **2Top**, **Unif**, **Prox**, **Qun**, **QProx**, **Near**; but **PTop** (1.8 (11) above) is a non-example; more examples are given in [45] and [54], [55].

Let $\mathbf{X} = T_o\mathbf{A}$ and let $t : 1_{\mathbf{A}} \to T$ be the reflection to **X**. A morphism $f : A \to B$ in **A** is called **relatively X**-*epi* if and only if, whenever $r, s : B \to C$ are **A**-morphisms with $rf = sf$ and $C \in \mathbf{X}$, then $r = s$. Let $\mathcal{E}$ be the class of relative **X**-epimorphisms in **A**. There exists a class $\mathcal{M}$ such that $(\mathcal{E}, \mathcal{M})$ is a factorization structure on Mor**A** precisely when the regular closure operator induced by **X** is weakly hereditary [18]. Also, a source in **X** is **X**-initial if and only if it is **A**-initial. Let Init**A** denote just the class of **A**-initial morphisms.

Proposition 3.6. *let* **R** *be a reflective subcategory of a universally topological category* **A** *with reflection* $r : 1_{\mathbf{A}} \to R$, *and let* $\mathbf{X} = T_o\mathbf{A}$. *Let* $\mathcal{E}$ *be the class of relative* **X**-*epimorphisms in* **A**, *and assume that the regular closure operator induced by* **X** *is weakly hereditary. Then,* **R** *is* Init**A**-*firmly* $\mathcal{E}$-*reflective in* **A** *if and only if* **R** *is* Emb**X**-*firmly epireflective in* **X**. *In this case,* $\mathbf{R} = Inj(\mathrm{Epi}\mathbf{X} \cap \mathrm{Emb}\mathbf{X}) = Inj(\mathcal{E} \cap \mathrm{Init}\mathbf{A})$, *and* R *preserves* **A**-*initial sources.*

The correspondence in the above result arises by factoring the morphism r_A through the $T_o\mathbf{A}$-reflection t_A. Thus, our examples of Emb**X**-firm epireflections in 1.8. (3)-(5), (7)-(9) give us Init**A**-firm $\mathcal{E}$-reflections in **A** =**Top**, **2Top**, **Unif**, **Prox**, **Qun**, **QProx**.

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A Stability Criterion for General Systems

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Abstract

The direct method of Liapunov characterizes stability properties of sets in dynamical systems in terms of the existence of corresponding real-valued "Liapunov functions". The traditional limitation of Liapunov functions to real values has blocked the extension of this approach to more general systems. In this paper, a stability concept analogous to classical uniform stability is defined as a relationship between a quasi-order and a uniformity on the same set, and it is shown that stability in this sense occurs if and only if there exists a Liapunov function taking values in a certain partially ordered uniform space associated with the given uniformity and called its retracted scale. A few general properties of scales and retracted scales are discussed, and the continuity of the Liapunov functions is briefly considered.

1. Introduction. Liapunov's "second" or direct method for investigating the stability of dynamical systems is based on a set of theorems relating various types of stability to the existence of appropriate norm-like functions, commonly called Liapunov functions (see [9]). The method has been developed very far for dynamical systems defined by ordinary differential equations, functional differential equations, etc., but the extension of Liapunov's approach to more general systems has lagged behind that of other parts of dynamics, and the literature on this problem is small (see [2], [11]).

The present paper introduces a stability criterion in the spirit of Liapunov's direct method in a setting where very little is assumed about the structures involved. An essential step is the introduction of the "scale" of a filter and in particular of the "retracted scale" of a uniformity; these concepts seem to have some intrinsic interest and should be useful in further studies in the same general direction.

2. The scale of a filter. By a *prefilter* on a nonempty set S we shall mean a nonempty collection α of nonempty subsets of S such that $X \in \alpha$ and $X \subset Y \subset S$ imply $Y \in \alpha$. For any nonempty subset X_0 of S, the collection $\{X \subset S : X \supset X_0\}$ is a prefilter; it will be denoted by $[X_0]$ and called the *principal* prefilter generated by X_0. We shall say that a prefilter α is *proper* if $\alpha \neq \{S\}$ or equivalently (since $S \in \alpha$ in any case) if α has at least two elements.

MATHEMATICAL SYSTEMS THEORY, Vol. 1, No. 1.
Published by Springer-Verlag New York Inc.

A prefilter α is a *filter* if it is multiplicative: $\{X,Y\} \subset \alpha$ implies $X \cap Y \in \alpha$. (For more on filters, see [3].)

Let $\mathscr{F}$ be a filter. We shall denote by $P(\mathscr{F})$, or simply by P where there is no risk of ambiguity, the set of all prefilters which are contained in $\mathscr{F}$. P may be partially ordered by the inverse of inclusion: for any $\{\alpha,\beta\} \subset P$ we shall say that $\alpha \leqslant \beta$ if and only if $\beta \subset \alpha$. The partially ordered set $(P,\leqslant)$ will be called the *scale* of the filter $\mathscr{F}$.

Some significant information about the order-theoretic structure of the scale of a filter is provided by the following theorem.

PROPOSITION 2.1. *The scale $(P,\leqslant)$ of a filter $\mathscr{F}$ has a first element 0. The subset $P^+ = P - \{0\}$ is directed downward ("filtrant à gauche"), and has a first element if and only if $\mathscr{F}$ is a proper principal filter.*

Proof. $\mathscr{F}$ itself is the first element of P: clearly $\mathscr{F} \in P$, and $\mathscr{F} \leqslant \alpha$ for every $\alpha \in P$ according to the definitions.

If $\{\alpha,\beta\} \subset P^+$, then $\gamma = \alpha \cup \beta \in P$, and $\gamma \leqslant \alpha$, $\gamma \leqslant \beta$. Let $X \in \mathscr{F} - \alpha$ and $Y \in \mathscr{F} - \beta$, and put $Z = X \cap Y \in \mathscr{F}$. Then $Z \notin \gamma$, for in the contrary case we would have $Z \in \alpha$ or $Z \in \beta$, whence (by the prefilter property) $X \in \alpha$ or $Y \in \beta$, contrary to the choice of X or Y. Hence $\gamma \neq \mathscr{F}$ and $\gamma \in P^+$. Thus P^+ is directed downward.

Now suppose that P^+ has a first element α. Then $\mathscr{F}$ is certainly proper, for otherwise P^+ is empty. Let $X \in \mathscr{F} - \alpha$ and set $\beta = [X]$.

First, $\alpha \cup \beta = \mathscr{F}$. If this were false, we would have $\alpha \cup \beta \leqslant \alpha$, $\alpha \cup \beta \neq \alpha$, $\alpha \cup \beta \in P^+$, and this would contradict the assumption that α is the first element of P^+.

Now, in fact, $\beta = \mathscr{F}$. In the contrary case, we could choose $Y \in \mathscr{F} - \beta$ and put $Z = X \cap Y$. Since $Z \in \mathscr{F} = \alpha \cup \beta$, we would have (i) $Z \in \alpha$ or (ii) $Z \in \beta$. In case (i), it would follow from $Z \subset X$ that $X \in \alpha$, but this would contradict the choice of X. In case (ii), it would follow from $Z \subset X$ and the definition of β that $Z = X$, whence $X = X \cap Y$, whence $X \subset Y$, whence $Y \in \beta$; and this would contradict the choice of Y. But $\mathscr{F} = \beta$ means that $\mathscr{F}$ is principal.

Conversely, if $\mathscr{F}$ is principal (say $\mathscr{F} = [X]$) and proper, it is easy to see that $\mathscr{F} - \{X\}$ is the first element of P^+. This completes the proof.

Remark. Proposition 2.1 asserts that when $\mathscr{F}$ is proper but not principal (and otherwise $\mathscr{F}$ is hardly interesting), $(P,\leqslant)$ is an *échelle généralisée* as defined by Colmez [6]; this is one reason for the term "scale."

3. The scale and the retracted scale of a uniformity. We shall be particularly concerned with the scale of a uniformity, i.e., of the filter of entourages for some uniform space. Recall that if E is a nonempty set, a filter $\mathscr{U}$ on $E \times E$ is a uniform structure or *uniformity* if it satisfies:

(U_1) every $V \in \mathscr{U}$ contains the diagonal set $\Delta = \{(x,x) : x \in E\}$;

(U_2) if $V \in \mathscr{U}$, then $V^{-1} = \{(x,y) : (y,x) \in V\} \in \mathscr{U}$;

(U_3) if $V \in \mathscr{U}$, there exists a $W \in \mathscr{U}$ such that $W^2 = \{(x,y) : (x,z) \in W$ and $(z,y) \in W$ for some $z\} \subset V$.

A uniformity is *Hausdorff* if $\cap\, \mathscr{U} = \Delta$. (For further information about uniformities and their uses, see [3], [4], or [8].)

Any uniformity $\mathscr{U}$, as a filter, has a scale P, which may be assigned a natural uniformity of its own. For each $V \in \mathscr{U}$, we define $\tilde{V} \subset P \times P$ by

$$\tilde{V} = \{(\alpha,\beta) \in P \times P : VS \in \beta \text{ for all } S \in \alpha \text{ and } VT \in \alpha \text{ for all } T \in \beta\}.$$

(Here and later the juxtaposition of the symbols for two relations denotes their composite; for example, $VS = \{(x,y) : (x,z) \in S \text{ and } (z,y) \in V \text{ for some } z \in E\}$.)

PROPOSITION 3.1. *The collection* $\{\tilde{V} : V \in \mathscr{U}\}$ *is a base for a uniformity* $\mathscr{V}$ *on* P.

That is, the collection $\mathscr{V}$ of all supersets in $P \times P$ of relations $\tilde{V}$ satisfies $(U_1)-(U_3)$. The only detail of the proof that may not be altogether straightforward is: If $W, V \in \mathscr{U}$ and $W^2 \subset V$, then $\tilde{W}^2 \subset \tilde{V}$; this essentially shows that $\mathscr{V}$ satisfies (U_3). It may be proved as follows. $(\alpha,\beta) \in \tilde{W}^2$ means that for some $\gamma \in P$, $(\alpha,\gamma) \in \tilde{W}$ and $(\gamma,\beta) \in \tilde{W}$. From these in turn,

$$WS \in \gamma \text{ for all } S \in \alpha;\ WT \in \alpha \text{ for all } T \in \gamma\ ; \tag{3.1}$$

$$WS \in \beta \text{ for all } S \in \gamma;\ WT \in \gamma \text{ for all } T \in \beta\ . \tag{3.2}$$

From the first parts of (3.1) and (3.2) we have $WWS \in \beta$ for all $S \in \alpha$, and from the second parts we have $WWT \in \alpha$ for all $T \in \beta$. Since $W^2 = WW \subset V$, it follows that $(\alpha,\beta) \in \tilde{V}$.

In some ways the scale of a uniformity suffers from an over-richness which may be offset by passing from the scale itself to a certain subset of the scale, which we shall obtain as the range of a map w defined as follows:

$$w(\alpha) = \{V \in \mathscr{U} : WV \in \alpha \text{ for all } W \in \mathscr{U}\}\ . \tag{3.3}$$

For every prefilter α, $w(\alpha)$ is also a prefilter; thus w is a map from P into P.

PROPOSITION 3.2. *For every* $\alpha \in P$, $w(\alpha) \leqslant \alpha$ *and* $w(w(\alpha)) = w(\alpha)$. *Moreover, w is order-preserving.*

Proof. The relation $w(\alpha) \leqslant \alpha$ follows from (3.3) and the prefilter property of α. In particular, $w(w(\alpha)) \leqslant w(\alpha)$. Now suppose that $V \in w(w(\alpha))$. Then $WV \in \alpha$ for all $W \in \mathscr{U}$; that is, $W'WV \in \alpha$ for all W, W' in $\mathscr{U}$. For any given $W_0 \in \mathscr{U}$ there exists, by (U_3), a $W_1 \in \mathscr{U}$ such that $W_1^2 \subset W_0$. Then if we take $W' = W = W_1$, we have $W_1^2 V \in \alpha$, whence $W_0 V \in \alpha$. This shows that $w(w(\alpha)) \subset w(\alpha)$, or $w(\alpha) \leqslant w(w(\alpha))$. The order-preserving property follows at once from (3.3). This completes the proof.

We shall say that two elements α and β of P are *equivalent* if $(\alpha,\beta) \in \cap\, \mathscr{V}$. The reflexivity, symmetry, and transitivity of this relation follow respectively from (U_1), (U_2), and (U_3). In terms of $\mathscr{U}$, α and β are equivalent if and only if, for all $V \in \mathscr{U}$, $VS \in \beta$ for all $S \in \alpha$, and $VT \in \alpha$ for all $T \in \beta$.

PROPOSITION 3.3. $w(\alpha) = w(\beta)$ *if and only if* α *and* β *are equivalent.*

Proof. Suppose that α and β are equivalent, and let $V \in w(\alpha)$. For any given $W \in \mathscr{U}$, let $W_1 \in \mathscr{U}$ satisfy $W_1^2 \subset W$. Then $W_1 V \in \alpha$ by the choice of

V, and in turn $W_1^2 V \in \beta$ because of the equivalence. Then $WV \in \beta$, so $V \in w(\beta)$, and we have $w(\alpha) \subset w(\beta)$. Similarly, $w(\beta) \subset w(\alpha)$, and equality follows. Conversely, let $w(\alpha) = w(\beta)$. If $S \in \alpha$, then $\alpha \subset w(\alpha) = w(\beta)$ implies that $VS \in \beta$ for all $V \in \mathscr{U}$. Similarly, $VT \in \alpha$ for all $T \in \beta$ and $V \in \mathscr{U}$, so α and β are equivalent.

We shall say that a prefilter $\alpha \in P$ is *regular* if $w(\alpha) = \alpha$, and denote the set of all regular prefilters in $\mathscr{U}$ by P_r; this, with the partial order $\leqslant$ defined as for P, namely as the inverse of inclusion, will be called the *retracted scale* of $\mathscr{U}$.

COROLLARY. *Every equivalence class of P has a unique regular member, which is also the least member of the class; it is $w(\alpha)$, where α is any member of the class.*

PROPOSITION 3.4. $w(\alpha) = 0$ *if and only if* $\alpha = 0$.

Proof. By Proposition 3.3, $w(0) \leqslant 0$. Hence $w(0) = 0$. If $w(\alpha) = 0$ $(=\mathscr{U})$, then by (3.3) for any $W \in \mathscr{U}$ we have $W^2 \in \alpha$. This implies, because of property (U_3) of uniformities and the prefilter property of $\mathscr{U}$, that $\alpha = \mathscr{U}$, i.e., $\alpha = 0$.

PROPOSITION 3.5. *The retracted scale P_r of a uniformity $\mathscr{U}$ has a first element* 0. *The subset $P_r^+ = P_r - \{0\}$ is directed downward; it has a first element if and only if $\mathscr{U}$ is a proper principal filter.*

In short, P_r has the order properties ascribed to P itself by Proposition 2.1. Proposition 3.5 follows easily from Propositions 2.1, 3.3, and 3.5.

As a subset of P, the retracted scale P_r inherits a relative uniformity $\mathscr{V}_r$ from $\mathscr{V}$. Specifically,

$$\mathscr{V}_r = \{V \cap (P_r \times P_r) : V \in \mathscr{V}\} .$$

PROPOSITION 3.6. *The uniformity $\mathscr{V}_r$ is Hausdorff and is the quotient uniformity on P_r in the sense that it is the finest of the uniformities $\mathscr{V}'$ on P_r such that $w : P \to P_r$ is uniformly continuous relative to $\mathscr{V}$ and $\mathscr{V}'$.*

[The fact that $\mathscr{V}_r$ is Hausdorff accounts in large measure for the introduction of the retracted scale, for $\mathscr{V}$ itself tends not to be Hausdorff. Recall that by definition a map $f : E_1 \to E_2$ is uniformly continuous relative to uniformities $\mathscr{U}_1$ and $\mathscr{U}_2$ on E_1 and E_2 respectively if for every $V_2 \in \mathscr{U}_2$ there exists a $V_1 \in \mathscr{U}_1$ such that $(x,y) \in V_1$ implies $(f(x), f(y)) \in V_2$.]

Proof. If α and β belong to P_r and $(\alpha,\beta) \in \cap \mathscr{V}_r$, then since $\cap \mathscr{V}_r \subset \cap \mathscr{V}$, α and β are equivalent and accordingly, by the corollary to Proposition 3.3, equal. Thus $\mathscr{V}_r$ is Hausdorff.

Let $V \in \mathscr{U}$, and let $W^2 \subset V$. Suppose that α and β belong to P and $(\alpha,\beta) \in \tilde{W}$. Then

$$WS \in \beta \text{ for all } S \in \alpha , \tag{3.4}$$

$$WT \in \alpha \text{ for all } T \in \beta . \tag{3.5}$$

Let $X \in w(\alpha)$. Then by (3.3), $WX \in \alpha$, and from (3.4), $W^2X \in \beta \subset w(\beta)$,

whence $VX \in w(\beta)$. Similarly, by (3.5), $VY \in w(\alpha)$ when $Y \in w(\beta)$. Thus $(w(\alpha), w(\beta)) \in \tilde{V}$. This implies that w is uniformly continuous relative to $\mathscr{V}$ and $\mathscr{V}_r$.

Now suppose that $\mathscr{V}'$ is a uniformity on P_r such that w is uniformly continuous relative to $\mathscr{V}$ and $\mathscr{V}'$. Then for every $W \in \mathscr{V}'$,

$$w^{-1}(W) = \{(\alpha,\beta) \in P \times P : (w(\alpha), w(\beta)) \in W\} \in \mathscr{V} .$$

That is, for some $V \in \mathscr{V}$, $(\alpha,\beta) \in V$ if and only if $(w(\alpha), w(\beta)) \in W$. Since $w(\alpha) = \alpha$ for all $\alpha \in P_r$, we have $V \cap (P_r \times P_r) = W$; thus $W \in \mathscr{V}_r$. This shows that $\mathscr{V}' \subset \mathscr{V}_r$, and completes the proof.

Remark. The subspace P_r of P is essentially the standard "Hausdorffization" or "associated separated space" [3] of P, but peculiarities of P have enabled us to construct it in a way which is more concrete and more useful for our purposes than the usual constructions.

4. Stability in quasi-ordered uniform spaces. Many recently proposed abstract models for the study of control theory, differential equations with or without uniqueness of solutions, contingent equations, finite automata, etc. ([1], [5], [7], [10], [11], and [12] constitute a sample of this literature), although differing considerably from one another, have at least the following elements in common: a set E of "events", often interpreted as pairs (t,x), where t represents a time variable and x a "state" of some system represented; and a quasi-order (reflexive and transitive binary relation on E) ρ, so interpreted that $(e_1,e_2) \in \rho$ means that the event e_2 could follow e_1 in some feasible history of the system. In other words, for any $e \in E$, $\rho(e)$ is the set of all events which may follow, or are attainable from, e.

We shall consider a concept of stability in this very general setting. To formulate it we assume that there is given also a uniformity $\mathscr{U}$ on E. A set $A \subset E$ will be called *stable* relative to ρ and $\mathscr{U}$ if, for every $V \in \mathscr{U}$, there exists a $W \in \mathscr{U}$ such that

$$\rho W(A) \subset V(A) , \tag{4.1}$$

which in terms of the interpretation sketched above means that any event attainable from an event sufficiently close to A is itself arbitrarily close to A. As stability concepts go, it is rather simple, but the discussion that follows will suggest how other stability concepts might be investigated in similar ways.

Stability is usually defined only for invariant sets, i.e., sets A satisfying $\rho(A) \subset A$. The following observation shows that this discrepancy is not severe.

PROPOSITION 4.1. *A set $A \subset E$ is stable relative to ρ and $\mathscr{U}$ if and only if its closure $\bar{A}$ in the uniform topology is stable; and any closed stable set is invariant.*

Proof. Recall that for any $A \subset E$,

$$\bar{A} = \bigcap \{V(A) : V \in \mathscr{U}\} \tag{4.2}$$

(e.g. [4], Theorem 44). Suppose that A is stable and let $V \in \mathscr{U}$. Choose $W \in \mathscr{U}$ so that $\rho W(A) \subset V(A)$ and, using (U_3), choose $W_1 \in \mathscr{U}$ so that $W_1^2 \subset W$. Then

$$\rho W_1(\bar{A}) \subset \rho W_1 W_1(A) \subset \rho W(A) \subset V(A) \subset V(\bar{A}) ,$$

so $\bar{A}$ is stable. On the other hand, let $\bar{A}$ be stable and let $V \in \mathscr{U}$. Choose $V_1 \in \mathscr{U}$ so that $V_1^2 \subset V$. For V_1 there exists a $W \in \mathscr{U}$ such that $\rho W(\bar{A}) \subset V_1(\bar{A})$. Then

$$\rho W(A) \subset \rho W(\bar{A}) \subset V_1(\bar{A}) \subset V_1^2(A) \subset V(A) ,$$

so A is stable.

Suppose now that A is both stable and closed: $\bar{A} = A$. From the stability, $\rho(A) \subset V(A)$ for any $V \in \mathscr{U}$; hence

$$\rho(A) \subset \bigcap \{V(A) : V \in \mathscr{U}\} = \bar{A} = A ,$$

so A is invariant. This completes the proof.

5. Liapunov functions and the stability criterion. Let $(M,\leqslant)$ be a partially ordered set with first element 0. The set $M - \{0\}$, assumed nonempty, will be denoted by M^+.

Suppose again that E is a nonempty set with quasi-order ρ and uniformity $\mathscr{U}$. A set $N \subset E$ is a *uniform neighborhood* of A if there exists a $V \in \mathscr{U}$ such that $V(A) \subset N$.

Generalizing an existing definition (see [2]), we shall say that a function $v : N \to M$, where N is some invariant uniform neighborhood of A, is a *Liapunov function for A* if it satisfies:

(L_1) for any $e_1 \in N$, $(e_1,e_2) \in \rho$ implies $v(e_1) \geqslant v(e_2)$;

(L_2) for any $\lambda \in M^+$ there exists a $V \in \mathscr{U}$ such that $v(e) \ngeqslant \lambda$ if $e \in V(A) \cap N$;

(L_3) for any $V \in \mathscr{U}$, there exists a $\lambda \in M^+$ such that $v(e) \geqslant \lambda$ if $e \in N - V(A)$.

Thus, informally, a Liapunov function is one which (L_1) decreases relative to ρ, (L_2) "approaches 0" uniformly as e approaches A, and (L_3) is bounded away from 0 outside any uniform neighborhood of A.

PROPOSITION 5.1. *If v is a Liapunov function for A, then $v(e) = 0$ if and only if $e \in \bar{A}$.*

The easy proof, which is based on (4.2), (L_2), and (L_3), is omitted.

Liapunov's direct method in stability theory characterizes stability properties by the existence of Liapunov functions of corresponding types. It seems that hitherto the partially ordered set $(M,\leqslant)$ has always been taken to be the set of non-negative reals with the standard order, or a very mild extension thereof, and the types of systems $(E,\rho,\mathscr{U})$ for which Liapunov stability criteria have been found have been restricted accordingly. The introduction of the (retracted) scale of $\mathscr{U}$ at this point wipes away the restrictions.

THEOREM. *A is stable if and only if there exists a Liapunov function for A into the retracted scale of $\mathscr{U}$.*

Proof. Let v be a Liapunov function for A into the retracted scale P_r of $\mathscr{U}$, defined on some invariant uniform neighborhood N of A. Let $V \in \mathscr{U}$ be given; we may assume that $V(A) \subset N$. If $V(A) = N$, it follows from the invariance of N that $\rho V(A) \subset V(A)$, and there is no more to do; thus we may also assume that $N - V(A) \neq \varnothing$.

Put $\alpha = w(\cup \{v(e) : e \in N - V(A)\})$, where w is the map discussed in §3. According to (L_3), there exists a $\lambda \in P_r^+$ such that $e \in N - V(A)$ implies $v(e) \geqslant \lambda$. But $v(e) \geqslant \lambda$ means $v(e) \subset \lambda$, whence $\alpha \subset \lambda$. Since $\lambda \neq 0$, we have $\alpha \neq 0$. Therefore, by (L_2), there exists a $W \in \mathscr{U}$ such that $e \in W(A) \cap N$ implies $v(e) \not\geqslant \alpha$. We may suppose that $W(A) \subset N$, whence the invariance of N gives $\rho W(A) \subset N$.

Now according to (L_1) it follows from $(e_1,e_2) \in \rho$ and $e_1 \in W(A)$ that $v(e_2) \not\geqslant \alpha$. That is,

$$e \in \rho W(A) \text{ implies } v(e) \not\geqslant \alpha \,. \tag{5.1}$$

Also, if $e \in N - V(A)$, the definition of α gives $v(e) \subset \alpha$, or $\alpha \leqslant v(e)$. Contrapositively,

$$e \in N \text{ and } v(e) \not\geqslant \alpha \text{ imply } e \in V(A) \,. \tag{5.2}$$

Combining (5.1) and (5.2) gives the desired conclusion $\rho W(A) \subset V(A)$.

Conversely, let A be stable. We take $N = E$ and for each $e \in E$ we define

$$v(e) = w(\{V \in \mathscr{U} : \rho(e) \subset V(A)\}) \,. \tag{5.3}$$

The set $\{V \in \mathscr{U} : \rho(e) \subset V(A)\}$ is not empty—it includes at least $E \times E$—and is a prefilter. We shall show that v is a Liapunov function for A.

(L_1). Let $(e_1,e_2) \in \rho$ and $\rho(e_1) \subset V(A)$. By the transitivity of ρ, $\rho(e_2) \subset \rho^2(e_1) = \rho(e_1)$, so $\rho(e_2) \subset V(A)$. Thus

$$\{V \in \mathscr{U} : \rho(e_2) \subset V(A)\} \subseteq \{V \in \mathscr{U} : \rho(e_1) \subset V(A)\} \,,$$

and the order-preserving property of w gives $v(e_2) \leqslant v(e_1)$.

(L_2). Let $\lambda \in P_r^+$ and let $V \in \mathscr{U} - \lambda$. Choose $W \in \mathscr{U}$ so that $\rho W(A) \subset V(A)$. Now suppose that for some $e \in W(A)$ we had $v(e) \geqslant \lambda$; then we would have

$$V \in \{V \in \mathscr{U} : \rho(e) \subset V(A)\} \subset v(e) \subset \lambda \,,$$

in contradiction to the choice of V.

(L_3). Let $V \in \mathscr{U}$. If $V(A) = E$, (L_3) holds vacuously, so suppose that $V(A) \neq E$ and let $W^2 \subset V$. Put

$$\alpha = \cup \{v(e) : e \notin V(A)\} \,. \tag{5.4}$$

The collection α is a prefilter. We shall show that $W \notin \alpha$. In the contrary case we would have $W \in v(e)$ for some $e \notin V(A)$. If $W \in v(e)$, then by (5.3) and (3.3) $\rho(e) \subset UW(A)$ for all $U \in \mathscr{U}$. In particular, $\rho(e) \subset W^2(A) \subset$

$V(A)$. Since ρ is reflexive, this implies $e \in V(A)$, a contradiction. Since $W \notin \alpha$, we have $\alpha \neq 0$, whence (Proposition 3.4) $\lambda = w(\alpha) \neq 0$. Now if $e \notin V(A)$, then $v(e) \subset \alpha \subset \lambda$, i.e., $v(e) \geqslant \lambda$, and λ meets the requirements.

This completes the proof of the theorem.

6. The continuity of v and perstability. Liapunov functions are sometimes expected to be continuous, but nothing has been said yet about the continuity of the Liapunov functions of §5. In fact, the Liapunov function v constructed in the proof of the theorem of §5 may fail to be continuous relative to the uniform topologies defined by $\mathscr{U}$ and $\mathscr{V}_r$. A necessary and sufficient condition for the continuity of v is what I venture to call perstability: the set $A \subset E$ is *perstable* (relative to the quasi-order ρ and the uniformity $\mathscr{U}$) if it has a neighborhood Q such that for every $e \in Q$ and every $V \in \mathscr{U}$ there exists a $W \in \mathscr{U}$ which satisfies: if e_1 and e_2 are any elements of $W(e)$ and $S \in \mathscr{U}$ satisfies $\rho(e_1) \subset \overline{S(A)}$, then $\rho(e_2) \subset \overline{VS(A)}$. Such a Q may be called a *zone of perstability* for A. If the W may be chosen independently of e, A is *uniformly* perstable. Loosely interpreted, perstability says that the greatest distance from $\rho(e)$ to A is a continuous function of e on Q. Thus it may be regarded as a kind of semicontinuity of ρ.

PROPOSITION 6.1. *The function v defined by (5.3) is [uniformly] continuous on a neighborhood Q of A if and only if Q is a zone of [uniform] perstability for A.*

Proof. (The proof for uniform continuity is analogous to that for simple continuity and will be omitted.) Suppose that Q is a zone of perstability for A. Let $e \in Q$ and $V \in \mathscr{U}$; then let $V_1 \in \mathscr{U}$ satisfy $V_1^2 \subset V$, and let W correspond to V_1 as in the definition of perstability. That is, $e_1, e_2 \in W(e) \cap Q$ and $\rho(e_1) \subset \overline{S(A)}$ imply $\rho(e_2) \subset \overline{V_1 S(A)}$. *A fortiori*, $\rho(e_1) \subset S(A)$ implies $\rho(e_2) \subset V_1^2 S(A) \subset VS(A)$. Putting $e_1 = e$ and $e_2 = e'$ and *vice versa*, where e' is an arbitrary element of $W(e) \cap Q$, now gives:

$$\rho(e) \subset S(A) \text{ implies } \rho(e') \subset VS(A) \ ;$$

$$\rho(e') \subset T(A) \text{ implies } \rho(e) \subset VT(A) \ .$$

That is, if $v'(e) = \{S \in \mathscr{U} : \rho(e) \subset S(A)\}$ we have

$$v'(e') \in \tilde{V}(v'(e)) \ ,$$

whence v' is continuous at e. Since v is the composite of w and v', it follows that v is continuous.

Conversely, suppose that v is continuous on some neighborhood Q of A. Let $e \in Q$ and $V \in \mathscr{U}$ be given. Again, choose $V_1 \in \mathscr{U}$ so that $V_1^2 \subset V$ and then choose a $W \in \mathscr{U}$ such that $v(e') \in \tilde{V}_1(v(e))$ if $e' \in W(e) \cap Q$. Now by (3.3) and (5.3),

$$\begin{aligned} v(e) &= w(\{S \in \mathscr{U} : \rho(e) \subset S(A)\}) \\ &= \{T \in \mathscr{U} : \rho(e) \subset UT(A) \text{ for all } U \in \mathscr{U}\} \ . \end{aligned}$$

Since $UT(A) = U(T(A))$ and $\overline{T(A)} = \cap\{U(T(A)) : U \in \mathscr{U}\}$, however, this gives

$$v(e) = \{T \in \mathscr{U} : \rho(e) \subset \overline{T(A)}\} ,$$

and similarly for e'. Accordingly, $v(e') \in \tilde{V}_1$ means that

$$\rho(e') \subset \overline{S(A)} \text{ implies } \rho(e) \subset \overline{V_1 S(A)} \tag{6.1}$$

and

$$\rho(e) \subset \overline{T(A)} \text{ implies } \rho(e') \subset \overline{V_1 T(A)} . \tag{6.2}$$

Now let $e_1, e_2 \in W(e) \cap Q$ and suppose that $\rho(e_1) \subset \overline{S(A)}$. Then by (6.1), $\rho(e) \subset \overline{V_1 S(A)}$. Then (6.2), with $V_1 S$ in place of T and e_2 in place of e', gives $\rho(e_2) \subset \overline{V_1^2 T(A)} \subset \overline{V(A)}$. In short, $\rho(e_1) \subset \overline{S(A)}$ implies $\rho(e_2) \subset \overline{VS(A)}$, and the proof is complete.

The stability of A is required neither in the definition of perstability nor in Proposition 6.1. In spite of the terminology, perstability does not imply stability (see Example 1 below). With Proposition 4.1, the following statement shows that it does not miss by far.

PROPOSITION 6.2. *If A has a uniform zone of uniform perstability and the closure of A is invariant, then A is stable.*

Proof. Suppose that Q is at the same time a uniform neighborhood and a zone of uniform perstability for A, and suppose that $\bar{A}$ is invariant. Let $V \in \mathscr{U}$ be given, and choose $V_1 \in \mathscr{U}$ so that $V_1^3 \subset V$. Then there exists a $W \in \mathscr{U}$ so that $W(A) \subset Q$ and, for all $e \in Q$ and $S \in \mathscr{U}$,

$$\rho(e) \subset \overline{S(A)} \text{ implies } \rho(W(e) \cap Q) \subset \overline{V_1 S(A)} . \tag{6.3}$$

If $e \in A$, then from the invariance of $\bar{A}$ we have

$$\rho(e) \subset \rho(A) \subset \rho(\bar{A}) \subset \bar{A} \subset \overline{S(A)} ,$$

so the antecedent of (6.3) holds automatically and

$$\rho(W(A)) \subset \overline{V_1 S(A)} \subset V_1^2 S(A)$$

for all $S \in \mathscr{U}$. Now if we take $S = V_1$ we obtain $\rho(W(A)) \subset V(A)$ and the stability condition is satisfied.

Example 1. Let $E = \{(x,y) : -\pi/2 \leq x < \pi/2\}$ in the real plane with the standard metric uniformity, and let ρ be the equivalence relation whose equivalence classes are the line $A = \{(x,y) : x = -\pi/2\}$ and the curves $\{(x,y) : y - \tan x = c\}$, $-\infty < c < +\infty$. The whole E is a uniform zone of perstability for the closed invariant set A, which however is not stable.

Example 2. Let E be the real plane with the standard metric uniformity and take ρ to be the equivalence relation whose equivalence classes are the singletons $\{(x,y)\}$ with irrational y and the lines $\{(x,y) : y = b\}$ with rational b. Let A be the x-axis and let Q be any neighborhood of A. Let $e = (0,\epsilon)$ be an interior point of Q, where $\epsilon > 0$. Finally, let

$$V = \{((x_1,y_1),(x_2,y_2)) : (x_1-x_2)^2 + (y_1-y_2)^2 < \epsilon^2/9\},$$

and choose $S \in \mathscr{U}$ so that

$$S(A) = \{(x,y) : |x| < \epsilon/2 \text{ or } |y| < \epsilon/2\}.$$

Then for any neighborhood $W(e)$, there exist points e_1 and e_2 of $W(e) \cap Q$ such that $\rho(e_1) \subset \overline{S(A)}$ but $\rho(e_2) \not\subset \overline{VS(A)}$: it suffices to take e_1 with an irrational ordinate and e_2 with a rational ordinate. Thus A is not perstable, but it is stable.

As an immediate corollary of Proposition 6.1, we have: *If A is [uniformly] perstable and stable, there exists a [uniformly] continuous Liapunov function for A into P_r.* I do not know whether the converse holds, but I suspect not.

Acknowledgment. A preliminary version of this paper used the full scale of $\mathscr{U}$ exclusively. The idea of using the retracted scale was suggested by D. C. Kent, who is also responsible for several of the results in §3, though in a somewhat different form. I am grateful for his willingness to have these improvements incorporated into this paper. Further work on full and retracted scales of uniformities will appear elsewhere.

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(Received 17 March 1966)

Applied Categorical Structures **11:** 267–286, 2003.
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Topological Features of Lax Algebras

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(Received: 29 May 2001; accepted: 2 October 2002)

Abstract. Having as starting point Barr's description of topological spaces as lax algebras for the ultrafilter monad, in this paper we present further topological examples of lax algebras – such as quasi-metric spaces, approach spaces and quasi-uniform spaces – and show that, in a suitable setting, the categories of lax algebras have indeed a topological nature. Furthermore, we generalize to this setting known properties of special categories of lax algebras and, extending the construction of Manes, we describe the Čech–Stone compactification of lax algebras.

Mathematics Subject Classifications (2000): 18C20, 18A40, 18B30, 54B30, 54A20, 54E15.

Key words: (lax) monad, (lax) algebra, ultrafilter monad, topological space, approach space, quasi-uniform space, Čech–Stone compactification.

1. Introduction

Monads, or triples, and the algebras they define, proved to be very important in several fields of mathematics, homological algebra being one of the fields where they have major applications.

In this paper we explore one topological facet of monads, which leads to a unified treatment of dispersed topological structures. Indeed, guided by Barr's [2] description of topological spaces and continuous maps, we show that several *topological structures* and their corresponding *continuous maps* may be described relaxing the axioms of algebras and algebra homomorphisms for a monad. Among these we obtain the category **Ap** of approach spaces and non-expansive maps, giving a new perspective of the relationship between **Top** and **Ap**. While the structure of a topological space is defined by saying whether a point is or is not a limit point of an ultrafilter, in an approach space this information is "numerified", so that we are given a value in $[0, \infty]$ which measures how far away a point is from being a limit point of an ultrafilter. This is reflected in our setting by the choice of the 2-category: **Top** can be obtained as the category of reflexive and transitive lax algebras for the extension of the ultrafilter monad to the 2-category **Rel** of

⋆ The authors acknowledge partial financial assistance by Centro de Matemática da Universidade de Coimbra.

relations, and **Ap** can be obtained as the category of (reflexive and transitive) lax algebras of the extension of the ultrafilter monad to the 2-category **NRel** of numerical relations. One further natural example is the category of quasi-metric spaces and non-expansive maps. Indeed, Lawvere's [9] description of these spaces shows clearly that they form exactly the category of reflexive and transitive lax algebras for the identity monad in **NRel**. Also quasi-uniform spaces are described as reflexive and transitive lax algebras for the identity monad in a convenient 2-category.

The occurrence of all these topological examples of lax algebras is not a coincidence: we show that, under natural assumptions, categories of lax algebras for a lax extension of a monad in **X** are topological over **X**. In order to compensate deficits of the category **Top**, one defines several supercategories by dropping axioms, like **PsTop** or **URS**, which have better properties. The same can be done in our setting, and we show that these supercategories are nice improvements. For instance, we show that, in a convenient setting, regular epimorphisms in categories of (reflexive) lax algebras are better behaved than in categories of reflexive and transitive lax algebras: they are pullback-stable.

Finally, extending the construction of Manes [12], we generalize the Čech–Stone compactification of topological spaces; that is, we construct the reflection of a reflexive and transitive lax algebra with respect to a lax monad into the corresponding category of algebras for the monad.

2. Categories of Lax Algebras

Throughout:

- **Set** $\hookrightarrow$ **Y** is a (non-full) embedding, bijective on objects, of the category of sets into a 2-category **Y** with thin 2-cells; we will denote its objects by $X, Y, \ldots$, its morphisms by $r : X \nrightarrow Y$, and, for simplicity, for $r, s \in \mathbf{Y}(X, Y)$ we will write $r \sqsubseteq s$ if there is a 2-cell $r \to s$.
- $\mathbb{T} = (T, e, m)$ is a monad in **Set**; that is, $T : \mathbf{Set} \to \mathbf{Set}$ is a functor, and $e : 1_{\mathbf{Set}} \to T$ and $m : T^2 \to T$ are natural transformations making the following diagrams commute

$$\begin{tikzcd} T^3 \arrow[r, "Tm"] \arrow[d, "mT"'] & T^2 \arrow[d, "m"] \\ T^2 \arrow[r, "m"] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "Te"] \arrow[dr, "1"'] & T^2 \arrow[d, "m"] & T \arrow[l, "eT"'] \arrow[dl, "1"] \\ & T & \end{tikzcd} \tag{1}$$

- The monad (T, e, m) can be lax extended into **Y**; that is:
  - there exists a lax functor $T : \mathbf{Y} \to \mathbf{Y}$ that extends T to **Y**, i.e., it coincides with the given functor $T : \mathbf{Set} \to \mathbf{Set}$ when restricted to **Set** and, for $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$ in **Y**, there is a two-cell $Tr \cdot Ts \to T(r \cdot s)$;
  - the natural transformations e and m become now op-lax:

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[d, "r"'] \arrow[dr, phantom, "\sqsubseteq"] & TX \arrow[d, "Tr"] \\ Y \arrow[r, "e_Y"'] & TY \end{tikzcd} \qquad \begin{tikzcd} T^2X \arrow[r, "m_X"] \arrow[d, "T^2r"'] \arrow[dr, phantom, "\sqsubseteq"] & TX \arrow[d, "Tr"] \\ T^2Y \arrow[r, "m_Y"'] & TY; \end{tikzcd} \tag{2}$$

this means that there are 2-cells $e_Y \cdot r \to Tr \cdot e_X$ and $m_Y \cdot T^2 r \to Tr \cdot m_X$.

An *algebra* for the monad $\mathbb{T}$ is a pair (X, a) where X is a set and $a : TX \nrightarrow X$ is a map making the following diagram commutative

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[dr, "1_X"'] & TX \arrow[d, "a"] & T^2X \arrow[l, "Ta"'] \arrow[d, "m_X"] \\ & X & TX. \arrow[l, "a"] \end{tikzcd} \tag{3}$$

A (homo)morphism $f : (X, a) \to (Y, b)$ of $\mathbb{T}$-algebras is a map $f : X \to Y$ such that the diagram

$$\begin{tikzcd} TX \arrow[r, "Tf"] \arrow[d, "a"'] & TY \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{4}$$

commutes. As usual we denote by $\mathbf{Set}^{\mathbb{T}}$ the category of $\mathbb{T}$-algebras and their morphisms. (For detailed information on monads see [11].)

A *lax algebra* for T is just a pair (X, a) where $a : TX \nrightarrow X$ is a morphism in $\mathbf{Y}$, while a *lax (homo)morphism* from (X, a) to (Y, b) is a **Set**-map $f : X \to Y$ such that

$$\begin{tikzcd} TX \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\sqsubseteq"] & TY \arrow[d, "b"] \\ X \arrow[r, "f"] & Y, \end{tikzcd} \tag{5}$$

that is, there exists a 2-cell $f \cdot a \to b \cdot Tf$. This way one defines the category $\mathrm{Alg}(T)$ of lax algebras and lax (homo)morphisms.

Relaxing the axioms of algebras for a monad, among lax algebras one may consider those that are *reflexive*, that is, those (X, a) for which there is a 2-cell $1_X \to a \cdot e_X$:

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[dr, "1_X"'] & TX \arrow[d, "a"] \arrow[dl, phantom, "\sqsubseteq"] \\ & X, \end{tikzcd} \tag{6}$$

and those that are *transitive*, that is, those (X, a) for which there is a 2-cell $a \cdot Ta \to a \cdot m_X$:

$$\begin{tikzcd} TX \arrow[d, "a"'] \arrow[dr, phantom, "\sqsubseteq"] & T^2X \arrow[l, "Ta"'] \arrow[d, "m_X"] \\ X & TX. \arrow[l, "a"] \end{tikzcd} \tag{7}$$

This way we define the (full) subcategories $\mathrm{Alg}(T, e)$ and $\mathrm{Alg}(T, e, m)$, of reflexive and of reflexive and transitive lax algebras, of $\mathrm{Alg}(T)$:

$$\mathrm{Alg}(T, e, m) \hookrightarrow \mathrm{Alg}(T, e) \hookrightarrow \mathrm{Alg}(T). \tag{8}$$

3. Examples

In this section we show that the structures described above encompass a great deal of topological life.

3.1. LAX ALGEBRAS ON **Rel**

In case that **Y** is the category **Rel** of relations, Barr [2] showed that, for the *identity monad*, $\mathrm{Alg}(\mathrm{Id}, 1, 1)$ is the category of preordered sets and monotone maps. He showed also that, for the *ultrafilter monad* (U, e, m) in **Set**, $\mathrm{Alg}(U, e, m)$ is the category of topological spaces and continuous maps.

We recall that $U : \mathbf{Set} \to \mathbf{Set}$ assigns to a set X its set of ultrafilters UX, and to each map $f : X \to Y$ the map $Uf : UX \to UY$ which assigns to an ultrafilter $\mathfrak{x} \in UX$ the ultrafilter $f(\mathfrak{x})$ generated by $\{f(A) \mid A \in \mathfrak{x}\}$. Moreover, for each set X,

$$\begin{array}{rcl} e_X : X & \to & UX \\ x & \mapsto & \dot{x} \end{array} \quad \text{and} \quad \begin{array}{rcl} m_X : UUX & \to & UX \\ \mathfrak{X} & \mapsto & m_X(\mathfrak{X}) = \bigcup\limits_{\mathcal{A} \in \mathfrak{X}} \bigcap\limits_{\mathfrak{x} \in \mathcal{A}} \mathfrak{x}, \end{array}$$

where $\dot{x}$ denotes the principal ultrafilter defined by x. It has a unique lax extension (which turns out to be strictly a functor) $U : \mathbf{Rel} \to \mathbf{Rel}$: for $r : X \nrightarrow Y$ and $\mathfrak{x} \in UX$ and $\mathfrak{y} \in UY$,

$$\mathfrak{x}(Ur)\mathfrak{y} :\Leftrightarrow \forall B \in \mathfrak{y}\{x \in X \mid \exists y \in B : xry\} \in \mathfrak{x},$$

such that the natural transformations e and m become op-lax (in fact, m remains strict).

Furthermore, one can easily show that:

– If one considers the identity monad in **Rel**, an (reflexive, reflexive and transitive, respectively) algebra (X, a) is just a *relation* (*reflexive relation*, *reflexive and transitive relation*) on X, and a lax morphism is exactly a map that preserves the relation; hence (8) becomes

$$\mathbf{Prord} \hookrightarrow \mathbf{SGrph} \hookrightarrow \mathbf{Grph},$$

where **Grph** is the category of (thin) graphs, **SGrph** is the category of reflexive (thin) graphs (also called spatial graphs), and **Prord** is the category of preordered sets.

– For the lax ultrafilter monad (U, e, m) in **Rel**, an (reflexive, reflexive and transitive, respectively) algebra (X, a) is exactly an *ultrarelational* – or grizzly – *space* (see [3] and [5]) (*pseudotopological space*, *topological space*, respectively), and a map is a lax morphism if and only if it is continuous, obtaining in (8)

$$\mathbf{Top} \hookrightarrow \mathbf{PsTop} \hookrightarrow \mathbf{URS}.$$

3.2. LAX ALGEBRAS ON NUMERICAL RELATIONS

Consider now the 2-category $\mathbf{Y} = \mathbf{NRel}$ whose:

– objects are sets;
– morphisms $d : X \nrightarrow Y$ from X to Y are maps $d : X \times Y \to [0, \infty]$;
– identity of X is

$$\begin{aligned} \Delta_X : X \times X &\to [0, \infty] \\ (x, x') &\mapsto \begin{cases} 0 & \text{if } x = x', \\ \infty & \text{otherwise;} \end{cases} \end{aligned}$$

– 2-cells: for $d_1, d_2 \in \mathbf{NRel}(X, Y)$, $d_1 \sqsubseteq d_2$ if, for each pair $(x, y) \in X \times Y$, $d_2(x, y) \leq d_1(x, y)$;
– composition law is defined as follows: for $d : X \nrightarrow Y$ and $d' : Y \nrightarrow Z$, $d' \cdot d : X \nrightarrow Z$ is given by

$$(d' \cdot d)(x, z) = \inf\{d(x, y) + d'(y, z) \mid y \in Y\}.$$

The reflexive and transitive lax algebras for the identity monad in **NRel** are the *quasi-metric spaces* (where the distance can also be ∞), while, for the natural extension of the ultrafilter monad, the reflexive and transitive lax algebras are the *approach spaces* (see [10]), as we show next.

THEOREM 3.1. *For the identity monad in* **NRel**, Alg(Id, 1, 1) *is isomorphic to the category of quasi-metric spaces and non-expansive maps.*

Proof. A lax algebra $a : X \times X \to [0, \infty]$ is reflexive if and only if

$$\Delta_X \sqsubseteq a \cdot 1_X \Leftrightarrow \forall x \in X \; a(x, x) = 0.$$

It is transitive if and only if

$$\begin{aligned} a \cdot a \sqsubseteq a \cdot 1_X \;\Leftrightarrow\; & (\forall x, z \in X) \\ & a(x, z) \leq (a \cdot a)(x, z) = \inf\{a(x, y) + a(y, z) \mid y \in X\} \\ \Leftrightarrow\; & (\forall x, y, z \in X) \; a(x, z) \leq a(x, y) + a(y, z). \end{aligned}$$

Hence a map $a : X \times X \to [0, \infty]$ is a reflexive and transitive lax algebra if and only if it is a quasi-metric space (also called *quasi-pseudometric space*).

Furthermore, given two lax algebras $a : X \nrightarrow X$ and $b : Y \nrightarrow Y$, a map $f : X \to Y$ is a lax morphism if and only if it is non-expansive, since

$$\begin{aligned} f \cdot a \sqsubseteq b \cdot f \;&\Leftrightarrow\; (\forall x \in X)\ (\forall y \in Y)\ (b \cdot f)(x, y) \le (f \cdot a)(x, y) \\ &\Leftrightarrow\; (\forall x \in X)\ (\forall y \in Y) \\ &\qquad \inf_{y' \in Y} (f(x, y') + b(y', y)) \le \inf_{x' \in X} (a(x, x') + f(x', y)) \\ &\Leftrightarrow\; (\forall x \in X)\ (\forall y \in Y) \\ &\qquad b(f(x), y) \le \inf\{a(x, x') \mid x' \in X,\ f(x') = y\} \\ &\Leftrightarrow\; (\forall x \in X)\ (\forall x' \in X)\ b(f(x), f(x')) \le a(x, x'). \end{aligned}$$

□

We will now define a lax extension into **NRel** of the ultrafilter monad (U, e, m) in **Set**. In order to do this we will make use of the following result, which depends of choice (it follows in particular from the Prime Ideal Theorem: see Section 2 of [8]).

LEMMA 3.2. *If $r : X \nrightarrow Y$ is a relation, $\mathfrak{f}$ is a filter in X and $\mathfrak{y}$ is an ultrafilter in Y such that $r(\mathfrak{f}) \subseteq \mathfrak{y}$, then there exists an ultrafilter $\mathfrak{x}$ in X containing $\mathfrak{f}$ and such that $r(\mathfrak{x}) \subseteq \mathfrak{y}$.*

To define the lax functor U : **NRel** $\to$ **NRel**, for each $d : X \times Y \to [0, \infty]$ and each $\alpha \in [0, \infty]$, we consider the relation $d_\alpha : X \nrightarrow Y$ defined by $x d_\alpha y$ if $d(x, y) \le \alpha$. As usual, for each subset A of X and each subset $\mathcal{A}$ of the powerset PX of X, we define

$$d_\alpha(A) := \{y \in Y \mid \exists x \in A : d(x, y) \le \alpha\},$$

and $d_\alpha(\mathcal{A}) := \{d_\alpha(A) \mid A \in \mathcal{A}\}$.

PROPOSITION 3.3. *The assignment*

$$\begin{aligned} U(d) : UX \times UY \;&\to\; [0, \infty] \\ (\mathfrak{x}, \mathfrak{y}) \;&\mapsto\; \inf\{\alpha \in [0, \infty] \mid d_\alpha(\mathfrak{x}) \subseteq \mathfrak{y}\}, \end{aligned} \tag{9}$$

*defines a lax extension U : **NRel** $\to$ **NRel** of the ultrafilter monad (U, e, m) in* **Set**.

Proof. The only non-trivial part of the proof is to show that, for $d : X \times Y \to [0, \infty]$ and $d' : Y \times Z \to [0, \infty]$,

$$U(d') \cdot U(d) \sqsubseteq U(d' \cdot d);$$

that is, for $\mathfrak{x} \in UX$ and $\mathfrak{z} \in UZ$,

$$\alpha_1 := U(d' \cdot d)(\mathfrak{x}, \mathfrak{z}) \le U(d') \cdot U(d)(\mathfrak{x}, \mathfrak{z}) =: \alpha_2.$$

Let $\varepsilon > 0$. By definition of composition, there exists $\mathfrak{y} \in UY$ such that

$$U(d)(\mathfrak{x}, \mathfrak{y}) + U(d')(\mathfrak{y}, \mathfrak{z}) \leq \alpha_2 + \varepsilon.$$

Let $\beta_1 := U(d)(\mathfrak{x}, \mathfrak{y})$ and $\beta_2 := U(d')(\mathfrak{y}, \mathfrak{z})$. By definition of U, for each $A \in \mathfrak{x}$,

$$d'_{\beta_2+\varepsilon}(d_{\beta_1+\varepsilon}(A)) \in \mathfrak{z}.$$

From

$$\begin{aligned} d'_{\beta_2+\varepsilon}(d_{\beta_1+\varepsilon}(A)) &= d'_{\beta_2+\varepsilon}(\{y \in Y \mid \exists x \in A : d(x, y) \leq \beta_1 + \varepsilon\}) \\ &= \{z \in Z \mid \exists y \in Y \exists x \in A : d(x, y) \leq \beta_1 + \varepsilon \text{ and} \\ &\qquad d'(y, z) \leq \beta_2 + \varepsilon\} \\ &\subseteq \{z \in Z \mid \exists x \in A : (d' \cdot d)(x, z) \leq \beta_1 + \beta_2 + 2\varepsilon\} \\ &= (d' \cdot d)_{\beta_1+\beta_2+2\varepsilon}(A), \end{aligned}$$

it follows that $(d' \cdot d)_{\beta_1+\beta_2+2\varepsilon}(A) \in \mathfrak{z}$. Therefore

$$\alpha_1 \leq \beta_1 + \beta_2 + 2\varepsilon \leq \alpha_2 + 2\varepsilon,$$

hence $\alpha_1 \leq \alpha_2$. □

A lax algebra for this monad is a pair (X, a), where X is a set and $a : UX \times X \to [0, \infty]$ is a map. Moreover, (X, a) is reflective if and only if

$$\begin{aligned} \Delta_X \sqsubseteq a \cdot e_X \;&\Leftrightarrow\; (\forall x, x' \in X)\; a \cdot e_X(x, x') \leq \Delta_X(x, x') \\ &\Leftrightarrow\; (\forall x \in X)\; a(\dot{x}, x) = 0. \end{aligned}$$

It is transitive if and only if

$$\begin{aligned} a \cdot Ua \sqsubseteq m_X \cdot a \;&\Leftrightarrow\; (\forall x \in X)\; (\forall \mathfrak{X} \in UUX) \\ &\qquad (a \cdot m_X)\; (\mathfrak{X}, x) \leq (a \cdot Ua)(\mathfrak{X}, x) \\ &\Leftrightarrow\; (\forall \mathfrak{X} \in UUX)\; (\forall \mathfrak{x} \in UX)\; (\forall x \in X) \\ &\qquad a(m_X(\mathfrak{X}), x) \leq Ua(\mathfrak{X}, \mathfrak{x}) + a(\mathfrak{x}, x). \end{aligned}$$

A lax morphism $f : (X, a) \to (Y, b)$ is a map $f : X \to Y$ such that

$$\begin{aligned} f \cdot a \sqsubseteq b \cdot Uf \;&\Leftrightarrow\; (\forall x \in X)\; (\forall \mathfrak{x} \in UX)\; b \cdot Tf(\mathfrak{x}, x) \leq f \cdot a(\mathfrak{x}, x) \\ &\Leftrightarrow\; (\forall x \in X)\; (\forall \mathfrak{x} \in UX)\; b(f(\mathfrak{x}), x) \leq a(\mathfrak{x}, x). \end{aligned}$$

In order to show that $\mathrm{Alg}(U, e, m)$ is isomorphic to the category of approach spaces, we first recall the definition of the latter. (For more detailed information on approach spaces see [10].)

DEFINITION 3.4. The category **Ap** is the category with objects *approach spaces*, that is, pairs (X, δ) consisting of a set X and a map $\delta : PX \times X \to [0, \infty]$ satisfying the following axioms, for each $x \in X$, $A, B \in PX$ and $\alpha \in [0, \infty]$:

(A1) $\delta(\{x\}, x) = 0$,
(A2) $\delta(\emptyset, x) = \infty$,
(A3) $\delta(A \cup B, x) = \min\{\delta(A, x), \delta(B, x)\}$,
(A4) $\delta(A, x) \leq \delta(A^{(\alpha)}, x) + \alpha$,

where $A^{(\alpha)} = \{x \in X \mid \delta(A, x) \leq \alpha\}$, and with morphisms $f : (X, \delta) \to (Y, \gamma)$ *contrations* (=*non-expansive maps*), i.e., maps $f : X \to Y$ such that, for each subset A and each point x of X,

$$\gamma(f(A), f(x)) \leq \delta(A, x).$$

Each map $\delta : P(X) \times X \to [0, \infty]$ defines a lax algebra

$$\begin{aligned} a_\delta : UX \times X &\to [0, \infty] \\ (\mathfrak{x}, x) &\mapsto \sup\{\delta(A, x) \mid A \in \mathfrak{x}\}, \end{aligned}$$

and each lax algebra $a : UX \times X \to [0, \infty]$ induces a map

$$\begin{aligned} \delta_a : PX \times X &\to [0, \infty] \\ (A, x) &\mapsto \inf\{a(\mathfrak{x}, x) \mid A \in \mathfrak{x}\}, \end{aligned}$$

that satisfies obviously (A2) and (A3).

We will split the proof that the above assignments define an isomorphism between **Ap** and $\mathrm{Alg}(U, e, m)$ in several steps (see also [10]):

1. *If $f : (X, \delta) \to (Y, \gamma)$ is a contraction, then $f : (X, a_\delta) \to (Y, a_\gamma)$ is a lax morphism.*
   If $\mathfrak{x} \in UX$ and $x \in X$, from $\gamma(f(A), f(x)) \leq \delta(A, x)$ it follows that $a_\gamma(f(\mathfrak{x}), f(x)) \leq a_\delta(\mathfrak{x}, x)$.
2. *If $f : (X, a) \to (Y, b)$ is a lax morphism, then $f : (X, \delta_a) \to (Y, \delta_b)$ is a contraction.*
   Let $A \in PX$ and $x \in X$. For each $\mathfrak{y} \in UY$ with $f(A) \in \mathfrak{y}$ there is an ultrafilter $\mathfrak{x} \in UX$ such that $f(\mathfrak{x}) = \mathfrak{y}$ and the result follows.
3. *If $\delta : PX \times X \to [0, \infty]$ satisfies axioms* (A1) *and* (A2)*, then $\delta = \delta_{a_\delta}$.*
   (a) $\delta(A, x) \leq \delta_{a_\delta}(A, x)$: for each ultrafilter $\mathfrak{x} \in UX$ with $A \in \mathfrak{x}$, one has $a_\delta(\mathfrak{x}, x) \geq \delta(A, x)$, hence
   $$\delta(A, x) \leq \inf\{a_\delta(\mathfrak{x}, x) \mid A \in \mathfrak{x}\}.$$
   (b) $\delta_{a_\delta}(A, x) \leq \delta(A, x)$: in case $\delta(A, x) = \infty$, the inequality trivially holds; otherwise, the set $\mathcal{I} = \{B \in P(X) \mid \delta(B, x) > \delta(A, x)\}$ is an ideal disjoint from the filter $\dot{A}$, hence there exists an ultrafilter $\mathfrak{x}$ with $\dot{A} \subseteq \mathfrak{x}$

disjoint from $\mathcal{I}$; by definition of a_δ we then have that $a_\delta(\mathfrak{x}, x) \leq \delta(A, x)$, hence $\delta_{a_\delta}(A, x) \leq a_\delta(\mathfrak{x}, x) \leq \delta(A, x)$.

4. *If $a : UX \times X \to [0, \infty]$ is a transitive lax algebra, then $a = a_{\delta_a}$.*

   (a) $a \sqsubseteq a_{\delta_a}$: Given $\mathfrak{x} \in UX$, $A \in \mathfrak{x}$ and $x \in X$, $\delta_a(A, x) \leq a(\mathfrak{x}, x)$ by definition of δ_a; hence

   $$a(\mathfrak{x}, x) \geq \sup\{\delta_a(A, x) \mid A \in \mathfrak{x}\} = a_{\delta_a}(\mathfrak{x}, x);$$

   (b) $a_{\delta_a} \sqsubseteq a$: Let $\mathfrak{x} \in UX$, $x \in X$, $\alpha = a_{\delta_a}(\mathfrak{x}, x)$ and $\varepsilon > 0$. By definition of δ_a, for each $A \in \mathfrak{x}$ there exists an ultrafilter $\mathfrak{y}_A$ in X with $A \in \mathfrak{y}_A$ and $a(\mathfrak{x}_A, x) \leq \alpha + \varepsilon$; that is, the set

   $$\mathcal{A}_A := \{\mathfrak{y} \in UX \mid A \in \mathfrak{y} \text{ and } a(\mathfrak{y}, x) \leq \alpha + \varepsilon\}$$

   is non-empty. By construction, the filterbase $\mathcal{F} = \{\mathcal{A}_A \mid A \in \mathfrak{x}\}$ is such that $a_{\alpha+\varepsilon}(\mathcal{F}) \subseteq \dot{x}$. Therefore $\mathcal{F}$ can be refined to an ultrafilter $\mathfrak{X}$ on UX such that $a_{\alpha+\varepsilon}(\mathfrak{X}) \subseteq \dot{x}$, hence $Ua(\mathfrak{X}, \dot{x}) \leq \alpha + \varepsilon$. We remark that, by construction, $m_X(\mathfrak{X})$ is exactly $\mathfrak{x}$. We therefore obtain, due to the transitivity of the lax algebra a,

   $$a(\mathfrak{x}, x) = a(m_X(\mathfrak{X}), x) \leq Ua(\mathfrak{X}, \dot{x}) + a(\dot{x}, x) \leq \alpha + \varepsilon.$$

5. *If (X, δ) is an approach space, then (X, a_δ) is a reflexive and transitive lax algebra.*

   It is clear that a_δ is reflexive. To show its transitivity, consider $\mathfrak{X} \in UUX$, $\mathfrak{x} \in UX$ and $x \in X$, and put $\alpha_1 := Ua_\delta(\mathfrak{X}, \mathfrak{x})$ and $\alpha_2 := a_\delta(\mathfrak{x}, x)$. Now let $B \in m_X(\mathfrak{X})$ and $\varepsilon > 0$. By definition of m_X,

   $$\mathcal{B} = \{\mathfrak{y} \in UX \mid B \in \mathfrak{y}\} \in \mathfrak{X},$$

   and therefore $(a_\delta)_{\alpha_1+\varepsilon}(\mathcal{B}) \in \mathfrak{x}$. Since

   $$\begin{aligned}(a_\delta)_{\alpha_1+\varepsilon}(\mathcal{B}) &= \{x \in X \mid \exists \mathfrak{y} \in \mathcal{B} : a_\delta(\mathfrak{y}, x) \leq \alpha_1 + \varepsilon\} \\ &\subseteq \{x \in X \mid \delta(B, x) \leq \alpha_1 + \varepsilon\} \\ &= B^{(\alpha_1+\varepsilon)},\end{aligned}$$

   also $B^{(\alpha_1+\varepsilon)}$ belongs to $\mathfrak{x}$. Hence $\delta(B^{(\alpha_1+\varepsilon)}, x) \leq \alpha_2$, and by (A4) we obtain

   $$\delta(B, x) \leq \delta(B^{(\alpha_1+\varepsilon)}, x) + \alpha_1 + \varepsilon \leq \alpha_2 + \alpha_1 + \varepsilon.$$

   Finally, by definition of a_δ,

$$\begin{aligned} a_\delta(m_X(\mathfrak{X}), x) &= \sup\{\delta(B, x) \mid B \in m_X(\mathfrak{X})\} \\ &\leq \alpha_1 + \alpha_2 \\ &\leq Ua_\delta(\mathfrak{x}, \mathfrak{x}) + a_\delta(\mathfrak{x}, x). \end{aligned}$$

6. *If (X, a) is a reflexive and transitive algebra, then (X, δ_a) is an approach space.* Reflexivity of a clearly implies that δ_a satisfies (A1). It remains to be shown that it also satisfies (A4). To show this, let $A \subseteq X$, $x \in X$, $\alpha \in [0, \infty]$ and $\varepsilon > 0$. By definition of δ_a, there exists $\mathfrak{x}_0 \in UX$ such that $A^{(\alpha)} \in \mathfrak{x}_0$ and

$$a(\mathfrak{x}_0, x) \leq \delta_a(A^{(\alpha)}, x) + \varepsilon.$$

Now, by definition of $A^{(\alpha)}$, for each $a \in A^{(\alpha)}$ there exists an ultrafilter $\mathfrak{x}_a$ such that $A \in \mathfrak{x}_a$ and $a(\mathfrak{x}_a, a) \leq \alpha + \varepsilon$. For $\mathcal{A} = \{\mathfrak{x} \in UX \mid A \in \mathfrak{x}\}$, one has

$$a_{\alpha+\varepsilon}(\mathcal{A}) \supseteq A^{(\alpha)} \in \mathfrak{x}_0.$$

Hence, by Lemma 3.2, there exists an ultrafilter $\mathfrak{X}$ in UX such that $\mathcal{A} \in \mathfrak{X}$ and $a_{\alpha+\varepsilon}(\mathfrak{X}) \subseteq \mathfrak{x}_0$, which implies that

$$Ua(\mathfrak{X}, \mathfrak{x}_0) \leq \alpha + \varepsilon.$$

We finally have that

$$\begin{aligned} \delta_a(A, x) &\leq a(m_X(\mathfrak{X}), x) \\ &\leq Ua(\mathfrak{X}, \mathfrak{x}_0) + a(\mathfrak{x}_0, x) \\ &\leq \alpha + \varepsilon + \delta_a(A^{(\alpha)}, x) + \varepsilon \\ &\leq \delta_a(A^{(\alpha)}, x) + \alpha + 2\varepsilon. \end{aligned}$$

We have therefore shown that:

THEOREM 3.5. *For the lax extension* (9) *to* **NRel** *of the ultrafilter monad,* $\mathrm{Alg}(U, e, m)$ *is isomorphic to the category of approach spaces and contraction maps.*

3.3. "QUASI-UNIFORM" STRUCTURES AS LAX ALGEBRAS

Having as starting point the category **Rel** of relations, one can define a 2-category **Y** as follows:

– objects are sets;

- morphisms $\mathfrak{r} : X \nrightarrow Y$ are (possibly improper) filters in $\mathbf{Rel}(X, Y)$;
- identity of X is $\uparrow \mathrm{id}_X$, where id_X is the identity map on the set X;
- 2-cells: for $\mathfrak{r}, \mathfrak{r}' \in \mathbf{Y}(X, Y)$, $\mathfrak{r} \sqsubseteq \mathfrak{r}'$ if $\mathfrak{r}' \subseteq \mathfrak{r}$;
- composition law: for $\mathfrak{r} : X \nrightarrow Y$ and $\mathfrak{s} : Y \nrightarrow Z$, $\mathfrak{s} \cdot \mathfrak{r} : X \nrightarrow Z$ is the filter generated by $\{s \cdot r \mid s \in \mathfrak{s} \text{ and } r \in \mathfrak{r}\}$.

We remark that relations (in particular, maps) $r : X \nrightarrow Y$ are identified with the $\mathbf{Y}$-morphisms

$$\uparrow r = \{s \in \mathbf{Rel}(X, Y) \mid s \geq r\}.$$

THEOREM 3.6. *If* $\mathbf{Y}$ *is the* 2*-category described above, then* Alg(Id, 1, 1) *is isomorphic to the category of quasi-uniform spaces and uniformly continuous maps.*

Proof. A lax algebra $\mathfrak{r} : X \nrightarrow X$ is reflexive if and only if

$$\begin{aligned} \uparrow \mathrm{id}_X \sqsubseteq \mathfrak{r} \;&\Leftrightarrow\; \mathfrak{r} \subseteq \uparrow \mathrm{id}_X \\ &\Leftrightarrow\; (\forall r \in \mathfrak{r})(\forall x \in X)\; xrx. \end{aligned}$$

It is transitive if and only if

$$\begin{aligned} \mathfrak{r} \cdot \mathfrak{r} \sqsubseteq \mathfrak{r} \;&\Leftrightarrow\; \mathfrak{r} \subseteq \mathfrak{r} \cdot \mathfrak{r} \\ &\Leftrightarrow\; (\forall r \in \mathfrak{r})\; (\exists s, s' \in \mathfrak{r}) : s \cdot s' \leq r \\ &\Leftrightarrow\; (\forall r \in \mathfrak{r})\; (\exists s \in \mathfrak{r}) : s \cdot s \leq r. \end{aligned}$$

That is, $\mathfrak{r} : X \nrightarrow X$ is a reflexive and transitive lax algebra if and only if it is a quasi-uniformity on X.

Given two lax algebras $\mathfrak{r} : X \nrightarrow X$ and $\mathfrak{s} : Y \nrightarrow Y$, a map $f : X \to Y$ is a lax morphism if and only if

$$\begin{aligned} \uparrow f \cdot \mathfrak{r} \sqsubseteq \mathfrak{s} \cdot \uparrow f \;&\Leftrightarrow\; \mathfrak{s} \cdot \uparrow f \subseteq \uparrow f \cdot \mathfrak{r} \\ &\Leftrightarrow\; (\forall s \in \mathfrak{s})\; (\exists r \in \mathfrak{r}) : s \cdot f \geq f \cdot r \\ &\Leftrightarrow\; (\forall s \in \mathfrak{s})\; (\exists r \in \mathfrak{r}) : (\forall x, x' \in X)\; xrx' \Rightarrow f(x)sf(x'). \end{aligned}$$

Hence, $f : (X, \mathfrak{r}) \to (Y, \mathfrak{s})$ is a lax morphism if and only if it is a uniformly continuous map. □

As in the previous examples, if we now replace the category **Rel** by the category **NRel** and proceed as above, we obtain a natural notion of *quasi-uniform approach space*. A convenient construction of quasi-uniform structures and their properties will be the subject of a subsequent paper [4].

4. Properties of the Categories of Lax Algebras

In all the previous examples we are given:

– a category $\mathbf{Y}$ such that:

  (a) for every $X, Y \in \mathbf{Y}$, $\mathbf{Y}(X, Y)$ is a complete preordered set;
  (b) there exists a pseudo-involution $(\,)^\circ : \mathbf{Y} \to \mathbf{Y}^{\mathrm{op}}$;

– a category $\mathbf{X}$ having as objects the objects of $\mathbf{Y}$ and as morphisms the $\mathbf{Y}$-morphisms satisfying $f \cdot f^\circ \sqsubseteq 1_Y$ and $1_X \sqsubseteq f^\circ \cdot f$ (i.e., an $\mathbf{Y}$-morphism belongs to $\mathbf{X}$ if and only if it is a left adjoint), and
– a lax monad $(T, e, m) : \mathbf{Y} \to \mathbf{Y}$ such that:
  (c) T commutes with $(\,)^\circ$, i.e., $T(r^\circ) = (Tr)^\circ$.

Under these assumptions, one can easily show:

LEMMA 4.1. (1) *For $X, Y \in \mathbf{X}$, $\mathbf{X}(X, Y)$ is a discrete category, hence $\mathbf{X}$ is just an ordinary category.*
(2) *If $f \in \mathbf{X}(X, Y)$ and $r \in \mathbf{Y}(Y, Z)$, then $T(r \cdot f) = Tr \cdot Tf$.*
(3) *T maps $\mathbf{X}$-morphisms into $\mathbf{X}$-morphisms.*

We assume further that:

(d) The natural transformations e and m are pointwise in $\mathbf{X}$, defining then a monad (T, e, m) in $\mathbf{X}$.

From now on we assume we are given a 2-category $\mathbf{Y}$ and a lax monad $T : \mathbf{Y} \to \mathbf{Y}$ satisfying conditions (a)–(d), and consider the category $\mathbf{X}$ defined as above.

In order to study the properties of the categories of lax algebras of (8), we first state some auxiliary results.

LEMMA 4.2. *If $a : TX \nrightarrow X$ is a morphism in $\mathbf{Y}$, then:*

(1) *a is a reflexive lax algebra if and only if $e_X^\circ \sqsubseteq a$, which in turn implies $a \sqsubseteq a \cdot Ta \cdot m_X^\circ$;*
(2) *a is a transitive lax algebra if and only if $a \cdot Ta \cdot m_X^\circ \sqsubseteq a$.*

Proof. (1) If $1_X \sqsubseteq a \cdot e_X$, then $e_X^\circ \sqsubseteq a \cdot e_X \cdot e_X^\circ \sqsubseteq a$, since e_X belongs to $\mathbf{X}$. Conversely, if $e_X^\circ \sqsubseteq a$, then $1_X \sqsubseteq e_X^\circ \cdot e_X \sqsubseteq a \cdot e_X$.

Finally, if $e_X^\circ \sqsubseteq a$, then

$$a^\circ = m_X \cdot Te_X \cdot a^\circ \sqsubseteq m_X \cdot Ta^\circ \cdot a^\circ = (a \cdot Ta \cdot m_X^\circ)^\circ,$$

from which it follows that $a \sqsubseteq a \cdot Ta \cdot m_X^\circ$.

The proof of (2) is similar. □

Pure routine computation shows that:

LEMMA 4.3. *If, in the diagram*

$$\begin{tikzcd} TX \arrow[r, "Tf"] \arrow[d, "a"'] & TY \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{10}$$

$f : X \to Y$ is an **X***-morphism, then the following conditions are equivalent:*

(i) *$f : (X, a) \to (Y, b)$ is a lax morphism, that is $f \cdot a \sqsubseteq b \cdot Tf$;*
(ii) *$a \sqsubseteq f^\circ \cdot b \cdot Tf$;*
(iii) *$f \cdot a \cdot Tf^\circ \sqsubseteq b$;*
(iv) *$Tf \cdot a^\circ \sqsubseteq b^\circ \cdot f$.* □

THEOREM 4.4. *The full embeddings*

$$\mathrm{Alg}(T, e, m) \hookrightarrow \mathrm{Alg}(T, e) \hookrightarrow \mathrm{Alg}(T)$$

are reflective. Moreover, the reflections have underlying identity morphisms.

Proof. The inclusion $I : \mathrm{Alg}(T, e) \hookrightarrow \mathrm{Alg}(T)$ has as left adjoint

$$\begin{aligned} F : \mathrm{Alg}(T) &\longrightarrow \mathrm{Alg}(T, e) \\ (X, a) &\longmapsto (X, a \vee e_X^\circ) \\ f : (X, a) \to (Y, b) &\longmapsto f : (X, a \vee e_X^\circ) \to (b \vee e_Y^\circ); \end{aligned}$$

that F is well-defined follows from the observation that, if $a \sqsubseteq f^\circ \cdot b \cdot Tf$, then, since

$$f \cdot e_X^\circ \sqsubseteq e_Y^\circ \cdot e_Y \cdot f \cdot e_X^\circ = e_Y^\circ \cdot Tf \cdot e_X \cdot e_X^\circ \sqsubseteq e_Y^\circ \cdot Tf,$$

one has

$$e_X^\circ \sqsubseteq f^\circ \cdot f \cdot e_X^\circ \sqsubseteq f^\circ \cdot e_Y^\circ \cdot Tf.$$

Therefore

$$a \vee e_X^\circ \sqsubseteq f^\circ \cdot (b \vee e_Y^\circ) \cdot Tf,$$

and so F is a functor, that is obviously left adjoint to I.

The left adjoint to $\mathrm{Alg}(T, e, m) \hookrightarrow \mathrm{Alg}(T, e)$ is obtained by iteration of the (pointed) endofunctor

$$\begin{aligned} G : \mathrm{Alg}(T, e) &\longrightarrow \mathrm{Alg}(T, e) \\ (X, a) &\longmapsto (X, a \cdot Ta \cdot m_X^\circ) \\ f : (X, a) \to (Y, b) &\longmapsto f : (X, a \cdot Ta \cdot m_X^\circ) \to (Y, b \cdot Tb \cdot m_Y^\circ); \end{aligned}$$

this defines a functor, since, from $f \cdot a \sqsubseteq b \cdot Tf$ it follows that

$$\begin{aligned} f \cdot (a \cdot Ta \cdot m_X^\circ) &\sqsubseteq b \cdot Tf \cdot Ta \cdot m_X^\circ \\ &\sqsubseteq b \cdot T(f \cdot a) \cdot m_X^\circ \\ &\sqsubseteq b \cdot T(b \cdot Tf) \cdot m_X^\circ \\ &= b \cdot Tb \cdot T^2 f \cdot m_X^\circ \\ &\sqsubseteq (b \cdot Tb \cdot m_X^\circ) \cdot Tf; \end{aligned}$$

Lemma 4.2 assures that $1_X : (X, a) \to (X, a \cdot Ta \cdot m_X^\circ)$ is a lax morphism whenever (X, a) is a reflexive lax algebra, hence $\alpha = (1_X : (X, a) \to (X, a \cdot Ta \cdot m_X^\circ))_{(X,a)\in \mathrm{Alg}(T,e)} : 1 \to G$ is a natural transformation as claimed.

Finally, since there is only a set of possible lax algebra structures on each object X of $\mathbf{X}$, the iteration of G will eventually stop, defining the claimed left adjoint. □

THEOREM 4.5. *The canonical forgetful functors from* $\mathrm{Alg}(T)$, $\mathrm{Alg}(T, e)$ *and* $\mathrm{Alg}(T, e, m)$ *into* $\mathbf{X}$ *are topological functors.*

Proof. Since the following diagram is commutative

$$\begin{tikzcd} \mathrm{Alg}(T,e,m) \arrow[r, shift right, "\perp"'] \arrow[dr] & \mathrm{Alg}(T,e) \arrow[l, shift right, "\perp"'] \arrow[r, shift right, "\perp"'] \arrow[d] & \mathrm{Alg}(T) \arrow[l, shift right, "\perp"'] \arrow[dl] \\ & \mathbf{X} & \end{tikzcd} \tag{11}$$

it is enough to show that the functor $|\ \ | : \mathrm{Alg}(T) \to \mathbf{X}$ is topological (see [1]).

Given $(f_i : X \to (Y_i, b_i))_{i\in I}$, each of the lax algebra structures $a_i := f_i^\circ \cdot b_i \cdot Tf_i$ makes $f_i : (X, a_i) \to (Y_i, b_i)$ a morphism. Furthermore

$$a := \bigwedge_{i\in I} f_i^\circ \cdot b_i \cdot Tf_i,$$

clearly gives the $|\ \ |$-initial lifting. □

5. Pullback Stability of Regular Epimorphisms in $\mathrm{Alg}(T, e)$

One reason for considering **PsTop** a good improvement of **Top** is its pullback-stability of regular epimorphisms. We are now going to show that this property is shared by every category $\mathrm{Alg}(T, e)$ in our setting, provided that it fulfills, in addition to conditions (a)–(d) of the previous section, the following conditions:

(e) $\mathbf{X}$ has pullbacks and the functors $\mathbf{X} \hookrightarrow \mathbf{Y}$ and $T : \mathbf{X} \to \mathbf{Y}$ have *Beck–Chevalley Property* (BCP); that is, if

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "k"'] & W \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{12}$$

is a pullback diagram, then $g^\circ \cdot f = h \cdot k^\circ$ and $Tg^\circ \cdot Tf = Th \cdot Tk^\circ$;

(f) in $\mathbf{X}$ regular epimorphisms are stable under pullback, and every regular epimorphism $f : X \to Y$ satisfies the equality $f \cdot f^\circ = 1_Y$;

(g) for every diagram in $\mathbf{Y}$, with $f, g \in \mathbf{X}$,

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "r"'] & W \arrow[d, "s"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{13}$$

$$f \cdot r \cdot g^\circ \wedge s \sqsubseteq f \cdot (r \wedge f^\circ \cdot s \cdot g) \cdot g^\circ.$$

We first point out that these conditions are fulfilled by the category **Set** as well as by the monads considered in the examples. (To prove this in the case of the ultrafilter monad one makes use of Lemma 3.2.)

We also remark that:

- from (BCP) it follows that every monomorphism $f : X \to Y$ in $\mathbf{X}$ satisfies $f^\circ \cdot f = 1_X$, hence an $\mathbf{X}$-morphism is a monomorphism in $\mathbf{X}$ if and only if it is a split monomorphism in $\mathbf{Y}$;
- condition (f) – which is in fact an equality since the other implication is trivially satisfied – follows from *Freyd Modular Laws*

$$s \cdot r \wedge t \leq s \cdot (r \wedge s^\circ \cdot t) \text{ and } s \cdot r \wedge t \leq (s \wedge t \cdot r^\circ) \cdot r \tag{FML}$$

(see [6]), but it is not equivalent. In fact, in the category $\mathbf{Y}$ of Example 2.2, (f) holds true although (FML) fails.

PROPOSITION 5.1. *A lax morphism $f : (X, a) \to (Y, b)$ in* $\mathrm{Alg}(T, e)$ *is a regular epimorphism if and only if*

(1) *f is a regular epimorphism in $\mathbf{X}$, and*
(2) $b = f \cdot a \cdot Tf^\circ$.

Proof. Let $f : (X, a) \to (Y, b)$ be a regular epimorphism in $\mathrm{Alg}(T, e)$. Since the forgetful functor $\mathrm{Alg}(T, e) \to \mathbf{X}$ preserves colimits, f is necessarily a regular epimorphism in $\mathbf{X}$. To prove that condition (2) is also necessary, it is enough to show that $f \cdot a \cdot Tf^\circ : TY \nrightarrow Y$ is a reflexive algebra provided that $a : TX \nrightarrow X$ is: if $e_X^\circ \sqsubseteq a$, then

$$e_Y^\circ = f \cdot f^\circ \cdot e_Y^\circ = f \cdot e_X^\circ \cdot Tf^\circ \sqsubseteq f \cdot a \cdot Tf^\circ.$$

Now, assuming (1) and (2), $b = f \cdot a \cdot Tf^\circ$ is clearly the final structure for $f : (X, a) \to Y$, and the result follows. □

THEOREM 5.2. *The regular epimorphisms are pullback-stable in* $\mathrm{Alg}(T, e)$.

Proof. Consider the diagram

$$\begin{tikzcd}
 & TW \arrow[rr, "Th"] \arrow[dl, "d"'] \arrow[dd, "Tk"'] & & TZ \arrow[dl, "c"'] \arrow[dd, "Tg"] \\
W \arrow[rr, "h"] \arrow[dd, "k"'] & & Z \arrow[dd, "g"'] & \\
 & TX \arrow[rr, "Tf"] \arrow[dl, "a"'] & & TY \arrow[dl, "b"'] \\
X \arrow[rr, "f"] & & Y &
\end{tikzcd} \tag{14}$$

where the front square is a pullback and $f : (X, a) \to (Y, b)$ is a regular epimorphism. By the construction of initial and final structures,

$$d = (k^\circ \cdot a \cdot Tk) \wedge (h^\circ \cdot c \cdot Th),$$

and $b = f \cdot a \cdot Tf^\circ$. We then have

$$\begin{aligned}
c &\sqsubseteq g^\circ \cdot b \cdot Tg \quad \text{(because } g \text{ is a morphism)} \\
&= g^\circ \cdot f \cdot a \cdot Tf^\circ \cdot Tg \\
&= h \cdot k^\circ \cdot a \cdot Tk \cdot Th^\circ \quad \text{(by (BCP));}
\end{aligned}$$

hence

$$\begin{aligned}
c &\sqsubseteq h \cdot k^\circ \cdot a \cdot Tk \cdot Th^\circ \wedge c \\
&\sqsubseteq h \cdot (k^\circ \cdot a \cdot Tk \wedge h^\circ \cdot c \cdot Th) \cdot Th^\circ \quad \text{(by condition (f))} \\
&= h \cdot d \cdot Th^\circ.
\end{aligned}$$

Therefore $c = h \cdot d \cdot Th^\circ$, hence $h : (W, d) \to (Z, c)$ is a regular epimorphism in $\mathrm{Alg}(T, e)$ by Proposition 5.1. □

We remark that in Proposition 5.1 and in Theorem 5.2 we can replace $\mathrm{Alg}(T, e)$ by $\mathrm{Alg}(T)$, since the entire argumentation is also valid for this category.

6. The Čech–Stone Compactification of Reflexive and Transitive Lax Algebras

It is well-known that for each topological space X there exists a compact Hausdorff space βX and a continuous map $\beta_X : X \to \beta X$ through which every continuous map from X to a compact Hausdorff space factors in a unique way. Observing that the category of compact Hausdorff spaces is isomorphic to the category of algebras for the ultrafilter monad in **Set**, and **Top** is the category of reflexive and transitive lax algebras for the same monad, one may ask whether the former result is a particular instance of a more general one. That is:

Is the category $\mathbf{X}^{\mathbb{T}}$ *a reflective subcategory of* $\mathrm{Alg}(T, e, m)$*?*

Barr proved this result in the case $\mathbf{Y} = \mathbf{Rel}$ (and hence $\mathbf{X} = \mathbf{Set}$, see [2]), while Manes constructed the corresponding reflections in [12]. The aim of this section is to show that we can extend the *Čech–Stone compactification* to our setting, provided that it satisfies some further assumptions. Therefore, throughout we will be working in the setting considered in the previous section, assuming, in addition, that $\mathbf{X}$ is a complete category. We will denote its terminal object by 1.

We start showing that:

PROPOSITION 6.1. *The category* $\mathbf{X}$ *is regular-cowellpowered.*

Proof. For $X \in \mathbf{X}$, to conclude that the class $\mathrm{Quot}(X)$ of regular epimorphisms with domain X has a representative set, we consider the map

$$\begin{array}{ccc} \mathrm{Quot}(X) & \longrightarrow & \mathbf{Y}(X, X) \\ X \xrightarrow{q} Q & \longmapsto & X \xrightarrow{q} Q \xrightarrow{q^\circ} X \end{array}$$

into the set $\mathbf{Y}(X, X)$ and show that any two elements of $\mathrm{Quot}(X)$ with the same image are isomorphic. For $p, q \in \mathrm{Quot}(X)$ with $p^\circ \cdot p = q^\circ \cdot q$, the following diagram commutes

$$\begin{tikzcd} & P \arrow[d, shift left, "q \cdot p^\circ"] \\ X \arrow[ur, "p"] \arrow[dr, "q"'] & \\ & Q. \arrow[uu, shift left, "p \cdot q^\circ"] \end{tikzcd} \tag{15}$$

Hence, if $f := q \cdot p^\circ$, then $f^\circ = p \cdot q^\circ$. Since p and q are epimorphisms in $\mathbf{Y}$, we obtain

$$f \cdot f^\circ = 1_P \quad \text{and} \quad f^\circ \cdot f = 1_Q,$$

hence, as a monic regular epimorphism in $\mathbf{X}$, f is an $\mathbf{X}$-isomorphism.

A necessary condition for the existence of a left adjoint of the functor $\mathbf{X}^{\mathbb{T}} \hookrightarrow \mathrm{Alg}(T, e, m)$ is its preservation of limits. The preservation of equalizers is guaranteed by (BCP), since, for every monomorphism $m : (X, a) \to (Y, b)$ in $\mathbf{X}^{\mathbb{T}}$, one has $a = m^\circ \cdot m \cdot a = m^\circ \cdot b \cdot Tm$, that is a is the initial structure in $\mathrm{Alg}(T, e, m)$. Next we analyze the closedness under products of $\mathbf{X}^{\mathbb{T}}$ in $\mathrm{Alg}(T, e, m)$.

LEMMA 6.2. *The following conditions are equivalent:*

(i) $\mathbf{X}^{\mathbb{T}}$ *is closed under products in* $\mathrm{Alg}(T, e, m)$*;*
(ii) *For each product* $(X, (\pi_i : X \to X_i)_{i \in I})$ *in* $\mathbf{X}$*,*

$$\bigwedge_{i \in I} (\pi_i^\circ \cdot \pi_i) = \mathrm{id}_X;$$

(iii) *If* $(f_i : X \to X_i)_{i\in I}$ *is a* (*small*) *monosource in* $\mathbf{X}$, $r : Y \to X$ *is an* $\mathbf{X}$-*morphism and* $s : Y \nrightarrow X$ *is an* $\mathbf{Y}$-*morphism, then*

$$(\forall i \in I f_i \cdot s \sqsubseteq f_i \cdot r) \Rightarrow s \sqsubseteq r.$$

Proof. (i) $\Leftrightarrow$ (ii) A product $((X, a), (\pi_i : (X, a) \to (X_i, a_i))_{i\in I})$ in $\mathbf{X}^{\mathbb{T}}$ is still a product in $\mathrm{Alg}(T, e, m)$ if and only if a is the initial structure for π_i; that is,

$$a = \bigwedge_{i\in I} (\pi_i^\circ \cdot a_i \cdot T\pi_i).$$

Since a is a map

$$\bigwedge_{i\in I} (\pi_i^\circ \cdot a_i \cdot T\pi_i) = \bigwedge_{i\in I} (\pi_i^\circ \cdot \pi_i \cdot a) = \left(\bigwedge_{i\in I} \pi_i^\circ \cdot \pi_i \right) \cdot a \sqsupseteq a,$$

hence we have that

$$\begin{aligned} a = \bigwedge_{i\in I} (\pi_i^\circ \cdot a_i \cdot T\pi_i) \;&\Leftrightarrow\; a = \left(\bigwedge_{i\in I} \pi_i^\circ \cdot \pi_i \right) \cdot a \\ &\Leftrightarrow\; \mathrm{id}_X = \bigwedge_{i\in I} (\pi_i^\circ \cdot \pi_i) \\ &\qquad\qquad \text{(since } a \text{ is a split epimorphism).} \end{aligned}$$

(ii) $\Leftrightarrow$ (iii) Condition (ii) follows from (iii) in the case $m_i = \pi_i$, $r = \mathrm{id}_X$ and $s = \bigwedge_{i\in I} \pi_i^\circ \cdot \pi_i$.

The converse follows from the fact that every (small) monosource factors through a product source followed by a monomorphism. □

We remark that, in the lemma above, only the first condition depends on $\mathbb{T}$, and that in fact the second one is a particular case of the former one for $\mathbb{T}$ the identity monad.

LEMMA 6.3. *Each of the* (*equivalent*) *conditions of the lemma above implies that, for each object* X *of* $\mathbf{X}$, *the* $\mathbf{X}$-*morphism* $(!_X : X \to 1)$ *is the top element of the preordered set* $\mathbf{Y}(X, 1)$.

Proof. We first remark that condition (ii) of Lemma 6.2 means $\mathrm{id}_1 = \max \mathbf{Y}(1, 1)$ in case $I = \emptyset$. Now, if $a : X \to 1$ is a morphism in $\mathbf{Y}$ such that $a \sqsupseteq !_X$, then $a^\circ \sqsupseteq !_X^\circ$, and therefore $a^\circ \cdot a \sqsupseteq !_X^\circ \cdot !_X \sqsupseteq \mathrm{id}_X$. Furthermore, $a \cdot a^\circ \sqsubseteq \max \mathbf{Y}(1, 1) = \mathrm{id}_1$. Hence a, as a map from X to 1, must coincide with $!_X$.

THEOREM 6.4. *Let* $\mathbf{X}$ *have* (*regular epi, mono*)-*factorizations. Given a monad* $\mathbb{T} = (T, e, m)$ *in* $\mathbf{X}$ *such that* T *preserves regular epimorphisms, the following conditions are equivalent:*

(i) $\mathbf{X}^{\mathbb{T}}$ *is a reflective subcategory of* $\mathrm{Alg}(T, e, m)$;

(ii) *for every product* $(X, (\pi_i : X \to X_i)_{i\in I})$, $\bigwedge_{i\in I}(\pi_i^\circ \cdot \pi_i) = \mathrm{id}_X$.

Proof. We first point out that the preservation of regular epimorphisms by T assures that the (regular epi, mono)-factorization system in $\mathbf{X}$ as well as the regular-cowellpoweredness of $\mathbf{X}$ can be lifted to $\mathbf{X}^{\mathbb{T}}$.

Let (X, a) be a reflexive and transitive lax algebra. Since (TX, m_X) is a free $\mathbb{T}$-algebra, although $e_X : X \to TX$ is not a lax morphism, every lax morphism $f : (X, a) \to (Y, b)$ into a $\mathbb{T}$-algebra (Y, b) factors uniquely through e_X.

Now, a morphism $h : (TX, m_X) \to (Y, b)$ in $\mathbf{X}^{\mathbb{T}}$ makes $h \cdot e_X : (X, a) \to (Y, b)$ a lax morphism if and only if $h \cdot a^\circ \cdot a \sqsubseteq h$. In fact, it is a lax morphism if and only if

$$h \cdot e_X \cdot a \sqsubseteq b \cdot Th \cdot Te_X = h \cdot m_X \cdot Te_X = h;$$

hence, from $h \cdot a^\circ \cdot a \sqsubseteq h$ it follows that $h \cdot e_X \cdot a \sqsubseteq h$ since $e_X \cdot a \sqsubseteq a^\circ \cdot a$, by the fact that (X, a) is a reflexive algebra. Conversely, if $h \cdot e_X \cdot a \sqsubseteq h$, then

$$\begin{aligned} h \cdot a^\circ \cdot a &= h \cdot m_X \cdot Te_X \cdot a^\circ \cdot a \\ &= b \cdot Th \cdot Te_X \cdot a^\circ \cdot a \sqsubseteq b \cdot b^\circ \cdot h \cdot e_X \cdot a \sqsubseteq h \cdot e_X \cdot a \sqsubseteq h. \end{aligned}$$

Furthermore, if

$$(TX, m_X) \xrightarrow{h} (Y, b) = (TX, m_X) \xrightarrow{q} (Z, c) \xrightarrow{u} (Y, b)$$

is the (regular epi, mono)-factorization of h in $\mathbf{X}^{\mathbb{T}}$, then $h \cdot a^\circ \cdot a \sqsubseteq h$ if and only if $q \cdot a^\circ \cdot a \sqsubseteq q$. Hence, in order to construct the reflection of (X, a) into $\mathbf{X}^{\mathbb{T}}$, we just have to consider the family of regular epimorphisms $(q_i : (TX, m_X) \to (Y_i, b_i))_{i\in I}$ in $\mathbf{X}^{\mathbb{T}}$ such that $q_i \cdot a^\circ \cdot a \sqsubseteq a$ for every $i \in I$. By the regular-cowellpoweredness of $\mathbf{X}^{\mathbb{T}}$ we may assume that I is a set. It is even non-empty since $!_{TX} : (TX, m_X) \to (1, !_{T1})$ verifies $!_X \cdot a^\circ \cdot a \sqsubseteq !_X$ by Lemma 6.3. Let

$$(TX, m_X) \xrightarrow{q_i} (Y, b) = (TX, m_X) \xrightarrow{q} (Q, d) \xrightarrow{f_i} (Y_i, b_i),$$

be the (regular epi, mono)-factorization of $(q_i)_{i\in I}$. Since $(f_i)_{i\in I}$ is a monosource, from $q_i \cdot a^\circ \cdot a \sqsubseteq q_i$ for every $i \in I$ it follows that $q \cdot a^\circ \cdot a \sqsubseteq q$. Therefore, $q \cdot e_X : (X, a) \to (Q, d)$ is a lax morphism and by construction it is the desired reflection. □

Acknowledgement

We are indebted to George Janelidze for suggesting us the study of this subject and also for his valuable comments.

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arXiv:0704.3976v1 [math.CT] 30 Apr 2007

Lawvere completeness in Topology

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October 31, 2018

Abstract

It is known since 1973 that Lawvere's notion of (Cauchy-)complete enriched category is meaningful for metric spaces: it captures exactly Cauchy-complete metric spaces. In this paper we introduce the corresponding notion of Lawvere completeness for $(\mathbb{T}, \mathsf{V})$-categories and show that it has an interesting meaning for topological spaces and quasi-uniform spaces: for the former ones means weak sobriety while for the latter means Cauchy completeness. Further, we show that V has a canonical $(\mathbb{T}, \mathsf{V})$-category structure which plays a key role: it is Lawvere-complete under reasonable conditions on the setting; permits us to define a Yoneda embedding in the realm of $(\mathbb{T}, \mathsf{V})$-categories.

Mathematics Subject Classification (2000): 18A05, 18D15, 18D20, 18B35, 18C15, 54E15, 54E50.

Key words: V-category, bimodule, monad, $(\mathbb{T}, \mathsf{V})$-category, completeness.

0 Introduction

Lawvere in his 1973 paper *Metric spaces, generalized logic, and closed categories* formulates a notion of *complete* V-category and shows that for (generalised) metric spaces it means Cauchy completeness. This notion of completeness deserved the attention of the categorical community, and the notion of *Cauchy-complete category*, or *Freyd-Karoubi complete category* is well-known, mostly in the context of Ab-enriched categories. However, it never got the attention of the topological community. In this paper we interpret Lawvere's completeness in topological settings. We extend Lawvere's notion of complete V-category to the (topological) setting of $(\mathbb{T}, \mathsf{V})$-categories (for a symmetric and unital quantale V), and show that it encompasses well-known notions in topological categories, meaning *weakly sober space* in the category of topological spaces and continuous maps, *weakly sober approach space* in the category of approach spaces and non-expansive maps, and *Cauchy-completeness* in the category of quasi-uniform spaces and uniformly continuous maps.

We present also a first step towards a possible construction of completion. Indeed, in the setting of V-categories, it is well-known that the completion of a V-category may be built out of the Yoneda embedding $X \to \mathsf{V}^{X^{\mathrm{op}}}$. In the $(\mathbb{T}, \mathsf{V})$-setting, we could prove that V has a canonical $(\mathbb{T}, \mathsf{V})$-categorical structure and that every $(\mathbb{T}, \mathsf{V})$-category X has a canonical dual

*The authors acknowledge partial financial assistance by Centro de Matemática da Universidade de Coimbra/FCT and Unidade de Investigação e Desenvolvimento Matemática e Aplicações da Universidade de Aveiro/FCT.

X^{op}. Using this structure and the free Eilenberg-Moore algebra structure $|X|$ on TX, we get two "Yoneda-like" morphisms

$$X \to \mathsf{V}^{X^{\mathrm{op}}} \quad \text{and} \quad X \to \mathsf{V}^{|X|}.$$

For the latter one we prove a Yoneda Lemma (see 4.2).

Furthermore, we show that, under suitable conditions, V is a Lawvere-complete $(\mathbb{T}, \mathsf{V})$-category, a first step towards a completion construction which will be the subject of a forthcoming paper.

In order to make the presentation of this paper smoother, in Section 1 we recall the notions and properties of V-categories we will generalize throughout. First we introduce V-categories and V-bimodules, and define Lawvere-complete V-categories, for a commutative and unital quantale V. V is then naturally equipped with the V-categorical structure hom. We give a direct proof of Lawvere completeness of the V-category $(V, \hom)$.

In Section 2 we describe our basic setting for the study of $(\mathbb{T}, \mathsf{V})$-categories and introduce them. We describe Kleisli composition in the category $\mathsf{V}\text{-}\mathsf{Mat}$ of V-valued matrices and define $(\mathbb{T}, \mathsf{V})$-bimodule. Although $(\mathbb{T}, \mathsf{V})$-bimodules do not compose in general, one can still formulate and study the notion of Lawvere-complete $(\mathbb{T}, \mathsf{V})$-category.

Similarly to what was done in V-categories, we define a canonical $(\mathbb{T}, \mathsf{V})$-categorical structure on V, as the composition of hom with the (canonical) $\mathbb{T}$-algebra structure on V described by Manes in [21]. This is the subject of Section 3. In addition we also prove that, under some conditions, the $(\mathbb{T}, \mathsf{V})$-category V is Lawvere-complete.

In Section 1 we present the Yoneda embedding for V-categories as a subproduct of the fact that a V-matrix $\psi : X \nrightarrow Y$ between V-categories (X, a) and (Y, b) is a V-bimodule if and only if, as a map $\psi : X^{\mathrm{op}} \otimes Y \to \mathsf{V}$, is a V-functor (Theorem 1.5); then the monoidal-closed structure of $\mathsf{V}\text{-}\mathsf{Cat}$ gives us the *Yoneda Functor* $X \to \mathsf{V}^{X^{\mathrm{op}}}$. In the $(\mathbb{T}, \mathsf{V})$-setting this construction becomes more elaborated (see Theorem 3.3): a V-matrix $\psi : TX \nrightarrow Y$ is a $(\mathbb{T}, \mathsf{V})$-bimodule $\psi : (X, a) \multimap (Y, b)$ if and only if both $\psi : |X| \otimes Y \to \mathsf{V}$ and $\psi : X^{\mathrm{op}} \otimes Y \to \mathsf{V}$ are $(\mathbb{T}, \mathsf{V})$-functors. Thus, given a $(\mathbb{T}, \mathsf{V})$-category (X, a), the $(\mathbb{T}, \mathsf{V})$-bimodule $a : X \multimap X$ gives rise to two *Yoneda* $(\mathbb{T}, \mathsf{V})$*-functors* $X \to \mathsf{V}^{X^{\mathrm{op}}}$ and $X \to \mathsf{V}^{|X|}$.

In Section 5 we present the announced topological examples, with the exception of quasi-uniform spaces, which are presented in the Appendix, due to the fact that their presentation as lax algebras does not fit in the $(\mathbb{T}, \mathsf{V})$-setting (as shown in [20]).

Acknowledgments. We are grateful to Francis Borceux for his enlightening proof of completeness of the V-category V. The work on this paper was started while both authors were visiting Walter Tholen in May 2004, at York University, Canada, and benefited from further visits in January 2006 and February 2007. We thank Walter Tholen for fruitful discussions on the subject of this paper.

1 The category of V-categories

Although the material of this section can be found essentially on [17], we find that its inclusion here may enlighten the corresponding – but more technical – notions and results for $(\mathbb{T}, \mathsf{V})$-

categories presented in the forthcoming sections.

1.1 V**.** Throughout V is a (commutative and unital) *quantale*. In other words, V is a complete lattice equipped with a symmetric and associative tensor product $\otimes$, with unit k, and with right adjoint hom; that is, for each $u, v, w \in \mathsf{V}$,

$$u \otimes v \leq w \iff v \leq \hom(u, w).$$

Considered as a (thin) category, V is said to be *symmetric monoidal-closed*. If k is the bottom element $\perp$ of V, then $\mathsf{V} = 1$ is the trivial lattice. Throughout this paper *we assume that* V *is non-trivial, i.e.* $k \neq \perp$.

Every non-trivial Heyting algebra – with $\otimes = \wedge$ and $k = \top$ the top element – is an example of such a lattice, in particular the two-element chain $2 = \{\text{false} \models \text{true}\}$, with the monoidal structure given by "&" (and) and "true". The complete real half-line $\mathsf{P} = [0, \infty]$, with the categorical structure induced by the relation $\geq$ (i.e., $a \to b$ means $a \geq b$), admits several interesting monoidal structures. First of all, with $\wedge = \max$ it is a Heyting algebra $\mathsf{P}_{\max}$. Another possible choice for $\otimes$ is $+$; we denote P equipped with this tensor by P_+. Note that in this example the right adjoint hom is given by truncated minus: $\hom(u, v) = \max\{v - u, 0\}$.

1.2 V**-Mat.** The category V-Mat of V-*matrices* [3, 10] has sets as objects, and a morphism $r : X \nrightarrow Y$ in V-Mat is a map $r : X \times Y \to \mathsf{V}$. Composition of V-matrices $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$ is defined as matrix multiplication:

$$s \cdot r(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z).$$

The identity arrow $1_X : X \nrightarrow X$ in V-Mat is the V-matrix which sends all diagonal elements (x, x) to k and all other elements to the bottom element $\perp$ of V. In fact, each Set-map $f : X \to Y$ can be interpreted as the V-matrix

$$f : X \nrightarrow Y,\ f(x, y) = \begin{cases} k & \text{if } f(x) = y, \\ \perp & \text{else.} \end{cases}$$

To keep notation simple, in the sequel we will write $f : X \to Y$ rather then $f : X \nrightarrow Y$ for a V-matrix induced by a map. The formula for matrix composition becomes considerably easier if one of the V-matrices is a Set-map:

$$s \cdot f(x, z) = s(f(x), z), \qquad g \cdot r(x, z) = \bigvee_{y \in g^{-1}(z)} r(x, y)$$

for maps $f : X \to Y$ and $g : Y \to Z$ and V-matrices $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$.

The complete order on V induces a complete order on $\mathsf{V}\text{-}\mathsf{Mat}(X, Y) = \mathsf{V}^{X \times Y}$: for V-matrices $r, r' : X \nrightarrow Y$ we define

$$r \leq r' \ :\iff \ \forall x \in X\ \forall y \in Y\ r(x, y) \leq r'(x, y).$$

The *transpose* $r^\circ : Y \nrightarrow X$ of a V-matrix $r : X \nrightarrow Y$ is defined by $r^\circ(y, x) = r(x, y)$. It is easy to see that $(\,)^\circ : V\text{-}\mathsf{Mat}(X, Y) \to V\text{-}\mathsf{Mat}(Y, X)$ is order-preserving, and

$$1_X^\circ = 1_X, \qquad (s \cdot r)^\circ = r^\circ \cdot s^\circ, \qquad r^{\circ\circ} = r.$$

For each Set-map $f : X \to Y$ we have $1_X \leq f^\circ \cdot f$ and $f \cdot f^\circ \leq 1_Y$, i.e. f is left adjoint to f° and we write $f \dashv f^\circ$. In general, given V-matrices $r : X \nrightarrow Y$ and $s : Y \nrightarrow X$, we say that r is left adjoint to s (and that s is right adjoint to r) if $1_X \leq s \cdot r$ and $1_Y \geq r \cdot s$.

Lemma. *Let V be a quantale and $r, r' : X \nrightarrow Y$ and $s, s' : Y \nrightarrow X$ be V-matrices such that $r \dashv s$ and $r' \dashv s'$. Then $r \leq r'$ if and only if $s' \leq s$. Consequently, if $r \leq r'$ and $s \leq s'$, then $r = r'$ and $s = s'$.*

As another consequence of the lemma above we have that left and right adjoints are uniquely determined by s (respectively r). Therefore we say that r is left adjoint if it has a right adjoint s, and likewise, s is right adjoint if it has a left adjoint r. In pointwise notation, we have $r \dashv s$ if and only if

$$\forall x \in X \bigvee_{y \in Y} r(x,y) \otimes s(y,x) \geq k,$$

$$\forall x \in X\ \forall y, y' \in Y\ s(y,x) \otimes r(x,y') \leq \begin{cases} \bot & \text{if } y \neq y', \\ k & \text{if } y = y' \end{cases}$$

which, by symmetry of $\otimes$, is equivalent to

$$\forall x \in X \bigvee_{y \in Y} r(x,y) \otimes s(y,x) = k,$$

$$\forall x \in X\ \forall y, y' \in Y\ (y \neq y' \ \Rightarrow\ s(y,x) \otimes r(x,y') = \bot).$$

Our next example shows that there exist indeed left adjoint V-matrices which are not induced by Set-maps.

Example. Consider a set X and the Boolean algebra $\mathsf{V} = PX$ the powerset of X. Define a V-matrix $r : 1 \nrightarrow X$ by putting $r(\star, x) = \{x\}$ for $x \in X$. Then

$$r^\circ \cdot r(\star,\star) = \bigcup_{x \in X} \{x\} = X \qquad \text{and} \qquad r \cdot r^\circ(x,y) = \{x\} \cap \{y\} = \begin{cases} \emptyset & \text{if } x \neq y, \\ \{x\} & \text{if } x = y, \end{cases}$$

hence $r \dashv r^\circ$. But r is not a Set-map unless X has at most one element.

We wish to characterise those quantales V where the class of left adjoint V-matrices coincides with the class of Set-maps. In order to do so we introduce some notation. Let $u, v \in \mathsf{V}$. We say that v is a $\otimes$-*complement* of u if

$$u \vee v = k \qquad \text{and} \qquad u \otimes v = \bot.$$

Clearly, each $u \in \mathsf{V}$ has at most one $\otimes$-complement. Moreover, if u is $\otimes$-complemented (i.e. has a $\otimes$-complement v), then

$$u = u \otimes k = u \otimes (u \vee v) = (u \otimes u) \vee (u \otimes v) = u \otimes u,$$

that is, u is idempotent. Our next result generalises [14, 2.14].

Proposition. *Let* V *be a quantale. Each left adjoint* V*-matrix is a* Set*-map if and only if* k *and* $\perp$ *are the only* $\otimes$*-complemented elements of* V *and*

$$\forall u, v \in \mathsf{V}\ (u \otimes v = k \quad \Rightarrow \quad u = k = v).$$

Proof. Assume first that each left adjoint V-matrix is a Set-map. Let $u, v \in \mathsf{V}$. If $u \otimes v = k$, then $u \dashv v$, and we have $u = v = k$. Suppose that $u \vee v = k$ and $u \otimes v = \perp$. Let $X = \{u, v\}$ and define $r : 1 \longrightarrow\!\!\!\!\!\!+\ X$ with $r(\star, u) = u$ and $r(\star, v) = v$. Then $r \dashv r^\circ$ and, by assumption, $u = k$ or $v = k$.
Let $r : X \longrightarrow\!\!\!\!\!\!+\ Y$ and $s : Y \longrightarrow\!\!\!\!\!\!+\ X$ be V-matrices such that $r \dashv s$. Let $x \in X$. There is some $y \in Y$ such that $r(x, y) \otimes s(y, x) > \perp$[1]. Then

$$k = (r(x, y) \otimes s(y, x)) \vee \bigvee_{y' \neq y} (r(x, y') \otimes s(y', x))$$

and

$$r(x, y) \otimes s(y, x) \otimes \bigvee_{y' \neq y} r(x, y') \otimes s(y', x) = \bigvee_{y' \neq y} r(x, y) \otimes s(y, x) \otimes r(x, y') \otimes s(y', x) = \perp.$$

Hence, by assumption, $r(x, y) = k = s(y, x)$ and $r(x, y') \otimes s(y', x) = \perp$ for all $y' \neq y$. We have shown that, for each $x \in X$, there exists exactly one $y \in Y$ with $r(x, y) = k = s(y, x)$. Consider now $f : X \to Y$ which assigns to x this unique y. Clearly, $f \leq r$, but also $f^\circ \leq s$ since

$$f^\circ(y, x) = k \quad \Rightarrow \quad f(x) = y \quad \Rightarrow \quad s(y, x) = k.$$

The assertion follows now from the previous lemma. □

1.3 V-categories. V-enriched categories were introduced and studied in [11, 17] in the more general context of symmetric monoidal-closed categories. For a very nice presentation of this material we refer to [18]. In the next subsections we recall some well-known facts about V-categories, which will serve as a guideline for our study of $(\mathbb{T}, \mathsf{V})$-categories.

A V*-enriched category* (or simply V*-category*) is a pair (X, a) with X a set and $a : X \longrightarrow\!\!\!\!\!\!+\ X$ a V-matrix such that

$$1_X \leq a \cdot a \quad \text{and } a \cdot a \leq a;$$

equivalently, the map $a : X \times X \to \mathsf{V}$ satisfies the following conditions:

(R) for each $x \in X$, $k \leq a(x, x)$;

(T) for each $x, x', x'' \in X$, $a(x, x') \otimes a(x', x'') \leq a(x, x'')$.

Given V-categories (X, a) and (Y, b), a V*-functor* $f : (X, a) \to (Y, b)$ is a map $f : X \to Y$ such that, for each $x, x' \in X$, $a(x, x') \leq b(f(x), f(x'))$. V-categories and V-functors are the objects and morphisms of the category V-Cat. Finally, given a V-category $X = (X, a)$, the *dual category* X^{op} of X is defined by $X^{\mathrm{op}} = (X, a^\circ)$.

We remark that V-Cat is actually a *closed category* since the tensor product on V can be naturally transported to V-Cat. More precisely, for V-categories $X = (X, a)$ and $Y = (Y, b)$, we

[1] Since $\perp < k$. The assertion of the proposition is trivially true if $k = \perp$

put $X \otimes Y = (X \times Y, a \otimes b)$ where $a \otimes b((x,y),(x',y')) = a(x,x') \otimes b(y,y')$ for all $x, x' \in X$ and $y, y' \in Y$. Then, for each V-category $X = (X, a)$, the functor $X \otimes _ : \mathsf{V}\text{-}\mathsf{Cat} \to \mathsf{V}\text{-}\mathsf{Cat}$ has a right adjoint $_^X$ defined by $Y^X = (\mathsf{V}\text{-}\mathsf{Cat}(X,Y), d)$ with $d(f,g) = \bigwedge_{x \in X} a(f(x), g(x))$.

Being monoidal-closed, V has a natural structure as V-category:

$$\hom : \mathsf{V} \nrightarrow \mathsf{V}.$$

Indeed, for $u, v, w \in \mathsf{V}$,

$$k \otimes v = v \;\Rightarrow\; k \leq \hom(v,v),$$

$$u \otimes (\hom(u,v) \otimes \hom(v,w)) \leq v \otimes \hom(v,w) \leq w \;\Rightarrow\; \hom(u,v) \otimes \hom(v,w) \leq \hom(u,w),$$

that is, $1_{\mathsf{V}} \leq \hom$ and $\hom \cdot \hom \leq \hom$.

For $\mathsf{V} = \mathsf{2}$, with the usual notation $x \leq x' :\Longleftrightarrow a(x,x') = \text{true}$, axioms (R) and (T) read as

$$\forall x \in X \text{ true} \models x \leq x \qquad \text{and} \qquad \forall x, x', x'' \in X \; x \leq x' \;\&\; x' \leq x'' \models x \leq x'',$$

that is, $(X, \leq)$ is an ordered set[2]. A $\mathsf{2}$-functor is a map $f : (X, \leq) \to (Y, \leq)$ between ordered sets such that

$$\forall x, x' \in X \; x \leq x' \;\models\; f(x) \leq f(x');$$

that is, f is a monotone map. Hence $\mathsf{2}\text{-}\mathsf{Cat}$ is equivalent to the category Ord of ordered sets and monotone maps.

A P_+-category is a set X endowed with a map $a : X \times X \to \mathsf{P}_+$ such that

$$\forall x \in X \; 0 \geq a(x,x) \qquad \text{and} \qquad \forall x, x', x'' \in X \; a(x,x') + a(x',x'') \geq a(x,x'');$$

that is, $a : X \times X \to \mathsf{P}_+$ is a (generalised) metric on X. A P_+-functor is a map $f : (X,a) \to (Y,b)$ between metric spaces satisfying the following inequality:

$$\forall x, x' \in X \; a(x,x') \geq b(f(x), f(x')),$$

which means precisely that f is a non-expansive map. Therefore the category $\mathsf{P}_+\text{-}\mathsf{Cat}$ coincides with the category Met of metric spaces and non-expansive maps. (For more details, see [18, 10].)

For $\mathsf{V} = \mathsf{P}_{\max}$, the transitivity axiom (T) reads as

$$\max\{a(x,x'), a(x',x'')\} \geq a(x,x''),$$

hence the category $\mathsf{P}_{\max}\text{-}\mathsf{Cat}$ coincides with the category UMet of *(generalised) ultrametric spaces and non-expansive maps.*

1.4 V-bimodules. Given V-categories (X,a) and (Y,b), a *bimodule*[3] $\psi : (X,a) \multimap (Y,b)$ is a V-matrix $\psi : X \nrightarrow Y$ such that $\psi \cdot a \leq \psi$ and $b \cdot \psi \leq \psi$; that is, for each $x, x' \in X$ and $y, y' \in Y$,

$$a(x,x') \otimes \psi(x',y) \leq \psi(x,y) \qquad \text{and} \qquad \psi(x,y') \otimes b(y',y) \leq \psi(x,y).$$

It is easy to verify that bimodules compose and that V-categorical structures are themselves bimodules. In fact, they are the identities for the composition of bimodules, that is, for any bimodule $\psi : (X,a) \multimap (Y,b)$, $\psi \cdot a = \psi$ and $b \cdot \psi = \psi$. Therefore, V-categories and V-bimodules constitute a category, which we will denote by $\mathsf{V}\text{-}\mathsf{Mod}$. The category $\mathsf{V}\text{-}\mathsf{Mod}$ inherits the bicategorical structure of $\mathsf{V}\text{-}\mathsf{Mat}$ via the forgetful functor $\mathsf{V}\text{-}\mathsf{Mod} \to \mathsf{V}\text{-}\mathsf{Mat}$.

[2]Note that we do *not* require $\leq$ to be anti-symmetric.

[3]Also known as *profunctor* or *distributor* (see [2, 5, 26]).

1.5 V-functors as V-bimodules. Any V-functor $f : (X,a) \to (Y,b)$ defines a pair of matrices $f_* : (X,a) \longrightarrow\!\!\!\!\!\!\!+\;\; (Y,b)$ and $f^* : (Y,b) \longrightarrow\!\!\!\!\!\!\!+\;\; (X,a)$, with $f_* = b \cdot f$ and $f^* = f^\circ \cdot b$, that is $f_*(x,y) = b(f(x),y)$ and $f^*(y,x) = b(y,f(x))$, which are in fact bimodules: for every $x, x' \in X$ and $y, y' \in Y$,

$$\begin{aligned} a(x,x') \otimes f_*(x',y) &= a(x,x') \otimes b(f(x'),y) \leq b(f(x),f(x')) \otimes b(f(x'),y) \leq b(f(x),y), \\ f_*(x,y') \otimes b(y'y) &= b(f(x),y') \otimes b(y',y) \leq b(f(x),y), \end{aligned}$$

and similarly for f^*.

Moreover, the bimodules f_* and f^* form an adjunction, as we show next. We recall first that, given bimodules $\varphi : (X,a) \longrightarrow\!\!\!\!\!\!\!\circ\;\; (Y,b)$ and $\psi : (Y,b) \longrightarrow\!\!\!\!\!\!\!\circ\;\; (X,a)$, φ is left adjoint to ψ, $\varphi \dashv \psi$, if $1_{(X,a)} \leq \psi \cdot \varphi$ and $\varphi \cdot \psi \leq 1_{(Y,b)}$, i.e. $a \leq \psi \cdot \varphi$ and $\varphi \cdot \psi \leq b$. It is now straightforward to check that $f_* \dashv f^*$, since, for $x, x' \in X$ and $y, y' \in Y$, the inequality

$$a(x,x') \leq \bigvee_{y \in Y} f_*(x,y) \otimes f^*(y,x') = \bigvee_{y \in Y} b(f(x),y) \otimes b(y,f(x')) = b(f(x),f(x'))$$

follows from V-functoriality of f, while

$$\bigvee_{x \in X} f^*(y,x) \otimes f_*(x,y') = \bigvee_{x \in X} b(y,f(x)) \otimes b(f(x),y') \leq b(y,y')$$

follows from the associativity axiom for V-categories. A quite different connection between functors and bimodules offers the following

Theorem. *For V-categories $X = (X,a)$ and $Y = (Y,b)$ and a V-matrix $\psi : X \longrightarrow\!\!\!\!\!\!\!+\;\; Y$, the following conditions are equivalent:*

(i) *$\psi : X \longrightarrow\!\!\!\!\!\!\!\circ\;\; Y$ is a bimodule;*

(ii) *$\psi : X^{\mathrm{op}} \otimes Y \to \mathsf{V}$ is a V-functor.*

Proof. (i) $\Rightarrow$ (ii): For $x, x' \in X$ and $y, y' \in Y$,

$$\begin{aligned} \psi(x,y) \otimes a^\circ(x,x') \otimes b(y,y') &= a(x',x) \otimes \psi(x,y) \otimes b(y,y') \\ &\leq \psi(x',y) \otimes b(y,y') \\ &\leq \psi(x',y'), \end{aligned}$$

hence

$$a^\circ(x,x') \otimes b(y,y') \leq \hom(\psi(x,y), \psi(x',y')).$$

(ii) $\Rightarrow$ (i): For $x, x' \in X$ and $y, y' \in Y$,

$$\begin{aligned} a(x,x') \otimes \psi(x',y) &\leq \psi(x',y) \otimes a^\circ(x',x) \otimes b(y,y) \\ &\leq \psi(x,y), \end{aligned}$$

that is $a \cdot \psi \leq \psi$, and

$$\begin{aligned} \psi(x,y') \otimes b(y',y) &\leq \psi(x,y') \otimes a^\circ(x,x) \otimes b(y',y) \\ &\leq \psi(x,y), \end{aligned}$$

that is $\psi \cdot b \leq \psi$. $\square$

Corollary. *There is a* V*-functor* $\ulcorner a \urcorner : X \to \mathsf{V}^{X^{\mathrm{op}}}$*. Moreover, for each* $x \in X$ *and* $f \in \mathsf{V}^{X^{\mathrm{op}}}$*, we have*

$$d(a(-,x), f) = f(x).$$

Proof. Note that $d(a(-,x), f) = \bigwedge_y \hom(a(y,x), f(y)) \leq f(x)$. On the other hand, for each $y \in Y$,

$$\begin{aligned} a(y,x) \leq \hom(f(x), f(y)) \quad &\Longleftrightarrow \quad f(x) \otimes a(y,x) \leq f(y) \\ &\Longleftrightarrow \quad f(x) \leq \hom(a(y,x), f(y)). \end{aligned}$$

□

1.6 Lawvere-complete V-categories.

Definition. A V-category (X, a) is said to be *Lawvere-complete* if, for any V-category (Y, b), for every pair of adjoint bimodules

$$\begin{tikzcd} b \arrow[loop left, "Y"] \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\perp"] & X \arrow[loop right, "a"] \arrow[l, bend left, "\psi"] \end{tikzcd}$$

φ is in the image of $(\ )_* : \mathsf{V}\text{-}\mathsf{Cat} \to \mathsf{V}\text{-}\mathsf{Mod}$, i.e. there exists a V-functor $f : (X, a) \to (Y, b)$ such that $f_* = \varphi$ and $f^* = \psi$.

It is interesting to notice that, in order to check Lawvere completeness, we can restrict to the case (Y, b) is the V-category $(1, p)$, where $1 = \{\star\}$ is a singleton and $p(\star, \star) = k$.

Proposition. *For a* V*-category* (X, a)*, the following conditions are equivalent:*

(i) (X, a) *is Lawvere-complete;*

(ii) *for each pair of adjoint bimodules* $(\varphi : (1,p) \multimap (X,a)) \dashv (\psi : (X,a) \multimap (1,p))$*, there exists a* V*-functor* $f : (1,p) \to (X,a)$ *such that* $\varphi = f_*$ *and* $\psi = f^*$ *(in this situation we say that* $f(\star)$ *represents the adjunction* $\varphi \dashv \psi$*).*

Proof. It is a special case of Proposition 2.7. We omit the proof here because it follows, step by step, the proof of Proposition 2.7. □

Theorem. *The* V*-category* $(\mathsf{V}, \hom)$ *is Lawvere-complete.*

Proof. Although this fact can be deduced from more general categorical results, we prefer to give here a direct proof, which provides guidance for the more general corresponding result for the $(\mathbb{T}, \mathsf{V})$-categorical structure of V we will study later.

Consider

$$\begin{tikzcd} p \arrow[loop left, "1"] \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\perp"] & V \arrow[loop right, "\hom"] \arrow[l, bend left, "\psi"] \end{tikzcd}$$

From the above theorem it follows that

$$\begin{aligned} (1) \qquad \varphi \text{ is a bimodule} \quad &\Longleftrightarrow \quad \varphi : (\mathsf{V}, \hom) \to (\mathsf{V}, \hom) \text{ is a } \mathsf{V}\text{-functor} \\ &\Longleftrightarrow \quad \forall u, v \in \mathsf{V} \ \hom(u, v) \leq \hom(\varphi(u), \varphi(v)); \end{aligned}$$

$$(2) \qquad \psi \text{ is a bimodule} \iff \psi : (\mathsf{V}, \hom^{\mathrm{op}}) \to (\mathsf{V}, \hom) \text{ is a } \mathsf{V}\text{-functor}$$
$$\iff \forall u, v \in \mathsf{V} \ \hom(u, v) \leq \hom(\psi(v), \psi(u));$$

the conditions for the adjunction read as:

$$(3) \qquad \varphi \dashv \psi \iff \forall u, v \in \mathsf{V} \quad \psi(u) \otimes \varphi(v) \leq \hom(u, v) \quad \& \quad k \leq \bigvee_{u \in V} \varphi(u) \otimes \psi(u)$$

We will show that the adjunction $\varphi \dashv \psi$ is represented by $\psi(k)$, i.e, $\varphi(v) = \hom(\psi(k), v)$ and $\psi(v) = \hom(v, \psi(k))$, for every $v \in \mathsf{V}$. First we notice that from (3) it follows that $\psi(k) \otimes \varphi(v) \leq \hom(k, v) = v$, hence $\varphi(v) \leq \hom(\psi(k), v)$. Now the proof consists of checking three equalities:

(1st) $\psi(k) = \bigvee_{u \in \mathsf{V}} \psi(u) \otimes u$:

It is immediate that $\psi(k) = \psi(k) \otimes k \leq \bigvee_{u \in \mathsf{V}} \psi(u) \otimes u$, and, moreover, for every $v \in \mathsf{V}$,

$$\begin{aligned} \psi(u) \otimes u = \psi(u) \otimes \hom(k, u) &\leq \psi(u) \otimes \hom(\psi(u), \psi(k)) \qquad \text{(by (2))} \\ &\leq \psi(k). \end{aligned}$$

(2nd) $\forall v \in \mathsf{V} \ \hom(v, \psi(k)) = \bigvee_{u \in \mathsf{V}} \hom(v, u) \otimes \psi(u)$:

To show "$\geq$" we just observe that

$$v \otimes (\hom(v, u) \otimes \psi(u)) \leq u \otimes \psi(u) \leq \psi(k);$$

for "$\leq$", we have

$$\begin{aligned} \hom(v, \psi(k)) &\leq \hom(v, \psi(k)) \otimes \bigvee_{u \in V} \varphi(u) \otimes \psi(u) \qquad \text{(by (3))} \\ &= \bigvee_{u \in V} \hom(v, \psi(k)) \otimes \varphi(u) \otimes \psi(u) \\ &\leq \bigvee_{u \in V} \hom(v, \psi(k)) \otimes \hom(\psi(k), u) \otimes \psi(u) \qquad (\text{since } \varphi \leq \hom(\psi(k), -)) \\ &\leq \bigvee_{u \in V} \hom(v, u) \otimes \psi(u). \end{aligned}$$

(3rd) Since $\psi = \psi \cdot \hom$ we have $\forall v \in \mathsf{V} \ \psi(v) = \bigvee_{u \in V} \hom(v, u) \otimes \psi(u)$. □

A new insight on Lawvere completeness for V-categories may be found in [25].

2 Basic properties of $(\mathbb{T}, \mathsf{V})$-categories

In the first part of this section we present the setting for the study of $(\mathbb{T}, \mathsf{V})$-categories, or (Eilenberg-Moore) lax algebras, that can be studied in more detail in [7, 10, 8].

2.1 $\mathbb{T}$ and its extension. Recall that a *monad* $\mathbb{T} = (T, e, m)$ on Set consists of a functor $T : \mathsf{Set} \to \mathsf{Set}$ together with natural transformations $e : \mathrm{Id}_{\mathsf{Set}} \to T$ (unit) and $m : TT \to T$ (multiplication) such that

$$m \cdot Tm = m \cdot m_T \qquad \text{and} \qquad m \cdot Te = 1_T = m \cdot e_T.$$

There are two *trivial monads* on Set, one sending all sets X to the terminal set 1, and the other with $T\varnothing = \varnothing$ and $TX = 1$ for $X \neq \varnothing$. Any other monad is called *non-trivial*.

By a *lax extension* of a Set-monad $\mathbb{T} = (T, e, m)$ to V-Mat we mean an extension of the endofunctor $T : \mathsf{Set} \to \mathsf{Set}$ to V-matrices acting on Set-maps as T and satisfying

(a) $(Ta)^\circ = T(a^\circ)$ (and we write Ta°),

(b) $Tb \cdot Ta \leq T(b \cdot a)$,

(c) $a \leq a' \;\Rightarrow\; Ta \leq Ta'$,

(d) $e_Y \cdot a \leq Ta \cdot e_X$,

(e) $m_Y \cdot T^2 a \leq Ta \cdot m_X$,

for all $a, a' : X \longrightarrow\!\!\!\!\!\!\!+\; Y$ and $b : Y \longrightarrow\!\!\!\!\!\!\!+\; Z$ [4]. Note that we have automatically equality in (b) if $a = f$ is a Set-map. A Set-monad $\mathbb{T} = (T, e, m)$ admitting a lax extension to V-Mat is called V-*admissible*. Although $\mathbb{T}$ may have many lax extensions to V-Mat, in the sequel we usually have a fixed extension in mind when talking about a V-admissible monad. Trivially, the identity monad $\mathbb{1} = (\mathrm{Id}, 1, 1)$ on Set can be extended to the identity monad on V-Mat. In [1] M. Barr shows how to extend Set-monads to $\mathsf{Rel} = \mathsf{2}$-$\mathsf{Mat}$: first observe that each relation $r : X \longrightarrow\!\!\!\!\!\!\!+\; Y$ can be written as $r = p \cdot q^\circ$ where $q : R \to X$ and $p : R \to Y$ are the projection maps, then put $Tr = Tp \cdot Tq^\circ$. All conditions above but the second one are satisfied, and this extension satisfies (b) if and only if the Set-functor T has (BC) (that is, sends pullbacks to weak pullbacks). In [8] we showed how to make the step from Rel to V-Mat, provided that in addition V is *constructively completely distributive (ccd)*[5]. Given a monad $\mathbb{T} = (T, e, m)$ and a V-matrix $a : X \longrightarrow\!\!\!\!\!\!\!+\; Y$, we define relations $a_v : X \longrightarrow\!\!\!\!\!\!\!+\; Y$ $(v \in \mathsf{V})$ by $a_v(x, y) = \text{true} \iff a(x, y) \geq v$ and put, for $\mathfrak{x} \in TX$ and $\mathfrak{y} \in TY$,

$$Ta(\mathfrak{x}, \mathfrak{y}) = \bigvee \{v \in \mathsf{V} \mid Ta_v(\mathfrak{x}, \mathfrak{y}) = \text{true}\}.$$

Then the formula above defines an extension of $T : \mathsf{Set} \to \mathsf{Set}$ provided that either $k = \top$ or $T\varnothing = \varnothing$. Moreover, all five conditions above are satisfied. In addition we have

(f) $Tb \cdot Ta = T(b \cdot a)$ provided that $\otimes = \wedge$,

(g) $Tg \cdot Ta = T(g \cdot a)$,

for all V-matrices $a : X \longrightarrow\!\!\!\!\!\!\!+\; Y$ and $b : Y \longrightarrow\!\!\!\!\!\!\!+\; Z$ and all maps $g : Y \to Z$. In some occasions we will need that the (Set-based) natural transformation $m : TT \to T$ has (BC) (that is, each naturality square is a weak pullback); this guarantees that m is also a (strict) natural transformation for the extension of T to V-Mat described above.

[4] The conditions for our extension are stronger than Seal's in [24].

[5] Recall that a lattice Y is (ccd) if $\bigvee : 2^{Y^{\mathrm{op}}} \to Y$ has a left adjoint; for more details see [27].

2.2 $(\mathbb{T},\mathsf{V})$**-categories.** Let $\mathbb{T}=(T,e,m)$ be a V-admissible monad. A $(\mathbb{T},\mathsf{V})$-*category* is a pair (X,a) consisting of a set X and a V-matrix $a : TX \nrightarrow X$ such that:

$$1_X \leq a\cdot e_X \quad \text{and } a\cdot Ta \leq a\cdot m_X;$$

that is, the map $a : TX\times X \to \mathsf{V}$ satisfies the conditions:

(R) for each $x\in X$, $k \leq a(e_X(x),x)$;

(T) for each $\mathfrak{X}\in T^2X$, $\mathfrak{x}\in TX$, $x\in X$, $Ta(\mathfrak{X},\mathfrak{x})\otimes a(\mathfrak{x},x)\leq a(m_X(\mathfrak{X}),x)$.

Given $(\mathbb{T},\mathsf{V})$-categories (X,a) and (Y,b), a $(\mathbb{T},\mathsf{V})$-*functor* $f:(X,a)\to(Y,b)$ is a map $f: X\to Y$ such that, for each $\mathfrak{x}\in TX$ and $x\in X$, $a(\mathfrak{x},x)\leq b(Tf(\mathfrak{x}),f(x))$. $(\mathbb{T},\mathsf{V})$-categories and $(\mathbb{T},\mathsf{V})$-functors are the objects and morphisms of the category $(\mathbb{T},\mathsf{V})$-Cat.

Note that each Eilenberg-Moore algebra for $\mathbb{T}$ can be viewed as a $(\mathbb{T},\mathsf{V})$-category; in fact, we have an embedding

$$\mathsf{Set}^{\mathbb{T}} \hookrightarrow (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}.$$

In particular, for each set X we have the $(\mathbb{T},\mathsf{V})$-category (TX,m_X) which we denote by $|X|$.

Obviously, each V-category is a $(\mathbb{T},\mathsf{V})$-category for $\mathbb{T}=\mathbb{1}$ the identity monad "identically" extended to V-Mat. A further class of interesting examples involves the ultrafilter monad $\mathbb{U}=(U,e,m)$. The extension of $U:\mathsf{Set}\to\mathsf{Set}$ to V-Mat of 2.1 can be equivalently described by

$$Ur(\mathfrak{x},\mathfrak{y}) = \bigwedge_{(A\in\mathfrak{x},B\in\mathfrak{y})}\ \bigvee_{(x\in A,y\in B)} r(x,y),$$

for all $r: X\nrightarrow Y$ in V-Mat, $\mathfrak{x}\in TX$ and $\mathfrak{y}\in TY$. The main result of [1] states that $(\mathbb{U},2)$-$\mathsf{Cat}\cong\mathsf{Top}$. In [7] it is shown that $(\mathbb{U},\mathsf{P}_+)$-$\mathsf{Cat}\cong\mathsf{App}$, the category of approach spaces and non-expansive maps (see [19] for details.)

2.3 The dual $(\mathbb{T},\mathsf{V})$**-category.** We have the canonical forgetful functor

$$\begin{aligned} E : (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat} &\to \mathsf{V}\text{-}\mathsf{Cat},\\ (X,a) &\mapsto (X, a\cdot e_X)\end{aligned}$$

with left adjoint

$$\begin{aligned} E^\circ : \mathsf{V}\text{-}\mathsf{Cat} &\to (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}.\\ (X,a) &\mapsto (X, e_X^\circ\cdot Ta)\end{aligned}$$

Furthermore, (the extension of) T induces an endofunctor

$$\begin{aligned} T : V\text{-}\mathsf{Cat} &\to V\text{-}\mathsf{Cat}.\\ (X,a) &\mapsto (TX, Ta)\end{aligned}$$

If m is a (strict) natural transformation, we can represent this functor as the composite

$$\begin{tikzcd} & (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat} \arrow[dr, "M^\circ"] & \\ \mathsf{V}\text{-}\mathsf{Cat} \arrow[ur, "E^\circ"] \arrow[rr, "T"'] & & \mathsf{V}\text{-}\mathsf{Cat}, \end{tikzcd}$$

where $M^\circ : (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat} \to \mathsf{V}\text{-}\mathsf{Cat}$ is given by $(X, a) \mapsto (TX, Ta \cdot m_X^\circ)$. In fact, given a V-category (X, a), we have

$$T(e_X^\circ \cdot Ta) \cdot m_X^\circ = Te_X^\circ \cdot T^2 a \cdot m_X^\circ = Te_X^\circ \cdot m_X^\circ \cdot Ta = Ta.$$

The functors M° and E° are the keys to define the *dual* $(\mathbb{T}, \mathsf{V})$*-category* X^{op} of a $(\mathbb{T}, \mathsf{V})$-category $X = (X, a)$: we put $X^{\mathrm{op}} = E^\circ(M^\circ(X)^{\mathrm{op}})$. We point out that if X is a V-category interpreted as a $(\mathbb{T}, \mathsf{V})$-category, i.e. $X = (X, e_X^\circ \cdot Ta)$ for a given V-category structure $a : X \longrightarrow\!\!\!

2.5 $(\mathbb{T},\mathsf{V})$**-bimodules.** Given $(\mathbb{T},\mathsf{V})$-categories (X,a) and (Y,b), a $(\mathbb{T},\mathsf{V})$-*bimodule* (or simply a *bimodule*) $\psi : (X,a) \relbar\!\!\circ\!\!\rightarrow (Y,b)$ is a V-matrix $\psi : TX \relbar\!\!\!\rightarrow Y$ such that $\psi * a \leq \psi$ and $b * \psi \leq \psi$. This means that $\psi \cdot Ta \cdot m_X^\circ \leq \psi$ and $b \cdot T\psi \cdot m_X^\circ \leq \psi$; that is, for $\mathfrak{X} \in T^2X$, $\mathfrak{x} \in TX$, $\mathfrak{y} \in TY$ and $y \in Y$,

$$Ta(\mathfrak{X},\mathfrak{x}) \otimes \psi(\mathfrak{x},y) \leq \psi(m_X(\mathfrak{X}),y),$$

$$T\psi(\mathfrak{X},\mathfrak{y}) \otimes b(\mathfrak{y},y) \leq \psi(m_X(\mathfrak{X}),y).$$

Whenever the Kleisli composition is associative (in particular if $T : \mathsf{V}\text{-}\mathsf{Mat} \to \mathsf{V}\text{-}\mathsf{Mat}$ is a functor and m is a natural transformation: see [15]), bimodules compose. The identities for the composition law are again the $(\mathbb{T},\mathsf{V})$-categorical structures, and we can consider the category $(\mathbb{T},\mathsf{V})$-Mod of $(\mathbb{T},\mathsf{V})$-categories and $(\mathbb{T},\mathsf{V})$-bimodules.

2.6 $(\mathbb{T},\mathsf{V})$**-functors as** $(\mathbb{T},\mathsf{V})$**-bimodules.** Analogously to the situation in V-categories, each $(\mathbb{T},\mathsf{V})$-functor $f : (X,a) \to (Y,b)$ defines a pair of bimodules $f_* : (X,a) \relbar\!\!\circ\!\!\rightarrow (Y,b)$ and $f^* : (Y,b) \relbar\!\!\circ\!\!\rightarrow (X,a)$ as indicated in the following diagram

$$\begin{tikzcd}
TX \arrow[r, "Tf"] \arrow[d, shift right, "a"'] \arrow[d, shift left, "\dashv"] \arrow[dr, bend right=60, "f^*"'] & TY \arrow[d, shift right, "\dashv"'] \arrow[d, shift left, "b"] \arrow[dl, bend left=60, "f_*"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

that is, $f_* := b \cdot Tf$ and $f^* := f^\circ \cdot b$. In fact, the following assertions are equivalent for $(\mathbb{T},\mathsf{V})$-categories (X,a) and (Y,b) and a function $f : X \to Y$.

(i) $f : (X,a) \to (Y,b)$ is a $(\mathbb{T},\mathsf{V})$-functor.

(ii) $f_* : (X,a) \relbar\!\!\circ\!\!\rightarrow (Y,b)$ is a $(\mathbb{T},\mathsf{V})$-bimodule.

(iii) $f^* : (Y,b) \relbar\!\!\circ\!\!\rightarrow (X,a)$ is a $(\mathbb{T},\mathsf{V})$-bimodule.

We point out that, although in general bimodules do not compose, if $f : (X,a) \to (Y,b)$ is a functor, then, for any bimodules $\varphi : (Y,b) \relbar\!\!\circ\!\!\rightarrow (Z,c)$ and $\psi : (Z,c) \relbar\!\!\circ\!\!\rightarrow (Y,b)$,

$$\begin{tikzcd}
X \arrow[loop left, "a"] \arrow[r, bend left, "f_*"] \arrow[r, "f"] & Y \arrow[loop above, "b"] \arrow[l, bend left, "f^*"] \arrow[r, bend left, "\varphi"] & Z \arrow[loop right, "c"] \arrow[l, bend left, "\psi"]
\end{tikzcd}$$

$\varphi * f_*$ and $f^* * \psi$ are bimodules, as we show next. First note that

$$\varphi * f_* = \varphi \cdot Tf \qquad \text{and} \qquad f^* * \psi = f^\circ \cdot \psi.$$

The latter equality follows from

$$f^* * \psi = f^\circ \cdot b \cdot T\psi \cdot m_Z^\circ = f^\circ \cdot \psi,$$

and V-functoriality of f implies

$$\begin{aligned}
\varphi * f_* = \varphi * (b \cdot Tf) &= \varphi \cdot T(b \cdot Tf) \cdot m_X^\circ \\
&\geq \varphi \cdot T(f \cdot a) \cdot m_X^\circ && \text{(by functoriality of } f) \\
&\geq \varphi \cdot Tf \cdot Ta \cdot m_X^\circ \\
&\geq \varphi \cdot Tf \cdot Ta \cdot Te_X && \text{(since } m_X^\circ \geq Te_X) \\
&\geq \varphi \cdot Tf, && \text{(since } a \cdot e_X \geq 1_X)
\end{aligned}$$

whereby φ bimodule gives us

$$\varphi * f_* = \varphi * (b \cdot Tf) = \varphi \cdot Tb \cdot T^2 f \cdot m_X^\circ \leq \varphi \cdot Tb \cdot m_Y^\circ \cdot Tf = \varphi \cdot Tf.$$

The bimodule properties of $\varphi * f_*$ and $f^* * \psi$ follow now from

$$\begin{aligned}
&c * (\varphi * f_*) = c * (\varphi \cdot Tf) \leq (c * \varphi) \cdot Tf = \varphi * f_*, \\
&(\varphi * f_*) * a = \varphi \cdot Tf \cdot Ta \cdot m_X^\circ \leq \varphi \cdot Tb \cdot T^2 f \cdot m_X^\circ \leq \varphi \cdot Tb \cdot m_Y^\circ \cdot Tf = \varphi * f_*, \\
&a * (f^* * \psi) = a \cdot T(f^\circ \cdot \psi) \cdot m_Z^\circ = a \cdot Tf^\circ \cdot T\psi \cdot m_Z^\circ \leq f^\circ \cdot b \cdot T\psi \cdot m_Z^\circ = f^\circ \cdot \psi = f^* * \psi, \\
&(f^* * \psi) * c = f^\circ \cdot \psi \cdot Tc \cdot m_Z^\circ = f^\circ \cdot (\psi * c) = f^\circ \cdot \psi = f^* * \psi.
\end{aligned}$$

Therefore we can define the "whiskering" functors

$$\begin{aligned}
- * f_* : (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Mod}(Y, Z) &\longrightarrow (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Mod}(X, Z), \text{ and} \\
\varphi &\longmapsto \varphi \cdot Tf
\end{aligned}$$

$$\begin{aligned}
f^* * - : (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Mod}(Z, Y) &\longrightarrow (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Mod}(Z, X) \\
\psi &\longmapsto f^\circ \cdot \psi.
\end{aligned}$$

Moreover, given a pair of adjoint bimodules $(\varphi : (Y, b) \multimap (Z, c)) \dashv (\psi : (Z, c) \multimap (Y, b))$, we have

$$\varphi * f_* \dashv f^* * \psi,$$

provided that the diagram

$$\begin{tikzcd}
T^2 X \arrow[r, "m_X"] \arrow[d, "T^2 f"'] & TX \arrow[d, "Tf"] \\
T^2 Y \arrow[r, "m_Y"] & TY
\end{tikzcd}$$

satisfies (BC): $(\varphi * f_*) * (f^* * \psi) \leq c$ is always true, since

$$(\varphi \cdot Tf) * (f^\circ \cdot \psi) = \varphi \cdot Tf \cdot Tf^\circ \cdot T\psi \cdot m_Z^\circ \leq \varphi \cdot T\psi \cdot m_Z^\circ = \varphi * \psi \leq c,$$

while to conclude that $a \leq (f^* * \psi) * (\varphi * f_*)$ we need the condition above:

$$a \leq f^\circ \cdot b \cdot Tf \leq f^\circ \cdot \psi \cdot T\varphi \cdot m_Y^\circ \cdot Tf = f^\circ \cdot \psi \cdot T\varphi \cdot T^2 f \cdot m_X^\circ = (f^\circ \cdot \psi) * (\varphi \cdot Tf).$$

2.7 Lawvere-complete $(\mathbb{T}, \mathsf{V})$-categories.

Definition. A $(\mathbb{T},\mathsf{V})$-category (X,a) is called *Lawvere-complete* if, for each $(\mathbb{T},\mathsf{V})$-category (Y,b) and each pair of adjoint bimodules

$$\begin{tikzcd} Y \arrow[loop left, "b"] \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\perp"] & X \arrow[loop right, "a"] \arrow[l, bend left, "\psi"] \end{tikzcd}$$

there exists a functor $f:(Y,b)\to(X,a)$ such that $f_*=\varphi$ and $f^*=\psi$.

Analogously to the V-categorical case, Lawvere completeness is fully tested by left adjoint bimodules with domain $(1,p)$, where $p=e_1^\circ$, hence $p(\overset{\bullet}{\star},\star)=k$ and $p(\mathfrak{x},\star)=\perp$ for $\mathfrak{x}\neq\overset{\bullet}{\star}$ in $T1$.

Proposition. *Assume that either $T1=1$ or that the (Set-based) natural transformation m satisfies* (BC)*. Then, for a $(\mathbb{T},\mathsf{V})$-category (X,a), the following conditions are equivalent:*

(i) *(X,a) is Lawvere-complete;*

(ii) *each pair of adjoint bimodules* $\begin{tikzcd} (1,p) \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\perp"] & (X,a) \arrow[l, bend left, "\psi"] \end{tikzcd}$ *is induced by a functor $f:(1,p)\to(X,a)$;*

(iii) *each pair of adjoint bimodules* $\begin{tikzcd} (1,p) \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\perp"] & (X,a) \arrow[l, bend left, "\psi"] \end{tikzcd}$ *is induced by a map $f:1\to X$ (so that $\varphi=a\cdot Tf$ and $\psi=f^\circ\cdot a$).*

Proof. (iii) $\Rightarrow$ (i): Let $\begin{tikzcd} (Y,b) \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\perp"] & (X,a) \arrow[l, bend left, "\psi"] \end{tikzcd}$ be a pair of adjoint bimodules. For each $y\in Y$, let $g_y:(1,p)\to(Y,b)$ be the functor that picks y. This functor induces a pair of adjoint bimodules $(g_y)_*\dashv(g_y)^*$, whence we have

$$\begin{tikzcd} 1 \arrow[loop left, "p"] \arrow[r, bend left, "(g_y)_*"] & Y \arrow[loop above, "b"] \arrow[l, bend left, "(g_y)^*"] \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\perp"] & X \arrow[loop right, "a"] \arrow[l, bend left, "\psi"] \end{tikzcd}$$

If m satisfies (BC), we know already that $\varphi_y\dashv\psi_y$, where $\varphi_y=\varphi*(g_y)_*=\varphi\cdot Tg_y$ and $\psi_y=(g_y)^**\psi=g_y^\circ\cdot\psi$. The same happens whenever $T1=1$, as it is easily checked. By hypothesis, there exists a map $f_y:1\to X$ such that $\varphi_y=b\cdot Tf_y$ and $\psi_y=f_y^\circ\cdot b$. Gluing together the maps $(f_y)_{y\in Y}$ we obtain a map $f:Y\to X$. Then, for $\mathfrak{x}\in TX$ and $y\in Y$,

$$\psi(\mathfrak{x},y)=\psi_y(\mathfrak{x},\star)=f_y^\circ\cdot a(\mathfrak{x},\star)=a(\mathfrak{x},f_y(\star))=a(\mathfrak{x},f(y)),$$

that is, $\psi=f^*=f^\circ\cdot a$. We can show now that f is necessarily a functor:

$$b\cdot Tf^\circ\leq b\cdot Tf^\circ\cdot Ta\cdot Te_X\leq b\cdot Tf^\circ\cdot Ta\cdot m_X^\circ=b*\psi\leq\psi=f^\circ\cdot a.$$

This concludes the proof since, by unicity of adjoints, φ is necessarily f_*. $\square$

3 V as a $(\mathbb{T}, \mathsf{V})$-category

3.1 The $\mathbb{T}$-algebra structure of V. Our next goal is to explore the notions introduced in the previous section. In particular we are aiming for results which extend known facts about V-categories (as Theorem 1.5 or Theorem 1.6). To do so, from now on *we will always assume that the extension $T : \mathsf{V}\text{-}\mathsf{Mat} \to \mathsf{V}\text{-}\mathsf{Mat}$ is constructed as in [8] and consequently we assume V to be constructively completely distributive. Furthermore, we assume that $\mathbb{T} = (T, e, m)$ is non-trivial and that T and m satisfy (BC).*

Under these conditions, as Manes essentially showed in [21],

$$\begin{aligned} \xi : T\mathsf{V} &\longrightarrow \mathsf{V} \\ \mathfrak{x} &\longmapsto \bigvee\{v \in \mathsf{V} \mid \mathfrak{x} \in T(\uparrow v)\} \end{aligned}$$

is a $\mathbb{T}$-algebra structure on V, where $\uparrow v = \{u \in \mathsf{V} \mid v \leq u\}$.

There is an interesting link between this $\mathbb{T}$-algebra structure and the image under the lax functor $T : \mathsf{V}\text{-}\mathsf{Mat} \to \mathsf{V}\text{-}\mathsf{Mat}$ of the identity $1_{\mathsf{V}} : \mathsf{V} \to \mathsf{V}$ considered as a matrix $i : 1 \nrightarrow \mathsf{V}$, with $i(\star, v) = v$. Let us compute $Ti : T1 \nrightarrow T\mathsf{V}$. We consider, for each $v \in \mathsf{V}$, the relation

$$\begin{aligned} i_v : 1 \times V &\longrightarrow 2 \\ (\star, u) &\longmapsto \begin{cases} \text{true} & \text{if } v \leq u, \\ \text{false} & \text{elsewhere,} \end{cases} \end{aligned}$$

hence the diagram

$$\begin{tikzcd} 1 \arrow[rr, "i_v", "\shortmid"{marking}] \arrow[dr, "q_v^\circ"'] & & V, \\ & \uparrow v \arrow[ur, "p_v"'] & \end{tikzcd}$$

commutes where p_v and q_v are the projections. Now, for each $\mathfrak{x} \in T1$ and $\mathfrak{y} \in T\mathsf{V}$,

$$\begin{aligned} Ti(\mathfrak{x}, \mathfrak{y}) &= \bigvee\{v \in \mathsf{V} \mid T(i_v)(\mathfrak{x}, \mathfrak{y}) = \text{true}\} \\ &= \bigvee\{v \in \mathsf{V} \mid Tp_v Tq_v^\circ(\mathfrak{x}, \mathfrak{y}) = \text{true}\} \\ &= \bigvee\{v \in \mathsf{V} \mid \exists \mathfrak{z} \in \uparrow v \,:\, Tq_v(\mathfrak{z}) = \mathfrak{x} \text{ and } Tp_v(\mathfrak{z}) = \mathfrak{y}\}, \end{aligned}$$

hence, since T preserves injections and considering Tp_v as an inclusion, we can write

$$Ti(\mathfrak{x}, \mathfrak{y}) = \bigvee\{v \in \mathsf{V} \mid \mathfrak{y} \in T(\uparrow v) \text{ and } Tq_v(\mathfrak{y}) = \mathfrak{x}\} \leq \xi(\mathfrak{y}),$$

by definition of ξ. In particular, if $\mathfrak{x} = Tq(\mathfrak{y})$, for $q : \mathsf{V} \to 1$, then $Ti(\mathfrak{x}, \mathfrak{y}) = \xi(\mathfrak{y})$. Whenever $T1 = 1$, $Tq(\mathfrak{y}) = \overset{\bullet}{\star}$ for every $\mathfrak{y} \in T\mathsf{V}$, whence

$$Ti(\overset{\bullet}{\star}, \mathfrak{y}) = \bigvee\{v \in \mathsf{V} \mid \mathfrak{y} \in T(\uparrow v)\} = \xi(\mathfrak{y}).$$

This link between the extension of T and the $\mathbb{T}$-algebra structure ξ is more general. Whenever necessary, in the sequel we denote the Set-endofunctor T by T_{o}, and keep T for its extension to $\mathsf{V}\text{-}\mathsf{Mat}$. Each V-matrix $r : X \nrightarrow Y$ can be considered also as a map $r : X \times Y \to \mathsf{V}$. The interplay between $T_{\mathrm{o}} r$ and Tr is stated in the following result, whose proof is straightforward.

Proposition. *For any* V*-matrix* $r : X \longrightarrow\!\!\!\!\!\!+\; Y$*, each* $\mathfrak{x} \in TX$ *and* $\mathfrak{y} \in TY$*,*

$$Tr(\mathfrak{x}, \mathfrak{y}) = \bigvee_{\mathfrak{w}:\ \substack{T_{\mathrm{o}}\pi_X(\mathfrak{w})=\mathfrak{x} \\ T_{\mathrm{o}}\pi_Y(\mathfrak{w})=\mathfrak{y}}} \xi \cdot T_{\mathrm{o}} r(\mathfrak{w}),$$

that is the following diagram

$$\begin{tikzcd} T(X \times Y) \arrow[r, "T_{\mathrm{o}} r"] & T\mathsf{V} \arrow[d, "\xi"] \\ TX \times TY \arrow[u, "<T_{\mathrm{o}}\pi_X, T_{\mathrm{o}}\pi_Y>"] \arrow[r, "Tr"] & \mathsf{V} \end{tikzcd}$$

commutes.

Remark. Besides being the structure map of an Eilenberg-Moore algebra, $\xi : T\mathsf{V} \to \mathsf{V}$ satisfies also the inequalities

$$\otimes \cdot \langle \xi \cdot T_{\mathrm{o}}\pi_1, \xi \cdot T_{\mathrm{o}}\pi_2 \rangle \leq \xi \cdot T_{\mathrm{o}}(\otimes) \qquad \text{and} \qquad k \cdot ! \leq \xi \cdot T_{\mathrm{o}} k.$$

$$\begin{tikzcd} T(\mathsf{V} \times \mathsf{V}) \arrow[r, "T_{\mathrm{o}}(\otimes)"] \arrow[d, "\langle \xi \cdot T_{\mathrm{o}}\pi_1, \xi \cdot T_{\mathrm{o}}\pi_2 \rangle"'] \arrow[dr, phantom, "\leq"] & T\mathsf{V} \arrow[d, "\xi"] \\ \mathsf{V} \times \mathsf{V} \arrow[r, "\otimes"'] & \mathsf{V} \end{tikzcd} \qquad \begin{tikzcd} T1 \arrow[r, "T_{\mathrm{o}} k"] \arrow[d, "!"'] \arrow[dr, phantom, "\leq"] & T\mathsf{V} \arrow[d, "\xi"] \\ 1 \arrow[r, "k"'] & \mathsf{V} \end{tikzcd}$$

Recall that we assume $Tf = T_{\mathrm{o}} f$ for each Set-map $f : X \to Y$; this condition requires and implies equality in the latter inequality (see [16]).

3.2 The canonical $(\mathbb{T}, \mathsf{V})$-categorical structure of V. The composition of the natural V-categorical and $\mathbb{T}$-algebra structures of V defines an interesting structure, $\hom_\xi$, of a $(\mathbb{T}, \mathsf{V})$-category on V

$$T\mathsf{V} \overset{\hom_\xi}{\longrightarrow\!\!\!\!\!\!\circ} \mathsf{V} = (T\mathsf{V} \overset{\xi}{\to} \mathsf{V} \overset{\hom}{\longrightarrow\!\!\!\!\!\!\circ} \mathsf{V}),$$

as we show next.

Proposition. $\xi : (T\mathsf{V}, T\hom) \to (\mathsf{V}, \hom)$ *is a* V*-functor.*

Proof. We have to show that $\xi \cdot T\hom \leq \hom \cdot \xi$, or, equivalently, $T\hom \leq \xi^\circ \cdot \hom \cdot \xi$. This means that, for $\mathfrak{x}, \mathfrak{y} \in T\mathsf{V}$,

$$T\hom(\mathfrak{x}, \mathfrak{y}) \leq \hom(\xi(\mathfrak{x}), \xi(\mathfrak{y})).$$

We consider again the matrix $i : 1 \longrightarrow\!\!\!\!\!\!+\; V$, and compute $1 \overset{i}{\to} \mathsf{V} \overset{\hom}{\longrightarrow\!\!\!\!\!\!\circ} \mathsf{V}$:

$$\hom \cdot i(\star, v) = \bigvee_{u \in V} i(\star, u) \otimes \hom(u, v) = \bigvee_{u \in \mathsf{V}} u \otimes \hom(u, v) \leq v;$$

that is $\hom \cdot i \leq i$. Hence $T\hom \cdot Ti \leq T(\hom \cdot i) \leq Ti$, and so, for $\mathfrak{x}, \mathfrak{y} \in T\mathsf{V}$ and $\mathfrak{z} = Tq(\mathfrak{x})$ as in Section 3.1, we have

$$\xi(\mathfrak{x}) \otimes T\hom(\mathfrak{x}, \mathfrak{y}) = Ti(\mathfrak{z}, \mathfrak{x}) \otimes T\hom(\mathfrak{x}, \mathfrak{y}) \leq Ti(\mathfrak{z}, \mathfrak{y}) \leq \xi(\mathfrak{y}),$$

and therefore

$$T\hom(\mathfrak{x}, \mathfrak{y}) \leq \hom(\xi(\mathfrak{x}), \xi(\mathfrak{y}))$$

as claimed. $\square$

Corollary. $(\mathsf{V}, \hom_\xi)$ *is a* $(\mathbb{T}, \mathsf{V})$*-category.*

Proof. Follows from the proposition above and Lemma 2.2. □

3.3 The tensor product. The tensor product in V defines in a natural way a (not necessarily closed) product structure in $(\mathbb{T}, \mathsf{V})$-Cat. Given $(\mathbb{T}, \mathsf{V})$-categories $X = (X, a)$ and $Y = (Y, b)$, we put $X \otimes Y = (X \times Y, a \otimes b)$ where $a \otimes b(\mathfrak{w}, (x, y)) = a(T\pi_X(\mathfrak{w}), x) \otimes b(T\pi_Y(\mathfrak{w}), y)$ for all $\mathfrak{w} \in T(X \times Y)$, $x \in X$ and $y \in Y$. One easily verifies reflexivity of $a \otimes b$, while transitivity holds if and only if $\otimes \cdot \langle \xi \cdot T_o\pi_1, \xi \cdot T_o\pi_2 \rangle = \xi \cdot T_o(\otimes)$ (see Remark 3.1 and [16]) *which we assume from now on.* We remark that this condition guarantees that $\mathbb{T}$ is a *(lax) Hopf monad* on V-Mat (see [22]) where the tensor product in V is naturally extended to V-Mat. However, we will not develop this aspect here.

It is well-known that in general the functor $X \otimes _ : (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat} \to (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat}$ has no right adjoint as, for example, Top is not Cartesian closed. The problem of characterising those $(\mathbb{T}, \mathsf{V})$-categories $X = (X, a)$ such that tensoring with X has a right adjoint is studied in [16].

Theorem. *Let m be a natural transformation. For $(\mathbb{T}, \mathsf{V})$-categories (X, a) and (Y, b) and a V-matrix $\psi : TX \nrightarrow Y$, the following assertions are equivalent.*

(i) $\psi : (X, a) \multimap (Y, b)$ *is a* $(\mathbb{T}, \mathsf{V})$*-bimodule.*

(ii) *Both* $\psi : |X| \otimes Y \to \mathsf{V}$ *and* $\psi : X^{\mathrm{op}} \otimes Y \to \mathsf{V}$ *are* $(\mathbb{T}, \mathsf{V})$*-functors.*

Proof. Assume that $\psi : (X, a) \multimap (Y, b)$ is a $(\mathbb{T}, \mathsf{V})$-bimodule. First observe that, for $\mathfrak{W} \in T(TX \times Y)$,

$$\xi \cdot T_o\psi(\mathfrak{W}) \leq T\psi(T_o\pi_{TX}(\mathfrak{W}), T_o\pi_Y(\mathfrak{W})).$$

Let $\mathfrak{W} \in T(TX \times Y)$, $\mathfrak{x} \in TX$ and $y \in Y$. To see that $\psi : |X| \otimes Y \to \mathsf{V}$ is a $(\mathbb{T}, \mathsf{V})$-functor, note that the structure c on $|X| \otimes Y$ is given by

$$c(\mathfrak{W}, (\mathfrak{x}, y)) = \begin{cases} \bot & \text{if } \mathfrak{x} \neq m_X(T_o\pi_{TX}(\mathfrak{W})), \\ b(T\pi_Y(\mathfrak{W}), y) & \text{if } \mathfrak{x} = m_X(T_o\pi_{TX}(\mathfrak{W})). \end{cases}$$

Assume $\mathfrak{x} = m_X(T_o\pi_{TX}(\mathfrak{W}))$. Since

$$b(T_o\pi_Y(\mathfrak{W}), y) \leq \hom(\xi \cdot T_o\psi(\mathfrak{W}), \psi(\mathfrak{x}, y))$$

is equivalent to

$$\xi \cdot T_o\psi(\mathfrak{W}) \otimes b(T_o\pi_Y(\mathfrak{W}), y) \leq \psi(\mathfrak{x}, y),$$

the assertion follows at once. We show now that $\psi : X^{\mathrm{op}} \otimes Y \to \mathsf{V}$ is a $(\mathbb{T}, \mathsf{V})$-functor. As above we have that (with $a^{\mathrm{op}} = e^\circ_{TX} \cdot Tm_X \cdot T^2a^\circ$ the structure on X^{op})

$$a^{\mathrm{op}}(T_o\pi_{TX}(\mathfrak{W}), \mathfrak{x}) \otimes b(T_o\pi_Y(\mathfrak{W}), y) \leq \hom(\xi \cdot T_o\psi(\mathfrak{W}), \psi(\mathfrak{x}, y))$$

is equivalent to

$$\xi \cdot T_o\psi(\mathfrak{W}) \otimes a^{\mathrm{op}}(T_o\pi_{TX}(\mathfrak{W}), \mathfrak{x}) \otimes b(T_o\pi_Y(\mathfrak{W}), y) \leq \psi(\mathfrak{x}, y).$$

Now

$$\begin{aligned}
&\xi \cdot T_{\mathrm{o}}\psi(\mathfrak{W}) \otimes a^{\mathrm{op}}(T_{\mathrm{o}}\pi_{TX}(\mathfrak{W}), \mathfrak{x}) \otimes b(T_{\mathrm{o}}\pi_Y(\mathfrak{W}), y) \\
&\qquad \le T^2 a \cdot T m_X^{\circ} \cdot e_{TX}(\mathfrak{x}, T_{\mathrm{o}}\pi_{TX}(\mathfrak{W})) \otimes T\psi(T_{\mathrm{o}}\pi_{TX}(\mathfrak{W}), T_{\mathrm{o}}\pi_Y(\mathfrak{W})) \otimes b(T_{\mathrm{o}}\pi_Y(\mathfrak{W}), y) \\
&\qquad \le b \cdot T\psi \cdot T^2 a \cdot T m_X^{\circ} \cdot m_X^{\circ}(\mathfrak{x}, y) \\
&\qquad \le b \cdot T\psi \cdot m_X \cdot T a \cdot m_X^{\circ}(\mathfrak{x}, y) \\
&\qquad = \psi \cdot T a \cdot m_X^{\circ}(\mathfrak{x}, y) = \psi(\mathfrak{x}, y).
\end{aligned}$$

Now assume that $\psi : |X| \otimes Y \to \mathsf{V}$ and $\psi : X^{\mathrm{op}} \otimes Y \to \mathsf{V}$ are $(\mathbb{T}, \mathsf{V})$-functors. Functoriality of $\psi : |X| \otimes Y \to \mathsf{V}$ implies, for all $\mathfrak{x} \in TX$ and $y \in Y$,

$$\begin{aligned}
\psi(\mathfrak{x}, y) &\ge \bigvee_{\substack{\mathfrak{X} \in TTX: \\ m_X(\mathfrak{X}) = \mathfrak{x}; \\ \mathfrak{y} \in TY}} \bigvee \Big\{ \xi \cdot T_{\mathrm{o}}\psi(\mathfrak{W}) \otimes b(\mathfrak{y}, y) \;\Big|\; \mathfrak{W} \in T(TX \times Y) : \mathfrak{W} \mapsto \mathfrak{X}, \mathfrak{W} \mapsto \mathfrak{y} \Big\} \\
&= \bigvee_{\substack{\mathfrak{X} \in TTX: \\ m_X(\mathfrak{X}) = \mathfrak{x}; \\ \mathfrak{y} \in TY}} T\psi(\mathfrak{X}, \mathfrak{y}) \otimes b(\mathfrak{y}, y) \\
&= \bigvee_{\substack{\mathfrak{X} \in TTX: \\ m_X(\mathfrak{X}) = \mathfrak{x}}} b \cdot T\psi(\mathfrak{X}, y) \\
&= b \cdot T\psi \cdot m_X^{\circ}(\mathfrak{x}, y).
\end{aligned}$$

On the other hand, by functoriality of $\psi : X^{\mathrm{op}} \otimes Y \to \mathsf{V}$, for all $\mathfrak{x} \in TX$ and $y \in Y$ we have

$$\begin{aligned}
\psi(\mathfrak{x}, y) &\ge \bigvee_{\substack{\mathfrak{X} \in TTX, \\ \mathfrak{y} \in TY}} \bigvee \Big\{ \xi \cdot T_{\mathrm{o}}\psi(\mathfrak{W}) \otimes b(\mathfrak{y}, y) \otimes a^{\mathrm{op}}(\mathfrak{X}, \mathfrak{x}) \;\Big|\; \mathfrak{W} \in T(TX \times Y) : \mathfrak{W} \mapsto \mathfrak{X}, \mathfrak{W} \mapsto \mathfrak{y} \Big\} \\
&= \bigvee_{\substack{\mathfrak{X} \in TTX, \\ \mathfrak{y} \in TY}} T\psi(\mathfrak{X}, \mathfrak{y}) \otimes b(\mathfrak{y}, y) \otimes T^2 a \cdot T m_X^{\circ} \cdot e_{TX}(\mathfrak{x}, \mathfrak{X}) \\
&= b \cdot T\psi \cdot T^2 a \cdot T m_X^{\circ} \cdot e_{TX}(\mathfrak{x}, \mathfrak{X}) \\
&\ge b \cdot e_Y \cdot \psi \cdot T a \cdot m_X^{\circ}(\mathfrak{x}, y) \\
&\ge \psi \cdot T a \cdot m_X^{\circ}(\mathfrak{x}, y).
\end{aligned}$$

□

3.4 V is Lawvere-complete.

Theorem. *Assume that $T1 = 1$. Then $(V, \hom_{\xi})$ is a Lawvere-complete $(\mathbb{T}, \mathsf{V})$-category provided that $a := M^{\circ}(\hom_{\xi}) = \xi^{\circ} \cdot \hom \cdot \xi$ (i.e., for $\mathfrak{v}, \mathfrak{w} \in T\mathsf{V}$, $a(\mathfrak{v}, \mathfrak{w}) = \hom(\xi(\mathfrak{v}), \xi(\mathfrak{w}))$).*

Proof. Let

$$\begin{tikzcd}
p \arrow[loop left, "1"] & 1 \arrow[r, bend left, "\varphi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\perp"] & V \arrow[loop right, "\hom_{\xi}"]
\end{tikzcd}$$

be a pair of adjoint bimodules. By the previous theorem we know that:

$$\begin{aligned}
(4) \qquad \varphi \text{ bimodule} &\iff \varphi : (\mathsf{V}, \hom_{\xi}) \to (\mathsf{V}, \hom_{\xi}) \text{ is a } (\mathbb{T}, \mathsf{V})\text{-functor} \\
&\iff \forall \mathfrak{v} \in T\mathsf{V} \;\; \forall v \in \mathsf{V} \;\; \hom(\xi(\mathfrak{v}), v) \le \hom(\xi \cdot T\varphi(\mathfrak{v}), \varphi(v)).
\end{aligned}$$

In particular, for every $\mathfrak{v} \in T\mathsf{V}$, $k \leq \hom(\xi(\mathfrak{v}), \xi(\mathfrak{v})) \leq \hom(\xi \cdot T\varphi(\mathfrak{v}), \varphi \cdot \xi(\mathfrak{v}))$, hence $\xi \cdot T\varphi(\mathfrak{v}) \leq \varphi \cdot \xi(\mathfrak{v})$.

$$\psi \text{ bimodule} \iff \psi : (T\mathsf{V}, a^\circ) \to (V, \hom) \text{ is a } \mathsf{V}\text{-functor} \tag{5}$$
$$\iff \forall \mathfrak{v}, \mathfrak{w} \in T\mathsf{V}\ a(\mathfrak{v}, \mathfrak{w}) \leq \hom(\psi(\mathfrak{w}), \psi(\mathfrak{v})).$$

Finally,

$$\varphi \dashv \psi \iff \begin{cases} (a) & \varphi * \psi \leq \hom \cdot \xi \iff \forall \mathfrak{v} \in T\mathsf{V}\, \forall v \in \mathsf{V}\ \psi(\mathfrak{v}) \otimes \varphi(v) \leq \hom(\xi(\mathfrak{v}), v), \\ (b) & p \leq \psi * \varphi \iff k \leq \bigvee_{\mathfrak{u} \in TV} \psi(\mathfrak{u}) \otimes \xi(T_{\mathrm{o}}\varphi(\mathfrak{u})). \end{cases} \tag{6}$$

We will show that the adjunction $\varphi \dashv \psi$ is represented by $\psi(\overset{\bullet}{k}\,)$, where $\overset{\bullet}{k} = e_{\mathsf{V}}(k)$. Similarly to the proof of Theorem 1.6, we split our argument in three steps:

(1st) $\psi(\overset{\bullet}{k}\,) = \bigvee_{\mathfrak{v} \in TV} \psi(\mathfrak{v}) \otimes \xi(\mathfrak{v})$:

"$\leq$" is immediate; for "$\geq$" we argue as follows:

$$\begin{aligned} \psi(\mathfrak{v}) \otimes \xi(\mathfrak{v}) &= \psi(\mathfrak{v}) \otimes \hom(\xi(\overset{\bullet}{k}\,), \xi(\mathfrak{v})) \\ &= \psi(\mathfrak{v}) \otimes a(\overset{\bullet}{k}\,, \mathfrak{v}) && \text{(by hypothesis)} \\ &\leq \psi(\mathfrak{v}) \otimes \hom(\psi(\mathfrak{v}), \psi(\overset{\bullet}{k}\,)) && \text{(by (5))} \\ &\leq \psi(\overset{\bullet}{k}\,). \end{aligned}$$

(2nd) $\forall \mathfrak{v} \in TV\ \hom_{\xi}(\mathfrak{v}, \psi(\overset{\bullet}{k}\,)) = \bigvee_{\mathfrak{u} \in TV} \hom(\xi(\mathfrak{v}), \xi(\mathfrak{u})) \otimes \psi(\mathfrak{u})$:

To check "$\geq$" we just observe that

$$\xi(\mathfrak{v}) \otimes (\hom(\xi(\mathfrak{v}), \xi(\mathfrak{u})) \otimes \psi(\mathfrak{u})) \leq \xi(\mathfrak{u}) \otimes \psi(\mathfrak{u}) \leq \psi(\overset{\bullet}{k}\,).$$

For "$\leq$", first note that

$$\psi(\overset{\bullet}{k}\,) \otimes \varphi(\xi(\mathfrak{u})) \leq \hom(\xi(\overset{\bullet}{k}\,), \xi(\mathfrak{u})) = \hom(k, \xi(\mathfrak{u})) = \xi(\mathfrak{u})$$

from which follows

$$\xi(T\varphi(\mathfrak{u})) \leq \varphi(\xi(\mathfrak{u})) \leq \hom(\psi(\overset{\bullet}{k}\,), \xi(\mathfrak{u})). \tag{7}$$

From that we conclude that

$$\begin{aligned} \hom(\xi(\mathfrak{v}), \psi(\overset{\bullet}{k}\,)) &\leq \hom(\xi(\mathfrak{v}), \psi(\overset{\bullet}{k}\,)) \otimes \bigvee_{\mathfrak{u} \in TV} \psi(\mathfrak{u}) \otimes \xi(T\varphi(\mathfrak{u})) && \text{(by (6b))} \\ &= \bigvee_{\mathfrak{u} \in TV} \hom(\xi(\mathfrak{v}), \psi(\overset{\bullet}{k}\,)) \otimes \psi(\mathfrak{u}) \otimes \xi(T\varphi(\mathfrak{u})) \\ &\leq \bigvee_{\mathfrak{u} \in TV} \hom(\xi(\mathfrak{v}), \psi(\overset{\bullet}{k}\,)) \otimes \hom(\psi(\overset{\bullet}{k}\,), \xi(\mathfrak{u})) \otimes \psi(\mathfrak{u}) && \text{(by (7))} \\ &\leq \bigvee_{\mathfrak{u} \in TV} \hom(\xi(\mathfrak{v}), \xi(\mathfrak{u})) \otimes \psi(\mathfrak{u}). \end{aligned}$$

(3rd) $\forall \mathfrak{v} \in T\mathsf{V}\ \psi(\mathfrak{v}) = \bigvee_{\mathfrak{u} \in T\mathsf{V}} a(\mathfrak{v}, \mathfrak{u}) \otimes \psi(\mathfrak{u})$:

For "$\leq$" take $\mathfrak{u} = \mathfrak{v}$; for "$\geq$" we use (5): $a^\circ(\mathfrak{u}, \mathfrak{v}) \otimes \psi(\mathfrak{u}) \leq \hom(\psi(\mathfrak{u}), \psi(\mathfrak{v})) \otimes \psi(\mathfrak{u}) \leq \psi(\mathfrak{v})$. $\square$

Lemma. *Assume that* $T1 = 1$. *Then* $T(\hom_\xi) \cdot m_{\mathsf{V}}^\circ = \xi^\circ \cdot \hom \cdot \xi$ *provided that*

$$\xi \cdot T_{\mathrm{o}} \hom(u, _) \leq \hom(u, _) \cdot \xi, \tag{8}$$

$$\begin{tikzcd} T\mathsf{V} \arrow[r, "T_{\mathrm{o}}(\hom(u,_))"] \arrow[d, "\xi"'] \arrow[dr, phantom, "\leq"] & T\mathsf{V} \arrow[d, "\xi"] \\ \mathsf{V} \arrow[r, "\hom(u,_)"'] & \mathsf{V} \end{tikzcd}$$

for each $u \in \mathsf{V}$. *The inequality* (8) *is surely true if* $\hom(u, _)$ *preserves non-empty suprema.*

Proof. First observe that

$$\begin{aligned} T(\hom_\xi) \cdot m_V^\circ &= T \hom \cdot T\xi \cdot m_{\mathsf{V}}^\circ \\ &\leq \xi^\circ \cdot \hom \cdot \xi \cdot T\xi \cdot m_{\mathsf{V}}^\circ \qquad \text{(because } \xi \text{ is a } \mathsf{V}\text{-functor, by Proposition 3.2)} \\ &= \xi^\circ \cdot \hom \cdot \xi \cdot m_{\mathsf{V}} \cdot m_V^\circ \\ &\leq \xi^\circ \cdot \hom \cdot \xi. \end{aligned}$$

On the other hand, for $\mathfrak{u}, \mathfrak{v} \in T\mathsf{V}$, we have

$$\begin{aligned} a(\mathfrak{u}, \mathfrak{v}) &\geq T \hom_\xi(\dot{\mathfrak{u}}, \mathfrak{v}) \\ &= T \hom(T_{\mathrm{o}}\xi(\dot{\mathfrak{u}}), \mathfrak{v}) \\ &= T \hom(\xi(\dot{\mathfrak{u}}), \mathfrak{v}) \\ &= \xi \cdot T_{\mathrm{o}} \hom \cdot T_{\mathrm{o}}\langle \xi(\mathfrak{u}), 1_{\mathsf{V}} \rangle(\mathfrak{v}) \qquad (*) \\ &\geq \hom(\xi(\mathfrak{u}), _) \cdot \xi(\mathfrak{v}) = \hom(\xi(\mathfrak{u}), \xi(\mathfrak{v})). \end{aligned}$$

To see (*), just observe that $T_{\mathrm{o}}\langle \xi(\mathfrak{u}), 1_{\mathsf{V}} \rangle(\mathfrak{v})$ is the only element of $T(\mathsf{V} \times \mathsf{V})$ which projects to both $\xi(\dot{\mathfrak{u}})$ and $\mathfrak{v}$. Assume now that $\hom(u, _)$ preserves non-empty suprema and let $u \in \mathsf{V}$ and $\mathfrak{u} \in T\mathsf{V}$. Then

$$\begin{aligned} \hom(u, \xi(\mathfrak{u})) &= \hom(u, \bigvee\{v \in \mathsf{V} \mid \mathfrak{u} \in T(\uparrow v)\}) \\ &= \bigvee\{\hom(u, v) \mid v \in \mathsf{V}, \mathfrak{u} \in T(\uparrow v)\}) \\ &\leq \bigvee\{v \in \mathsf{V} \mid T_{\mathrm{o}} \hom(u, _)(\mathfrak{u}) \in T(\uparrow v)\}). \end{aligned}$$

□

4 A Yoneda Lemma for $(\mathbb{T}, \mathsf{V})$-categories

4.1 Function spaces. In this section we wish to obtain the analogue result to Corollary 1.5 in the setting of $(\mathbb{T}, \mathsf{V})$-categories. This in turn requires an understanding of the right adjoint to $X \otimes _ : (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat} \to (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat}$, a problem studied in [16]. From there we import the following result.

Proposition. *Let* $X = (X, a)$ *be a* $(\mathbb{T}, \mathsf{V})$*-category. Then* $X \otimes _ : (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat} \to (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat}$ *has a right adjoint* $_^X$ *provided that* $a \cdot Ta = a \cdot m_X$.

Certainly, each (Eilenberg-Moore) $\mathbb{T}$-algebra, considered as a $(\mathbb{T}, \mathsf{V})$-category, satisfies the condition above. Moreover, the $(\mathbb{T}, \mathsf{V})$-categorical structure (X, a) induced by any V-category $X = (X, r)$ (see 2.3) satisfies this condition if $Te_X \cdot e_X = m_X^\circ \cdot e_X$.

Let $X = (X, a)$ and (Y, b) be $(\mathbb{T}, \mathsf{V})$-categories, and assume that $a \cdot Ta = a \cdot m_X$. Then Y^X has as underlying set

$$\{h : (X, a) \otimes (1, p) \to (Y, b) \mid h \text{ is a } (\mathbb{T}, \mathsf{V})\text{-functor}\},$$

thanks to the bijection (with $P = (1, p)$)

$$\frac{X \otimes P \to Y}{P \to Y^X}.$$

The structure $[\![a, b]\!]$ on Y^X is the largest structure making the evaluation map

$$\mathrm{ev} : X \otimes Y^X \to Y,\ (x, h) \mapsto h(x)$$

a $(\mathbb{T}, \mathsf{V})$-functor: for $\mathfrak{p} \in T(Y^X)$ and $h \in Y^X$ we have

$$[\![a, b]\!](\mathfrak{p}, h) = \bigvee \left\{ v \in \mathsf{V} \;\middle|\; \forall \mathfrak{q} \in T\pi_{Y^X}^{-1}(\mathfrak{p}), x \in X \;\; a(T\pi_X(\mathfrak{q}), x) \otimes v \leq b(T\mathrm{ev}(\mathfrak{q}), h(x)) \right\}.$$

4.2 The Yoneda Embedding. By Theorem 3.3, the bimodule $a : X \multimap X$ gives rise to $(\mathbb{T}, \mathsf{V})$-functors $a : |X| \otimes X \to \mathsf{V}$ and $a : X^{\mathrm{op}} \otimes X \to \mathsf{V}$. According to the considerations above, we obtain a $(\mathbb{T}, \mathsf{V})$-functor $y = \ulcorner a \urcorner : X \to \mathsf{V}^{|X|}$. Our next result should be compared with Corollary 1.5.

Theorem (Yoneda)**.** *Let $X = (X, a)$ be a $(\mathbb{T}, \mathsf{V})$-category. Then the following assertions hold.*

(a) For all $\mathfrak{x} \in TX$ and $\varphi \in \mathsf{V}^{|X|}$, $[\![m_X, \hom_\xi]\!](Ty(\mathfrak{x}), \varphi) \leq \varphi(\mathfrak{x})$.

(b) Let $\varphi \in \mathsf{V}^{|X|}$. Then

$$\forall \mathfrak{x} \in TX \;\; \varphi(\mathfrak{x}) \leq [\![m_X, \hom_\xi]\!](Ty(\mathfrak{x}), \varphi) \qquad \Longleftrightarrow \qquad \varphi : X^{\mathrm{op}} \to \mathsf{V} \text{ is a } (\mathbb{T}, \mathsf{V})\text{-functor.}$$

Proof. Note that the diagrams

$$\begin{tikzcd} & \mathsf{V} \\ TX \times X \arrow[ur, "a"] \arrow[r, "1_{TX} \times y"'] & TX \times \mathsf{V}^{|X|} \arrow[u, "\mathrm{ev}"'] \end{tikzcd} \qquad \begin{tikzcd} TX \times X \arrow[r, "1_{TX} \times y"] \arrow[d, "\pi_2"'] & TX \times \mathsf{V}^{|X|} \arrow[d, "\pi_2"] \\ X \arrow[r, "y"'] & \mathsf{V}^{|X|} \end{tikzcd}$$

commute, where the right-hand side diagram is even a pullback. Let $\mathfrak{x} \in TX$ and $\varphi \in \mathsf{V}^{|X|}$.

Hence

$$
\begin{aligned}
&[\![m_X, \hom_\xi]\!](T_{\mathrm{o}} y(\mathfrak{x}), \varphi) \\
&\qquad = \bigvee \{ v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX, \mathfrak{Y} \in m_X^{-1}(\mathfrak{y}), \mathfrak{W} \in T(TX \times \mathsf{V}^{|X|}) \\
&\qquad\qquad (T_{\mathrm{o}}\pi_1(\mathfrak{W}) = \mathfrak{Y} \,\&\, T_{\mathrm{o}}\pi_2(\mathfrak{W}) = T_{\mathrm{o}} y(\mathfrak{x})) \;\Rightarrow\; v \le \hom(\xi \cdot T_{\mathrm{o}}\,\mathrm{ev}(\mathfrak{W}), \varphi(\mathfrak{y})) \} \\
&\qquad = \bigvee \{ v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX, \mathfrak{Y} \in m_X^{-1}(\mathfrak{y}), \mathfrak{W} \in T(TX \times X) \\
&\qquad\qquad (T_{\mathrm{o}}\pi_1(\mathfrak{W}) = \mathfrak{Y} \,\&\, T_{\mathrm{o}}\pi_2(\mathfrak{W}) = \mathfrak{x}) \;\Rightarrow\; v \le \hom(\xi \cdot T_{\mathrm{o}} a(\mathfrak{W}), \varphi(\mathfrak{y})) \} \\
&\qquad = \bigvee \{ v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX, \mathfrak{Y} \in m_X^{-1}(\mathfrak{y}) \;\; v \le \bigwedge_{\substack{\mathfrak{W} \in T(TX \times X) \\ T_{\mathrm{o}}\pi_1(\mathfrak{W}) = \mathfrak{Y} \\ T_{\mathrm{o}}\pi_2(\mathfrak{W}) = \mathfrak{x}}} \hom(\xi \cdot T_{\mathrm{o}} a(\mathfrak{W}), \varphi(\mathfrak{y})) \} \\
&\qquad = \bigvee \{ v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX, \mathfrak{Y} \in m_X^{-1}(\mathfrak{y}) \;\; v \le \hom(\bigvee_{\substack{\mathfrak{W} \in T(TX \times X) \\ T_{\mathrm{o}}\pi_1(\mathfrak{W}) = \mathfrak{Y} \\ T_{\mathrm{o}}\pi_2(\mathfrak{W}) = \mathfrak{x}}} \xi \cdot T_{\mathrm{o}} a(\mathfrak{W}), \varphi(\mathfrak{y})) \} \\
&\qquad = \bigvee \{ v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX, \mathfrak{Y} \in m_X^{-1}(\mathfrak{y}) \;\; v \le \hom(Ta(\mathfrak{Y}, \mathfrak{x}), \varphi(\mathfrak{y})) \} \\
&\qquad = \bigvee \{ v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX \;\; Ta \cdot m_X^{\circ}(\mathfrak{y}, \mathfrak{x}) \otimes v \le \varphi(\mathfrak{y}) \}.
\end{aligned}
$$

In particular we have

$$v = k \otimes v \le Ta \cdot m_X^{\circ}(\mathfrak{x}, \mathfrak{x}) \otimes v \le \varphi(\mathfrak{x}),$$

which proves (a). On the other hand, $\varphi : (TX, Ta \cdot m_X^{\circ}) \to (\mathsf{V}, \hom)$ is a V-functor if and only if

$$Ta \cdot m_X^{\circ}(\mathfrak{y}, \mathfrak{x}) \otimes \varphi(\mathfrak{x}) \le \varphi(\mathfrak{y})$$

for all $\mathfrak{y}, \mathfrak{x} \in TX$, from which follows (b). □

We put $\hat{X} = (\hat{X}, \hat{a})$ where $\hat{X} := \{\varphi \in \mathsf{V}^{|X|} \mid \varphi : X^{\mathrm{op}} \to \mathsf{V} \text{ is a } (\mathbb{T}, \mathsf{V})\text{-functor}\}$ considered as a subcategory of $\mathsf{V}^{|X|}$. Recall that $a : X^{\mathrm{op}} \otimes X \to \mathsf{V}$ is a $(\mathbb{T}, \mathsf{V})$-functor, and therefore $a(_, x) : X^{\mathrm{op}} \otimes P \to \mathsf{V}$ is a $(\mathbb{T}, \mathsf{V})$-functor for each $x \in X$. If $T1 = 1$, then $P = (1, p) = (1, k)$ is the neutral element for $\otimes$ and we can restrict the Yoneda functor y to $\hat{X}$.

Corollary. *Assume $T1 = 1$. Then the Yoneda functor $y : X \to \hat{X}$ is full and faithful.*

If $Te_X \cdot e_X = m_X^{\circ} \cdot e_X$, we might also consider the transpose $y_0 = \ulcorner a \urcorner : X \to \mathsf{V}^{X^{\mathrm{op}}}$ of $a : X^{\mathrm{op}} \otimes X \to \mathsf{V}$ as below. However, unlike the situation for V-categories, in general we do not have $\hat{X} \cong \mathsf{V}^{X^{\mathrm{op}}}$ (see example below).

Proposition (Yoneda II)**.** *Assume that $Te_X \cdot e_X = m_X^{\circ} \cdot e_X$ and let $X = (X, a)$ be a $(\mathbb{T}, \mathsf{V})$-category. Then the following assertions hold.*

(a) For all $\mathfrak{x} \in TX$ and $\varphi \in \mathsf{V}^{X^{\mathrm{op}}}$, $[\![a^{\mathrm{op}}, \hom_\xi]\!](Ty_0(\mathfrak{x}), \varphi) \ge \varphi(\mathfrak{x})$.

(b) Let $\mathfrak{x} \in TX$ such that $Ta \cdot e_{TX}(\mathfrak{x}, \mathfrak{x}) \ge k$. Then $[\![a^{\mathrm{op}}, \hom_\xi]\!](Ty_0(\mathfrak{x}), \varphi) \le \varphi(\mathfrak{x})$.

Proof. Let $\mathfrak{x} \in TX$ and $\varphi \in \mathsf{V}^{X^{\mathrm{op}}}$. As above, we obtain

$$\begin{aligned}
&[\![a^{\mathrm{op}}, \hom_\xi]\!](T_{\mathrm{o}}y(\mathfrak{x}), \varphi) \\
&= \bigvee\{v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX, \mathfrak{Y} \in T^2X, \mathfrak{W} \in T(TX \times \mathsf{V}^{X^{\mathrm{op}}}) \\
&\qquad (T_{\mathrm{o}}\pi_1(\mathfrak{W}) = \mathfrak{Y} \,\&\, T_{\mathrm{o}}\pi_2(\mathfrak{W}) = T_{\mathrm{o}}y(\mathfrak{x})) \Rightarrow a^{\mathrm{op}}(\mathfrak{Y}, \mathfrak{x}) \otimes v \leq \hom(\xi \cdot T_{\mathrm{o}}\,\mathrm{ev}(\mathfrak{W}), \varphi(\mathfrak{y}))\} \\
&= \bigvee\{v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX, \mathfrak{Y} \in T^2Y \,.\, Ta(\mathfrak{Y}, \mathfrak{x}) \otimes a^{\mathrm{op}}(\mathfrak{Y}, \mathfrak{y}) \otimes v \leq \varphi(\mathfrak{y})\} \\
&= \bigvee\{v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX \;\; a^{\mathrm{op}} \cdot Ta^{\circ}(\mathfrak{x}, \mathfrak{y}) \otimes v \leq \varphi(\mathfrak{y})\}.
\end{aligned}$$

Furthermore, we have

$$a^{\mathrm{op}} \cdot Ta^{\circ} = e^{\circ}_{TX} \cdot Tm^{\circ}_X \cdot TTa^{\circ} \cdot Ta^{\circ} = e^{\circ}_{TX} \cdot Ta^{\circ} \leq m_X \cdot Ta^{\circ}.$$

Hence $\varphi(\mathfrak{x}) \leq [\![a^{\mathrm{op}}, \hom_\xi]\!](T_{\mathrm{o}}y_0(\mathfrak{x}), \varphi)$ and, if $k \leq Ta \cdot e_{TX}(\mathfrak{x}, \mathfrak{x}) = a^{\mathrm{op}} \cdot Ta^{\circ}(\mathfrak{x}, \mathfrak{x})$, we also have $[\![a^{\mathrm{op}}, \hom_\xi]\!](T_{\mathrm{o}}y_0(\mathfrak{x}), \varphi) \leq \varphi(\mathfrak{x})$. □

Example. Unlike y, the functor y_0 does not need to be full and faithful. In fact, consider $X = \mathbb{N}$ as a $(\mathbb{U}, 2)$-category, i.e. a topological space, equipped with the discrete topology $a = e^{\circ}_{\mathbb{N}}$. Then $\mathbb{N}^{\mathrm{op}}$ is the discrete space $\mathbb{N}^{\mathrm{op}} = (U\mathbb{N}, e^{\circ}_{U\mathbb{N}})$. Let $\mathfrak{x}$ be a free ultrafilter on $\mathbb{N}$. Then, for each $\mathfrak{y} \in U\mathbb{N}$, $a^{\mathrm{op}} \cdot Ua^{\circ}(\mathfrak{x}, \mathfrak{y}) = e^{\circ}_{\mathbb{N}} \cdot Ue_{\mathbb{N}}(\mathfrak{x}, \mathfrak{y}) =$ false and therefore $Uy_0(\mathfrak{x}) \to \varphi$ for each $\varphi \in 2^{\mathbb{N}^{\mathrm{op}}}$. On the other hand, for $\varphi = a(_, x)$ (x any element of $\mathbb{N}$) we have $\varphi(\mathfrak{x}) =$ false. In particular we see that $y_0 : \mathbb{N} \to 2^{\mathbb{N}^{\mathrm{op}}}$ is not full and faithful.

5 Examples

5.1 Ordered sets. Recall that $2\text{-}\mathsf{Cat} = \mathsf{Ord}$. Given an ordered set $X = (X, \leq)$, by Theorem 1.5 we have that a bimodule $\phi : 1 \multimap X$ is an order-preserving map $\phi : X \to 2$, while a bimodule $\psi : X \multimap 1$ is an order-preserving map $X^{\mathrm{op}} \to 2$. We can identify φ with the upclosed set $A = \varphi^{-1}(\text{true})$ and ψ with the downclosed set $B = \psi^{-1}(\text{true})$. Under this identification, $\varphi \dashv \psi$ translates to

$$A \cap B \neq \varnothing \qquad \text{and} \qquad \forall x \in A \;\; \forall y \in B \;\; y \leq x.$$

Then any $z \in A \cap B$ is simultaneously a smallest element of A and a largest element of B, therefore z represents $\varphi \dashv \psi$. Hence, by Proposition 1.6, each ordered set is Lawvere-complete. Note that the proof of Proposition 1.6 makes use of the Axiom of Choice; in fact, as pointed out in [6], here we have no choice.

Theorem. *The following assertions are equivalent.*

(i) *Each ordered set is Lawvere-complete.*

(ii) *The Axiom of Choice.*

Proof. To see (ii)⇒(i), let $f : X \to Y$ be a surjective map. We equip Y with the discrete order Δ_Y and X with the kernel relation of f; then we have not only $f_* \dashv f^*$ but also $f^* \dashv f_*$. Hence there exists some $g : Y \to X$ which represents $f^* \dashv f_*$, and such g necessarily satisfies $f \cdot g = 1_Y$. □

5.2 Metric spaces. For $\mathsf{V} = \mathsf{P}_+$ we have $\mathsf{P}_+\text{-}\mathsf{Cat} \cong \mathsf{Met}$. Let $X = (X, d)$ be a metric space. A pair of adjoint bimodules $\varphi \dashv \psi$ corresponds to a pair of non-expansive maps $\varphi : X \to \mathsf{P}_+$ and $\psi : X^{\mathrm{op}} \to \mathsf{P}_+$ which satisfy

$$\inf_{x \in X} \varphi(x) + \psi(x) = 0 \qquad \text{and} \qquad \forall x, y \in X \;\; \psi(y) + \varphi(x) \geq d(y, x).$$

As observed in [18], pairs of adjoint bimodules on X correspond exactly to equivalence classes of Cauchy sequences. To see this, recall first that a sequence $s = (x_n)_{n \in \mathbb{N}}$ is called *Cauchy* if

$$\inf_{k \in \mathbb{N}} \sup_{n, n' \geq k} d(x_n, x_{n'}) = 0.$$

Given a Cauchy sequence $s = (x_n)_{n \in \mathbb{N}}$, we have

$$\inf_{m \in \mathbb{N}} \sup_{n \geq m} d(x_n, x) = \sup_{m \in \mathbb{N}} \inf_{n \geq m} d(x_n, x)$$

as well as

$$\inf_{m \in \mathbb{N}} \sup_{n \geq m} d(x, x_n) = \sup_{m \in \mathbb{N}} \inf_{n \geq m} d(x, x_n),$$

and s gives rise to non-expansive maps

$$\begin{aligned} \varphi_s : X &\to \mathsf{P}_+ \\ x &\mapsto \sup_{m \in \mathbb{N}} \inf_{n \geq m} d(x_n, x) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \psi_s : X^{\mathrm{op}} &\to \mathsf{P}_+ . \\ x &\mapsto \sup_{m \in \mathbb{N}} \inf_{n \geq m} d(x, x_n) \end{aligned}$$

One sees easily that $\varphi_s \dashv \psi_s$; moreover, two equivalent Cauchy sequences induce the same maps.

On the other hand, given an adjunction $\varphi \dashv \psi$, we may define $s = (x_n)_{n \in \mathbb{N}}$ such that $\varphi(x_n) + \psi(x_n) \leq \frac{1}{n}$, hence $d(x_n, x_m) \leq \frac{1}{n} + \frac{1}{m}$, and therefore s is a Cauchy sequence. Any two such sequences are equivalent. Furthermore, $\varphi \leq \varphi_s$ as well as $\psi \leq \psi_s$, therefore, since $\varphi \dashv \psi$ and $\varphi_s \dashv \psi_s$, we have even equality. Starting with a Cauchy sequence $s = (x_n)_{n \in \mathbb{N}}$, then for any sequence $t = (y_n)_{n \in \mathbb{N}}$ chosen for $\varphi \dashv \psi$ as above we have

$$\inf_{m \in \mathbb{N}} \inf_{k \in \mathbb{N}} \sup_{n \geq k} d(x_n, y_m) = 0 \qquad \text{and} \qquad \inf_{m \in \mathbb{N}} \inf_{k \in \mathbb{N}} \sup_{n \geq k} d(y_m, x_n) = 0,$$

hence s and t are equivalent. Finally, $s = (x_n)_{n \in \mathbb{N}}$ converges to x (i.e. s is equivalent to $(x)_{n \in \mathbb{N}}$) if and only if $\varphi_s \dashv \psi_s$ is represented by x.

The same argumentation applies also to the case $\mathsf{V} = \mathsf{P}_{\max}$: pairs of adjoint bimodules $\varphi \dashv \psi : 1 \to X$ with X an ultrametric space correspond precisely to Cauchy sequences in X, and convergence to representability.

Remark. A notion of non-symmetric Cauchy-sequence was introduced and studied in [3].

5.3 Topological spaces. We consider now $\mathbb{T} = \mathbb{U} = (U, e, m)$ the *ultrafilter monad* and $\mathsf{V} = 2$. As already stated, Proposition 3.1 describes our extension U in terms of $U_{\mathrm{o}} : \mathsf{Set} \to \mathsf{Set}$ (for a direct calculation of U, see [7, Example 6.4]). Then $(\mathbb{U}, 2)\text{-}\mathsf{Cat} = \mathsf{Top}$, as it was shown by Barr [1]. By Theorem 3.3, a bimodule $\varphi : U1 \multimap X$ from the one-element space 1 to a topological space X is essentially a continuous map $\varphi : X \to 2$ from X into the Sierpinski space 2, hence we can identify it with a closed subset $A \subseteq X$. A bimodule $\psi : UX \multimap 1$ is basically a map $\psi : UX \to 2$ such that $\mathcal{A} = \psi^{-1}(\text{true})$ is closed in $|X|$ as well as in X^{op}. The topology on $|X|$

is given by the Zariski closure, that is, $\mathfrak{x} \in UX$ is in the closure of $\mathcal{M} \subseteq UX$ if $\bigcap \mathcal{M} \subseteq \mathfrak{x}$. To understand the structure of X^{op}, observe first that the order on $M^\circ X$ is given by

$$\begin{aligned} \mathfrak{x} \leq \mathfrak{y} &\iff \exists \mathfrak{X} \in U^2X \; m_X(\mathfrak{X}) = \mathfrak{x} \text{ and } \mathfrak{X} \to \mathfrak{y} \\ &\iff \forall A \in \mathfrak{x}, B \in \mathfrak{y} \;\; \exists \mathfrak{a} \in UA, y \in B \; \mathfrak{a} \to y \\ &\iff \forall A \in \mathfrak{x}, B \in \mathfrak{y} \;\; \overline{A} \cap B \neq \varnothing. \end{aligned}$$

Denoting the filter base $\{\overline{A} \mid A \in \mathfrak{x}\}$ by $\overline{\mathfrak{x}}$, we have

$$\mathfrak{x} \leq \mathfrak{y} \iff \overline{\mathfrak{x}} \subseteq \mathfrak{y}.$$

Hence bimodules $\psi : UX \nrightarrow 1$ can be identified with subsets $\mathcal{A} \subseteq UX$ which are Zariski closed and down-closed for the order described above. Now $\varphi \dashv \psi$ translates to

$$\exists \mathfrak{x}_0 \in UX \;\; \mathfrak{x}_0 \in \mathcal{A} \;\&\; A \in \mathfrak{x}_0 \qquad \text{and} \qquad \forall \mathfrak{x} \in \mathcal{A}, x \in A \;\; \mathfrak{x} \to x.$$

Clearly, each $\mathfrak{x} \in \mathcal{A}$ converges to all points of A. On the other hand, for any $\mathfrak{x} \in UX$ with this property we have $\mathfrak{x} \leq \mathfrak{x}_0$ and therefore $\mathfrak{x} \in \mathcal{A}$. We conclude that

$$\mathcal{A} = \{\mathfrak{x} \in UX \mid \forall x \in A \;\; \mathfrak{x} \to x\}.$$

A closed subset $A \subseteq X$ admits an ultrafilter $\mathfrak{x}_0 \in UA$ which converges to all $x \in A$ if and only if $\{V \subseteq X \mid V \text{ open}, V \cap A \neq \varnothing\}$ is a filter base. In the language of closed sets this is expressed by saying that A is not the union of two proper closed subsets, i.e. A is *irreducible*. Finally, ψ (and hence φ) is representable if and only if $\mathfrak{x}_0$ can be chosen principal, that is, if and only if there exists some point $x_0 \in A$ which converges to all $x \in A$. In conclusion, we have

Theorem. *The following assertions are equivalent for a topological space X.*

(i) *X is Lawvere-complete.*

(ii) *Each irreducible closed subset $A \subseteq X$ is of the form $A = \overline{\{x\}}$ for some $x \in A$, i.e. X is weakly sober.*

5.4 Approach spaces. Recall that $\mathsf{App} = (\mathbb{U}, \mathsf{P}_+)\text{-}\mathsf{Cat}$ is the category of approach spaces and non-expansive maps. We fix an approach space $X = (X, a)$. As above, a bimodule $\varphi : U1 \nrightarrow X$ is a non-expansive map $\varphi : X \to \mathsf{P}_+$, by Theorem 3.3. There is a bijective correspondence between maps $\varphi : X \to \mathsf{P}_+$ and families $(A_v)_{v \in \mathsf{P}_+}$ of subsets $A_v \subseteq X$ satisfying

$$A_v = \bigcap_{u > v} A_u, \tag{9}$$

where $\varphi \mapsto (\varphi^{-1}([0, v]))_{v \in \mathsf{P}_+}$ and a family $(A_v)_{v \in \mathsf{P}_+}$ defines the map $x \mapsto \inf\{v \in \mathsf{P}_+ \mid x \in A_v\}$. Under this bijection, non-expansive maps correspond precisely to those families $(A_v)_{v \in \mathsf{P}_+}$ which satisfy in addition

$$\forall u, v \in \mathsf{P}_+ \; \forall x \in X \; (d(A_u, x) \leq v \Rightarrow x \in A_{u+v}), \tag{10}$$

where $d(A, x) = \inf\{a(\mathfrak{x}, x) \mid \mathfrak{x} \in UA\}$.

We may think of the family $A = (A_v)_{v \in \mathsf{P}_+}$ satisfying (9) as a *variable set*[6]; we call A *closed* if it satisfies (10). Now it is not difficult to see that a right adjoint $\psi : X \multimap 1$ to $\varphi : 1 \multimap X$ is determined by the variable set $\mathcal{A} = (\mathcal{A}_v)_{v \in \mathsf{P}_+}$ given by

$$\mathcal{A}_v = \{\mathfrak{x} \in UX \mid \forall u \in \mathsf{P}_+ \ \ \forall x \in A_u \ \ a(\mathfrak{x}, x) \leq u + v\},$$

for each $v \in \mathsf{P}_+$. Furthermore, given $\varphi : 1 \multimap X$, the variable set $\mathcal{A}$ defined as above corresponds to a right adjoint of φ if and only if

$$\forall u \in \mathsf{P}_+ \ (u > 0 \Rightarrow UA_u \cap \mathcal{A}_u \neq \varnothing). \tag{11}$$

In analogy to the situation in Top, we call a variable set A *irreducible* if it satisfies (11). Finally, we remark that the bimodule $\varphi : 1 \multimap X$ is represented by $x \in X$ precisely if the corresponding variable set A is of the form

$$A_v = \{y \in X \mid d(x, y) \leq v\},$$

for each $v \in \mathsf{P}_+$. Naturally, we say that such a variable set is *representable* (by x).

Theorem. *The following assertions are equivalent for an approach space X.*

(i) *X is Lawvere-complete.*

(ii) *Each irreducible closed variable set A is representable.*

We point out that this setting satisfies the conditions of Theorem 3.4, therefore it assures that P_+ is Lawvere-complete.

Remark. The notion of approach frame and its connection with approach spaces was recently studied by Christophe Van Olmen in his PhD thesis [23]. In particular, the concept of *sober approach space* as a fixed point of the dual adjunction between App and the category AFrm of approach frames and homomorphisms was introduced. As confirmed by the author of [23], these are precisely the approach spaces where each irreducible closed variable set is uniquely representable.

6 Appendix: Lawvere-complete quasi-uniform spaces

6.1 Cauchy-complete quasi-uniform spaces. We recall that a *quasi-uniformity* U on a set X is a set of binary relations on X such that:

$\forall u \in U \ \ \Delta \subseteq u$;

$\forall u \in U \ \ \exists v \in U \ \ v \cdot v \subseteq u$.

The pair (X, U) is called a *quasi-uniform space*; it is a *uniform space* when, for all $u \in U$, $u^{-1} \in U$. Given quasi-uniform spaces (X, U) and (Y, V), a map $f : X \to Y$ is *uniformly continuous* if

$$\forall v \in V \ \ \exists u \in U \ \ \forall x, y \in X \ \ x\,u\,y \ \Rightarrow \ f(x)\,v\,f(y).$$

[6] In fact, we may consider $A : \mathsf{P}_+ \to \mathsf{Set}$ as a sheaf where, for each $u \in \mathsf{P}_+$, $\{v < u\}$ is a cover of u.

Definition. Let (X, U) be a quasi-uniform space.

1. A pair $(\mathfrak{f}, \mathfrak{g})$ is a *filter* in (X, U) if $\mathfrak{f}$ and $\mathfrak{g}$ are filters in X such that
$$\forall F \in \mathfrak{f} \ \ \forall G \in \mathfrak{g} \ \ F \cap G \neq \varnothing.$$
2. A filter $(\mathfrak{f}, \mathfrak{g})$ in (X, U) is a *Cauchy filter* if
$$\forall u \in U \ \ \exists F \in \mathfrak{f} \ \ \exists G \in \mathfrak{g} \ \ F \times G \subseteq X_u := \{(x, x') \mid x\, u\, x'\}.$$
3. A filter $(\mathfrak{f}, \mathfrak{g})$ in (X, U) *converges to* $x_0 \in X$ if
$$\forall u \in U \ \ \exists F \in \mathfrak{f} \ \ \exists G \in \mathfrak{g} \ \ F \times G \subseteq X_{-ux_0} \times X_{x_0u-},$$
where $X_{-ux_0} := \{x \in X \mid x\, u\, x_0\}$ and $X_{x_0u-} := \{x \in X \mid x_0\, u\, x\}$.

Lemma. *Given a quasi-uniformity U in X and $x_0 \in X$, the neighbourhood filter of x_0*
$$(\{X_{-ux_0} \mid u \in U\}, \{X_{x_0u-} \mid u \in U\})$$
is a minimal Cauchy filter in (X, U).

Proposition. *For a quasi-uniform space (X, U), the following conditions are equivalent.*

(i) *Every Cauchy filter converges.*

(ii) *Every minimal Cauchy filter is the neighbourhood filter of a point x_0.*

A quasi-uniform space is said to be *Cauchy-complete* if it satisfies any of the equivalent conditions of the Proposition.

For further information see [12] and [13].

6.2 Quasi-uniform spaces as lax algebras. In order to describe quasi-uniform spaces as lax algebras, we turn back to the setting described in [7] and substitute the bicategory $\mathsf{V}\text{-}\mathsf{Mat}$ of 2.1 by the bicategory Y having sets as objects and (possibly improper) filters in $\mathsf{Rel}(X, Y)$ as morphisms, where Rel is the bicategory of relations. The composition of two filters $R : X \nrightarrow Y$ and $S : Y \nrightarrow Z$ is the filter obtained by pointwise composition of relations $R \cdot S = \{s \cdot r \mid s \in S$ and $r \in R\}$, while $R \leq R'$ whenever $R' \subseteq R$ (as sets).

We define a *lax algebra* now exactly like a V-category: it is a Y-morphism $A : X \nrightarrow X$ such that
$$1_X \leq A \quad \text{and} \quad A \cdot A \leq A,$$
or, equivalently,
$$\forall x \in X \ \ \forall a \in A \ \ x\, a\, x \quad \text{and} \quad \forall a \in A \ \exists a' \in A \ \ a' \cdot a' \leq a.$$
A *lax morphism* $f : (X, A) \to (Y, B)$ between lax algebras is a map $f : X \to Y$ such that $f \cdot A \leq B \cdot f$, i.e.
$$\forall b \in B \ \ \exists a \in A \ \ f \cdot a \leq b \cdot f.$$
It was shown in [7, Theorem 3.6] that this category of lax algebras and lax morphisms is equivalent to the category of quasi-uniform spaces and uniformly continuous maps.

6.3 Adjoint pairs of bimodules in quasi-uniform spaces. A *bimodule* $\Psi : (X, A) \multimap (Y, B)$ between lax algebras is a Y-morphism $\Psi : X \longrightarrow\!\!\!

equivalent to this pair being a Cauchy filter. But we did not show yet that Φ and Ψ are bimodules. For any $a \in A$,

$$(\{\bigcup_{x\in X_{-\psi\star}} X_{-ax} \,|\, \psi \in \Psi,\, a \in A\},\ \{\bigcup_{y\in X_{\star\varphi-}} X_{ya-} \,|\, \varphi \in \Phi, a \in A\})$$

is a Cauchy filter contained in the former one, as we show next. First,

$$\bigcup_{x\in X_{-\psi\star}} X_{-ax} \cap \bigcup_{y\in X_{\star a-}} X_{ya-} \ \supseteq\ X_{-\psi\star} \cap X_{\star\varphi-} \neq \varnothing.$$

To prove the other condition, let $a \in A$, and consider $b \in A$ such that $b \cdot b \cdot b \leq a$. There exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that $X_{-\psi\star} \times X_{\star\varphi-} \subseteq X_b$, and this implies that

$$\bigcup_{x\in X_{-\psi\star}} X_{-bx} \times \bigcup_{y\in X_{\star\varphi-}} X_{yb-} \subseteq X_a,$$

since

$$x' \in \bigcup_{x\in X_{-\psi\star}} X_{-bx} \ \Rightarrow\ \exists x \in X_{-\psi\star}\ \ (x', x) \in X_b,$$

$$y' \in \bigcup_{y\in X_{\star\varphi-}} X_{yb-} \ \Rightarrow\ \exists y \in X_{\star\varphi-}\ \ (y, y') \in X_b;$$

hence, since also $(x, y) \in X_b$, we conclude that $(x', y') \in X_a$ as claimed. □

6.4 Lawvere-complete=Cauchy-complete. It is now straightforward to prove that the two notions of completeness coincide.

Theorem. *For a quasi-uniform space (X, A) the following conditions are equivalent.*

(i) *(X, A) is a Lawvere-complete lax algebra.*

(ii) *(X, A) is a Cauchy-complete quasi-uniform space.*

Proof. (i) $\Rightarrow$ (ii): Each minimal Cauchy filter in (X, A) defines an adjoint pair of bimodules $(\Phi : 1 \multimap (X, A)) \dashv (\Psi : (X, A) \multimap 1)$, which, by (i), is induced by a map $f : 1 \to X$, $\star \mapsto x_0$. Hence $\Phi = \{\varphi_b = b \cdot f \,|\, b \in B\}$ and $\Psi = \{\psi_b = f^\circ \cdot b \,|\, b \in B\}$. Moreover, $x \in X_{\star\varphi_b-}$ exactly when $b(x_0, x) = \top$, that is $X_{\star\varphi_b-} = X_{x_0 b-}$, and $x \in X_{-\psi_b\star}$ exactly when $b(x, x_0) = \top$, which means $X_{-\psi_b\star} = X_{-bx_0}$.

(ii) $\Rightarrow$ (i): Given an adjoint pair of bimodules $(\Phi : 1 \multimap (X, A)) \dashv (\Psi : (X, A) \multimap 1)$, by (ii) the minimal Cauchy filter it induces is the neighbourhood filter of a point x_0. It is straightforward to check that $\Phi = A \cdot f$ and $\Psi = f^\circ \cdot A$ for $f : 1 \to X$, $\star \mapsto x_0$. □

Final remark. The results of this section can be investigated in the more general setting introduced in [9], i.e., in proalgebras; here, for simplicity, we decided to state them only at the level of quasi-uniform structures, which are proalgebras for the identity monad.

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arXiv:0807.4123v1 [math.CT] 25 Jul 2008

RELATIVE INJECTIVITY AS COCOMPLETENESS FOR A CLASS OF DISTRIBUTORS

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Dedicated to Walter Tholen on the occasion of his sixtieth birthday

Abstract. Notions and techniques of enriched category theory can be used to study topological structures, like metric spaces, topological spaces and approach spaces, in the context of topological theories. Recently in [D. Hofmann, Injective spaces via adjunction, arXiv:math.CT/0804.0326] the construction of a Yoneda embedding allowed to identify injectivity of spaces as cocompleteness and to show monadicity of the category of injective spaces and left adjoints over Set. In this paper we generalise these results, studying cocompleteness with respect to a given class of distributors. We show in particular that the description of several semantic domains presented in [M. Escardó and B. Flagg, Semantic domains, injective spaces and monads, Electronic Notes in Theoretical Computer Science 20 (1999)] can be translated into the V-enriched setting.

Introduction

This work continues the research line of previous papers, aiming to use categorical tools in the study of topological structures. Indeed, the perspective proposed in [2, 6] of looking at topological structures as (Eilenberg-Moore) lax algebras and, simultaneously, as a monad enrichment of V-enriched categories, has shown to be very effective in the study of special morphisms – like effective descent and exponentiable ones – at a first step [3, 4], and recently in the study of (Lawvere/Cauchy-)completeness and injectivity [5, 11, 10]. The results we present here complement this study of injectivity. More precisely, in the spirit of Kelly-Schmitt [12] we generalise the results of [10], showing that injectivity and cocompleteness – when considered relative to a class of distributors – still coincide. Suitable choices of this class of distributors allow us to recover, in the V-enriched setting, results on injectivity of Escardó-Flagg [7].

The starting point of our study of injectivity is the notion of distributor (or bimodule, or profunctor), which allowed the study of weighted colimit, presheaf category, and the Yoneda embedding. It was then a natural step to 'relativize' these ingredients and to consider *cocompleteness with respect to a class of distributors* Φ. Namely, we introduce the notion of Φ-cocomplete category, we construct the Φ-presheaf category, and we prove that Φ-cocompleteness is equivalent to the existence of a left adjoint of the Yoneda embedding into the Φ-presheaf category. Furthermore, the class Φ determines a class of embeddings so that the injective $\mathcal{T}$-categories with respect to this class are precisely the Φ-cocomplete categories. This result links our work with [7], where the authors study systematically semantic domains and injectivity characterisations with the help of Kock-Zöberlein monads.

2000 *Mathematics Subject Classification.* 18A05, 18D15, 18D20, 18B35, 18C15, 54B30, 54A20.

Key words and phrases. Quantale, V-category, monad, topological theory, distributor, Yoneda lemma, weighted colimit.

The authors acknowledge partial financial assistance by Centro de Matemática da Universidade de Coimbra/FCT and Unidade de Investigação e Desenvolvimento Matemática e Aplicações da Universidade de Aveiro/FCT.

1. The Setting

Throughout this paper we consider a (strict) *topological theory* as introduced in [9]. Such a theory $\mathcal{T} = (\mathbb{T}, \mathsf{V}, \xi)$ consists of:

(1) a commutative quantale $\mathsf{V} = (\mathsf{V}, \otimes, k)$,
(2) a Set-monad $\mathbb{T} = (T, e, m)$, where T and m satisfy (BC); that is, T sends pullbacks to weak pullbacks and each naturality square of m is a weak pullback, and
(3) a $\mathbb{T}$-algebra structure $\xi : T\mathsf{V} \longrightarrow \mathsf{V}$ on V such that:
   (a) $\otimes : \mathsf{V} \times \mathsf{V} \longrightarrow \mathsf{V}$ and $k : 1 \longrightarrow \mathsf{V}$, $* \longmapsto k$, are $\mathbb{T}$-algebra homomorphisms making (V, ξ) a monoid in $\mathsf{Set}^{\mathbb{T}}$; that is, the following diagrams

$$\begin{tikzcd} T1 \arrow[r, "Tk"] \arrow[d, "!"'] & T\mathsf{V} \arrow[d, "\xi"] \\ 1 \arrow[r, "k"'] & \mathsf{V} \end{tikzcd} \qquad \begin{tikzcd} T(\mathsf{V} \times \mathsf{V}) \arrow[r, "T(\otimes)"] \arrow[d, "\langle \xi \cdot T\pi_1, \xi \cdot T\pi_2 \rangle"'] & T\mathsf{V} \arrow[d, "\xi"] \\ \mathsf{V} \times \mathsf{V} \arrow[r, "\otimes"'] & \mathsf{V} \end{tikzcd}$$

   are commutative;
   (b) For each set X, $\xi_X : \mathsf{V}^X \longrightarrow \mathsf{V}^{TX}$, $(X \xrightarrow{\varphi} \mathsf{V}) \longmapsto (TX \xrightarrow{T\varphi} T\mathsf{V} \xrightarrow{\xi} \mathsf{V})$, defines a natural transformation $(\xi_X)_X : P \longrightarrow PT : \mathsf{Set} \longrightarrow \mathsf{Ord}$.

Here $P : \mathsf{Set} \longrightarrow \mathsf{Ord}$ is the V-powerset functor defined as follows. We put $PX = \mathsf{V}^X$ with the pointwise order. Each map $f : X \longrightarrow Y$ defines a monotone map $\mathsf{V}^f : \mathsf{V}^Y \longrightarrow \mathsf{V}^X$, $\varphi \longmapsto \varphi \cdot f$. Since V^f preserves all infima and all suprema, it has a left adjoint Pf. Explicitly, for $\varphi \in \mathsf{V}^X$ we have $Pf(\varphi)(y) = \bigvee\{\varphi(x) \mid x \in X,\ f(x) = y\}$.

Examples. Throughout this paper we will keep in mind the following topological theories:

(1) The identity theory $\mathcal{I} = (\mathbb{1}, \mathsf{V}, 1_{\mathsf{V}})$, for each quantale V, where $\mathbb{1} = (\mathrm{Id}, 1, 1)$ denotes the identity monad.
(2) $\mathcal{U}_2 = (\mathbb{U}, 2, \xi_2)$, where $\mathbb{U} = (U, e, m)$ denotes the ultrafilter monad and ξ_2 is essentially the identity map.
(3) $\mathcal{U}_{\mathsf{P}_+} = (\mathbb{U}, \mathsf{P}_+, \xi_{\mathsf{P}_+})$ where $\mathsf{P}_+ = ([0, \infty]^{\mathrm{op}}, +, 0)$ and
$$\xi_{\mathsf{P}_+} : U\mathsf{P}_+ \longrightarrow \mathsf{P}_+, \ \ \mathfrak{x} \longmapsto \inf\{v \in \mathsf{P}_+ \mid [0, v] \in \mathfrak{x}\}.$$
(4) The word theory $(\mathbb{L}, \mathsf{V}, \xi_\otimes)$, for each quantale V, where $\mathbb{L} = (L, e, m)$ is the word monad and
$$\begin{aligned} \xi_\otimes : L\mathsf{V} &\longrightarrow \mathsf{V}. \\ (v_1, \ldots, v_n) &\longmapsto v_1 \otimes \ldots \otimes v_n \\ () &\longmapsto k \end{aligned}$$

Every topological theory $\mathcal{T} = (\mathbb{T}, \mathsf{V}, \xi)$ encompasses several interesting ingredients.

I. *The quantaloid* V-Rel with *sets* as objects and V-*relations* (also called V-*matrices*, see [1]) $r : X \times Y \longrightarrow \mathsf{V}$ as morphisms. We use the usual notation for relations, denoting the V-relation $r : X \times Y \longrightarrow \mathsf{V}$ by $r : X \longrightarrow\!\!\!\!\!\!+\ Y$. Since every map $f : X \longrightarrow Y$ can be thought of as a V-relation $f : X \times Y \longrightarrow \mathsf{V}$ through its graph, there is an injective on objects and faithful functor $\mathsf{Set} \longrightarrow \mathsf{V}$-$\mathsf{Rel}$, unless V is degenerate (i.e. k is the bottom element). Moreover, V-Rel has an involution $(-)^\circ : \mathsf{V}\text{-}\mathsf{Rel} \longrightarrow \mathsf{V}\text{-}\mathsf{Rel}$, assigning to $r : X \longrightarrow\!\!\!\!\!\!+\ Y$ the V-relation $r^\circ : Y \longrightarrow\!\!\!\!\!\!+\ X$, with $r^\circ(y, x) := r(x, y)$. For each V-relation $r : X \longrightarrow\!\!\!\!\!\!+\ Y$, the maps

$$(-) \cdot r : \mathsf{V}\text{-}\mathsf{Rel}(Y, Z) \longrightarrow \mathsf{V}\text{-}\mathsf{Rel}(X, Z) \text{ and } r \cdot (-) : \mathsf{V}\text{-}\mathsf{Rel}(Z, X) \longrightarrow \mathsf{V}\text{-}\mathsf{Rel}(Z, Y)$$

preserve suprema; hence they have right adjoints,

$$(-) \bullet\!\!- \, r : \mathsf{V}\text{-}\mathsf{Rel}(X,Z) \longrightarrow \mathsf{V}\text{-}\mathsf{Rel}(Y,Z) \text{ and } r \, -\!\!\bullet (-) : \mathsf{V}\text{-}\mathsf{Rel}(Z,Y) \longrightarrow \mathsf{V}\text{-}\mathsf{Rel}(Z,X).$$

II. *The* Set*-functor* T *extends to a 2-functor* $T_\xi : \mathsf{V}\text{-}\mathsf{Rel} \longrightarrow \mathsf{V}\text{-}\mathsf{Rel}$. To each V-relation $r : X \times Y \longrightarrow \mathsf{V}$, T_ξ assigns a V-relation $T_\xi r : TX \times TY \longrightarrow \mathsf{V}$, which is the smallest (order-preserving) map $s : TX \times TY \longrightarrow \mathsf{V}$ such that $\xi \cdot Tr \leq s \cdot \langle T\pi_1, T\pi_2 \rangle$.

$$\begin{tikzcd} T(X \times Y) \arrow[rr, "\langle T\pi_1, T\pi_2\rangle"] \arrow[dr, "\xi_{X\times Y}(r)=\xi\cdot Tr"'] & \leq & TX \times TY \arrow[dl, dotted, "T_\xi r"] \\ & \mathsf{V} & \end{tikzcd}$$

Hence, for $\mathfrak{x} \in TX$ and $\mathfrak{y} \in TY$,

$$T_\xi r(\mathfrak{x}, \mathfrak{y}) = \bigvee \left\{ \xi \cdot Tr(\mathfrak{w}) \;\middle|\; \mathfrak{w} \in T(X \times Y), T\pi_1(\mathfrak{w}) = \mathfrak{x}, T\pi_2(\mathfrak{w}) = \mathfrak{y} \right\}.$$

This 2-functor T_ξ preserves the involution, i.e. $T_\xi(r^\circ) = T_\xi(r)^\circ$ (and we write $T_\xi r^\circ$) for each V-relation $r : X \nrightarrow Y$, m becomes a natural transformation $m : T_\xi T_\xi \longrightarrow T_\xi$ and e an op-lax natural transformation $e : \mathrm{Id} \longrightarrow T_\xi$, i.e. $e_Y \circ r \leq T_\xi r \circ e_X$ for all $r : X \nrightarrow Y$ in $\mathsf{V}\text{-}\mathsf{Rel}$.

III. A V-relation of the form $\alpha : TX \nrightarrow Y$, called a $\mathcal{T}$*-relation* and denoted by $\alpha : X \rightarrowtail Y$, will play an important role here. Given two $\mathcal{T}$-relations $\alpha : X \rightarrowtail Y$ and $\beta : Y \rightarrowtail Z$, their *Kleisli convolution* $\beta \circ \alpha : X \rightarrowtail Z$ is defined as

$$\beta \circ \alpha = \beta \cdot T_\xi \alpha \cdot m_X^\circ.$$

This operation is associative and has the $\mathcal{T}$-relation $e_X^\circ : X \rightarrowtail X$ as a lax identity: $a \circ e_X^\circ = a$ and $e_Y^\circ \circ a \geq a$ for any $a : X \rightarrowtail Y$.

IV. $\mathcal{T}$-relations satisfying the usual unit and associativity categorical rules define $\mathcal{T}$-categories: a $\mathcal{T}$-*category* is a pair (X, a) consisting of a set X and a $\mathcal{T}$-relation $a : X \rightarrowtail X$ on X such that

$$e_X^\circ \leq a \qquad \text{and} \qquad a \circ a \leq a.$$

Expressed elementwise, these conditions become

$$k \leq a(e_X(x), x) \qquad \text{and} \qquad T_\xi a(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x) \leq a(m_X(\mathfrak{X}), x)$$

for all $\mathfrak{X} \in TTX$, $\mathfrak{x} \in TX$ and $x \in X$. A function $f : X \longrightarrow Y$ between $\mathcal{T}$-categories (X, a) and (Y, b) is a $\mathcal{T}$*-functor* if $f \cdot a \leq b \cdot Tf$, which in pointwise notation reads as

$$a(\mathfrak{x}, x) \leq b(Tf(\mathfrak{x}), f(x))$$

for all $\mathfrak{x} \in TX$, $x \in X$. The category of $\mathcal{T}$-categories and $\mathcal{T}$-functors is denoted by $\mathcal{T}\text{-}\mathsf{Cat}$.

V. In particular, the quantale V is a $\mathcal{T}$-category $\mathsf{V} = (\mathsf{V}, \hom_\xi)$, where

$$\hom_\xi : T\mathsf{V} \times \mathsf{V} \longrightarrow \mathsf{V}, \; (\mathfrak{v}, v) \longmapsto \hom(\xi(\mathfrak{v}), v).$$

VI. *The forgetful functor* $\mathrm{O} : \mathcal{T}\text{-}\mathsf{Cat} \longrightarrow \mathsf{Set}$, $(X, a) \longmapsto X$, *is topological*, hence it has a left and a right adjoint. In particular, the free $\mathcal{T}$-category on a one-element set is given by $G = (1, e_1^\circ)$.

VII. A V-relation $\varphi : X \rightarrowtail Y$ between $\mathcal{T}$-categories $X = (X, a)$ and $Y = (Y, b)$ is a $\mathcal{T}$*-distributor*, denoted as $\varphi : X \nrightarrow Y$, if $\varphi \circ a \leq \varphi$ and $b \circ \varphi \leq \varphi$. Note that we always have $\varphi \circ a \geq \varphi$ and $b \circ \varphi \geq \varphi$, so that the $\mathcal{T}$-distributor conditions above are in fact equalities. $\mathcal{T}$-categories and $\mathcal{T}$-distributors form a 2-category, denoted by $\mathcal{T}\text{-}\mathsf{Mod}$, with Kleisli convolution as composition and with the 2-categorical structure inherited from $\mathsf{V}\text{-}\mathsf{Rel}$.

VIII. *Each $\mathcal{T}$-functor $f : (X, a) \longrightarrow (Y, b)$ induces an adjunction $f_* \dashv f^*$ in* $\mathcal{T}$-Mod, with $f_* : X \longrightarrow\!\!\!

We can now define the process of *dualizing a $\mathcal{T}$-category* as the composition of the following functors

$$\begin{tikzcd} \mathcal{T}\text{-}\mathsf{Cat} \arrow[r, dotted, "(\ )^{\mathrm{op}}"] \arrow[d, "M"'] & \mathcal{T}\text{-}\mathsf{Cat} \\ \mathsf{V}\text{-}\mathsf{Cat} \arrow[r, "(\ )^{\mathrm{op}}"'] & \mathsf{V}\text{-}\mathsf{Cat} \arrow[u, "A"'] \end{tikzcd}$$

that is, the *dual of a $\mathcal{T}$-category* (X, a) is defined as

$$X^{\mathrm{op}} = \mathrm{A}(\mathrm{M}(X)^{\mathrm{op}}),$$

which is a structure on TX. If $\mathbb{T}$ is the identity monad, then X^{op} is indeed the dual V-category of X.

XI. The tensor product on V can be transported to $\mathcal{T}$-Cat by putting

$$(X, a) \otimes (Y, b) = (X \times Y, c),$$

with

$$c(\mathfrak{w}, (x, y)) = a(T\pi_1(\mathfrak{w}), x) \otimes b(T\pi_2(\mathfrak{w}), y),$$

where $\mathfrak{w} \in T(X \times Y)$, $x \in X$, $y \in Y$. The $\mathcal{T}$-category $E = (1, k)$ is a $\otimes$-neutral object, where 1 is a singleton set and $k : T1 \times 1 \longrightarrow \mathsf{V}$ the constant relation with value $k \in \mathsf{V}$. For each set X, the functor $|X| \otimes (-) : \mathcal{T}\text{-}\mathsf{Cat} \longrightarrow \mathcal{T}\text{-}\mathsf{Cat}$ has a right adjoint $(-)^{|X|} : \mathcal{T}\text{-}\mathsf{Cat} \longrightarrow \mathcal{T}\text{-}\mathsf{Cat}$. Explicitly, the structure $[\![-, -]\!]$ on $\mathsf{V}^{|X|}$ is given by the formula

$$[\![\mathfrak{p}, \psi]\!] = \bigwedge_{\substack{\mathfrak{q} \in T(|X| \times \mathsf{V}^{|X|}) \\ \mathfrak{q} \longmapsto \mathfrak{p}}} \hom(\xi \cdot T\,\mathrm{ev}(\mathfrak{q}), \psi(m_X \cdot T\pi_1(\mathfrak{q}))),$$

for each $\mathfrak{p} \in T\mathsf{V}^{|X|}$ and $\psi \in \mathsf{V}^{|X|}$.

Theorem. [5] *For $\mathcal{T}$-categories (X, a) and (Y, b), and a $\mathcal{T}$-relation $\psi : X \longrightarrow\!\!\!\!\!\!+\ Y$, the following assertions are equivalent.*

(i) *$\psi : (X, a) \multimapinv (Y, b)$ is a $\mathcal{T}$-distributor.*
(ii) *Both $\psi : |X| \otimes Y \longrightarrow \mathsf{V}$ and $\psi : X^{\mathrm{op}} \otimes Y \longrightarrow \mathsf{V}$ are $\mathcal{T}$-functors.*

XII. Hence, *each $\mathcal{T}$-distributor $\varphi : X \multimapinv Y$ provides a $\mathcal{T}$-functor*

$$\ulcorner\varphi\urcorner : Y \longrightarrow \mathsf{V}^{|X|}$$

which factors through the embedding $\mathcal{P}X \hookrightarrow \mathsf{V}^{|X|}$, where $\mathcal{P}X = \{\psi \in \mathsf{V}^{|X|} \mid \psi : X \multimapinv G\}$ is the $\mathcal{T}$-category of *contravariant presheafs on X*:

$$\begin{tikzcd} Y \arrow[r, "\ulcorner\varphi\urcorner"] \arrow[dr, "\ulcorner\varphi\urcorner"'] & \mathsf{V}^{|X|} \\ & \mathcal{P}X \arrow[u, hook] \end{tikzcd}$$

In particular, for each $\mathcal{T}$-category $X = (X, a)$, the V-relation $a : TX \times X \longrightarrow \mathsf{V}$ is a $\mathcal{T}$-distributor $a : X \multimapinv X$, and therefore we have the *Yoneda functor*

$$y_X = \ulcorner a \urcorner : X \longrightarrow \mathcal{P}X.$$

Theorem. [10] *Let $\psi : X \multimapinv Z$ and $\varphi : X \multimapinv Y$ be $\mathcal{T}$-distributors. Then, for all $\mathfrak{z} \in TZ$ and $y \in Y$,*

$$[\![T\ulcorner\psi\urcorner(\mathfrak{z}), \ulcorner\varphi\urcorner(y)]\!] = (\varphi \circ\!\!- \psi)(\mathfrak{z}, y).$$

Corollary. [10] *For each $\varphi \in \hat{X}$ and each $\mathfrak{x} \in TX$, $\varphi(\mathfrak{x}) = [\![T\, y_X(\mathfrak{x}), \varphi]\!]$, that is, $(y_X)_* : X \longrightarrow\!\!\!\!\!\!\circ\; \hat{X}$ is given by the evaluation map* $\mathrm{ev} : TX \times \hat{X} \longrightarrow \mathsf{V}$. *As a consequence, $y_X : X \longrightarrow \hat{X}$ is fully faithful.*

XIII. Transporting the order-structure on hom-sets from $\mathcal{T}$-Mod to $\mathcal{T}$-Cat via the functor $(-)^* : \mathcal{T}\text{-}\mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathcal{T}\text{-}\mathsf{Mod}$, $\mathcal{T}$-Cat *becomes a 2-category.* That is, for $\mathcal{T}$-functors $f, g : X \longrightarrow Y$ we define $f \leq g$ whenever $f^* \leq g^*$, which in turn is equivalent to $g_* \leq f_*$. We call $f, g : X \longrightarrow Y$ *equivalent*, and write $f \cong g$, if $f \leq g$ and $g \leq f$. Hence, $f \cong g$ if and only if $f^* = g^*$ if and only if $f_* = g_*$. A $\mathcal{T}$-category X is called *separated* (see [11] for details) whenever $f \cong g$ implies $f = g$, for all $\mathcal{T}$-functors $f, g : Y \longrightarrow X$ with codomain X. One easily verifies that the $\mathcal{T}$-*category* $\mathsf{V} = (\mathsf{V}, \hom_\xi)$ *is separated*, and so is each $\mathcal{T}$-category of the form $\mathcal{P}X$ for a $\mathcal{T}$-category X. The full subcategory of $\mathcal{T}$-Cat consisting of all separated $\mathcal{T}$-categories is denoted by $\mathcal{T}\text{-}\mathsf{Cat}_{\mathrm{sep}}$. The 2-categorical structure on $\mathcal{T}$-Cat allows us to consider adjoint $\mathcal{T}$-functors: $\mathcal{T}$-functor $f : X \longrightarrow Y$ is *left adjoint* if there exists a $\mathcal{T}$-functor $g : Y \longrightarrow X$ such that $1_X \leq g \cdot f$ and $1_Y \geq f \cdot g$. Considering the corresponding $\mathcal{T}$-distributors, f is left adjoint to g if and only if $g_* \dashv f_*$, that is, if and only if $f_* = g^*$.

A more complete study of this subject can be found in [9, 10].

2. The results

In the sequel we consider a class Φ of $\mathcal{T}$-distributors subject to the following axioms.

(Ax 1).: For each $\mathcal{T}$-functor f, $f^* \in \Phi$.

(Ax 2).: For all $\varphi \in \Phi$ and all $\mathcal{T}$-functors $f : A \longrightarrow X$ we have

$$f^* \circ \varphi \in \Phi, \qquad \varphi \circ f^* \in \Phi, \qquad f_* \in \Phi \Rightarrow \varphi \circ f_* \in \Phi;$$

whenever the compositions are defined.

(Ax 3).: For all $\varphi : X \longrightarrow\!\!\!\!\!\!\circ\; Y \in \mathcal{T}$-$\mathsf{Mod}$,

$$(\forall y \in Y \,.\, y^* \circ \varphi \in \Phi) \Rightarrow \varphi \in \Phi$$

where y^* is induced by $y : 1 \longrightarrow Y$, $* \longmapsto y$.

Condition (Ax 2) requires that Φ is closed under certain compositions. In fact, in most examples Φ will be closed under arbitrary compositions. Furthermore, there is a largest and a smallest such class of $\mathcal{T}$-distributors, namely the class $\mathcal{P}$ of all $\mathcal{T}$-distributors and the class $\mathcal{R} = \{f^* \mid f : X \longrightarrow Y\}$ of all representable $\mathcal{T}$-distributors.

We call a $\mathcal{T}$-functor $f : X \longrightarrow Y$ Φ-*dense* if $f_* \in \Phi$. Certainly, if f is a left adjoint $\mathcal{T}$-functor, with $f \dashv g$, then $f_* = g^* \in \Phi$, i.e. f is Φ-dense. A $\mathcal{T}$-category X is called Φ-*injective* if, for all $\mathcal{T}$-functors $f : A \longrightarrow X$ and fully faithful Φ-dense $\mathcal{T}$-functors $i : A \longrightarrow B$, there exists a $\mathcal{T}$-functor $g : B \longrightarrow X$ such that $g \cdot i \cong f$. Furthermore, X is called Φ-*cocomplete* if each weighted diagram

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circ"] & X \\ Z & \end{tikzcd}$$

with $\varphi \in \Phi$ has a colimit $g \cong \mathrm{colim}(\varphi, h) : Z \longrightarrow X$. A $\mathcal{T}$-functor $f : X \longrightarrow Y$ is Φ-*cocontinuous* if f preserves all existing Φ-weighted colimits. Note that in both cases it is enough to consider diagrams where $h = 1_X$. We denote by $\mathcal{T}\text{-}\mathsf{Cocont}^\Phi$ the 2-category of all Φ-cocomplete $\mathcal{T}$-categories and Φ-cocontinuous $\mathcal{T}$-functors, and by $\mathcal{T}\text{-}\mathsf{Cocont}^\Phi_{\mathrm{sep}}$ its full subcategory of all Φ-cocomplete and separated $\mathcal{T}$-categories.

If Φ is the class $\mathcal{P}$ of all $\mathcal{T}$-distributors, then $\mathcal{T}\text{-}\mathsf{Cocont}^\Phi$ is the category of cocomplete $\mathcal{T}$-categories and left adjoint $\mathcal{T}$-functors (as shown in [10, Prop. 2.12]).

Lemma. *Consider the (up to isomorphism) commutative triangle*

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "h"] \arrow[dr, phantom, "\cong"] & \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

of $\mathcal{T}$-functors. Then the following assertions hold.

(1) *If g and f are Φ-dense, then so is h.*
(2) *If h is Φ-dense and g is fully faithful, then f is Φ-dense.*
(3) *If h is Φ-dense and f is dense, then g is Φ-dense.*

Proof. The proof is straightforward: (1) $h_* = g_* \circ f_* \in \Phi$ by (Ax 2), since $g_*, f_* \in \Phi$; (2) $f_* = g^* \circ g_* \circ f_* = g^* \circ h_* \in \Phi$ by (Ax 2), since $h_* \in \Phi$; (3) $g_* = g_* \circ f_* \circ f^* = h_* \circ f^* \in \Phi$ by (Ax 2), since $h_* \in \Phi$. □

We put now

$$\Phi X = \{\psi \in \mathcal{P}X \mid \psi \in \Phi\}$$

considered as a subcategory of $\mathcal{P}X$. We have the restriction

$$y_X^\Phi : X \longrightarrow \Phi X$$

of the Yoneda map, and each $\psi \in \Phi X$ is a Φ-weighted colimit of representables (see [10, Proposition 2.5]).

Lemma. *The following assertions hold.*

(1) *$y_X^\Phi : X \longrightarrow \Phi X$ is Φ-dense.*
(2) *For each $\mathcal{T}$-distributor $\varphi : X \multimap Y$, $\varphi \in \Phi$ if and only if $\ulcorner\varphi\urcorner : Y \longrightarrow \mathcal{P}X$ factors through the embedding $\Phi X \hookrightarrow \mathcal{P}X$.*

Proof. By the Yoneda Lemma (Corollary 1), for any $\psi \in \Phi X$ we have $\psi^* \circ (y_X^\Phi)_* = \psi \in \Phi$, therefore $(y_X^\Phi)_* \in \Phi$ by (Ax 3) and the assertion (1) follows. To see (2), just observe that $\ulcorner\varphi\urcorner(y) = y^* \circ \varphi$, and use again (Ax 3). □

Our next result extends Theorem 2.6 of [10]. We omit its proof because it uses exactly the same arguments.

Theorem. *The following assertions are equivalent, for a $\mathcal{T}$-category X.*

(i) *X is Φ-injective.*
(ii) *$y_X^\Phi : X \longrightarrow \Phi X$ has a left inverse $\mathrm{Sup}_X^\Phi : \Phi X \longrightarrow X$.*
(iii) *$y_X^\Phi : X \longrightarrow \Phi X$ has a left adjoint $\mathrm{Sup}_X^\Phi : \Phi X \longrightarrow X$.*
(iv) *X is Φ-cocomplete.*

Recall from [10] that, for a given $\mathcal{T}$-functor $f : X \longrightarrow Y$, we have an adjoint pair of $\mathcal{T}$-functors $\mathcal{P}f \dashv f^{-1}$ where

$$\begin{aligned} \mathcal{P}f : \mathcal{P}X &\longrightarrow \mathcal{P}Y \\ \psi &\longmapsto \psi \circ f^* \end{aligned} \qquad \text{and} \qquad \begin{aligned} f^{-1} : \mathcal{P}Y &\longrightarrow \mathcal{P}X. \\ \psi &\longmapsto \psi \circ f_* \end{aligned}$$

By (Ax 1) and (Ax 2), the $\mathcal{T}$-functor $\mathcal{P}f : \mathcal{P}X \longrightarrow \mathcal{P}Y$ restricts to a $\mathcal{T}$-functor $\Phi f : \Phi X \longrightarrow \Phi Y$. On the other hand, $f^{-1} : \mathcal{P}Y \longrightarrow \mathcal{P}X$ restricts to $f^{-1} : \Phi Y \longrightarrow \Phi X$ provided that f is Φ-dense.

Proposition. *The following conditions are equivalent for a $\mathcal{T}$-functor $f : X \longrightarrow Y$.*

(i) *f is Φ-dense.*
(ii) *Φf is left adjoint.*
(iii) *Φf is Φ-dense.*

Proof. (i) $\Rightarrow$ (ii): If f is Φ-dense, then $\Phi f \dashv f^{-1} : \Phi Y \longrightarrow \Phi X$ defined above. (ii) $\Rightarrow$ (iii): If $\Phi f \dashv g$, then $(\Phi f)_* = g^* \in \Phi$, i.e. Φf is Φ-dense. (iii) $\Rightarrow$ (i): Consider the diagram

$$\begin{tikzcd} X \arrow[r, "y_X^\Phi"] \arrow[d, "f"'] & \Phi X \arrow[d, "\Phi f"] \\ Y \arrow[r, "y_Y^\Phi"'] & \Phi Y \end{tikzcd}$$

If Φf is Φ-dense, then $y_Y^\Phi \cdot f = \Phi f \cdot y_X^\Phi$ is Φ-dense, and so by 2(2) f is Φ-dense because y_Y^Φ is fully faithful. $\square$

In particular, for each $\mathcal{T}$-category X, $\Phi\, y_X^\Phi : \Phi X \longrightarrow \Phi\Phi X$ has a right adjoint, $(y_X^\Phi)^{-1}$. We show next that $(y_X^\Phi)^{-1}$ has also a right adjoint, $y_{\Phi X}^\Phi : \Phi X \longrightarrow \Phi\Phi X$, so that:

$$\Phi\, y_X^\Phi \dashv (y_X^\Phi)^{-1} = \mathrm{Sup}_{\Phi X}^\Phi \dashv y_{\Phi X}^\Phi\,.$$

Proposition. *For each $\mathcal{T}$-category X, ΦX is Φ-cocomplete where* $\mathrm{Sup}_{\Phi X}^\Phi = (y_X^\Phi)^{-1}$.

Proof. Since y_X^Φ is Φ-dense, we may define $\mathrm{Sup}_{\Phi X}^\Phi := (y_X^\Phi)^{-1}$. We have to show that $\mathrm{Sup}_{\Phi X}^\Phi$ is a left inverse for $y_{\Phi X}^\Phi$; that is, $(y_X^\Phi)^{-1} \cdot y_{\Phi X}^\Phi = 1_{\Phi X}$: for each $\psi \in \Phi X$, $((y_X^\Phi)^{-1} \cdot y_{\Phi X}^\Phi)(\psi) = \psi^* \circ (y_X^\Phi)_* = \psi$. $\square$

In [10] we constructed $\mathcal{P}f$ as the colimit $\mathcal{P}f \cong \mathrm{colim}((y_X)_*, y_Y \cdot f)$, and a straightforward calculation shows that also $\Phi f \cong \mathrm{colim}((y_X^\Phi)_*, y_Y^\Phi \cdot f)$, for each $\mathcal{T}$-functor $f : X \longrightarrow Y$. To see this, we consider the commutative diagrams

$$\begin{tikzcd} X \arrow[r, "y_X^\Phi"] \arrow[rr, bend left, "y_X"] \arrow[d, "f"'] & \Phi X, \arrow[r, "i_X"] \arrow[d, "\Phi f"] & \mathcal{P}X \arrow[d, "\mathcal{P}f"] \\ Y \arrow[r, "y_Y^\Phi"'] \arrow[rr, bend right, "y_Y"'] & \Phi Y \arrow[r, "i_Y"'] & \mathcal{P}Y \end{tikzcd}$$

and obtain

$$\begin{aligned} (\Phi f)_* &= i_Y^* \circ i_{Y*} \circ (\Phi f)_* \\ &= i_Y^* \circ (\mathcal{P}f)_* \circ i_{X*} \\ &= i_Y^* \circ ((y_{Y*} \circ f_*) \circ\!\!- y_{X*}) \circ i_{X*} && \text{since } \mathcal{P}f \cong \mathrm{colim}((y_X)_*, y_Y \cdot f) \\ &= (i_Y^* \circ y_{Y*} \circ f_*) \circ\!\!- (i_X{}^* \circ y_{X*}) && \text{by Lemma 1} \\ &= (y_{Y*}^\Phi \circ f_*) \circ\!\!- y_{X*}^\Phi. \end{aligned}$$

Proposition. *Let $f : X \longrightarrow Y$ a $\mathcal{T}$-functor where X and Y are Φ-cocomplete.*

(1) *The following assertions are equivalent.*
  (a) *f is Φ-cocontinuous.*

(b) *We have* $f \cdot \mathrm{Sup}_X^\Phi \cong \mathrm{Sup}_Y^\Phi \cdot \Phi f$.

$$\begin{tikzcd} \Phi X \arrow[r, "\Phi f"] \arrow[d, "\mathrm{Sup}_X^\Phi"'] \arrow[dr, phantom, "\cong"] & \Phi Y \arrow[d, "\mathrm{Sup}_Y^\Phi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

(2) *If f is Φ-cocontinuous, then f is Φ-dense if and only it is a left adjoint.*

Proof. (1) (a) $\Rightarrow$ (b): Recall that

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "(y_X^\Phi)_*"', "\circ"] & X \\ \Phi X \arrow[ur, dotted, "(\mathrm{Sup}_X^\Phi)_* = 1_X \circ\!\!- (y_X^\Phi)_*"'] & \end{tikzcd}$$

Hence

$$\begin{aligned} (f \cdot \mathrm{Sup}_X^\Phi)_* &= f_* \circ\!\!- (y_X^\Phi)_* \\ &= ((\mathrm{Sup}_Y^\Phi)_* \circ (y_Y^\Phi)_* \circ f_*) \circ\!\!- (y_X^\Phi)_* \\ &= (\mathrm{Sup}_Y^\Phi)_* \circ ((y_Y^\Phi)_* \circ f_* \circ\!\!- (y_X^\Phi)_*) \\ &= (\mathrm{Sup}_Y^\Phi)_* \circ \Phi f_*. \end{aligned}$$

(b)$\Rightarrow$ (a): Consider

$$\begin{tikzcd} X \arrow[r, "1_X^*", "\circ"'] \arrow[d, "\varphi"', "\circ"] & X \arrow[r, "f"] & Y \\ A \arrow[ur, dotted, "(\mathrm{Sup}_X^\Phi \cdot \ulcorner\varphi\urcorner)_*"'] & & \end{tikzcd}$$

Then

$$\begin{aligned} (f \cdot \mathrm{Sup}_X^\Phi \cdot \ulcorner\varphi\urcorner) &= \mathrm{Sup}_Y^\Phi \cdot \Phi f \cdot \ulcorner\varphi\urcorner \\ &= \mathrm{Sup}_Y^\Phi \cdot \ulcorner\varphi \cdot f^*\urcorner \\ &\cong \mathrm{colim}(\varphi, f) \end{aligned}$$

(2) If f is Φ-cocontinuous and Φ-dense, from the commutative diagram of (1)(b) we have $f \dashv \mathrm{Sup}_X^\Phi \cdot f^{-1} \cdot y_Y^\Phi$ since $f \cdot \mathrm{Sup}_X^\Phi = \mathrm{Sup}_Y^\Phi \cdot \Phi f \dashv f^{-1} \cdot y_Y^\Phi$ and $\mathrm{Sup}_X^\Phi \cdot y_X^\Phi = 1_X$. The converse is trivially true. □

Corollary. *ΦX is closed in $\mathcal{P}X$ under Φ-weighted colimits.*

Proof. We show that the inclusion functor $i : \Phi X \longrightarrow \mathcal{P}X$ is Φ-cocontinuous, which, by the proposition above, is equivalent to the commutativity of the diagram

$$\begin{tikzcd} \Phi\Phi X \arrow[r, "\Phi i"] \arrow[d, "\mathrm{Sup}_{\Phi X}^\Phi"'] & \Phi\mathcal{P}X \arrow[d, "\mathrm{Sup}_{\mathcal{P}X}^\Phi"] \\ \Phi X \arrow[r, "i"'] & \mathcal{P}X. \end{tikzcd}$$

In Proposition 2 we observed $\mathrm{Sup}_{\Phi X}^\Phi = (y_X^\Phi)^{-1}$, and from Theorem 2 and [10, Theorem 2.8] follows that $\mathrm{Sup}_{\mathcal{P}X}^\Phi$ is the restriction of $y_X^{-1} : \mathcal{P}\mathcal{P}X \longrightarrow \mathcal{P}X$ to $\Phi\mathcal{P}X$. Let $\Psi \in \Phi\Phi X$. Then

$$i \cdot (y_X^\Phi)^{-1}(\Psi) = \Psi \circ (y_X^\Phi)_*$$

and

$$y_X^{-1} \cdot \Phi i(\Psi) = y_X^{-1}(\Psi \circ i^*) = \Psi \circ i^* \circ (y_X)_* = \Psi \circ (y_X^{\Phi})_*,$$

and the assertion follows. □

Theorem 2 says in particular that, for each $\mathcal{T}$-functor $f : A \longrightarrow X$, Φ-injective $\mathcal{T}$-category X and fully faithful Φ-dense $\mathcal{T}$-functor $i : A \longrightarrow B$, we have a canonical extension $g : B \longrightarrow X$ of f along i, namely $g \cong \mathrm{colim}(i_*, f)$, giving us an alternative description of Φf.

Theorem. *Composition with $y_X^{\Phi} : X \longrightarrow \Phi X$ defines an equivalence*

$$\mathcal{T}\text{-}\mathsf{Cocont}^{\Phi}(\Phi X, Y) \longrightarrow \mathcal{T}\text{-}\mathsf{Cat}(X, Y)$$

of ordered sets, for each Φ-cocomplete $\mathcal{T}$-category Y.

The series of results above tell us that $\mathcal{T}\text{-}\mathsf{Cocont}^{\Phi}_{\mathrm{sep}}$ is actually a (non-full) reflective subcategory of $\mathcal{T}\text{-}\mathsf{Cat}$, with left adjoint $\Phi : \mathcal{T}\text{-}\mathsf{Cat} \longrightarrow \mathcal{T}\text{-}\mathsf{Cocont}^{\Phi}_{\mathrm{sep}}$. In fact, Φ is a 2-functor and one verifies as in [10] that the induced monad $\mathbb{I}^{\Phi} = (\Phi, y^{\Phi}, (y^{\Phi})^{-1})$ on $\mathcal{T}\text{-}\mathsf{Cat}$ is of Kock-Zöberlein type. Theorem 2 and Proposition 2 imply that $\mathcal{T}\text{-}\mathsf{Cocont}^{\Phi}_{\mathrm{sep}}$ is equivalent to the category of Eilenberg-Moore algebras of $\mathbb{I}^{\Phi}$.

Finally, we wish to study monadicity of the canonical forgetful functor

$$G : \mathcal{T}\text{-}\mathsf{Cocont}^{\Phi}_{\mathrm{sep}} \longrightarrow \mathsf{Set}.$$

Certainly,

(a): G has a left adjoint given by the composite

$$\mathsf{Set} \xrightarrow{\mathrm{disc}} \mathcal{T}\text{-}\mathsf{Cat} \xrightarrow{\Phi} \mathcal{T}\text{-}\mathsf{Cocont}^{\Phi}_{\mathrm{sep}},$$

where $\mathrm{disc}(X) = (X, e_X^{\circ})$, and $\mathrm{disc}(f) = f$.

In order to prove monadicity of G we will impose, in addition to (Ax 1)-(Ax 3),

(Ax 4).: For each surjective $\mathcal{T}$-functor f, $f_* \in \Phi$.

Hence, any bijective $f : X \longrightarrow Y$ in $\mathcal{T}\text{-}\mathsf{Cocont}^{\Phi}_{\mathrm{sep}}$ is Φ-dense and therefore left adjoint. By [10, Lemma 2.16], f is invertible and we have seen that

(b): G reflects isomorphisms.

In order to conclude that G is monadic, it is left to show that

(c): $\mathcal{T}\text{-}\mathsf{Cocont}^{\Phi}_{\mathrm{sep}}$ has and G preserves coequaliser of G-equivalence relations

(see, for instance, [14, Corollary 2.7]). To do so, let $\pi_1, \pi_2 : R \rightrightarrows X$ in $\mathcal{T}\text{-}\mathsf{Cocont}^{\Phi}_{\mathrm{sep}}$ be an equivalence relation in Set, where π_1 and π_2 are the projection maps, and let $q : X \longrightarrow Q$ be its coequaliser in $\mathcal{T}\text{-}\mathsf{Cat}$. The proof in [10, Section 2.6] rests on the observation that

$$\mathcal{P}R \underset{\mathcal{P}\pi_2}{\overset{\mathcal{P}\pi_1}{\rightrightarrows}} \mathcal{P}X \xrightarrow{\mathcal{P}q} \mathcal{P}Q$$

is a split fork in $\mathcal{T}\text{-}\mathsf{Cat}_{\mathrm{sep}}$. Naturally, we wish to show that, in our setting,

$$\Phi R \underset{\Phi\pi_2}{\overset{\Phi\pi_1}{\rightrightarrows}} \Phi X \xrightarrow{\Phi q} \Phi Q$$

gives rise to a split fork in $\mathcal{T}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ as well. Since π_1, π_2 and q are surjective, the $\mathcal{T}$-functors π_1, π_2 and q are Φ-dense and therefore we have $\mathcal{T}$-functors $q^{-1} : \Phi Q \longrightarrow \Phi X$ and $\pi_1^{-1} : \Phi X \longrightarrow \Phi R$. Furthermore, $\Phi q \cdot q^{-1} = 1_{\Phi X} = \Phi\pi_1 \cdot \pi_1^{-1}$. It is left to show that

$$q^{-1} \cdot \Phi q = \Phi\pi_2 \cdot \pi_1^{-1},$$

which can be shown with the same calculation as in [10], based on the following proposition.

Proposition. *Consider the following diagram in* $\mathcal{T}$-Cat

$$\begin{tikzcd} R \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X \arrow[r, "q"] & Q \end{tikzcd}$$

with $\pi_1, \pi_2 : R \rightrightarrows X$ *in* $\mathcal{T}$-$\mathsf{Cocont}^{\Phi}_{\mathrm{sep}}$, (π_1, π_2) *an equivalence relation in* Set, *and* $q : X \longrightarrow Q$ *its coequaliser in* $\mathcal{T}$-Cat.

(1) *If* π_1, π_2 *are left adjoints, then* q *is proper.*
(2) *The diagram*

$$\begin{tikzcd} \Phi R \arrow[r, shift left, "\Phi\pi_1"] \arrow[r, shift right, "\Phi\pi_2"'] & \Phi X \arrow[r, "\Phi q"] \arrow[l, bend right, "\pi_1^{-1}"'] & \Phi Q \arrow[l, bend right, "q^{-1}"'] \end{tikzcd}$$

is a split fork in $\mathcal{T}$-Cat.

Proof. (1) As in [10, Lemma 2.19 and Corollary 2.20].
(2) Analogous to the proof presented in [10, Section 2.6]. □

Finally, we conclude that:

Theorem. *Under (Ax 1)-(Ax 4), the forgetful functor* $G : \mathcal{T}$-$\mathsf{Cocont}^{\Phi}_{\mathrm{sep}} \longrightarrow \mathsf{Set}$ *is monadic.*

Proof. In order to show that $\mathcal{T}$-$\mathsf{Cocont}^{\Phi}_{\mathrm{sep}}$ has and G preserves coequaliser of G-equivalence relations, consider again the first diagram of Proposition 2. We have seen that

$$\begin{tikzcd} \Phi R \arrow[r, shift left, "\Phi\pi_1"] \arrow[r, shift right, "\Phi\pi_2"'] & \Phi X \arrow[r, "\Phi q"] \arrow[l, bend right, "\pi_1^{-1}"'] & \Phi Q \arrow[l, bend right, "q^{-1}"'] \end{tikzcd}$$

is a split fork and hence a coequaliser diagram in $\mathcal{T}$-Cat. Since π_1 and π_2 are Φ-cocontinuous, there is a $\mathcal{T}$-functor $\mathrm{Sup}^{\Phi}_{Q} : \Phi Q \longrightarrow Q$ which, since $q : X \longrightarrow Q$ is the coequaliser of $\pi_1, \pi_2 : R \rightrightarrows X$ in $\mathcal{T}$-Cat, satisfies $\mathrm{Sup}^{\Phi}_{Q} \cdot y^{\Phi}_{Q} = 1_Q$. The situation is depicted in the following diagram.

$$\begin{tikzcd} R \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] \arrow[d, "y^{\Phi}_{R}"'] & X \arrow[r, "q"] \arrow[d, "y^{\Phi}_{X}"] & Q \arrow[d, "y^{\Phi}_{Q}"] \arrow[dd, bend left, "1_Q"] \\ \Phi R \arrow[r, shift left, "\Phi\pi_1"] \arrow[r, shift right, "\Phi\pi_2"'] \arrow[d, "\mathrm{Sup}^{\Phi}_{R}"'] & \Phi X \arrow[r, "\Phi q"] \arrow[d, "\mathrm{Sup}^{\Phi}_{X}"] & \Phi Q \arrow[d, dotted, "\mathrm{Sup}^{\Phi}_{Q}"] \\ R \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X \arrow[r, "q"] & Q \end{tikzcd}$$

We conclude that Q is separated and Φ-cocomplete, and $q : X \longrightarrow Q$ is Φ-cocontinuous. Finally, to see that $q : X \longrightarrow Q$ is the coequaliser of $\pi_1, \pi_2 : R \rightrightarrows X$ in $\mathcal{T}$-$\mathsf{Cocont}^{\Phi}_{\mathrm{sep}}$, let $h : X \longrightarrow Y$ be in $\mathcal{T}$-$\mathsf{Cocont}^{\Phi}_{\mathrm{sep}}$ with $h \cdot \pi_1 = h \cdot \pi_2$. Then, since Φq is the coequaliser of $\Phi\pi_1, \Phi\pi_2 : \Phi R \rightrightarrows \Phi X$ in $\mathcal{T}$-$\mathsf{Cocont}^{\Phi}_{\mathrm{sep}}$, there is a Φ-cocontinuous $\mathcal{T}$-functor $f : \Phi Q \longrightarrow Y$ such that $f \cdot \Phi q = h \cdot \mathrm{Sup}^{\Phi}_{X}$. Then

$$f \cdot y^{\Phi}_{Q} \cdot q = f \cdot \Phi q \cdot y^{\Phi}_{X} = h \cdot \mathrm{Sup}^{\Phi}_{X} \cdot y^{\Phi}_{X} = h$$

and

$$\mathrm{Sup}_Y^\Phi \cdot \Phi f \cdot \Phi\, y_Q^\Phi \cdot \Phi q = f \cdot \mathrm{Sup}_{\Phi Q}^\Phi \cdot \Phi\, y_Q^\Phi \cdot \Phi q = f \cdot \Phi q = h \cdot \mathrm{Sup}_X^\Phi$$
$$= f \cdot y_Q^\Phi \cdot q \cdot \mathrm{Sup}_X^\Phi = f \cdot y_Q^\Phi \cdot \mathrm{Sup}_Q^\Phi \cdot \Phi q,$$

hence $\mathrm{Sup}_Y \cdot \Phi(f \cdot y_Q^\Phi) = f \cdot y_Q^\Phi \cdot \mathrm{Sup}_Q^\Phi$, that is, $f \cdot y_Q^\Phi$ is Φ-cocontinuous. □

3. The examples

3.1. **All distributors.** The class $\Phi = \mathcal{P}$ of all distributors satisfies obviously all four axioms. In fact, this is the situation studied in [10].

3.2. **Representable distributors.** The smallest possible choice is $\Phi = \mathcal{R}$ being the class of all representable $\mathcal{T}$-distributors $\mathcal{R} = \{f^* \mid f \text{ is a } \mathcal{T}\text{-functor}\}$. Clearly, $\mathcal{R}$ satisfies (Ax 1), (Ax 2) and (Ax 3) but not (Ax 4). We have $\mathcal{R}(X) = \{x^* \mid x \in X\}$, each $\mathcal{T}$-category is $\mathcal{R}$-cocomplete and each $\mathcal{T}$-functor is $\mathcal{R}$-cocontinuous, and therefore $\mathcal{T}\text{-}\mathsf{Cocont}^{\mathcal{R}}_{\mathrm{sep}} = \mathcal{T}\text{-}\mathsf{Cat}_{\mathrm{sep}}$. This case is certainly not very interesting; however, our results tell us that the inclusion functor $\mathcal{T}\text{-}\mathsf{Cat}_{\mathrm{sep}} \hookrightarrow \mathcal{T}\text{-}\mathsf{Cat}$ is monadic. In particular, *the category* Top_0 *of topological T_0-spaces and continuous maps is a monadic subcategory of* Top.

3.3. **Almost representable distributors.** We can modify slightly the example above and consider $\Phi = \mathcal{R}_0$ the class of all almost representable $\mathcal{T}$-distributors, where a $\mathcal{T}$-distributor $\varphi : X \multimap Y$ is called *almost representable* whenever, for each $y \in Y$, either $y^* \circ \varphi = \bot$ or $y^* \circ \varphi = x^*$ for some $x \in X$. As above, $\mathcal{R}_0$ satisfies (Ax 1), (Ax 2) and (Ax 3) but not (Ax 4).

By definition, for a $\mathcal{T}$-category X we have

$$\mathcal{R}_0(X) = \{\psi \in \mathcal{P}X \mid \psi \in \mathcal{R}_0\} = \{x^* \mid x \in X\} \cup \{\bot\},$$

with the structure inherited from $\mathcal{P}X$. Furthermore, a $\mathcal{T}$-functor $f : (X, a) \longrightarrow (Y, b)$ is $\mathcal{R}_0$-dense whenever, for each $y \in Y$,

$$\exists \mathfrak{x} \in TX \,.\, b(Tf(\mathfrak{x}), y) > \bot \;\Rightarrow\; \exists x \in X\, \forall \mathfrak{x} \in TX \,.\, b(Tf(\mathfrak{x}), y) = a(\mathfrak{x}, x).$$

Hence, with

$$Y_0 = \{y \in Y \mid \exists \mathfrak{x} \in TX \,.\, b(Tf(\mathfrak{x}), y) > \bot\}$$

we can factorise an $\mathcal{R}_0$-dense $\mathcal{T}$-functor $f : X \longrightarrow Y$ as

$$X \xrightarrow{f} Y_0 \hookrightarrow Y,$$

where $Y_0 \hookrightarrow Y$ is fully faithful and $X \xrightarrow{f} Y_0$ is left adjoint. If we consider $f : X \longrightarrow Y$ in Top, then $Y_0 = \overline{f(X)}$ is the closure of the image of f, so that each $\mathcal{R}_0$-dense continuous map factors as a left adjoint continuous map followed by a closed embedding. Consequently, *for a topological space X, the following assertions are equivalent:*

(i) *X is injective with respect to $\mathcal{R}_0$-dense fully faithful continuous maps.*
(ii) *X is injective with respect to closed embeddings.*

Note that in this example we are working with the dual order, compared with [7, Section 11].

3.4. **Right adjoint distributors.** Now we consider $\Phi = \mathcal{L}$ the class of all right adjoint $\mathcal{T}$-distributors. This class contains all distributors of the form f^*, for a $\mathcal{T}$-functor f, and it is closed under composition. Since adjointness of a $\mathcal{T}$-distributor $\varphi : X \longrightarrow\!\!\!\!\!\!\circ\; Y$ can be tested pointwise in Y, the axioms (Ax 1), (Ax 2) and (Ax 3) are satisfied. By definition, $\mathcal{L}(X) = \{\psi \in \mathcal{P}X \mid \psi \text{ is right adjoint}\}$, and a $\mathcal{T}$-category is $\mathcal{L}$-cocomplete if each pair $\varphi \dashv \psi$, $\varphi : Y \longrightarrow\!\!\!\!\!\!\circ\; X$, $\psi : X \longrightarrow\!\!\!\!\!\!\circ\; Y$, of adjoint $\mathcal{T}$-distributors is of the form $f_* \dashv f^*$, for a $\mathcal{T}$-functor $f : Y \longrightarrow X$. For V-categories, this is precisely the well-known notion of Cauchy-completeness as introduced by Lawvere in [13] as a generalisation of the classical notion for metric spaces. However, Lawvere never proposed the name "Cauchy-complete", and, while working on this notion in the context of $\mathcal{T}$-categories in [5] and [11], we used instead Lawvere-complete and L-complete, respectively. Furthermore, one easily verifies that each $\mathcal{T}$-functor is $\mathcal{L}$-cocontinuous, i.e. (right adjoint)-weighted colimits are absolute, so that $\mathcal{T}\text{-}\mathsf{Cocont}^{\mathcal{L}}_{\text{sep}} = \mathcal{T}\text{-}\mathsf{Cat}_{\text{cpl}}$ is the full subcategory of $\mathcal{T}\text{-}\mathsf{Cat}$ consisting of all separated and Lawvere complete $\mathcal{T}$-categories.

On the other hand, for a surjective $\mathcal{T}$-functor f, f_* does not need to be right adjoint, so that (Ax 4) is in general not satisfied. This is not a surprise, since natural instances of this example fail Theorem 2. Indeed, in the category of ordered sets and monotone maps, any ordered set is Lawvere-complete, hence the category of Lawvere-complete and separated ordered sets coincides with the category of anti-symmetric ordered sets. The canonical forgetful functor from this category to Set is surely not monadic. Also, the canonical forgetful functor from the category of Lawvere-complete and separated topological spaces (= sober spaces) and continuous maps to Set is also not monadic.

3.5. **Inhabited distributors.** Another class of distributors considered in [10] is $\Phi = \mathcal{I}$ the class of all inhabited $\mathcal{T}$-distributors. Here a $\mathcal{T}$-distributor $\varphi : X \longrightarrow\!\!\!\!\!\!\circ\; Y$ is called *inhabited* if

$$\forall y \in Y \,.\, k \leq \bigvee_{\mathfrak{x} \in TX} \varphi(\mathfrak{x}, y).$$

(Ax 3) is satisfied by definition, and in [10] we showed already the validity of (Ax 1) and (Ax 2). Furthermore, one easily verifies that (Ax 4) is satisfied. Hence, as already observed in [10], all results stated in Section 2 are available for this class of distributors. Let us recall that, specialised to Top, inhabited-dense continuous maps are precisely the topologically dense continuous maps, and the injective spaces with respect to topologically dense embeddings are known as *Scott domains* [8].

3.6. **"closed" distributors.** A further interesting class of distributors is given by

$$\Phi = \{\varphi : X \longrightarrow\!\!\!\!\!\!\circ\; Y \mid \forall y \in Y, \mathfrak{x} \in TX \,.\, \varphi(\mathfrak{x}, y) \leq \bigvee_{x \in X} a(\mathfrak{x}, x) \otimes \varphi(e_X(x), y)\},$$

that is, $\varphi \in \Phi$ if and only if $\varphi \leq \varphi \cdot e_X \cdot a$. Clearly, (Ax 3) is satisfied. For each $\mathcal{T}$-functor $g : (Y, b) \longrightarrow (X, a)$ we have

$$g^* \cdot e_X \cdot a = g^\circ \cdot a \cdot e_X \cdot a \geq g^\circ \cdot a = g^*,$$

hence $g^* \in \Phi$. Furthermore, given $\mathcal{T}$-distributors $\varphi : X \longrightarrow\!\!\!\!\!\!\circ\; Y$ and $\psi : Y \longrightarrow\!\!\!\!\!\!\circ\; Z$ in Φ, then

$$\begin{aligned}\psi \circ \varphi = \psi \cdot T_\xi \varphi \cdot m_X^\circ \leq \psi \cdot e_Y \cdot b \cdot T_\xi \varphi \cdot m_X^\circ = \psi \cdot e_Y \cdot \varphi \leq \psi \cdot e_Y \cdot \varphi \cdot e_X \cdot a \\ \leq \psi \cdot T_\xi \varphi \cdot e_{TX} \cdot e_X \cdot a \leq \psi \cdot T_\xi \varphi \cdot m_X^\circ \cdot e_X \cdot a = (\psi \circ \varphi) \cdot e_X \cdot a\end{aligned}$$

and therefore also $\psi \circ \varphi \in \Phi$. We have seen that this class of distributors satisfies (Ax 1), (Ax 2) and (Ax 3). On the other hand, (Ax 4) is not satisfied.

By definition, a $\mathcal{T}$-functor $f : (X, a) \longrightarrow (Y, b)$ is Φ-dense whenever, for all $\mathfrak{x} \in TX$ and $y \in Y$,

$$b(Tf(\mathfrak{x}), y) \leq \bigvee_{x \in X} a(\mathfrak{x}, x) \otimes b(e_Y(f(x)), y).$$

Hence, each proper $\mathcal{T}$-functor (see [3]) is Φ-dense. In fact, Φ-dense $\mathcal{T}$-functors can be seen as "proper over V-Cat", and the condition above states exactly properness of f if the underlying V-category $\mathrm{S}\,Y$ of $Y = (Y, b)$ is discrete. Furthermore, each surjective Φ-dense $\mathcal{T}$-functor is final with respect to the forgetful functor $\mathrm{S} : \mathcal{T}\text{-}\mathsf{Cat} \longrightarrow \mathsf{V}\text{-}\mathsf{Cat}$. To see this, let $f : (X, a) \longrightarrow (Y, b)$ be a surjective Φ-dense $\mathcal{T}$-functor, $Z = (Z, c)$ a $\mathcal{T}$-category and $g : \mathrm{S}\,Y \longrightarrow \mathrm{S}\,Z$ a V-functor such that gf is a $\mathcal{T}$-functor. We have to show that g is a $\mathcal{T}$-functor. Let $\mathfrak{y} \in TY$ and $y \in Y$. Since Tf is surjective, there is some $\mathfrak{x} \in TX$ with $Tf(\mathfrak{x}) = \mathfrak{y}$. We conclude

$$\begin{aligned} b(\mathfrak{y}, y) &= b(Tf(\mathfrak{x}), y) \\ &\leq \bigvee_{x \in X} a(\mathfrak{x}, x) \otimes b(e_Y(f(x)), y) \\ &\leq \bigvee_{x \in X} c(T(gf)(\mathfrak{x}), gf(x) \otimes c(e_Z(gf(x)), g(y)) \\ &\leq c(Tg(\mathfrak{y}), g(y)). \end{aligned}$$

3.7. **Further examples.** A wide class of examples of injective topological spaces is described in [7], where the authors consider injectivity with respect to a class of embeddings $f : X \longrightarrow Y$ such that the induced frame morphism $f_* : \Omega X \longrightarrow \Omega Y$ preserves certain suprema. A similar construction can be done in our setting; to do so *we assume from now on* $T1 = 1$. For a $\mathcal{T}$-category X, the V-*category of covariant presheafs* V^X is defined as

$$\mathsf{V}^X = \{\alpha : 1 \,{-\!\!\circ\!\!\rightharpoonup}\, X \mid \alpha \text{ is a } \mathcal{T}\text{-distributor}\} = \{\alpha : X \longrightarrow \mathsf{V} \mid \alpha \text{ is a } \mathcal{T}\text{-functor}\},$$

and the V-categorical structure $[\alpha, \beta] \in \mathsf{V}$ is given as the lifting

$$\begin{tikzcd} X & 1 \arrow[l, "\beta"', "\circ"] \\ 1 \arrow[u, "\alpha"] \arrow[ur, dotted, "\circ", "\alpha{-\!\circ}\beta =: [\alpha,\beta]"'] & \end{tikzcd}$$

for all $\alpha, \beta \in \mathsf{V}^X$. Since $e_1 : 1 \longrightarrow T1$ is an isomorphism, this lifting of $\mathcal{T}$-distributors does exist and can be calculated as the corresponding lifting of V-distributors

$$\begin{tikzcd} X & 1 \arrow[l, "\beta"', "\circ"] \arrow[dl, dotted, "\circ"] \\ 1 \arrow[u, "\alpha"] & \end{tikzcd}$$

Each $\mathcal{T}$-distributor $\varphi : X \,{-\!\!\circ\!\!\rightharpoonup}\, Y$ induces a V-functor

$$\varphi \circ (-) : \mathsf{V}^X \longrightarrow \mathsf{V}^Y,\ \alpha \longmapsto \varphi \circ \alpha,$$

which is right adjoint if φ is a right adjoint $\mathcal{T}$-distributor. Given now a class Ψ of V-distributors, we may consider the class Φ of all those $\mathcal{T}$-distributors φ for which $\varphi \circ (-)$ preserves Ψ-weighted limits. This class of $\mathcal{T}$-distributors is certainly closed under composition, and contains all right adjoint $\mathcal{T}$-distributors, hence it includes all representable ones. Finally, if Ψ-weighted limits are calculated pointwise in V^X, then also (Ax 3) is fulfilled. As particular examples we have the class Φ of all $\mathcal{T}$-distributors $\varphi : X \,{-\!\!\circ\!\!\rightharpoonup}\, Y$ for which $\varphi \circ (-)$ preserves

(1) the top element of V^X, that is, for which $\varphi \circ \top = \top$. In pointwise notation, this reads as

$$\forall y \in Y \,.\, \top = \bigvee_{\mathfrak{x} \in TX} \varphi(\mathfrak{x}, y) \otimes \top.$$

If $k = \top$, then this class of $\mathcal{T}$-distributors coincides with the class of inhabited $\mathcal{T}$-distributors considered in 3.5.

(2) cotensors, that is, for each $u \in \mathsf{V}$ and each $\alpha \in \mathsf{V}^X$, $\varphi \circ \hom(u, \alpha) = \hom(u, \varphi \circ \alpha)$.

(3) finite infima (cf. [7, Section 6]).

(4) arbitrary infima (cf. [7, Section 7]).

(5) codirected infima (cf. [7, Section 8]).

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Applied Categorical Structures **12:** 127–154, 2004.
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One Setting for All: Metric, Topology, Uniformity, Approach Structure

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(Received: 21 November 2002; accepted: 4 December 2002)

Abstract. For a complete lattice **V** which, as a category, is monoidal closed, and for a suitable **Set**-monad T we consider $(\mathsf{T}, \mathbf{V})$-algebras and introduce $(\mathsf{T}, \mathbf{V})$-proalgebras, in generalization of Lawvere's presentation of metric spaces and Barr's presentation of topological spaces. In this lax-algebraic setting, uniform spaces appear as proalgebras. Since the corresponding categories behave functorially both in T and in **V**, one establishes a network of functors at the general level which describe the basic connections between the structures mentioned by the title. Categories of $(\mathsf{T}, \mathbf{V})$-algebras and of $(\mathsf{T}, \mathbf{V})$-proalgebras turn out to be topological over **Set**.

Mathematics Subject Classifications (2000): 18C20, 18B30, 54E15.

Key words: **V**-matrix, **V**-promatrix, $(\mathsf{T}, \mathbf{V})$-algebra, $(\mathsf{T}, \mathbf{V})$-proalgebra, co-Kleisli composition, ordered set, metric space, topological space, uniform space, approach space, prometric space, protopological space, proapproach space, topological category.

Introduction

Since the late sixties it has been known that monads over the category **Set**, via their Eilenberg–Moore construction [9], describe precisely the varieties of general algebras (with arbitrarily many infinitary operations and free algebras, see for example [13, 10]), and therefore provide a common categorical description not only of the standard categories of algebra, such as groups, rings, R-modules, R-algebras, etc., but also of some categories outside the realm of algebra, such as the category of compact Hausdorff spaces. The equational description of this latter category by Manes [16] in terms of the "operation" that sends an ultrafilter to a point of convergence satisfying two basic "equations" fully explained many of the algebraic properties of **CompHaus** and enjoyed wide-spread attention. By comparison, there was only moderate interest in Barr's subsequent observation that, when relaxing the operation to a relation and the equalities to inequalities,

the Eilenberg–Moore construction actually describes precisely the category **Top** of all topological spaces, in terms of two simple axioms on a convergence relation between ultrafilters and points [2]. It is the aim of this paper to show that, with one additional ingredient to Barr's presentation, one is able to describe uniformly all structures that seem to matter in topology, namely metrics, topologies, uniformities and approach structures (as introduced by [14]), and to display the basic functors connecting them in general terms.

This one additional ingredient is an arbitrary complete lattice $\mathbf{V}$ with a monoidal-closed structure that takes the place of the 2-element chain which implicitly governs the axioms defining topological spaces. Lawvere in his fundamental paper [12] considered for $\mathbf{V}$ the extended real half-line $\overline{\mathbb{R}}_+ = [0, \infty]$ (ordered opposite to the natural order) and displayed individual metric spaces as $\mathbf{V}$-categories. In [7], we combined Barr's and Lawvere's ideas and introduced $(\mathsf{T}, \mathbf{V})$-algebras, for a monad T on **Set** and a symmetric monoidal-closed category $\mathbf{V}$, obtaining topological spaces for $\mathsf{T} = \mathsf{U}$ the ultrafilter monad and $\mathbf{V} = \mathbf{2}$, premetric ($= \infty pq$-metric) spaces for $\mathsf{T} = 1$ the identity monad and $\mathbf{V} = \overline{\mathbb{R}}_+$, and approach spaces via the natural combination of the previous two structures, with $\mathsf{T} = \mathsf{U}$ and $\mathbf{V} = \overline{\mathbb{R}}_+$. However, we were not able to include uniformities in this setting, although their inclusion in the setting given in [4] indicated that it should be possible to do so.

This paper fills the gap. Instead of considering sets X with a single $\mathbf{V}$-relational Eilenberg–Moore structure $TX \nrightarrow X$, we define $(\mathsf{T}, \mathbf{V})$-*proalgebras* as sets which come with a directed set of $\mathbf{V}$-relational structures $TX \nrightarrow X$. The category of *quasi-uniform spaces* is equivalent to the category of $(1, \mathbf{2})$-proalgebras, denoted here by **ProOrd** since $(1, \mathbf{2})$-algebras form precisely the category **Ord** of preordered sets. Likewise, the category of $(1, \overline{\mathbb{R}}_+)$-proalgebras is denoted by **ProMet** since $(1, \overline{\mathbb{R}}_+)$-algebras form precisely the category **Met** of premetric spaces; it is closely related to the category of *approach-uniform spaces* considered in [15]. By further exploiting the fact that the formation of the categories of $(\mathsf{T}, \mathbf{V})$-algebras and of $(\mathsf{T}, \mathbf{V})$-proalgebras behaves functorially in both T and $\mathbf{V}$, we arrive at a commutative diagram which, together with the various adjoints to the embeddings, not only comprehensively describes the basic connections between the fundamental topological structures already mentioned, but also introduces two new players: *protopological spaces* ($\mathsf{T} = \mathsf{U}$, $\mathbf{V} = \mathbf{2}$), not to be confused though closely related with pretopological and pseudotopological spaces as discussed in [11], as well as *proapproach spaces* ($\mathsf{T} = \mathsf{U}$, $\mathbf{V} = \overline{\mathbb{R}}_+$). They turn out to be useful when describing some of the connections between the previous categories.

$$\begin{tikzcd}
 & \mathbf{ProTop} \arrow[rr, hook] & & \mathbf{ProApp} \\
\mathbf{Top} \arrow[ur, hook] \arrow[rr, hook] & & \mathbf{App} \arrow[ur, hook] & \\
 & \mathbf{ProOrd} \arrow[uu, hook] \arrow[rr, hook] & & \mathbf{ProMet} \arrow[uu, hook] \\
\mathbf{Ord} \arrow[uu, hook] \arrow[ur, hook] \arrow[rr, hook] & & \mathbf{Met} \arrow[uu, hook] \arrow[ur, hook] &
\end{tikzcd}$$

The horizontal embeddings in the diagram above are both reflective and coreflective, a fact that was observed in [14] for **Top** $\hookrightarrow$ **App**, all arising from the both reflective and coreflective embedding $\mathbf{2} \to \overline{\mathbb{R}}_+$. The vertical embeddings are coreflective, with the coreflector always induced by the monad morphism $1 \to \mathsf{U}$. The diagonal embeddings are coreflective as well, with the coreflector always given by "meet". The "induced topology" functor of (quasi-)uniform spaces factors through the coreflector of **Top** $\hookrightarrow$ **ProTop**.

For our $(\mathsf{T}, \mathbf{V})$-algebras and $(\mathsf{T}, \mathbf{V})$-proalgebras, the two basic axioms of an Eilenberg–Moore algebra, namely the unit and the associativity laws, look more like reflexivity and transitivity conditions. We show that, with a formally inverted Kleisli-composition law, the two axioms may also be presented as extensitivity and idempotency conditions. For a topological space, this is exactly the transition from its convergence structure to its Kuratowski closure operation. We extend a result of [4] and give a general proof that the categories occuring in the diagram above are topological over **Set**, by showing the existence of initial structures w.r.t. the underlying **Set**-functors. Furthermore, in the examples considered here we describe explicitly the 2-categorical structure of categories of $(\mathsf{T}, \mathbf{V})$-algebras as given in [7] and extend it naturally to categories of $(\mathsf{T}, \mathbf{V})$-proalgebras.

Finally we point out that, like $(\mathsf{T}, \mathbf{V})$-algebras, also $(\mathsf{T}, \mathbf{V})$-proalgebras may be considered more generally when $\mathbf{V}$ is an arbitrary symmetric monoidal category with coproducts preserved by tensor in each variable, not just a lattice, for the price that one then has to deal with a considerable number of coherence issues which make the treatment considerably more cumbersome (as indicated in [7] and [6] in the case of $(\mathsf{T}, \mathbf{V})$-algebras). But even in this more general context it is possible to prove significant results. For example, the paper [6] shows the local Cartesian closedness of categories of $(\mathsf{T}, \mathbf{V})$-algebras which are only reflexive, not necessarily transitive, and thereby provides an important step towards a characterization of exponentiable maps in the category of all $(\mathsf{T}, \mathbf{V})$-algebras. Another type of maps which is notoriously difficult to describe, namely the class of effective descent morphisms, is characterized in [5] for certain cases.

1. Categories of V-matrices

1.1. HYPOTHESIS

Let $\mathbf{V}$ be a complete lattice which, when considered as a category, is symmetric monoidal-closed. Hence, there are a distinguished element $k \in \mathbf{V}$ and an associative and commutative binary operation $\otimes$ on $\mathbf{V}$ for which k is neutral and which preserves suprema in each variable:

$$a \otimes \bigvee_{i \in I} b_i = \bigvee_{i \in I} a \otimes b_i.$$

1.2. EXAMPLES

(1) Each frame (= complete lattice in which binary meets distribute over arbitrary joins) is symmetric monoidal-closed, with $\otimes$ given by binary meet and $k = \top$ the top element. In particular, the two-element chain

$$\mathbf{2} = \{\bot, \top\}$$

carries this structure.

(2) ([12]) Let $\overline{\mathbb{R}}_+ = [0, \infty]$ be the extended real (half-)line, provided with the order opposite to the natural order (so that $\bigvee_{i\in I} a_i = \inf_{i\in I} a_i$ is the natural infimum of the elements a_i), and with $\otimes = +$ the addition (extended by $a + \infty = \infty + a = \infty$) and $k = 0$. In this way we consider $\overline{\mathbb{R}}_+$ as a symmetric monoidal-closed lattice.

For future reference we remark that the embedding (considered as a functor between "thin" categories, i.e. preordered sets)

$$E : \mathbf{2} \to \overline{\mathbb{R}}_+,\ \bot \mapsto \infty,\ \top \mapsto 0,$$

has both a right adjoint retraction

$$R : \overline{\mathbb{R}}_+ \to \mathbf{2},\ (0 < x < \infty) \mapsto \bot,$$

and a left adjoint retraction

$$L : \overline{\mathbb{R}}_+ \to \mathbf{2},\ (0 < x < \infty) \mapsto \top.$$

1.3. **V**-MATRICES

The category Mat(**V**) of **V**-*matrices* has as its objects sets, and its morphisms $r : X \nrightarrow Y$ are functions $r : X \times Y \to \mathbf{V}$, often written as families $r = (r(x, y))_{x\in X, y\in Y}$; the composite arrow of r followed by $s : Y \nrightarrow Z$ is given by matrix multiplication

$$(sr)(x, z) = \bigvee_{y\in Y} r(x, y) \otimes s(y, z),$$

and the identity arrow $1_X : X \nrightarrow X$ is the diagonal matrix with values k in the main diagonal and all other values $\bot$, the bottom element of **V**.

The hom-sets of Mat(**V**) are partially ordered by

$$r \le r' \Leftrightarrow \forall x \in X\ \forall y \in Y : r(x, y) \le r'(x, y),$$

compatibly with composition. Hence, Mat(**V**) is actually a 2-*category*. In addition, Mat(**V**) has an order-preserving *involution*, given by matrix transposition: the transpose $r^\circ : Y \nrightarrow X$ of $r : X \nrightarrow Y$ is defined by $r^\circ(y, x) = r(x, y)$, $x \in X$, $y \in Y$, and satisfies

$$(sr)^\circ = r^\circ s^\circ, \qquad (1_X)^\circ = 1_X.$$

Finally, there is a functor

$$\mathbf{Set} \to \mathrm{Mat}(\mathbf{V})$$

which maps objects identically and treats $f : X \to Y$ in **Set** as a matrix $f : X \nrightarrow Y$, putting

$$f(x, y) = \begin{cases} k & \text{if } f(x) = y, \\ \bot & \text{else.} \end{cases}$$

When we write $f : X \to Y$ in $\mathrm{Mat}(\mathbf{V})$, it is understood that f is a set map considered as a matrix in this way. In the 2-category $\mathrm{Mat}(\mathbf{V})$ f plays the role of a map (in the sense of Lawvere), satisfying the inequalities $1_X \leq f^\circ f$ and $ff^\circ \leq 1_Y$. We also note that the matrix composition becomes a lot simpler when one of the players is a map:

$$(sf)(x, z) = s(f(x), z), \qquad (gr)(x, z) = \bigvee_{y:g(y)=z} r(x, y)$$

for $f : X \to Y$, $s : Y \nrightarrow Z$, $r : X \nrightarrow Y$ and $g : Y \to Z$; furthermore, with $t : X \nrightarrow Z$ one has the adjunction rules

$$\frac{t \leq sf}{tf^\circ \leq s} \qquad \frac{gr \leq t}{r \leq g^\circ t}. \tag{1}$$

1.4. EXAMPLES

(1) For $\mathbf{V} = \mathbf{2}$, $\mathrm{Mat}(\mathbf{V})$ is the 2-category $\mathrm{Rel}(\mathbf{Set})$ whose objects are sets and whose morphisms are relations $r : X \nrightarrow Y$ which, when we write xry instead of $r(x, y) = \top$, compose as usual as

$$x(sr)z \Leftrightarrow \exists y : xry \;\&\; ysz.$$

(2) For $\mathbf{V} = \overline{\mathbb{R}}_+$, the morphisms $r : X \nrightarrow Y$ of $\mathrm{Mat}(\mathbf{V})$ are functions providing for $x \in X$ and $y \in Y$ a (generalized) "distance" $r(x, y) \in \overline{\mathbb{R}}_+$, with composite distances given by

$$(sr)(x, z) = \inf_{y \in Y} (r(x, y) + s(y, z));$$

1_X puts a discrete structure on X. Alternatively, one may think of r as of a *fuzzy relation* from X to Y.

1.5. CHANGING $\mathbf{V}$

Let $\mathbf{W}$ be, like $\mathbf{V}$, a symmetric monoidal-closed complete lattice, and let $F : \mathbf{V} \to \mathbf{W}$ be a lax morphism of monoidal categories, i.e. a monotone function satisfying

$$Fx \otimes Fy \leq F(x \otimes y) \quad \text{and} \quad l \leq Fk$$

for all $x, y \in \mathbf{V}$, with l denoting the $\otimes$-neutral element in $\mathbf{W}$. Then F induces a lax functor

$$\mathrm{Mat}(\mathbf{V}) \to \mathrm{Mat}(\mathbf{W})$$

which maps objects identically and sends $r : X \times Y \to \mathbf{V}$ to Fr. Hence, for all $r : X \nrightarrow Y, s : Y \nrightarrow Z$,

$$(Fs)(Fr) \leq F(sr) \quad \text{and} \quad 1_X \leq F1_X$$

in $\mathrm{Mat}(\mathbf{W})$. More generally, $f \leq Ff$ for every **Set**-map $f : X \to Y$, and the triangle

$$\begin{tikzcd} & \mathbf{Set} \arrow[dl] \arrow[dr] & \\ \mathrm{Mat}(\mathbf{V}) \arrow[rr] & & \mathrm{Mat}(\mathbf{W}) \end{tikzcd}$$

commutes if $Fk = l$. But even without the latter condition, one always has $(Fs)(Ff) = F(sf)$, whereas the more general equality $(Fs)(Fr) = F(sr)$ would require $F : \mathbf{V} \to \mathbf{W}$ to preserve the tensor product strictly as well as suprema. Of course, the lax extension of F commutes with the involution:

$$F(r^\circ) = (Fr)^\circ.$$

The functors E, L, R of 1.2(2) preserve the tensor product, and E, L preserve joins, but R not so.

2. Categories of V-promatrices

2.1. PREAMBLE

The completion $\mathrm{Pro}\, A$ of a partially ordered set A under down-directed infima is given by its down-directed subsets $D \subseteq A$ (so that every finite subset of D has a lower bound in D, in particular $D \neq \emptyset$), preordered by

$$D \leq E :\Leftrightarrow \forall e \in E\ \exists d \in D : d \leq e.$$

(This is a special case of the well-known construction of the *procategory* $\mathrm{Pro}\,\mathbf{A}$ for a category $\mathbf{A}$; see, for example, [17].) There is a natural embedding

$$A \to \mathrm{Pro}\, A, \qquad x \mapsto \{x\},$$

which has a right adjoint if and only if A has all down-directed infima.

2.2. **V**-PROMATRICES

For **V** as in 1.1 one constructs the category

$$\mathrm{ProMat}(\mathbf{V})$$

having objects sets, with hom-sets given by the formula

$$\mathrm{ProMat}(\mathbf{V})(X, Y) = \mathrm{Pro}(\mathrm{Mat}(\mathbf{V})(X, Y)).$$

Hence, a morphism $R : X \nrightarrow Y$ in ProMat(**V**) is a down-directed set of morphisms $r : X \nrightarrow Y$ in Mat(**V**); its composite with $S : Y \nrightarrow Z$ is the set

$$SR = \{sr \mid r \in R, s \in S\}$$

of composites sr taken in Mat(**V**). The composition is compatible with the preorder of the hom-sets, whence ProMat(**V**) *is a* 2-*category*. There is a natural 2-functor

$$\mathrm{Mat}(\mathbf{V}) \to \mathrm{ProMat}(\mathbf{V})$$

which maps objects identically and interprets $r : X \nrightarrow Y$ as $\{r\} : X \nrightarrow Y$. Its right adjoints at the hom-level (see 2.1) define a lax functor

$$\Lambda : \mathrm{ProMat}(\mathbf{V}) \to \mathrm{Mat}(\mathbf{V})$$

which sends $R : X \nrightarrow Y$ to its meet $\bigwedge R$, taken pointwise.

Trivially, the involution of Mat(**V**) extends to ProMat(**V**) via

$$R^\circ = \{r^\circ \mid r \in R\}.$$

2.3. CHANGING **V**

For a morphism $F : \mathbf{V} \to \mathbf{W}$ as in 1.5, its lax extension $\mathrm{Mat}(\mathbf{V}) \to \mathrm{Mat}(\mathbf{W})$ extends further to a lax functor

$$\mathrm{ProMat}(\mathbf{V}) \to \mathrm{ProMat}(\mathbf{W}),$$

sending $R : X \nrightarrow Y$ to $FR := \{Fr \mid r \in R\} : X \nrightarrow Y$ and commuting with the involution.

3. $(\mathsf{T}, \mathbf{V})$-algebras

3.1. **V**-ADMISSIBLE MONADS

Recall that a monad $\mathsf{T} = (T, e, m)$ of **Set** (or any other category) is given by an endofunctor T together with natural transformations $e : \mathrm{Id} \to T$, $m : TT \to T$ satisfying the unit and associativity laws

$$m(Te) = 1 = m(eT), \qquad m(Tm) = m(mT).$$

For **V** as in 1.1 we call the monad **V**-*admissible* if T : **Set** $\to$ **Set** admits a lax extension

$$T : \mathrm{Mat}(\mathbf{V}) \to \mathrm{Mat}(\mathbf{V})$$

along **Set** $\to$ Mat(**V**) which makes the transformations e and m op-lax in Mat(**V**) and commutes with the involution. Explicitly, the **Set**-monad T allows for an extension

$$(r : X \nrightarrow Y) \mapsto (Tr : TX \nrightarrow TY)$$

which preserves the partial order described in 1.3 and must satisfy

(0) $(Tr)^\circ \leq T(r^\circ)$,
(1) $e_Y r \leq (Tr)e_X$,
(2) $m_Y(T^2 r) \leq (Tr)m_X$,
(3) $(Ts)(Tr) \leq T(sr)$

for all $r : X \nrightarrow Y$, $s : Y \nrightarrow Z$. We hasten to remark that in (0) we have in fact an equality (as one easily sees applying the inequality (0) to r° in lieu of r). In pointwise notation, (1)–(3) mean

(1′) $r(x, y) \leq Tr(e_X(x), e_Y(y))$,
(2′) $T^2 r(\mathfrak{X}, \mathfrak{Y}) \leq Tr(m_X(\mathfrak{X}), m_Y(\mathfrak{Y}))$,
(3′) $Tr(\mathfrak{x}, \mathfrak{y}) \otimes Ts(\mathfrak{y}, \mathfrak{z}) \leq T(sr)(\mathfrak{x}, \mathfrak{z})$,

for all $x \in X$, $y \in Y$, $\mathfrak{X} \in T^2X$, $\mathfrak{Y} \in T^2Y$, $\mathfrak{x} \in TX$, $\mathfrak{y} \in TY$, $\mathfrak{z} \in TZ$. *When* $r = f$ *is a* **Set**-*map one has*

$$T(sf) \leq T(sf)(Tf^\circ)(Tf) \leq T(sff^\circ)(Tf) \leq (Ts)(Tf),$$

hence (3) *becomes an equality in this case*, reading as $Ts(Tf(\mathfrak{x}), \mathfrak{z}) = T(sf)(\mathfrak{x}, \mathfrak{z})$ in pointwise notation. Likewise, when $s = g$ is a **Set**-map one has the equality

$$(Tg^\circ)(Tr) = T(g^\circ r).$$

3.2. REMARK

A lax extension to Mat(**V**) of a **V**-admissible **Set**-monad T need not be unique. For example, the identity monad admits a non-identical lax extension I to Mat(**3**), where **3** is the 3-element chain, as follows:

$$(Ir)(x, y) = \begin{cases} \bot & \text{if } r(x, y) = \bot, \\ \top & \text{else,} \end{cases}$$

for all $r : X \nrightarrow Y$, $x \in X$, $y \in Y$.

Hence, when talking about a **V**-admissible **Set**-monad T, we always have a fixed lax extension of T to Mat(**V**) in mind.

3.3. $(\mathsf{T}, \mathbf{V})$-ALGEBRAS

For a $\mathbf{V}$-admissible monad $\mathsf{T} = (T, e, m)$ one forms the category

$$\mathrm{Alg}(\mathsf{T}, \mathbf{V})$$

of (*reflexive and transitive*) $(\mathsf{T}, \mathbf{V})$-*algebras*, as follows: its objects are pairs (X, a) with a set X and a structure $a : TX \nrightarrow X$ in $\mathrm{Mat}(\mathbf{V})$ satisfying the reflexivity and transitivity laws

(4) $1_X \leq ae_X$,
(5) $a(Ta) \leq am_X$,

which, when expressed pointwise, read as

(4′) $k \leq a(e_X(x), x)$,
(5′) $Ta(\mathfrak{X}, \mathfrak{y}) \otimes a(\mathfrak{y}, z) \leq a(m_X(\mathfrak{X}), z)$,

for all $x, z \in X$, $\mathfrak{y} \in TX$ and $\mathfrak{X} \in T^2X$.

A morphism $f : (X, a) \to (Y, b)$ in $\mathrm{Alg}(\mathsf{T}, \mathbf{V})$ is a *lax homomorphism*, i.e. a **Set**-map $f : X \to Y$ satisfying

(6) $fa \leq b(Tf)$,

the pointwise version of which reads as

(6′) $a(\mathfrak{x}, y) \leq b(Tf(\mathfrak{x}), f(y))$

for all $\mathfrak{x} \in TX$, $y \in X$. Composition is as in **Set**.

3.4. CO-KLEISLI COMPOSITION

There is another way of thinking of the two fundamental conditions (4), (5). First of all, there is a least $(\mathsf{T}, \mathbf{V})$-algebra structure on each set X, namely e°_X, which in fact defines the left adjoint to the forgetful functor

$$\mathrm{Alg}(\mathsf{T}, \mathbf{V}) \to \mathbf{Set}.$$

Now, (4) of 3.3 reads as the *extensivity law*

(4″) $e^\circ_X \leq a$.

With the *co-Kleisli composition*

$$a * b := a(Tb)m^\circ_X$$

for all $a, b : TX \nrightarrow X$, condition (5) presents itself as

(5″) $a * a \leq a$.

Since the co-Kleisli composition is monotone in each variable, so that (4″) implies $a = a * e_X^\circ \leq a * a$, (5) in the presence of (4) has become equivalent to the *idempotency* condition $a * a = a$.

Of course, the co-Kleisli composition is in fact the Kleisli composition for the lax comonad (T, e°, m°) of the selfdual 2-category Mat($\mathbf{V}$).

3.5. ORDERING HOMOMORPHISMS

We recall from [7] that Alg(T, $\mathbf{V}$) *actually carries the structure of a* 2*-category* since its ordinary hom-sets Alg(T, $\mathbf{V}$)$((X, a), (Y, b))$ may be compatibly preordered by

$$f \leq g :\Leftrightarrow gf^\circ \leq be_Y \Leftrightarrow 1_X \leq g^\circ be_Y f,$$

which, in pointwise notation, read as

$$f \leq g \Leftrightarrow \forall x \in X : k \leq b(e_Y f(x), g(x)).$$

Reflexivity follows immediately from (4). For transitivity we observe that when $f \leq g$ and $g \leq h$, with (1) and (5) one obtains $f \leq h$:

$$\begin{aligned} hf^\circ &\leq (hg^\circ)(gf^\circ) \\ &\leq (be_Y)(be_Y) \\ &\leq b(Tb)e_{TY}e_Y \\ &\leq b(m_Y e_{TY})e_Y \\ &= be_Y. \end{aligned}$$

3.6. CHANGE-OF-BASE FUNCTORS

If the monad T is both $\mathbf{V}$- and $\mathbf{W}$-admissible, so that T extends to an endofunctor of both Mat($\mathbf{V}$) and Mat($\mathbf{W}$), for a morphism $F : \mathbf{V} \to \mathbf{W}$ of monoidal categories as in 1.5 we call T F*-admissible* if the extension F : Mat($\mathbf{V}$) $\to$ Mat($\mathbf{W}$) satisfies

(7) $TFr \leq FTr$

for all $r : X \nrightarrow Y$. In this case F *induces a* 2*-functor*

$$\overline{F} : \mathrm{Alg}(\mathsf{T}, \mathbf{V}) \to \mathrm{Alg}(\mathsf{T}, \mathbf{W})$$

which maps an object (X, a) to (X, Fa) and leaves morphisms unchanged. This is due to the fact that F preserves the co-Kleisli composition laxly:

(8) $(Fa) * (Fb) \leq F(a * b)$;

indeed,

$$(Fa)(TFb)m_X^\circ \leq (Fa)(FTb)(Fm_X^\circ) \leq F(a(Tb)m_X^\circ).$$

Consequently, $(Fa) * (Fa) \leq F(a * a) \leq Fa$, which shows preservation of (5″) by $\overline{F}$. Also, $e_X^\circ \leq a$ gives immediately $e_X^\circ \leq Fe_X^\circ \leq Fa$, hence preservation of (4″) follows. Similarly one deals with the homomorphism condition (6) and preservation of the preorder 3.5.

3.7. ALGEBRAIC FUNCTORS

Let us now consider **V**-admissible monads $\mathsf{T} = (T, e, m)$ and $\mathsf{S} = (S, d, n)$ of **Set** with a morphism $j : \mathsf{S} \to \mathsf{T}$ of monads, i.e. a natural transformation $j : S \to T$ satisfying

$$jd = e \quad \text{and} \quad jn = mj^2 \quad (\text{with } j^2 = Tj \cdot jS = jT \cdot Sj).$$

If the extensions of T and S to Mat(**V**) make j op-lax, so that

(9) $j_Y(Sr) \leq (Tr)j_X$

for all $r : X \nrightarrow Y$ in Mat(**V**), which in pointwise notation reads as

(9′) $Sr(\mathfrak{x}, \mathfrak{y}) \leq Tr(j_X(\mathfrak{x}), j_Y(\mathfrak{y}))$

for all $\mathfrak{x} \in SX$, $\mathfrak{y} \in SY$, then j *induces a* 2*-functor*

$$J : \mathrm{Alg}(\mathsf{T}, \mathbf{V}) \to \mathrm{Alg}(\mathsf{S}, \mathbf{V}),$$

sending (X, a) to (X, aj_X) and mapping morphisms identically. Since $1_X \leq ae_X = (aj_X)d_X$, (X, aj_X) remains reflexive, while its transitivity follows from (3), (9), (5); indeed,

$$\begin{aligned}(aj_X)S(aj_X) &= aj_X(Sa)(Sj_X)\\ &\leq a(Ta)j_{TX}(Sj_X)\\ &\leq am_X j_{TX}(Sj_X)\\ &= (aj_X)n_X.\end{aligned}$$

A morphism $f : (X, a) \to (Y, b)$ in Alg(T, **V**) becomes a morphism $f : (X, aj_X) \to (Y, bj_Y)$ since

$$f(aj_X) \leq b(Tf)j_X = (bj_Y)(Sf).$$

One easily sees that the preorder on the hom-sets is preserved as well.

Often we consider the case $\mathsf{S} = 1 = (\mathrm{Id}, 1, 1)$; then necessarily $j = e$, and we obtain a 2-functor

$$J : \mathrm{Alg}(\mathsf{T}, \mathbf{V}) \to \mathrm{Alg}(1, \mathbf{V}),\ (X, a) \mapsto (X, ae_X).$$

3.8. PROPOSITION

For every morphism $F : \mathbf{V} \to \mathbf{W}$ of monoidal lattices as in 1.5 *and every F-admissible* **Set**-*monad* T *there is a commutative diagram of* 2-*functors*

$$\begin{tikzcd}
(X,a) \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & (X,Fa) \arrow[ddd, mapsto] \\
& \mathrm{Alg}(\mathsf{T},\mathbf{V}) \arrow[r, "\overline{F}"] \arrow[d, "J"'] \arrow[dr] & \mathrm{Alg}(\mathsf{T},\mathbf{W}) \arrow[dl] \arrow[d, "J"] & \\
& \mathrm{Alg}(1,\mathbf{V}) \arrow[r, "\overline{F}"'] \arrow[ur] & \mathrm{Alg}(1,\mathbf{W}) \arrow[ul] & \\
(X,ae_X) \arrow[rrr, mapsto] & & & (X,F(ae_X) = (Fa)e_X).
\end{tikzcd}$$

An example of this situation is considered in Section 5.

4. $(\mathsf{T}, \mathbf{V})$-proalgebras

4.1. MONAD EXTENSION

For $\mathbf{V}$ as in 1.1 and a $\mathbf{V}$-admissible **Set**-monad $\mathsf{T} = (T, e, m)$, the lax extension $T : \mathrm{Mat}(\mathbf{V}) \to \mathrm{Mat}(\mathbf{V})$ admits a further extension

$$\begin{aligned} T : \mathrm{ProMat}(\mathbf{V}) &\to \mathrm{ProMat}(\mathbf{V}) \\ (R : X \nrightarrow Y) &\mapsto (TR := \{Tr \mid r \in R\} : X \nrightarrow Y), \end{aligned}$$

which automatically satisfies the conditions

(0) $(TR)^\circ \leq T(R^\circ)$,
(1) $e_Y R \leq (TR)e_X$,
(2) $m_Y(T^2R) \leq (TR)m_X$,
(3) $(TS)(TR) \leq T(SR)$

for all $R : X \nrightarrow Y$, $S : Y \nrightarrow Z$, *with equality holding when R is a map* (more precisely: a singleton set $\{f\}$ containing a map f; here notationally we don't distinguish between $\{f\}$ and f).

4.2. $(\mathsf{T}, \mathbf{V})$-PROALGEBRAS

For $\mathbf{V}$ as in 1.1 and a $\mathbf{V}$-admissible monad $\mathsf{T} = (T, e, m)$ of **Set**, a $(\mathsf{T}, \mathbf{V})$-*proalgebra* (X, A) is a set X with a morphism $A : TX \nrightarrow X$ in $\mathrm{ProMat}(\mathbf{V})$ satisfying the reflexivity and transitivity conditions

(4) $1_X \leq Ae_X$,
(5) $A(TA) \leq Am_X$.

This means that A is a down-directed set of morphisms $TX \nrightarrow X$ in Mat($\mathbf{V}$) satisfying the conditions

(4′) $\forall a \in A : 1_X \leq ae_X$,
(5′) $\forall a \in A\ \exists b \in A : b(Tb) \leq am_X$,

which are expressed pointwise as in 3.3, and in terms of the co-Kleisli composition as

(4″) $\forall a \in A : e_X^\circ \leq a$,
(5″) $\forall a \in A\ \exists b \in A : b * b \leq a$.

A *lax homomorphism* $f : (X, A) \to (Y, B)$ of $(\mathsf{T}, \mathbf{V})$-proalgebras is a **Set**-map $f : X \to Y$ satisfying

(6) $fA \leq B(Tf)$,

meaning that

(6′) $\forall b \in B\ \exists a \in A : fa \leq b(Tf)$,

to be expressed pointwise as in 3.3. With composition of **Set**-maps, this defines the ordinary category

$$\mathrm{ProAlg}(\mathsf{T}, \mathbf{V}).$$

4.3. ORDERING AND COREFLECTION

When we preorder the hom-sets $\mathrm{ProAlg}(\mathsf{T}, \mathbf{V})((X, A), (Y, B))$ by

$$f \leq g :\Leftrightarrow \forall b \in B\ gf^\circ \leq be_Y$$

it is easy to see that $\mathrm{ProAlg}(\mathsf{T}, \mathbf{V})$ *becomes a* 2*-category*: the only slightly critical part is to check that $f \leq g$ implies $hf \leq hg$ for every morphism $h : (Y, B) \to (Z, C)$; but for all $c \in C$ there is $b \in B$ such that

$$1_X \leq g^\circ be_Y f \leq g^\circ h^\circ hbe_Y f \leq (hg)^\circ c(Th)e_Y f = (hg)^\circ ce_Z(hf),$$

as desired. Furthermore, the full embedding

$$\mathrm{Alg}(\mathsf{T}, \mathbf{V}) \hookrightarrow \mathrm{ProAlg}(\mathsf{T}, \mathbf{V})$$

is obviously a 2-functor. More importantly, *there is a 2-functor*

$$\Lambda : \mathrm{ProAlg}(\mathsf{T}, \mathbf{V}) \to \mathrm{Alg}(\mathsf{T}, \mathbf{V}),\ (X, A) \mapsto (X, \Lambda A),$$

which is right adjoint to the embedding, with $\Lambda A = \bigwedge A$ (see 2.2). Indeed, $1_X \leq Ae_X$ implies $1_X \leq (\Lambda A)e_X$, and from $A(TA) \leq Am_X$ one obtains for all $a \in A$ some $b \in A$ with

$$(\Lambda A)(T\Lambda A) \leq (\Lambda A)(\Lambda TA) \leq b(Tb) \leq am_X,$$

hence $(\Lambda A)(T\Lambda A) \leq \Lambda(Am_X) = (\Lambda A)m_X$. Similarly one shows that a morphism $f : (X, A) \to (Y, B)$ becomes a morphism $f : (X, \Lambda A) \to (Y, \Lambda B)$, with the preorder being preserved, as well as right adjointness.

4.4. CHANGE-OF-BASE FUNCTORS

Let now $F : \mathbf{V} \to \mathbf{W}$ be as in 1.5, and consider an F-admissible monad T as in 3.6. Then the extension 2.3 $F : \mathrm{ProMat}(\mathbf{V}) \to \mathrm{ProMat}(\mathbf{V})$ satisfies

(7) $TFR \leq FTR$

for all $R : X \nrightarrow Y$. This condition enables us to extend $\overline{F}$ of 3.6 along the embedding $\mathrm{Alg}(\mathsf{T}, \mathbf{V}) \hookrightarrow \mathrm{ProAlg}(\mathsf{T}, \mathbf{W})$ to *obtain a* 2*-functor*

$$\overline{F} : \mathrm{ProAlg}(\mathsf{T}, \mathbf{V}) \to \mathrm{ProAlg}(\mathsf{T}, \mathbf{W}),$$

which maps an object (X, A) to (X, FA) and leaves morphisms unchanged. The verifications are as in 3.4. We obtain a commutative diagram of 2-functors:

$$\begin{tikzcd}
\mathrm{ProAlg}(\mathsf{T}, \mathbf{V}) \arrow[rr, "\overline{F}"] \arrow[dr] & & \mathrm{ProAlg}(\mathsf{T}, \mathbf{W}) \arrow[dl] \\
& \mathbf{Set} & \\
\mathrm{Alg}(\mathsf{T}, \mathbf{V}) \arrow[uu, hook] \arrow[ur] \arrow[rr, "\overline{F}"'] & & \mathrm{Alg}(\mathsf{T}, \mathbf{W}) \arrow[uu, hook] \arrow[ul]
\end{tikzcd}$$

The diagram remains commutative if the vertical embeddings are replaced by their right adjoints Λ, provided that $F : \mathbf{V} \to \mathbf{W}$ preserves infima.

4.5. ALGEBRAIC FUNCTORS

For a morphism $j : \mathsf{S} \to \mathsf{T}$ of $\mathbf{V}$-admissible monads satisfying 3.5(8) we automatically have

(8) $j_Y(SR) \leq (TR)j_X$

for all $R : X \nrightarrow Y$ in $\mathrm{ProMat}(\mathbf{V})$. The 2-functor $J : \mathrm{Alg}(\mathsf{T}, \mathbf{V}) \to \mathrm{Alg}(\mathsf{S}, \mathbf{V})$ may therefore be extended to a 2-functor

$$J : \mathrm{ProAlg}(\mathsf{T}, \mathbf{V}) \to \mathrm{ProAlg}(\mathsf{S}, \mathbf{V})$$

sending (X, A) to (X, Aj_X) and leaving morphisms unchanged. As in 3.8, the case $\mathsf{S} = 1 = (\mathrm{Id}, 1, 1)$ with $j = e$ is of particular importance, and with 4.3 and 4.4, we can extend the commutative diagram obtained in 3.8, as follows:

4.6. THEOREM

For every morphism $F : \mathbf{V} \to \mathbf{W}$ of monoidal lattices as in 1.5 *and every F-admissible* **Set**-*monad* T *there is a commutative diagram of* 2-*functors which also commutes with the underlying* **Set**-*functors:*

$$\begin{tikzcd}
 & \mathrm{ProAlg}(\mathsf{T},\mathbf{V}) \arrow[rr, "\overline{F}"] \arrow[dd, "J"] & & \mathrm{ProAlg}(\mathsf{T},\mathbf{W}) \arrow[dd, "J"] \\
\mathrm{Alg}(\mathsf{T},\mathbf{V}) \arrow[ur, hook] \arrow[rr, "\overline{F}"] \arrow[dd, "J"'] & & \mathrm{Alg}(\mathsf{T},\mathbf{W}) \arrow[ur, hook] \arrow[dd, "J"] & \\
 & \mathrm{ProAlg}(1,\mathbf{V}) \arrow[rr, "\overline{F}"] & & \mathrm{ProAlg}(1,\mathbf{W}) \\
\mathrm{Alg}(1,\mathbf{V}) \arrow[ur, hook] \arrow[rr, "\overline{F}"] & & \mathrm{Alg}(1,\mathbf{W}) \arrow[ur, hook] &
\end{tikzcd}$$

If the (*diagonal*) *embeddings are replaced by their right adjoints* Λ, *the vertical faces remain commutative while the top- and bottom faces commute if F preserves infima.*

An example of this situation is considered in the next section.

5. Examples

5.1. ORDERED SETS

For $\mathbf{V} = \mathbf{2}$ and $\mathsf{T} = 1$, conditions 3.2(4′), (5′) translate into the reflexivity and transitivity conditions for a relation a on X, and (6′) expresses preservation of the relation. Hence, $\mathrm{Alg}(1, \mathbf{2})$ is the category **Ord** of preordered sets. Denoting the preorders by $\leq$, we see that 3.5 puts the pointwise preorder on the hom-sets:

$$f \leq g \Leftrightarrow \forall x \in X : f(x) \leq g(x).$$

5.2. METRIC SPACES

For $\mathbf{V} = \overline{\mathbb{R}}_+$ and $\mathsf{T} = 1$, the $(\mathsf{T}, \mathbf{V})$-algebra structure $a : X \times X \to [0, \infty]$ must satisfy the conditions

$$a(x, x) = 0 \quad \text{and} \quad a(x, z) \leq a(x, y) + a(y, z)$$

for all $x, y, z \in X$. A lax homomorphism $f : (X, a) \to (Y, b)$ is a non-expansive map:

$$b(f(x), f(y)) \leq a(x, y)$$

for all $x, y \in X$. Hence, $\mathrm{Alg}(1, \overline{\mathbb{R}}_+)$ is the category **Met** of *premetric spaces* (called metric spaces in [12] and ∞pq-metric spaces in [14]). The hom-sets are preordered via 3.5 by

$$f \leq g \Leftrightarrow \forall x \in X : b(f(x), g(x)) = 0.$$

The embedding $E : \mathbf{2} \to \overline{\mathbb{R}}_+$ of 1.2(2) gives with 3.6 the 2-functor

$$\overline{E} : \mathbf{Ord} \to \mathbf{Met}$$

which maps $(X, \leq)$ to the premetric space (X, d) with $d(x, y) = 0$ if $x \leq y$ and $d(x, y) = \infty$ otherwise. The two adjoints $L \dashv E \dashv R$ give adjoints $\overline{L} \dashv \overline{E} \dashv \overline{R}$, providing a premetric space (X, d) with the preorders given by

$$\overline{L} : x \leq y \Leftrightarrow d(x, y) < \infty, \qquad \overline{R} : x \leq y \Leftrightarrow d(x, y) = 0.$$

5.3. UNIFORM SPACES

An object in $\mathrm{ProAlg}(1, \mathbf{2})$ is a set X which comes with a down-directed (w.r.t. $\subseteq$) set A of relations on X which are reflexive and satisfy the transitivity condition 4.2(5″)

$$\forall a \in A\ \exists b \in A : bb \subseteq a$$

(with the usual relational product, see 1.4(1)); a morphism $f : (X, A) \to (Y, B)$ satisfies the condition

$$\forall b \in B\ \exists a \in A : (f \times f)(a) \subseteq b,$$

as 4.2(6′) and 3.3(6′) show. Hence, $\mathrm{ProAlg}(1, \mathbf{2})$ is the category **ProOrd** of *pro-ordered sets* which is obviously equivalent to the category **QUnif** of *quasi-uniform spaces*. (A quasi-uniformity A on X is usually required to be not just a filter base but a filter on $X \times X$; in this paper we do not distinguish between **ProOrd** and **QUnif**.) The preorder on the hom-sets in **ProOrd** is given by

$$f \leq g \Leftrightarrow \forall b \in B : (f \times g)\Delta_X \subseteq b,$$

with Δ_X the diagonal in $X \times X$. According to 4.3, the embedding

$$\mathbf{Ord} \to \mathbf{ProOrd}, \quad (X, a) \mapsto (X, \{a\}),$$

has a right adjoint Λ which preorders a quasi-uniform space (X, A) by

$$x \leq y \Leftrightarrow \forall a \in A : (x, y) \in a.$$

5.4. PROMETRIC SPACES

An object in $\mathrm{ProAlg}(1, \overline{\mathbb{R}}_+)$ equips a set X with an up-directed (w.r.t. the pointwise natural order of $[0, \infty]$-valued functions) set A of ("distance") functions $a : X \times X \to [0, \infty]$ satisfying the conditions

$$\forall a \in A\ \forall x \in X : a(x, x) = 0,$$

$$\forall a \in A\ \exists b \in A\ \forall x, y, z \in X : a(x, z) \leq b(x, y) + b(y, z);$$

a morphism $f : (X, A) \to (Y, B)$ satisfies

$$\forall b \in B\ \exists a \in A\ \forall x, y \in X : b(f(x), f(y)) \leq a(x, y).$$

The resulting category **ProMet** $= \mathrm{ProAlg}(1, \overline{\mathbb{R}}_+)$ of *prometric spaces* contains the category **AQUnif** of *approach-quasi-uniform spaces* as considered by Lowen and Windels [15] (which satisfy an additional saturation condition for the structure A) as a full subcategory. Its hom-sets are preordered by

$$f \leq g \Leftrightarrow \forall b \in B\ \forall x \in X : b(f(x), g(x)) = 0.$$

The right adjoint Λ to the embedding

$$\mathbf{Met} \to \mathbf{ProMet}, \quad (X, d) \mapsto (X, \{d\}),$$

provides a prometric space (X, A) with the premetric

$$d(x, y) := \sup\{a(x, y) \mid a \in A\}.$$

E of 1.2(2) induces the 2-functor

$$\overline{E} : \mathbf{ProOrd} \to \mathbf{ProMet}$$

which equips a quasi-uniform space (X, A) with the set $\overline{A} = \{\overline{a} \mid a \in A\}$ of distance functions $\overline{a}$ with $\overline{a}(x, y) = 0$ if $(x, y) \in a$ and $\overline{a}(x, y) = \infty$ otherwise. $\overline{E}$ is both a full reflective and coreflective embedding, with adjoints $\overline{L} \dashv \overline{E} \dashv \overline{R}$, induced by $L \dashv E \dashv R$, where $\overline{L}$ assigns to a prometric space (X, A) the quasi-uniformity $\{\{(x, y) \mid a(x, y) < \infty\} \mid a \in A\}$, and $\overline{R}$ the quasi-uniformity $\{\{(x, y) \mid a(x, y) = 0\} \mid a \in A\}$.

We have thus described the diagram

$$\begin{tikzcd}
\mathbf{ProOrd} \arrow[r, hook, "\overline{E}"] \arrow[d, hook, "\Lambda"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathbf{ProMet} \arrow[l, dotted, bend right, "\overline{L}"'] \arrow[l, dashed, bend left, "\overline{R}"] \arrow[d, hook, "\Lambda"] \\
\mathbf{Ord} \arrow[r, hook, "\overline{E}"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathbf{Met} \arrow[l, dotted, bend right, "\overline{L}"'] \arrow[l, dashed, bend left, "\overline{R}"]
\end{tikzcd}$$

which commutes with respect to both the solid and the dashed arrows.

5.5. TOPOLOGICAL SPACES

The ultrafilter functor $U : \mathbf{Set} \to \mathbf{Set}$ assigns to a set X the set UX of ultrafilters on X; for $f : X \to Y$, the map $Uf : UX \to UY$ takes an ultrafilter $\mathfrak{x}$ on X to its image $f(\mathfrak{x})$ defined by $(B \in f(\mathfrak{x}) \Leftrightarrow f^{-1}(B) \in \mathfrak{x})$. Since U preserves finite

coproducts, there is a uniquely determined monad structure e, m on U (see [3]), given by

$$(A \in e_X(x) \Leftrightarrow x \in A), \qquad (A \in m_X(\mathfrak{X}) \Leftrightarrow A^\sharp \in \mathfrak{X}),$$

with $A^\sharp := \{\mathfrak{x} \in UX \mid A \in \mathfrak{x}\}$, for all $x \in X$, $A \subseteq X$, $\mathfrak{X} \in UUX$. As G. Janelidze observed, the monad $\mathsf{U} = (U, e, m)$ is naturally induced by the adjunction

$$(\mathbf{BooA})^{\mathrm{op}} \underset{\longleftarrow}{\overset{\longrightarrow}{\top}} \mathbf{Set}$$

with both adjoints represented by 2, the two-element set or Boolean algebra. The lax extension of U to $U : \mathrm{Rel}(\mathbf{Set}) \to \mathrm{Rel}(\mathbf{Set})$ transforms $r : X \nrightarrow Y$ into $Ur : UX \nrightarrow UY$ defined by

$$\mathfrak{x}(Ur)\mathfrak{y} \Leftrightarrow \forall A \in \mathfrak{x}\, \forall B \in \mathfrak{y}\, \exists x \in A\, \exists y \in B : xry$$

for all $\mathfrak{x} \in UX$, $\mathfrak{y} \in UY$. Briefly, the **Set**-monad U is **2**-admissible.

A $(\mathsf{U}, \mathbf{2})$-algebra is a set X with a relation $a : UX \nrightarrow X$ which, when we write $(\mathfrak{x} \to x :\Leftrightarrow \mathfrak{x}ax)$ and $(\mathfrak{X} \to \mathfrak{x} \Leftrightarrow \mathfrak{X}(Ua)\mathfrak{x})$, must satisfy the reflexivity and transitivity conditions

$$\dot{x} := e_X(x) \to x, \quad (\mathfrak{X} \to \mathfrak{y} \;\&\; \mathfrak{y} \to z \Rightarrow m_X(\mathfrak{X}) \to z)$$

for all $x, z \in X$, $\mathfrak{y} \in UX$, $\mathfrak{X} \in UUX$. These are exactly the convergence structures defining a topology on X. Morphisms in $\mathrm{Alg}(\mathsf{U}, \mathbf{2})$ preserve the convergence structures, i.e. are continuous maps. Hence, $\mathrm{Alg}(\mathsf{U}, \mathbf{2})$ is (isomorphic to) the category **Top** of topological spaces (see [2]). It may be considered as a 2-category when we preorder its hom-sets by

$$\begin{aligned} f \leq g \;&\Leftrightarrow\; \forall x \in X : f\dot{(x)} \to g(x) \\ &\Leftrightarrow\; \forall x \in X : g(x) \in \overline{f(x)}. \end{aligned}$$

The unique monad morphism $j = e : 1 \to \mathsf{U}$ induces the 2-functor

$$J : \mathbf{Top} \to \mathbf{Ord}$$

which provides a topological space X with the "specialization order" given by $(x \leq y \Leftrightarrow \dot{x} \to y \Leftrightarrow y \in \overline{x})$ for all $x, y \in X$. J has a left adjoint which embeds **Ord** as a full coreflective subcategory into **Top**: it provides a preordered set $(X, \leq)$ with the topology whose open sets are generated by the down sets $\downarrow x = \{z \in X \mid z \leq x\}$, $x \in X$.

5.6. PROTOPOLOGICAL SPACES

We wish to give an easy description of the category $\mathrm{ProAlg}(\mathsf{U}, \mathbf{2})$. Recall that a *pretopology* (or Čech closure operation) on a set X is an extensive and finitely-additive function $c : PX \to PX$; hence, $M \subseteq c(M)$, $c(\emptyset) = \emptyset$, $c(M \cup N) =$

$c(M) \cup c(N)$ for all $M, N \subseteq X$. Pretopologies on X are ordered pointwise: $c \leq d$ if $c(M) \subseteq d(M)$ for all $M \subseteq X$. A *protopology* on X is a down-directed set C of pretopologies on X with the transitivity property

$$\forall c \in C\ \exists d \in C : dd \leq c.$$

Continuity of a map $f : (X, C) \to (Y, D)$ means

$$\forall d \in D\ \exists c \in C\ \forall M \subseteq X : f(c(M)) \subseteq d(f(M)).$$

This defines the category **ProTop** of protopological spaces which can be made into a 2-category by

$$f \leq g :\Leftrightarrow \forall d \in D\ \forall x \in X : g(x) \in d(\{f(x)\}).$$

In order to see that **ProTop** is equivalent to ProAlg(U, **2**) one follows the same procedure that shows **Top** $\cong$ Alg(U, **2**): every pretopology c on X defines a "convergence relation" $a : UX \nrightarrow X$ via

$$\mathfrak{x}ax :\Leftrightarrow \forall M \in \mathfrak{x} : x \in c(M),$$

which satisfies the reflexivity but not necessarily the transitivity condition; hence a is a *pseudotopology* on X. Conversely, every pseudotopology a defines a pretopology c via

$$x \in c(M) :\Leftrightarrow \exists \mathfrak{x} \in UX(M \in \mathfrak{x}\ \&\ \mathfrak{x}ax).$$

The resulting maps

$$\{\text{pretopologies on } X\} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \{\text{pseudotopologies on } X\}$$

which satisfy $\psi\varphi = \mathrm{id}$ and $\mathrm{id} \leq \varphi\psi$ (and describe the category of pretopological spaces as a full reflective subcategory of the category of pseudotopological spaces), have an important algebraic property: *they are homomorphisms with respect to the ordinary composition of pretopologies* (*as closure operations*) *and to the co-Kleisli composition of pseudotopologies as introduced in* 3.4:

$$\varphi(\mathrm{id}) = e_X^\circ, \qquad \varphi(cd) = \varphi(c) * \varphi(d),$$
$$\psi(e_X^\circ) = \mathrm{id}, \qquad \psi(a * b) = \psi(a)\psi(b)$$

(see [8] for details). This homomorphic behaviour helps to prove that ϕ and ψ induce a category equivalence

$$\textbf{ProTop} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathrm{ProAlg}(\mathsf{U}, \mathbf{2}).$$

The only non-trivial point is to see that, for $(X, A) \in \mathrm{ProAlg}(\mathsf{U}, \mathbf{2})$, the identity map is actually a morphism $\Phi\Psi(X, A) \to (X, A)$. Indeed, for every $a \in A$, we

have $b \in A$ with $b*b \leq a$; now it is not difficult to show that every pseudotopology b satisfies $\phi(\psi(b)) \leq b * b$.

The full embedding **Top** $\hookrightarrow$ **ProTop** has a left adjoint Λ which provides a protopological space (X, C) with a topology whose Kuratowski closure operation is given by

$$\overline{M} = \bigcap \{c(M) \mid c \in C\}$$

for all $M \subseteq X$. The 2-functor J : **Top** $\to$ **Ord** of 5.5 can be extended to a 2-functor

$$J : \mathbf{ProTop} \to \mathbf{ProOrd}$$

which provides a protopological space (X, C) with the quasi-uniformity given by the sets $\{(x, y) \mid y \in c(\{x\})\}$, $c \in C$. J has a left adjoint which embeds **ProOrd** as a full reflective subcategory into **ProTop**, as follows: for a quasi-uniformity A on X consider the protopology $\hat{A} = \{\hat{a} \mid a \in A\}$ on X with

$$\hat{a}(M) = \{y \in X \mid \exists x \in M : (x, y) \in a\}$$

for all $M \subseteq X$.

In summary, in 5.5 and 5.6 we have described the diagram

$$\begin{tikzcd}
\mathbf{ProTop} \arrow[r, shift right, "J", "\perp"'] \arrow[d, dashed, shift right, "\Lambda"] & \mathbf{ProOrd} \arrow[l, dotted, shift right] \arrow[d, dashed, shift right, "\Lambda"] \\
\mathbf{Top} \arrow[u, hook, shift right] \arrow[r, shift right, "J", "\perp"'] & \mathbf{Ord} \arrow[l, dotted, shift right] \arrow[u, hook, shift right]
\end{tikzcd}$$

which commutes with respect to the solid arrows; also, the two full embeddings **Ord** $\hookrightarrow$ **ProTop** described by the diagram coincide. Let us also remark that the composite

$$\mathbf{ProOrd} \hookrightarrow \mathbf{ProTop} \xrightarrow{\Lambda} \mathbf{Top}$$

is nothing but the induced-topology functor of quasi-uniform spaces which provides a quasi-uniform space (X, A) with the Kuratowski closure operation given by

$$y \in \overline{M} \Leftrightarrow \forall a \in A\ \exists x \in M : (x, y) \in a.$$

5.7. APPROACH SPACES

The objects of $\mathrm{Alg}(\mathsf{U}, \overline{\mathbb{R}}_+)$ are sets X which come with a function $a : UX \times X \to [0, \infty]$ satisfying the reflexivity and transitivity conditions

$$a(\dot{x}, x) = 0, \qquad a(m_X(\mathfrak{X}), z) \leq Ua(\mathfrak{X}, \mathfrak{y}) + a(\mathfrak{y}, z)$$

for all $x, z \in X$, $\mathfrak{y} \in UX$, $\mathfrak{X} \in UUX$, with

$$Ua(\mathfrak{X}, \mathfrak{y}) = \sup_{\substack{\mathcal{A}\in\mathfrak{X} \\ B\in\mathfrak{y}}} \inf_{\substack{\mathfrak{x}\in\mathcal{A} \\ y\in B}} a(\mathfrak{x}, y).$$

A morphism $f : (X, a) \to (Y, b)$ must satisfy

$$b(f(\mathfrak{x}), f(x)) \leq a(\mathfrak{x}, x)$$

for all $x \in X$, $\mathfrak{x} \in UX$. As observed in [4, 7], this is precisely Lowen's category of approach spaces which becomes a 2-category via

$$f \leq g \Leftrightarrow \forall x \in X : b(f(\dot{x}), g(x)) = 0.$$

The monad morphism $j = e : 1 \to \mathsf{U}$ induces the 2-functor

$$J : \mathbf{App} \to \mathbf{Met},$$

providing an approach space (X, a) with the premetric d given by $d(x, y) = a(\dot{x}, y)$. It is the right adjoint to the full embedding $\mathbf{Met} \hookrightarrow \mathbf{App}$ described by [14] which puts on a premetric space (X, d) the approach structure

$$a(\mathfrak{x}, y) = \inf_{F\in\mathfrak{x}} \sup_{x\in F} d(x, y).$$

The full reflective and coreflective embedding

$$\overline{E} : \mathbf{Top} \hookrightarrow \mathbf{App}$$

is also described by [14]; as observed in [7], it is induced by $L \dashv E \dashv R$ although the situation is more complicated than in 5.4. $\overline{E}$ provides a topological space X with the approach structure a defined by $a(\mathfrak{x}, x) = 0$ if $\mathfrak{x}$ converges to x, and $a(\mathfrak{x}, x) = \infty$ otherwise. Its right adjoint $\overline{R}$ puts on an approach space (X, a) the topology which lets $\mathfrak{x}$ converge to x precisely when $a(\mathfrak{x}, x) = 0$. But while U is R-admissible, it fails to be L-admissible. Nevertheless, it is useful to consider for an approach space (X, a) the adjoint (X, La) as in 3.4, which tries to let $\mathfrak{x}$ converge to x precisely when $a(\mathfrak{x}, x) < \infty$. This structure satisfies the reflexivity but not the transitivity condition for topologies defined via convergence. In other words, (X, La) is just a pseudotopological space to which, however, one may apply the reflector of $\mathbf{Top} \hookrightarrow \mathbf{PsTop}$ to obtain the topological space $\hat{L}(X, a)$. The resulting functor $\hat{L}$ is left adjoint to $\overline{E}$, as observed in [7].

Incidently, the reflector $\mathbf{PsTop} \to \mathbf{Top}$ is obtained by iterating the endofunctor $(X, b) \mapsto (X, b * b)$ transfinitely (see [4]), another useful application of the co-Kleisli composition 3.4.

We have thus described the diagram below which commutes with respect to both the solid and the dotted arrows, but not the dashed arrows; also, the two full embeddings **Ord** $\hookrightarrow$ **App** described by it coincide.

$$\begin{tikzcd}
\mathbf{Top} \arrow[r, hook, "\perp"] \arrow[d, "J", "\dashv"'] & \mathbf{App} \arrow[l, dotted, bend right, "\hat{L}"'] \arrow[l, dashed, bend left, "\overline{R}", "\perp"'] \arrow[d, "J", "\dashv"'] \\
\mathbf{Ord} \arrow[r, hook, "\perp"] \arrow[u, dotted, hook, "\dashv"] & \mathbf{Met} \arrow[l, dotted, bend right, "\overline{L}"'] \arrow[l, dashed, bend left, "\overline{R}", "\perp"'] \arrow[u, dotted, hook, "\dashv"]
\end{tikzcd}$$

5.8. PROAPPROACH SPACES

We call the objects (X, A) of $\mathrm{ProAlg}(\mathsf{U}, \overline{\mathbb{R}}_+) = \mathbf{ProApp}$ *proapproach spaces*. These are sets with an up-directed (w.r.t. the pointwise natural order of $[0, \infty]$-valued functions) set A of functions $a : UX \times X \to [0, \infty]$ satisfying the conditions

$$\begin{aligned}
&\forall a \in A\ \forall x \in X : a(\dot{x}, x) = 0,\\
&\forall a \in A\ \exists b \in B\ \forall \mathfrak{X} \in UUX,\ \mathfrak{y} \in UX,\ z \in X :\\
&\quad a(m_X(\mathfrak{X}), z) \leq Ub(\mathfrak{X}, \mathfrak{y}) + b(\mathfrak{y}, z),
\end{aligned}$$

with

$$Ub(\mathfrak{X}, \mathfrak{y}) = \sup_{\substack{\mathcal{A} \in \mathfrak{X} \\ B \in \mathfrak{y}}} \inf_{\substack{\mathfrak{x} \in \mathcal{A} \\ y \in B}} b(\mathfrak{x}, y);$$

a morphism $f : (X, A) \to (Y, B)$ satisfies

$$\forall b \in B\ \exists a \in A\ \forall \mathfrak{x} \in UX,\ y \in X : b(f(\mathfrak{x}), f(y)) \leq a(\mathfrak{x}, y).$$

Its hom-sets are preordered by

$$f \leq g \Leftrightarrow \forall b \in B\ \forall x \in X : b(f(\dot{x}), g(x)) = 0.$$

The right adjoint Λ to the full embedding **App** $\hookrightarrow$ **ProApp** provides a proapproach space (X, A) with the approach structure given by

$$d(\mathfrak{x}, y) = \sup\{a(\mathfrak{x}, y) \mid a \in A\}.$$

The 2-functor $J : (X, A) \mapsto (X, \{a(e_X \times 1_X) \mid a \in A\})$ is right adjoint to the full embedding

ProMet $\hookrightarrow$ **ProApp**

which extends the embedding **Met** $\hookrightarrow$ **App** "structure by structure". This describes the commutative diagram which "lifts" the diagram given in 5.6:

$$\begin{tikzcd}
\mathbf{ProApp} \arrow[r, shift right, "J", "\perp"'] \arrow[d, shift right, "\Lambda"] \arrow[d, phantom, "\dashv"] & \mathbf{ProMet} \arrow[l, dotted, shift right] \arrow[d, shift right, "\Lambda"] \arrow[d, phantom, "\dashv"] \\
\mathbf{App} \arrow[r, shift right, "J", "\perp"'] \arrow[u, hook, shift right] & \mathbf{Met} \arrow[l, dotted, shift right] \arrow[u, hook, shift right]
\end{tikzcd}$$

Finally we consider the full embedding

$$\overline{E} : \mathbf{ProTop} \hookrightarrow \mathbf{ProApp}$$

induced by $E : \mathbf{2} \to [0, \infty]$; it extends $\overline{E} : \mathbf{Top} \to \mathbf{App}$, again "structure by structure", providing a protopological space (X, C) with the proapproach structure $\tilde{C} = \{\tilde{c} \mid c \in C\}$, where $\tilde{c}(\mathfrak{x}, x) = 0$ if $\mathfrak{x}$ converges to x in the pretopology c, and $\tilde{c}(\mathfrak{x}, x) = \infty$ otherwise. Its right adjoint $\overline{R}$ defines for a proapproach space (X, A) a protopology $A^* = \{a^* \mid a \in A\}$ with

$$a^*(M) = \{x \in X \mid \exists \mathfrak{x} \in UX(M \in \mathfrak{x} \;\&\; a(\mathfrak{x}, x) = 0)\}$$

for all $M \subseteq X$.

$\overline{E}$ also has a left adjoint $\hat{L}$ whose construction we can only sketch, as follows: *pseudo-protopological spaces* are sets with a down-directed set of pseudotopologies; with morphisms as in ProAlg(U, **2**), they form the category **PsProTop** in which **ProTop** $\simeq$ ProAlg(U, **2**) is reflective. Using the definition of $\overline{L}$ as in 4.3 one obtains a functor $\overline{L}$: **ProApp** $\to$ **PsProTop** which, when composed with the reflector of **ProTop** $\hookrightarrow$ **PsProTop**, gives us $\hat{L}$.

In extension of the diagrams given in 5.4 and 5.7 we obtain the following diagrams which commute to the same extent as their predecessors:

$$\begin{tikzcd}
\mathbf{ProTop} \arrow[r, shift right, "\perp"'] \arrow[d, shift right, "\Lambda"] \arrow[d, phantom, "\dashv"] & \mathbf{ProApp} \arrow[l, dotted, shift right, "\hat{L}"'] \arrow[l, dashed, shift left, "\perp", "R"'] \arrow[d, shift right, "\Lambda"] \arrow[d, phantom, "\dashv"] \\
\mathbf{Top} \arrow[r, shift right, "\perp"'] \arrow[u, hook, shift right] & \mathbf{App} \arrow[l, dotted, shift right, "\hat{L}"'] \arrow[l, dashed, shift left, "\perp", "\overline{R}"'] \arrow[u, hook, shift right]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{ProTop} \arrow[r, shift right, "\perp"'] \arrow[d, shift left, "J"] \arrow[d, phantom, "\dashv"] & \mathbf{ProApp} \arrow[l, dotted, shift right, "\hat{L}"'] \arrow[l, dashed, shift left, "\perp", "\overline{R}"'] \arrow[d, shift left, "J"] \arrow[d, phantom, "\dashv"] \\
\mathbf{ProOrd} \arrow[r, shift right, "\perp"'] \arrow[u, dotted, shift left] & \mathbf{ProMet} \arrow[l, dotted, shift right, "\overline{L}"'] \arrow[l, dashed, shift left, "\perp", "\overline{R}"'] \arrow[u, dotted, shift left]
\end{tikzcd}$$

We also note that, in analogy to the induced-topology functor of quasi-uniform spaces, one has the induced-approach functor of approach-uniform spaces and, more generally, of prometric spaces, given by

$$\mathbf{ProMet} \hookrightarrow \mathbf{ProApp} \xrightarrow{\Lambda} \mathbf{App}.$$

It assigns to a prometric space (X, A) the approach structure d given by

$$d(\mathfrak{x}, y) = \sup_{a \in A} \inf_{F \in \mathfrak{x}} \sup_{x \in F} a(x, y).$$

6. Alg(T, **V**) as a Topological Category

6.1. INITIAL STRUCTURES

Recall that, in order to show topologicity of the underlying **Set**-functor of Alg(T, **V**) (with T and **V** as in 3.1), by definition we must, for every family $(Y_i, b_i)_{i \in I}$ of (T, **V**)-algebras (*with no size restriction on* I) and every family $(f_i : X \to Y_i)_{i \in I}$ of **Set**-maps, provide a (T, **V**)-algebra structure a on the fixed set X (the so-called *initial* structure) such that, for any (T, **V**)-algebra (Z, c), a **Set**-map $h : Z \to X$ is actually a morphism in Alg(T, **V**) when all composites $f_i h$ are. First we note the following simple but useful lemma.

6.2. LEMMA

(1) *For any morphisms* $r_i : X \nrightarrow Y$ *in* Mat(**V**),

$$T\left(\bigwedge_i r_i\right) \leq \bigwedge_i T r_i.$$

(2) *For any morphisms* $f : X \to Y$, $a : TX \nrightarrow X$, $b : TY \nrightarrow Y$ *in* Mat(**V**), *if* $fa \leq b(Tf)$ *one has*

$$(Tf)(Ta) \leq (Tb)(T^2 f).$$

Proof. (1) This inequality follows trivially from the fact that T preserves the preorder on the hom-sets of Mat(**V**).

(2) Monotonicity of T leads from $fa \leq b(Tf)$ to

$$(Tf)(Ta) \leq T(fa) \leq T(b(Tf)) = (Tb)(T^2 f)$$

when we use the fact that T preserves composition in Mat(**V**) strictly whenever the first factor is a map. □

6.3. PROPOSITION

In the notation of 6.1, *the initial structure* a *on* X *can be constructed as* $a = \bigwedge_i b_i^*$, *with* $b_i^* := f_i^\circ b_i(Tf_i)$, *which, in pointwise notation, reads as*

$$b_i^*(\mathfrak{x}, y) = b_i(Tf_i(\mathfrak{x}), f_i(y))$$

for all $\mathfrak{x} \in TX$, $y \in X$ *and* $i \in I$.

Proof. 3.3(4) follows from

$$1_X \leq \bigwedge_i 1_X \leq \bigwedge_i f_i^\circ f_i \leq \bigwedge_i f_i^\circ b_i e_{Y_i} f_i = \bigwedge_i f_i^\circ b_i (Tf_i) e_X = \left(\bigwedge_i b_i^*\right) e_X.$$

In order to show (5) of 3.3, we apply Lemma 6.2 and obtain:

$$\begin{aligned} a(Ta) &\leq \left(\bigwedge_i b_i^*\right)\left(\bigwedge_j Tb_j^*\right) \\ &\leq \bigwedge_i b_i^*(Tb_i^*) \\ &\leq \bigwedge_i f_i^\circ b_i (Tf_i)(Tf_i^\circ)(Tb_i)(T^2 f_i) \\ &\leq \bigwedge_i f_i^\circ b_i (Tb_i)(T^2 f_i) \\ &\leq \bigwedge_i f_i^\circ b_i m_{Y_i} (T^2 f_i) \\ &\leq \left(\bigwedge_i f_i^\circ b_i (Tf_i)\right) m_X. \end{aligned}$$

Trivially, for every $i \in I$, $f_i : (X, a) \to (Y_i, b_i)$ is a morphism in $\mathrm{Alg}(\mathsf{T}, \mathbf{V})$. Given (Z, c) and $h : Z \to X$ as in 6.1, such that $f_i h c \leq b_i (Tf_i)(Th)$ for all $i \in I$, we obtain

$$hc \leq \bigwedge_i f_i^\circ f_i h c \leq \bigwedge_i f_i^\circ b_i (Tf_i)(Th) = a(Th),$$

as desired. □

6.4. THEOREM

For T *and* $\mathbf{V}$ *as in* 3.1, *the forgetful functor* $\mathrm{Alg}(\mathsf{T}, \mathbf{V}) \to \mathbf{Set}$ *is topological, in the sense that it admits all initial structures. It therefore admits also all final structures, has both a left and a right adjoint, and makes* $\mathrm{Alg}(\mathsf{T}, \mathbf{V})$ *a complete, cocomplete, wellpowered and cowellpowered category with a generator and a cogenerator. Furthermore, for every monad morphism* $j : \mathsf{S} \to \mathsf{T}$ *as in* 3.5, *the functor* $J : \mathrm{Alg}(\mathsf{T}, \mathbf{V}) \to \mathrm{Alg}(\mathsf{S}, \mathbf{V})$ *has a left adjoint, and for every lax monoidal morphism* $F : \mathbf{V} \to \mathbf{W}$ *as in* 3.4, *the functor* $\overline{F} : \mathrm{Alg}(\mathsf{T}, \mathbf{V}) \to \mathrm{Alg}(\mathsf{T}, \mathbf{W})$ *has a left adjoint if* $F : \mathbf{V} \to \mathbf{W}$ *preserves infima.*

Proof. Existence of initial structures was shown in 6.3 and implies existence of final structures (defined dually, see [1]). The initial and final structures for empty families define the left- and right adjoint of $\mathrm{Alg}(\mathsf{T}, \mathbf{V}) \to \mathbf{Set}$, respectively (see also 3.4). Limits and colimits in $\mathrm{Alg}(\mathsf{T}, \mathbf{V})$ are therefore constructed by putting the

initial and final structures on the limits and colimits formed in **Set**, respectively. Wellpoweredness and cowellpoweredness get lifted from **Set** to $\mathrm{Alg}(\mathsf{T}, \mathbf{V})$ since every set X admits only a (small) set of structures $a : TX \nrightarrow X$ (smallness of $\mathbf{V}$ is crucial here). To obtain a generator in $\mathrm{Alg}(\mathsf{T}, \mathbf{V})$, one applies the left adjoint of the underlying **Set**-functor to a generator of **Set**, and proceeds dually to have a cogenerator. With the explicit formula for the initial structures given in 6.3, it is easy to see that these are always preserved by J, while $\overline{F}$ preserves them when F preserves infima. Consequently, in this case J and $\overline{F}$ preserve all limits, hence have left adjoints by Freyd's Special Adjoint Functor Theorem. □

7. $\mathrm{ProAlg}(\mathsf{T}, \mathbf{V})$ as a Topological Category

7.1. PREAMBLE

In this section we show topologicity of the underlying **Set**-functor $P : \mathrm{ProAlg}(\mathsf{T}, \mathbf{V}) \to \mathbf{Set}$ and first confirm the existence of initial structures for *down-directed* structured cones. Hence, we are given a down-directed class I and a functor $D : I \to \mathrm{ProAlg}(\mathsf{T}, \mathbf{V})$ which provides us with objects $D_i = (Y_i, B_i)$ and morphisms $g_{i,j} : (Y_i, B_i) \to (Y_j, B_j)$ whenever $i \leq j$ in I; furthermore, we have a cone $f : \Delta X \to PD$, i.e. a family of compatible maps $f_i : X \to Y_i$. Putting

$$A := \{b_i^* \mid b \in B_i,\ i \in I\},$$

with b_i^* defined as in 6.3 by $b_i^* = f_i^\circ b(Tf_i)$ we obtain:

7.2. PROPOSITION

A is the initial structure on X w.r.t. $(f_i)_{i\in I}$.

Proof. First of all, A is in fact a (small) set since when $\mathbf{V}$ is small also the hom-set $\mathrm{Mat}(\mathbf{V})(TX, X)$ is small. Next we show that A is down-directed. Since $I \neq \emptyset$, also $A \neq \emptyset$. Having $b \in B_i$, $c \in B_j$ with $i, j \in I$, we can find $t \leq i, j$ in I and then $d \in B_t$ such that

$$g_{t,i}d \leq b(Tg_{t,i}), \qquad g_{t,j}d \leq c(Tg_{t,j}).$$

Hence,

$$d_t^* = f_t^\circ d(Tf_t) \leq f_t^\circ g_{t,i}^\circ b(Tg_{t,i})(Tf_t) = f_i^\circ b(Tf_i) = b_i^*,$$

and likewise $d_t^* \leq c_j^*$.

The proof of conditions (4), (5) of 4.2 now proceeds with the same arguments as in the proof of 6.3 and is therefore omitted; likewise the trivial verification that A does indeed have the initiality property. □

We arrive at the following theorem whose additional assertions are obvious.

7.3. THEOREM

For T *and* $\mathbf{V}$ *as in* 3.1*, like* $\mathrm{Alg}(\mathsf{T}, \mathbf{V}) \to \mathbf{Set}$ *also the forgetful functor* $\mathrm{ProAlg}(\mathsf{T}, \mathbf{V}) \to \mathbf{Set}$ *is topological, and all assertions of Theorem* 6.4 *remain true mutatis mutandis.*

Proof. After Proposition 7.2 it suffices to show the existence of initial structures for *finite* families $(f_i : X \to Y_i)_{i \in I}$ of **Set**-maps with $(\mathsf{T}, \mathbf{V})$-proalgebras (Y_i, B_i). Indeed, given any such family of arbitrary size, we obtain the initial structure on X by applying 7.2 to the cone

$$(f_F : X \to Y_F)_{F \subseteq I \text{finite}},$$

with $f_F = \langle f_i \rangle_{i \in F} : X \to Y_F := \prod_{i \in F} Y_i$.

The empty case is taken care of by the left adjoint to P which is constructed just like the left adjoint to $\mathrm{Alg}(\mathsf{T}, \mathbf{V}) \to \mathbf{Set}$ (see 3.4). In case $I = \{1, 2\}$ one lets

$$A := \{b_1^* \wedge b_2^* \mid b_1 \in B_1,\ b_2 \in B_2\},$$

with $b_i^* = f_i^\circ b_i(Tf_i)$. Since B_1 and B_2 are down-directed, also A is down-directed, and all other verifications proceed as in 6.3. □

7.4. COROLLARY

Each of the categories appearing in the cubic diagram of the Introduction is topological over **Set**.

Acknowledgements

The authors acknowledge partial financial assistance by Centro de Matemática da Universidade de Coimbra/FCT, Centro de Matemática da Universidade de Aveiro/FCT and NSERC.

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arXiv:0804.0326v1 [math.CT] 2 Apr 2008

INJECTIVE SPACES VIA ADJUNCTION

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Abstract. Our work over the past years shows that not only the collection of (for instance) all topological spaces gives rise to a category, but also each topological space can be seen individually as a category by interpreting the convergence relation $\mathfrak{x} \longrightarrow x$ between ultrafilters and points of a topological space X as arrows in X. Naturally, this point of view opens the door to the use of concepts and ideas from (enriched) Category Theory for the investigation of (for instance) topological spaces. In this paper we study cocompleteness, adjoint functors and Kan extensions in the context of topological theories. We show that the cocomplete spaces are precisely the injective spaces, and they are algebras for a suitable monad on Set. This way we obtain enriched versions of known results about injective topological spaces and continuous lattices.

Introduction

The title of the present article is clearly reminiscent of the chapter *Ordered sets via adjunctions* by R. Wood [Woo04], where the theory of ordered sets is developed elegantly employing consequently the concept of adjunction. One of the fundamental aspects of our recent research is described by the slogan *topological spaces are categories*, and therefore can be studied using notions and techniques from (enriched) Category Theory. We hope to be able to show in this paper that concepts like module, colimit and adjointness can be a very useful tool for the study of topological spaces too.

We should explain what is meant by "spaces are categories". In his famous 1973 paper [Law73] F.W. Lawvere considers the points of a (generalised) metric space X as the objects of a category X and lets the distance

$$d(x,y) \in [0,\infty]$$

play the role of the hom-set of x and y. In fact, the basic laws

$$0 \geq d(x,x) \qquad \text{and} \qquad d(x,y)+d(y,z) \geq d(x,z)$$

remind us immediately to the operations "choosing the identity" and "composition"

$$1 \longrightarrow \hom(x,x) \qquad \text{and} \qquad \hom(x,y)\times\hom(y,z) \longrightarrow \hom(x,z)$$

of a category. Motivated by Lawvere's approach, we consider the points of a topological space X as the objects of our category, and interprete the convergence $\mathfrak{x} \longrightarrow x$ of an ultrafilter $\mathfrak{x}$ on X to a point $x \in X$ as a morphism in X. With this interpretation, the convergence relation

$$(*) \qquad \longrightarrow : UX \times X \longrightarrow 2$$

becomes the "hom-functor" of X. Clearly, we have to make here the concession that a morphism in X does not have just an object but rather an ultrafilter (of objects) as domain. This intuition is supported by

2000 *Mathematics Subject Classification.* 18A05, 18D15, 18D20, 18B35, 18C15, 54B30, 54A20.

Key words and phrases. Quantale, V-category, monad, topological theory, module, Yoneda lemma, weighted colimit.

The author acknowledges partial financial assistance by Unidade de Investigação e Desenvolvimento Matemática e Aplicações da Universidade de Aveiro/FCT.

the observation (due to M. Barr [Bar70]) that a relation $\mathfrak{x} \longrightarrow x$ between ultrafilters and points of a set X is the convergence relation of a (unique) topology on X if and only if

$$(\dagger) \qquad e_X(x) \longrightarrow x \qquad \text{and} \qquad (\mathfrak{X} \longrightarrow \mathfrak{x} \;\&\; \mathfrak{x} \longrightarrow x) \models m_X(\mathfrak{X}) \longrightarrow x,$$

for all $x \in X$, $\mathfrak{x} \in UX$ and $\mathfrak{X} \in UUX$, where $m_X(\mathfrak{X})$ is the filtered sum of the filters in $\mathfrak{X}$ and $e_X(x) = \dot{x}$ the principal ultrafilter generated by $x \in X$. In the second axiom we use the natural extension of a relation between ultrafilters and points to a relation between ultrafilters of ultrafilters and ultrafilters, so that $\mathfrak{X} \longrightarrow \mathfrak{x}$ is a meaningful expression. In our interpretation, the first condition postulates the existence of an "identity arrow" on X, whereby the second one requires the existence of a "composite" of "composable pairs of arrows". Furthermore, a function $f : X \longrightarrow Y$ between topological spaces is continuous whenever $\mathfrak{x} \longrightarrow x$ in X implies $f(\mathfrak{x}) \longrightarrow f(x)$ in Y, that is, f associates to each object in X an object in Y and to each arrow in X an arrow in Y between the corresponding (ultrafilter of) objects in Y. It is now a little step to admit that the hom-functor $(*)$ of such a category X takes values in a quantale V other than the two-element Boolean algebra $\mathsf{2}$, and that the domain $\mathfrak{x}$ of an arrow $\mathfrak{x} \longrightarrow x$ in X is an element of a set TX other than the set UX of all ultrafilters of X. As one can see immediately, we need T to be a functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ in order to define the notion of functor between such categories, moreover, we need T to be part of a Set-monad $\mathbb{T} = (T, e, m)$ in order to formulate the axioms $(\dagger)$ of a category in this context. Eventually, we reach the notion of a $(\mathbb{T}, \mathsf{V})$-category (also called $(\mathbb{T}, \mathsf{V})$-algebra or lax algebras), for a Set-monad $\mathbb{T}$ and quantale V, as introduced in [CH03, CT03, CHT04]. A different but related approach to this kind of categories was presented by Burroni [Bur71].

Though the initial paper [CH03] focused on the topological features of this approach, already in [CT03] the emphasis was put on the categorical description of $(\mathbb{T}, \mathsf{V})$-algebras. The theory of categories enriched in a monoidal closed category V is by now classical [Ben63, Ben65, EK66, Kel82, Law73]. We have a wide range of concepts and theorems at our disposal, it includes such things as modules (also called distributors, profunctors), weighted (co)limits, the Yoneda Lemma, Kan extensions, adjoint functors, and many more. Naturally, we wish to lift these notions and results to the $(\mathbb{T}, \mathsf{V})$-setting. A first step in this direction was done in [CH07], where the notion of module is introduced into the realm of (now called) $(\mathbb{T}, \mathsf{V})$-categories. As in the case of V-categories, this concept is fundamental for the further development of the theory; for instance, completeness properties of $(\mathbb{T}, \mathsf{V})$-categories are formulated in terms of modules. In fact, in [CH07] the categorical notion of Cauchy-completeness (the name Lawvere-completeness respectively L-completeness is proposed in [CH07, HT08]) is introduced and studied. A further achievement of [CH07] is the formulation and proof of a $(\mathbb{T}, \mathsf{V})$-version of the famous Yoneda lemma, a result which turns out to be crucial for the study of $(\mathbb{T}, \mathsf{V})$-categories in the same way as the classical result is for the development of the theory of V-categories. This can be judged by looking at the results and proofs of the subsequent paper [HT08] and also the present one. However, in order to proceed with our "spaces as categories" project, further conditions on the monad $\mathbb{T}$ and the quantale V are needed. As a result of our work on this subject emerged the notion of a *topological theory* $\mathcal{T} = (\mathbb{T}, \mathsf{V}, \xi)$ introduced in [Hof07], where we add a map $\xi : T\mathsf{V} \longrightarrow \mathsf{V}$ compatible with the monad and the quantale structure to our setting. Our experience shows so far that this concept is broad enough to include our principal examples, and at the same time restrictive enough to allow us to introduce categorical ideas into the realm of $(\mathbb{T}, \mathsf{V})$-categories (which we now call $\mathcal{T}$-categories).

The particular topic of this paper is the study of weighted colimits, cocomplete $\mathcal{T}$-categories and adjoint $\mathcal{T}$-functors. We start by recalling the definition of the principal players, namely $\mathcal{T}$-categories, $\mathcal{T}$-functors and $\mathcal{T}$-modules, and then proceed introducing adjoint $\mathcal{T}$-functors and weighted colimits for $\mathcal{T}$-categories precisely as for V-categories. Furthermore, we show that the development of many basic

properties does not go much beyond the V-category case, as soon as we have $\mathcal{T}$-substitutes for dual category, presheaf-construction and the Yoneda lemma available. Finding useful equivalents to these notions and results we see as one of the main challenges here, fortunately, most of these problems are already solved in [CH07]. However, in this paper we give a different approach to the Yoneda lemma, by proving a more general result (Theorem 1.10) more suitable for our purpose. Moreover, our proof does not need anymore the restrictive condition $T1 = 1$. The achievements of this paper can then be summarised as follows. We characterise cocomplete $\mathcal{T}$-categories as precisely the injective ones with respect to fully faithful $\mathcal{T}$-functors, and as those $\mathcal{T}$-categories X for which the Yoneda functor $y_X : X \longrightarrow \hat{X}$ into the presheaf $\mathcal{T}$-category $\hat{X}$ has a left adjoint. We deduce cocompleteness of the presheaf $\mathcal{T}$-category $\hat{X}$, and show the existence of Kan-extensions in our setting, that is, any $\mathcal{T}$-functor $f : X \longrightarrow Y$ into a cocomplete $\mathcal{T}$-category has an (up to equivalence) unique extension to a left adjoint $\mathcal{T}$-functor $f_L : \hat{X} \longrightarrow Y$. As a consequence, we see that the category $\mathcal{T}$-$\mathsf{Cocont}_{\mathrm{sep}}$ of separated and cocomplete (=injective) $\mathcal{T}$-categories and left adjoint $\mathcal{T}$-functors is a reflective subcategory of $\mathcal{T}$-Cat (and of $\mathcal{T}$-$\mathsf{Cat}_{\mathrm{sep}}$), the category of (separated) $\mathcal{T}$-categories and $\mathcal{T}$-functors. Furthermore, we show that the induced monad on $\mathcal{T}$-$\mathsf{Cat}_{\mathrm{sep}}$ is of Kock-Zöberlein type and the inclusion functor is even monadic. We also prove that the forgetful functors from $\mathcal{T}$-$\mathsf{Cocont}_{\mathrm{sep}}$ to Set and to V-$\mathsf{Cat}_{\mathrm{sep}}$ are monadic. At this point we notice that our categorical approach has led us to a well-known result for topological spaces: injective T_0-spaces (together with suitable morphisms) are the Eilenberg–Moore algebras for the "filter on open subsets" monad on Top_0, the category of T_0-spaces and continuous maps, as well as for the filter monad on Set (see [Day75, Esc97] for details). We have now generalised these facts to $\mathcal{T}$-categories, but to do so we used (almost) only standard arguments from Category Theory!

Finally, we wish to highlight a possible application of our work. One of the nice features of domain theory is the strong interaction between topological and order-theoretic ideas. For instance, continuous lattices [Sco72] can be described purely in order theoretic terms as well as in topological terms: as ordered sets with certain completeness properties, or as injective topological T_0-spaces with respect to embeddings. There exist many interesting attempts in the literature to introduce *continuous metric spaces*, or, more general, *continuous* V*-categories*; all of them are (more or less) based on the order-theoretic approach to continuous lattices ([Wag94, BvBR98, Was02]). We are not aware of any attempt using injectivity properties in a suitable category. The results of our work indicate that, for instance, R. Lowen's approach spaces ([Low97]) can serve as a useful tool for the introduction and study of continuous metric spaces. In fact, as a particular instance of our work we deduce that the injective T_0-approach spaces can be described as the cocomplete T_0-approach spaces, but also as the Eilenberg–Moore algebras for suitable monads on sets respectively metric spaces. Looking at it from the other end, we obtain a metric equivalent to the filter monad, whose algebras are precisely the injective T_0-approach spaces.

1. The Setting

1.1. **Topological theories.** Throughout this paper we consider a (strict) *topological theory* as introduced in [Hof07]. Such a theory $\mathcal{T} = (\mathbb{T}, \mathsf{V}, \xi)$ consists of a commutative quantale $\mathsf{V} = (\mathsf{V}, \otimes, k)$, a Set-monad $\mathbb{T} = (T, e, m)$ where T and m satisfy (BC) (that is, T sends pullbacks to weak pullbacks and each naturality square of m is a weak pullback) and a map $\xi : T\mathsf{V} \longrightarrow \mathsf{V}$ such that

(1) the monoid V in Set lifts to a monoid (V, ξ) in $(\mathsf{Set}^{\mathbb{T}}, \times, 1)$, that is, $\xi : T\mathsf{V} \longrightarrow \mathsf{V}$ is a $\mathbb{T}$-algebra structure on V and $\otimes : \mathsf{V} \times \mathsf{V} \longrightarrow \mathsf{V}$ and $k : 1 \longrightarrow \mathsf{V}$ are $\mathbb{T}$-algebra homomorphisms. In orther

words, we require the following diagrams to commute.

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[dr, "1_X"'] & TX \arrow[d, "\xi"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} TTX \arrow[r, "T\xi"] \arrow[d, "m_X"'] & TX \arrow[d, "\xi"] \\ TX \arrow[r, "\xi"'] & X \end{tikzcd}$$

$$\begin{tikzcd} T1 \arrow[r, "Tk"] \arrow[d, "!"'] & T\mathsf{V} \arrow[d, "\xi"] \\ 1 \arrow[r, "k"'] & \mathsf{V} \end{tikzcd} \qquad \begin{tikzcd} T(\mathsf{V}\times\mathsf{V}) \arrow[r, "T(\otimes)"] \arrow[d, "\langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle"'] & T\mathsf{V} \arrow[d, "\xi"] \\ \mathsf{V}\times\mathsf{V} \arrow[r, "\otimes"'] & \mathsf{V} \end{tikzcd}$$

(2) $\xi_X := \xi \cdot T(-)$ defines a natural transformation $(\xi_X)_X : P_\mathsf{V} \longrightarrow P_\mathsf{V}T : \mathsf{Set} \longrightarrow \mathsf{Ord}$.

Here $P_\mathsf{V} : \mathsf{Set} \longrightarrow \mathsf{Ord}$ is the V-powerset functor defined as follows. We put $P_\mathsf{V}(X) = \mathsf{V}^X$ with the pointwise order. For a function $f : X \longrightarrow Y$, we have a monotone map $\mathsf{V}^f : \mathsf{V}^Y \longrightarrow \mathsf{V}^X$, $\varphi \longmapsto \varphi \cdot f$. It is easy to see that V^f preserves all infima and all suprema, hence has in particular a left adjoint denoted as $P_\mathsf{V}(f)$. Explicitly, for $\varphi \in \mathsf{V}^X$ we have $P_\mathsf{V}(f)(\varphi)(y) = \bigvee\{\varphi(x) \mid x \in X,\ f(x) = y\}$.

Examples 1.1. (1) The identity theory $\mathcal{I} = (\mathbb{1}, \mathsf{V}, 1_\mathsf{V})$, for each quantale V, where $\mathbb{1} = (\mathrm{Id}, 1, 1)$ denotes the identity monad.

(2) $\mathcal{U}_2 = (\mathbb{U}, 2, \xi_2)$, where $\mathbb{U} = (U, e, m)$ denotes the ultrafilter monad and ξ_2 is essentially the identity map.

(3) $\mathcal{U}_{\mathsf{P}_+} = (\mathbb{U}, \mathsf{P}_+, \xi_{\mathsf{P}_+})$ where $\mathsf{P}_+ = ([0,\infty]^{\mathrm{op}}, +, 0)$ and

$$\xi_{\mathsf{P}_+} : U\mathsf{P}_+ \longrightarrow \mathsf{P}_+, \quad \mathfrak{x} \longmapsto \inf\{v \in \mathsf{P}_+ \mid [0, v] \in \mathfrak{x}\}.$$

(4) The word theory $(\mathbb{L}, \mathsf{V}, \xi_\otimes)$, for each quantale V, where $\mathbb{L} = (L, e, m)$ is the word monad and

$$\begin{aligned} \xi_\otimes : L\mathsf{V} &\longrightarrow \mathsf{V}. \\ (v_1, \ldots, v_n) &\longmapsto v_1 \otimes \ldots \otimes v_n \\ () &\longmapsto k \end{aligned}$$

1.2. **V-relations.** The quantaloid V-Rel [BCSW83] has sets as objects, and an arrow $r : X \longrightarrow\!\!\!

sending a map $f : X \longrightarrow Y$ to its graph $f : X \longrightarrow\!\!\!

as composition. In fact, $\mathcal{T}$-URel is a locally completely 2-category, where the 2-categorical structure is inherited from V-Rel. Furthermore, for a $\mathcal{T}$-relation $\alpha : X \longrightarrow\!\!\!

Clearly, if $\mathcal{T} = \mathcal{I}_{\mathsf{V}}$ is the identity theory $\mathcal{I}_{\mathsf{V}} = (\mathbb{1}, \mathsf{V}, 1_{\mathsf{V}})$, then X^{op} is the usual dual V-category of X. It is by no means obvious why the definition above provides us with a "good" generalisation of this construction. We take Theorem 1.9 as well as the Yoneda lemma for $\mathcal{T}$-categories (see Theorem 1.10 and Corollary 1.11) as a reason to believe so.

As studied in [Hof07], the tensor product of V can be transported to $\mathcal{T}$-Cat by putting $(X, a) \otimes (Y, b) = (X \times Y, c)$ with

$$c(\mathfrak{w}, (x, y)) = a(\mathfrak{x}, x) \otimes b(\mathfrak{y}, y),$$

where $\mathfrak{w} \in T(X \times Y)$, $x \in X$, $y \in Y$, $\mathfrak{x} = T\pi_1(\mathfrak{w})$ and $\mathfrak{y} = T\pi_2(\mathfrak{w})$. The $\mathcal{T}$-category $E = (1, k)$ is a $\otimes$-neutral object, where 1 is a singleton set and $k : T1 \times 1 \longrightarrow \mathsf{V}$ the constant relation with value $k \in \mathsf{V}$. In general, this constructions does not result in a closed structure on $\mathcal{T}$-Cat; however, the results of [Hof07] give us the following

Proposition 1.3. *For each $\mathbb{T}$-algebra X, $X \otimes - : \mathcal{T}\text{-}\mathsf{Cat} \longrightarrow \mathcal{T}\text{-}\mathsf{Cat}$ has a right adjoint $(-)^X : \mathcal{T}\text{-}\mathsf{Cat} \longrightarrow \mathcal{T}\text{-}\mathsf{Cat}$. In particular, the structure $[\![-,-]\!]$ on $\mathsf{V}^{|X|}$ is given by the formula*

$$[\![\mathfrak{p}, \psi]\!] = \bigwedge_{\substack{\mathfrak{q} \in T(|X| \times \mathsf{V}^{|X|}) \\ \mathfrak{q} \longmapsto \mathfrak{p}}} \hom(\xi \cdot T\,\mathrm{ev}(\mathfrak{q}), \psi(m_X \cdot T\pi_1(\mathfrak{q}))),$$

for each $\mathfrak{p} \in T\mathsf{V}^{|X|}$ and $\psi \in \mathsf{V}^{|X|}$. Moreover, for $\mathfrak{p} = e_{\mathsf{V}^{|X|}}(\varphi)$ we have

$$[\![e_{\mathsf{V}^{|X|}}(\varphi), \psi]\!] = \bigwedge_{\mathfrak{x} \in TX} \hom(\varphi(\mathfrak{x}), \psi(\mathfrak{x})).$$

Furthermore, several maps obtained from the quantale structure on V become now $\mathcal{T}$-functors.

Proposition 1.4. *The following assertions hold.*

(1) *Both $k : E \longrightarrow \mathsf{V}$ and $\otimes : \mathsf{V} \otimes \mathsf{V} \longrightarrow \mathsf{V}$ are $\mathcal{T}$-functors, hence V is even a monoid in $(\mathcal{T}\text{-}\mathsf{Cat}, \otimes, E)$.*
(2) *$\xi : |\mathsf{V}| \longrightarrow \mathsf{V}$ is a $\mathcal{T}$-functor.*
(3) *$\bigvee : \mathsf{V}^{|X|} \longrightarrow \mathsf{V}$ is a $\mathcal{T}$-functor, for each set X.*

Proof. (1) and (2) are easy to prove, (3) is a consequence of [Hof07, Proposition 6.11]. □

1.5. **$\mathcal{T}$-modules.** Let $X = (X, a)$ and $Y = (Y, b)$ be $\mathcal{T}$-categories and $\varphi : X \longrightarrow\!\!\!

Let now $X = (X, a)$ and $Y = (Y, b)$ be $\mathcal{T}$-categories and $f : X \longrightarrow Y$ be a function. We define $\mathcal{T}$-relations $f_* : X \longrightarrow\!\!\!\!\!\!+\; Y$ and $f^* : Y \longrightarrow\!\!\!\!\!\!+\; X$ by putting $f_* = b \cdot Tf$ and $f^* = f^\circ \cdot b$ respectively. Hence, for $\mathfrak{x} \in TX$, $\mathfrak{y} \in TY$, $x \in X$ and $y \in Y$, we have $f_*(\mathfrak{x}, y) = b(Tf(\mathfrak{x}), y)$ and $f^*(\mathfrak{y}, x) = b(\mathfrak{y}, f(x))$. Given now $\mathcal{T}$-modules φ and ψ, we obtain

$$\varphi \circ f_* = \varphi \cdot Tf \qquad\qquad \text{and} \qquad\qquad f^* \circ \psi = f^\circ \cdot \psi.$$

In particular, $b \circ f_* = f_*$ and $f^* \circ b = f^*$, as well as $f_* \circ f^* = b \cdot Tf \cdot Tf^\circ \cdot T_\xi b \cdot m_Y^\circ \leq b$. The following lemma can be easily verified.

Lemma 1.6. *The following assertions are equivalent.*

(i) $f : X \longrightarrow Y$ *is a* $\mathcal{T}$*-functor.*
(ii) f_* *is a* $\mathcal{T}$*-module* $f_* : X \longrightarrow\!\!\!\!\!\!\circ\; Y$.
(iii) f^* *is a* $\mathcal{T}$*-module* $f^* : Y \longrightarrow\!\!\!\!\!\!\circ\; X$.
(iv) $a \leq f^* \circ f_*$.

As a consequence, for each $\mathcal{T}$-functor $f : (X, a) \longrightarrow (Y, b)$ we have an adjunction $f_* \dashv f^*$ in $\mathcal{T}$-Mod. Moreover, given also a $\mathcal{T}$-functor $g : (Y, b) \longrightarrow (Z, c)$,

$$g_* \circ f_* = c \cdot Tg \cdot Tf = c \cdot T(g \cdot f) = (g \cdot f)_*$$

and

$$f^* \circ g^* = f^\circ \cdot g^\circ \cdot c = (g \cdot f)^\circ \cdot c = (g \cdot f)^*.$$

Since also $(1_X)_* = (1_X)^* = a$, we obtain functors

$$(-)_* : \mathcal{T}\text{-Cat} \longrightarrow \mathcal{T}\text{-Mod} \qquad\qquad \text{and} \qquad\qquad (-)^* : \mathcal{T}\text{-Cat}^{\mathrm{op}} \longrightarrow \mathcal{T}\text{-Mod},$$

where $X_* = X = X^*$, for each $\mathcal{T}$-category X.

Lemma 1.7. *A* $\mathcal{T}$*-functor* $f : (X, a) \longrightarrow (Y, b)$ *is fully faithful if and only if* $1_X^* = f^* \circ f_*$.

Lemma 1.8. *Consider* $\mathcal{T}$*-modules* $\varphi : X \longrightarrow\!\!\!\!\!\!\circ\; Y$, $\psi : X \longrightarrow\!\!\!\!\!\!\circ\; Z$ *and* $\alpha : Y \longrightarrow\!\!\!\!\!\!\circ\; B$, *where* α *is right adjoint. Then*

$$\alpha \circ (\varphi \circ\!\!- \psi) = (\alpha \circ \varphi) \circ\!\!- \psi.$$

Proof. Let $\beta : B \longrightarrow\!\!\!\!\!\!\circ\; Y$ be the left adjoint of α. We have to show that the diagram

$$\begin{tikzcd}
\mathcal{T}\text{-Mod}(X, Y) \arrow[r, "(-)\circ\!- \psi"] \arrow[d, "\alpha\circ -"'] & \mathcal{T}\text{-Mod}(Z, Y) \arrow[d, "\alpha\circ -"] \\
\mathcal{T}\text{-Mod}(X, B) \arrow[r, "(-)\circ\!- \psi"'] & \mathcal{T}\text{-Mod}(Z, B)
\end{tikzcd}$$

of right adjoints commutes. But the diagram

$$\begin{tikzcd}
\mathcal{T}\text{-Mod}(X, Y) & \mathcal{T}\text{-Mod}(Z, Y) \arrow[l, "-\circ\psi"'] \\
\mathcal{T}\text{-Mod}(X, B) \arrow[u, "\beta\circ -"] & \mathcal{T}\text{-Mod}(Z, B) \arrow[l, "-\circ\psi"] \arrow[u, "\beta\circ -"']
\end{tikzcd}$$

of the corresponding left adjoints commutes since Kleisli convolution is associative, and the assertion follows. □

Theorem 1.9 ([CH07]). *For $\mathcal{T}$-categories (X,a) and (Y,b), and a $\mathcal{T}$-relation $\psi : X \longrightarrow\!\!\!

2. Cocomplete $\mathcal{T}$-categories

2.1. $\mathcal{T}$-Cat **as an ordered category.** We can transport the order-structure on hom-sets from $\mathcal{T}$-Mod to $\mathcal{T}$-Cat via the functor $(-)^* : \mathcal{T}\text{-Cat}^{\mathrm{op}} \longrightarrow \mathcal{T}\text{-Mod}$, that is, we define $f \leq g$ whenever $f^* \leq g^*$. Clearly, we have $f \leq g$ if and only if $g_* \leq f_*$. With this definition we turn $\mathcal{T}$-Cat into a 2-category, and therefore the (representable) forgetful functor $\mathrm{O} : \mathcal{T}\text{-Cat} \longrightarrow \mathsf{Set}$ factors through $\mathrm{O} : \mathcal{T}\text{-Cat} \longrightarrow \mathsf{Ord}$. As usual, we call $\mathcal{T}$-functors $f, g : X \longrightarrow Y$ *equivalent*, and write $f \cong g$, if $f \leq g$ and $g \leq f$. Hence, $f \cong g$ if and only if $f^* = g^*$, which in turn is equivalent to $f_* = g_*$. We call a $\mathcal{T}$-category X *L-separated* (see [HT08] for details) whenever $f \cong g$ implies $f = g$, for all $\mathcal{T}$-functors $f, g : Y \longrightarrow X$ with codomain X. The $\mathcal{T}$-category $\mathsf{V} = (\mathsf{V}, \hom_\xi)$ is L-separated, and so is each $\mathcal{T}$-category of the form $\hat{X}$, for a $\mathcal{T}$-category X. The full subcategory of $\mathcal{T}$-Cat consisiting of all L-separated $\mathcal{T}$-categories is denoted by $\mathcal{T}\text{-Cat}_{\mathrm{sep}}$. A $\mathcal{T}$-category X is called *injective* if, for all $\mathcal{T}$-functors $f : A \longrightarrow X$ and fully faithful $\mathcal{T}$-functors $i : A \longrightarrow B$, there exists a $\mathcal{T}$-functor $g : B \longrightarrow X$ such that $g \cdot i \cong f$. Clearly, for a L-separated $\mathcal{T}$-category X we have then $g \cdot i = f$.

Lemma 2.1. *The following assertions hold.*

(1) *Let $f, g : X \longrightarrow Y$ be $\mathcal{T}$-functors between $\mathcal{T}$-categories $X = (X, a)$ and $Y = (Y, b)$. Then*

$$f \leq g \iff \forall x \in X \,.\, k \leq b(e_Y(f(x)), g(x)).$$

In particular, for $\mathcal{T}$-functors $f, g : Y \longrightarrow \mathsf{V}^{|X|}$ we have

$$f \leq g \iff \forall y \in Y, \mathfrak{x} \in TX \,.\, f(y)(\mathfrak{x}) \leq g(y)(\mathfrak{x}).$$

(2) *A $\mathcal{T}$-category X is L-separated if and only if the underlying V-category $\mathrm{S}\,X$ is L-separated.*

(3) *With X also $\mathrm{S}\,X$ is injective with respect to fully faithful functors, for each $\mathcal{T}$-category X.*

Proof. (1) can be found in [HT08], (2) follows immediately from (1), and (3) follows from the facts that $\mathrm{S} : \mathcal{T}\text{-Cat} \longrightarrow \mathsf{V}\text{-Cat}$ is actually a 2-functor and it's left adjoint $\mathrm{A} : \mathsf{V}\text{-Cat} \longrightarrow \mathcal{T}\text{-Cat}$ sends fully faithful V-functors to fully faithful $\mathcal{T}$-functors. □

One of the most important concepts in a 2-category is that of *adjointness*. Here, a $\mathcal{T}$-functor $f : X \longrightarrow Y$ is *left adjoint* if there exists a $\mathcal{T}$-functor $g : Y \longrightarrow X$ such that $1_X \leq g \cdot f$ and $1_Y \geq f \cdot g$. Passing to $\mathcal{T}$-Mod, f is left adjoint to g if and only if $g_* \dashv f_*$, that is, if and only if $f_* = g^*$. Bearing in mind Lemma 1.6, we have

Proposition 2.2. *A $\mathcal{T}$-functor $f : X \longrightarrow Y$ is left adjoint if and only if there exists a function $g : Y \longrightarrow X$ such that $f_* = g^*$, that is,*

$$b(Tf(\mathfrak{x}), y) = a(\mathfrak{x}, g(y)),$$

for all $\mathfrak{x} \in TX$ and $y \in Y$.

2.2. **Cocomplete $\mathcal{T}$-categories.** Let now $X = (X, a)$ be a $\mathcal{T}$-category. Given a $\mathcal{T}$-functor $h : Y \longrightarrow X$ and a *weight* $\psi : Y \longrightarrow\!\!\!\!\!\circ\; Z$ in $\mathcal{T}$-Mod,

$$\begin{tikzcd} Y \arrow[r, "h_*", "\circ"{marking}] \arrow[d, "\psi"', "\circ"{marking}] & X \\ Z \arrow[ur, dotted, "h_* \circ\!\!- \psi"'] & \end{tikzcd}$$

we call a $\mathcal{T}$-functor $g : Z \longrightarrow X$ *a ψ-weighted colimit of h*, and write $g \cong \mathrm{colim}(\psi, h)$, if g represents $h_* \circ\!\!- \psi$, i.e. if $h_* \circ\!\!- \psi = g_*$. Clearly, if such g exists, it is unique up to equivalence and therefore we call g "the" ψ-weighted colimit of h. We say that a $\mathcal{T}$-functor $f : X \longrightarrow Y$ *preserves the ψ-weighted colimit of h* if $f \cdot \mathrm{colim}(\psi, h) \cong \mathrm{colim}(\psi, f \cdot h)$, that is, if $(f \cdot g)_* = (f \cdot h)_* \circ\!\!- \psi$. A $\mathcal{T}$-functor $f : X \longrightarrow Y$ is

cocontinuous if f preserves all weighted colimits which exist in X, and a $\mathcal{T}$-category X is *cocomplete* if each "weighted diagram" has a colimit in X. A straightforward calculation shows that we only need to consider $f = 1_X$.

Lemma 2.3. *Let $f : Y \longrightarrow X$ be a $\mathcal{T}$-functor and $\psi : Y \multimap Z$ be a $\mathcal{T}$-module. Then* $\mathrm{colim}(\psi, f) \cong \mathrm{colim}(\psi \circ f^*, 1_X)$. *In particular, X is cocomplete if and only if $1_X^* \circ\!\!- \psi$ is representable by some $\mathcal{T}$-functor $g : Z \longrightarrow X$, for each $\mathcal{T}$-module $\psi : X \multimap Z$. Furthermore, a $\mathcal{T}$-functor $f : X \longrightarrow Y$ is cocontinuous if and only if f preserves all ψ-weighted colimits of 1_X.*

Remark 2.4. When studying V-categories, one can go even one step further and show that cocompleteness reduces to the case $Z = G$. More precise, a V-category X is cocomplete if and only if $(1_X)^* \bullet\!\!- \psi$ is representable by some V-functor, for each V-module $\psi : X \multimap G$. However, for a general theory $\mathcal{T}$ I am not able to prove this.

We let $\mathcal{T}$-Cocont denote the 2-category of all cocomplete $\mathcal{T}$-categories and left adjoint $\mathcal{T}$-functors between them. Correspondingly, $\mathcal{T}$-$\mathsf{Cocont}_{\mathrm{sep}}$ denotes the full subcategory of $\mathcal{T}$-Cocont consisting of all L-separated cocomplete $\mathcal{T}$-categories.

Proposition 2.5. *The following assertions hold.*

(1) *Each $\ulcorner\psi\urcorner \in \hat{X}$ is a colimit of represantables. More precisely, we have $y_* \circ\!\!- \psi = \ulcorner\psi\urcorner_*$.*

$$\begin{tikzcd} X \arrow[r, "y_*", "\circ"'] \arrow[d, "\psi"', "\circ"] & \hat{X} \arrow[dl, dotted, "y_* \circ\!\!- \psi"] \\ G & \end{tikzcd}$$

(2) *A left adjoint $\mathcal{T}$-functor $f : X \longrightarrow Y$ between $\mathcal{T}$-categories is cocontinuous.*

Proof. (1) Let $\mathfrak{a} \in T1$ and $h \in \hat{X}$. Then, by Theorem 1.10,

$$(y_* \circ\!\!- \psi)(\mathfrak{a}, h) = [\![T\ulcorner\psi\urcorner(\mathfrak{a}), h]\!] = \ulcorner\psi\urcorner_*(\mathfrak{a}, h).$$

(2) Let $h : A \longrightarrow X$ be in $\mathcal{T}$-Cat, $\psi : A \multimap B$ in $\mathcal{T}$-Mod, and $g \cong \mathrm{colim}(\psi, h)$. Then, since f_* is a right adjoint $\mathcal{T}$-module, from Lemma 1.8 we deduce

$$(f \cdot h)_* \circ\!\!- \psi = f_* \circ (h_* \circ\!\!- \psi) = f_* \circ g_* = (f \cdot g)_*. \qquad \square$$

Theorem 2.6. *Let $X = (X, a)$ be a $\mathcal{T}$-category. The following assertions are equivalent.*

(i) *X is injective.*
(ii) *$y_X : X \longrightarrow \hat{X}$ has a left inverse, i.e. there exists a $\mathcal{T}$-functor $\mathrm{Sup}_X : \hat{X} \longrightarrow X$ such that $\mathrm{Sup}_X \cdot y_X \cong 1_X$.*
(iii) *$y_X : X \longrightarrow \hat{X}$ has a left adjoint $\mathrm{Sup}_X : \hat{X} \longrightarrow X$.*
(iv) *X is cocomplete.*

Proof. (i)⇒(ii) Follows immediately from the fact that $y_X : X \longrightarrow \hat{X}$ is fully faithful (see Corollary 1.11).
(ii)⇒(iii) Since $\mathrm{Sup}_X \cdot y_X \cong 1_X$ by hypothesis, it is enough to show $1_{\hat{X}} \leq y_X \cdot \mathrm{Sup}_X$. Let $\psi \in \hat{X}$ and $\mathfrak{x} \in TX$. Then, by Corollary 1.11 and Lemma 2.1, we have

$$\psi(\mathfrak{x}) = [\![T\, y_X(\mathfrak{x}), \psi]\!] \leq a(T(\mathrm{Sup}_X \cdot y)(\mathfrak{x}), \mathrm{Sup}_X(\psi)) = a(\mathfrak{x}, \mathrm{Sup}_X(\psi)) = [\![T\, y_X(\mathfrak{x}), y_X \cdot \mathrm{Sup}_X(\psi)]\!] = y_X \cdot \mathrm{Sup}_X(\psi)(\mathfrak{x}).$$

(iii)⇒(iv) Assume $\mathrm{Sup}_X \dashv y_X$ and let $\psi : X \longrightarrow\!\!\!\!\!\!\circ\; Y$ in $\mathcal{T}$-Mod. By Theorem 1.10, for all $\mathfrak{y} \in TY$ and $x \in X$ we have

$$1_X^* \circ\!\!- \psi(\mathfrak{y}, x) = [\![T \ulcorner\psi\urcorner(\mathfrak{y}), y_X(x)]\!] = y_X^\circ \cdot \ulcorner\psi\urcorner_*(\mathfrak{y}, x) = y_X^* \circ \ulcorner\psi\urcorner_*(\mathfrak{y}, x)$$
$$= (\mathrm{Sup}_X)_* \circ \ulcorner\psi\urcorner_*(\mathfrak{y}, x) = (\mathrm{Sup}_X \cdot \ulcorner\psi\urcorner)_*(\mathfrak{y}, x),$$

hence $\mathrm{Sup}_X \cdot \ulcorner\psi\urcorner \cong \mathrm{colim}(\psi, 1_X)$.

(iv)⇒(i) Let $i : A \longrightarrow B$ be a fully faithful $\mathcal{T}$-functor. Let $f : A \longrightarrow X$ be a $\mathcal{T}$-functor. Hence, by cocompleteness of X, $f_* \circ\!\!- i_* = g_*$ for some $\mathcal{T}$-functor $g : B \longrightarrow X$. Hence $(g \cdot i)_* = g_* \circ i_* \leq f_*$. On the other hand, from $f_* = f_* \circ i^* \circ i_*$ we deduce $f_* \circ i^* \leq f_* \circ\!\!- i_* = g_*$, hence $f_* \leq g_* \circ i_*$. □

Remarks 2.7. As it happens often, the proof of the theorem above gives us some further information. Firstly, any left inverse $S : \hat{X} \longrightarrow X$ to the Yoneda embedding $y_X : X \longrightarrow \hat{X}$ is actually left adjoint to y_X. I learned this useful fact in the context of quantaloid-enriched categories from Isar Stubbe. Secondly, the ψ-weighted colimit of $1_X : X \longrightarrow X$ in a cocomplete $\mathcal{T}$-category X can be calculated as $\mathrm{Sup}_X \cdot \ulcorner\psi\urcorner$. Finally, if X is injective, then any $\mathcal{T}$-functor $f : A \longrightarrow X$ has not only an extension along a fully faithful $\mathcal{T}$-functor $i : A \longrightarrow B$, but even a smallest one with respect to the order on hom-sets in $\mathcal{T}$-Cat.

Let $f : X \longrightarrow Y$ be a function. We define $f^{-1} : \mathsf{V}^{|Y|} \longrightarrow \mathsf{V}^{|X|}$ to be the mate of the composite

$$|X| \otimes \mathsf{V}^{|Y|} \xrightarrow{|f| \otimes 1_{\mathsf{V}^{|Y|}}} |Y| \otimes \mathsf{V}^{|Y|} \xrightarrow{\mathrm{ev}} \mathsf{V}$$

of $\mathcal{T}$-functors. Explicitly, for any $\psi \in \mathsf{V}^{|Y|}$ and $\mathfrak{x} \in TX$, $f^{-1}(\psi)(\mathfrak{x}) = \psi(Tf(\mathfrak{x}))$. Hence, if f is a $\mathcal{T}$-functor and $\psi \in \hat{Y}$, then $f^{-1}(\psi) = \psi \circ f_* \in \hat{X}$, so hat f^{-1} restricts to a $\mathcal{T}$-functor

$$f^{-1} : \hat{Y} \longrightarrow \hat{X}.$$

Theorem 2.8. *For each $\mathcal{T}$-category X, $\hat{X}$ is cocomplete where* $\mathrm{Sup}_{\hat{X}} = y_X^{-1}$.

Proof. According to Theorem 2.6, we have to show $y_X^{-1} \cdot y_{\hat{X}} = 1_{\hat{X}}$. To do so, let $\psi \in \hat{X}$ and $\mathfrak{x} \in TX$. Then, by the Yoneda Lemma (Corollary 1.11), we have

$$y_X^{-1}(y_{\hat{X}}(\psi))(\mathfrak{x}) = y_{\hat{X}}(\psi)(T\, y_X(\mathfrak{x})) = [\![T\, y_X(\mathfrak{x}), \psi]\!] = \psi(\mathfrak{x}),$$

and the assertion follows. □

Note that the Theorem above applies in particular to the discrete $\mathcal{T}$-category $X = (X, e_X^\circ)$, hence $\mathsf{V}^{|X|}$ is cocomplete for each set X. Clearly, if $T1 = 1$, then $\mathsf{V}^{|1|} \cong \mathsf{V}$ and therefore the $\mathcal{T}$-category V is cocomplete and hence injective in $\mathcal{T}$-Cat. A different proof of this property of V can be found in [HT08, Lemma 3.18]. Note that also in the proof of [HT08] the condition $T1 = 1$ is crucial.

2.3. **Kan extension.** From Theorem 2.6 we know that each $\mathcal{T}$-functor $f : X \longrightarrow Y$ into a cocomplete $\mathcal{T}$-category Y has a smallest extension along $y_X : X \longrightarrow \hat{X}$. We will see now that this extension is particularly nice (compare with [Kel82, Theorem 5.35]).

Theorem 2.9. *Composition with $y_X : X \longrightarrow \hat{X}$ defines an equivalence*

$$\mathcal{T}\text{-Cocont}(\hat{X}, Y) \longrightarrow \mathcal{T}\text{-Cat}(X, Y)$$

of ordered sets, for each cocomplete $\mathcal{T}$-category Y. That is, for each $\mathcal{T}$-functor $f : X \longrightarrow Y$ into a cocomplete $\mathcal{T}$-category Y, there exists a (up to equivalence) unique left adjoint $\mathcal{T}$-functor $f_L : \hat{X} \longrightarrow Y$

such that $f_L \cdot y_X \cong f$*; and, if* $f \leq f'$*, then* $f_L \leq f'_L$*. Moreover, the right adjoint to* f_L *is given by* $\ulcorner f_* \urcorner$*.*

$$\begin{tikzcd} X \arrow[r, "y_X"] \arrow[dr, "f"'] & \hat{X} \arrow[d, "f_L"', "\cong"] \arrow[d, phantom, shift left=6ex, "\dashv"] \\ & Y \arrow[u, bend right, "\ulcorner f_* \urcorner"'] \end{tikzcd}$$

Proof. Let $f_L : \hat{X} \longrightarrow Y$ be the extension of f where $(f_L)_* = f_* \circ\!\!- (y_X)_*$. Then, by Theorem 1.10, for any $\mathfrak{p} \in T\hat{X}$ and $y \in Y$, we have

$$(f_L)_*(\mathfrak{p}, y) = f_* \circ\!\!- (y_X)_*(\mathfrak{p}, y) = [\![\mathfrak{p}, \ulcorner f_* \urcorner(y)]\!] = \ulcorner f_* \urcorner^*(\mathfrak{p}, y),$$

hence $f_L \dashv \ulcorner f_* \urcorner$. Unicity of f_L follows from Proposition 2.5. Assume now $f \leq f'$. Then $f'_* \leq f_*$ and therefore $(f'_L)_* \circ (y_X)_* \leq f'_* \leq f_*$. Hence $(f'_L)_* \leq (f_L)_*$, that is, $f_L \leq f'_L$. □

The theorem above tells us that both inclusion functors $\mathcal{T}$-$\mathsf{Cocont}_{\mathrm{sep}} \hookrightarrow \mathcal{T}$-$\mathsf{Cat}_{\mathrm{sep}}$ and $\mathcal{T}$-$\mathsf{Cocont}_{\mathrm{sep}} \hookrightarrow \mathcal{T}$-$\mathsf{Cat}$ have a left adjoint defined by $X \longmapsto \hat{X}$ which, moreover, is a 2-functor. In particular, if $f : X \longrightarrow Y$ is a $\mathcal{T}$-functor, then $y_Y \cdot f : X \longrightarrow \hat{Y}$ has a left adjoint extension $\hat{f} : \hat{X} \longrightarrow \hat{Y}$ along $y_X : X \longrightarrow \hat{X}$.

$$\begin{tikzcd} X \arrow[r, "y_X"] \arrow[d, "f"'] & \hat{X} \arrow[d, "\hat{f}"] \\ Y \arrow[r, "y_Y"'] & \hat{Y} \end{tikzcd}$$

Furthermore, by Theorem 2.9, the right adjoint of $\hat{f}$ is given by $\ulcorner (y_Y \cdot f) \urcorner_* : \hat{Y} \longrightarrow \hat{X}$. Explicitly, for each $\psi \in \hat{Y}$ and each $\mathfrak{x} \in TX$ we have

$$\ulcorner (y_Y \cdot f) \urcorner_*(\psi)(\mathfrak{x}) = (y_Y)_* \circ f_*(\mathfrak{x}, \psi) = (y_Y)_* \cdot Tf(\mathfrak{x}, \psi) = (y_Y)_*(Tf(\mathfrak{x}), \psi) = \psi(Tf(\mathfrak{x})),$$

that is, $f^{-1} = \ulcorner (y_Y \cdot f) \urcorner_*$ and $\hat{f} \dashv f^{-1}$. Passing to the underlying ordered sets, $f^{-1} : \hat{Y} \longrightarrow \hat{X}$ corresponds to $- \circ f_*$, therefore the underlying (order-preserving) map of $\hat{f}$ is given by $- \circ f^*$ (see Remark 1.5). Hence, for $\psi \in \hat{X}$ and $\mathfrak{y} \in TY$ we have

$$\psi \circ f^* = \psi \circ (f^\circ \cdot b) = \psi \cdot Tf^\circ \cdot T_\xi b \cdot m_Y^\circ = \psi \cdot Tf^\circ \cdot s$$

and

$$\psi \circ f^*(\mathfrak{y}) = \bigvee_{\mathfrak{x} \in TX} \psi(\mathfrak{x}) \otimes s(\mathfrak{y}, Tf(\mathfrak{x})),$$

where b denotes the structure on Y and $s = T_\xi b \cdot m_Y$.

Consider now the discrete $\mathcal{T}$-category $X_D = (X, e_X^\circ)$. Then, for any $\mathcal{T}$-category X, the identity map $j_X : X_D \longrightarrow X,\ x \longmapsto x$ is a $\mathcal{T}$-functor, and we obtain a left adjoint $\mathcal{T}$-functor $\widehat{j_X} : \widehat{X_D} = \mathsf{V}^{|X|} \longrightarrow \hat{X}$. In the sequel we find it convenient to write R_X instead. One easily verifies that its right adjoint $j_X^{-1} : \hat{X} \longrightarrow \mathsf{V}^{|X|}$ is given by the inclusion map $i_X : \hat{X} \hookrightarrow \mathsf{V}^{|X|}$.

Corollary 2.10. *For each* $\mathcal{T}$*-category* $X = (X, a)$*, the inclusion functor* $i_X : \hat{X} \longrightarrow \mathsf{V}^{|X|}$ *has a left adjoint given by*

$$R_X : \mathsf{V}^{|X|} \longrightarrow \hat{X},\ \psi \longmapsto \left(\mathfrak{x} \longmapsto \bigvee_{\mathfrak{y} \in TX} \psi(\mathfrak{y}) \otimes r(\mathfrak{x}, \mathfrak{y}) \right),$$

where $r = T_\xi a \cdot m_X^\circ$*.*

Corollary 2.11. *For each function $f : X \longrightarrow Y$, the left adjoint to $f^{-1} : \mathsf{V}^{|Y|} \longrightarrow \mathsf{V}^{|X|}$ is given by*

$$\mathsf{V}^{|X|} \longrightarrow \mathsf{V}^{|Y|},\ \psi \longmapsto \left(\mathfrak{y} \longmapsto \bigvee_{\mathfrak{x}:T f(\mathfrak{x})=\mathfrak{y}} \psi(\mathfrak{x}) \right).$$

For a $\mathcal{T}$-functor $f : X \longrightarrow Y$, let us write temporarily $f_D : (X, e_X^\circ) \longrightarrow (Y, e_Y^\circ)$ for the same map between the discrete $\mathcal{T}$-categories. Since obviously $j_Y \cdot f_D = f \cdot j_X$, we have a commutative diagram

$$\begin{tikzcd} \mathsf{V}^{|X|} \arrow[r, "\widehat{f_D}"] \arrow[d, "R_X"'] & \mathsf{V}^{|Y|} \arrow[d, "R_Y"] \\ \hat{X} \arrow[r, "\hat{f}"'] & \hat{Y} \end{tikzcd}$$

of $\mathcal{T}$-functors. Furthermore, we have $\widehat{f} \cdot f^{-1} = 1_{\hat{X}}$ provided that f is L-dense, i.e. $f_* \circ f^* = 1_X^*$. Satisfying (BC), the functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ sends surjections to surjections, and therefore each surjective $\mathcal{T}$-functor f is L-dense.

2.4. Cocomplete $\mathcal{T}$-categories as Eilenberg–Moore algebras.

Proposition 2.12. *Let $f : X \longrightarrow Y$ be a $\mathcal{T}$-functor between cocomplete $\mathcal{T}$-categories. Then the following assertions are equivalent.*

(i) *f is left adjoint.*
(ii) *f is cocontinuous, that is, f preserves all weighted colimits.*
(iii) *We have $f \cdot \mathrm{Sup}_X \cong \mathrm{Sup}_Y \cdot \hat{f}$, where $\mathrm{Sup}_X \dashv y_X$ and $\mathrm{Sup}_Y \dashv y_Y$.*

$$\begin{tikzcd} \hat{X} \arrow[r, "\hat{f}"] \arrow[d, "\mathrm{Sup}_X"'] \arrow[dr, phantom, "\cong"] & \hat{Y} \arrow[d, "\mathrm{Sup}_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Proof. The implication (i)⇒(ii) we proved already in Proposition 2.5. To see that (ii)⇒(iii), recall that $\mathrm{Sup}_X \cong \mathrm{colim}((y_X)_*, 1_X)$ and therefore $f \cdot \mathrm{Sup}_X \cong \mathrm{colim}((y_X)_*, f)$. With the help of Lemma 1.8, we get

$$(f \cdot \mathrm{Sup}_X)_* = f_* \circ\!\!- (y_X)_* = (y_Y^* \circ (y_Y \cdot f)_*) \circ\!\!- (y_X)_* = y_Y^* \circ ((y_Y \cdot f)_* \circ\!\!- (y_X)_*) = y_Y^* \circ \hat{f}_* = (\mathrm{Sup}_Y \cdot \hat{f})_*.$$

Finally, to obtain (iii)⇒(i), we show that $f \dashv \mathrm{Sup}_X \cdot f^{-1} \cdot y_Y$. In fact,

$$(\mathrm{Sup}_X \cdot f^{-1} \cdot y_Y)^* = y_Y^* \circ f^{-1^*} \circ \mathrm{Sup}_X^* = \mathrm{Sup}_{Y*} \circ \hat{f}_* \circ \mathrm{Sup}_X^* = f_* \circ \mathrm{Sup}_{X*} \circ \mathrm{Sup}_X^* = f_* \circ y_X^* \circ \mathrm{Sup}_X^* = f_*. \quad \square$$

Example 2.13. Recall from Subsection 2.9 that, for each $\mathcal{T}$-functor $f : X \longrightarrow Y$, we have an adjunction $\hat{f} \dashv f^{-1}$ in $\mathcal{T}$-Cat. The underlying (order-preserving) maps of $\hat{f}$ and f^{-1} are given by $- \circ f^*$ and $- \circ f_*$ respectively. Furthermore, we have $\hat{\hat{f}} \dashv \widehat{f^{-1}}$. Since $y_Y \cdot f = \hat{f} \cdot y_X$, we obtain $\widehat{y_Y} \cdot \hat{f} = \hat{\hat{f}} \cdot \widehat{y_X}$ and therefore $y_X^{-1} \cdot \widehat{f^{-1}} = f^{-1} \cdot y_Y^{-1}$. Hence, by Theorem 2.8 and Proposition 2.12, f^{-1} has a right adjoint $f_\bullet : \hat{X} \longrightarrow \hat{Y}$ in $\mathcal{T}$-Cat. The underlying order-preserving map of $f_\bullet$ we identified in Remark 1.5 as $(-) \circ\!\!- f_*$.

The pair of adjoint functors $\mathcal{T}\text{-}\mathsf{Cocont}_{\mathrm{sep}} \hookrightarrow \mathcal{T}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ and $\widehat{(-)} : \mathcal{T}\text{-}\mathsf{Cat}_{\mathrm{sep}} \hookrightarrow \mathcal{T}\text{-}\mathsf{Cocont}_{\mathrm{sep}}$ induces monad on $\mathcal{T}\text{-}\mathsf{Cat}_{\mathrm{sep}}$, denoted as $\mathbb{I} = (\widehat{(-)}, y, \mu)$. By Theorem 2.9, we have that $f \leq g$ implies $\hat{f} \leq \hat{g}$, so that $\widehat{(-)}$ is a 2-functor. Furthermore, since obviously $y_{\hat{X}} \cdot y_X = y_{\hat{X}} \cdot y_X$, we have $(y_{\hat{X}})_* \leq (\widehat{y_X})_*$, that is, $\widehat{y_X} \leq y_{\hat{X}}$. In general, a monad $\mathbb{S} = (S, d, l)$ on a locally thin 2-category X is of *Kock-Zöberlein type* (see [Koc95]) if S is a 2-functor and $S\,d_X \leq d_{S\,X}$, for all $X \in \mathsf{X}$. In fact, in [Koc95] it is shown that

Theorem 2.14. *Let $\mathbb{S} = (S, d, l)$ be a monad on a locally thin 2-category X where S is a 2-functor. Then the following assertions are equivalent.*

(i) $S\,d_X \leq d_{S\,X}$ *for all* $X \in \mathsf{X}$.
(ii) $S\,d_X \dashv l_X$ *for all* $X \in \mathsf{X}$.
(iii) $l_X \dashv d_{S\,X}$ *for all* $X \in \mathsf{X}$.
(iv) *For all* $X \in \mathsf{X}$, *a* X*-morphism* $h : S\,X \longrightarrow X$ *is the structure morphism of a* $\mathbb{S}$*-algebra if and only if* $h \dashv d_X$ *with* $h \cdot d_X = 1_X$.

The considerations above tell us that the monad $\mathbb{I} = (\widehat{(-)}, y, \mu)$ on $\mathcal{T}$-$\mathsf{Cat}_{\mathrm{sep}}$ is of Kock-Zöberlein type. Furthermore, by Theorem 2.6 and Proposition 2.12 we have

Theorem 2.15. $(\mathcal{T}\text{-}\mathsf{Cat}_{\mathrm{sep}})^{\mathbb{I}} \cong \mathcal{T}\text{-}\mathsf{Cocont}_{\mathrm{sep}}$. *Hence, in particular,* $\mathcal{T}$-$\mathsf{Cocont}_{\mathrm{sep}}$ *is complete.*

Theorem 2.14 also helps us to compute the multiplication μ of $\mathbb{I}$: for any (L-separated) $\mathcal{T}$-category X we have $\widehat{y_X} \dashv \mu_X$ and $\widehat{y_X} \dashv y_X^{-1}$, hence $\mu_X = y_X^{-1}$.

2.5. **Example: topological spaces.** We consider now $\mathcal{T} = \mathcal{U}_2 = (\mathbb{U}, 2, \xi_2)$. Hence $\mathcal{T}$-$\mathsf{Cat} = \mathsf{Top}$ is the category of topological spaces and continuous maps, and $\mathcal{T}$-$\mathsf{Cat}_{\mathrm{sep}} = \mathsf{Top}_0$ its full subcategory of T_0-spaces (see also [CH07, HT08]). Then $M(X) = (UX, \leq)$ is the ordered set with

$$\mathfrak{x} \leq \mathfrak{y} \iff \{\overline{A} \mid A \in \mathfrak{x}\} \subseteq \mathfrak{y},$$

and the topology on $|X|$ is given by the Zariski-closure defined by

$$\mathfrak{x} \in \mathrm{cl}\,\mathcal{A} :\iff \bigcap \mathcal{A} \subseteq \mathfrak{x} \iff \mathfrak{x} \subseteq \bigcup \mathcal{A}.$$

In [HT08] we observed already that the down-closure as well as the up-closure of a Zariski-closed set is again Zariski-closed. A presheaf $\psi \in \hat{X}$ can be identified with the Zariski-closed and down-closed subset $\mathcal{A} = \psi^{-1}(1) \subseteq UX$, and we consider

$$\hat{X} = \{\mathcal{A} \subseteq UX \mid \mathcal{A} \text{ is Zariski-closed and down-closed}\}.$$

The topology on $\hat{X}$ is the *compact-open topology*, which has as basic open sets

$$B(\mathcal{B}, \{0\}) = \{\mathcal{A} \in \hat{X} \mid \mathcal{A} \cap \mathcal{B} = \emptyset\}, \qquad \mathcal{B} \subseteq UX \text{ Zariski-closed.}$$

The Yoneda map $y_X : X \longrightarrow \hat{X}$ is given by $y_X(x) = \{\mathfrak{x} \in UX \mid \mathfrak{x} \to x\}$. For $\mathfrak{x} \in UX$, $U\,y_X(\mathfrak{x})$ is the ultrafilter generated by the sets

$$\{\{\mathfrak{a} \mid \mathfrak{a} \to x\} \mid x \in A\} \qquad (A \in \mathfrak{x}),$$

and the Yoneda lemma (Corollary 1.11) states that it converges to $\mathcal{A} \in \hat{X}$ precisely if $\mathfrak{x} \in \mathcal{A}$.

We have maps

$$\Phi_X : P(UX) \longrightarrow FX,\ \mathcal{A} \longmapsto \bigcap \mathcal{A} \qquad \text{and} \qquad \Pi_X : FX \longrightarrow P(UX),\ \mathfrak{f} \longmapsto \{\mathfrak{x} \in UX \mid \mathfrak{f} \subseteq \mathfrak{x}\}.$$

where $P(UX)$ denotes the powerset of UX and FX the set of all (possibly improper) filters on X. Clearly, we have $\mathfrak{f} = \Phi_X(\Pi_X(\mathfrak{f}))$ and $\mathcal{A} \subseteq \Pi_X(\Phi_X(\mathcal{A}))$ for $\mathfrak{f} \in FX$ and $\mathcal{A} \in P(UX)$. Furthermore, $\mathcal{A} = \Pi_X(\Phi_X(\mathcal{A}))$ if and only if $\mathcal{A}$ is Zariski-closed. We let F_0X denote the set of all filters on the lattice τ of open sets of a topological space X, and F_1X the set of all filters on the lattice σ of closed sets of X. For each filter $\mathfrak{f}$ on X we can consider $\mathfrak{f} \cap \tau \in F_0X$ and $\mathfrak{f} \cap \sigma \in F_1X$, and $\mathfrak{f}$ is determined by this restriction precisely if $\mathfrak{f}$ has a basis of open respectively closed sets. In [HT08] we showed that $\mathfrak{f} = \bigcap \mathcal{A}$ has a basis of open sets if and only if $\mathcal{A}$ is down-closed, and $\mathfrak{f}$ has a basis of closed sets if and only if $\mathcal{A}$ is up-closed. Hence

$$\hat{X} \cong F_0X \qquad \text{and} \qquad \{\mathcal{A} \subseteq UX \mid \mathcal{A} \text{ is Zariski-closed and up-closed}\} \cong F_1X,$$

and the first homeomorphism we also denote as $\Phi_X : \hat{X} \longrightarrow F_0X$, $\mathcal{A} \longmapsto (\bigcap \mathcal{A}) \cap \tau$. Let $B(\mathcal{B}, \{0\})$ be a basic open set of the topology of $\hat{X}$. Since $B(\mathcal{B}, \{0\}) = B(\uparrow\mathcal{B}, \{0\})$, we can assume that $\mathcal{B}$ is up-closed. Hence, under the bijections above, $F_0(X)$ has

$$\{\mathfrak{f} \in F_0(X) \mid \exists A \in \mathfrak{f}, B \in \mathfrak{g} \,.\, A \cap B = \emptyset\} \qquad\qquad (\mathfrak{g} \in F_1(X))$$

as basic open sets. Clearly, it is enough to consider $\mathfrak{g} = \dot{B}$ the principal filter induced by a closed set B, so that all sets

$$\{\mathfrak{f} \in F_0(X) \mid \exists A \in \mathfrak{f} \,.\, A \cap B = \emptyset\} = \{\mathfrak{f} \in F_0(X) \mid X \setminus B \in \mathfrak{f}\} \qquad\qquad (B \subseteq X \text{ closed})$$

form a basis for the topology on $F_0(X)$. We have shown that our presheaf space $\hat{X}$ is homeomorphic to the filter space $F_0(X)$ considered in [Esc97]. Furthermore, for a continuous map $f : X \longrightarrow Y$, $f^{-1} : \hat{Y} \longrightarrow \hat{X}$ corresponds to $f^{-1} : F_0Y \longrightarrow F_0X$, $\mathfrak{g} \longmapsto \{f^{-1}(B) \mid B \in \mathfrak{g}\}$ in the sense that the diagram

$$\begin{tikzcd} \hat{Y} \arrow[r, "\Phi_Y"] \arrow[d, "f^{-1}"'] & F_0Y \arrow[d, "f^{-1}"] \\ \hat{X} \arrow[r, "\Phi_X"'] & F_0X \end{tikzcd}$$

commutes. Hence, since $\hat{f} \dashv f^{-1}$ as well as $F_0 f \dashv f^{-1}$, $\Phi = (\Phi_X)_X$ is a natural isomorphism from $\widehat{(-)} : \mathsf{Top}_0 \longrightarrow \mathsf{Top}_0$ to $F_0 : \mathsf{Top}_0 \longrightarrow \mathsf{Top}_0$. Since $\Phi_X(y(x)) = \{U \in \tau \mid x \in U\}$ is the neighborhood filter of $x \in X$, the monad $\mathbb{I} = (\widehat{(-)}, y, y^{-1})$ is isomorphic to the filter monad on Top_0 considered in [Esc97].

2.6. **Cocomplete $\mathcal{T}$-categories are algebras over Set and V-$\mathsf{Cat}_{\mathrm{sep}}$.** We are now aiming to prove that the forgetful functor

$$G : \mathcal{T}\text{-}\mathsf{Cocont}_{\mathrm{sep}} \longrightarrow \mathsf{Set}$$

is monadic. Clearly, G has a left adjoint given by the composite

$$\mathsf{Set} \xrightarrow{\text{discrete}} \mathcal{T}\text{-}\mathsf{Cat}_{\mathrm{sep}} \xrightarrow{\widehat{(-)}} \mathcal{T}\text{-}\mathsf{Cocont}_{\mathrm{sep}}.$$

Furthermore, we have the following elementary facts.

Lemma 2.16. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ be $\mathcal{T}$-functors with $f \dashv g$ where X, Y are L-separated.*

(1) *The following assertions are equivalent.*
  (i) *f is an epimorphism in $\mathcal{T}$-$\mathsf{Cat}_{\mathrm{sep}}$.*
  (ii) *$f \cdot g = 1_Y$.*
  (iii) *f is a split epimorphism in $\mathcal{T}$-$\mathsf{Cat}_{\mathrm{sep}}$.*
(2) *The following assertions are equivalent.*
  (i) *f is a monomorphism in $\mathcal{T}$-$\mathsf{Cat}_{\mathrm{sep}}$.*
  (ii) *$g \cdot f = 1_X$.*
  (iii) *f is a split monomorphism in $\mathcal{T}$-$\mathsf{Cat}_{\mathrm{sep}}$.*

Proof. From $f \dashv g$ we obtain $f \cdot g \cdot f = f$. If f is an epimorphism in $\mathcal{T}$-$\mathsf{Cat}_{\mathrm{sep}}$, then $f \cdot g = 1_Y$; if f is a monomorphism in $\mathcal{T}$-$\mathsf{Cat}_{\mathrm{sep}}$, then $g \cdot f = 1_X$. □

Corollary 2.17. *G reflects isomorphisms.*

Proof. If $f : X \longrightarrow Y$ in $\mathcal{T}$-$\mathsf{Cocont}_{\mathrm{sep}}$ is bijective, then f is an isomorphism in $\mathcal{T}$-$\mathsf{Cat}_{\mathrm{sep}}$ and therefore also in $\mathcal{T}$-$\mathsf{Cocont}_{\mathrm{sep}}$. □

In order to conclude that G is monadic, it is left to show that $\mathcal{T}$-$\mathsf{Cocont}_{\mathrm{sep}}$ has and G preserves coequaliser of G-equivalence relations (see, for instance, [MS04, Corollary 2.7]). Hence, let $\pi_1, \pi_2 : R \rightrightarrows X$ in $\mathcal{T}$-$\mathsf{Cocont}_{\mathrm{sep}}$ be an equivalence relation in Set, where π_1 and π_2 are the projection maps. Let $q : X \longrightarrow Q$ be its coequaliser in $\mathcal{T}$-Cat. The following fact will be crucial in the sequel:

$$(\ddagger) \qquad \hat{R} \underset{\widehat{\pi_2}}{\overset{\widehat{\pi_1}}{\rightrightarrows}} \hat{X} \xrightarrow{\hat{q}} \hat{Q} \quad \text{is a split fork in } \mathcal{T}\text{-}\mathsf{Cat}_{\mathrm{sep}}.$$

The splitting here is given by $q^{-1} : \hat{Q} \longrightarrow \hat{X}$ and $\pi_1^{-1} : \hat{X} \longrightarrow \hat{R}$. First note that, since both π_1 and q are surjective, we have $\hat{q} \cdot q^{-1} = 1$ and $\widehat{\pi_1} \cdot \pi_1^{-1} = 1$. Hence, in order to obtain ($\ddagger$), we need to show

$$q^{-1} \cdot \hat{q} = \widehat{\pi_2} \cdot \pi_1^{-1}.$$

Note that we have $\hat{q} = \hat{q} \cdot \widehat{\pi_1} \cdot \pi_1^{-1} = \hat{q} \cdot \widehat{\pi_2} \cdot \pi_1^{-1}$, and therefore

$$q^{-1} \cdot \hat{q} = q^{-1} \cdot \hat{q} \cdot \widehat{\pi_2} \cdot \pi_1^{-1} \geq \widehat{\pi_2} \cdot \pi_1^{-1}.$$

We will give a proof for ($\ddagger$) at the end of this subsection, and show first how ($\ddagger$) can be used to prove monadicity of G. Observe first that, being a split fork,

$$\hat{R} \underset{\widehat{\pi_2}}{\overset{\widehat{\pi_1}}{\rightrightarrows}} \hat{X} \xrightarrow{\hat{q}} \hat{Q}$$

is a coequaliser diagram in $\mathcal{T}$-Cat and $\mathcal{T}$-$\mathsf{Cat}_{\mathrm{sep}}$. Hence, there is a $\mathcal{T}$-functor $\mathrm{Sup}_Q : \hat{Q} \longrightarrow Q$ with $\mathrm{Sup}_Q \cdot \hat{q} = q \cdot \mathrm{Sup}_X$ and $\mathrm{Sup}_Q \cdot y_Q = 1_Q$. The situation is depicted below.

$$\begin{tikzcd}
R \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] \arrow[d, "y_R"'] & X \arrow[r, "q"] \arrow[d, "y_X"'] & Q \arrow[d, "y_Q"] \arrow[dd, bend left, "1_Q"] \\
\hat{R} \arrow[r, shift left, "\widehat{\pi_1}"] \arrow[r, shift right, "\widehat{\pi_2}"'] \arrow[d, "\mathrm{Sup}_R"'] & \hat{X} \arrow[r, "\hat{q}"] \arrow[d, "\mathrm{Sup}_X"'] & \hat{Q} \arrow[d, dotted, "\mathrm{Sup}_Q"] \\
R \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X \arrow[r, "q"] & Q
\end{tikzcd}$$

We conclude that Q is L-separated and cocomplete, and $q : X \longrightarrow Q$ is cocontinuous. Next we show that

$$R \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X \xrightarrow{q} Q$$

is indeed a coqualiser diagram in $\mathcal{T}$-$\mathsf{Cocont}_{\mathrm{sep}}$. Note that

$$\hat{R} \underset{\widehat{\pi_2}}{\overset{\widehat{\pi_1}}{\rightrightarrows}} \hat{X} \xrightarrow{\hat{q}} \hat{Q}$$

is a coequaliser diagram in $\mathcal{T}$-$\mathsf{Cocont}_{\mathrm{sep}}$ since $\widehat{(-)} : \mathcal{T}\text{-}\mathsf{Cat} \longrightarrow \mathcal{T}\text{-}\mathsf{Cocont}_{\mathrm{sep}}$ is left adjoint. Let $h : X \longrightarrow Y$ be a cocontinuous $\mathcal{T}$-functor with cocomplete codomain such that $h \cdot \pi_1 = h \cdot \pi_2$. Then there exists a cocontinuous $\mathcal{T}$-functor $f : \hat{Q} \longrightarrow Y$ such that $f \cdot \hat{q} = h \cdot \mathrm{Sup}_X$. We consider now $f \cdot y_Q : Q \longrightarrow Y$. Then

$$f \cdot y_Q \cdot q = f \cdot \hat{q} \cdot y_X = h \cdot \mathrm{Sup}_X \cdot y_X = h.$$

Furthermore,

$$\begin{aligned}
\mathrm{Sup}_Y \cdot \hat{f} \cdot \widehat{y_Q} \cdot \hat{q} &= f \cdot \mathrm{Sup}_{\hat{Q}} \cdot \widehat{y_Q} \cdot \hat{q} \\
&= f \cdot \hat{q} && (\mathrm{Sup}_{\hat{Q}} = \mu_Q \text{ the multiplication of the monad } \mathbb{I}) \\
&= h \cdot \mathrm{Sup}_X \\
&= f \cdot y_Q \cdot q \cdot \mathrm{Sup}_X \\
&= f \cdot y_Q \cdot \mathrm{Sup}_Q \cdot \hat{q},
\end{aligned}$$

and therefore $\mathrm{Sup}_Y \cdot \widehat{f \cdot y_Q} = f \cdot y_Q \cdot \mathrm{Sup}_Q$, i.e. $f \cdot y_Q$ is cocontinuous.

Remark 2.18. Being cocontinuous, $f \cdot y_Q$ is left adjoint. In fact, one can directly show $f \cdot y_Q \dashv q \cdot l$, where $l : Y \longrightarrow X$ is right adjoint to $h : X \longrightarrow Y$. To do so, let $g : Y \longrightarrow \hat{Q}$ be right adjoint to $f : \hat{Q} \longrightarrow Y$. Then $y_X \cdot l = q^{-1} \cdot g$, and therefore

$$g = \hat{q} \cdot y_X \cdot l \qquad \text{and} \qquad l = \mathrm{Sup}_X \cdot q^{-1} \cdot g.$$

Hence, we have

$$f \cdot y_Q \cdot q \cdot l = f \cdot \hat{q} \cdot y_X \cdot l = f \cdot g \leq 1_Y$$

and

$$q \cdot l \cdot f \cdot y_Q = q \cdot \mathrm{Sup}_X \cdot q^{-1} \cdot g \cdot f \cdot y_Q \geq q \cdot \mathrm{Sup}_X \cdot q^{-1}\, y_Q = \mathrm{Sup}_Q \cdot \hat{q} \cdot q^{-1}\, y_Q = 1_Q.$$

Finally, we prove (‡). Let $\pi_1, \pi_2 : R \rightrightarrows X$ be an equivalence relation in Set, and $q : X \longrightarrow Q$ its quotient. We typically write $x \sim x'$ for $(x, x') \in R$. Furthermore, for $\mathfrak{x}, \mathfrak{x}' \in TX$ we write $\mathfrak{x} \sim \mathfrak{x}'$ whenever the pair $(\mathfrak{x}, \mathfrak{x}')$ belongs to the kernel relation of Tq. Since T has (BC), we have

$$\mathfrak{x} \sim \mathfrak{x}' \iff \exists \mathfrak{w} \in TR \,.\, (T\pi_1(\mathfrak{w}) = \mathfrak{x}) \,\&\, (T\pi_2(\mathfrak{w}) = \mathfrak{x}').$$

Furthermore, we have to warn the reader that, when talking about an equivalence relation $\pi_1, \pi_2 : R \rightrightarrows X$ in $\mathcal{T}$-Cat or $\mathcal{T}$-$\mathsf{Cat}_{\mathrm{sep}}$, we always include that the canonical map $R \hookrightarrow X \times X$ is an embedding (and not just a monomorphism). Clearly, a sub-$\mathcal{T}$-category $R \hookrightarrow X \times X$ is an equivalence relation in $\mathcal{T}$-Cat respectively in $\mathcal{T}$-$\mathsf{Cat}_{\mathrm{sep}}$ if and only if it is an equivalence relation in Set.

Lemma 2.19. *Let $X = (X, a)$ be a L-separated $\mathcal{T}$-category and $\pi_1, \pi_2 : R \rightrightarrows X$ be an equivalence relation in $\mathcal{T}$-$\mathsf{Cat}_{\mathrm{sep}}$. In addition, assume that $\pi_2 \dashv \rho_2$*[1]*. Then, for all $\mathfrak{x}, \mathfrak{x}' \in TX$ with $\mathfrak{x} \sim \mathfrak{x}'$ and all $x' \in X$, there exists $x \in X$ such that $x \sim x'$ and $a(\mathfrak{x}', x') \leq a(\mathfrak{x}, x)$.*

Proof. Since π_2 is surjective, we have $\pi_2 \cdot \rho_2 = 1_X$. Let $\mathfrak{w} \in TR$ such that $T\pi_1(\mathfrak{w}) = \mathfrak{x}$ and $T\pi_2(\mathfrak{w}) = \mathfrak{x}'$. Then

$$\begin{aligned}
a(\mathfrak{x}', x') &= a(T\pi_2(\mathfrak{w}), x') \\
&= a \times a(\mathfrak{w}, \rho_2(x')) && (\rho_2(x') = (x, x') \text{ for some } x \sim x') \\
&= a(\mathfrak{x}, x) \wedge a(\mathfrak{x}', x'),
\end{aligned}$$

hence $a(\mathfrak{x}', x') \leq a(\mathfrak{x}, x)$. □

[1]Note that, since R is symmetric, π_1 is left adjoint precisely if π_2 is so.

Our next goal is to describe the quotient $q : X \longrightarrow Q$ of $\pi_1, \pi_2 : R \rightrightarrows X$ in $\mathcal{T}$-Cat. In general, the quotient structure in $\mathcal{T}$-Cat is difficult to handle, see [Hof05] for details. The situation is much better in $\mathcal{T}$-Gph, the category of $\mathcal{T}$-graphs and $\mathcal{T}$-graph morphisms. Here a $\mathcal{T}$-graph is a pair (X, a) consisting of a set X and a $\mathcal{T}$-relation $a : X \longrightarrow\!\!\!

Theorem 2.22. *The forgetful functor* $G : \mathcal{T}\text{-}\mathsf{Cocont}_{\mathrm{sep}} \longrightarrow \mathsf{Set}$ *is monadic. As a consequence,* $\mathcal{T}\text{-}\mathsf{Cocont}_{\mathrm{sep}}$ *is cocomplete.*

Theorem 2.23. *The forgetful functor* $\mathrm{S} : \mathcal{T}\text{-}\mathsf{Cocont}_{\mathrm{sep}} \longrightarrow \mathsf{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ *is monadic.*

Proof. Clearly, S has a left adjoint and reflects isomorphisms. We show that S preserves coequalisers of S-contractible equivalence relations (see [MS04, Theorem 2.7]). Hence, let $\pi_1, \pi_2 : R \rightrightarrows X$ in $\mathcal{T}\text{-}\mathsf{Cocont}_{\mathrm{sep}}$ be a contractible equivalence relation in $\mathsf{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$. Then $\pi_1, \pi_2 : R \rightrightarrows X$ is also an equivalence relation in Set, and hence its coequaliser $q : X \longrightarrow Q$ in Set underlies its coequaliser $q : X \longrightarrow Q$ in $\mathcal{T}\text{-}\mathsf{Cocont}_{\mathrm{sep}}$, moreover, $q : X \longrightarrow Q$ is a proper $\mathcal{T}$-functor. Consequentely, the underlying V-functor $q : X \longrightarrow Q$ is proper as well, and therefore a coequaliser of $\pi_1, \pi_2 : R \rightrightarrows X$ in $\mathsf{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$. □

2.7. **Densely injective $\mathcal{T}$-categories.** Another well-known result in Topology is

Theorem 2.24. *The algebras for the proper filter monad on* Top_0 *are precisely the* T_0*-spaces which are injective with respect to dense embeddings.*

In the language of convergence, a continuous map $f : X \longrightarrow Y$ is dense whenever

$$\forall y \in Y\ \exists \mathfrak{x} \in TX\,.\, Uf(\mathfrak{x}) \to y,$$

and we observe that $Uf(\mathfrak{x}) \to y \iff \mathfrak{x}\ f_*\ y$. This suggests the following

Definition 2.25. A $\mathcal{T}$-module $\varphi : X \longrightarrow\!\!\!\!\!\!\circ\ Y$ is called *inhabited* if

$$k \leq \bigwedge_{y \in Y} \bigvee_{\mathfrak{x} \in TX} \varphi(\mathfrak{x}, y).$$

A $\mathcal{T}$-functor $f : X \longrightarrow Y$ is called *dense* if f_* is inhabited.

We hasten to remark that f^* is inhabited, for each $\mathcal{T}$-functor $f : X \longrightarrow Y$. Hence

Proposition 2.26. *Each left adjoint* $\mathcal{T}$*-functor is dense.*

By definition, $\varphi : X \longrightarrow\!\!\!\!\!\!\circ\ Y$ is inhabited if and only if $k \leq \varphi \circ k$, where k denotes the constant $\mathcal{T}$-relation $k : T1 \times Z \longrightarrow \mathsf{V}$ with value $k \in \mathsf{V}$, for a set Z. Consequentely, with $\varphi : X \longrightarrow\!\!\!\!\!\!\circ\ Y$ and $\psi : Y \longrightarrow\!\!\!\!\!\!\circ\ Z$ also $\psi \circ \varphi$ is inhabited. Furthermore, if φ is inhabited and $\varphi \leq \varphi'$, then φ' is inhabited too. Note also that each surjective $\mathcal{T}$-functor is dense.

Proposition 2.27. *Consider the (up to ≅) commutative triangle*

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "g"] \arrow[dr, phantom, shift right=3ex, "\cong"] & \\ Y \arrow[r, "h"'] & Z \end{tikzcd}$$

of $\mathcal{T}$*-functors. Then the following assertions hold.*

(1) *If* h *and* f *are dense, then so is* g*.*
(2) *If* g *is dense and* h *is fully faithful, then* f *is dense.*
(3) *If* g *is dense, then* h *is dense.*

Proof. (1) is obvious since inhabited $\mathcal{T}$-modules compose. To see (2), note that from $h_* \circ f_* = g_*$ follows $f_* = h^* \circ g_*$, hence f_* is inhabited and therefore f is dense. (3) can be shown in a similar way. □

By the Yoneda Lemma (Corollary 1.11), for each $\psi \in \hat{X}$ we have

$$\bigvee_{\mathfrak{x}\in TX} (y_X)_*(\mathfrak{x},\psi) = \bigvee_{\mathfrak{x}\in TX} \psi(\mathfrak{x}).$$

Hence, with

$$X^+ = \{\psi \in \hat{X} \mid \psi \text{ is inhabited}\}$$

and the structure being inherited from $\hat{X}$, the restriction $y_X : X \longrightarrow X^+$ of the Yoneda embedding is dense. Furthermore, for a $\mathcal{T}$-module $\varphi : X \longrightarrow\!\!\!\!\!\circ\; Y$ we have

$$\varphi \text{ is inhabited} \iff \ulcorner\varphi\urcorner : Y \longrightarrow \hat{X} \text{ factors through } X^+ \hookrightarrow \hat{X}.$$

We call a $\mathcal{T}$-category X *densely injective* if, for all $\mathcal{T}$-functors $f : A \longrightarrow X$ and fully faithful and dense $\mathcal{T}$-functors $i : A \longrightarrow B$, there exists a $\mathcal{T}$-functor $g : B \longrightarrow X$ such that $g \cdot i \cong f$. A $\mathcal{T}$-category X is called *inhabited-cocomplete* if X has all φ-weighted colimits where φ is inhabited. Note that, when passing from

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\varphi"', "\circ"] & X \\ B & \end{tikzcd} \qquad \text{to} \qquad \begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "\varphi\circ f^*"', "\circ"] & X, \\ B & \end{tikzcd}$$

with φ also $\varphi \circ f^*$ is inhabited, so that it is enough to consider $f = 1_X$ in the definition of inhabited-cocomplete. A $\mathcal{T}$-functor $f : X \longrightarrow Y$ is *inhabited-cocontinuous* if f preserves all φ-weighted colimits where φ is inhabited. Let $\mathcal{T}$-$\mathsf{ICocont}$ denote the category of inhabited-cocomplete $\mathcal{T}$-categories and inhabited-cocontinuous $\mathcal{T}$-functors between them, and $\mathcal{T}$-$\mathsf{ICocont}_{\text{sep}}$ denotes its full subcategory of L-separated $\mathcal{T}$-categories.

Lemma 2.28. *For each $\mathcal{T}$-category X, X^+ is closed under inhabited colimits in $\hat{X}$. In particular, X^+ is inhabited-cocomplete.*

Proof. We consider the diagram

$$\begin{tikzcd} X^+ \arrow[r, "\iota"] \arrow[d, "\varphi"', "\circ"] & \hat{X}, \\ Y & \end{tikzcd}$$

with $\iota : X^+ \hookrightarrow \hat{X}$ being the inclusion functor and φ inhabited. Its colimit in $\hat{X}$ is given by

$$y_X^{-1} \cdot \ulcorner\varphi \circ \iota^*\urcorner : Y \longrightarrow \hat{X}.$$

Hence, for any $y \in Y$ and $\mathfrak{x} \in TX$,

$$y_X^{-1} \cdot \ulcorner\varphi \circ \iota^*\urcorner(y)(\mathfrak{x}) = \varphi \circ \iota^*(T\, y_X(\mathfrak{x}), y) \geq \varphi \cdot T\iota^\circ(T\, y_X(\mathfrak{x}), y) = \varphi(T\, y_X(\mathfrak{x}), y) = \varphi \circ (y_X)_*(\mathfrak{x}, y),$$

where in the last two expessions we consider $y_X : X \longrightarrow X^+$. Since $\varphi \circ (y_X)_*$ is inhabited, the $\mathcal{T}$-functor $y_X^{-1} \cdot \ulcorner\varphi \circ \iota^*\urcorner : Y \longrightarrow \hat{X}$ takes values in X^+ and the assertion follows. □

From the observations made so far it is now clear that we have the same series of results for densely injective and inhabited-cocomplete $\mathcal{T}$-categories as we proved for injective and cocomplete $\mathcal{T}$-categories.

Theorem 2.29. *Let X be $\mathcal{T}$-category.*

(1) *Each $\psi \in X^+$ is an inhabited colimit of representables.*
(2) *The following assertions are equivalent.*
    (i) *X is densely injective.*

(ii) $y_X : X \longrightarrow X^+$ *has a left inverse* $\mathrm{Sup}_X^+ : X^+ \longrightarrow X$.
(iii) $y_X : X \longrightarrow X^+$ *has a left adjoint* $\mathrm{Sup}_X^+ : X^+ \longrightarrow X$.
(iv) X *is inhabited-cocomplete.*

(3) *Composition with* $y_X : X \longrightarrow X^+$ *defines an equivalence*

$$\mathcal{T}\text{-}\mathsf{ICocont}(X^+, Y) \longrightarrow \mathcal{T}\text{-}\mathsf{Cat}(X, Y)$$

of ordered sets, for each inhabited-cocomplete $\mathcal{T}$*-category* Y.

We have just seen that the inclusion functor $\mathcal{T}\text{-}\mathsf{ICocont}_{\mathrm{sep}} \hookrightarrow \mathcal{T}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ has a left adjoint $(-)^+ : \mathcal{T}\text{-}\mathsf{Cat}_{\mathrm{sep}} \longrightarrow \mathcal{T}\text{-}\mathsf{ICocont}_{\mathrm{sep}}$. In fact, since for each $\mathcal{T}$-functor $f : X \longrightarrow Y$ and each $\psi \in X^+$ we have $\hat{f}(\psi) \in Y^+$, the $\mathcal{T}$-functor $f^+ : X^+ \longrightarrow Y^+$ is just the restriction of $\hat{f}$ to X^+ and Y^+. With a similar proof as for Proposition 2.12 one shows

Proposition 2.30. *Let* $f : X \longrightarrow Y$ *be a* $\mathcal{T}$*-functor between inhabited-cocomplete* $\mathcal{T}$*-categories. Then the following assertions are equivalent.*

(i) f *is inhabited-cocontinuous.*
(ii) *We have* $f \cdot \mathrm{Sup}_X^+ \cong \mathrm{Sup}_Y^+ \cdot \hat{f}$.

$$\begin{tikzcd} X^+ \arrow[r, "f^+"] \arrow[d, "\mathrm{Sup}_X^+"'] \arrow[dr, phantom, "\cong"] & Y^+ \arrow[d, "\mathrm{Sup}_Y^+"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

The induced monad on $\mathcal{T}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ we denote as $\mathbb{I}^+ = ((-)^+, y, \mu)$. With the same arguments used in 2.4 one verifies that $\mathbb{I}^+$ is of Kock-Zöberlein type. We conclude

Theorem 2.31. $\mathcal{T}\text{-}\mathsf{ICocont}_{\mathrm{sep}} \cong (\mathcal{T}\text{-}\mathsf{Cat}_{\mathrm{sep}})^{\mathbb{I}^+}$.

Finally, we consider a $\mathcal{T}$-functor $f : X \longrightarrow Y$. Then $\hat{f} : \hat{X} \longrightarrow \hat{Y}$ has a right adjoint $f^{-1} : \hat{Y} \longrightarrow \hat{X}$ given by $f^{-1}(\psi) = \psi \circ f_*$. Clearly, if f is dense, then f^{-1} can be restricted to $f^{-1} : Y^+ \longrightarrow X^+$ and we have $f^+ \dashv f^{-1}$. In particular, $y_X^+ : X^+ \longrightarrow X^{++}$ is left adjoint to $y_X^{-1} : X^{++} \longrightarrow X^+$, which tells us that the multiplication μ_X of $\mathbb{I}^+$ is also given by y_X^{-1}.

Proposition 2.32. *The following are equivalent for a* $\mathcal{T}$*-functor* $f : X \longrightarrow Y$.

(i) f *is dense.*
(ii) f^+ *is left adjoint.*
(iii) f^+ *is dense.*

If f *is a inhabited-cocontinuous* $\mathcal{T}$*-functor between inhabited cocomplete* $\mathcal{T}$*-categories, then any of the conditions above is equivalent to*

(iv) f *is left adjoint.*

Proof. The implication (i)⇒(ii) we proved above, (ii)⇒(iii) and (iv)⇒(i) follow from Proposition 2.26 and (iii)⇒(i) from Proposition 2.27. Finally, (ii)⇒(iv) can be shown as (iii)⇒(i) of Proposition 2.12. □

Finally, thanks to the considerations made above, also

$$\begin{tikzcd} R^+ \arrow[r, shift left, "\pi_1^+"] \arrow[r, shift right, "\pi_2^+"'] & X^+ \arrow[r, "q^+"] & Q^+ \end{tikzcd}$$

is a split fork in $\mathcal{T}\text{-}\mathsf{Cat}_{\mathrm{sep}}$. Consequentely, with the same proof as in 2.6, we conclude that the forgetful functor $\mathcal{T}\text{-}\mathsf{ICocont}_{\mathrm{sep}} \longrightarrow \mathsf{Set}$ is monadic.

Remark 2.33. The results of this subsection suggest that in the future one should consider cocompleteness with respect to a class Φ of $\mathcal{T}$-modules, i.e. use [KS05]. Besides the classes considered in this paper, another reasonable choice is Φ being the class of all right adjoint $\mathcal{T}$-modules. In fact, this case is studied in [CH07, HT08] where the Φ-cocomplete $\mathcal{T}$-categories are called L-complete (resp. Cauchy-complete). Furthermore, it is easy to see that any $\mathcal{T}$-functor preserves colimits indexed by a right adjoint weight, so that the category of L-separated and Φ-cocomplete $\mathcal{T}$-categories and Φ-cocontinuous $\mathcal{T}$-functors is precisely the full subcategory $\mathcal{T}\text{-}\mathsf{Cat}_{\mathrm{cpl}}$ of L-complete and L-separated $\mathcal{T}$-categories of $\mathcal{T}\text{-}\mathsf{Cat}$. But be aware that, thought with the same techniques we obtain monadicity of $\mathcal{T}\text{-}\mathsf{Cat}_{\mathrm{cpl}}$ over $\mathcal{T}\text{-}\mathsf{Cat}_{\mathrm{sep}}$, the proof in 2.6 does not work here. The problem is that the $\mathcal{T}$-functor $q^{-1} : \hat{Q} \longrightarrow \hat{X}$ does not restrict to $\tilde{Q}$ and $\tilde{X}$[2] since q_* is in general not right adjoint. This is not a surprise, since, for instance, any ordered set is L-complete, hence the category of L-complete and L-separated ordered set coincides with the category of anti-symmetric ordered sets (and monotone maps). The canonical forgetful functor from this category to Set is surely not monadic. Furthermore, the canonical forgetful functor from the category of L-complete and L-separated topological spaces (= sober spaces) and continuous maps to Set is also not monadic.

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Appl Categor Struct (2010) 18:259–287
DOI 10.1007/s10485-008-9169-9

Lawvere Completion and Separation Via Closure

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Received: 30 December 2007 / Accepted: 1 October 2008 / Published online: 1 November 2008
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Abstract For a quantale V, first a closure-theoretic approach to completeness and separation in V-categories is presented. This approach is then generalized to $\mathcal{T}$-categories, where $\mathcal{T}$ is a topological theory that entails a set monad $\mathbb{T}$ and a compatible $\mathbb{T}$-algebra structure on V.

Keywords Quantale · V-category · Monad · Topological theory · Module · Yoneda lemma · Closure operator · Completeness

Mathematics Subject Classifications (2000) 18A05 · 18D15 · 18D20 · 18B35 · 18C15 · 54B30 · 54A20

1 Introduction

Bill Lawvere's 1973 milestone paper "Metric spaces, generalized logic, and closed categories" helped us to detect categorical structures in previously unexpected

Dedicated to Bill Lawvere at the occasion of his seventieth birthday.

The first author acknowledges partial financial assistance by Unidade de Investigação e Desenvolvimento Matemática e Aplicações da Universidade de Aveiro/FCT.

The second author acknowledges partial financial assistance by the Natural Sciences and Engineering Council of Canada.

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surroundings. His revolutionary idea was not only to regard individual metric spaces as categories (enriched over the monoidal-closed category given by the non-negative extended real half-line, with arrows provided by $\geq$ and tensor by $+$), but also to expose the purely categorical nature of the key concept of the theory, Cauchy completeness. The first step to this end was to disregard metric conditions that actually obscure the categorical intuition. In fact, once one has dropped the symmetry requirement it seems much more natural to regard the metric d of a space X as the categorical hom and, given a Cauchy sequence (a_n) in X, to associate with it the pair of functions

$$\varphi(x) = \lim d(a_n, x) \qquad \text{and} \qquad \psi(x) = \lim d(x, a_n).$$

Lawvere's great insight was to expose these functions as pairs of adjoint (bi)modules whose representability as

$$\varphi(x) = d(a, x) \qquad \text{and} \qquad \psi(x) = d(x, a)$$

is facilitated precisely by a limit a for (a_n). Hence, a new notion of completeness for categories enriched over any symmetric monoidal-closed category V was born. Also in the enriched category context it is often referred to as Cauchy completeness. But since Lawvere's brilliant notion entails no sequences at all, just the representability requirement for bimodules, this name seems to be far-fetched and, contrary to popular belief, was in fact not proposed in his paper. Hence, here we use *L-completeness* instead.

In the first part of this paper we give a quick introduction to V-category theory (see [12]) in the special case of a commutative unital quantale V, focussing on the themes of L-completion and *L-separation*. We are not aware of an explicit prior occurrence of the latter notion, and both themes are treated with the help of a new closure operator that arises most naturally in the 2-category V-Cat, as follows. Call a V-functor $m : M \longrightarrow X$ *L-dense* if $f \cdot m = g \cdot m$ implies $f \cong g$ for all V-functors $f, g : X \longrightarrow Y$; the *L-closure* of a subobject M of X is then the largest subobject $\overline{M}$ of X for which $M \longrightarrow \overline{M}$ is L-dense. For L-separated V-categories, L-dense simply means epimorphism. The L-separated reflection of a V-category X is its image under the Yoneda functor $y : X \longrightarrow \mathsf{V}^{X^{\mathrm{op}}} = \hat{X}$, and its L-completion is the L-closure of that image in $\hat{X}$.

The main part of the paper is devoted to a substantial generalization of the first part which, however, without the reader's recalling of the more familiar V-category context, may be hard to motivate, especially in view of the considerable additional "technical" difficulties. The quantale V gets augmented by a *topological theory* $\mathcal{T} = (\mathbb{T}, \mathsf{V}, \xi)$ which now entails also a Set-monad $\mathbb{T}$ and a $\mathbb{T}$-algebra structure ξ on V, with suitable compatibility conditions (see [11]). While a V-category X comes with a V-relation $a : X \nrightarrow X$ (given by a function $a : X \times X \longrightarrow \mathsf{V}$), $\mathcal{T}$-categories come with a V-relation $a : TX \nrightarrow X$ making X a lax $\mathbb{T}$-algebra. For $\mathbb{T}$ the ultrafilter monad and $\mathsf{V} = \mathsf{2}$, $\mathcal{T}$-Cat provides Barr's [2] relational description of the category of topological spaces (which, in turn, was based on Manes' [17] description of compact Hausdorff spaces); for the same monad but with V the Lawvere half-line, one obtains Lowen's approach spaces [14], as shown by Clementino and Hofmann [4].

The V-to-$\mathcal{T}$ generalization must necessarily entail the provision of a Yoneda functor for a $\mathcal{T}$-category X. But what is X^{op} supposed to be in this highly asymmetric context? Fortunately, this problem was solved in [5]: the underlying set of X^{op} is

TX, provided with a suitable $\mathcal{T}$-structure. This structure needs to be considered in addition to the free $\mathbb{T}$-algebra structure on TX, leading to the surprising fact that the $\mathcal{T}$-equivalent of the Yoneda functor of the familiar V-context has now two equally important facets. Once one has fully understood this "technical" part of the general theory, it is in fact rather straightforward to extend the V-categorical results on L-completion and L-separation to $\mathcal{T}$-categories, again with the help of the L-closure. We could therefore often keep the proofs in the $\mathcal{T}$-context quite short, especially when no new ideas beyond the initial "Yoneda investment" are needed.

Completeness of V-categories and the induced topology was also investigated by Flagg [9, 10] (who called them V-continuity spaces). Its generalization to (essentially) $\mathcal{T}$-categories was introduced by Clementino and Hofmann [5]. We also refer the reader to Burroni [3], wo presented an alternative approach to the categories of interest in this paper.

2 Preliminaries

2.1 The Quantale V

Throughout the paper we consider a commutative and unital quantale $\mathsf{V} = (\mathsf{V}, \otimes, k)$. Hence, V is a complete lattice with a commutative binary operation $\otimes$ and neutral element k, such that $u \otimes (-)$ preserves suprema, for all $u \in \mathsf{V}$. Consequently, V has an "internal hom" $u \multimap (-)$, given by

$$z \leq u \multimap v \iff z \otimes u \leq v,$$

for all $z, u, v \in \mathsf{V}$. Sometimes we write $v \multimapinv u$ instead of $u \multimap v$. The quantale is *trivial* when $\mathsf{V} = 1$; equivalently, when $k = \bot$ is the bottom element of V. Non-trivial examples of quantales are the two-element chain $\mathsf{2} = (\{0, 1\}, \wedge, 1)$, the extended positive half-line $\mathsf{P}_+ = ([0, \infty]^{\mathrm{op}}, +, 0)$, and $\mathsf{P}_{\max} = ([0, \infty]^{\mathrm{op}}, \max, 0)$; here $[0, \infty]^{\mathrm{op}} = ([0, \infty], \geq)$, with the natural $\geq$. (We will use $\bigvee$, $\bigwedge$ to denote suprema, infima in V, but use sup, inf, max, etc. when we work in $[0, \infty]$ and refer to the natural order $\leq$.)

2.2 V-relations

The category V-Rel has sets as objects, and a morphism $r : X \longrightarrow\!\!\!\!\!\!+\; Y$ is simply a function $r : X \times Y \longrightarrow \mathsf{V}$; its composite with $s : Y \longrightarrow\!\!\!\!\!\!+\; Z$ is given by

$$s \cdot r(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z).$$

There is a functor

$$\mathsf{Set} \longrightarrow \mathsf{V}\text{-}\mathsf{Rel}$$

which maps objects identically and interprets a map $f : X \longrightarrow Y$ as a V-relation $f_\circ : X \longrightarrow\!\!\!\!\!\!+\; Y$:

$$f_\circ(x, y) = \begin{cases} k & \text{if } f(x) = y, \\ \bot & \text{otherwise;} \end{cases}$$

we normally write f instead of $f_\circ$. The functor is faithful precisely when $k > \bot$. The hom-sets of V-Rel carry the pointwise order of V, so that V-Rel becomes an ordered category. In fact, V-Rel is Sup-enriched (with Sup the category if complete lattices and suprema-preserving maps), hence it is a quantaloid. Consequently, for every $r : X \nrightarrow Y$, composition by r in V-Rel from either side has a right adjoint, given by *extensions* and *liftings* respectively:

$$(-) \cdot r \dashv (-) \bullet\!\!- r \qquad\qquad r \cdot (-) \dashv r -\!\!\bullet (-)$$

$$\frac{t \cdot r \leq s}{t \leq s \bullet\!\!- r} \qquad\qquad \frac{r \cdot t \leq s}{t \leq r -\!\!\bullet s}$$

$$\begin{tikzcd} X \arrow[d, "r"'] \arrow[dr, "s", "\leq"'] & \\ Y \arrow[r, "t"'] & Z \end{tikzcd} \qquad\qquad \begin{tikzcd} Y & \\ X \arrow[u, "r"] & Z \arrow[ul, "s"', "\leq"] \arrow[l, "t"] \end{tikzcd}$$

$$s \bullet\!\!- r(y, z) = \bigwedge_{x \in X} s(x, z) \circ\!\!- r(x, y) \qquad r -\!\!\bullet s(z, x) = \bigwedge_{y \in Y} r(x, y) -\!\!\circ s(z, y)$$

V-Rel has a contravariant involution

$$(\text{V-Rel})^{\text{op}} \longrightarrow \text{V-Rel}$$

which maps objects identically and assigns to $r : X \nrightarrow Y$ its opposite relation $r^\circ : Y \nrightarrow X$. When applied to a map $f = f_\circ$, one obtains $f \dashv f^\circ$ in the 2-category V-Rel.

2.3 V-categories

A V-*category* $X = (X, a)$ is a set X with a V-relation $a : X \nrightarrow X$ satisfying $1_X \leq a$, $a \cdot a \leq a$; equivalently,

$$k \leq a(x, x), \qquad\qquad a(x, y) \otimes a(y, z) \leq a(x, z)$$

for all $x, y, z \in X$. A V-functor $f : (X, a) \longrightarrow (Y, b)$ must satisfy $f \cdot a \leq b \cdot f$; equivalently,

$$a(x, y) \leq b(f(x), f(y))$$

for all $x, y \in X$. The resulting category V-Cat is the category Ord of (pre)ordered sets if $\mathsf{V} = \mathsf{2}$, Lawvere's category Met of (pre)metric spaces if $\mathsf{V} = \mathsf{P}_+$ (see [13]), and the category UMet of (pre)ultrametric spaces if $\mathsf{V} = \mathsf{P}_{\max}$. For the trivial quantale one has $\mathsf{1}\text{-Cat} = \mathsf{Set}$. Furthermore, $\mathsf{V} = (\mathsf{V}, -\!\!\circ)$ with its internal hom becomes a V-category.

V-Cat is a symmetric monoidal closed category, with tensor product

$$(X, a) \otimes (Y, b) = (X \times Y, a \otimes b), \quad a \otimes b((x, y), (x', y')) = a(x, x') \otimes b(y, y'),$$

and internal hom

$$(X, a) -\!\!\circ (Y, b) = (\text{V-Cat}(X, Y), [a, b]), \quad [a, b](f, g) = \bigwedge_{x \in X} b(f(x), g(x)).$$

The $\otimes$-neutral object is $E = (E, k)$ (with a singleton set E), which generally must be distinguished from the terminal object $1 = (1, \top)$ in V-Cat. The internal hom describes the pointwise order if $\mathsf{V} = \mathsf{2}$, and the non-symmetrized sup-metric if $\mathsf{V} = \mathsf{P}_+$ or $\mathsf{V} = \mathsf{P}_{\max}$.

2.4 V-modules

The category V-Mod has V-categories as objects and V-modules as morphisms; here a V-module $\varphi : (X, a) \multimap (Y, b)$ is a V-relation $\varphi : X \nrightarrow Y$ with $\varphi \cdot a \leq \varphi$ and $b \cdot \varphi \leq \varphi$. Since always $\varphi = \varphi \cdot 1_X \leq \varphi \cdot a$ and $\varphi = 1_Y \cdot \varphi \leq b \cdot \varphi$, one actually has $\varphi \cdot a = \varphi$ and $b \cdot \varphi = \varphi$ for a V-module $\varphi : X \multimap Y$. In particular, the V-module $a : X \multimap X$ assumes the role of the identity morphism on X in V-Mod, and we write $a = 1_X^*$, in order not to confuse it with 1_X in V-Cat. This notation is extended to arbitrary maps $f : X \longrightarrow Y$ by

$$f_* = b \cdot f \qquad \text{and} \qquad f^* = f^\circ \cdot b,$$

and one easily verifies:

Lemma 2.1 *The following are equivalent for a map $f : X \longrightarrow Y$ between V-categories X and Y:*

(i) *$f : X \longrightarrow Y$ is a V-functor.*
(ii) *f_* is a V-module $f_* : X \multimap Y$.*
(iii) *f^* is a V-module $f^* : Y \multimap X$.*

Hence there are functors which make the following diagram commute.

$$\begin{tikzcd}
\mathsf{V}\text{-}\mathsf{Cat} \arrow[r, "(-)_*"] & \mathsf{V}\text{-}\mathsf{Mod} & (\mathsf{V}\text{-}\mathsf{Cat})^{\mathrm{op}} \arrow[l, "(-)^*"'] \\
\mathsf{Set} \arrow[u] \arrow[r, "(-)_\circ"'] & \mathsf{V}\text{-}\mathsf{Rel} \arrow[u] & \mathsf{Set}^{\mathrm{op}} \arrow[u] \arrow[l, "(-)^\circ"]
\end{tikzcd}$$

Here the vertical full embeddings are given by $X \longmapsto (X, 1_X)$. Just like V-Rel also V-Mod is a quantaloid, with the same pointwise order structure. But not just suprema of V-modules formed in V-Rel are again V-modules, also extensions and liftings. For example, for $\varphi : (X, a) \multimap (Y, b)$, $\psi : (Z, c) \multimap (Y, b)$, the lifting $\varphi \multimapdotbothA \psi$ formed in V-Rel is indeed a V-module $\varphi \multimapdotbothA \psi : (Z, c) \multimap (Y, b)$: from $\psi \cdot c \leq \psi$ and $\varphi \cdot (\varphi \multimapdotbothA \psi) \leq \psi$ one obtains $\varphi \cdot (\varphi \multimapdotbothA \psi) \cdot c \leq \psi$ and then $(\varphi \multimapdotbothA \psi) \cdot c \leq \varphi \multimapdotbothA \psi$; similarly $a \cdot (\varphi \multimapdotbothA \psi) \leq \varphi \multimapdotbothA \psi$. Also the contravariant involution of V-Rel extends to V-Mod (e.g., if $\varphi : X \multimap Y$, then $\varphi : X^{\mathrm{op}} \multimap Y^{\mathrm{op}}$, where $X^{\mathrm{op}} = (X, a^\circ)$ is the usual opposite V-category), and one has the commutative diagram

$$\begin{tikzcd}
(\mathsf{V}\text{-}\mathsf{Mod})^{\mathrm{op}} \arrow[r, "(-)^{\mathrm{op}}"] & \mathsf{V}\text{-}\mathsf{Mod} \\
\mathsf{V}\text{-}\mathsf{Cat} \arrow[u, "(-)^*"] \arrow[r, "(-)^{\mathrm{op}}"'] & \mathsf{V}\text{-}\mathsf{Cat}. \arrow[u, "(-)_*"']
\end{tikzcd}$$

As a quantaloid, V-Mod is in particular a 2-category, and for all $f : X \longrightarrow Y$ in V-Cat one has

$$f_* \dashv f^*$$

in V-Mod. V-Cat inherits its 2-categorical structure from V-Mod via

$$\begin{aligned} f \leq f' :\iff\ & f^* \leq (f')^* \iff \forall x \in X, y \in Y \,.\, b(y, f(x)) \leq b(y, f'(x)) \\ \iff\ & f'_* \leq f_* \iff \forall x \in X, y \in Y \,.\, b(f'(x), y) \leq b(f(x), y) \\ \iff\ & 1^*_X \leq (f')^* \cdot f_* \iff \forall x \in X \,.\, k \leq b(f(x), f'(x)). \end{aligned}$$

Hence, the previous diagram actually shows commuting 2-functors when we add dualization w.r.t. 2-cells (indicated by co) appropriately:

$$\begin{tikzcd} (\mathsf{V\text{-}Mod})^{\mathrm{op}} \arrow[r, "(-)^{\mathrm{op}}"] & \mathsf{V\text{-}Mod} \\ \mathsf{V\text{-}Cat} \arrow[u, "(-)^*"] \arrow[r, "(-)^{\mathrm{op}}"'] & \mathsf{V\text{-}Cat}^{\mathrm{co}} \arrow[u, "(-)_*"'] \end{tikzcd}$$

Of course, V-Cat being a 2-category, there is also a notion of adjointness in V-Cat:

$$\begin{aligned} f \dashv g \text{ in } \mathsf{V\text{-}Cat} \iff\ & f \cdot g \leq 1 \text{ and } 1 \leq g \cdot f \text{ in } \mathsf{V\text{-}Cat} \\ \iff\ & g^* \cdot f^* \leq 1^* \text{ and } 1^* \leq f^* \cdot g^* \text{ in } \mathsf{V\text{-}Mod} \\ \iff\ & g^* \dashv f^* \text{ in } \mathsf{V\text{-}Mod} \\ \iff\ & f_* = g^* \quad (\text{since } f_* \dashv f^* \text{ in } \mathsf{V\text{-}Mod}) \\ \iff\ & \forall x \in X, y \in Y \,.\, a(x, g(y)) = b(f(x), y). \end{aligned}$$

2.5 Yoneda

V-modules give rise to V-functors, as follows.

Proposition 2.2 *The following are equivalent for* V*-relations* $\varphi : X \longrightarrow\!\!\!\!\!\!+\ Y$ *between* V*-categories:*

(i) $\varphi : X \longrightarrow\!\!\!\!\!\!\circ\ Y$ *is a* V*-module.*
(ii) $\varphi : X^{\mathrm{op}} \otimes Y \longrightarrow \mathsf{V}$ *is a* V*-functor.*

With $\varphi = a = 1^*_X : X \longrightarrow\!\!\!\!\!\!\circ\ X$ we obtain in particular the V-functor $a : X^{\mathrm{op}} \otimes X \longrightarrow \mathsf{V}$ whose mate $\ulcorner a \urcorner$ is the Yoneda-V-functor

$$y : X \longrightarrow \hat{X} := (X^{\mathrm{op}} \multimap \mathsf{V}),\ x \longmapsto a(-, x).$$

The structure $\hat{a}$ of $\hat{X}$ is given by

$$\hat{a}(f, f') = \bigwedge_{x \in X} f(x) \multimap f'(x).$$

Lemma 2.3 *For all $x \in X$ and $f \in \hat{X}$, $\hat{a}(y(x), f) = f(x)$.*

One calls a V-functor $f : (X, a) \longrightarrow (Y, b)$ *fully faithful* if $a(x, x') = b(f(x), f(x'))$ for all $x, x' \in X$; equivalently, if $1_X^* = f^* \cdot f_*$ (since $f^* \cdot f_* = f^\circ \cdot b \cdot f$), or just $1_X^* \geq f^* \cdot f_*$ (since the other inequality comes for free).

Corollary 2.4 *$y : X \longrightarrow \hat{X}$ is fully faithful.*

2.6 L-separation

For V-functors $f, g : Z \longrightarrow X$ we write $f \cong g$ if $f \leq g$ and $g \leq f$; equivalently, if $f^* = g^*$, or $f_* = g_*$. We call X *L-separated* if $f \cong g$ implies $f = g$, for all $f, g : Z \longrightarrow X$. The full subcategory of V-Cat consisting of all L-separated V-categories is denoted by V-$\mathsf{Cat}_{\mathrm{sep}}$. Obviously, it suffices to consider $Z = E$ (the $\otimes$-neutral object) here: writing $x : E \longrightarrow X$ in V-Cat instead of $x \in X$, we just note that $f_* = g_*$ implies

$$(f \cdot x)_* = f_* \cdot x_* = g_* \cdot x_* = (g \cdot x)_*.$$

This proves the equivalence of (i),(ii) of the following proposition.

Proposition 2.5 *The following statements are equivalent for a V-category $X = (X, a)$.*

(i) *X is L-separated.*
(ii) *$x \cong y$ implies $x = y$, for all $x, y \in X$.*
(iii) *For all $x, y \in X$, if $a(x, y) \geq k$ and $a(y, x) \geq k$, then $x = y$.*
(iv) *The Yoneda functor $y : X \longrightarrow \hat{X}$ is injective.*

Proof For (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv) one observes

$$\begin{aligned} y(x) = y(y) &\iff x^\circ \cdot a = y^\circ \cdot a \\ &\iff x^* = y^* \\ &\iff x \leq y \text{ and } y \leq x \\ &\iff k \leq a(x, y) \text{ and } k \leq a(y, x). \end{aligned}$$

□

Corollary 2.6 *The V-category V is L-separated. For all V-categories X, Y, if Y is L-separated, $X \multimap Y$ is also L-separated. In particular, $\hat{X}$ is L-separated, for every X.*

Proof $k \leq u \multimap v$ and $k \leq v \multimap u$ means $u \leq v$ and $v \leq u$ in V, hence $u = v$. For $Y = (Y, b)$ and $X = (X, a)$, $k \leq [a, b](f, g)$ in $X \multimap Y$ means $k \leq b(f(x), g(x))$ for all $x \in X$, which makes the second statement obvious. □

2.7 L-completeness

Following Lawvere [13] we call a V-category X *L-complete* if every adjunction $\varphi \dashv \psi : X \multimapinv Z$ in V-Mod is of the form $f_* \dashv f^*$, for a V-functor $f : Z \longrightarrow X$. Clearly, if X is L-separated, such a presentation is unique. As in Section 2.6, it suffices to consider $Z = E$ here; but we need the Axiom of Choice for that.

Proposition 2.7 *The following statements are equivalent for a* V*-category* X.

(i) *X is L-complete.*
(ii) *Each left adjoint* V*-module* $\varphi : E \multimap X$ *is of the form* $\varphi = x_*$ *for some* x *in* X.
(iii) *Each right adjoint* V*-module* $\psi : X \multimap E$ *is of the form* $\psi = x^*$ *for some* x *in* X.

Elements in $\hat{X}$ are V-functors $X^{\mathrm{op}} \cong X^{\mathrm{op}} \otimes E \longrightarrow \mathsf{V}$ which, by Proposition 2.2, may be considered as V-modules $\psi : X \multimap E$. Suppose such a V-module has a left adjoint $\varphi : E \multimap X$. From $\varphi \cdot \psi \leq 1_X^*$ one obtains $\varphi \leq 1_X^* \bullet\!\!- \psi$ (see Section 2.2), and from $(1_X^* \bullet\!\!- \psi) \cdot \psi \leq 1_X^*$ and $\psi \cdot \varphi \geq 1_E^*$ one has $1_X^* \bullet\!\!- \psi \leq \varphi$. Hence, if ψ is right adjoint, its left adjoint must necessarily be $1_X^* \bullet\!\!- \psi$; moreover $(1_X^* \bullet\!\!- \psi) \cdot \psi \leq 1_X^*$ always holds. Therefore:

Proposition 2.8 *A* V*-module* $\psi : X \multimap E$ *(with* $X = (X, a)$*) is right adjoint if, and only if,* $1_E^* \leq \psi \cdot (1_X^* \bullet\!\!- \psi)$*, that is, if*

$$k \leq \bigvee_{y \in X} \psi(y) \otimes \left(\bigwedge_{x \in X} a(x, y) \circ\!\!- \psi(x) \right). \qquad (*)$$

Note that $\bigwedge_{x \in X} a(x, y) \circ\!\!- \psi(x) = \hat{a}(\psi, y(y))$. We call a V-functor $\psi : X^{\mathrm{op}} \longrightarrow \mathsf{V}$ *tight* ([18]) if, as a V-module $X \multimap E$, it is right adjoint, that is, if it satisfies $(*)$. We consider

$$\tilde{X} = \{\psi \in \hat{X} \mid \psi \text{ tight}\}$$

as a full V-subcategory of $\hat{X}$. Our goal is to exhibit $\tilde{X}$ as an "L-completion" of X.

Example 2.9

(1) $\mathsf{V} = \mathsf{2}$. A V-functor $X^{\mathrm{op}} \longrightarrow \mathsf{2}$ is the characteristic function of a down-closed set A in the (pre)ordered set X. Condition $(*)$ then reads as

$$\exists y \in A \forall x \in A \,.\, x \leq y,$$

so that $A = {\downarrow} y$. In other words, $\tilde{X}$ is simply the image of the Yoneda functor $y : X \longrightarrow \hat{X},\ y \longmapsto {\downarrow} y$.

(2) $\mathsf{V} = \mathsf{P}_+$. A tight V-functor $X^{\mathrm{op}} \longrightarrow \mathsf{V}$ is given by a function $\psi : X \longrightarrow [0, \infty]$ with

$$\psi(y) \leq \psi(x) \ \Rightarrow\ \psi(x) - \psi(y) \leq a(x, y) \quad (x, y \in X),$$

$$\inf_{y \in X} (\psi(y) + \sup_{\substack{x \in X, \\ \psi(x) \leq a(x, y)}} (a(x, y) - \psi(x))) = 0,$$

here a is the metric on X. If a is symmetric (so that $a = a^\circ$), these conditions are more conveniently described as

$$|\psi(x) - \psi(y)| \leq a(x, y) \leq \psi(x) + \psi(y) \quad (x, y \in X),$$

$$\inf_{x \in X} \psi(x) = 0.$$

These are precisely the *supertight* maps on X considered in [16], where the reader finds the necessary details.

(3) $\mathsf{V} = \mathsf{P}_{\max}$. Here the two conditions of (2) change to

$$\psi(y) < \psi(x) \;\Rightarrow\; \psi(x) \leq a(x, y) \quad (x, y \in X),$$

$$\inf_{y \in Y}(\max(\psi(y), \sup_{\substack{x \in X, \\ \psi(x) < a(x,y)}} (a(x, y)))) = 0.$$

2.8 L-injectivity

A V-functor $f : (X, a) \longrightarrow (Y, b)$ is called *L-dense* if $f_* \cdot f^* = 1_Y^*$; that is, if $b = b \cdot f \cdot f^\circ \cdot b$, or

$$b(y, y') = \bigvee_{x \in X} b(y, f(x)) \otimes b(f(x), y')$$

for all $y, y' \in Y$. L-dense V-functors have good composition-cancellation properties.

Lemma 2.10 *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be V-functors. Then the following assertions hold:*

(1) *f, g L-dense $\Rightarrow g \cdot f$ L-dense.*
(2) *$g \cdot f$ L-dense $\Rightarrow g$ L-dense.*
(3) *$g \cdot f$ L-dense, g fully faithful $\Rightarrow f$ L-dense.*
(4) *$g \cdot f$ fully faithful, f L-dense $\Rightarrow g$ fully faithful.*

A fully faithful L-dense V-functor is an *L-equivalence*. Hence, f is an L-equivalence if, and only if, f_* (or f^*) is an isomorphism in V-Mod. A V-category Z is *pseudo-injective* if, for every fully faithful V-functor $f : X \longrightarrow Y$ and for all V-functors $h : X \longrightarrow Z$ there is a V-functor $g : Y \longrightarrow Z$ with $g \cdot f \cong h$; if strict equality is obtainable, we call Z injective. Z is *L-injective* if this extension property is required only for L-equivalences f. Hence, injectivity implies pseudo-injectivity, and every pseudo-injective V-category is also L-injective.

Lemma 2.11 *The V-category V is injective, hence in particular L-injective.*

Proof Let $f : X \longrightarrow Y$ be fully faithful and $\varphi : X \longrightarrow \mathsf{V}$ be any V-functor. Then the V-module $\varphi : E \longrightarrow\!\!\!\!\circ\; X$ factors as $\varphi = f^* \cdot \psi$, with $\psi = f_* \cdot \varphi$. But the V-module $f^* \cdot \psi$ corresponds to the V-functor $\psi \cdot f$, hence $\psi \cdot f = \varphi$.

$$\begin{tikzcd}
 & \mathsf{V} & Y \arrow[r, "f^*", "\circ"{marking}] & X \\
X \arrow[ur, "\varphi"] \arrow[r, "f"'] & Y \arrow[u, "\psi"'] & E \arrow[u, "\psi = f_* \cdot \varphi"] \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

□

Note that the V-functor ψ has been constructed effectively, with

$$\psi(y) = \bigvee_{x \in X} \varphi(x) \otimes b(f(x), y).$$

In case $\mathsf{V} = \mathsf{2}$, this means

$$\psi(y) = \top \iff \exists x \in X . (\varphi(x) = \top \text{ and } f(x) \leq y),$$

and for $\mathsf{V} = \mathsf{P}_+$ we have

$$\psi(y) = \inf_{x \in X} (\varphi(x) + b(f(x), y)).$$

Proposition 2.12 *For all V-categories X, Y, if Y is pseudo-injective or L-injective, $X \multimap Y$ has the respective property. In particular, $\hat{X}$ is injective.*

Proof Let $f : A \longrightarrow B$ be fully faithful, and consider any V-functor $\varphi : A \longrightarrow (X \multimap Y)$, with Y pseudo-injective. Since $f \otimes 1_X$ is fully faithful, the mate $\llcorner\varphi\lrcorner : A \otimes X \longrightarrow Y$ factors (up to $\cong$) as $\llcorner\varphi\lrcorner \cong \llcorner\psi\lrcorner \cdot (f \otimes 1_X)$, with $\llcorner\psi\lrcorner : B \otimes X \longrightarrow Y$ corresponding to a V-functor $\psi : B \longrightarrow (X \multimap Y)$. Since $\llcorner\psi\lrcorner \cdot (f \otimes 1_X)$ corresponds to $\psi \cdot f$, $\varphi \cong \psi \cdot f$ follows. The proof works *mutatis mutandis* for L-injectivity. □

Our goal is to show that L-injectivity and L-completeness are equivalent properties.

3 L-closure

3.1 L-dense V-functors

We first show that L-dense V-functors are characterized as "epimorphisms up to $\cong$".

Proposition 3.1 *A V-functor $m : M \longrightarrow X$ is L-dense if, and only if, for all V-functors $f, g : X \longrightarrow Y$ with $f \cdot m = g \cdot m$ one has $f \cong g$.*

Proof The necessity of the condition is clear since from $f_* \cdot m_* = g_* \cdot m_*$ one obtains $f_* = g_*$ when $m_* \cdot m^* = 1_X^*$. To show the converse implication, by Lemma 2.10 we may assume that m is a full embedding $M \hookrightarrow X$ and consider its cokernel pair

$$(X, a) \underset{g}{\overset{f}{\rightrightarrows}} (Y, b),$$

given by the disjoint union

$$Y = \{f(x) = g(x) \mid x \in M\} \cup \{f(x) \mid x \in X \setminus M\} \cup \{g(x) \mid x \in X \setminus M\},$$

where both f and g are full embeddings, and

$$b(f(x), g(y)) = \bigvee_{z \in Z} a(x, z) \otimes a(z, y)$$

for all $y, x \in X \setminus M$. Since $f_* = g_*$ by hypothesis, we obtain

$$a(x, y) = b\,(g(x), g(y)) = b\,(f(x), g(y)) = m_* \cdot m^*(x, y)$$

for all $x, y \in X \setminus M$. But this identity holds trivially when $x \in M$ or $y \in M$. Hence $m_* \cdot m^* = 1^*_X$. □

Since $f \cong g$ precisely when $f \cdot x \cong g \cdot x$ for all $x \in X$ (considered as $x : E \longrightarrow X$), it is now easy to identify the largest subset of X which contains M as an L-dense subset.

3.2 L-closure

For a V-category X and $M \subseteq X$, we define the *L-closure* of M in X by

$$\overline{M} = \{x \in X \mid \forall f, g : X \longrightarrow Y\,.\,(f|_M = g|_M \;\Rightarrow\; f \cdot x \cong g \cdot x)\}$$

and prove:

Proposition 3.2 *Let $X = (X, a)$ be a V-category, $M \subseteq X$ and $x \in X$. Then the following assertions are equivalent.*

(i) $x \in \overline{M}$.
(ii) $a(x, x) \leq \bigvee_{y \in M} a(x, y) \otimes a(y, x)$.
(iii) $k \leq \bigvee_{y \in M} a(x, y) \otimes a(y, x)$.
(iv) $1^*_E \leq x^* \cdot m_* \cdot m^* \cdot x_*$, *where m denotes the full embedding $m : M \hookrightarrow X$.*
(v) $m^* \cdot x_* \dashv x^* \cdot m_*$.
(vi) $x_* : E \multimap X$ *factors through* $m_* : M \multimap X$ *by a map* $\varphi : E \multimap M$ *in* V-Mod.

Proof (i)⇒(ii) follows from the L-density of $M \hookrightarrow \overline{M}$. (ii)⇒(iii) is clear since $k \leq a(x, x)$. To see (iii)⇒(iv), just observe

$$x^* \cdot m_* \cdot m^* \cdot x_*(\star, \star) = \bigvee_{y \in M} a(x, y) \otimes a(y, x).$$

The second inequality needed for the adjunction (v) comes for free: $m^* \cdot x_* \cdot x^* \cdot m_* \leq m^* \cdot m_* = 1^*_M$. Hence,(iv)⇒(v)follows. Assuming (v), we have $m_* \cdot m^* \cdot x_* \dashv x^* \cdot m_* \cdot x^*$ as well as $m_* \cdot m^* \cdot x_* \leq x_*$ and $x^* \cdot m_* \cdot x^* \leq x^*$, which implies $m_* \cdot m^* \cdot x_* = x_*$. This shows (v)⇒(vi). Finally, assume (vi) and let $f, g : X \longrightarrow Y$ with $f|_M = g|_M$. Then

$$f_* \cdot x_* = f_* \cdot m_* \cdot \varphi = g_* \cdot m_* \cdot \varphi = g_* \cdot x_*,$$

which proves (i). □

V-functors respect the L-closure, as we show next.

Proposition 3.3 *For a V-functor $f : X \longrightarrow Y$ and $M, M' \subseteq X$, $N \subseteq Y$, we have:*

(1) $M \subseteq \overline{M}$ *and* $M \subseteq M'$ *implies* $\overline{M} \subseteq \overline{M'}$.
(2) $\overline{\varnothing} = \varnothing$ *and* $\overline{\overline{M}} = \overline{M}$.

(3) $f(\overline{M}) \subseteq \overline{f(M)}$ *and* $f^{-1}(\overline{N}) \supseteq \overline{f^{-1}(N)}$.
(4) *If* k *is* $\vee$*-irreducible (so that* $k \leq u \vee v$ *implies* $k \leq u$ *or* $k \leq v$*), then* $\overline{M \cup M'} = \overline{M} \cup \overline{M'}$.

Proof (1), (2) are obvious. For (3), applying Lemma 2.10 to

$$\begin{tikzcd} M \arrow[r] \arrow[d, tail] & f(M) \arrow[d, tail] \\ \overline{M} \arrow[r] & f(\overline{M}) \end{tikzcd}$$

one sees that $f(M) \longrightarrow f(\overline{M})$ is L-dense, hence $f(\overline{M}) \subseteq \overline{f(M)}$. With $M = f^{-1}(N)$, this implies $\overline{f^{-1}(N)} \subseteq f^{-1}(\overline{N})$. To see (4), we just need to show that $x \in \overline{M \cup M'}$ implies $x \in \overline{M}$ or $x \in \overline{M'}$. But this follows from

$$k \leq \bigvee_{y \in M \cup M'} a(x, y) \otimes a(y, x) = \left(\bigvee_{y \in M} a(x, y) \otimes a(y, x) \right) \vee \left(\bigvee_{y \in M'} a(x, y) \otimes a(y, x) \right),$$

assuming that k is $\vee$-irreducible. □

Corollary 3.4 *If* k *is* $\vee$*-irreducible in* V*, then the L-closure operator defines a topology on* X *such that every* V*-functor becomes continuous. Hence, L-closure defines a functor* L : V-Cat $\longrightarrow$ Top.

Example 3.5

(1) For $X = (X, \leq)$ in 2-Cat = Ord and $M \subseteq X$, one has $x \in \overline{M}$ precisely when $x \leq z \leq x$ for some $z \in M$. Also, $M \subseteq X$ is open in LX if every $x \in M$ satisfies

$$\forall z \in X \,.\, (x \leq z \leq x \;\Rightarrow\; z \in M).$$

(2) In Met, $\overline{M} = \{x \in X = (X, a) \mid \inf_{z \in M}(a(x, z) + a(z, x)) = 0\}$, and in UMet

$$\overline{M} = \left\{x \in X = (X, a) \mid \inf_{z \in M}(\max(a(x, z), a(z, x))) = 0\right\},$$

which for symmetric (ultra)metric spaces describes the ordinary topological closure.

3.3 L-separatedness via L-closure

Proposition 3.6 *Let* $X = (X, a)$ *be a* V*-category and* $\Delta \subseteq X \times X$ *the diagonal. Then*

$$\overline{\Delta} = \{(x, y) \in X \times X \mid x \cong y\}.$$

Proof Let first $(x, y) \in \overline{\Delta}$. With $\pi_1, \pi_2 : X \times X \longrightarrow X$ denoting the projection maps, we have $\pi_1|_\Delta = \pi_2|_\Delta$ and therefore $x = \pi_1(x, y) \cong \pi_2(x, y) = y$. Assume now $x \cong y$. Note that the canonical functor V-Cat $\longrightarrow$ Ord preserves products, hence

$$(x_1, y_1) \cong (x_2, y_2) \iff x_1 \cong x_2 \text{ and } y_1 \cong y_2,$$

for all $(x_1, y_1), (x_2, y_2) \in X \times X$. Therefore we have $(x, y) \cong (x, x)$. Let now $f, g : X \times X \longrightarrow Y$ be V-functors with $f|_\Delta = g|_\Delta$. Then $f(x, y) \cong f(x, x) = g(x, x) \cong g(x, y)$. □

Corollary 3.7 *A* V*-category* X *is L-separated if and only if the diagonal* Δ *is closed in* $X \times X$.

Theorem 3.8 $\mathsf{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ *is an epi-reflective subcategory of* $\mathsf{V}\text{-}\mathsf{Cat}$*, where the reflection map is given by* $y_X : X \longrightarrow y_X(X)$*, for each* V*-category* X*. Hence, limits of L-separated* V*-categories are formed in* $\mathsf{V}\text{-}\mathsf{Cat}$*, while colimits are obtained by reflecting the colimit formed in* $\mathsf{V}\text{-}\mathsf{Cat}$*. The epimorphisms in* $\mathsf{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ *are precisely the L-dense* V*-functors.*

3.4 L-completeness via L-closure

Lemma 3.9 *Let* $X = (X, a)$ *be a* V*-category and* $M \subseteq X$.

(1) *Assume that* X *is L-complete and* M *is L-closed. Then* M *is L-complete.*
(2) *Assume that* X *is L-separated and* M *is L-complete. Then* M *is L-closed.*

Proof (1) follows immediately from Proposition 3.2. To see (2), let $x \in X$ be such that $m^* \cdot x_* \dashv x^* \cdot m_*$. Since M is L-complete, there is some $y \in M$ such that $y_* = m^* \cdot x_*$ and $y^* = x^* \cdot m_*$. Hence $m(y)_* = m_* \cdot y_* \leq x_*$ and $m(y)^* = y^* \cdot m^* \leq x^*$ and therefore, $m(y)_* = x_*$. L-separation of X gives now $m(y) = x$, i.e. $x \in M$. □

Theorem 3.10 *Let* $X = (X, b)$ *be a* V*-category. The following assertions are equivalent.*

(i) X *is L-complete.*
(ii) X *is L-injective.*
(iii) $y : X \longrightarrow \tilde{X}$ *has a pseudo left-inverse* V*-functor* $R : \tilde{X} \longrightarrow X$*, i.e.* $R \cdot y \cong 1_X$.

Proof To see (i)⇒(ii), let $i : A \longrightarrow B$ be a fully faithful dense V-functor and $f : A \longrightarrow X$ be a V-functor. Since $i_* \dashv i^*$ is actually an equivalence of V-modules, we have $f_* \cdot i^* \dashv i_* \cdot f^*$. Hence, since X is L-complete, there is a V-functor $g : B \longrightarrow X$ such that $g_* = f_* \cdot i^*$, hence $g_* \cdot i_* = f_*$.

The implication (ii)⇒(iii) is surely true since $y : X \longrightarrow \tilde{X}$ is L-dense and fully faithful.

Finally, to see (iii)⇒(i), let $R : \tilde{X} \longrightarrow X$ be a left inverse of $y : X \longrightarrow \tilde{X}$. Then $y \cdot R = 1_{\tilde{X}}$ since $y : X \longrightarrow \tilde{X}$ is dense and $\tilde{X}$ is L-separated. Hence, for each right adjoint V-module $\psi : X \multimap E$, we have $\psi = R(\psi)^*$. □

Proposition 3.11 *For a* V*-category* X*, as a set* $\tilde{X}$ *(see Section 2.7) coincides with the L-closure of* $y(X)$ *in* $\hat{X}$*. Hence,* $y : X \longrightarrow \tilde{X}$ *is fully faithful and L-dense, and* $\tilde{X}$ *is L-complete.*

Proof By Proposition 3.2, a V-functor $\psi : X^{\mathrm{op}} \longrightarrow \mathsf{V}$ lies in the L-closure of $y(X)$ in $\hat{X}$ if, and only if,

$$k \leq \bigvee_{y \in X} \hat{a}(\psi, y(y)) \otimes \hat{a}(y(y), \psi).$$

Since $\hat{a}(y(y), \psi) = \psi(y)$ by Lemma 2.3, this means precisely that ψ must be tight. □

Theorem 3.12 *The full subcategory* $\mathsf{V}\text{-}\mathsf{Cat}_{\mathrm{cpl}}$ *of* $\mathsf{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ *of L-complete* V*-categories is an epi-reflective subcategory of* $\mathsf{V}\text{-}\mathsf{Cat}_{\mathrm{sep}}$. *The reflection map of an L-separated* V*-category X is given by any L-dense embedding of X into an L-complete and L-separated* V*-category, for instance by* $y : X \longrightarrow \tilde{X}$.

4 The $\mathcal{T}$-setting

4.1 The Theory $\mathcal{T}$

From now on we assume that the quantale V is part of a *strict topological theory* $\mathcal{T} = (\mathbb{T}, \mathsf{V}, \xi)$ as introduced in [11]. Here $\mathbb{T} = (T, e, m)$ is a Set-monad where T and m satisfy (BC) (that is, T sends pullbacks to weak pullbacks and each naturality square of m is a weak pullback) and $\xi : T\mathsf{V} \longrightarrow \mathsf{V}$ is a map such that

$$1_{\mathsf{V}} = \xi \cdot e_{\mathsf{V}}, \qquad \xi \cdot T\xi = \xi \cdot m_{\mathsf{V}},$$

the diagrams

$$\begin{tikzcd} T(\mathsf{V} \times \mathsf{V}) \arrow[r, "T(\otimes)"] \arrow[d, "\langle \xi \cdot T\pi_1, \xi \cdot T\pi_2 \rangle"'] & T\mathsf{V} \arrow[d, "\xi"] \\ \mathsf{V} \times \mathsf{V} \arrow[r, "\otimes"'] & \mathsf{V}, \end{tikzcd} \qquad \begin{tikzcd} T1 \arrow[r, "Tk"] \arrow[d, "!"'] & T\mathsf{V} \arrow[d, "\xi"] \\ 1 \arrow[r, "k"'] & \mathsf{V}, \end{tikzcd}$$

commute and $(\xi_X)_X : P_{\mathsf{V}} \longrightarrow P_{\mathsf{V}}T$ is a natural transformation, where P_{V} is the V-powerset functor considered as a functor from Set to Ord and the X-component $\xi_X : P_{\mathsf{V}}(X) \longrightarrow P_{\mathsf{V}}T(X)$ is given by $\varphi \longmapsto \xi \cdot T\varphi$.

Explicitly, $P_{\mathsf{V}}(X) = \mathsf{V}^X$, and for a function $f : X \longrightarrow Y$ we have a canonical map $f^{-1} : \mathsf{V}^Y \longrightarrow \mathsf{V}^X$, $\varphi \longmapsto \varphi \cdot f$. Now $P_{\mathsf{V}}(f)$ is defined as the left adjoint to f^{-1}, explicitly, for $\varphi \in \mathsf{V}^X$ we have $P_{\mathsf{V}}(\varphi)(y) = \bigvee_{x \in f^{-1}(y)} \varphi(x)$. Furthermore, we assume $T1 = 1$.

Example 4.1

(1) For each quantale V, $(\mathbb{1}, \mathsf{V}, 1_{\mathsf{V}})$ is a strict topological theory, where $\mathbb{1} = (\mathrm{Id}, 1, 1)$ denotes the identity monad.
(2) $\mathcal{U}_2 = (\mathbb{U}, 2, \xi_2)$ is a strict topological theory, where $\mathbb{U} = (U, e, m)$ denotes the ultrafilter monad and ξ_2 is essentially the identity map.

(3) $\mathcal{U}_{\mathsf{P}_+} = (\mathbb{U}, \mathsf{P}_+, \xi_{\mathsf{P}_+})$ is a strict topological theory, where

$$\xi_{\mathsf{P}_+} : U\mathsf{P}_+ \longrightarrow \mathsf{P}_+, \quad \mathfrak{x} \longmapsto \inf\{v \in \mathsf{P}_+ \mid [0, v] \in \mathfrak{x}\}.$$

As shown in [11, Lemma 3.2], the right adjoint $\multimap$ of the tensor product $\otimes$ in V is automatically compatible with the map $\xi : T\mathsf{V} \longrightarrow \mathsf{V}$ in the sense that

$$\xi \cdot T(\multimap) \leq \multimap \cdot \langle \xi \cdot T\pi_1, \xi \cdot T\pi_2 \rangle.$$

$$\begin{tikzcd}
T(\mathsf{V} \times \mathsf{V}) \arrow[r, "T(\multimap)"] \arrow[d, "\langle \xi \cdot T\pi_1, \xi \cdot T\pi_2 \rangle"'] \arrow[dr, phantom, "\geq"] & T\mathsf{V} \arrow[d, "\xi"] \\
\mathsf{V} \times \mathsf{V} \arrow[r, "\multimap"'] & \mathsf{V}
\end{tikzcd}$$

Furthermore, our condition $T1 = 1$ implies $m_X^\circ \cdot e_X = e_{TX} \cdot e_X$ for each set X. In fact, $m_X^\circ \cdot e_X \geq e_{TX} \cdot e_X$ is true for each monad since $m_X^\circ \geq e_{TX}$. Let now $\mathfrak{X} \in TTX$ and $x \in X$ such that $m_X(\mathfrak{X}) = e_X(x)$. We consider the commutative diagram

$$\begin{tikzcd}
TT1 \arrow[r, "TTx"] \arrow[d, "m_1"'] & TTX \arrow[d, "m_X"] \\
T1 \arrow[r, "Tx"'] & TX,
\end{tikzcd}$$

where $x : 1 \longrightarrow X$. Since m satisfies (BC), there is some $\mathfrak{Y} \in TT1 = 1$ with $TTx(\mathfrak{Y}) = \mathfrak{X}$, that is, $\mathfrak{X} = e_{TX} \cdot e_X(x)$.

The functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ can be extended to a 2-functor $T_\xi : \mathsf{V}\text{-}\mathsf{Rel} \longrightarrow \mathsf{V}\text{-}\mathsf{Rel}$ as follows. Given a V-relation $r : X \times Y \longrightarrow \mathsf{V}$, we define $T_\xi r : TX \times TY \longrightarrow \mathsf{V}$ as the left Kan-extension

$$\begin{tikzcd}
T(X \times Y) \arrow[rr, "\mathrm{can}"] \arrow[dr, "\xi_{X \times Y}(r) = \xi \cdot Tr"'] & & TX \times TY \arrow[dl, dotted, "T_\xi r"] \\
& \mathsf{V} &
\end{tikzcd}$$

in Ord (where TX, TY, $T(X \times Y)$ are discrete), i.e. the smallest (order-preserving) map $s : TX \times TY \longrightarrow \mathsf{V}$ such that $\xi \cdot Tr \leq s \cdot \mathrm{can}$. Elementwise one has

$$T_\xi r(\mathfrak{x}, \mathfrak{y}) = \bigvee \left\{ \xi \cdot Tr(\mathfrak{w}) \;\middle|\; \mathfrak{w} \in T(X \times Y), T\pi_1(\mathfrak{w}) = \mathfrak{x}, T\pi_2(\mathfrak{w}) = \mathfrak{y} \right\}$$

for each $\mathfrak{x} \in TX$ and $\mathfrak{y} \in TY$. We obtain now the following properties.

Proposition 4.2 ([11]) *The following assertions hold:*

(1) *For each* V*-matrix* $r : X \nrightarrow Y$, $T_\xi(r^\circ) = T_\xi(r)^\circ$ *(and we write* $T_\xi r^\circ$*).*
(2) *For each function* $f : X \longrightarrow Y$, $Tf = T_\xi f$ *(and therefore also* $Tf^\circ = T_\xi f^\circ$*).*

(3) $e_Y \cdot r \leq T_\xi r \cdot e_X$ *for all* $r : X \longrightarrow\!\!\!

for all $\mathfrak{x} \in TX$ and $x \in X$. The resulting category of $\mathcal{T}$-categories and $\mathcal{T}$-functors is denoted by $\mathcal{T}$-Cat (see also [4, 6, 7]). Note that the quantale V becomes in a natural way a $\mathcal{T}$-category $\mathsf{V} = (\mathsf{V}, \hom_\xi)$ where $\hom_\xi : T\mathsf{V} \times \mathsf{V} \longrightarrow \mathsf{V},\ (\mathfrak{v}, v) \longmapsto (\xi(\mathfrak{v}) \multimap v)$.

Example 4.3

(1) For each quantale V, $\mathcal{I}_\mathsf{V}$-categories are precisely V-categories and $\mathcal{I}_\mathsf{V}$-functors are V-functors.
(2) The main result of [2] states that $\mathcal{U}_2$-Cat is isomorphic to the category Top of topological spaces. The $\mathcal{U}_2$-category $\mathsf{V} = \mathsf{2}$ is the Sierpinski space with $\{0\}$ open and $\{1\}$ closed. In [4] it is shown that $\mathcal{U}_{\mathsf{P}_+}$-Cat is isomorphic to the category App of approach spaces (see [15] for more details about App).

A $\mathcal{T}$-category $X = (X, a)$ can also be thought of as a lax Eilenberg–Moore algebra, since the two conditions above can be equivalently expressed as

$$1_X \leq a \cdot e_X, \qquad a \cdot T_\xi a \leq a \cdot m_X.$$

As a consequence, each $\mathbb{T}$-algebra (X, α) can be considered as a $\mathcal{T}$-category by simply regarding the function $\alpha : TX \longrightarrow X$ as a $\mathcal{T}$-relation $\alpha : X \rightarrowtail\!\!\!

where $\mathfrak{w} \in T(X \times Y)$, $x \in X$, $y \in Y$, $\mathfrak{x} = T\pi_1(\mathfrak{w})$ and $\mathfrak{y} = T\pi_2(\mathfrak{w})$. The $\mathcal{T}$-category $E = (E, k)$ is a $\otimes$-neutral object, where E is a singleton set and k the constant relation with value $k \in \mathsf{V}$. Unlike in the V-case, in general this does not result in a closed structure on $\mathcal{T}$-Cat. However, as shown in [11], if a $\mathcal{T}$-category $X = (X, a)$ satisfies $a \cdot T_\xi a = a \cdot m_X$, then $X \otimes _ : \mathcal{T}\text{-}\mathsf{Cat} \longrightarrow \mathcal{T}\text{-}\mathsf{Cat}$ has a right adjoint $X \multimap _ : \mathcal{T}\text{-}\mathsf{Cat} \longrightarrow \mathcal{T}\text{-}\mathsf{Cat}$. Explicitly, for a $\mathcal{T}$-category $Y = (Y, b)$, the exponential $X \multimap Y$ is given by the set

$$\{f : X \longrightarrow Y \mid f \text{ is a } \mathcal{T}\text{-functor}\},$$

equipped with the structure-relation $[[a, b]]$ defined by

$$[[a, b]](\mathfrak{p}, h) = \bigvee \left\{ v \in \mathsf{V} \;\middle|\; \forall \mathfrak{q} \in T\pi_2^{-1}(\mathfrak{p}), x \in X \,.\, a(T\pi_1(\mathfrak{q}), x) \otimes v \leq b\,(T\mathrm{ev}(\mathfrak{q}), h(x)) \right\},$$

where $\mathfrak{p} \in T(Y^X)$, $h \in Y^X$, $\pi_1 : X \times (X \multimap Y) \longrightarrow X$ and $\pi_2 : X \times (X \multimap Y) \longrightarrow Y^X$. Using the adjunction $u \otimes _ \dashv u \multimap _$ in V, we see that

$$[[a, b]](\mathfrak{p}, h) = \bigwedge_{\substack{\mathfrak{q} \in T(X \times (X \multimap Y)), x \in X \\ \mathfrak{q} \longrightarrow \mathfrak{p}}} a(T\pi_1(\mathfrak{q}), x) \multimap b\,(T\mathrm{ev}(\mathfrak{q}), h(x)).$$

Lemma 4.5 *Let $X = (X, a)$, $Y = (Y, b)$ be $\mathcal{T}$-categories with $a \cdot T_\xi a = a \cdot m_X$ and $h, h' \in (X \multimap Y)$. Then*

$$[[a, b]](e_{Y^X}(h), h') = \bigwedge_{x \in X} b\,(e_Y(h(x)), h'(x)).$$

4.4 $\mathcal{T}$-modules

Let $X = (X, a)$ and $Y = (Y, b)$ be $\mathcal{T}$-categories and $\varphi : X \longrightarrow\!\!\!\!\!\!+\; Y$ be a $\mathcal{T}$-relation. We call φ a $\mathcal{T}$-*module*, and write $\varphi : X \multimapinv Y$, if $\varphi \circ a \leq \varphi$ and $b \circ \varphi \leq \varphi$. Note that we always have $\varphi \circ a \geq \varphi$ and $b \circ \varphi \geq \varphi$, so that the $\mathcal{T}$-module condition above actually implies equality. It is easy to see that the extension as well as the lifting (if it exists) in $\mathcal{T}$-URel of $\mathcal{T}$-modules is again a $\mathcal{T}$-module. Furthermore, we have $a : X \multimapinv X$ for each $\mathcal{T}$-category $X = (X, a)$; in fact, a is the identity $\mathcal{T}$-module on X for the Kleisli convolution. The category of $\mathcal{T}$-categories and $\mathcal{T}$-modules, with Kleisli convolution as composition, is denoted by $\mathcal{T}$-Mod. In fact, $\mathcal{T}$-Mod is an ordered category, with the structure on hom-sets inherited from $\mathcal{T}$-URel.

Let now $X = (X, a)$ and $Y = (Y, b)$ be $\mathcal{T}$-categories and $f : X \longrightarrow Y$ be a Set-map. We define $\mathcal{T}$-relations $f_* : X \longrightarrow\!\!\!\!\!\!+\; Y$ and $f^* : Y \longrightarrow\!\!\!\!\!\!+\; X$ by putting $f_* = b \cdot Tf$ and $f^* = f^\circ \cdot b$ respectively. Hence, for $\mathfrak{x} \in TX$, $\mathfrak{y} \in TY$, $x \in X$ and $y \in Y$, $f_*(\mathfrak{x}, y) = b\,(Tf(\mathfrak{x}), y)$ and $f^*(\mathfrak{y}, x) = b\,(\mathfrak{y}, f(x))$. Given now $\mathcal{T}$-modules φ and ψ, we have

$$\varphi \circ f_* = \varphi \cdot Tf \qquad \text{and} \qquad f^* \circ \psi = f^\circ \cdot \psi.$$

The latter equality follows from

$$f^* \circ \psi = f^\circ \cdot b \cdot T_\xi \psi \cdot m_Z^\circ = f^\circ \cdot \psi,$$

whereas the first equality follows from

$$\varphi \circ f_* = \varphi \circ (b \cdot Tf) = \varphi \cdot T_\xi b \cdot T^2 f \cdot m_X^\circ = \varphi \cdot T_\xi b \cdot m_Y^\circ \cdot Tf = \varphi \cdot Tf.$$

In particular we have $b \circ f_* = f_*$ and $f^* \circ b = f^*$, as well as $f_* \circ f^* = b \cdot Tf \cdot Tf^\circ \cdot T_\xi b \cdot m_Y^\circ \leq b$. The latter inequality becomes even an equality provided that f is surjective. As before, one easily verifies:

Proposition 4.6 *The following assertions are equivalent for a* Set*-map f between $\mathcal{T}$-categories:*

(i) $f : X \longrightarrow Y$ *is a $\mathcal{T}$-functor.*
(ii) f_* *is a $\mathcal{T}$-module* $f_* : X \multimap Y$.
(iii) f^* *is a $\mathcal{T}$-module* $f^* : Y \multimap X$.

As in the V-case, there are functors

$$\mathcal{T}\text{-Cat} \xrightarrow{(-)_*} \mathcal{T}\text{-Mod} \xleftarrow{(-)^*} \mathcal{T}\text{-Cat}^{\mathrm{op}}.$$

We can transport the order on the hom-sets from $\mathcal{T}$-Mod to $\mathcal{T}$-Cat via the functor $(_)^* : \mathcal{T}\text{-Cat}^{\mathrm{op}} \longrightarrow \mathcal{T}\text{-Mod}$. That is, we can define $f \leq g$ if $f^* \leq g^*$, or equivalently, if $g_* \leq f_*$. With this definition we turn $\mathcal{T}$-Cat into an *ordered category*. As usual, we call $\mathcal{T}$-functors $f, g : X \longrightarrow Y$ *equivalent*, and write $f \cong g$, if $f \leq g$ and $g \leq f$. Hence, $f \cong g$ if and only if $f^* = g^*$, which in turn is equivalent to $f_* = g_*$.

Lemma 4.7 *Let $f, g : X \longrightarrow Y$ be $\mathcal{T}$-functors between $\mathcal{T}$-categories $X = (X, a)$ and $Y = (Y, b)$. Then*

$$f \leq g \iff \forall x \in X \,.\, k \leq b\,(e_Y(f(x)), g(x)).$$

Proof If $g_* \leq f_*$, then

$$k \leq g_*(e_X(x), g(x)) \leq f_*(e_X(x), g(x)) = b\,(e_Y(f(x)), g(x)).$$

On the other hand, if $k \leq b\,(e_Y(g(x)),\ f(x))$ for each $x \in X$, then

$$\begin{aligned} f^*(\mathfrak{y}, x) &= b\,(\mathfrak{y},\ f(x)) \leq T_\xi b\,(e_{TY}(\mathfrak{y}), e_Y(f(x))) \otimes b\,(e_Y(f(x)), g(x)) \\ &\leq b\,(\mathfrak{y}, g(x)) = g^*(\mathfrak{y}, x). \end{aligned}$$

□

Hence, we obtain the same order relation as in [[7], Theorem 12 (page 46)]. In particular, for $\mathcal{T}$-functors $f, g : X \longrightarrow \mathsf{V}$, we have $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in X$. Assume now that $X = (X, a)$, $Y = (Y, b)$ and $Z = (Z, c)$ are $\mathcal{T}$-categories where $a \cdot T_\xi a = a \cdot m_X$. By combining the previous lemma with Lemma 4.5, we obtain $f \leq g \iff \ulcorner f \urcorner \leq \ulcorner g \urcorner$ for all $\mathcal{T}$-functors $f, g : X \otimes Y \longrightarrow Z$, where $\ulcorner f \urcorner, \ulcorner g \urcorner : Y \longrightarrow Z^X$.

4.5 Yoneda

Also $\mathcal{T}$-modules give rise to $\mathcal{T}$-functors, but in addition to X^{op} we must also take the $\mathcal{T}$-category $|X|$ (see Section 4.3) into consideration.

Theorem 4.8 ([5]) *For $\mathcal{T}$-categories (X, a) and (Y, b), and a $\mathcal{T}$-relation $\psi : X \nrightarrow Y$, the following assertions are equivalent:*

(i) $\psi : (X, a) \multimap (Y, b)$ *is a $\mathcal{T}$-module.*
(ii) *Both* $\psi : |X| \otimes Y \longrightarrow \mathsf{V}$ *and* $\psi : X^{\mathrm{op}} \otimes Y \longrightarrow \mathsf{V}$ *are $\mathcal{T}$-functors.*

Since we have $a : X \longrightarrow\!\!\!\!\!\!\circ\; X$ for each $\mathcal{T}$-category $X = (X, a)$, the theorem above provides us with two $\mathcal{T}$-functors

$$a : |X| \otimes X \longrightarrow \mathsf{V} \qquad \text{and} \qquad a : X^{\mathrm{op}} \otimes X \longrightarrow \mathsf{V}.$$

We refer to the mate $y = \ulcorner a \urcorner : X \longrightarrow (|X| \multimap \mathsf{V})$ of the first $\mathcal{T}$-functor as the Yoneda functor of X.

Theorem 4.9 ([5]) *Let $X = (X, a)$ be a $\mathcal{T}$-category. Then the following assertions hold:*

(1) *For all $\mathfrak{x} \in TX$ and $\varphi \in (|X| \multimap \mathsf{V})$, $[[m_X, \hom_\xi]](Ty(\mathfrak{x}), \varphi) \leq \varphi(\mathfrak{x})$.*
(2) *Let $\varphi \in (|X| \multimap \mathsf{V})$. Then*

$$\forall \mathfrak{x} \in TX \,.\, \varphi(\mathfrak{x}) \leq [[m_X, \hom_\xi]](Ty(\mathfrak{x}), \varphi) \quad \Longleftrightarrow \quad \varphi : X^{\mathrm{op}} \longrightarrow \mathsf{V} \textit{ is a } \mathcal{T}\textit{-functor}.$$

Consequently, we put $\hat{X} = (\hat{X}, \hat{a})$ where

$$\hat{X} = \{\psi \in (|X| \multimap \mathsf{V}) \mid \psi : X^{\mathrm{op}} \longrightarrow \mathsf{V} \text{ is a } \mathcal{T}\text{-functor}\}$$

considered as a subcategory of $|X| \multimap \mathsf{V}$, so that $\hat{a}$ is the restriction of $[[m_X, \hom_\xi]]$ to $\hat{X}$ (see Section 4.3). In particular, $y : X \longrightarrow \hat{X}$ is full and faithful.

Example 4.10 For $X \in \mathcal{U}_2\text{-}\mathsf{Cat} \cong \mathsf{Top}$, every $\psi \in \hat{X}$ is the characteristic function of a Zariski-closed and down-closed subset $\mathcal{A} \subseteq UX$ (see Example 4.4). Using the ψ-$\mathcal{A}$-exchange, we will now give an alternative description of $\hat{X}$, as the set $F_0(X)$ of (possibly improper) filters on the lattice τ of open sets of X, in terms of the bijective maps

$$\hat{X} \xrightarrow{\Phi} F_0(X) \qquad \text{and} \qquad F_0(X) \xrightarrow{\Pi} \hat{X},$$

where $\Phi(\mathcal{A}) = \bigcap \mathcal{A} \cap \tau$ and $\Pi(\mathfrak{f}) = \{\mathfrak{x} \in UX \mid \mathfrak{f} \subseteq \mathfrak{x}\}$. Clearly, $\mathcal{A} = \Pi(\mathfrak{f})$ is Zariski-closed. If $\mathfrak{x} \leq \mathfrak{y}$ for some $\mathfrak{x} \in UX$ and $\mathfrak{y} \in \mathcal{A}$, then, for each $A \in \mathfrak{x}$ and $B \in \mathfrak{f}$, one has

$$\overline{A} \cap B \neq \emptyset$$

which, since B is open, gives $A \cap B \neq \emptyset$. Hence $\mathfrak{f} \subseteq \mathfrak{x}$, that is, $\mathfrak{x} \in \mathcal{A}$. Furthermore, one easily proves $\mathfrak{f} = \Phi\Pi(\mathfrak{f})$ and $\mathcal{A} \subseteq \Pi\Phi(\mathcal{A})$. On the other hand, for $\mathfrak{x} \supseteq \bigcap \mathcal{A} \cap \tau$ and $A \in \mathfrak{x}$ we have $X \setminus \overline{A} \notin \bigcap \mathcal{A}$, and therefore $X \setminus \overline{A} \notin \mathfrak{x}$ for some $\mathfrak{x} \in \mathcal{A}$, hence $\overline{A} \in \mathfrak{x}$. Consequently, $\overline{A} \subseteq \bigcup \mathcal{A}$ and, since $\mathcal{A}$ is Zariski-closed, $\mathfrak{x} \leq \mathfrak{y}$ for some $\mathfrak{y} \in \mathcal{A}$. But $\mathcal{A}$ is also down-closed, hence $\mathfrak{x} \in \mathcal{A}$. Similarly and, in fact, more easily one can show that there are bijective maps

$$\check{X} \xrightarrow{\Phi'} F_1(X) \qquad \text{and} \qquad F_1(X) \xrightarrow{\Pi'} \check{X},$$

where $\check{X} = \{\mathcal{A} \subseteq UX \mid \mathcal{A} \text{ is Zariski-closed and up-closed}\}$, $F_1(X)$ is the set of all (possibly improper) filters on the lattice σ of closed sets of X, $\Phi'(\mathcal{A}) = \bigcap \mathcal{A} \cap \sigma$ and $\Pi'(\mathfrak{f}) = \{\mathfrak{x} \in UX \mid \mathfrak{f} \subseteq \mathfrak{x}\}$. Furthermore, for any Zariski-closed $\mathcal{A} \subseteq UX$, its

down-closure $\downarrow\mathcal{A}$ is Zariski-closed as well. To see this, let $\mathfrak{x} \in \mathrm{cl}(\downarrow\mathcal{A})$. Hence $\mathfrak{x} \in \bigcup \downarrow\mathcal{A}$ and therefore, for any $A \in \mathfrak{x}$, we have $\overline{A} \in \bigcup \mathcal{A}$. The set

$$\mathfrak{j} = \{B \subseteq X \mid \forall \mathfrak{a} \in \mathcal{A} \,.\, B \notin \mathfrak{a}\}.$$

is an ideal, and $\mathfrak{j} \cap \{\overline{A} \mid A \in \mathfrak{x}\} = \varnothing$. Hence, there is some $\mathfrak{y} \in UX$ such that $\mathfrak{x} \leq \mathfrak{y}$ and $\mathfrak{j} \cap \mathfrak{y} = \varnothing$. But the latter fact gives us $\mathfrak{y} \subseteq \bigcup \mathcal{A}$, that is, $\mathfrak{y} \in \mathrm{cl}\,\mathcal{A} = \mathcal{A}$. We conclude $\mathfrak{x} \in \downarrow\mathcal{A}$. Similarly one can show that $\uparrow\mathcal{A}$ is Zariski-closed for each Zariski-closed subset $\mathcal{A} \subseteq UX$ (but now use $\mathfrak{x} \in \mathrm{cl}(\uparrow\mathcal{A}) \iff \bigcap \uparrow\mathcal{A} \subseteq \mathfrak{x}$).

The topology of $\hat{X}$ is the *compact-open topology*, which has as basic open sets

$$B(\mathcal{B}, \{0\}) = \{\mathcal{A} \in \hat{X} \mid \mathcal{A} \cap \mathcal{B} = \varnothing\} \qquad (\mathcal{B} \subseteq UX \text{ Zariski-closed}).$$

Since $B(\mathcal{B}, \{0\}) = B(\uparrow\mathcal{B}, \{0\})$, it is enough to consider Zariski-closed and up-closed subsets $\mathcal{B} \subseteq UX$. Hence, using the bijections $\hat{X} \cong F_0(X)$ and $\check{X} \cong F_1(X)$, $F_0(X)$ has

$$\{\mathfrak{f} \in F_0(X) \mid \exists A \in \mathfrak{f}, B \in \mathfrak{g} \,.\, A \cap B = \varnothing\} \qquad (\mathfrak{g} \in F_1(X))$$

as basic open sets. Clearly, it is enough to consider $\mathfrak{g} = \dot{B}$ the principal filter induced by a closed set B, so that all sets

$$\{\mathfrak{f} \in F_0(X) \mid \exists A \in \mathfrak{f} \,.\, A \cap B = \varnothing\} = \{\mathfrak{f} \in F_0(X) \mid X \setminus B \in \mathfrak{f}\} \quad (B \subseteq X \text{ closed})$$

form a basis for the topology on $F_0(X)$. But this is precisely the topology on $F_0(X)$ considered in [8].

4.6 L-separation

We call a $\mathcal{T}$-category $X = (X, a)$ *L-separated* whenever, for every $\mathcal{T}$-category Y, the ordered set $\mathcal{T}\text{-}\mathsf{Cat}(Y, X)$ is separated, that is: its preorder is anti-symmetric. The full subcategory of $\mathcal{T}\text{-}\mathsf{Cat}$ consisting of all L-separated $\mathcal{T}$-categories is denoted by $\mathcal{T}\text{-}\mathsf{Cat}_{\mathrm{sep}}$.

Proposition 4.11 *Let $X = (X, a)$ be a $\mathcal{T}$-category. Then the following assertions are equivalent:*

(i) *X is L-separated.*
(ii) *$x \cong y$ implies $x = y$, for all $x, y \in X$.*
(iii) *For all $x, y \in X$, if $a(e_X(x), y) \geq k$ and $a(e_X(y), x) \geq k$, then $x = y$.*
(iv) *$y : X \longrightarrow \hat{X}$ is injective.*

Proof As for Proposition 2.5. □

Corollary 4.12

(1) *The $\mathcal{T}$-category $\mathsf{V} = (\mathsf{V}, \hom_\xi)$ is separated.*
(2) *For all $\mathcal{T}$-categories $Y = (Y, b)$ and $X = (X, a)$ where Y is L-separated and $a \cdot T_\xi a = a \cdot m_X$, Y^X is L-separated. In particular, $|X| \multimap \mathsf{V}$ is L-separated, for each $\mathcal{T}$-category X.*
(3) *Any subcategory of an L-separated $\mathcal{T}$-category is L-separated. In particular, $\hat{X}$ is L-separated, for every $\mathcal{T}$-category X.*

Example 4.13 A topological space is L-separated if, and only if, it is T_0. An approach space $X = (X, d)$ with distance function $d : PX \times X \longrightarrow \mathsf{P}_+$ is L-separated if and only if

$$d(\{x\}, y) = 0 = d(\{y\}, x) \;\Rightarrow\; x = y$$

for all $x, y \in X$.

4.7 L-completeness

As in Section 2.7, we call a $\mathcal{T}$-category $X = (X, a)$ *L-complete* if every adjunction $\varphi \dashv \psi$ with $\varphi : Z \multimapinv X$ and $\psi : X \multimapinv Z$ is of the form $f_* \dashv f^*$ for a $\mathcal{T}$-functor $f : Z \longrightarrow X$. Of course, Up to equivalence, f is uniquely determined by $\varphi \dashv \psi$, and is indeed unique if X is L-separated. As before, it is enough to consider $Z = E$ (see also [5]).

Proposition 4.14 *Let $X = (X, a)$ be a $\mathcal{T}$-category. The following assertions are equivalent:*

(i) *X is L-complete.*
(ii) *Each left adjoint $\mathcal{T}$-module $\varphi : E \multimapinv X$ is of the form $\varphi = x_*$ for some x in X.*
(iii) *Each right adjoint $\mathcal{T}$-module $\psi : X \multimapinv E$ is of the form $\psi = x^*$ for some x in X.*

A topological space is L-complete precisely if it is weakly-sober, that is, if every irreducible closed set is the closure of a point. A similar result holds for approach spaces: L-completeness is equivalent to the condition that every irreducible closed variable set A be representable (see [5] for details). Furthermore, in both cases we obtain that L-complete and L-separated objects are precisely the fixed objects of the dual adjunction between topological (approach) spaces and (approach) frames, with dualizing object $\mathsf{V} = \mathsf{2}$ ($\mathsf{V} = \mathsf{P}_+$ respectively; see [19] for details).

For a pair $\psi : X \multimapinv Y$ and $\varphi : Y \multimapinv X$ of adjoint $\mathcal{T}$-modules $\varphi \dashv \psi$, the same calculation as in Section 2.7 shows that $\varphi = 1_X^* \circ\!\!- \psi$. Since for each $\mathcal{T}$-module $\psi : X \multimapinv Y$ one obtains $(1_X^* \circ\!\!- \psi) \circ \psi \leq 1_X^*$, ψ is right adjoint if and only if $\psi \circ (1_X^* \circ\!\!- \psi) \geq (1_Y)_*$. Considering in particular $Y = E$, a $\mathcal{T}$-module $\psi : X \multimapinv E$ is right adjoint if, and only if,

$$k \leq \bigvee_{\mathfrak{x} \in TX} \psi(\mathfrak{x}) \otimes \xi \cdot T\varphi(\mathfrak{x})$$

where $\varphi = 1_X^* \circ\!\!- \psi$. Note that $\bigvee\{\xi \cdot T\psi(\mathfrak{X}) \mid \mathfrak{X} \in TTX, m_X(\mathfrak{X}) = \mathfrak{x}\} = \psi(\mathfrak{x})$ since $\psi : |X| \longrightarrow \mathsf{V}$ is a $\mathcal{T}$-functor Hence, with the help of Lemma 4.5, we see that

$$\begin{aligned}
\varphi(x) &= \bigwedge_{\mathfrak{x} \in TX} \left(\left(\bigvee_{\substack{\mathfrak{X} \in TTX, \\ m_X(\mathfrak{X}) = \mathfrak{x}}} \xi \cdot T\psi(\mathfrak{X}) \right) \multimap a(\mathfrak{x}, x) \right) \\
&= \bigwedge_{\mathfrak{x} \in TX} (\psi(\mathfrak{x}) \multimap a(\mathfrak{x}, x)) \\
&= \hat{a}(e_{\hat{X}}(\psi), y(x)).
\end{aligned}$$

Lemma 4.15 *Let $\psi : X \multimap E$ be a $\mathcal{T}$-module and put $\varphi = 1_X^* \circ\!- \psi$. Then, for every $\mathfrak{x} \in TX$,*

$$\xi \cdot T\varphi(\mathfrak{x}) = T_\xi \hat{a}(e_{T\hat{X}} \cdot e_{\hat{X}}(\psi), Ty(\mathfrak{x})).$$

Proof Since $\xi \cdot T\varphi(\mathfrak{x}) = T_\xi \varphi(\mathfrak{x})$, one obtains the result by applying T_ξ to the equality above. □

Hence, we now have:

Proposition 4.16 *Let $X = (X, a)$ be $\mathcal{T}$-category. A $\mathcal{T}$-module $\psi : X \multimap E$ is right adjoint if and only if*

$$k \leq \bigvee_{\mathfrak{x} \in TX} \psi(\mathfrak{x}) \otimes T_\xi \hat{a}(e_{T\hat{X}} \cdot e_{\hat{X}}(\psi), Ty(\mathfrak{x})). \tag{†}$$

Given a $\mathcal{T}$-category $X = (X, a)$, we call a $\mathcal{T}$-functor $\psi : |X| \longrightarrow \mathsf{V}$ *tight* if $\psi : X^{\mathrm{op}} \longrightarrow \mathsf{V}$ is a $\mathcal{T}$-functor and if, considered as a $\mathcal{T}$-module $\psi : X \multimap E$, it is right adjoint, that is, if it satisfies (†).

Example 4.17 For a topological space X and $\psi \in \hat{X}$, as in Example 4.10 we can identify ψ with a Zariski-closed and down-closed subset $\mathcal{A} \subseteq UX$, and then $1_X^* \circ\!- \psi$ with

$$A = \{x \in X \mid \forall \mathfrak{a} \in \mathcal{A} \,.\, \mathfrak{a} \longrightarrow x\}.$$

Then ψ is tight if, and only if, there exists some $\mathfrak{a} \in \mathcal{A}$ with $A \in \mathfrak{a}$. Furthermore, under the bijection $\hat{X} \cong F_0(X)$ (see Example 4.10), a tight map ψ corresponds to a filter $\mathfrak{f} \in F_0(X)$ with $(\operatorname{Lim} \mathfrak{f}) \# \mathfrak{f}$, where $\operatorname{Lim} \mathfrak{f}$ denotes the set of all limit points of $\mathfrak{f}$, and where $A \# \mathfrak{g}$ if for all $B \in \mathfrak{f} \,.\, A \cap B \neq \varnothing$. Furthermore, for each $\mathfrak{f} \in F_0(X)$ one has

$$(\operatorname{Lim} \mathfrak{f}) \# \mathfrak{f} \iff \mathfrak{f} \text{ is completely prime},$$

that is: if $\bigcup_{i \in I} U_i \in \mathfrak{f}$, then $U_i \in \mathfrak{f}$ for some $i \in I$. In fact, if $(\operatorname{Lim} \mathfrak{f}) \# \mathfrak{f}$ and $\bigcup_{i \in I} U_i \in \mathfrak{f}$ for some family of open subsets of X, then $(\operatorname{Lim} \mathfrak{f}) \cap \bigcup_{i \in I} U_i \neq \varnothing$. Therefore, for some $i \in I$, U_i contains a limit point of $\mathfrak{f}$. Hence $U_i \in \mathfrak{f}$. Conversely, assume that $\mathfrak{f}$ is completely prime. Suppose that $U \in \mathfrak{f}$ does not contain a limit point of $\mathfrak{f}$. Then, for each $x \in U$, there is an open neighborhood U_x of x with $U_x \notin \mathfrak{f}$. But $\bigcup_{x \in X} U_x \in \mathfrak{f}$ and, since $\mathfrak{f}$ is completely prime, $U_x \in \mathfrak{f}$ for some $x \in U$, a contradiction.

4.8 L-injectivity

The notions of L-dense $\mathcal{T}$-functor, L-equivalence as well as L-injective $\mathcal{T}$-category can now be introduced as in Section 2.8. More precise, we call a $\mathcal{T}$-functor $f : (X, a) \longrightarrow (Y, b)$ *L-dense* if $f_* \circ f^* = 1_X^*$, which amounts to $b \cdot Tf \cdot Tf^\circ \cdot T_\xi b \cdot m_Y^\circ = b$. L-dense $\mathcal{T}$-functors have the same composition-cancellation properties as V-functors (see Section 2.8). A fully faithful L-dense $\mathcal{T}$-functor is an *L-equivalence*, which can be equivalently expressed by saying that f_* is an isomorphism in $\mathcal{T}$-Mod. A $\mathcal{T}$-category Z is called *pseudo-injective* if, for all $\mathcal{T}$-functors $f : X \longrightarrow Z$ and fully faithful $\mathcal{T}$-functors $i : X \longrightarrow Y$, there exists a $\mathcal{T}$-functor $g : Y \longrightarrow Z$ such that $g \cdot i \cong f$. Z is called *L-injective* if this extension property is only required along

L-equivalences $i : X \longrightarrow Y$. Of course, for an L-separated $\mathcal{T}$-category Z, $g \cdot i \cong f$ implies $g \cdot i = f$, and then pseudo-injectivity coincides with the usual notion of injectivity. The following two results can be proven as in Section 2.8.

Lemma 4.18 *The $\mathcal{T}$-category V is injective.*

Proposition 4.19 *For all $\mathcal{T}$-categories $Y = (Y, b)$ and $X = (X, a)$ where Y is L-injective (pseudo-injective) and $a \cdot T_\xi a = a \cdot m_X$, Y^X is L-injective (pseudo-injective).*

In particular, we obtain the injectivity of the $\mathcal{T}$-category $|X| \multimap \mathsf{V}$. Lateron we will see that $\hat{X}$ and $\tilde{X}$ are also L-injective.

5 L-closure

5.1 L-dense $\mathcal{T}$-functors

As in Section 3.1, L-dense $\mathcal{T}$-functors can be characterized as "epimorphisms up to $\cong$". However, we will use here a different proof.

Lemma 5.1 *Let $X = (X, a)$ be a $\mathcal{T}$-category, $M \subseteq X$ and $i : M \hookrightarrow X$ the embedding of M into X. Then i is dense if, and only if,*

$$k \leq \bigvee_{\mathfrak{a} \in TM} a(\mathfrak{a}, x) \otimes T_\xi a(Te_X \cdot e_X(x), \mathfrak{a}) \tag{‡}$$

for all $x \in X$.

Proof Recall that i is L-dense whenever $i_* \circ i^* \geq a$, that is,

$$a(\mathfrak{x}, x) \leq \bigvee_{\mathfrak{a} \in TM} \bigvee_{\substack{\mathfrak{X} \in TTX \\ m_X(\mathfrak{X}) = \mathfrak{x}}} a(\mathfrak{a}, x) \otimes T_\xi a(\mathfrak{X}, \mathfrak{a})$$

for all $\mathfrak{x} \in TX$ and $x \in X$. If i is dense, then (‡) follows from the inequality above by putting $\mathfrak{x} = e_X(x)$ and using $m_X^\circ \cdot e_X = e_{TX} \cdot e_X$ (see Section 4.1). On the other hand, from (‡) we obtain

$$\begin{aligned}
a(\mathfrak{x}, x) &\leq \bigvee_{\mathfrak{a} \in TM} a(\mathfrak{a}, x) \otimes T_\xi a(Te_X \cdot e_X(x), \mathfrak{a}) \otimes a(\mathfrak{x}, x) \\
&\leq \bigvee_{\mathfrak{a} \in TM} a(\mathfrak{a}, x) \otimes T_\xi T_\xi a(e_{TTX} \cdot e_{TX}(\mathfrak{x}), e_{TX} \cdot e_X(x)) \otimes T_\xi a(Te_X \cdot e_X(x), \mathfrak{a}) \\
&\leq \bigvee_{\mathfrak{a} \in TM} a(\mathfrak{a}, x) \otimes T_\xi a(e_{TX}(\mathfrak{x}), \mathfrak{a}) \\
&\leq \bigvee_{\mathfrak{a} \in TM} \bigvee_{\substack{\mathfrak{X} \in TTX \\ m_X(\mathfrak{X}) = \mathfrak{x}}} a(\mathfrak{a}, x) \otimes T_\xi a(\mathfrak{X}, \mathfrak{a}).
\end{aligned}$$

□

Proposition 5.2 *For a $\mathcal{T}$-functor $i : M \longrightarrow X$, the following assertions are equivalent:*

(i) *$i : M \longrightarrow X$ is L-dense.*
(ii) *For all $\mathcal{T}$-functors $f, g : X \longrightarrow Y$, with $f \cdot i = g \cdot i$ one has $f \cong g$.*
(iii) *For all $\mathcal{T}$-functors $f, g : X \longrightarrow \mathsf{V}$, with $f \cdot i = g \cdot i$ one has $f = g$.*

Proof Assuming (i), so that $i : M \longrightarrow X$ is L-dense, from $f \cdot i = g \cdot i$ we obtain $f_* = g_*$ and therefore (ii) since $i_* \circ i^* = 1^*_X$. The implication (ii)⇒(iii) holds trivially since V is L-separated. Now assume (iii). According to the remarks made above, we can assume that $i : M \longrightarrow X$ is the embedding of a subset $M \subseteq X$. For $x \in X$, First note that the map

$$\varphi : X \longrightarrow \mathsf{V}, \;\; y \longmapsto a(e_X(x), y)$$

is a $\mathcal{T}$-functor since $a : |X| \otimes X \longrightarrow \mathsf{V}$ is one. Using the same argument as in [11, Lemma 6.8], we see that also

$$\psi : X \longrightarrow \mathsf{V}, \;\; y \longmapsto \bigvee_{\mathfrak{x} \in TM} T_{\xi} a(Te_X \cdot e_X(x), \mathfrak{x}) \otimes a(\mathfrak{x}, y)$$

is a $\mathcal{T}$-functor. Clearly, for each $y \in X$ we have $\psi(y) \leq \varphi(y)$. If $y \in M$, we can choose $\mathfrak{x} = e_X(y) \in TM$ and therefore, using $Te_X \cdot e_X = e_{TX} \cdot e_X$ and op-laxness of e, obtain $\varphi(y) \leq \psi(y)$. Hence $\varphi|_M = \psi|_M$, and from our assumption (iii) we deduce $k \leq \varphi(x) = \psi(x)$. □

5.2 L-closure

For a $\mathcal{T}$-category $X = (X, a)$ and $M \subseteq X$, we define the *L-closure* of M in X by

$$\overline{M} = \{x \in X \mid \forall f, g : X \longrightarrow Y \,.\, (f|_M = g|_M \;\Rightarrow\; f(x) \cong g(x))\}.$$

Hence $\overline{M}$ is the largest subset N of X making the inclusion map $i : M \hookrightarrow N$ dense.

Proposition 5.3 *Let $X = (X, a)$ be a $\mathcal{T}$-category, $M \subseteq X$ and $x \in X$. Then the following assertions are equivalent.*

(i) *$x \in \overline{M}$.*
(ii) *$k \leq \bigvee_{\mathfrak{x} \in TM} a(\mathfrak{x}, x) \otimes T_{\xi} a(Te_X \cdot e_X(x), \mathfrak{x})$.*
(iii) *$i^* \circ x_* \dashv x^* \circ i_*$, where $i : M \hookrightarrow X$ is the inclusion map.*
(iv) *$1^*_E \leq x^* \circ i_* \circ i^* \circ x_*$,*
(v) *$i^* \circ x_* \dashv x^* \circ i_*$.*
(vi) *$x_* : E \multimap X$ factors through $i_* : M \multimap X$ by a map $\varphi : E \multimap M$ in $\mathcal{T}$-Mod.*

Proof Using Lemma 5.1 one can proceed as in Proposition 3.2. □

We can now proceed as in Section 3.2.

Proposition 5.4 *For a $\mathcal{T}$-functor $f : X \longrightarrow Y$ and $M, M' \subseteq X$, $N \subseteq Y$, one has:*

(1) *$M \subseteq \overline{M}$ and $M \subseteq M'$ implies $\overline{M} \subseteq \overline{M'}$.*
(2) *$\overline{\overline{M}} = \overline{M}$ and, if $T\varnothing = \varnothing$, then $\overline{\varnothing} = \varnothing$.*

(3) $f(\overline{M}) \subseteq \overline{f(M)}$ *and* $f^{-1}(\overline{N}) \supseteq \overline{f^{-1}(N)}$.
(4) *If k is $\vee$-irreducible and T preserves binary sums, then* $\overline{M \cup M'} = \overline{M} \cup \overline{M'}$.

Corollary 5.5 *If k is $\vee$-irreducible in* V *and T preserves finite sums, then the L-closure operator defines a topology on X such that every $\mathcal{T}$-functor becomes continuous. Hence, L-closure defines a functor $L : \mathcal{T}\text{-}\mathsf{Cat} \longrightarrow \mathsf{Top}$.*

Example 5.6 For a topological space X, $x \in X$ lies in the L-closure of $A \subseteq X$ precisely if there exists some ultrafilter $\mathfrak{x} \in UA$ with $\overline{x} \in \mathfrak{x}$ and which converges to x; in other words, for every neighborhood U of x we have $U \cap \overline{x} \cap A \neq \varnothing$. Hence the L-closure of a topological space X coincides with the so called *b-closure* [1].

5.3 L-separation via L-closure

Proposition 5.7 *Let $X = (X, a)$ be a $\mathcal{T}$-category and $\Delta \subseteq X \times X$ the diagonal. Then*

$$\overline{\Delta} = \{(x, y) \in X \times X \mid x \cong y\}.$$

Proof As for Proposition 3.6. □

Corollary 5.8 *A $\mathcal{T}$-category X is L-separated if and only if the diagonal Δ is closed in $X \times X$.*

Theorem 5.9 $\mathcal{T}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ *is an epi-reflective subcategory of* $\mathcal{T}\text{-}\mathsf{Cat}$, *where the reflection map is given by $y_X : X \longrightarrow y_X(X)$, for each $\mathcal{T}$-category X. Hence, limits of L-separated $\mathcal{T}$-categories are formed in* $\mathcal{T}\text{-}\mathsf{Cat}$, *while colimits are obtained by reflecting the colimit formed in* $\mathcal{T}\text{-}\mathsf{Cat}$. *The epimorphisms in* $\mathcal{T}\text{-}\mathsf{Cat}_{\mathrm{sep}}$ *are precisely the L-dense $\mathcal{T}$-functors.*

5.4 L-completeness via L-closure

Lemma 5.10 *Let $X = (X, a)$ be a $\mathcal{T}$-category and $M \subseteq X$.*

(1) *Assume that X is L-complete and M be L-closed. Then M is L-complete.*
(2) *Assume that X is L-separated and M is L-complete. Then M is L-closed.*

Proof As for Lemma 3.9. □

Theorem 5.11 *Let $X = (X, b)$ be a $\mathcal{T}$-category. The following assertions are equivalent.*

(i) *X is L-complete.*
(ii) *X is L-injective.*
(iii) *$y : X \longrightarrow \tilde{X}$ has a pseudo left-inverse $\mathcal{T}$-functor $R : \tilde{X} \longrightarrow X$, i.e. $R \cdot y \cong 1_X$.*

Proof As for Theorem 3.10. □

Therefore we have that $|X| \multimap \mathsf{V}$ is L-complete. Our next result shows that $\hat{X}$ is also L-complete.

Proposition 5.12 *$\hat{X}$ is L-closed in $|X| \multimap \mathsf{V}$, for each $\mathcal{T}$-category X.*

Proof Let $X = (X, a)$ be a $\mathcal{T}$-category and assume that $\varphi \in (|X| \multimap \mathsf{V})$ belongs to the closure of $\hat{X}$, that is,

$$k \leq \bigvee_{\mathfrak{u} \in T\hat{X}} [[m_X, \hom_\xi]](\mathfrak{u}, \varphi) \otimes T_\xi[[m_X, \hom_\xi]](Te_{|X| \multimap \mathsf{V}} \cdot e_{|X| \multimap \mathsf{V}}(\varphi), \mathfrak{u}).$$

We wish to show that $r(\mathfrak{x}, \mathfrak{y}) \otimes \varphi(\mathfrak{y}) \leq \varphi(\mathfrak{x})$ for all $\mathfrak{x}, \mathfrak{y} \in TX$, where $r = T_\xi a \cdot m_X^\circ$.

First note that, for all $\alpha, \beta \in (|X| \multimap \mathsf{V})$,

$$e^\circ_{|X| \multimap \mathsf{V}} \cdot [[m_X, \hom_\xi]](\alpha, \beta) = \bigwedge_{\mathfrak{x} \in TX} (\alpha(\mathfrak{x}) \multimap \beta(\mathfrak{x})).$$

Hence, with $h_\mathfrak{x} : (|X| \multimap \mathsf{V}) \nrightarrow (|X| \multimap \mathsf{V})$, $h_\mathfrak{x}(\alpha, \beta) = (\alpha(\mathfrak{x}) \multimap \beta(\mathfrak{x}))$, we have $Te^\circ_{|X| \multimap \mathsf{V}} \cdot T_\xi[[m_X, \hom_\xi]] \leq T_\xi h_\mathfrak{x}$. Since the diagram

$$\begin{tikzcd}
(|X| \multimap \mathsf{V}) \times (|X| \multimap \mathsf{V}) \arrow[r, "\mathrm{ev}_\mathfrak{x} \times \mathrm{ev}_\mathfrak{x}"] \arrow[dr, "h_\mathfrak{x}"'] & \mathsf{V} \times \mathsf{V} \arrow[d, "\multimap"] \\
& \mathsf{V}
\end{tikzcd}$$

commutes, and since in the diagram

$$\begin{tikzcd}
T((|X| \multimap \mathsf{V}) \times (|X| \multimap \mathsf{V})) \arrow[r, "T(\mathrm{ev}_\mathfrak{x} \times \mathrm{ev}_\mathfrak{x})"] \arrow[d, "\mathrm{can}"'] & T(\mathsf{V} \times \mathsf{V}) \arrow[r, "T(\multimap)"] \arrow[d, "\mathrm{can}"'] & T\mathsf{V} \arrow[dd, "\xi"] \\
T(|X| \multimap \mathsf{V}) \times T(|X| \multimap \mathsf{V}) \arrow[r, "T\,\mathrm{ev}_\mathfrak{x} \times T\,\mathrm{ev}_\mathfrak{x}"'] & T\mathsf{V} \times T\mathsf{V} \arrow[d, "\xi \times \xi"'] \arrow[ur, phantom, "\geq"] & \\
& \mathsf{V} \times \mathsf{V} \arrow[r, "\multimap"'] & \mathsf{V}
\end{tikzcd}$$

the left hand rectangle commutes whereas in the right hand rectangle the "lower path" is greater or equal to the "upper path", we obtain

$$T_\xi h_\mathfrak{x}(\mathfrak{u}, \mathfrak{v}) \leq (\mathfrak{u}(\mathfrak{x}) \multimap \mathfrak{v}(\mathfrak{x}))$$

for every $\mathfrak{x} \in TX$ and $\mathfrak{u}, \mathfrak{v} \in T(|X| \multimap \mathsf{V})$, where $\mathfrak{u}(\mathfrak{x}) = \xi \cdot T\,\mathrm{ev}_\mathfrak{x}(\mathfrak{u})$. Accordingly, $e_{|X| \multimap \mathsf{V}}(\varphi)(\mathfrak{x}) = \varphi(\mathfrak{x})$, and we obtain

$$\forall \mathfrak{x} \in TX \,.\, T_\xi[[m_X, \hom_\xi]](Te_{|X| \multimap \mathsf{V}} \cdot e_{|X| \multimap \mathsf{V}}(\varphi), \mathfrak{u}) \leq (\varphi(\mathfrak{x}) \multimap \mathfrak{u}(\mathfrak{x})).$$

Furthermore, for all $\mathfrak{x}, \mathfrak{y} \in TX$ one has

$$\begin{tikzcd}
\hat{X} \arrow[r, "\Delta"] \arrow[d, "!"'] \arrow[dr, phantom, "\leq"] & \hat{X} \times \hat{X} \arrow[r, "\mathrm{ev}_{\mathfrak{x}} \times \mathrm{ev}_{\mathfrak{y}}"] & \mathsf{V} \times \mathsf{V} \arrow[d, "\multimap"] \\
1 \arrow[rr, "r(\mathfrak{x},\mathfrak{y})"'] & & \mathsf{V}
\end{tikzcd}$$

and obtains

$$r(\mathfrak{x}, \mathfrak{y}) \leq \xi \cdot T(\multimap) \cdot T(\mathrm{ev}_{\mathfrak{x}} \times \mathrm{ev}_{\mathfrak{y}}) \cdot T\Delta(\mathfrak{u}) \leq (\mathfrak{u}(\mathfrak{y}) \multimap \mathfrak{u}(\mathfrak{x}))$$

for all $\mathfrak{u} \in T\hat{X}$. We conclude that

$$\begin{aligned}
r(\mathfrak{x}, \mathfrak{y}) \otimes \varphi(\mathfrak{y}) &\leq \bigvee_{\mathfrak{u} \in T\hat{X}} r(\mathfrak{x}, \mathfrak{y}) \otimes \varphi(\mathfrak{y}) \\
&\quad \otimes [[m_X, \hom_\xi]](\mathfrak{u}, \varphi) \otimes T_\xi[[m_X, \hom_\xi]](Te_{|X| \multimap \mathsf{V}} \cdot e_{|X| \multimap \mathsf{V}}(\varphi), \mathfrak{u}) \\
&\leq \bigvee_{\mathfrak{u} \in T\hat{X}} r(\mathfrak{x}, \mathfrak{y}) \otimes \varphi(\mathfrak{y}) \otimes (\varphi(\mathfrak{y}) \multimap \mathfrak{u}(\mathfrak{y})) \otimes [[m_X, \hom_\xi]](\mathfrak{u}, \varphi) \\
&\leq \bigvee_{\mathfrak{u} \in T\hat{X}} r(\mathfrak{x}, \mathfrak{y}) \otimes \mathfrak{u}(\mathfrak{y}) \otimes [[m_X, \hom_\xi]](\mathfrak{u}, \varphi) \\
&\leq \bigvee_{\mathfrak{u} \in T\hat{X}} \mathfrak{u}(\mathfrak{x}) \otimes [[m_X, \hom_\xi]](\mathfrak{u}, \varphi) \leq \bigvee_{\mathfrak{u} \in T\hat{X}} \mathfrak{u}(\mathfrak{x}) \otimes (\mathfrak{u}(\mathfrak{x}) \multimap \varphi(\mathfrak{x})) \leq \varphi(\mathfrak{x}).
\end{aligned}$$

□

Proposition 5.13 *Let $X = (X, a)$ be a $\mathcal{T}$-category and $\psi \in \hat{X}$. Then ψ is a right adjoint $\mathcal{T}$-module if and only if $\psi \in \overline{y(X)}$.*

Proof By Proposition 4.16 and Theorem 4.9, ψ is right adjoint if, and only if,

$$k \leq \bigvee_{\mathfrak{x} \in TX} \hat{a}(Ty(\mathfrak{x}), \psi) \otimes T_\xi \hat{a}(Te_{\hat{X}} \cdot e_{\hat{X}}(\psi), Ty(\mathfrak{x})),$$

which means precisely that $\psi \in \overline{y(X)}$. □

The proposition above identifies $\tilde{X}$ as the L-closure of $y(X)$ in $\hat{X}$, and therefore as an L-complete $\mathcal{T}$-category. Furthermore, $y : X \longrightarrow \tilde{X}$ is fully faithful and L-dense. Hence we can state:

Theorem 5.14 *The full subcategory $\mathcal{T}$-$\mathsf{Cat}_{\mathrm{cpl}}$ of $\mathcal{T}$-$\mathsf{Cat}_{\mathrm{sep}}$ of L-complete $\mathcal{T}$-categories is an epi-reflective subcategory of $\mathcal{T}$-$\mathsf{Cat}_{\mathrm{sep}}$. The reflection map of an L-separated $\mathcal{T}$-category X is given by any full L-dense embedding of X into an L-complete and L-separated $\mathcal{T}$-category, for instance by $y : X \longrightarrow \tilde{X}$.*

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Reprints in Theory and Applications of Categories, No. 10, 2005.

BASIC CONCEPTS OF ENRICHED CATEGORY THEORY

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Received by the editors 2004-10-30.
Transmitted by Steve Lack, Ross Street and RJ Wood. Reprint published on 2005-04-23.
Several typographical errors corrected 2012-05-13.
2000 Mathematics Subject Classification: 18-02, 18D10, 18D20.
Key words and phrases: enriched categories, monoidal categories.
Originally published by Cambridge University Press, London Mathematical Society Lecture Notes Series 64, 1982
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Acknowledgements for the Reprint:

From the author:

I want first to express my very deep gratitude to the volunteer typists whose efforts have allowed the production of this attractive reprint of 'Basic Concepts of Enriched Category Theory'. They are: Richard Garner, who had begun to re-type the book for his own use and offered several chapters, along with Toby Bartels, William Boshuck, Maria Manuel Clementino, Robert Dawson, Steve Lack, Tom Leinster, Francisco Marmolejo, Shane O'Conchuir, Craig Pastro, Mark Weber and Ralph Wojtowicz – each of whom handled several sections. Thanks also to the many other colleagues who offered to help with the typing.

I am in addition particularly indebted to Donovan Van Osdol for his careful proof-reading of the text; to Michael Barr for his coordination of the volunteer effort, for his elegant typesetting of the diagrams, and for his fine work on the index; and to Bob Rosebrugh, the Managing Editor, for his constant encouragement and for many kindnesses: to these, too, I offer my very sincere thanks.

G. Max Kelly
Sydney, April 2005

From the Editors:

It is a pleasure to add this book to the Reprints in Theory and Applications of Categories. There have been requests for its inclusion since we began the series.

As we did for Jon Beck's thesis, we asked for volunteers to retype the text and were again overwhelmed by the response – nearly 30 colleagues volunteered within a day of sending the request. We warmly thank everyone who volunteered and especially those we asked to do the work.

Serendipitously, Richard Garner had begun to re-type the book for his own use and kindly offered his excellent work – much of the first three chapters. We then selected other volunteers to type several sections each. The typists we thank, in addition to Richard Garner, are: Toby Bartels, William Boshuck, Maria Manuel Clementino, Robert Dawson, Steve Lack, Tom Leinster, Francisco Marmolejo, Shane O'Conchuir, Craig Pastro, Mark Weber and Ralph Wojtowicz. Everyone did their part promptly and followed instructions that simplified coordination.

We also thank Donovan Van Osdol for providing a superb proof-reading job and Michael Barr for coordinating the many different files, for rendering the diagrams beautifully using his own diagxy front end for xy-pic, and for the difficult task of creating the index.

The few typographical errors from the original of which the author was aware, and a few noted by the typists, have been corrected without comment.

The Editors of *Theory and Applications of Categories*

To my wife and children

Contents

Introduction **1**

1 The elementary notions **7**

1.1 Monoidal categories . . . 7
1.2 The 2-category $\mathcal{V}$-**CAT** for a monoidal $\mathcal{V}$ . . . 8
1.3 The 2-functor $(\ )_0 \colon \mathcal{V}\text{-}\mathbf{CAT} \longrightarrow \mathbf{CAT}$ . . . 10
1.4 Symmetric monoidal categories: tensor product and duality . . . 11
1.5 Closed and biclosed monoidal categories . . . 13
1.6 $\mathcal{V}$ as a $\mathcal{V}$-category; representable $\mathcal{V}$-functors . . . 15
1.7 Extraordinary $\mathcal{V}$-naturality . . . 17
1.8 The $\mathcal{V}$-naturality of the canonical maps . . . 19
1.9 The (weak) Yoneda lemma for $\mathcal{V}$-**CAT** . . . 21
1.10 Representability of $\mathcal{V}$-functors; $\mathcal{V}$-functor of the passive variables . . . 22
1.11 Adjunctions and equivalences in $\mathcal{V}$-**CAT** . . . 23

2 Functor categories **27**

2.1 Ends in $\mathcal{V}$ . . . 27
2.2 The functor-category $[\mathcal{A}, \mathcal{B}]$ for small $\mathcal{A}$ . . . 29
2.3 The isomorphism $[\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] \cong [\mathcal{A}, [\mathcal{B}, \mathcal{C}]]$ . . . 31
2.4 The (strong) Yoneda lemma for $\mathcal{V}$-**CAT**; the Yoneda embedding . . . 33
2.5 The free $\mathcal{V}$-category on a **Set**-category . . . 35
2.6 Universe-enlargement $\mathcal{V} \subset \mathcal{V}'$; $[\mathcal{A}, \mathcal{B}]$ as a $\mathcal{V}'$-category . . . 35

3 Indexed limits and colimits **37**

3.1 Indexing types; limits and colimits; Yoneda isomorphisms . . . 37
3.2 Preservation of limits and colimits . . . 39
3.3 Limits in functor categories; double limits and iterated limits . . . 40
3.4 The connexion with classical conical limits when $\mathcal{V} = \mathbf{Set}$ . . . 42
3.5 Full subcategories and limits; the closure of a full subcategory . . . 44
3.6 Strongly generating functors . . . 46
3.7 Tensor and cotensor products . . . 48
3.8 Conical limits in a $\mathcal{V}$-category . . . 49
3.9 The inadequacy of conical limits . . . 50
3.10 Ends and coends in a general $\mathcal{V}$-category; completeness . . . 52
3.11 The existence of a limit-preserving universe-enlargement $\mathcal{V} \subset \mathcal{V}'$ . . . 54

3.12 The existence of a limit- and colimit-preserving universe-enlargement $\mathcal{V} \subset \mathcal{V}'$ 56

4 Kan extensions **59**
4.1 The definition of Kan extensions; their expressibility by limits and colimits 59
4.2 Elementary properties and examples 61
4.3 A universal property of $\mathrm{Lan}_K G$; its inadequacy as a definition 64
4.4 Iterated Kan extensions; Kan adjoints; $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ as the free cocompletion . . 66
4.5 Initial diagrams as the left Kan extensions into $\mathcal{V}$ 68
4.6 Filtered categories when $\mathcal{V} = \mathbf{Set}$ 71
4.7 Factorization into an initial functor and a discrete op-fibration 73
4.8 The general representability and adjoint-functor theorems 75
4.9 Representability and adjoint-functor theorems when $\mathcal{V} = \mathbf{Set}$ 77
4.10 Existence and characterization of the left Kan extension 80

5 Density **85**
5.1 Definition of density, and equivalent formulations 85
5.2 Composability and cancellability properties of dense functors 87
5.3 Examples of density; comparison with strong generation 89
5.4 Density of a fully-faithful K in terms of K-absolute colimits 92
5.5 Functor categories; small projectives; Morita equivalence 94
5.6 Left Kan extensions and adjoint functor theorems involving density 96
5.7 The free completion of $\mathcal{A}$ with respect to 98
5.8 Cauchy completion as the idempotent-splitting completion when $\mathcal{V} = \mathbf{Set}$. 100
5.9 The finite-colimit completion when $\mathcal{V} = \mathbf{Set}$ 101
5.10 The filtered-colimit completion when $\mathcal{V} = \mathbf{Set}$ 102
5.11 The image under $[K, 1]\colon [\mathcal{C}, \mathcal{B}] \longrightarrow [\mathcal{A}, \mathcal{B}]$ 104
5.12 Description of the image in terms of K-comodels 106
5.13 A dense $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ as an essentially-algebraic theory 109
5.14 The image under $[K, 1]\colon [\mathcal{C}, \mathcal{B}] \longrightarrow [\mathcal{A}, \mathcal{B}]$ 110

6 Essentially-algebraic theories defined by reguli and by sketches **113**
6.1 Locally-bounded categories; the local boundedness of $\mathcal{V}$ 113
6.2 The reflectivity in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ of the category of algebras for a regulus 115
6.3 The category of algebras for a sketch 118
6.4 The $\mathcal{F}$-theory generated by a small sketch 122
6.5 Symmetric monoidal closed structure on $\mathcal{F}$-cocomplete categories 124

Bibliography **125**

Index **131**

Introduction

(i) Although numerous contributions from divers authors, over the past fifteen years or so, have brought enriched category theory to a developed state, there is still no connected account of the theory, or even of a substantial part of it. As the applications of the theory continue to expand – some recent examples are given below – the lack of such an account is the more acutely felt.

The present book is designed to supply the want in part, by giving a fairly complete treatment of the limited area to which the title refers. The *basic* concepts of category theory certainly include the notion of functor-category, of limit and colimit, of Kan extension, and of density; with their applications to completions, perhaps including those relative completions given by categories of algebras for limit-defined theories. If we read "$\mathcal{V}$-category" for "category" here, this is essentially the list of our chapter-headings below, after the first chapter introducing $\mathcal{V}$-categories.

In fact our scope is wider than this might suggest; for what we give is also a self-contained account of basic category theory as described above, assuming as prior knowledge only the most elementary categorical concepts, and treating the ordinary and enriched cases together from Chapter 3 on.

(ii) In order to include the enriched case we begin in Chapter 1 by introducing symmetric monoidal closed categories $\mathcal{V}$, examining their elementary properties, and defining the 2-category $\mathcal{V}$-**CAT** of $\mathcal{V}$-categories, $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations, together with the forgetful 2-functor $(-)_0 \colon \mathcal{V}\text{-}\mathbf{CAT} \longrightarrow \mathbf{CAT}$; this much is easy and brief. Next, developing the basic structure of $\mathcal{V}$-**CAT** – tensor products of $\mathcal{V}$-categories, $\mathcal{V}$-functors of two variables, extraordinary $\mathcal{V}$-natural transformations, $\mathcal{V}$ itself as a $\mathcal{V}$-category, representable $\mathcal{V}$-functors, and the Yoneda lemma – requires verifications of diagram commutativity, whose analogues for $\mathcal{V} = \mathbf{Set}$ reduce to fairly trivial equations between functions. This seems to be an inevitable cost of the extra generality; but we have been at pains so to arrange the account that the reader should find the burden a light one. With this done, the discussion of representability, adjunction, and equivalences in $\mathcal{V}$-**CAT**, which closes Chapter 1, is simple and direct.

The short Chapter 2 takes up the closed structure of $\mathcal{V}$-**CAT**, given by the $\mathcal{V}$-functor-category $[\mathcal{B}, \mathcal{C}]$. Since the hom $[\mathcal{B}, \mathcal{C}](T, S)$ is to be an object of $\mathcal{V}$ and not merely a set, more work is once again required than in the $\mathcal{V} = \mathbf{Set}$ case. However this hom for small $\mathcal{B}$ is quite a simple limit in $\mathcal{V}_0$ – now supposed complete – of the special kind called an *end*, precisely adapted to the extraordinary $\mathcal{V}$-natural transformations; and from this definition flow easily the desired properties.

The indexed limits and colimits of Chapter 3, along with their various special cases,

are constructs of the greatest importance even when $\mathcal{V} = \mathbf{Set}$; and the relations between double and iterated indexed limits express a rich content. This importance is scarcely lessened by their reducibility when $\mathcal{V} = \mathbf{Set}$ to classical limits – any more than the importance of these classical ones is lessened by their reducibility in good cases to products and equalizers. Even in the case $\mathcal{V} = \mathbf{Ab}$ of additive categories, the indexed limits – although here they *exist* when all the classical ones do – are no longer *directly* reducible to the latter; while for a general $\mathcal{V}$ the indexed limits are essential, and the classical ones no longer suffice. Chapter 3 ends by showing how to expand $\mathcal{V}$ into a bigger universe, without disturbing limits and colimits, so as to allow the free use of functor-categories $[\mathcal{B}, \mathcal{C}]$ with large $\mathcal{B}$.

The remaining chapters 4, 5, and 6, dealing respectively with Kan extensions, density, and algebras defined by limits (or more generally by "topologies"), make use of these limit- and colimit-notions to complete the development of our chosen area of basic category theory. Most of the results apply equally to categories and to $\mathcal{V}$-categories, without a word's being changed in the statement or the proof; so that scarcely a word would be saved if we restricted ourselves to ordinary categories alone. Certainly this requires proofs adapted to the case of a general $\mathcal{V}$; but these almost always turn out to be the best proofs in the classical case $\mathcal{V} = \mathbf{Set}$ as well. This reflects the hopes with which Eilenberg and the author set out when writing [26], and Lawvere's observation in [54] that the relevant segment of classical logic "applies directly to structures valued in an arbitrary [symmetric monoidal] closed category $\mathcal{V}$".

Because of the special properties of **Set**, there are of course certain results peculiar to the case $\mathcal{V} = \mathbf{Set}$; and we accordingly devote an occasional section to this case alone. Some examples are the commutativity in **Set** of filtered colimits with finite limits; the notions of initial functor and of discrete op-fibration; and the classical adjoint-functor theorems. We treat all of these for completeness – partly to keep the account self-contained, and partly to compare them, where appropriate, with analogues for a general $\mathcal{V}$.

The little prior knowledge that we do assume is easily available – for instance from [60] – and to have included it here, with examples to enliven it, would have involved an unjustifiable increase in the length. For Chapter 1, it consists of the basic notions of category, functor and natural transformation, including functors of two or more variables and contravariant functors; representable functors and the Yoneda lemma; adjunction and equivalence; and what is meant by a faithful or a fully-faithful functor. From Chapter 2 on, we also need the classical notions of limit and colimit; with the names of such special limits as product, equalizer, pullback, and terminal object; and the meaning of monomorphism and epimorphism. In the rare places where more is assumed but not expounded here, references are given.

(iii) We now turn to what we have omitted from the present book; the list includes many important notions, well meriting an extended treatment, whose inclusion would however have disturbed the essential simplicity of an initial account such as this.

First (to return to the title), the basic concepts of *category* theory concern categories – in our case, $\mathcal{V}$-categories – as the objects of discussion. These form, or live in, a 2-category $\mathcal{V}$-**CAT**; but 2-categories are not yet the formal objects of discussion, any more than categories are when we study group theory. Category theory casts light on group theory,

as does 2-category theory on category theory. Hence the step to this next level, where weaker notions of natural transformation naturally arise, is an important one; but it is quite properly deferred pending experience of some particular 2-categories, and we do not take it here. We make a start on some aspects of it in the forthcoming [46].

Closely connected to this is our decision not to discuss the "change of base-category" given by a symmetric monoidal functor $\mathcal{V} \longrightarrow \mathcal{V}'$, and the induced 2-functor $\mathcal{V}$-**CAT** $\longrightarrow \mathcal{V}'$-**CAT**. We do, as we said, consider the forgetful 2-functor $(-)_0 \colon \mathcal{V}\text{-}\mathbf{CAT} \longrightarrow \mathbf{CAT}$ induced by the canonical $\mathcal{V} \longrightarrow \mathbf{Set}$; but this is entirely elementary, not involving even the definition of a symmetric monoidal functor. The general change of base, important though it is, is yet logically secondary to the basic $\mathcal{V}$-category theory it acts on. To treat it properly needs a careful analysis of the 2-category of symmetric monoidal categories, symmetric monoidal functors, and symmetric monoidal natural transformations – including adjunctions therein and the dual concept of op-monoidal functor. There is evidence in [43] that this itself is best studied in the more general context of categories with essentially-algebraic structure, which draws on the matter of this book together with [45] and [46].

Since the natural setting for the important work of Day ([12], [14], [16]) on the *construction* of symmetric monoidal closed categories as functor-categories, or as reflective subcategories of these, involves the 2-category of symmetric monoidal categories, this too has been omitted.

One thing we particularly regret omitting is the theory of monads; it could certainly be seen as basic category theory, yet there was no convenient place to put it, and it would have required an extra chapter to itself. Luckily, besides the account of Mac Lane [60] in the classical case, we have the articles of Linton [57], Dubuc [22], and Kock [51], covering various aspects of the enriched case. We also have the elegant 2-categorical treatment by Street [70], which provides some argument for deferring the topic until 2-categories have been more closely studied. A consequence is our failure to discuss the completeness and cocompleteness of the 2-category $\mathcal{V}$-**Cat** of small $\mathcal{V}$-categories – which is most easily referred back to the completeness and the cocompleteness of the algebras for a monad.

Finally, our account covers only what can be said for *every* well-behaved $\mathcal{V}$ (except for those things special to $\mathcal{V} = \mathbf{Set}$). Results valid only for a special class of $\mathcal{V}$ are best treated in separate articles, of which the author's forthcoming [45] is one example.

(iv) Our concern being to provide a development of basic category theory that covers the enriched case, we have given illustrations of many of the results in isolation, but have not thought this the place to discuss detailed applications to particular areas of mathematics. In fact, applications needing the enriched case are rapidly proliferating; the following is a selection of recent ones that have come to the author's notice.

The discussion of dualities in topological algebra, taking for $\mathcal{V}$ a suitable category of topological or quasitopological spaces, and initiated by Dubuc and Porta [23], has been continued in numerous articles by Dubuc, Day, and Nel. Pelletier [67] has made use of Kan extensions where $\mathcal{V}$ is Banach spaces. Recent Soviet work in representation theory (see the articles by Kleiner and Roiter, and by Golovaschuk, Ovsienko, and Roiter, in [64]) has differential graded modules for $\mathcal{V}$. The present author, as part of a study of essentially-algebraic structures borne by categories, has extended in [45] the classical

results on "cartesian theories" (finite-limit theories) from the case where $\mathcal{V}$ is sets to those where it is groupoids or categories.

The study of homotopy limits by Gray [32] takes him into the area of change of base-categories, involving the connexions between the closed categories of categories and of simplicial sets. The work of May on infinite loop spaces, involving from the beginning [61] the base category $\mathcal{V}$ of compactly-generated spaces, has of late led him [62] to very general considerations on categories $\mathcal{V}$ with two symmetric monoidal structures and a distributive law between them. Mitchell [63] has related monoidal structures on $[G, \mathcal{V}]$ for suitable $\mathcal{V}$ to low-dimensional cohomology groups of the group G. These latter applications go beyond the basic theory presented here, but of course presuppose the relevant parts of it.

Recent work of Walters ([75], [76]) does concern the basic theory, but with $\mathcal{V}$ generalized to a symmetric monoidal closed *bicategory*; in this context he exhibits the sheaves on a site as the symmetric Cauchy-complete $\mathcal{V}$-categories. It is plain that continued expansion of the applications is to be expected.

(v) Writing in text-book style, we have not interrupted the development to assign credit for individual results. To do so with precision and justice would in any case be a daunting task; much has been written, and many insights were arrived at independently by several authors. Such references as do occur in the text are rather intended to show where more detail may be found. What we can do, though, is to list here some of the works to which we are particularly indebted.

Perhaps the first to advocate in print the study of enriched categories was Mac Lane [59]; although published in 1965, this represents basically his Colloquium Lectures at Boulder in 1963. There were some early attempts by Linton [56] and Kelly [40] to make a start; Bénabou ([4], [5]) went further; while Eilenberg and Kelly wrote a voluminous article [26] covering only the "elementary" notions, but including change of base-category. Our present Chapter 1 draws mainly on [26] and on the author's article [42].

The principal source for Chapter 2 is Day and Kelly [20]; but see also Bunge [10]. The ideas in Chapter 3 of cotensor products, of ends, and of limits in a $\mathcal{V}$-category, go back to [42] and [20]; and the concept of their *pointwise* existence in a functor-category to Dubuc [22]. However, the general notion of indexed limit, which makes Chapter 3 possible in its present form, was discovered independently by Street [72], Auderset [1], and Borceux and Kelly [9]. The last two sections of Chapter 3 call on Day's articles [12] and [14].

Chapter 4, on Kan extensions, is certainly indebted to the three articles [20], [22], [9] already mentioned; in particular it was Dubuc in [22] who pointed out the importance of the "pointwise" existence of Kan extensions, which we make part of our definition. The chapter also contains many more-or-less classical results for $\mathcal{V} = \mathbf{Set}$; all but the best known are credited in the text to their authors.

It is especially in Chapter 5, on density, that we have been heavily influenced by writers who themselves dealt only with $\mathcal{V} = \mathbf{Set}$; namely Gabriel and Ulmer [31] and Diers [21]. The term "density presentation" we have taken from Day [16], although modifying its meaning somewhat. Again, writers on particular aspects are given credit in the text.

Chapter 6 shows its debt to Ehresmann [24], as well as to Gabriel and Ulmer. The form of the transfinite construction used here to prove the reflectivity of the algebras,

although it is taken from the author's article [44], ultimately depends on an idea from an unpublished manuscript of Barr (reference [3] of [44]), which makes cowellpoweredness an unnecessary hypothesis, and so enables us to include the important example of quasitopological spaces.

The author's original contributions to the book are perhaps most visible in the arrangement of topics and in the construction of proofs that apply equally to the classical and enriched cases. For instance, many readers will find the way of introducing Kan extensions quite novel. Beyond this, the work in 5.11–5.13 on Kan extensions along a non-fully-faithful dense functor seems to be quite new even when $\mathcal{V} = \mathbf{Set}$, as is its application in 6.4; while the whole of Chapter 6 is new in the enriched setting.

(vi) In the early chapters the formal definition-theorem-proof style seemed out of place, and apt to lengthen an essentially simple account. Deciding to avoid it, we have accordingly made reference to a result either by quoting the section in which it occurs, or (more commonly) by quoting the number of the displayed formula most closely associated with it. In the later chapters, now more concerned with applying the logic than developing it, it seemed best to return to formally-numbered propositions and theorems.

The end-result is as follows. A symbol 5.6 without parentheses refers to section 6 of Chapter 5. All other references occur in a single series, consisting of displayed formulae and proposition- or theorem-numbers, with the first digit denoting the chapter. Thus in Chapter 5, formula (5.51) is followed by Theorem 5.52, itself followed by formula (5.53), which in fact occurs in the statement of the theorem. To assist cross-referencing, each left-hand page bears at its head the number of the last section to which that page belongs, and each right-hand page bears the numbers of the formulae, propositions, and theorems that occur on the two-page spread (in parentheses, even if they are theorem-numbers).[1]

(vii) It remains to thank those who have made the book possible. Over the last twelve months I have had the opportunity to present early drafts of the material to acute and stimulating audiences at the Universities of Sydney, Trieste, and Hagen; their encouragement has been of inestimable value. Equally important has been the support of my colleagues in the Sydney Category Theory Seminar – including the visiting member André Joyal, whose presence was made possible by the Australian Research Grants Committee. I am indebted to Cambridge University Press, for suggesting this Lecture Notes Series as suited to such a book, and for their careful help with the technical details of publication. It was they who observed that the original typed manuscript was already suitable as camera-ready copy; I regard this as a compliment to the intelligence and taste of my secretary Helen Rubin, whose first mathematical job this was, and I thank her very sincerely. Finally I thank my wife and children, who have borne months of shameful neglect.

Max Kelly,
Sydney,
January 1981.

[1] This has not been done in this electronic version. Instead, we use a running section title on recto pages and a running chapter title on the verso.

Chapter 1

The elementary notions

1.1 Monoidal categories

A *monoidal category* $\mathcal{V} = (\mathcal{V}_0, \otimes, I, a, l, r)$ consists of a category $\mathcal{V}_0$, a functor $\otimes\colon \mathcal{V}_0 \times \mathcal{V}_0 \longrightarrow \mathcal{V}_0$, an object I of $\mathcal{V}_0$ and natural isomorphisms $a_{XYZ}\colon (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z)$, $l_X\colon I \otimes X \longrightarrow X$, $r_X\colon X \otimes I \longrightarrow X$, subject to two *coherence axioms* expressing the commutativity of the following diagrams:

$$\begin{tikzcd}
\big((W \otimes X) \otimes Y\big) \otimes Z \arrow[r, "a"] \arrow[d, "a\otimes 1"'] & (W \otimes X) \otimes (Y \otimes Z) \arrow[r, "a"] & W \otimes \big(X \otimes (Y \otimes Z)\big) \\
\big(W \otimes (X \otimes Y)\big) \otimes Z \arrow[rr, "a"'] & & W \otimes \big((X \otimes Y) \otimes Z\big), \arrow[u, "1\otimes a"']
\end{tikzcd} \tag{1.1}$$

$$\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[rr, "a"] \arrow[dr, "r\otimes 1"'] & & X \otimes (I \otimes Y) \arrow[dl, "1\otimes l"] \\
& X \otimes Y. &
\end{tikzcd} \tag{1.2}$$

It then follows (see [58] and [39]) that every diagram of natural transformations commutes, each arrow of which is obtained by repeatedly applying the functor $\otimes$ to instances of a, l, r, their inverses, and 1; here an "instance" of a is a natural transformation, such as $a_{W\otimes X,Y,Z}$ or $a_{X,I,Y}$ in the diagrams above, formed from a by repeated application of the functors $\otimes$ and I to its variables. For a precise formulation, see [58].

A special kind of example, called a *cartesian monoidal category*, is given by taking for $\mathcal{V}_0$ any category with finite products, taking for $\otimes$ and I the product $\times$ and the terminal object 1, and taking for a, l, r the canonical isomorphisms. Important particular cases of this are the categories **Set**, **Cat**, **Gpd**, **Ord**, **Top**, **CGTop**, **HCGTop**, **QTop**, **Shv** S, of (small) sets, categories, groupoids, ordered sets, topological spaces, compactly generated topological spaces, Hausdorff such, quasitopological spaces [69], and sheaves for a site S; here "small" is in reference to some chosen universe, which we suppose given once and for all. Other cartesian examples are obtained by taking for $\mathcal{V}_0$ an ordered set with finite intersections, such as the ordinal $\mathbf{2} = \{0, 1\}$. All of these examples are *symmetric* in the sense of 1.4 below, and all the named ones except **Top** are *closed* in the sense of 1.5 below.

A collection of non-cartesian (symmetric, closed) examples is given by the categories **Ab**, R-**Mod**, **G**-R-**Mod**, **DG**-R-**Mod** of (small) abelian groups, R-modules for a commutative ring R, graded R-modules, and differential graded R-modules, each with its usual tensor product as $\otimes$; the category **Ban** of Banach spaces and linear maps of norm $\leqslant 1$, with the projective tensor product; the category $\mathbf{CGTop}_*$ of pointed compactly-generated spaces with the smash product for $\otimes$; and the ordered set $\overline{\mathbb{R}}_+$ of extended non-negative reals, with the reverse of the usual order, and with $+$ as $\otimes$.

A non-symmetric example is the category of bimodules over a non-commutative ring R, with $\otimes_R$ as $\otimes$. Another is the category of endofunctors of a small category, with composition as $\otimes$; here a, l, r are identities, so that the monoidal category is called *strict*.

In general we call $\otimes$ the *tensor product*, and I the *unit object*.

We suppose henceforth given *a particular monoidal* $\mathcal{V}$ *such that* $\mathcal{V}_0$ *is locally small* (has small hom-sets). We then have the representable functor $\mathcal{V}_0(I,-)\colon \mathcal{V}_0 \longrightarrow \mathbf{Set}$, which we denote by V. In such cases as **Set**, **Ord**, **Top**, **Ab**, R-**Mod**, $\mathbf{CGTop}_*$, it is (to within isomorphism) the ordinary "underlying-set" functor; in the case of **Ban**, VX is the unit ball of X; in these cases V is faithful, while in some of them (**Set**, **Ab**, R-**Mod**, **Ban**) it is even conservative (= isomorphism-reflecting). Yet V is not faithful in general; in the cases **Cat** and **Gpd**, VX is the set of objects of X, and in the case **DG**-R-**Mod**, VX is the set of 0-cycles.

In spite of the failure of V to be faithful in general, it is convenient to call an element f of VX (that is, a map $f\colon I \longrightarrow X$ in $\mathcal{V}_0$) *an element* f *of* X.

1.2 The 2-category $\mathcal{V}$-**CAT** for a monoidal $\mathcal{V}$

A $\mathcal{V}$*-category* $\mathcal{A}$ consists of a set $\operatorname{ob}\mathcal{A}$ of *objects*, a *hom-object* $\mathcal{A}(A,B) \in \mathcal{V}_0$ for each pair of objects of $\mathcal{A}$, a *composition law* $M = M_{ABC}\colon \mathcal{A}(B,C)\otimes\mathcal{A}(A,B) \longrightarrow \mathcal{A}(A,C)$ for each triple of objects, and an *identity element* $j_A\colon I \longrightarrow \mathcal{A}(A,A)$ for each object; subject to the associativity and unit axioms expressed by the commutativity of

$$\begin{tikzcd}
\big(\mathcal{A}(C,D)\otimes\mathcal{A}(B,C)\big)\otimes\mathcal{A}(A,B) \arrow[rr, "a"] \arrow[d, "M\otimes 1"'] & & \mathcal{A}(C,D)\otimes\big(\mathcal{A}(B,C)\otimes\mathcal{A}(A,B)\big) \arrow[d, "1\otimes M"] \\
\mathcal{A}(B,D)\otimes\mathcal{A}(A,B) \arrow[dr, "M"'] & & \mathcal{A}(C,D)\otimes\mathcal{A}(A,C) \arrow[dl, "M"] \\
& \mathcal{A}(A,D), &
\end{tikzcd} \tag{1.3}$$

$$\begin{tikzcd}
\mathcal{A}(B,B)\otimes\mathcal{A}(A,B) \arrow[r, "M"] & \mathcal{A}(A,B) & \mathcal{A}(A,B)\otimes\mathcal{A}(A,A) \arrow[l, "M"'] \\
I\otimes\mathcal{A}(A,B) \arrow[u, "j\otimes 1"] \arrow[ur, "l"'] & & \mathcal{A}(A,B)\otimes I. \arrow[u, "1\otimes j"'] \arrow[ul, "r"]
\end{tikzcd} \tag{1.4}$$

Taking $\mathcal{V} =$ **Set**, **Cat**, **2**, **Ab**, **DG**-R-**Mod**, $\overline{\mathbb{R}}_+$, we re-find the classical notions of (*locally small*) ordinary category, 2-category, pre-ordered set, additive category (some

call it "pre-additive"), differential graded category, and (rather generalized) metric space. For a general reference on 2-categories, see [49]; and for the generalized metric spaces, see [54]. We call the $\mathcal{V}$-category $\mathcal{A}$ *small* if $\mathrm{ob}\,\mathcal{A}$ is small.

For $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, a $\mathcal{V}$-*functor* $T\colon \mathcal{A} \longrightarrow \mathcal{B}$ consists of a function

$$T\colon\ \mathrm{ob}\,\mathcal{A} \longrightarrow \mathrm{ob}\,\mathcal{B}$$

together with, for each pair $A,\, B \in \mathrm{ob}\,\mathcal{A}$, a map $T_{AB}\colon \mathcal{A}(A,B) \longrightarrow \mathcal{B}(TA,TB)$, subject to the compatibility with composition and with the identities expressed by the commutativity of

$$\begin{tikzcd} \mathcal{A}(B,C) \otimes \mathcal{A}(A,B) \arrow[r, "M"] \arrow[d, "T\otimes T"'] & \mathcal{A}(A,C) \arrow[d, "T"] \\ \mathcal{B}(TB,TC) \otimes \mathcal{B}(TA,TB) \arrow[r, "M"'] & \mathcal{B}(TA,TC), \end{tikzcd} \tag{1.5}$$

$$\begin{tikzcd} & \mathcal{A}(A,A) \arrow[dd, "T"] \\ I \arrow[ur, "j"] \arrow[dr, "j"'] & \\ & \mathcal{B}(TA,TA). \end{tikzcd} \tag{1.6}$$

In the six examples above we re-find the classical notions of functor, 2-functor, increasing function, additive functor, differential graded functor, and contracting map. The $\mathcal{V}$-functor T is *fully faithful* if each T_{AB} is an isomorphism; an example is the inclusion $T\colon \mathcal{A} \longrightarrow \mathcal{B}$ of a *full subcategory*, determined by a subset $\mathrm{ob}\,\mathcal{A}$ of $\mathrm{ob}\,\mathcal{B}$. Clearly $\mathcal{V}$-functors can be composed, to form a category.

For $\mathcal{V}$-functors $T, S\colon \mathcal{A} \longrightarrow \mathcal{B}$, a $\mathcal{V}$-*natural transformation* $\alpha\colon T \longrightarrow S\colon \mathcal{A} \longrightarrow \mathcal{B}$ is an $\mathrm{ob}\,\mathcal{A}$-indexed family of *components* $\alpha_A\colon I \longrightarrow \mathcal{B}(TA,SA)$ satisfying the $\mathcal{V}$-naturality condition expressed by the commutativity of

$$\begin{tikzcd} & I \otimes \mathcal{A}(A,B) \arrow[r, "\alpha_B\otimes T"] & \mathcal{B}(TB,SB) \otimes \mathcal{B}(TA,TB) \arrow[dr, "M"] & \\ \mathcal{A}(A,B) \arrow[ur, "l^{-1}"] \arrow[dr, "r^{-1}"'] & & & \mathcal{B}(TA,SB). \\ & \mathcal{A}(A,B) \otimes I \arrow[r, "S\otimes\alpha_A"'] & \mathcal{B}(SA,SB) \otimes \mathcal{B}(TA,SA) \arrow[ur, "M"'] & \end{tikzcd} \tag{1.7}$$

The "vertical composite" $\beta \cdot \alpha$ of $\alpha\colon T \longrightarrow S\colon \mathcal{A} \longrightarrow \mathcal{B}$ and $\beta\colon S \longrightarrow R\colon \mathcal{A} \longrightarrow \mathcal{B}$ has the component $(\beta \cdot \alpha)_A$ given by

$$I \cong I \otimes I \xrightarrow[\beta_A\otimes\alpha_A]{} \mathcal{B}(SA,RA) \otimes \mathcal{B}(TA,SA) \xrightarrow[M]{} \mathcal{B}(TA,RA). \tag{1.8}$$

The composite of α above with $Q\colon \mathcal{B} \longrightarrow \mathcal{C}$ has for its component $(Q\alpha)_A$ the composite

$$I \xrightarrow[\alpha_A]{} \mathcal{B}(TA,SA) \xrightarrow[Q]{} \mathcal{C}(QTA,QSA); \tag{1.9}$$

while the composite of α with $P\colon \mathcal{D} \longrightarrow \mathcal{A}$ has for its component $(\alpha P)_D$ simply α_{PD}.

It is now easy to verify that $\mathcal{V}$-categories, $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations constitute a 2-category $\mathcal{V}$-**CAT**; an "illegitimate" one, of course, unless some restriction is placed on the size, as in the legitimate 2-category $\mathcal{V}$-**Cat**, of *small* $\mathcal{V}$-categories. Of course $\mathcal{V}$-**CAT** reduces, when $\mathcal{V} = \mathbf{Set}$, to the 2-category **CAT** of *locally small* ordinary categories.

1.3 The 2-functor $(\ )_0\colon \mathcal{V}\text{-}\mathbf{CAT} \longrightarrow \mathbf{CAT}$

Denoting by $\mathcal{I}$ the *unit $\mathcal{V}$-category* with one object 0 and with $\mathcal{I}(0,0) = I$, we write $(-)_0\colon \mathcal{V}\text{-}\mathbf{CAT} \longrightarrow \mathbf{CAT}$ for the representable 2-functor $\mathcal{V}\text{-}\mathbf{CAT}(\mathcal{I}, -)$; which we now proceed to describe in more elementary terms.

A $\mathcal{V}$-functor $A\colon \mathcal{I} \longrightarrow \mathcal{A}$ may be identified with an object A of the $\mathcal{V}$-category $\mathcal{A}$; and a $\mathcal{V}$-natural $f\colon A \longrightarrow B\colon \mathcal{I} \longrightarrow \mathcal{A}$ consists of a single component $f\colon I \longrightarrow \mathcal{A}(A,B)$, the axiom (1.7) being trivially satisfied. Thus the ordinary category $\mathcal{A}_0 = \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{I}, \mathcal{A})$, which is called the *underlying category* of $\mathcal{A}$, has the same objects as $\mathcal{A}$, while a map $f\colon A \longrightarrow B$ in $\mathcal{A}_0$ is just an element $f\colon I \longrightarrow \mathcal{A}(A,B)$ of $\mathcal{A}(A,B)$, in the sense of 1.1. Otherwise put, $\mathcal{A}_0(A,B) = V\mathcal{A}(A,B)$. By (1.8), the composite gf in $\mathcal{A}_0$ is given by the composite

$$I \cong I \otimes I \xrightarrow[g\otimes f]{} \mathcal{A}(B,C) \otimes \mathcal{A}(A,B) \xrightarrow[M]{} \mathcal{A}(A,C) \tag{1.10}$$

in $\mathcal{V}_0$, while the identity in $\mathcal{A}_0(A,A)$ is clearly j_A.

How much information about $\mathcal{A}$ is retained by $\mathcal{A}_0$ depends upon how faithful V is. When $\mathcal{V} = \mathbf{Cat}$, V is not faithful; $\mathcal{A}$ is a 2-category, and $\mathcal{A}_0$ is the category obtained by discarding the 2-cells. When $\mathcal{V} = \mathbf{CGTop}$, V is faithful, and $\mathcal{A}_0$ has lost only the topology on the hom-objects of $\mathcal{A}$. When $\mathcal{V} = \mathbf{Ab}$ or R-**Mod**, V is even conservative, and $\mathcal{A}_0$ is still closer to $\mathcal{A}$.

The ordinary functor $T_0\colon \mathcal{A}_0 \longrightarrow \mathcal{B}_0$ induced by (or *underlying*) the $\mathcal{V}$-functor $T\colon \mathcal{A} \longrightarrow \mathcal{B}$ sends $A\colon \mathcal{I} \longrightarrow \mathcal{A}$ to TA and sends $f\colon A \longrightarrow B\colon \mathcal{I} \longrightarrow \mathcal{A}$ to Tf, which by (1.9) is the composite

$$I \xrightarrow[f]{} \mathcal{A}(A,B) \xrightarrow[T_{AB}]{} \mathcal{B}(TA,TB). \tag{1.11}$$

Thus we have

$$T_0A = TA, \qquad T_0f = Tf; \tag{1.12}$$

the latter of which means that

$$T_{0AB}\colon \mathcal{A}_0(A,B) \longrightarrow \mathcal{B}_0(TA,TB) \quad \text{is} \quad VT_{AB}\colon V\mathcal{A}(A,B) \longrightarrow V\mathcal{B}(TA,TB). \tag{1.13}$$

Clearly T_0 is fully faithful if T is; $T \mapsto T_0$ is injective when V is faithful, but not in general; and, when V is conservative, T is fully faithful if T_0 is.

The ordinary natural transformation $\alpha_0\colon T_0 \longrightarrow S_0\colon \mathcal{A}_0 \longrightarrow \mathcal{B}_0$ induced by the $\mathcal{V}$-natural $\alpha\colon T \longrightarrow S\colon \mathcal{A} \longrightarrow \mathcal{B}$ has for its A-component $\alpha_{0A} \in \mathcal{B}_0(TA,SA)$ precisely the A-component $\alpha_A\colon I \longrightarrow \mathcal{B}(TA,SA)$ of α; so that it is not usually necessary to distinguish α from α_0. The $\mathcal{V}$-naturality condition (1.7) for $\alpha\colon T \longrightarrow S$ becomes the usual naturality condition for

$\alpha\colon T_0 \longrightarrow S_0$ when V is applied to it. Hence the naturality of $\alpha\colon T_0 \longrightarrow S_0$, while weaker in general than the $\mathcal{V}$-naturality of $\alpha\colon T \longrightarrow S$, is equivalent to it when V is faithful.

In spite of such formulae as (1.12), both clarity and economy are served by carefully distinguishing $\mathcal{A}$ from $\mathcal{A}_0$ and T from T_0. For instance, completeness of $\mathcal{A}$, to be defined below, means something stronger than completeness of $\mathcal{A}_0$; and continuity of T means something stronger than continuity of T_0. Having a left adjoint for T is stronger than having a left adjoint for T_0; while the existence of a small dense subcategory of $\mathcal{A}$ neither implies nor is implied by the existence of a small dense subcategory of $\mathcal{A}_0$. As for economy, maintaining the distinction allows us, when $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{V}$-categories, to abbreviate "$\mathcal{V}$-functor $T\colon \mathcal{A} \longrightarrow \mathcal{B}$" to "functor (or map) $T\colon \mathcal{A} \longrightarrow \mathcal{B}$"; if we had meant a functor $T\colon \mathcal{A}_0 \longrightarrow \mathcal{B}_0$, we should have said so. Again, to speak of "a $\mathcal{V}$-functor $T\colon \mathcal{A} \longrightarrow \mathcal{B}$" carries the converse implication, that $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{V}$-categories. Similarly, when $T, S\colon \mathcal{A} \longrightarrow \mathcal{B}$ are $\mathcal{V}$-functors, by "map (or natural transformation) $\alpha\colon T \longrightarrow S$" we must mean a $\mathcal{V}$-natural one; for we did not speak of a map $\alpha\colon T_0 \longrightarrow S_0$. When only components are written, however, it may be necessary to say "$\alpha_A\colon TA \longrightarrow SA$ is $\mathcal{V}$-natural in A", since TA is also T_0A. Finally, since strictly speaking there are no "morphisms" in the $\mathcal{V}$-category $\mathcal{A}$, it is harmless to call a map $f\colon A \longrightarrow B$ in $\mathcal{A}_0$, which is an element $f\colon I \longrightarrow \mathcal{A}(A,B)$ of $\mathcal{A}(A,B)$, "a map $f\colon A \longrightarrow B$ in $\mathcal{A}$".

1.4 Symmetric monoidal categories: the tensor product and duality on $\mathcal{V}$-**CAT** for a symmetric monoidal $\mathcal{V}$

A *symmetry* c for a monoidal category $\mathcal{V}$ is a natural isomorphism $c_{XY}\colon X \otimes Y \longrightarrow Y \otimes X$ satisfying the *coherence axioms* expressed by the commutativity of

$$\begin{tikzcd} X \otimes Y \arrow[r, "c"] \arrow[dr, "1"'] & Y \otimes X \arrow[d, "c"] \\ & X \otimes Y, \end{tikzcd} \tag{1.14}$$

$$\begin{tikzcd} (X \otimes Y) \otimes Z \arrow[r, "a"] \arrow[d, "c\otimes 1"'] & X \otimes (Y \otimes Z) \arrow[r, "c"] & (Y \otimes Z) \otimes X \arrow[d, "a"] \\ (Y \otimes X) \otimes Z \arrow[r, "a"'] & Y \otimes (X \otimes Z) \arrow[r, "1\otimes c"'] & Y \otimes (Z \otimes X), \end{tikzcd} \tag{1.15}$$

$$\begin{tikzcd} I \otimes X \arrow[rr, "c"] \arrow[dr, "l"'] & & X \otimes I \arrow[dl, "r"] \\ & X. & \end{tikzcd} \tag{1.16}$$

Note that (1.16) defines l in terms of r, and then we need only the four coherence axioms (1.1), (1.2), (1.14), (1.15). It follows from [58] and [39] that every diagram of natural transformations commutes, each arrow of which is obtained by the repeated application of $\otimes$ to instances of a, l, r, c, their inverses, and 1; cf. 1.1, and for a precise formulation, see again [58].

A monoidal category – even a closed one in the sense of 1.5 below – may admit more than one symmetry. The classical example is $\mathcal{V} = \mathbf{G}\text{-}R\text{-}\mathbf{Mod}$, with $c(x \otimes y)$, for homogeneous elements x, y of degrees p, q, defined either as $y \otimes x$ or as $(-1)^{pq} y \otimes x$. However, if V is faithful and $\mathcal{V}$ is closed, there is at most one symmetry ([26] Chapter III, Proposition 6.1). For a cartesian $\mathcal{V}$, there is an evident canonical $c\colon X \otimes Y \longrightarrow Y \otimes X$. For the cases $\mathbf{Ab}$, $R\text{-}\mathbf{Mod}$, $\mathbf{Ban}$, and $\mathbf{CGTop}_*$, V is faithful and the unique symmetry is well-known. For $\mathbf{DG}\text{-}R\text{-}\mathbf{Mod}$ the classical symmetry is given by $c(x \otimes y) = (-1)^{pq} y \otimes x$; and for $\overline{\mathbb{R}}_+$ it is the equality $x + y = y + x$.

A monoidal category $\mathcal{V}$ together with a symmetry is called a *symmetric monoidal category*. We now suppose $\mathcal{V}$ to be such.

Then to each pair $\mathcal{A}$, $\mathcal{B}$ of $\mathcal{V}$-categories we can associate a *tensor product* $\mathcal{A} \otimes \mathcal{B}$, with object-set $\operatorname{ob}\mathcal{A} \times \operatorname{ob}\mathcal{B}$, and with $(A \otimes B)\big((A,B),(A',B')\big) = \mathcal{A}(A,A') \otimes \mathcal{B}(B,B')$. The composition-law is given by the top edge of

$$\begin{tikzcd}
\big(\mathcal{A}(A',A'') \otimes \mathcal{B}(B',B'')\big) \otimes \big(\mathcal{A}(A,A') \otimes \mathcal{B}(B,B')\big) \arrow[r, "M"] \arrow[d, "m"'] & \mathcal{A}(A,A'') \otimes \mathcal{B}(B,B'') \\
\big(\mathcal{A}(A',A'') \otimes \mathcal{A}(A,A')\big) \otimes \big(\mathcal{B}(B',B'') \otimes \mathcal{B}(B,B')\big), \arrow[ur, "M \otimes M"'] &
\end{tikzcd} \tag{1.17}$$

where $m\colon (W \otimes X) \otimes (Y \otimes Z) \cong (W \otimes Y) \otimes (X \otimes Z)$ is the *middle-four interchange* defined by any suitable composite of instances of a and of c. The identity element is the composite

$$I \cong I \otimes I \xrightarrow[j_A \otimes j_B]{} \mathcal{A}(A,A) \otimes \mathcal{B}(B,B), \tag{1.18}$$

and axioms (1.3) and (1.4) are easy to verify.

It is further easy to check that, with the obvious definitions of $T \otimes S\colon \mathcal{A} \otimes \mathcal{B} \longrightarrow \mathcal{A}' \otimes \mathcal{B}'$ and of $\alpha \otimes \beta\colon T \otimes S \longrightarrow T' \otimes S'$, we have a 2-functor $\otimes\colon \mathcal{V}\text{-}\mathbf{CAT} \times \mathcal{V}\text{-}\mathbf{CAT} \longrightarrow \mathcal{V}\text{-}\mathbf{CAT}$; and that we have coherent 2-natural isomorphisms $(\mathcal{A} \otimes B) \otimes C \cong \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})$, $\mathcal{I} \otimes \mathcal{A} \cong \mathcal{A} \otimes \mathcal{I} \cong \mathcal{A}$, and $\mathcal{A} \otimes \mathcal{B} \cong \mathcal{B} \otimes \mathcal{A}$ (where $\mathcal{I}$ is the unit $\mathcal{V}$-category of 1.3). Thus $\mathcal{V}\text{-}\mathbf{CAT}$ is, in an evident sense, a *symmetric monoidal 2-category*.

Moreover, to each $\mathcal{V}$-category $\mathcal{A}$ we can associate a *dual*, or *opposite*, $\mathcal{V}$-category $\mathcal{A}^{\mathrm{op}}$, with the same objects as $\mathcal{A}$, but with $\mathcal{A}^{\mathrm{op}}(A,B) = \mathcal{A}(B,A)$. The composition-law $M\colon \mathcal{A}^{\mathrm{op}}(B,C) \otimes \mathcal{A}^{\mathrm{op}}(A,B) \longrightarrow \mathcal{A}^{\mathrm{op}}(A,C)$ is just the composite

$$\mathcal{A}(C,B) \otimes \mathcal{A}(B,A) \xrightarrow[c]{} \mathcal{A}(B,A) \otimes \mathcal{A}(C,B) \xrightarrow[M]{} \mathcal{A}(C,A), \tag{1.19}$$

while the unit element $I \longrightarrow \mathcal{A}^{\mathrm{op}}(A,A)$ is that of $\mathcal{A}$. From $\alpha\colon T \longrightarrow S\colon \mathcal{A} \longrightarrow \mathcal{B}$ we get, with evident definitions, $\alpha^{\mathrm{op}}\colon S^{\mathrm{op}} \longrightarrow T^{\mathrm{op}}\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{B}^{\mathrm{op}}$; note that $(-)^{\mathrm{op}}$ reverses 2-cells but not 1-cells. Of course $(-)^{\mathrm{op}}$ is involutary: $(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{A}$; and it respects tensor products: $(\mathcal{A} \otimes \mathcal{B})^{\mathrm{op}} = \mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}^{\mathrm{op}}$. Thus, in an evident sense, $\mathcal{V}\text{-}\mathbf{CAT}$ is a *symmetric monoidal 2-category with a duality involution*.

A functor $\mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{B}$ may be called a *contravariant* functor from $\mathcal{A}$ to $\mathcal{B}$; while a functor $T\colon \mathcal{A} \otimes \mathcal{B} \longrightarrow \mathcal{C}$ may be thought of as a *functor of two variables*. Such a T gives rise to *partial functors* $T(A,-)\colon \mathcal{B} \longrightarrow \mathcal{C}$ for each $A \in \mathcal{A}$ and $T(-,B)\colon \mathcal{A} \longrightarrow \mathcal{C}$ for each $B \in \mathcal{B}$.

Here $T(A,-)$ is the composite

$$\mathcal{B} \cong \mathcal{I} \otimes \mathcal{B} \xrightarrow[A\otimes 1]{} \mathcal{A} \otimes \mathcal{B} \xrightarrow[T]{} \mathcal{C}, \tag{1.20}$$

from which can be read off the value of $T(A,-)_{BB'}$.

Suppose, conversely, that we are given a family of functors $T(A,-)\colon \mathcal{B} \longrightarrow \mathcal{C}$ indexed by $\operatorname{ob}\mathcal{A}$ and a family of functors $T(-,B)\colon \mathcal{A} \longrightarrow \mathcal{C}$ indexed by $\operatorname{ob}\mathcal{B}$, such that on objects we have $T(A,-)B = T(-,B)A, = T(A,B)$, say. Then there is a functor $T\colon \mathcal{A}\otimes\mathcal{B} \longrightarrow \mathcal{C}$ of which these are the partial functors if and only if we have commutativity in

$$\begin{tikzcd}
\mathcal{A}(A,A') \otimes \mathcal{B}(B,B') \arrow[r, "T(-,B')\otimes T(A,-)"] \arrow[dd, "c"'] & \mathcal{C}\big(T(A,B'),T(A',B')\big) \otimes \mathcal{C}\big(T(A,B),T(A,B')\big) \arrow[d, "M"] \\
& \mathcal{C}\big(T(A,B),T(A',B')\big) \\
\mathcal{B}(B,B') \otimes \mathcal{A}(A,A') \arrow[r, "T(A',-)\otimes T(-,B)"'] & \mathcal{C}\big(T(A',B),T(A',B')\big) \otimes \mathcal{C}\big(T(A,B),T(A',B)\big). \arrow[u, "M"']
\end{tikzcd} \tag{1.21}$$

Moreover T is then unique, and $T_{(A,B),(A',B')}$ is the diagonal of (1.21). The verification is easy, and the details can be found in ([26] Chapter III, §4).

It is further easy (using (1.21), and (1.3) for $\mathcal{C}$) to verify that, if T, $S\colon \mathcal{A}\otimes\mathcal{B} \longrightarrow \mathcal{C}$, a family $\alpha_{AB}\colon T(A,B) \longrightarrow S(A,B)$ in $\mathcal{C}_0$ constitutes a $\mathcal{V}$-natural transformation $T \longrightarrow S$ if and only if it constitutes, for each fixed A, a $\mathcal{V}$-natural $T(A,-) \longrightarrow S(A,-)$, and, for each fixed B, a $\mathcal{V}$-natural $T(-,B) \longrightarrow S(-,B)$. In other words, *$\mathcal{V}$-naturality may be verified variable-by-variable.*

In relation to the underlying ordinary categories, we have of course $(\mathcal{A}^{\mathrm{op}})_0 = (\mathcal{A}_0)^{\mathrm{op}}$ and $(T^{\mathrm{op}})_0 = (T_0)^{\mathrm{op}}$. However, $(\mathcal{A}\otimes\mathcal{B})_0$ is not $\mathcal{A}_0\times\mathcal{B}_0$; rather there is an evident canonical functor $\mathcal{A}_0 \times \mathcal{B}_0 \longrightarrow (\mathcal{A}\otimes\mathcal{B})_0$, and a similar one $1 \longrightarrow \mathcal{I}_0$. For $T\colon \mathcal{A}\otimes\mathcal{B} \longrightarrow \mathcal{C}$, the partial functors of the composite ordinary functor

$$\mathcal{A}_0 \times \mathcal{B}_0 \longrightarrow (\mathcal{A}\otimes\mathcal{B})_0 \xrightarrow[T_0]{} \mathcal{C}_0 \tag{1.22}$$

are precisely $T(A,-)_0$ and $T(-,B)_0$.

We could discuss, in the present setting, the *extraordinary $\mathcal{V}$-natural families* of 1.7 below. However, the techniques that make this simple arise more naturally, and at a higher level, when $\mathcal{V}$ is closed; which is the case of real interest. Moreover it is shown in [17] that any symmetric monoidal category can, by passing to a higher universe, be embedded in a closed one; see also 2.6 and 3.11 below.

1.5 Closed and biclosed monoidal categories

The monoidal category $\mathcal{V}$ (symmetric or not) is said to be *closed* (*cartesian closed*, when $\mathcal{V}$ is cartesian monoidal) if each functor $-\otimes Y\colon \mathcal{V}_0 \longrightarrow \mathcal{V}_0$ has a right adjoint $[Y,-]$; so that we have an adjunction

$$\pi\colon \mathcal{V}_0(X\otimes Y, Z) \cong \mathcal{V}_0(X,[Y,Z]) \tag{1.23}$$

with unit and counit (the latter called *evaluation*) say

$$d\colon X \longrightarrow [Y, X \otimes Y], \qquad e\colon [Y, Z] \otimes Y \longrightarrow Z. \tag{1.24}$$

Putting $X = I$ in (1.23), using the isomorphism $l\colon I \otimes Y \cong Y$, and recalling from 1.1 that $V = \mathcal{V}_0(I, -)\colon \mathcal{V}_0 \longrightarrow \mathbf{Set}$, we get a natural isomorphism

$$\mathcal{V}_0(Y, Z) \cong V[Y, Z]; \tag{1.25}$$

since $[Y, Z]$ is thus exhibited as a lifting through V of the hom-set $\mathcal{V}_0(Y, Z)$, it is called the *internal hom* of Y and Z.

Putting $Y = I$ in (1.23) and using the isomorphism $r\colon X \otimes I \cong X$, we deduce a natural isomorphism

$$i\colon Z \cong [I, Z]; \tag{1.26}$$

it is easy to verify that $Vi\colon VZ \cong V[I, Z]$ is just the case $Y = I$ of (1.25). Replacing X by $W \otimes X$ in (1.23), and using the isomorphism $a\colon (W \otimes X) \otimes Y \cong W \otimes (X \otimes Y)$, we deduce a natural isomorphism

$$p\colon [X \otimes Y, Z] \cong [X, [Y, Z]]; \tag{1.27}$$

and it easy to verify that Vp agrees with the π of (1.23) modulo the isomorphisms (1.25). In many concrete cases it is possible and convenient to replace V with an isomorph in such a way that (1.25) becomes an equality; this should be clear from the examples of closed categories given in 1.1, and we do not labour the point.

The monoidal category $\mathcal{V}$ is said to be *biclosed* if, not only does every $- \otimes Y$ have a right adjoint $[Y, -]$, but also every $X \otimes -$ has a right adjoint $[\![X, -]\!]$. When $\mathcal{V}$ is symmetric, it is biclosed if closed, with $[\![X, -]\!] = [X, -]$. The non-symmetric monoidal category of R-bimodules, mentioned in 1.1, is biclosed. It is part of a larger structure, the *biclosed bicategory* of left-R-, right-S-, bimodules, where R and S vary over all rings; and it seems to be typical that non-symmetric biclosed monoidal categories occur in nature as a small part of such a bicategory. (The notion of bicategory, for an exposition of which see [6], generalises that of 2-category, the strict associativity of 1-cells being replaced by associativity *to within coherent isomorphisms*; a one-object bicategory is a monoidal category, just as a one-object 2-category is a strict monoidal category.) Recent work of R.F.C. Walters ([75], [76]) suggests the importance of studying "$\mathcal{V}$-categories $\mathcal{A}$" where $\mathcal{V}$ is a biclosed bicategory with an appropriate kind of symmetry; but that goes beyond the scope of the present account.

There also do occur in nature important closed monoidal categories that are not biclosed. A typical example is given by the strict monoidal category of *small endofunctors* (in some suitable sense of "small") of a well-behaved large category such as **Set**; see [45]. For such a monoidal closed $\mathcal{V}$, we can develop part of the theory of $\mathcal{V}$-categories; we have the results of 1.2 and 1.3, but not those of 1.4; and we have the Yoneda Lemma of 1.9 below, but not its extra-variable form. However, we do not pursue this at all, for in practice it seems that the interest in such a $\mathcal{V}$ lies in $\mathcal{V}$ itself, not in $\mathcal{V}$-categories – which become interesting chiefly when $\mathcal{V}$ is symmetric monoidal closed.

We end this section with two further comments on examples. First, the symmetric (cartesian) monoidal **Top** is not closed: $- \times Y$ cannot have a right adjoint since it does not preserve regular epimorphisms; see [19]. Next, to the examples of symmetric monoidal closed category in 1.1, we add one more class. An ordered set $\mathcal{V}$ with finite intersections and finite unions is called a *Heyting algebra* if it is cartesian closed; a *boolean algebra* is a special case.

1.6 $\mathcal{V}$ as a $\mathcal{V}$-category for symmetric monoidal closed $\mathcal{V}$; representable $\mathcal{V}$-functors

From now on we suppose that our given $\mathcal{V}$ is *symmetric monoidal closed, with $\mathcal{V}_0$ locally small*. The structure of $\mathcal{V}$-**CAT** then becomes rich enough to permit of Yoneda-lemma arguments formally identical with those in **CAT**. Before giving the Yoneda Lemma and its extensions in 1.9, we collect in this section and the next two some necessary preliminaries: results many of which are almost trivial in **CAT**, but less so here.

The proof of each assertion of these sections is the verification of the commutativity of a more-or-less large diagram. This verification can be done wholesale, in that each diagram involved is trivially checked to be of the type proved commutative in the coherence theorems of [47] and [48]. Yet direct verifications, although somewhat tedious, are nevertheless fairly straightforward if the order below is followed.

The first point is that the internal-hom of $\mathcal{V}$ "makes $\mathcal{V}$ itself into a $\mathcal{V}$-category". More precisely, there is a $\mathcal{V}$-category, which we call $\mathcal{V}$, whose objects are those of $\mathcal{V}_0$, and whose hom-object $\mathcal{V}(X,Y)$ is $[X,Y]$. Its composition-law $M\colon [Y,Z]\otimes[X,Y]\longrightarrow[X,Z]$ corresponds under the adjunction (1.23) to the composite

$$([Y,Z]\otimes[X,Y])\otimes X\xrightarrow[a]{}[Y,Z]\otimes([X,Y]\otimes X)\xrightarrow[1\otimes e]{}[Y,Z]\otimes Y\xrightarrow[e]{}Z, \tag{1.28}$$

and its identity element $j_X\colon I\longrightarrow[X,X]$ corresponds under (1.23) to $l\colon I\otimes X\longrightarrow X$. Verification of the axioms (1.3) and (1.4) is easy when we recall that, because of the relation of e to the π of (1.23), the definition (1.28) of M is equivalent to $e(M\otimes 1)=e(1\otimes e)a$. It is further easily verified that (1.25) gives an isomorphism between $\mathcal{V}_0$ and the underlying ordinary category of the $\mathcal{V}$-category $\mathcal{V}$; we henceforth *identify* these two ordinary categories by this isomorphism, thus rendering the notation $\mathcal{V}_0$ consistent with the notation $\mathcal{A}_0$ of 1.3.

Next we observe that, for each $\mathcal{V}$-category $\mathcal{A}$ and each object $A\in\mathcal{A}$, we have the *representable* $\mathcal{V}$-functor $\mathcal{A}(A,-)\colon\mathcal{A}\longrightarrow\mathcal{V}$ sending $B\in\mathcal{A}$ to $\mathcal{A}(A,B)\in\mathcal{V}$, and with

$$\mathcal{A}(A,-)_{BC}\colon\mathcal{A}(B,C)\longrightarrow[\mathcal{A}(A,B),\mathcal{A}(A,C)] \tag{1.29}$$

corresponding under the adjunction (1.23) to $M\colon\mathcal{A}(B,C)\otimes\mathcal{A}(A,B)\longrightarrow\mathcal{A}(A,C)$. Axioms (1.5) and (1.6) for a $\mathcal{V}$-functor reduce easily to (1.3) and half of (1.4).

Replacing $\mathcal{A}$ by $\mathcal{A}^{\mathrm{op}}$ gives the *contravariant representable functor* $\mathcal{A}(-,B)\colon\mathcal{A}^{\mathrm{op}}\longrightarrow\mathcal{V}$. The families $\mathcal{A}(A,-)$ and $\mathcal{A}(-,B)$ are the partial functors of a functor $\mathrm{Hom}_{\mathcal{A}}\colon\mathcal{A}^{\mathrm{op}}\otimes\mathcal{A}\longrightarrow\mathcal{V}$ sending (A,B) to $\mathcal{A}(A,B)$; for the condition (1.21) again reduces easily to (1.3) for $\mathcal{A}$.

We write $\hom_{\mathcal{A}}\colon \mathcal{A}_0^{\mathrm{op}} \times \mathcal{A}_0 \longrightarrow \mathcal{V}_0$ for the ordinary functor

$$\mathcal{A}_0^{\mathrm{op}} \times \mathcal{A}_0 \longrightarrow (\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A})_0 \xrightarrow[(\mathrm{Hom}_{\mathcal{A}})_0]{} \mathcal{V}_0; \tag{1.30}$$

it sends (A, B) to $\mathcal{A}(A, B)$, and for its value on maps we write

$$\mathcal{A}(f, g)\colon \mathcal{A}(A, B) \longrightarrow \mathcal{A}(A', B'),$$

where $f\colon A' \longrightarrow A$ and $g\colon B \longrightarrow B'$. By 1.4, the partial functors of $\hom_{\mathcal{A}}$ are the functors underlying $\mathcal{A}(A, -)$ and $\mathcal{A}(-, B)$, so that our writing their values on morphisms as $\mathcal{A}(A, g)$ and $\mathcal{A}(f, B)$ is consistent with (1.12). Combining (1.11) with the definition (1.29) of $\mathcal{A}(A, -)_{BC}$, we see that $\mathcal{A}(A, g)\colon \mathcal{A}(A, B) \longrightarrow \mathcal{A}(A, C)$ is the composite

$$\mathcal{A}(A, B) \xrightarrow[l^{-1}]{} I \otimes \mathcal{A}(A, B) \xrightarrow[g\otimes 1]{} \mathcal{A}(B, C) \otimes \mathcal{A}(A, B) \xrightarrow[M]{} \mathcal{A}(A, C), \tag{1.31}$$

while $\mathcal{A}(f, B)\colon \mathcal{A}(D, B) \longrightarrow \mathcal{A}(A, B)$ is

$$\mathcal{A}(D, B) \xrightarrow[r^{-1}]{} \mathcal{A}(D, B) \otimes I \xrightarrow[1\otimes f]{} \mathcal{A}(D, B) \otimes \mathcal{A}(A, D) \xrightarrow[M]{} \mathcal{A}(A, B). \tag{1.32}$$

From these it follows easily that we have commutativity in

$$\begin{tikzcd} \mathcal{A}_0^{\mathrm{op}} \times \mathcal{A}_0 \arrow[r, "\hom_{\mathcal{A}}"] \arrow[dr, "\mathrm{Hom}_{\mathcal{A}_0}"'] & \mathcal{V}_0 \arrow[d, "V"] \\ & \mathbf{Set}. \end{tikzcd} \tag{1.33}$$

When $\mathcal{A} = \mathcal{V}$, we see at once that $\hom_{\mathcal{V}}$ is just the functor $[-, -]\colon \mathcal{V}_0^{\mathrm{op}} \times \mathcal{V}_0 \longrightarrow \mathcal{V}_0$.

Now we observe that there is also a $\mathcal{V}$-functor $\mathrm{Ten}\colon \mathcal{V} \otimes \mathcal{V} \longrightarrow \mathcal{V}$, sending

$$(X, Y) \longrightarrow \mathrm{Ten}(X, Y) = X \otimes Y,$$

and with $\mathrm{Ten}_{(X,Y),(X',Y')}\colon [X, X'] \otimes [Y, Y'] \longrightarrow [X \otimes Y, X' \otimes Y']$ corresponding under the adjunction (1.23) to the composite

$$([X, X'] \otimes [Y, Y']) \otimes (X \otimes Y) \xrightarrow[m]{} ([X, X'] \otimes X) \otimes ([Y, Y'] \otimes Y) \xrightarrow[e\otimes e]{} X' \otimes Y'; \tag{1.34}$$

when we observe that (1.34) is equivalent to $e(\mathrm{Ten} \otimes 1) = (e \otimes e)m$, verification of the $\mathcal{V}$-functor axioms (1.5) and (1.6) is easy. The ordinary functor

$$\mathcal{V}_0 \times \mathcal{V}_0 \longrightarrow (\mathcal{V} \otimes \mathcal{V})_0 \xrightarrow[\mathrm{Ten}_0]{} \mathcal{V}_0 \tag{1.35}$$

is at once seen to be $\otimes\colon \mathcal{V}_0 \times \mathcal{V}_0 \longrightarrow \mathcal{V}_0$; so that, by (1.4), the underlying ordinary functor of $\mathrm{Ten}(X, -)$ is $X \otimes -$.

For any $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ we have a commutative diagram

$$\begin{tikzcd}
(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}^{\mathrm{op}}) \otimes (\mathcal{A} \otimes \mathcal{B}) \arrow[r, "\mathrm{Hom}_{\mathcal{A}\otimes\mathcal{B}}"] \arrow[d, "\cong"'] & \mathcal{V} \\
(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A}) \otimes (\mathcal{B}^{\mathrm{op}} \otimes \mathcal{B}) \arrow[r, "\mathrm{Hom}_{\mathcal{A}}\otimes\mathrm{Hom}_{\mathcal{B}}"'] & \mathcal{V} \otimes \mathcal{V}; \arrow[u, "\mathrm{Ten}"']
\end{tikzcd} \tag{1.36}$$

in terms of the partial functors, this asserts the commutativity of

$$\begin{tikzcd}
\mathcal{A} \otimes \mathcal{B} \arrow[r, "(\mathcal{A}\otimes\mathcal{B})((A,B),-)"] \arrow[dr, "\mathcal{A}(A,-)\otimes\mathcal{B}(B,-)"'] & \mathcal{V} \\
& \mathcal{V} \otimes \mathcal{V}, \arrow[u, "\mathrm{Ten}"']
\end{tikzcd} \tag{1.37}$$

which is easily checked. At the level of the underlying functors hom, this gives

$$(A \otimes B)\big((f,g),(f',g')\big) = \mathcal{A}(f,f') \otimes \mathcal{B}(g,g'). \tag{1.38}$$

1.7 Extraordinary $\mathcal{V}$-naturality

The point to be made in the next section is that all the families of maps canonically associated to $\mathcal{V}$, or to a $\mathcal{V}$-category $\mathcal{A}$, or to a $\mathcal{V}$-functor T, or to a $\mathcal{V}$-natural α, such as $a\colon (X\otimes Y)\otimes Z \longrightarrow X\otimes(Y\otimes Z)$, or $e\colon [Y,Z]\otimes Y \longrightarrow Z$, or $M\colon \mathcal{A}(B,C)\otimes\mathcal{A}(A,B) \longrightarrow \mathcal{A}(A,C)$, or $T\colon \mathcal{A}(A,B) \longrightarrow \mathcal{B}(TA,TB)$, or $\alpha\colon I \longrightarrow \mathcal{B}(TA,SA)$, are themselves $\mathcal{V}$-natural in every variable. To deal with a variable like B in the case of M, we must now introduce the notion of *extraordinary $\mathcal{V}$-naturality*; which later plays an essential role in the definition of *$\mathcal{V}$-functor-category*.

First we observe that the formulae (1.31) and (1.32) allow us to write the "ordinary" $\mathcal{V}$-naturality condition (1.7) in the more compact form

$$\begin{tikzcd}
\mathcal{A}(A,B) \arrow[r, "T"] \arrow[d, "S"'] & \mathcal{B}(TA,TB) \arrow[d, "\mathcal{B}(1,\alpha_B)"] \\
\mathcal{B}(SA,SB) \arrow[r, "\mathcal{B}(\alpha_A,1)"'] & \mathcal{B}(TA,SB).
\end{tikzcd} \tag{1.39}$$

By *(extraordinary) $\mathcal{V}$-naturality* for an $\mathrm{ob}\,\mathcal{A}$-indexed family of maps $\beta_A\colon K \longrightarrow T(A,A)$

in $\mathcal{B}_0$, where $K \in \mathcal{B}$ and $T\colon \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathcal{B}$, we mean the commutativity of each diagram

$$\begin{tikzcd} \mathcal{A}(A,B) \arrow[r, "T(A,-)"] \arrow[d, "T(-,B)"'] & \mathcal{B}\big(T(A,A),T(A,B)\big) \arrow[d, "\mathcal{B}(\beta_A,1)"] \\ \mathcal{B}\big(T(B,B),T(A,B)\big) \arrow[r, "\mathcal{B}(\beta_B,1)"'] & \mathcal{B}\big(K,T(A,B)\big). \end{tikzcd} \tag{1.40}$$

Duality gives the notion, for the same T and K, of a $\mathcal{V}$-natural family $\gamma_A\colon T(A,A) \longrightarrow K$; namely the commutativity of

$$\begin{tikzcd} \mathcal{A}(A,B) \arrow[r, "T(B,-)"] \arrow[d, "T(-,A)"'] & \mathcal{B}\big(T(B,A),T(B,B)\big) \arrow[d, "\mathcal{B}(1,\gamma_B)"] \\ \mathcal{B}\big(T(B,A),T(A,A)\big) \arrow[r, "\mathcal{B}(1,\gamma_A)"'] & \mathcal{B}\big(T(B,A),K\big). \end{tikzcd} \tag{1.41}$$

Such extraordinary $\mathcal{V}$-naturality is of course fully alive in the case $\mathcal{V} = \mathbf{Set}$; it is exhibited there, for instance, by the "naturality in Y" of the d and the e of (1.24). In the case $\mathcal{V} = \mathbf{Set}$, (1.40) and (1.41) have more elementary forms obtained by evaluating them at an arbitrary $f \in \mathcal{A}(A,B)$. Clearly extraordinary $\mathcal{V}$-naturality implies extraordinary $\mathbf{Set}$-naturality: we only have to apply V to (1.40) and (1.41).

Just as for ordinary $\mathcal{V}$-naturality, it is clear that if $\beta_A\colon K \longrightarrow T(A,A)$ is $\mathcal{V}$-natural as above, and if $P\colon \mathcal{D} \longrightarrow \mathcal{A}$ and $Q\colon \mathcal{B} \longrightarrow \mathcal{C}$, then the maps $Q\beta_{PD}(= Q_0\beta_{PD})\colon QK \longrightarrow QT(PD,PD)$ constitute a $\mathcal{V}$-natural family $Q\beta P$.

If $T\colon (\mathcal{A}\otimes\mathcal{D})^{\mathrm{op}} \otimes (\mathcal{A}\otimes\mathcal{D}) \longrightarrow \mathcal{B}$, a family $\beta_{AD}\colon K \longrightarrow T(A,D,A,D)$ is $\mathcal{V}$-natural in (A,D) if and only if it is $\mathcal{V}$-natural in A for each fixed D and in D for each fixed A, with respect to the partial functors $T(-,D,-,D)$ and $T(A,-,A,-)$ respectively. The most direct proof involves translating (1.40) via (1.32) into a form analogous to (1.7); then, as in the proof for ordinary $\mathcal{V}$-natural transformations, it is a matter of combining (1.21) with (1.3) for $\mathcal{C}$. Thus here, too, $\mathcal{V}$-naturality can be verified variable-by-variable.

That being so, we can combine ordinary and extraordinary $\mathcal{V}$-naturality, and speak of a $\mathcal{V}$-natural family $\alpha_{ABC}\colon T(A,B,B) \longrightarrow S(A,C,C)$, where $T\colon \mathcal{A}\otimes\mathcal{B}^{\mathrm{op}}\otimes\mathcal{B} \longrightarrow \mathcal{D}$ and $S\colon \mathcal{A}\otimes\mathcal{C}^{\mathrm{op}}\otimes\mathcal{C} \longrightarrow \mathcal{D}$. If each of A, B and C stands for a finite number of variables, this is then the most general form.

The analogue of "vertical composition" for such families is the calculus of [25]. There are three basic cases of composability, in addition to that for ordinary $\mathcal{V}$-natural transformations; and two of these three are dual. Each of the three, however, is in fact a sequence of sub-cases indexed by the natural numbers. The reader will see the pattern if we just give the first two sub-cases of the only two essentially-different sequences; the proofs are easy using (1.39)–(1.41).

For T, $S\colon \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathcal{B}$, if $\alpha_A\colon K \longrightarrow T(A,A)$ is $\mathcal{V}$-natural in A and if

$$\beta_{AB}\colon T(A,B) \longrightarrow S(A,B)$$

is $\mathcal{V}$-natural in $\mathcal{A}$ and $\mathcal{B}$, the composite

$$K \xrightarrow[\alpha_A]{} T(A,A) \xrightarrow[\beta_{AA}]{} S(A,A) \tag{1.42}$$

is $\mathcal{V}$-natural in A. For $T\colon \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathcal{B}$ and $S\colon \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathcal{B}$, if $\alpha_{AB}\colon K \longrightarrow T(A,A,B,B)$ is $\mathcal{V}$-natural in A and B and if

$$\beta_{ABC}\colon T(A,B,B,C) \longrightarrow S(A,C)$$

is $\mathcal{V}$-natural in A, B and C, then the composite

$$K \xrightarrow[\alpha_{AA}]{} T(A,A,A,A) \xrightarrow[\beta_{AAA}]{} S(A,A) \tag{1.43}$$

is $\mathcal{V}$-natural in A. The next one in this series has $\alpha_{ABC}\colon K \longrightarrow T(A,A,B,B,C,C)$ and $\beta_{ABCD}\colon T(A,B,B,C,C,D) \longrightarrow S(A,D)$; and there is of course a dual series.

The self-dual series begins with $T\colon \mathcal{A} \longrightarrow \mathcal{B}$, $S\colon \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathcal{B}$, and $R\colon \mathcal{A} \longrightarrow \mathcal{B}$; if $\alpha_{AB}\colon TA \longrightarrow S(A,B,B)$ and $\beta_{AB}\colon S(A,A,B) \longrightarrow RB$ are $\mathcal{V}$-natural in A and B, the composite

$$TA \xrightarrow[\alpha_{AA}]{} S(A,A,A) \xrightarrow[\beta_{AA}]{} RA \tag{1.44}$$

is $\mathcal{V}$-natural in A. For the same T and R, but for $S\colon \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathcal{B}$, if $\alpha_{ABC}\colon TA \longrightarrow S(A,B,B,C,C)$ and $\beta_{ABC}\colon S(A,A,B,B,C) \longrightarrow RC$ are $\mathcal{V}$-natural in A, B and C, then the composite

$$TA \xrightarrow[\alpha_{AAA}]{} S(A,A,A,A,A) \xrightarrow[\beta_{AAA}]{} RA \tag{1.45}$$

is $\mathcal{V}$-natural in A.

In each case a handsome geometrical picture is produced if, to the diagrams (1.42)–(1.45), there are added curves above the diagram linking variables which must be set equal in α, and curves below the diagram linking variables which must be set equal in β; the geometrical union of these curves, thought of as meeting at the variables, then links the variables which must be set equal in the composite. Of course the associativity of this kind of composition, whenever it makes sense, is a triviality.

1.8 The $\mathcal{V}$-naturality of the canonical maps

We turn now to the $\mathcal{V}$-naturality of the canonical maps mentioned at the beginning of 1.7. Of course such an object as $\mathcal{A}(B,C) \otimes \mathcal{A}(A,B)$ is here thought of as the value $\mathrm{Ten}(\mathrm{Hom}_{\mathcal{A}}(B,C), \mathrm{Hom}_{\mathcal{A}}(A,B))$ of the appropriate $\mathcal{V}$-functor.

(a) First, it follows at once from (1.29) that $\alpha_A\colon I \longrightarrow \mathcal{B}(TA,SA)$ is $\mathcal{V}$-natural in the sense of (1.40) precisely when α is a $\mathcal{V}$-natural transformation $\alpha\colon T \longrightarrow S$ in the sense of (1.7).

(b) Next, for a $\mathcal{V}$-functor $T\colon \mathcal{A} \longrightarrow \mathcal{B}$, the map $T_{AB}\colon \mathcal{A}(A,B) \longrightarrow \mathcal{B}(TA,TB)$ is $\mathcal{V}$-natural in both variables A and B. It suffices to verify $\mathcal{V}$-naturality in B; and here the appropriate diagram of the form (1.39) reduces, in the light of (1.29), to (1.5) for T.

(c) This last gives, for a $\mathcal{V}$-functor $T\colon \mathcal{A}\otimes\mathcal{B}\longrightarrow\mathcal{C}$, the $\mathcal{V}$-naturality in A and A' of $T(-,B)\colon \mathcal{A}(A,A')\longrightarrow\mathcal{C}(T(A,B),T(A',B))$. However, this is also $\mathcal{V}$-natural in B; the appropriate instance of (1.40) reduces via (1.29) to (1.21).

(d) In particular, for a $\mathcal{V}$-category $\mathcal{A}$, the $\mathcal{A}(A,-)_{BC}\colon \mathcal{A}(B,C)\longrightarrow[\mathcal{A}(A,B),\mathcal{A}(A,C)]$ of (1.29) is $\mathcal{V}$-natural in all variables.

(e) Composing the $\mathcal{A}(A,-)_{BC}$ of (d) with $g\colon I\longrightarrow\mathcal{A}(B,C)$ gives, by (a), the $\mathcal{V}$-naturality in A of $\mathcal{A}(A,g)\colon \mathcal{A}(A,B)\longrightarrow\mathcal{A}(A,C)$.

(f) The $\mathcal{V}$-naturality in both variables of $e\colon [Y,Z]\otimes Y\longrightarrow Z$ is an easy consequence of (1.29) and (1.34).

(g) Since, by (1.29), $M\colon \mathcal{A}(B,C)\otimes\mathcal{A}(A,B)\longrightarrow\mathcal{A}(A,C)$ for a $\mathcal{V}$-category $\mathcal{A}$ is the composite

$$\mathcal{A}(B,C)\otimes\mathcal{A}(A,B)\xrightarrow[\mathcal{A}(A,-)\otimes 1]{}[\mathcal{A}(A,B),\mathcal{A}(A,C)]\otimes\mathcal{A}(A,B)\xrightarrow[e]{}\mathcal{A}(A,C),$$

it is $\mathcal{V}$-natural in all variables by (d), (e), (f), (1.42) and (1.44).

(h) The $\mathcal{V}$-naturality of $j_A\colon I\longrightarrow\mathcal{A}(A,A)$ for a $\mathcal{V}$-category $\mathcal{A}$ follows from (a), since $1\colon 1\longrightarrow 1\colon \mathcal{A}\longrightarrow\mathcal{A}$ is $\mathcal{V}$-natural.

(i) The $\mathcal{V}$-naturality in every variable of $a\colon (X\otimes Y)\otimes Z\longrightarrow X\otimes(Y\otimes Z)$, $l\colon I\otimes X\longrightarrow X$, $r\colon X\otimes I\longrightarrow X$, $c\colon X\otimes Y\longrightarrow Y\otimes X$, follows easily from (1.34).

(j) From the definition (1.27) of the isomorphism p and the definition (1.34) of $\mathrm{Ten}(-,Y)$, we easily see that $p^{-1}\colon [X,[Y,Z]]\longrightarrow[X\otimes Y,Z]$ is the composite

$$[X,[Y,Z]]\xrightarrow[\mathrm{Ten}(-,Y)]{}[X\otimes Y,[Y,Z]\otimes Y]\xrightarrow[{[1,e]}]{}[X\otimes Y,Z].$$

It follows from (c), (e), (f), and (1.44) that p^{-1} is $\mathcal{V}$-natural in all variables; whence the same is true of its inverse p.

(k) The composite

$$X\xrightarrow[l^{-1}]{}I\otimes X\xrightarrow[j\otimes 1]{}[X\otimes Y,X\otimes Y]\otimes X\xrightarrow[p\otimes 1]{}[X,[Y,X\otimes Y]]\otimes X\xrightarrow[e]{}[Y,X\otimes Y]$$

is easily verified to be the $d\colon X\longrightarrow[Y,X\otimes Y]$ of (1.24); the latter is therefore $\mathcal{V}$-natural in every variable by (i), (h), (j), (f), (e), (1.42), and (1.44).

(l) The composite

$$X\xrightarrow[d]{}[I,X\otimes I]\xrightarrow[{[1,r]}]{}[I,X]$$

is the isomorphism $i\colon X\longrightarrow[I,X]$, which is therefore $\mathcal{V}$-natural.

This completes the list of canonical maps, and we end this section with a general principle. Since a map $f\colon X\otimes Y\longrightarrow Z$ and its image $\overline{f}\colon X\longrightarrow[Y,Z]$ under the π of (1.23) are related by $\overline{f}=[1,f]d$ and $f=e(\overline{f}\otimes 1)$, it follows from (f), (k), (e) and (1.42)–(1.44) that, for $\mathcal{V}$-functors T, S, R of suitable variances with codomain $\mathcal{V}$, we have:

(m) A family $f\colon T(D,D,A,B)\otimes S(E,E,A,C)\longrightarrow R(F,F,B,C)$ is $\mathcal{V}$-natural in any of its variables if and only if the corresponding

$$\overline{f}\colon T(D,D,A,B)\longrightarrow[S(E,E,A,C),R(F,F,B,C)]$$

is so.

1.9 The (weak) Yoneda lemma for $\mathcal{V}$-**CAT**

The form of the Yoneda Lemma we give here is a weak one, in that it asserts a bijection of sets rather than an isomorphism of objects of $\mathcal{V}$; a stronger form will be given in 2.4 below.

Consider a $\mathcal{V}$-functor $F\colon\mathcal{A}\longrightarrow\mathcal{V}$ and an object K of $\mathcal{A}$. To each $\mathcal{V}$-natural transformation $\alpha\colon\mathcal{A}(K,-)\longrightarrow F$ we can assign the element $\eta\colon I\longrightarrow FK$ of FK given by the composite

$$I\xrightarrow[j_K]{}\mathcal{A}(K,K)\xrightarrow[\alpha_K]{}FK. \tag{1.46}$$

The Yoneda Lemma asserts that *this gives a bijection between the set* $\mathcal{V}$-nat$(\mathcal{A}(K,-),F)$ *of $\mathcal{V}$-natural transformations and the set $\mathcal{V}_0(I,FK)$ $(=VFK)$ of elements of FK; the component α_A being given in terms of η as the composite*

$$\mathcal{A}(K,A)\xrightarrow[F_{KA}]{}[FK,FA]\xrightarrow[{[\eta,1]}]{}[I,FA]\xrightarrow[i^{-1}]{}FA. \tag{1.47}$$

For the proof, we first note that, for any η, (1.47) is indeed $\mathcal{V}$-natural in A, by 1.8 (b), (e), and (l). Next, if α is defined by (1.47), $\alpha_K j_K$ is in fact η; this follows easily from (1.6) for F, the **Set**-naturality of j (a consequence, as we saw in (1.7), of its $\mathcal{V}$-naturality), the naturality of i, and the easy verification that $j_I=i_I\colon I\longrightarrow[I,I]$. It remains to show that (1.47) is in fact α_A, when η is given by (1.46). In the diagram

$$\begin{tikzcd}
\mathcal{A}(K,A) \arrow[r, "\mathcal{A}(K,-)"] \arrow[d, "F"'] & {[\mathcal{A}(K,K),\mathcal{A}(K,A)]} \arrow[r, "{[j,1]}"] \arrow[d, "{[1,\alpha_A]}"] & {[I,\mathcal{A}(K,A)]} \arrow[r, "i^{-1}"] \arrow[d, "{[1,\alpha_A]}"] & \mathcal{A}(K,A) \arrow[d, "\alpha_A"] \\
{[FK,FA]} \arrow[r, "{[\alpha_K,1]}"'] & {[\mathcal{A}(K,K),FA]} \arrow[r, "{[j,1]}"'] & {[I,FA]} \arrow[r, "i^{-1}"'] & FA,
\end{tikzcd}$$

the left square commutes by the $\mathcal{V}$-naturality of α in the form (1.39), the middle square trivially, and the right square by the naturality of i. The proof is complete when we observe that the top edge of the diagram is the identity; which follows from the definition (1.29) of $\mathcal{A}(K,-)$, the axiom (1.4) for $\mathcal{A}$, and the definition (1.26) of i in terms of r.

There is an extra-variable form of the result, involving functors $F\colon \mathcal{B}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathcal{V}$ and $K\colon \mathcal{B} \longrightarrow \mathcal{A}$, and a family $\alpha_{BA}\colon \mathcal{A}(KB, A) \longrightarrow F(B, A)$ which is $\mathcal{V}$-natural in A for each fixed B, so that it corresponds as above to a family $\eta_B\colon I \longrightarrow F(B, KB)$. It then follows from (1.46) and (1.42) that *η is $\mathcal{V}$-natural in B if α is*, and from (1.47) and (1.44) that *α is $\mathcal{V}$-natural in B if η is.*

It is worth noting an alternative way of writing (1.47) when F has the form $\mathcal{B}(B, T-)$ for a $\mathcal{V}$-functor $T\colon \mathcal{A} \longrightarrow \mathcal{B}$, so that $\eta\colon I \longrightarrow \mathcal{B}(B, TK)$ is then a map $\eta\colon B \longrightarrow TK$ in $\mathcal{B}_0$ (in fact, the image of 1_K under $V\alpha_K\colon \mathcal{A}_0(K, K) \longrightarrow \mathcal{B}_0(B, TK)$). Then α_A is the composite

$$\mathcal{A}(K, A) \xrightarrow[T_{KA}]{} \mathcal{B}(TK, TA) \xrightarrow[\mathcal{B}(\eta,1)]{} \mathcal{B}(B, TA), \tag{1.48}$$

which is easily seen by (1.32) to coincide with (1.47). In particular, *every $\mathcal{V}$-natural $\alpha\colon \mathcal{A}(K, -) \longrightarrow \mathcal{A}(L, -)$ is $\mathcal{A}(k, -)$ for a unique $k\colon L \longrightarrow K$; and clearly k is an isomorphism if and only if α is.*

1.10 Representability of $\mathcal{V}$-functors; the representing object as a $\mathcal{V}$-functor of the passive variables

We have called the $\mathcal{V}$-functors $\mathcal{A}(K, -)\colon \mathcal{A} \longrightarrow \mathcal{V}$ *representable*. More generally, however, a $\mathcal{V}$-functor $F\colon \mathcal{A} \longrightarrow \mathcal{V}$ is called representable if there is some $K \in \mathcal{A}$ and an *isomorphism* $\alpha\colon \mathcal{A}(K, -) \longrightarrow F$. Then the pair (K, α) is a *representation* of F; it is essentially unique if it exists, in the sense that any other representation $\alpha'\colon \mathcal{A}(K', -) \longrightarrow F$ has the form $\alpha' = \alpha.\mathcal{A}(k, -)$ for a unique isomorphism $k\colon K \longrightarrow K'$. The corresponding $\eta\colon I \longrightarrow FK$ is called the *unit* of the representation. (We call η the *counit* when we are representing a contravariant $F\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{V}$ in the form $\mathcal{A}(-, K)$.)

For a general $\mathcal{V}$, there is no simple criterion in terms of an $\eta\colon I \longrightarrow FK$ for the corresponding $\alpha\colon \mathcal{A}(K, -) \longrightarrow F$ to be an isomorphism. It is of course otherwise in the classical case $\mathcal{V} = \mathbf{Set}$; there we have the comma-category $1/F = \mathrm{el}\, F$ of "elements of F", whose objects are pairs (A, x) with $A \in \mathcal{A}$ and $x \in FA$; and α is invertible if and only if (K, η) is initial in $\mathrm{el}\, F$, so that F is representable if and only if $\mathrm{el}\, F$ has an initial object. However, any such kind of "universal property" criterion ultimately expresses a bijection of sets, and cannot suffice in general to characterize an isomorphism in $\mathcal{V}_0$.

Now let $F\colon \mathcal{B}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathcal{V}$ be such that each $F(B, -)\colon \mathcal{A} \longrightarrow \mathcal{V}$ admits a representation $\alpha_B\colon \mathcal{A}(KB, -) \longrightarrow F(B, -)$. Then *there is exactly one way of defining $K_{BC}\colon \mathcal{B}(B, C) \longrightarrow \mathcal{A}(KB, KC)$ that makes K a $\mathcal{V}$-functor for which*

$$\alpha_{BA}\colon \mathcal{A}(KB, A) \longrightarrow F(B, A)$$

is $\mathcal{V}$-natural in B as well as A.

For by 1.9, the $\mathcal{V}$-naturality in B of α_{BA} is equivalent to that of η_B, which by (1.40)

means the commutativity of

$$\begin{tikzcd} \mathcal{B}(B,C) \arrow[r, "K_{BC}"] \arrow[d, "F(-,KC)"'] & \mathcal{A}(KB,KC) \arrow[r, "F(B,-)"] & {[F(B,KB),F(B,KC)]} \arrow[d, "{[\eta_B,1]}"] \\ {[F(C,KC),F(B,KC)]} \arrow[rr, "{[\eta_C,1]}"'] & & {[I,F(B,KC)]}. \end{tikzcd} \tag{1.49}$$

Now the composite $[\eta_B, 1]F(B,-)$ in this diagram is, by (1.47), the composite *isomorphism*

$$\mathcal{A}(KB,KC) \xrightarrow[\alpha_{B,KC}]{} F(B,KC) \xrightarrow[i]{} [I,F(B,KC)];$$

so that (1.49) forces us to define K_{BC} as

$$K_{BC} = \alpha^{-1} i^{-1} [\eta_C, 1] F(-,KC). \tag{1.50}$$

This composite (1.50) is $\mathcal{V}$-natural in B, by 1.7 and 1.8; and from this the proof of 1.8(b), read backwards, gives the $\mathcal{V}$-functor axiom (1.5) for K; while the remaining $\mathcal{V}$-functor axiom (1.6) is a trivial consequence of (1.49).

1.11 Adjunctions and equivalences in $\mathcal{V}$-**CAT**

As in any 2-category (see [49]) an *adjunction* $\eta, \epsilon \colon S \dashv T \colon \mathcal{A} \longrightarrow \mathcal{B}$ in $\mathcal{V}$-**CAT** between $T \colon \mathcal{A} \longrightarrow \mathcal{B}$ (the *right adjoint*) and $S \colon \mathcal{B} \longrightarrow \mathcal{A}$ (the *left adjoint*) consists of $\eta \colon 1 \longrightarrow TS$ (the *unit*) and $\epsilon \colon ST \longrightarrow 1$ (the *counit*) satisfying the *triangular equations* $T\epsilon.\eta T = 1$ and $\epsilon S.S\eta = 1$. By the Yoneda Lemma of 1.9, such adjunctions in $\mathcal{V}$-**CAT** are in bijection with $\mathcal{V}$-natural *isomorphisms*

$$n \colon \mathcal{A}(SB,A) \cong \mathcal{B}(B,TA); \tag{1.51}$$

for by (1.48) a $\mathcal{V}$-natural *map* $n \colon \mathcal{A}(SB,A) \longrightarrow \mathcal{B}(B,TA)$ has the form $n = \mathcal{B}(\eta,1)T$ for a unique η, while a $\mathcal{V}$-natural map $\overline{n} \colon \mathcal{B}(B,TA) \longrightarrow \mathcal{A}(SB,A)$ has the form $\overline{n} = \mathcal{A}(1,\epsilon)S$ for a unique ϵ, and the equations $\overline{n}n = 1$ and $n\overline{n} = 1$ reduce by Yoneda precisely to the triangular equations.

The 2-functor $(-)_0 \colon \mathcal{V}$-**CAT** $\longrightarrow$ **CAT** carries such an adjunction into an ordinary adjunction $\eta, \epsilon \colon S_0 \dashv T_0 \colon \mathcal{A}_0 \longrightarrow \mathcal{B}_0$ in **CAT**. It is immediate from (1.33) that the corresponding isomorphism of hom-sets is

$$Vn \colon \mathcal{A}_0(SB,A) \cong \mathcal{B}_0(B,TA). \tag{1.52}$$

Note that the unit and counit of this ordinary adjunction are the same η and ϵ, now seen as natural rather than $\mathcal{V}$-natural.

By 1.10, a $\mathcal{V}$-functor $T \colon \mathcal{A} \longrightarrow \mathcal{B}$ has a left adjoint (which is then unique to within isomorphism, as in any 2-category) exactly when each $\mathcal{B}(B,T-)$ is representable. More generally, if those B for which $\mathcal{B}(B,T-)$ *is* representable constitute the full subcategory $\mathcal{B}'$ of $\mathcal{B}$, we get a $\mathcal{V}$-functor $S \colon \mathcal{B}' \longrightarrow \mathcal{A}$ called a *partial left adjoint* to T.

When V is conservative – faithfulness is not enough – the existence of a left adjoint S_0 for T_0 implies that of a left adjoint S for T: we set $SB = S_0B$, define a map n that is $\mathcal{V}$-natural in A by $n = \mathcal{B}(\eta, 1)T$ as above (using the unit η for $S_0 \dashv T_0$), and deduce that n is an isomorphism since the Vn of (1.52) is. Thus, taking $\mathcal{V} = \mathbf{Ab}$ for instance, an additive functor has a left adjoint exactly when its underlying functor has one.

Again by 1.10, in the extra-variable case of a $\mathcal{V}$-functor $T\colon \mathcal{C}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathcal{B}$, if each $T(C, -)$ has a left adjoint $S(-, C)$, then S automatically becomes a $\mathcal{V}$-functor $\mathcal{B} \otimes \mathcal{C} \longrightarrow \mathcal{A}$ in such a way that $n\colon \mathcal{A}(S(B, C), A) \cong \mathcal{B}(B, T(C, A))$ is $\mathcal{V}$-natural in all three variables. An evident example, with $\mathcal{A} = \mathcal{B} = \mathcal{C} = \mathcal{V}$, is the adjunction $p\colon [X \otimes Y, Z] \cong [X, [Y, Z]]$ of (1.27).

For a typical adjunction (1.51) as above, we have a commutative diagram

$$\begin{tikzcd} \mathcal{A}(A, A') \arrow[rr, "T_{AA'}"] \arrow[dr, "\mathcal{A}(\epsilon,1)"'] & & \mathcal{B}(TA, TA') \\ & \mathcal{A}(STA, A'); \arrow[ur, "n"'] & \end{tikzcd} \tag{1.53}$$

this follows from Yoneda on setting $A' = A$ and composing both legs with j, since $(Vn)(\epsilon_A) = 1$. Because n is an isomorphism, we conclude that T *is fully faithful if and only if* ϵ *is an isomorphism.* In consequence, a right adjoint T is fully faithful exactly when T_0 is so.

As in any 2-category, a $\mathcal{V}$-functor $T\colon \mathcal{A} \longrightarrow \mathcal{B}$ is called an *equivalence*, and we write $T\colon \mathcal{A} \simeq \mathcal{B}$, if there is some $S\colon \mathcal{B} \longrightarrow \mathcal{A}$ and isomorphisms $\eta\colon 1 \cong TS$ and $\rho\colon ST \cong 1$. Replacing ρ (still in any 2-category) by the isomorphism $\epsilon\colon ST \cong 1$ given by $\epsilon = \rho.S\eta^{-1}T.\rho^{-1}ST$, we actually have an *adjunction* $\eta, \epsilon\colon S \dashv T\colon \mathcal{A} \longrightarrow \mathcal{B}$. The word "equivalence" is sometimes used to mean an adjunction in which, as here η and ϵ are isomorphisms; we write $S \dashv T\colon \mathcal{A} \simeq \mathcal{B}$.

In the 2-category $\mathcal{V}$-$\mathbf{CAT}$, $T\colon \mathcal{A} \longrightarrow \mathcal{B}$ *is an equivalence if and only if* T *is fully faithful and essentially surjective on objects*; the latter means that every $B \in \mathcal{B}$ is isomorphic (in $\mathcal{B}_0$, of course) to TA for some $A \in \mathcal{A}$; we often omit the words "on objects".

For, if T is an equivalence, T is fully faithful since ϵ is an isomorphism, and is essentially surjective on objects since $\eta_B\colon B \longrightarrow TSB$ is an isomorphism. As for the converse, we *choose* for each $B \in \mathcal{B}$ an $SB \in \mathcal{A}$ and an isomorphism $\eta_B\colon B \longrightarrow TSB$. The corresponding $\mathcal{V}$-natural-in-A transformation $n = \mathcal{B}(\eta_B, 1)T\colon \mathcal{A}(SB, A) \longrightarrow \mathcal{B}(B, TA)$ is then an isomorphism because T is fully faithful, so that S becomes automatically a $\mathcal{V}$-functor left adjoint to T. Since η is an isomorphism, so is $T\epsilon$, by the triangular equation $T\epsilon.\eta T = 1$; whence ϵ is an isomorphism by (1.12), since a fully-faithful T_0 is certainly conservative.

In any 2-category (see [49]) adjunctions can be composed, forming a category, of which the equivalences are a subcategory. It follows from Yoneda that the composition of adjunctions in $\mathcal{V}$-$\mathbf{CAT}$ corresponds to the composition of the $\mathcal{V}$-natural isomorphisms in

$$\mathcal{A}(SS'B, A) \cong \mathcal{B}(S'B, TA) \cong \mathcal{B}(B, T'TA). \tag{1.54}$$

It is further true in any 2-category (see [49]) that, given adjunctions $\eta, \epsilon\colon S \dashv T\colon \mathcal{A} \longrightarrow \mathcal{B}$ and $\eta', \epsilon'\colon S' \dashv T'\colon \mathcal{A}' \longrightarrow \mathcal{B}'$ and 1-cells $P\colon \mathcal{A} \longrightarrow \mathcal{A}'$ and $Q\colon \mathcal{B} \longrightarrow \mathcal{B}'$, there

is a bijection (with appropriate naturality properties) between 2-cells $\lambda\colon QT \longrightarrow T'P$ and 2-cells $\mu\colon S'Q \longrightarrow PS$. In $\mathcal{V}$-**CAT** it follows from Yoneda that λ and μ determine one another through the commutativity of

$$\begin{tikzcd}
\mathcal{A}(SB, A) \arrow[r, "n", "\cong"'] \arrow[d, "P"'] & \mathcal{B}(B, TA) \arrow[d, "Q"] \\
\mathcal{A}'(PSB, PA) \arrow[d, "\mathcal{A}'(\mu,1)"'] & \mathcal{B}'(QB, QTA) \arrow[d, "\mathcal{B}'(1,\lambda)"] \\
\mathcal{A}'(S'QB, PA) \arrow[r, "\cong", "n'"'] & \mathcal{B}'(QB, T'PA);
\end{tikzcd} \tag{1.55}$$

such a pair λ, μ may be called *mates*.

In particular, when $\mathcal{A}' = \mathcal{A}$ with $P = 1$ and $\mathcal{B}' = \mathcal{B}$ with $Q = 1$, so that $S \dashv T$ and $S' \dashv T'$ are both adjunctions $\mathcal{A} \longrightarrow \mathcal{B}$, we have a bijection between the $\lambda\colon T \longrightarrow T'$ and the $\mu\colon S' \longrightarrow S$. The adjunctions become a 2-category when we define a 2-cell from $S \dashv T$ to $S' \dashv T'$ to be such a pair (λ, μ).

Any $T\colon \mathcal{A} \longrightarrow \mathcal{B}$ determines two full subcategories of $\mathcal{B}$; the *full image* $\mathcal{A}'$ of T, determined by those B of the form TA, and the *replete image* $\mathcal{A}''$ of T, determined by those B isomorphic to some TA. Clearly $\mathcal{A}'$ and $\mathcal{A}''$ are equivalent; and each is equivalent to $\mathcal{A}$ if T is fully faithful. A full subcategory which contains all the isomorphs of its objects is said to be *replete*; any full subcategory has a *repletion*, namely the replete image of its inclusion. Clearly a fully-faithful $T\colon \mathcal{A} \longrightarrow \mathcal{B}$ has a left adjoint if and only if the inclusion $\mathcal{A}' \longrightarrow \mathcal{B}$ has one, and if and only if the inclusion $\mathcal{A}'' \longrightarrow \mathcal{B}$ has one.

A full subcategory $\mathcal{A}$ of $\mathcal{B}$ is called *reflective* if the inclusion $T\colon \mathcal{A} \longrightarrow \mathcal{B}$ has a left adjoint. This implies, of course, that $\mathcal{A}_0$ is reflective in $\mathcal{B}_0$, but is in general strictly stronger. It follows – since it is trivially true for ordinary categories – that *every retract (in $\mathcal{B}_0$) of an object of the reflective $\mathcal{A}$ lies in the repletion of $\mathcal{A}$.*

When $\mathcal{A}$ is reflective we may so choose the left adjoint $S\colon \mathcal{B} \longrightarrow A$ that $\epsilon\colon ST \longrightarrow 1$ is the identity. Then $\eta\colon 1 \longrightarrow R = TS\colon \mathcal{B} \longrightarrow \mathcal{B}$ satisfies $R^2 = R$, $\eta R = R\eta = 1$. Such an (R, η) is called an *idempotent monad* on $\mathcal{B}$. Conversely, for any idempotent monad (R, η), both its full image and its replete image are reflective in $\mathcal{B}$. If R is formed as above, the full image is $\mathcal{A}$ again; but (R, η) is not uniquely determined by $\mathcal{A}$. The replete image of R is the repletion of $\mathcal{A}$; and there is an evident bijection between full replete reflective subcategories of $\mathcal{B}$ and isomorphism classes of idempotent monads on $\mathcal{B}$.

Chapter 2

Functor categories

2.1 Ends in $\mathcal{V}$

So far we have supposed that $\mathcal{V}$ is symmetric monoidal closed, and that the underlying ordinary category $\mathcal{V}_0$ is locally small. Henceforth we add the assumption that $\mathcal{V}_0$ is *complete*, in the sense that it admits all small limits; and from 2.5 on, we shall suppose as well that $\mathcal{V}_0$ is *cocomplete*.

Consider a $\mathcal{V}$-functor $T\colon \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathcal{V}$. If there exists a *universal* $\mathcal{V}$-natural family $\lambda_A\colon K \longrightarrow T(A,A)$, in the sense that every $\mathcal{V}$-natural $\alpha_A\colon X \longrightarrow T(A,A)$ is given by $\alpha_A = \lambda_A f$ for a unique $f\colon X \longrightarrow K$, we call (K,λ) the *end* of T; it is clearly unique to within a unique isomorphism when it exists. We write $\int_{A\in\mathcal{A}} T(A,A)$ for the object K, and by the usual abuse of language also call this *object* the end of T: then the universal $\mathcal{V}$-natural $\lambda_A\colon \int_A T(A,A) \longrightarrow T(A,A)$ may be called the *counit* of this end.

The $\mathcal{V}$-naturality condition (1.40), in the case $\mathcal{B} = \mathcal{V}$, transforms under the adjunction $\mathcal{V}_0(X,[Y,Z]) \cong \mathcal{V}_0(X\otimes Y, Z) \cong \mathcal{V}_0(Y,[X,Z])$ into

$$\begin{tikzcd} K \arrow[r, "\lambda_A"] \arrow[d, "\lambda_B"'] & T(A,A) \arrow[d, "\rho_{AB}"] \\ T(B,B) \arrow[r, "\sigma_{AB}"'] & {[\mathcal{A}(A,B), T(A,B)]}, \end{tikzcd} \tag{2.1}$$

where ρ_{AB} and σ_{AB} are the transforms of $T(A,-)_{AB}$ and $T(-,B)_{BA}$. It follows that, when $\mathcal{A}$ is small, the end $\int_{A\in\mathcal{A}} T(A,A)$ certainly exists, and is given as an equalizer

$$\int_{A\in\mathcal{A}} T(A,A) \xrightarrow[\lambda]{} \prod_{A\in\mathcal{A}} T(A,A) \underset{\sigma}{\overset{\rho}{\rightrightarrows}} \prod_{A,B\in\mathcal{A}} [\mathcal{A}(A,B), T(A,B)]. \tag{2.2}$$

If $\mathcal{A}$, while not necessarily small, is *equivalent* to a small $\mathcal{V}$-category, then the full image in $\mathcal{A}$ of the latter is a small full subcategory $\mathcal{A}'$ for which the inclusion $\mathcal{A}' \longrightarrow \mathcal{A}$ is an equivalence. Since it clearly suffices to impose (2.1) only for A, $B \in \mathcal{A}'$, it follows that $\int_{A\in\mathcal{A}} T(A,A)$ still exists, being $\int_{A\in\mathcal{A}'} T(A,A)$. In consequence, *we henceforth change the meaning of "small", extending it to include those $\mathcal{V}$-categories equivalent to small ones in the sense of 1.2.*

The universal property of the end – expressed, as it stands, by a bijection of sets – in fact lifts to an isomorphism in $\mathcal{V}_0$; in the sense that

$$[X, \lambda_A] \colon [X, \textstyle\int_A T(A,A)] \longrightarrow [X, T(A,A)]$$

expresses its domain as the end of its codomain:

$$[X, \textstyle\int_A T(A,A)] \cong \int_A [X, T(A,A)] \qquad (\textit{with counit } [X, \lambda_A]). \tag{2.3}$$

For $Y \longrightarrow [X, T(A,A)]$ is $\mathcal{V}$-natural if and only if $Y \otimes X \longrightarrow T(A,A)$ is, by 1.8(m). Note that the right side of (2.3) exists for all X if $\int_A T(A,A)$ exists; the converse of this is also true, as we see on taking $X = I$.

If $\alpha \colon T \longrightarrow T' \colon \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathcal{V}$, and if the ends of T and T' exist, the top leg of the diagram

$$\begin{tikzcd} \int_A T(A,A) \arrow[r, "\lambda_A"] \arrow[d, "\int_A \alpha_{AA}"'] & T(A,A) \arrow[d, "\alpha_{AA}"] \\ \int_A T'(A,A) \arrow[r, "\lambda'_A"'] & T'(A,A) \end{tikzcd} \tag{2.4}$$

is $\mathcal{V}$-natural by (1.42); so that, by the universal property of λ', there is a unique $\int_A \alpha_{AA}$ rendering the diagram commutative.

This "functoriality" of ends lifts, in the appropriate circumstances, to a $\mathcal{V}$-functoriality. Suppose namely that, for $T \colon \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \otimes \mathcal{B} \longrightarrow \mathcal{V}$, the end $HB = \int_A T(A,A,B)$ exists for each B. Then *there is exactly one way of making H into a $\mathcal{V}$-functor $H \colon \mathcal{B} \longrightarrow \mathcal{V}$ that renders $\lambda_{AB} \colon HB \longrightarrow T(A,A,B)$ $\mathcal{V}$-natural in B as well as in A.*

For, by (1.39), the $\mathcal{V}$-naturality in B is expressed by the commutativity of

$$\begin{tikzcd} \mathcal{B}(B,C) \arrow[r, "H_{BC}"] \arrow[d, "T(A,A,-)"'] & {[HB, HC]} \arrow[d, "{[1,\lambda_{AC}]}"] \\ {[T(A,A,B), T(A,A,C)]} \arrow[r, "{[\lambda_{AB},1]}"'] & {[HB, T(A,A,C)]}. \end{tikzcd} \tag{2.5}$$

The bottom leg is $\mathcal{V}$-natural in A by 1.8(c), 1.8(e), and (1.43). The right edge is itself an end by (2.3). Hence a unique H_{BC} renders the diagram commutative. The $\mathcal{V}$-functor axioms (1.5) and (1.6) for H then follow from the $\mathcal{V}$-naturality of M and j (1.8(g) and (h)), the composition-calculus of 1.7, and the universal property of the counit $[1, \lambda]$.

In these circumstances, if $R \colon \mathcal{B} \longrightarrow \mathcal{V}$, *a family $\beta_B \colon RB \longrightarrow \int_A T(A,A,B)$ is $\mathcal{V}$-natural in B if and only if the composite*

$$RB \xrightarrow[\beta_B]{} \textstyle\int_A T(A,A,B) \xrightarrow[\lambda_{AB}]{} T(A,A,B) \tag{2.6}$$

is so. One direction is trivial, while the other follows easily from (2.5) and the universal property of the counit $[1, \lambda]$. In fact, this result can also be seen – using 1.8(m) to view

(2.6) as a $\mathcal{V}$-natural family $I \longrightarrow [RB, T(A,A,B)]$ and using (2.3) – as a special case of the following.

Let $T\colon (\mathcal{A}\otimes\mathcal{B})^{\mathrm{op}}\otimes(\mathcal{A}\otimes\mathcal{B}) \longrightarrow \mathcal{V}$, recalling that $(\mathcal{A}\otimes\mathcal{B})^{\mathrm{op}} = \mathcal{A}^{\mathrm{op}}\otimes\mathcal{B}^{\mathrm{op}}$. Suppose that, for each B, $C\in\mathcal{B}$, the end $\int_{A\in\mathcal{A}} T(A,B,A,C)$ exists; then, as above, it is the value of a $\mathcal{V}$-functor $\mathcal{B}^{\mathrm{op}}\otimes\mathcal{B}\longrightarrow\mathcal{V}$. Every family $\alpha_{AB}\colon X\longrightarrow T(A,B,A,B)$ that is $\mathcal{V}$-natural in A factorizes uniquely as

$$X \xrightarrow[\beta_B]{} \textstyle\int_{A\in\mathcal{A}} T(A,B,A,B) \xrightarrow[\lambda_{ABB}]{} T(A,B,A,B); \tag{2.7}$$

and it follows again from (2.5) and the universal property of $[1,\lambda]$ that *the $\mathcal{V}$-naturality in B of α_{AB} is equivalent to the $\mathcal{V}$-naturality in B of β_B.*

Since, by 1.7, $\mathcal{V}$-naturality of α_{AB} in A and B separately coincides with its $\mathcal{V}$-naturality in $(A,B)\in\mathcal{A}\otimes\mathcal{B}$, we deduce the *Fubini Theorem: if every $\int_A T(A,B,A,C)$ exists, then*

$$\textstyle\int_{(A,B)\in\mathcal{A}\otimes\mathcal{B}} T(A,B,A,B) \cong \int_{B\in\mathcal{B}}\int_{A\in\mathcal{A}} T(A,B,A,B), \tag{2.8}$$

either side existing if the other does.

This in turn gives, since $\mathcal{A}\otimes\mathcal{B}\cong\mathcal{B}\otimes\mathcal{A}$, the following *interchange of ends theorem*, often itself called the Fubini Theorem: *if every $\int_A T(A,B,A,C)$ and every $\int_B T(A,B,D,B)$ exists, then*

$$\textstyle\int_{A\in\mathcal{A}}\int_{B\in\mathcal{B}} T(A,B,A,B) \cong \int_{B\in\mathcal{B}}\int_{A\in\mathcal{A}} T(A,B,A,B), \tag{2.9}$$

either side existing if the other does.

2.2 The functor-category $[\mathcal{A},\mathcal{B}]$ for small $\mathcal{A}$

For $\mathcal{V}$-functors T, $S\colon \mathcal{A}\longrightarrow\mathcal{B}$ we introduce the notation

$$[\mathcal{A},\mathcal{B}](T,S) = \textstyle\int_{A\in\mathcal{A}} \mathcal{B}(TA,SA) \tag{2.10}$$

for the end on the right, whenever it exists; and we write the counit as

$$E_A = E_{A,TS}\colon [\mathcal{A},\mathcal{B}](T,S) \longrightarrow \mathcal{B}(TA,SA). \tag{2.11}$$

We write $[\mathcal{A},\mathcal{B}]_0(T,S)$ for the set $V[\mathcal{A},\mathcal{B}](T,S)$ of *elements* $\alpha\colon I\longrightarrow[\mathcal{A},\mathcal{B}](T,S)$ of this end. These of course correspond to $\mathcal{V}$-natural families

$$\alpha_A = E_A\alpha\colon I\longrightarrow\mathcal{B}(TA,SA), \tag{2.12}$$

which by 1.8(a) are precisely the $\mathcal{V}$-natural transformations $\alpha\colon T\longrightarrow S\colon\mathcal{A}\longrightarrow\mathcal{B}$.

Note that, by (2.3) and the adjunction $[X,[Y,Z]]\cong[Y,[X,Z]]$, we have

$$[X,[\mathcal{A},\mathcal{V}](T,S)] \cong [\mathcal{A},\mathcal{V}](T,[X,S-]), \tag{2.13}$$

either side existing for all $X\in\mathcal{V}$ if the other does.

When $[\mathcal{A},\mathcal{B}](T,S)$ exists for all T, $S\colon\mathcal{A}\longrightarrow\mathcal{B}$, as it surely does by 2.1 if $\mathcal{A}$ is small, it is the hom-object of a $\mathcal{V}$-category $[\mathcal{A},\mathcal{B}]$ with all the $T\colon\mathcal{A}\longrightarrow\mathcal{B}$ as objects. The composition

law of $[\mathcal{A},\mathcal{B}]$ is, by 1.8(g), (1.43), and the universal property of $E_{A,TR}$, uniquely determined by the commutativity of

$$\begin{tikzcd} {[\mathcal{A},\mathcal{B}](S,R)\otimes[\mathcal{A},\mathcal{B}](T,S)} \arrow[r, "M"] \arrow[d, "E_A\otimes E_A"'] & {[\mathcal{A},\mathcal{B}](T,R)} \arrow[d, "E_A"] \\ \mathcal{B}(SA,RA)\otimes\mathcal{B}(TA,SA) \arrow[r, "M"'] & \mathcal{B}(TA,RA), \end{tikzcd} \tag{2.14}$$

and its identity elements are similarly determined by

$$E_A j_T = j_{TA}; \tag{2.15}$$

the axioms (1.3) and (1.4) follow at once from the corresponding axioms for $\mathcal{B}$ and the universal property of E_A.

We call $[\mathcal{A},\mathcal{B}]$ the *functor category*; its underlying category $[\mathcal{A},\mathcal{B}]_0$ is by (2.12) the ordinary category of all $\mathcal{V}$-functors $\mathcal{A}\longrightarrow\mathcal{B}$ and all $\mathcal{V}$-natural transformations between them – that is

$$[\mathcal{A},\mathcal{B}]_0 = \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{A},\mathcal{B}). \tag{2.16}$$

Even when $[\mathcal{A},\mathcal{B}]$ does not exist, it may sometimes be convenient to use $[\mathcal{A},\mathcal{B}]_0$ as an abbreviation for $\mathcal{V}\text{-}\mathbf{CAT}(\mathcal{A},\mathcal{B})$.

When $[\mathcal{A},\mathcal{B}]$ exists, the function $T\mapsto TA$ and the maps $E_{A,TS}$ of (2.11) constitute, by (2.14) and (2.15), a $\mathcal{V}$-functor $E_A\colon[\mathcal{A},\mathcal{B}]\longrightarrow\mathcal{B}$, called *evaluation at* A. By (2.12), the ordinary functor underlying E_A sends a map $\alpha\in[\mathcal{A},\mathcal{B}]_0(T,S)$ to its component α_A.

There is a $\mathcal{V}$-functor $E\colon[\mathcal{A},\mathcal{B}]\otimes\mathcal{A}\longrightarrow\mathcal{B}$, simply called *evaluation*, whose partial functors are given by

$$E(-,A)=E_A\colon[\mathcal{A},\mathcal{B}]\longrightarrow\mathcal{B},\qquad E(T,-)=T\colon\mathcal{A}\longrightarrow\mathcal{B}. \tag{2.17}$$

To show this we must verify the appropriate instance of (1.21); but, by the proof of 1.8(c) read backwards, this instance reduces via (1.29) to the instance of (1.40) expressing the $\mathcal{V}$-naturality in A of $E_{A,TS}$.

More generally, $[\mathcal{A},\mathcal{B}](T,S)$ may exist, not for *all* T, S, but for all those in some *subset* of the set of $\mathcal{V}$-functors $\mathcal{A}\longrightarrow\mathcal{B}$; a typical example would be the subset of those $\mathcal{A}\longrightarrow\mathcal{B}$ which are left Kan extensions, in the sense of 4.1 below, of their restrictions to some fixed small full subcategory of $\mathcal{A}$. Then we get as above a $\mathcal{V}$-category $[\mathcal{A},\mathcal{B}]'$ with these functors as objects, whose underlying category $[\mathcal{A},\mathcal{B}]'_0$ is the corresponding full subcategory of $\mathcal{V}\text{-}\mathbf{CAT}(\mathcal{A},\mathcal{B})$; and we have the evaluation functor $E\colon[\mathcal{A},\mathcal{B}]'\otimes\mathcal{A}\longrightarrow\mathcal{B}$.

Still more generally, we may wish to consider $[\mathcal{A},\mathcal{B}](T,S)$ just for certain particular pairs T, S for which it exists – as in the Yoneda isomorphism of 2.4 below. To discuss its functoriality in this generality, suppose we have functors $P\colon\mathcal{C}\otimes\mathcal{A}\longrightarrow\mathcal{B}$ and $Q\colon\mathcal{D}\otimes\mathcal{A}\longrightarrow\mathcal{B}$ such that $[\mathcal{A},\mathcal{B}]\big(P(C,-),Q(D,-)\big)$ exists for each C and D. Then there is, by 2.1, a unique $\mathcal{V}$-functor $H\colon\mathcal{C}^{\mathrm{op}}\otimes\mathcal{D}\longrightarrow\mathcal{V}$ having

$$H(C,D)=[\mathcal{A},\mathcal{B}]\big(P(C,-),Q(D,-)\big), \tag{2.18}$$

with respect to which $E_A\colon [\mathcal{A},\mathcal{B}]\big(P(C,-),Q(D,-)\big)\longrightarrow\mathcal{B}\big(P(C,A),Q(D,A)\big)$ is $\mathcal{V}$-natural in C and D as well as A. Moreover by (2.6) and (2.7), *in the situations (the second of which has $\mathcal{D}=\mathcal{C}$)*

$$\begin{tikzcd} R(C,D) \arrow[r, "\beta_{CD}"] \arrow[dr, "\alpha_{ACD}"'] & {[\mathcal{A},\mathcal{B}]\big(P(C,-),Q(D,-)\big)} \arrow[d, "E_A"] \\ & \mathcal{B}\big(P(C,A),Q(D,A)\big), \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\beta_C"] \arrow[dr, "\alpha_{AC}"'] & {[\mathcal{A},\mathcal{B}]\big(P(C,-),Q(C,-)\big)} \arrow[d, "E_A"] \\ & \mathcal{B}\big(P(C,A),Q(C,A)\big), \end{tikzcd} \tag{2.19}$$

α is $\mathcal{V}$-natural in $\mathcal{C}$ or $\mathcal{D}$ if and only if β is.

When $[\mathcal{A},\mathcal{B}]$, or even some $[\mathcal{A},\mathcal{B}]'$, exists, the H of (2.18) can be given more explicitly – see 2.3 below.

2.3 The isomorphism $[\mathcal{A}\otimes\mathcal{B},\mathcal{C}]\cong[\mathcal{A},[\mathcal{B},\mathcal{C}]]$

We return to the case where $[\mathcal{A},\mathcal{B}]$ exists. The $\mathcal{V}$-functor $E\colon[\mathcal{A},\mathcal{B}]\otimes\mathcal{A}\longrightarrow\mathcal{B}$ induces, for each $\mathcal{V}$-category $\mathcal{C}$, an ordinary functor

$$\mathcal{V}\text{-}\mathbf{CAT}(\mathcal{C},[\mathcal{A},\mathcal{B}])\longrightarrow\mathcal{V}\text{-}\mathbf{CAT}(\mathcal{C}\otimes\mathcal{A},\mathcal{B}), \tag{2.20}$$

sending $G\colon\mathcal{C}\longrightarrow[\mathcal{A},\mathcal{B}]$ to the composite

$$\mathcal{C}\otimes\mathcal{A}\xrightarrow[G\otimes 1]{}[\mathcal{A},\mathcal{B}]\otimes\mathcal{A}\xrightarrow[E]{}\mathcal{B}, \tag{2.21}$$

and sending $\beta\colon G\longrightarrow G'\colon\mathcal{C}\longrightarrow[\mathcal{A},\mathcal{B}]$ to the composite $E(\beta\otimes 1_{\mathcal{A}})$. *Clearly the map* (2.20) *is 2-natural in $\mathcal{C}$; and in fact it is an isomorphism of categories.*

To prove this we must first show that every $P\colon\mathcal{C}\otimes\mathcal{A}\longrightarrow\mathcal{B}$ has the form (2.21) for a unique G. Equating the partial functors of P and (2.21) gives

$$P(C,-)=GC,\qquad P(-,A)=E_AG. \tag{2.22}$$

The first of these determines G on objects, whereupon the second asserts the commutativity of

$$\begin{tikzcd} \mathcal{C}(C,D) \arrow[r, "G_{CD}"] \arrow[dr, "P(-,A)_{CD}"'] & {[\mathcal{A},\mathcal{B}]\big(P(C,-),P(D,-)\big)} \arrow[d, "E_A"] \\ & \mathcal{B}\big(P(C,A),P(D,A)\big). \end{tikzcd} \tag{2.23}$$

Since $P(-,A)_{CD}$ is $\mathcal{V}$-natural in A by 1.8(c), this determines a unique G_{CD} by the universal property of E_A. By this same universal property together with the composition-calculus of 1.7, the $\mathcal{V}$-functor axioms (1.5) and (1.6) for G follow at once from those for $P(-,A)$.

Before showing that (2.20) is bijective on maps as well as on objects, let us analyze the $\mathcal{V}$-functoriality (2.18) when $[\mathcal{A},\mathcal{B}]$ exists. If $P\colon \mathcal{C}\otimes\mathcal{A}\longrightarrow\mathcal{B}$ and $Q\colon \mathcal{D}\otimes\mathcal{A}\longrightarrow\mathcal{B}$ correspond by (2.20) to $G\colon \mathcal{C}\longrightarrow[\mathcal{A},\mathcal{B}]$ and $F\colon \mathcal{D}\longrightarrow[\mathcal{A},\mathcal{B}]$, we have the composite $\mathcal{V}$-functor

$$\mathcal{C}^{\mathrm{op}}\otimes\mathcal{D}\xrightarrow[G^{\mathrm{op}}\otimes F]{}[\mathcal{A},\mathcal{B}]^{\mathrm{op}}\otimes[\mathcal{A},\mathcal{B}]\xrightarrow[\mathrm{Hom}_{[\mathcal{A},\mathcal{B}]}]{}\mathcal{V}; \tag{2.24}$$

and with respect to this, $E_A\colon [\mathcal{A},\mathcal{B}](GC,FD)\longrightarrow\mathcal{B}(E_AGC,E_AFD)$ is $\mathcal{V}$-natural in $\mathcal{C}$ and $\mathcal{D}$, by 1.7 and 1.8(b). By (2.22), this E_A is

$$E_A\colon [\mathcal{A},\mathcal{B}]\big(P(C,-),Q(D,-)\big)\longrightarrow\mathcal{B}\big(P(C,A),Q(D,A)\big);$$

hence the H of (2.18), by its uniqueness, must be (2.24).

To complete the proof that (2.20) is an isomorphism of categories, it remains to verify that, if P and P' are the images under (2.20) of G and G', every $\alpha\colon P\longrightarrow P'$ is $E(\beta\otimes 1_{\mathcal{A}})$ for a unique $\beta\colon G\longrightarrow G'$. By (2.22) the equation $\alpha=E(\beta\otimes 1_{\mathcal{A}})$ has the component-form $\alpha_{CA}=E_A\beta_C$, which by (2.12) is

$$\alpha_{CA}=E_A\beta_C=(\beta_C)_A. \tag{2.25}$$

Thus $\beta_C\colon GC\longrightarrow G'C$ must be the $\mathcal{V}$-natural transformation $\alpha_{C-}\colon P(C,-)\longrightarrow P'(C,-)$; and the proof will be complete when we show that β_C, so defined, is $\mathcal{V}$-natural in C. If we look on β_C as a map $I\longrightarrow[\mathcal{A},\mathcal{B}](GC,G'C)$, this follows from 1.8(m) and (2.19), now that H is identified as (2.24).

More generally, if $[\mathcal{A},\mathcal{B}]$ does not exist but some full subcategory $[\mathcal{A},\mathcal{B}]'$ does, in the sense of 2.2, we have by the same arguments an isomorphism between the category $\mathcal{V}\text{-}\mathbf{CAT}(\mathcal{C},[\mathcal{A},\mathcal{B}]')$ and the full subcategory of $\mathcal{V}\text{-}\mathbf{CAT}(\mathcal{C}\otimes\mathcal{A},\mathcal{B})$ determined by those $P\colon \mathcal{C}\otimes\mathcal{A}\longrightarrow\mathcal{B}$ for which each $P(C,-)$ lies in $[\mathcal{A},\mathcal{B}]'$; for we have the analogue of (2.24) with $[\mathcal{A},\mathcal{B}]'$ replacing $[\mathcal{A},\mathcal{B}]$.

The 2-natural isomorphism (2.20) exhibits $[\mathcal{A},\mathcal{B}]$, when it exists, as a representing object for the 2-functor $\mathcal{V}\text{-}\mathbf{CAT}(-\otimes\mathcal{A},\mathcal{B})$. It is conversely true that, if this 2-functor is representable by some $\mathcal{D}$, then $[\mathcal{A},\mathcal{B}]$ exists and is isomorphic to $\mathcal{D}$ – at least if $\mathcal{V}_0$ has an initial object 0. For taking $\mathcal{C}$ to be $\mathcal{I}$ in (2.20), with $[\mathcal{A},\mathcal{B}]$ replaced by $\mathcal{D}$, we identify the objects of $\mathcal{D}$ with the $\mathcal{V}$-functors $\mathcal{A}\longrightarrow\mathcal{B}$; while taking $\mathcal{C}$ to be the $\mathcal{V}$-category with two objects 0 and 1, with $\mathcal{C}(0,0)=\mathcal{C}(1,1)=I$, with $\mathcal{C}(1,0)=0$, and with $\mathcal{C}(0,1)$ an arbitrary object X of $\mathcal{V}$, we easily see that $\mathcal{D}(T,S)=\int_A\mathcal{B}(TA,SA)$.

It follows from 1.10 that, so far as it exists, $[\mathcal{A},\mathcal{B}]$ is uniquely 2-functorial in $\mathcal{A}$ and $\mathcal{B}$ in such a way that (2.20) is 2-natural in every variable; which by 1.9 is equivalent to the 2-naturality in each variable of $E\colon [\mathcal{A},\mathcal{B}]\otimes\mathcal{A}\longrightarrow\mathcal{B}$. Using this latter criterion we easily see that, for $M\colon \mathcal{A}'\longrightarrow\mathcal{A}$ and $N\colon \mathcal{B}\longrightarrow\mathcal{B}'$, the $\mathcal{V}$-functor $[M,N]\colon [\mathcal{A},\mathcal{B}]\longrightarrow[\mathcal{A}',\mathcal{B}']$ sends T to NTM and is determined on hom-objects by

$$\begin{tikzcd}
{[\mathcal{A},\mathcal{B}](T,S)} \arrow[r, "{[M,N]_{TS}}"] \arrow[d, "E_{MA}"'] & {[\mathcal{A}',\mathcal{B}'](NTM,NSM)} \arrow[d, "E'_A"] \\
\mathcal{B}(TMA,SMA) \arrow[r, "N_{TMA,SMA}"'] & \mathcal{B}'(NTMA,NSMA);
\end{tikzcd} \tag{2.26}$$

while the $\mathcal{V}$-natural transformation $[\mu,\nu]\colon [M,N]\longrightarrow[M',N']$ is given by

$$[\mu,\nu]_T = \nu T\mu\colon NTM\longrightarrow N'TM'. \tag{2.27}$$

As for the interplay of functor categories with duality, we clearly have a 2-natural isomorphism

$$[\mathcal{A}^{\mathrm{op}},\mathcal{B}^{\mathrm{op}}]\cong[\mathcal{A},\mathcal{B}]^{\mathrm{op}}, \tag{2.28}$$

either side existing if the other does.

The 2-functor $-\otimes\mathcal{A}\colon \mathcal{V}\text{-}\mathbf{CAT}\longrightarrow\mathcal{V}\text{-}\mathbf{CAT}$ has a right adjoint $[\mathcal{A},-]$ when $[\mathcal{A},\mathcal{B}]$ exists for *all* $\mathcal{B}$; and by the above, only then, if $\mathcal{V}_0$ has an initial object. This right adjoint surely exists when $\mathcal{A}$ is small, in the extended sense of 2.1; and for many $\mathcal{V}$ it exists only then. For instance, a result of Freyd (see [73])[1] shows that, when $\mathcal{V}=\mathbf{Set}$, $\mathcal{A}$ is necessarily small if $[\mathcal{A},\mathcal{V}]$ exists. Yet when $\mathcal{V}_0$ is an ordered set, as in the cases $\mathcal{V}=2$ and $\mathcal{V}=\overline{\mathbb{R}}_+$, $[\mathcal{A},\mathcal{B}]$ exists for every $\mathcal{A}$ and $\mathcal{B}$.

In this sense the symmetric monoidal 2-category $\mathcal{V}$-**CAT** is "partially closed"; while the symmetric monoidal 2-category $\mathcal{V}$-**Cat** of *small* $\mathcal{V}$-categories is actually closed.

The partial closedness is sufficient to imply the isomorphisms (1.26) and (1.27) of 1.5, which here read:

$$[\mathcal{I},\mathcal{B}]\cong\mathcal{B},\qquad [\mathcal{C}\otimes\mathcal{A},\mathcal{B}]\cong[\mathcal{C},[\mathcal{A},\mathcal{B}]], \tag{2.29}$$

the latter in the sense that either side exists if the other does, provided $[\mathcal{A},\mathcal{B}]$ exists – which follows alternatively from the Fubini Theorem (2.8).

2.4 The (strong) Yoneda lemma for $\mathcal{V}$-**CAT**; the Yoneda embedding $\mathcal{A}\longrightarrow[\mathcal{A}^{\mathrm{op}},\mathcal{V}]$

We can now give a stronger version of the Yoneda Lemma of 1.9, no longer expressed by a mere bijection of sets, but by a *Yoneda isomorphism* in $\mathcal{V}_0$.

Given a $\mathcal{V}$-functor $F\colon\mathcal{A}\longrightarrow\mathcal{V}$ and an object K of $\mathcal{A}$ as in 1.9, we have the map $F_{KA}\colon\mathcal{A}(K,A)\longrightarrow[FK,FA]$, which is $\mathcal{V}$-natural in A by 1.8(b). The transform

$$\phi_A\colon FK\longrightarrow[\mathcal{A}(K,A),FA] \tag{2.30}$$

of F_{KA} under the adjunction $\mathcal{V}_0(X,[Y,Z])\cong\mathcal{V}_0(Y,[X,Z])$ is $\mathcal{V}$-natural in A by 1.8(m). The stronger Yoneda Lemma is the assertion that (2.30) *expresses FK as the end $\int_A[\mathcal{A}(K,A),FA]$, so that we have an isomorphism*

$$\phi\colon FK\cong[\mathcal{A},\mathcal{V}](\mathcal{A}(K,-),F]. \tag{2.31}$$

For the proof, consider any $\mathcal{V}$-natural $\alpha_A\colon X\longrightarrow[\mathcal{A}(K,A),FA]$; its transform $\overline{\alpha}_A\colon\mathcal{A}(K,A)\longrightarrow[X,FA]$ under the adjunction, being $\mathcal{V}$-natural in A by 1.8(m), is the composite

$$\mathcal{A}(K,A)\xrightarrow[F_{KA}]{}[FK,FA]\xrightarrow[{[\eta,1]}]{}[X,FA]$$

[1]For a more useful citation, see Peter Freyd and Ross Street, On the size of categories, Theory and Applications of Categories **1** (1995) 174-178.
`http://www.tac.mta.ca/tac/volumes/1995/n9/1-09abs.html`

for a unique $\eta\colon X \longrightarrow FK$, by the Yoneda Lemma of 1.9 in the form (1.48). This is equivalent to the assertion that $\alpha_A = \phi_A\eta$ for a unique η; which completes the proof.

The image under $V = \mathcal{V}_0(I,-)\colon \mathcal{V}_0 \longrightarrow \mathbf{Set}$ of (2.31) is a bijection

$$\mathcal{V}_0(I, FK) \longrightarrow [\mathcal{A},\mathcal{V}]_0(\mathcal{A}(K,-), F).$$

This sends $\eta\colon I \longrightarrow FK$ to $\phi\eta\colon \mathcal{A}(K,-) \longrightarrow F$, whose component $\mathcal{A}(K,A) \longrightarrow FA$ is $E_A\phi\eta$ by (2.12), or $\phi_A\eta$. By the definition (2.30) of ϕ_A, this is at once seen to be the composite (1.47). Hence the weak Yoneda Lemma of 1.9, which we have used to prove the present strong Yoneda Lemma, is subsumed under the latter as its underlying bijection.

The special case $F = \mathcal{A}(L,-)\colon \mathcal{A} \longrightarrow \mathcal{V}$ of the present Yoneda Lemma strengthens the last assertion of 1.9 to

$$\mathcal{A}(L,K) \cong [\mathcal{A},\mathcal{V}]\big(\mathcal{A}(K,-), \mathcal{A}(L,-)\big). \tag{2.32}$$

Since (2.31) is true without any smallness restriction on $\mathcal{A}$, its $\mathcal{V}$-naturality must in general be discussed (cf. 2.2) in terms of the $\mathcal{V}$-functor $\mathrm{Hom}_{\mathcal{A}}\colon \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathcal{V}$ and some $\mathcal{V}$-functor $P\colon \mathcal{C} \otimes \mathcal{A} \longrightarrow \mathcal{V}$. We assert that

$$\phi = \phi_{CK}\colon P(C,K) \cong [\mathcal{A},\mathcal{V}]\big(\mathcal{A}(K,-), P(C,-)\big) \tag{2.33}$$

is $\mathcal{V}$-natural in C and K, when the right side is made $\mathcal{V}$-functorial as in (2.18). This follows from (2.19); for $E_A\phi = \phi_A\colon P(C,K) \longrightarrow [\mathcal{A}(K,A), P(C,A)]$ is $\mathcal{V}$-natural in C and K by 1.8(m), since $\mathcal{A}(K,A) \longrightarrow [P(C,K), P(C,A)]$ is so by 1.8(b) and 1.8(c).

When $[\mathcal{A},\mathcal{V}]$ exists, $\mathrm{Hom}_{\mathcal{A}}\colon \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathcal{V}$ corresponds under (2.20) to a $\mathcal{V}$-functor

$$Y\colon \mathcal{A}^{\mathrm{op}} \longrightarrow [\mathcal{A},\mathcal{V}] \tag{2.34}$$

sending K to

$$YK = \mathcal{A}(K,-); \tag{2.35}$$

being fully faithful by (2.32), Y is called the *Yoneda embedding*. In these circumstances the $\mathcal{V}$-naturality (2.33) can be more globally expressed by taking $[\mathcal{A},\mathcal{V}]$ for $\mathcal{C}$ and $E\colon [\mathcal{A},\mathcal{V}] \otimes \mathcal{A} \longrightarrow \mathcal{V}$ for P; the right side of (2.33) is then by (2.24) the value of the $\mathcal{V}$-functor

$$\mathcal{A} \otimes [\mathcal{A},\mathcal{V}] \xrightarrow[Y^{\mathrm{op}}\otimes 1]{} [\mathcal{A},\mathcal{V}]^{\mathrm{op}} \otimes [\mathcal{A},\mathcal{V}] \xrightarrow[\mathrm{Hom}_{[\mathcal{A},\mathcal{V}]}]{} \mathcal{V}; \tag{2.36}$$

so that, with respect to this functoriality,

the Yoneda isomorphism (2.31) *is $\mathcal{V}$-natural in K and F.* (2.37)

The extra-variable result of 1.9 also has a lifting to the present context. Let $F\colon \mathcal{B}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathcal{V}$ and $K\colon \mathcal{B} \longrightarrow \mathcal{A}$. By (2.33) we have an isomorphism $F(C,KB) \cong [\mathcal{A},\mathcal{V}]\big(\mathcal{A}(KB,-), F(C,-)\big)$, $\mathcal{V}$-natural in C and B. The Fubini Theorem (2.8) now gives

$$\textstyle\int_B F(B,KB) \cong [\mathcal{B}^{\mathrm{op}} \otimes \mathcal{A},\mathcal{V}]\big(\mathcal{A}(K?,-), F(?,-)\big), \tag{2.38}$$

either side existing if the other does.

2.5 The free $\mathcal{V}$-category on a **Set**-category

We henceforth suppose, in addition to all our former assumptions on $\mathcal{V}$, that *the underlying category $\mathcal{V}_0$ is cocomplete.*

Then the ordinary functor $V = \mathcal{V}_0(I, -)\colon \mathcal{V}_0 \longrightarrow \mathbf{Set}$ has a left adjoint $(-)\cdot I$, sending the set E to the coproduct $E \cdot I$ of E copies of I in $\mathcal{V}_0$. This functor sends the cartesian product in **Set** to the tensor product in $\mathcal{V}_0$, in that we clearly have

$$(E \times F)\cdot I \cong (E\cdot I)\otimes(F\cdot I), \qquad 1\cdot I \cong I. \tag{2.39}$$

The 2-functor $(-)_0\colon \mathcal{V}\text{-}\mathbf{CAT} \longrightarrow \mathbf{CAT}$ now has a left adjoint $(-)_{\mathcal{V}}$. The free $\mathcal{V}$-category $\mathcal{L}_{\mathcal{V}}$ on the ordinary locally-small category $\mathcal{L}$ has the same objects as $\mathcal{L}$, and has hom-objects $\mathcal{L}_{\mathcal{V}}(K,L) = \mathcal{L}(K,L)\cdot\mathcal{I}$; its composition law and its identities are induced from those of $\mathcal{L}$ using the isomorphisms (2.39).

For the proof, we have first to verify that $\mathcal{L}_{\mathcal{V}}$, as defined, satisfies the $\mathcal{V}$-category axioms (1.3) and (1.4); this is trivial from the corresponding axioms for $\mathcal{L}$.

There is an evident functor $\psi\colon \mathcal{L} \longrightarrow (\mathcal{L}_{\mathcal{V}})_0$ which is the identity on objects, and which is defined on morphisms by the obvious map $\mathcal{L}(K,L) \longrightarrow \mathcal{V}_0(I, \mathcal{L}(K,L)\cdot I)$. This ψ is the unit of the 2-adjunction in question.

In fact it is easy to see that, for a $\mathcal{V}$-category $\mathcal{B}$ and a functor $T\colon \mathcal{L} \longrightarrow \mathcal{B}_0$, there is exactly one $\mathcal{V}$-functor $\overline{T}\colon \mathcal{L}_{\mathcal{V}} \longrightarrow \mathcal{B}$ with $\overline{T}_0\psi = T$: on objects we have $\overline{T}K = TK$, while $\overline{T}_{KL}\colon \mathcal{L}(K,L)\cdot I \longrightarrow \mathcal{B}(TK,TL)$ is the transform of $T_{KL}\colon \mathcal{L}(K,L) \longrightarrow \mathcal{B}_0(TK,TL)$ under the adjunction $(-)\cdot I \dashv V$. Verification of (1.5) and (1.6) for $\overline{T}$ is easy.

Finally, if $\alpha\colon T \longrightarrow S\colon \mathcal{L} \longrightarrow \mathcal{B}_0$, there is a unique $\mathcal{V}$-natural $\overline{\alpha}\colon \overline{T} \longrightarrow \overline{S}\colon \mathcal{L}_{\mathcal{V}} \longrightarrow \mathcal{B}$ with $\overline{\alpha}_0\psi = \alpha$; the component $\overline{\alpha}_K$ must be $\alpha_K \in \mathcal{B}_0(TK, SK)$, and again verification of (1.7) for $\overline{\alpha}$ is easy. This completes the proof.

If $\mathcal{L}$ is a small category, $\mathcal{L}_{\mathcal{V}}$ is clearly a small $\mathcal{V}$-category, and we can form the functor category $[\mathcal{L}_{\mathcal{V}}, \mathcal{B}]$. Since the underlying ordinary category $[\mathcal{L}_{\mathcal{V}}, \mathcal{B}]_0$ of this is $\mathcal{V}\text{-}\mathbf{CAT}(\mathcal{L}_{\mathcal{V}}, \mathcal{B})$ by (2.16), the 2-adjunction above gives

$$[\mathcal{L}_{\mathcal{V}}, \mathcal{B}]_0 \cong [\mathcal{L}, \mathcal{B}_0]. \tag{2.40}$$

2.6 Universe-enlargement $\mathcal{V} \subset \mathcal{V}'$ in concrete cases; $[\mathcal{A}, \mathcal{B}]$ as a $\mathcal{V}'$-category for large $\mathcal{A}$

The possible non-existence of a $\mathcal{V}$-functor category $[\mathcal{A}, \mathcal{B}]$ when $\mathcal{A}$ is large (= non-small) is somewhat tedious, forcing us into such circumlocutions as using (2.18) instead of (2.24), or (2.33) instead of (2.37) – which then propagate themselves throughout the applications below.

Yet when $\mathcal{V} = \mathbf{Set}$, we always do have $[\mathcal{A}, \mathcal{B}]$ as a category; albeit not in general a **Set**-category (= locally-small category) when $\mathcal{A}$ and $\mathcal{B}$ are **Set**-categories, unless $\mathcal{A}$ is small. It can however be seen as a $\mathbf{Set}'$-category, where $\mathbf{Set}'$ is the category of sets in some larger universe containing $\operatorname{ob}\mathcal{A}$ as an element – we are taking the view that $\operatorname{ob}\mathcal{A}$ is always an honest set, and supposing that every set belongs to some universe.

We may ask whether such a view can be imitated for a general $\mathcal{V}$. Consider first the various examples of a symmetric monoidal closed $\mathcal{V}$ given in 1.1. These are of two types: either $\mathcal{V}_0$ is a particular ordered set, such as 2 or $\overline{\mathbb{R}}_+$, in which case $[\mathcal{A}, \mathcal{B}]$ exists for all $\mathcal{A}$; or $\mathcal{V}_0$ consists of the *small* models of some theory – small sets, small categories, small abelian groups, small differential graded R-modules, and so on. In the latter case we can clearly consider the category $\mathcal{V}_0'$ of models in a larger $\mathbf{Set}'$, which gives a symmetric monoidal closed category $\mathcal{V}'$, in which $\mathcal{V}$ with its symmetric monoidal closed structure is faithfully embedded. Moreover $\mathcal{V}_0'$ admits all limits and colimits which are small by reference to $\mathbf{Set}'$; and the inclusion $\mathcal{V}_0 \longrightarrow \mathcal{V}_0'$ preserves all limits and colimits that exist in $\mathcal{V}_0$. (We are supposing that our *original* universe contains infinite sets; otherwise we do not have the colimit-preservation above.)

Then, for $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, we always have a $\mathcal{V}'$-category $[\mathcal{A}, \mathcal{B}]$, where $\mathbf{Set}'$ is taken large enough to include $\operatorname{ob}\mathcal{A}$. Since $\mathcal{V}_0 \longrightarrow \mathcal{V}_0'$ preserves limits, to say that $[\mathcal{A}, \mathcal{B}](T, S)$ exists in the sense of 2.2 is precisely to say that the object $[\mathcal{A}, \mathcal{B}](T, S)$ of $\mathcal{V}'$ actually lies in $\mathcal{V}$; and to say that $[\mathcal{A}, \mathcal{B}]$ exists in the sense of 2.2 – that is, as a $\mathcal{V}$-category – is to say that $[\mathcal{A}, \mathcal{B}](T, S) \in \mathcal{V}$ for all T, S.

That a similar thing is possible for a perfectly general $\mathcal{V}$ follows from Day [17]. Using his construction, one can show that there is always a suitable $\mathcal{V}'$ whose underlying ordinary category $\mathcal{V}_0'$ is a full reflective subcategory of $[\mathcal{V}_0^{\mathrm{op}}, \mathbf{Set}']$, into which $\mathcal{V}_0$ is embedded by Yoneda. Since the monoidal structure of $\mathcal{V}'$ is defined by *coends* (relative to $\mathbf{Set}'$), we defer a description of it until 3.11 and 3.12 below, after these have been introduced.

Chapter 3

Indexed limits and colimits

3.1 Indexing types; limits and colimits; Yoneda isomorphisms

For a discussion of completeness, or of Kan extensions, or of density, in the context of $\mathcal{V}$-categories, ordinary limits – defined in terms of the representability of cones – do not suffice; we need the wider notion of *indexed limit*. These latter fully deserve the name of "limit", since they enjoy all the formal properties expected of a limit-notion, and include the classical cone-type limits as a special case. When $\mathcal{V} = \mathbf{Set}$, in contrast to the situation for a general $\mathcal{V}$, an indexed limit may be expressed as an ordinary limit; yet even here, the indexed limit is a valuable concept in its own right – its *meaning* is scarcely clarified by its *construction* as an ordinary limit. Since the cone-type limits have no special position of dominance in the general case, we go so far as to call indexed limits simply "limits", where confusion seems unlikely.

We may think of a $\mathcal{V}$-functor $F\colon \mathcal{K} \longrightarrow \mathcal{V}$ as an *indexing type* – a *small* one when $\mathcal{K}$ is small, which will be the usual case, but not the only one. Then we may think of a $\mathcal{V}$-functor $G\colon \mathcal{K} \longrightarrow \mathcal{B}$ as a *diagram in* $\mathcal{B}$ *of type* F. For each $B \in \mathcal{B}$, we get from G a $\mathcal{V}$-functor $\mathcal{B}(B, G-)\colon \mathcal{K} \longrightarrow \mathcal{V}$; and we can consider the existence in $\mathcal{V}$ of the object $[\mathcal{K}, \mathcal{V}]\big(F, \mathcal{B}(B, G-)\big)$. If this exists for all B – as it surely does when $\mathcal{K}$ is small – it is as in (2.18) the value of a $\mathcal{V}$-functor $\mathcal{B}^{\mathrm{op}} \longrightarrow \mathcal{V}$. If this $\mathcal{V}$-functor not only exists but admits a representation

$$\mathcal{B}(B, \{F, G\}) \cong [\mathcal{K}, \mathcal{V}]\big(F, \mathcal{B}(B, G-)\big), \tag{3.1}$$

with counit say

$$\mu\colon F \longrightarrow \mathcal{B}(\{F, G\}, G-), \tag{3.2}$$

we call the representation $(\{F, G\}, \mu)$ the *limit of* G *indexed by* F. An evidently equivalent formulation not mentioning $[\mathcal{K}, \mathcal{V}]$ explicitly is the following: each component μ_K of μ induces by Yoneda a family

$$\mathcal{B}(B, \{F, G\}) \longrightarrow [FK, \mathcal{B}(B, GK)] \tag{3.3}$$

$\mathcal{V}$-natural in B and K; and $(\{F, G\}, \mu)$ is the limit precisely when (3.3) is an end over K for each B. In practice, by the usual abuse of notation, we call the representing object $\{F, G\}$ the limit, consigning to the background the counit μ – or equivalently the choice of the $\mathcal{V}$-natural isomorphism (3.1).

If we take the view of 2.6, that $[\mathcal{K},\mathcal{V}]$ always exists, if not as a $\mathcal{V}$-category, then as a $\mathcal{V}'$-category for some larger $\mathcal{V}'$, we need not be concerned with the existence of $[\mathcal{K},\mathcal{V}]\big(F,\mathcal{B}(B,G-)\big)$, but only with its representability in the form (3.1) for some $\{F,G\}\in\mathcal{B}$. If it is so representable, the right side of (3.1) in fact lies in $\mathcal{V}$, since the left side does. We shall in future take this view whenever it is convenient – usually without bothering to change "$\mathcal{V}$-functor" to "$\mathcal{V}'$-functor", and so on.

Applying V to the isomorphism (3.1) gives a bijection of sets

$$\mathcal{B}_0(B,\{F,G\})\cong[\mathcal{K},\mathcal{V}_0]\big(F,\mathcal{B}(B,G-)\big); \tag{3.4}$$

but the requirement that $(\{F,G\},\mu)$ induce a bijection (3.4) is, *except in the special case when V is conservative*, strictly weaker than the requirement that it induce an isomorphism (3.1), and *does not suffice to make* $(\{F,G\},\mu)$ *the limit*; see 3.7 below. However it does suffice to *detect* $(\{F,G\},\mu)$ *as the limit if the limit is known to exist*; for the right side of (3.4) admits at most one representation to within isomorphism. An element $\alpha\colon F\longrightarrow\mathcal{B}(B,G-)$ of the right side of (3.4) may be called an (F,B)-*cylinder over* G; then the bijection (3.4) is the assertion that every such cylinder factorizes through the counit-cylinder μ as $\alpha=\mathcal{B}(f,1)\mu$ for a unique $f\colon B\longrightarrow\{F,G\}$.

An *indexed colimit* in $\mathcal{B}$ is nothing but an indexed limit in $\mathcal{B}^{\mathrm{op}}$; but it is usual to replace $\mathcal{K}$ by $\mathcal{K}^{\mathrm{op}}$ in the definition, so that we still have $G\colon\mathcal{K}\longrightarrow\mathcal{B}$, but now have $F\colon\mathcal{K}^{\mathrm{op}}\longrightarrow\mathcal{V}$. Then the *colimit of G indexed by F* is the representing object $F\star G$ in

$$\mathcal{B}(F\star G,B)\cong[\mathcal{K}^{\mathrm{op}},\mathcal{V}]\big(F,\mathcal{B}(G-,B)\big), \tag{3.5}$$

with *unit*

$$\nu\colon F\longrightarrow\mathcal{B}(G-,F\star G). \tag{3.6}$$

When $\mathcal{B}=\mathcal{V}$, we have from (2.13) and (3.1)

$$\{F,G\}\cong[\mathcal{K},\mathcal{V}](F,G)\qquad \textit{for } F,\,G\colon\mathcal{K}\longrightarrow\mathcal{V}, \tag{3.7}$$

either side existing if the other does. Hence (3.1) and (3.5) can be written

$$\mathcal{B}(B,\{F,G\})\cong\{F,\mathcal{B}(B,G-)\},\qquad \mathcal{B}(F\star G,B)\cong\{F,\mathcal{B}(G-,B)\}. \tag{3.8}$$

Thus, just as classical limits and colimits in an ordinary category $\mathcal{B}$ are defined, via representability, in terms of the primitive limits – sets of cones – in **Set**, so indexed limits and colimits in a $\mathcal{V}$-category $\mathcal{B}$ are defined, via representability, in terms of indexed limits in $\mathcal{V}$; and thus ultimately, by 2.1 and 2.2, in terms of ordinary limits in $\mathcal{V}_0$.

Again, in the special case $\mathcal{B}=\mathcal{V}$, the $\mathcal{V}$-natural isomorphism $[FK,[GL,B]]\cong[GL,[FK,B]]$ gives by (3.5)

$$F\star G\cong G\star F\qquad \textit{for } F\colon\mathcal{K}^{\mathrm{op}}\longrightarrow\mathcal{V} \textit{ and } G\colon\mathcal{K}\longrightarrow\mathcal{V}. \tag{3.9}$$

The $\mathcal{V}$-natural Yoneda isomorphism (2.33) immediately gives the values of $\{F,G\}$ and $F\star G$ when F is representable, for any $G\colon\mathcal{K}\longrightarrow\mathcal{B}$; namely

$$\{\mathcal{K}(K,-),G\}\cong GK,\qquad \mathcal{K}(-,K)\star G\cong GK. \tag{3.10}$$

More precisely, the cylinder $G_{K-}\colon \mathcal{K}(K,-) \longrightarrow \mathcal{B}(GK, G-)$ as counit exhibits GK as $\{\mathcal{K}(K,-), G\}$. We refer to (3.10) as *Yoneda isomorphisms*; by (3.7), the first of them includes as a special case the original Yoneda isomorphism (2.31).

When $\mathcal{K}$ is small, the maps $(F,G) \mapsto \{F,G\}$ and $(F,G) \mapsto F \star G$ are by 1.10 the object-functions of $\mathcal{V}$-functors

$$\{\ ,\ \}\colon ([\mathcal{K},\mathcal{V}]^{\mathrm{op}} \otimes [\mathcal{K},\mathcal{B}])' \longrightarrow \mathcal{B}, \qquad \star\colon ([\mathcal{K}^{\mathrm{op}},\mathcal{V}] \otimes [\mathcal{K},\mathcal{B}])' \longrightarrow \mathcal{B}, \tag{3.11}$$

where the domain consists of those pairs (F,G) for which $\{F,G\}$ or $F \star G$ exists; and in such a way that (3.1) and (3.5) are $\mathcal{V}$-natural in (F,G). Then (2.37) gives:

The Yoneda isomorphisms (3.10) *are $\mathcal{V}$-natural in K and G.* (3.12)

3.2 Preservation of limits and colimits

We call the limit $\{F,G\}$ a *small limit* if the indexing-type $F\colon \mathcal{K} \longrightarrow \mathcal{V}$ is small; that is, if $\mathcal{K}$ is small. We call the $\mathcal{V}$-category $\mathcal{B}$ *complete* if it admits all small limits; and dually, *cocomplete* if it admits all small colimits. By (3.7) and 2.2, $\mathcal{V}$ *is complete*; we shall see in 3.10 below that $\mathcal{V}$ *is also cocomplete*.

A $\mathcal{V}$-functor $T\colon \mathcal{B} \longrightarrow \mathcal{C}$ is said to *preserve* the limit $\{F,G\}$ if the composite cylinder

$$F \xrightarrow[\mu]{} \mathcal{B}(\{F,G\}, G-) \xrightarrow[T]{} \mathcal{C}(T\{F,G\}, TG-), \tag{3.13}$$

where μ is the counit of $\{F,G\}$, exhibits $T\{F,G\}$ as $\{F,TG\}$; in other words, if $\{F,TG\}$ exists and if the evident *canonical map*

$$T\{F,G\} \longrightarrow \{F,TG\} \tag{3.14}$$

is an isomorphism.

We call $T\colon \mathcal{B} \longrightarrow \mathcal{C}$ *continuous* (the dual is *cocontinuous*) if it preserves all *small* limits that exist; of course this condition is of interest particularly when $\mathcal{B}$ is complete.

The representables $\mathcal{B}(B,-)\colon \mathcal{B} \longrightarrow \mathcal{V}$ are more than continuous: by (3.8) they preserve *every* limit that exists; while in their totality, by definition, they "detect" the existence of $\{F,G\}$.

For a very general result on the preservation of limits, consider a functor $Q\colon \mathcal{C}^{\mathrm{op}} \otimes \mathcal{B} \longrightarrow \mathcal{V}$, and let $\mathcal{B}' \subset \mathcal{B}$ be the full subcategory of those B for which $Q(-,B)$ is representable, say as $Q(C,B) \cong \mathcal{C}(C,TB)$; we shall refer to the objects of $\mathcal{B}'$ as "those B for which TB exists". Let $\{F,G\}$ be a limit in $\mathcal{B}$ which is preserved by $Q(C,-)\colon \mathcal{B} \longrightarrow \mathcal{V}$ for each C. Finally, let TG exist; that is, suppose that G takes its values in $\mathcal{B}'$. Then we clearly have

$$T\{F,G\} \cong \{F,TG\}, \tag{3.15}$$

either side existing if the other does; for, using (3.7), the two sides of (3.15) represent the isomorphic $\mathcal{V}$-functors $Q(C,\{F,G\})$ and $[\mathcal{K},\mathcal{V}]\big(F,\mathcal{C}(C,TG)\big) \cong [\mathcal{K},\mathcal{V}]\big(F,Q(C,G-)\big) \cong \{F,Q(C,G-)\}$.

If Q here is defined by $Q(C,B) = \mathcal{B}(SC,B)$ where $S\colon \mathcal{C} \longrightarrow \mathcal{B}$, then $T\colon \mathcal{B}' \longrightarrow \mathcal{C}$ is the partial right adjoint of S. Since $\mathcal{B}(SC,-)$ preserves all limits, we have (3.15) whenever $\{F,G\}$, $T\{F,G\}$, and TG exist. This conclusion may be expressed as: *partial right adjoints in so far as they are defined, preserve limits.*

In the special case where S in fact has a right adjoint $T\colon \mathcal{B} \longrightarrow \mathcal{C}$, the conclusion becomes: *right adjoints preserve all limits that exist.*

3.3 Limits in functor categories; double limits and iterated limits

We now consider limits in a functor category $[\mathcal{A},\mathcal{B}]$. Let $G\colon \mathcal{K} \longrightarrow [\mathcal{A},\mathcal{B}]$ correspond under the adjunction (2.20) to $P\colon \mathcal{K}\otimes\mathcal{A} \longrightarrow \mathcal{B}$. In order to avoid formulae containing various letters whose inter-relation must be explained in a subordinate clause, let us write (2.22) with cyphers for variables, in the form $P(-,?) = (G-)?$; so that, in particular, $(G-)A$ denotes $P(-,A)\colon \mathcal{K} \longrightarrow \mathcal{B}$. Let $F\colon \mathcal{K} \longrightarrow \mathcal{V}$.

Then, *if $\{F,(G-)A\}$ exists for each A, the limit $\{F,G\}$ in $[\mathcal{A},\mathcal{B}]$ exists, and we have*

$$\{F,G\}A = \{F,(G-)A\}. \tag{3.16}$$

For, if we *define* $\{F,G\}$ to be the $\mathcal{V}$-functor whose value at A is given by (3.16), we have

$$\begin{aligned}
[\mathcal{A},\mathcal{B}](H,\{F,G\}) &= \textstyle\int_A \mathcal{B}(HA,\{F,(G-)A\}) && \text{by (3.16)}\\
&\cong \textstyle\int_A [\mathcal{K},\mathcal{V}]\big(F,\mathcal{B}(HA,(G-)A)\big) && \text{by (3.1)}\\
&\cong [\mathcal{K},\mathcal{V}]\big(F,\textstyle\int_A \mathcal{B}(HA,(G-)A)\big) && \text{by (2.3)}\\
&= [\mathcal{K},\mathcal{V}]\big(F,[\mathcal{A},\mathcal{B}](H,G-)\big) && \text{by (2.10).}
\end{aligned}$$

In these circumstances we say that the limit $\{F,G\}$ in $[\mathcal{A},\mathcal{B}]$ *exists pointwise*; it comes to the same thing to say that $\{F,G\}$ *exists and is preserved by each* $E_A\colon [\mathcal{A},\mathcal{B}] \longrightarrow \mathcal{B}$. *Clearly if $\mathcal{B}$ is complete [respectively cocomplete] so is $[\mathcal{A},\mathcal{B}]$, all small limits [respectively colimits] existing pointwise in $[\mathcal{A},\mathcal{B}]$.*

When $\mathcal{B}$ is not complete, fortuitous non-pointwise limits may exist in $[\mathcal{A},\mathcal{B}]$, even when $\mathcal{V} = \mathbf{Set}$. Take $\mathcal{A} = \mathbf{2}$ and let $\mathcal{B}$ be the category generated by the graph

$$\begin{tikzcd} \cdot \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \cdot \arrow[r, "h"] & \cdot \end{tikzcd}$$

with the relation $hf = hg$. Then the map $(f,h)\colon g \longrightarrow h$ given by

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "g"'] & \cdot \arrow[d, "h"] \\ \cdot \arrow[r, "h"'] & \cdot \end{tikzcd}$$

in $[\mathbf{2},\mathcal{B}]$ is monomorphic, although h is not monomorphic in $\mathcal{B}$. Since x is monomorphic in any category precisely when

$$\begin{tikzcd} \cdot \arrow[r, "1"] \arrow[d, "1"'] & \cdot \arrow[d, "x"] \\ \cdot \arrow[r, "x"'] & \cdot \end{tikzcd}$$

is a pullback, the pullback of (f, h) by itself in $[\mathbf{2}, \mathcal{B}]$ is a non-pointwise limit. The existence of a non-pointwise limit in $[\mathcal{A}, \mathcal{B}]$ says something about $[\mathcal{A}, \mathcal{B}]$ as a category, ignoring its relation to $\mathcal{A}$ and $\mathcal{B}$; while the existence of a pointwise limit says something about $[\mathcal{A}, \mathcal{B}]$ seen precisely as the functor category, and is really a completeness assertion about $\mathcal{B}$.

Applying the above, in the colimit case, to $[\mathcal{K}, \mathcal{V}]$ where $\mathcal{K}$ is small, we get from (3.9) and (3.10) a pointwise colimit

$$F \star Y \cong F, \tag{3.17}$$

for any $F\colon \mathcal{K} \longrightarrow \mathcal{V}$, Y being the Yoneda embedding $\mathcal{K}^{\mathrm{op}} \longrightarrow [\mathcal{K}, \mathcal{V}]$ of (2.34). This gives *a canonical representation of any $F\colon \mathcal{K} \longrightarrow \mathcal{V}$ as an (indexed) colimit of representables.* In contrast, *Y preserves whatever limits exist in $\mathcal{K}^{\mathrm{op}}$, sending them to pointwise limits in* $[\mathcal{K}, \mathcal{V}]$; for $E_K Y \cong \mathcal{K}(-, K)\colon \mathcal{K}^{\mathrm{op}} \longrightarrow \mathcal{V}$.

We can now give the *general Fubini Theorem* relating repeated limits to double limits. *Suppose that $F\colon \mathcal{K} \longrightarrow \mathcal{V}$ and $G\colon \mathcal{K} \longrightarrow [\mathcal{A}, \mathcal{B}]$, the latter corresponding to $P\colon \mathcal{K} \otimes \mathcal{A} \longrightarrow \mathcal{B}$, are such that the limit $\{F, G\}$ exists pointwise; and let $H\colon \mathcal{A} \longrightarrow \mathcal{V}$. Writing $H \otimes F\colon \mathcal{K} \otimes \mathcal{A} \longrightarrow \mathcal{V}$ for the functor sending (K, A) to $HA \otimes FK$, we have a canonical isomorphism*

$$\{H, \{F, G\}\} \cong \{H \otimes F, P\}, \tag{3.18}$$

either side existing if the other does.[1]

For by (3.1) the left side of (3.18) represents

$$\begin{aligned}
[\mathcal{A}, \mathcal{V}]\big(H?, \mathcal{B}(B, \{F, G\}?)\big) &= [\mathcal{A}, \mathcal{V}]\big(H?, \mathcal{B}(B, \{F-, (G-)?\})\big) && \text{by (3.16)}\\
&\cong [\mathcal{A}, \mathcal{V}]\big(H?, [\mathcal{K}, \mathcal{V}](F-, \mathcal{B}(B, (G-)?))\big) && \text{by (3.1)}\\
&= \textstyle\int_A [HA, \int_K [FK, \mathcal{B}(B, (GK)A)]] && \text{by (2.10)}\\
&\cong \textstyle\int_A \int_K [HA, [FK, \mathcal{B}(B, (GK)A)]] && \text{by (2.3)}\\
&\cong \textstyle\int_A \int_K [HA \otimes FK, \mathcal{B}(B, P(K, A))] && \text{by (1.27);}
\end{aligned}$$

and by the Fubini Theorem (2.8) we can replace $\int_A \int_K$ here by $\int_{(A,K)}$, giving by (2.10) and (3.1) the object represented by the right side of (3.18).

The use of the cyphers "?" and "$-$" here to keep track of unexpressed variables is presumably self-explanatory. If we extend it by allowing $\{F-, G-\}$ as an alternative notation for $\{F, G\}$, we can write (3.18), suppressing the letter G, as

$$\{H?, \{F-, P(-, ?)\}\} \cong \{H? \otimes F-, P(-, ?)\}. \tag{3.19}$$

Then there is no need for explicit mention of the functor category $[\mathcal{A}, \mathcal{B}]$, which may exist only as a $\mathcal{V}'$-category in the sense of 2.6; nor for explicit mention of the pointwise existence of $\{F, G\}$, which is just the existence of $\{F-, P(-, A)\}$ for each value of A.

In this notation the general Fubini Theorem gives at once the *general interchange of limits theorem: Let $F\colon \mathcal{K} \longrightarrow \mathcal{V}$, $H\colon \mathcal{A} \longrightarrow \mathcal{V}$, and $P\colon \mathcal{K} \otimes \mathcal{A} \longrightarrow \mathcal{B}$; and suppose that $\{F-, P(-, A)\}$ exists for each A and that $\{H?, P(K, ?)\}$ exists for each K. Then*

$$\{F-, \{H?, P(-, ?)\}\} \cong \{H?, \{F-, P(-, ?)\}\}, \tag{3.20}$$

[1] The statement of this theorem differs from the original text which has $F \otimes H$ instead of $H \otimes F$. However, the proof proves the statement in the form given here and the form given in the text would appear to contradict, say, (1.27).

either side existing if the other does.

For the colimit form of (3.19) and (3.20), let $F\colon \mathcal{K}^{\mathrm{op}} \longrightarrow \mathcal{V}$, $H\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{V}$, and $P\colon \mathcal{K} \otimes \mathcal{A} \longrightarrow \mathcal{B}$; then, under similar hypotheses,

$$F- \star (H? \star P(-,?)) \cong (F \otimes H) \star P \cong H? \star (F- \star P(-,?)). \tag{3.21}$$

In fact (3.20) also follows alternatively from the general principle of (3.15); for the right side of (3.1) clearly preserves *pointwise* limits in the variable $G \in [\mathcal{K}, \mathcal{B}]$.

In the variable $F \in [\mathcal{K}, \mathcal{V}]$, the right side of (3.1) sends *arbitrary* colimits, by (3.8), to limits; of course all *small* colimits in $[\mathcal{K}, \mathcal{V}]$ are pointwise, since we have remarked that $\mathcal{V}$ will be shown to be cocomplete. Hence we get from the principle (3.15) the *general theorem of the continuity of a limit in its index: Let* $F\colon \mathcal{K}^{\mathrm{op}} \longrightarrow \mathcal{V}$, $G\colon \mathcal{K} \longrightarrow [\mathcal{A}, \mathcal{V}]$, *and* $T\colon \mathcal{A} \longrightarrow \mathcal{B}$; *and suppose that* $F \star G$ *exists and that each* $\{GK, T\}$ *exists. Then*

$$\{F \star G, T\} \cong \{F, \{G-, T\}\}, \tag{3.22}$$

either side existing if the other does. The corresponding form for colimits, with F and T as above but now with $G\colon \mathcal{K} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$, is

$$(F \star G) \star T \cong F \star (G \star T). \tag{3.23}$$

In future, when we equate limits in a general theorem, the phrase "either side existing if the other does" is to be understood.

3.4 The connexion with classical conical limits when $\mathcal{V} = \mathbf{Set}$

We now consider, in the special case $\mathcal{V} = \mathbf{Set}$, the relation of indexed limits to the ordinary classical limits.

Writing $\Delta B\colon \mathcal{K} \longrightarrow \mathcal{B}$ for the constant functor at $B \in \mathcal{B}$, we have a natural bijection

$$[\mathcal{K}, \mathcal{B}](\Delta B, G) \cong [\mathcal{K}, \mathbf{Set}](\Delta 1, \mathcal{B}(B, G-)) \tag{3.24}$$

between the set of (*projective*) *cones over* G *with vertex* $\mathcal{B}$ and the set of cones over $\mathcal{B}(B, G-)$ with vertex the one-point set 1. The *ordinary*, or *classical*, or *conical* limit [resp. colimit] of $G\colon \mathcal{K} \longrightarrow \mathcal{B}$ is the representing object $\lim G$ [resp. $\operatorname{colim} G$] in

$$\mathcal{B}(B, \lim G) \cong [\mathcal{K}, \mathcal{B}](\Delta B, G), \qquad \mathcal{B}(\operatorname{colim} G, B) \cong [\mathcal{K}, \mathcal{B}](G, \Delta B), \tag{3.25}$$

existing when the right side here *is* representable. Using (3.24) we conclude from (3.1) and (3.5) that

$$\lim G = \{\Delta 1, G\}, \qquad \operatorname{colim} G = \Delta 1 \star G. \tag{3.26}$$

Thus the classical limit of $G\colon \mathcal{K} \longrightarrow \mathcal{B}$, often called a limit indexed by $\mathcal{K}$, is in fact the limit of G indexed by $\Delta 1\colon \mathcal{K} \longrightarrow \mathbf{Set}$. Of course, the counit (3.2) now corresponds to the *limit-cone*

$$\mu_K\colon \lim G \longrightarrow GK. \tag{3.27}$$

When $\mathcal{B} = \mathbf{Set}$, taking $B = 1$ in the first equation of (3.25) gives

$$\lim G \cong [\mathcal{K}, \mathbf{Set}](\Delta 1, G) \quad \text{for } G\colon \mathcal{K} \longrightarrow \mathbf{Set}, \tag{3.28}$$

so that here the limit *is* the set of cones with vertex 1. Then, as we said in 3.1, all conical limits and colimits are defined representably in terms of these primitive ones in **Set**; for (3.24) and (3.28) allow us to rewrite (3.25) in the form (analogous to (3.8))

$$\mathcal{B}(B, \lim G) \cong \lim \mathcal{B}(B, G-), \qquad \mathcal{B}(\operatorname{colim} G, B) \cong \lim \mathcal{B}(G-, B). \tag{3.29}$$

Now consider an arbitrary $F\colon \mathcal{K} \longrightarrow \mathbf{Set}$. We have as in 1.10 the category $\operatorname{el} F$ of elements of F, whose objects are pairs (K, x) with $K \in \mathcal{K}$ and $x \in FK$, and whose maps $(K, x) \longrightarrow (K', x')$ are the $f\colon K \longrightarrow K'$ in $\mathcal{K}$ for which $(Ff)x = x'$. Write $d\colon \operatorname{el} F \longrightarrow \mathcal{K}$ for the functor sending (K, x) to K, and let $G\colon \mathcal{K} \longrightarrow \mathbf{Set}$ be arbitrary. Then we have

$$\lim(\operatorname{el} F \xrightarrow[d]{} \mathcal{K} \xrightarrow[G]{} \mathbf{Set}) \cong [\mathcal{K}, \mathbf{Set}](F, G). \tag{3.30}$$

For an element of the left side is by (3.28) a cone $\alpha_{(K,x)}\colon 1 \longrightarrow GK$; to give such a cone is to give for each $x \in FK$ an element $\beta_K x = \alpha_{(K,x)}$ of GK; and the naturality of α as a cone translates exactly into the naturality of $\beta_K\colon FK \longrightarrow GK$.

Since a map of a representable $\mathcal{K}(K, -)$ into F corresponds by Yoneda to an element $x \in FK$, we have a canonical inductive cone

$$\mu_{(K,x)}\colon \mathcal{K}(K, -) \longrightarrow F \tag{3.31}$$

in $[\mathcal{K}, \mathbf{Set}]$ whose base is indexed by $(\operatorname{el} F)^{\mathrm{op}}$. The classical result on the *expressibility of any* $F\colon \mathcal{K} \longrightarrow \mathbf{Set}$ *as a canonical (conical) colimit of representables* is the assertion that this is a colimit-cone; more precisely,

$$F \cong \operatorname{colim}((\operatorname{el} F)^{\mathrm{op}} \xrightarrow[d^{\mathrm{op}}]{} \mathcal{K}^{\mathrm{op}} \xrightarrow[Y]{} [\mathcal{K}, \mathbf{Set}]), \tag{3.32}$$

with unit (3.31), where Y is the Yoneda embedding. This follows from (3.30) and the characterization (3.29) of the colimit, since the left side of (3.30) is also $\lim [\mathcal{K}, \mathbf{Set}](Y d^{\mathrm{op}}-, G)$ by Yoneda.

Because indexed limits are continuous in their index in the sense of (3.22), the colimit-expression $F \cong \operatorname{colim} \mathcal{K}(K, -)$ of (3.32) gives, for any $G\colon \mathcal{K} \longrightarrow \mathcal{B}$, a limit-expression $\{F, G\} \cong \lim\{\mathcal{K}(K, -), G\}$; so by the Yoneda isomorphism (3.10) we have

$$\{F, G\} \cong \lim(\operatorname{el} F \xrightarrow[d]{} \mathcal{K} \xrightarrow[G]{} \mathcal{B}), \tag{3.33}$$

of which (3.30) is a special case by (3.7).

We conclude that, when $\mathcal{V} = \mathbf{Set}$, the classical conical limits are special cases of indexed limits by (3.26), while the general indexed limit can be expressed as a conical limit by (3.33). Since $\operatorname{el} F$ is small when $\mathcal{K}$ is small, *a* **Set**-*category* $\mathcal{B}$ *is complete in our sense of 3.2 above exactly when it admits all small conical limits* – that is, when it is

complete in the classical sense. It follows in the same way that *the functor* $T\colon \mathcal{B} \longrightarrow \mathcal{C}$ *is continuous in our sense of 3.2 above exactly when it preserves all small conical limits.*

For $F\colon \mathcal{K}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ and $G\colon \mathcal{K} \longrightarrow \mathcal{B}$, the corresponding colimit-form of (3.33) is

$$F \star G \cong \operatorname{colim}((\operatorname{el} F)^{\mathrm{op}} \xrightarrow[d^{\mathrm{op}}]{} \mathcal{K} \xrightarrow[G]{} \mathcal{B}). \tag{3.34}$$

We had in (3.17), for any $\mathcal{V}$, the general expression of any $F\colon \mathcal{K} \longrightarrow \mathcal{V}$ as a colimit $F \star Y$ of representables; observe that when we use (3.34) to evaluate $F \star Y$ in the case $\mathcal{V} = \mathbf{Set}$, we re-find (3.32). Note too that, when $\mathcal{B} = \mathbf{Set}$, there is a second expression (3.34) for $F \star G$, since this is then isomorphic to $G \star F$ by (3.9). Finally, since the colimit of $\Delta 1\colon \mathcal{K} \longrightarrow \mathbf{Set}$ is clearly the set $\pi(\mathcal{K})$ of *connected components of* $\mathcal{K}$, taking $G = \Delta 1$ in (3.34) and using (3.26) and (3.9) gives

$$\operatorname{colim} F = \pi(\operatorname{el} F) \quad \text{for } F\colon \mathcal{K} \longrightarrow \mathbf{Set}. \tag{3.35}$$

We conclude this section with some remarks about conservativeness and faithfulness for ordinary functors $T\colon \mathcal{A} \longrightarrow \mathcal{B}$. First, *if* $\mathcal{A}$ *has equalizers and* T *preserves equalizers,* T *is faithful if it is conservative*; for if $Tf = Tg$, the equalizer of f and g is sent by T to an isomorphism. Next, *if* $\eta, \epsilon\colon S \dashv T\colon \mathcal{A} \longrightarrow \mathcal{B}$ *is an adjunction,* T *is faithful if and only if each* $\epsilon_A\colon STA \longrightarrow A$ *is an epimorphism*; this is immediate from (1.53), for $T_{AA'}$ is a monomorphism exactly when each $\mathcal{A}(\epsilon_A, A')$ is. Finally, *if* T *is a right adjoint as above, and if* $\mathcal{A}$ *admits equalizers, then* T *is conservative if and only if each* ϵ_A *is an extremal epimorphism* (one that factorizes through no proper subobject of its codomain).

To prove this, first note that *any* $f\colon A \longrightarrow B$ in $\mathcal{A}$ which factorizes through no proper subobject of B is an epimorphism, because equalizers exist. Suppose then that T is conservative. If $\epsilon_A\colon STA \longrightarrow A$ factorizes through a monomorphism $i\colon B \longrightarrow A$, then (by the adjunction) $1\colon TA \longrightarrow TA$ factorizes through the monomorphism Ti; hence Ti is isomorphism, whence i is. Conversely, let each ϵ_A be an extremal epimorphism; then, by the remark above, T is faithful, so that $f\colon A \longrightarrow B$ is surely a monomorphism if Tf is an isomorphism. But in this case, by the naturality of ϵ, we have $\epsilon_B = f.\epsilon_A.(STf)^{-1}$; so that f is an isomorphism.

3.5 Full subcategories and limits; the closure of a full subcategory under a class of colimits

Returning to the case of a general $\mathcal{V}$, we now consider some properties of (indexed) limits in relation to fully faithful functors.

A *fully faithful* $J\colon \mathcal{A} \longrightarrow \mathcal{B}$ *reflects limits*; in the sense that, for $F\colon \mathcal{K} \longrightarrow \mathcal{V}$ and $G\colon \mathcal{K} \longrightarrow \mathcal{A}$, a cylinder $F \longrightarrow \mathcal{A}(A, G-)$ expresses A as the limit $\{F, G\}$ in $\mathcal{A}$ whenever the composite cylinder $F \longrightarrow \mathcal{A}(A, G-) \longrightarrow \mathcal{B}(JA, JG-)$ expresses JA as the limit $\{F, JG\}$ in $\mathcal{B}$. This is immediate; and in the case where $\mathcal{A}$ is a full subcategory of $\mathcal{B}$, it is the assertion that $\{F, JG\}$, if it lies in $\mathcal{A}$, is also $\{F, G\}$.

If the full subcategory $\mathcal{A}$ here is reflective, and $\{F, JG\}$ exists, then the latter is isomorphic to an object of $\mathcal{A}$, which is therefore $\{F, G\}$. To see this, let the adjunction

be $\eta, 1\colon S \dashv J\colon \mathcal{A} \longrightarrow \mathcal{B}$ and let the counit of the limit be $\mu\colon F \longrightarrow \mathcal{B}(\{F, JG\}, JG-)$. By (1.48) the isomorphism $\mathcal{B}(JS\{F, JG\}, JG-) \cong \mathcal{A}(S\{F, JG\}, G-) \cong \mathcal{B}(\{F, JG\}, JG-)$ is $\mathcal{B}(\eta, 1)$. By (3.4), the cylinder $\mathcal{B}(\eta, 1)^{-1}\mu$ factorizes as $\mathcal{B}(f, 1)\mu$ for a unique $f\colon JS\{F, JG\} \longrightarrow \{F, JG\}$; and by the uniqueness part of the same result, we have $f\eta = 1$. Hence $\{F, JG\}$ is a retract of the object $JS\{F, JG\}$ of $\mathcal{A}$, and so by 1.11 lies in the repletion of $\mathcal{A}$. Thus a *reflective full subcategory $\mathcal{A}$ of $\mathcal{B}$ is closed under limits, and in particular is complete if $\mathcal{B}$ is so.*

As for colimits in this reflective situation, if the colimit $F \star JG$ exists, S preserves it by 3.2, giving $S(F \star JG) \cong F \star SJG = F \star G$. In particular, *the reflective full subcategory $\mathcal{A}$ of $\mathcal{B}$ is cocomplete if $\mathcal{B}$ is.*

Consider now a family $\Phi = (F_\gamma\colon \mathcal{K}_\gamma^{\mathrm{op}} \longrightarrow \mathcal{V}, G_\gamma\colon \mathcal{K}_\gamma \longrightarrow \mathcal{B})_{\gamma\in\Gamma}$ of diagrams in a $\mathcal{V}$-category $\mathcal{B}$, such that each $F_\gamma \star G_\gamma$ exists. If $\mathcal{A}$ is a full subcategory of $\mathcal{B}$, consider the replete full subcategories $\mathcal{C}$ containing $\mathcal{A}$ with the property that, if any G_γ takes its values in $\mathcal{C}$, then $F_\gamma \star G_\gamma$ lies in $\mathcal{C}$. The smallest such $\mathcal{C}$ is a replete full subcategory $\overline{\mathcal{A}}$ called *the closure of $\mathcal{A}$ in $\mathcal{B}$ under the family Φ of colimits.* For instance, we might have a set $\mathcal{F} = \{F\colon \mathcal{K}^{\mathrm{op}} \longrightarrow \mathcal{V}\}$ of indexing types, and take Φ to consist of all colimits that exist in $\mathcal{B}$ with indexing-type in F; then $\overline{\mathcal{A}}$ is *the closure of $\mathcal{A}$ in $\mathcal{B}$ under colimits of type $\mathcal{F}$.*

We can construct $\overline{\mathcal{A}}$ by transfinite induction: let $\mathcal{A}_0$ be the repletion of $\mathcal{A}$, let $\mathcal{A}_{\alpha+1}$ be $\mathcal{A}_\alpha$ together with all the isomorphs of those $F_\gamma \star G_\gamma$ for which G_γ lands in $\mathcal{A}_\alpha$, and let $\mathcal{A}_\alpha$ be $\bigcup_{\beta<\alpha} \mathcal{A}_\beta$ when α is a limit ordinal; then the sequence is certainly stationary when the cardinal of α exceeds that of Γ, and gives $\overline{\mathcal{A}}$. In fact, if each $\operatorname{ob}\mathcal{K}_\gamma$ has cardinal less than the regular cardinal ρ, we have $\overline{\mathcal{A}} = \mathcal{A}_\rho$, since any $\mathcal{K}_\gamma \longrightarrow \mathcal{A}_\rho$ lands in some $\mathcal{A}_\alpha$ for $\alpha < \rho$; in particular, if each $\mathcal{K}_\gamma$ is small we have $\overline{\mathcal{A}} = \mathcal{A}_\infty$, where ∞ is the cardinal of the chosen universe. It is an important observation that, *when $\mathcal{A}$ is small, and each $\mathcal{K}_\gamma$ is small, and the number of different indexing-types F_γ in Φ is small, then $\overline{\mathcal{A}}$ is small.* For then the ρ above is small, and the $\mathcal{A}_\alpha$ for $\alpha < \rho$ are small by induction: since there is but a small set of functors $\mathcal{K}_\gamma \longrightarrow \mathcal{A}_\alpha$.

If $\mathcal{B}$ itself is the Φ-closure $\overline{\mathcal{A}}$ of $\mathcal{A}$, it is immediate by induction that a functor $\mathcal{A} \longrightarrow \mathcal{C}$ has, to within isomorphism, at most one extension $\mathcal{B} \longrightarrow \mathcal{C}$ that preserves the colimits in Φ. However there may well be no such extension; as is clear when $\mathcal{A}$ is all of $\mathcal{B}$ and the original $\mathcal{A} \longrightarrow \mathcal{C}$ does not preserve the colimits in Φ.

Proposition 3.36 *Let $\mathcal{B}$ be the Φ-closure of $\mathcal{A}$, and let $Q\colon \mathcal{B}^{\mathrm{op}} \otimes \mathcal{C} \longrightarrow \mathcal{V}$ be such that each $Q(-, C)\colon \mathcal{B}^{\mathrm{op}} \longrightarrow \mathcal{V}$ sends colimits belonging to Φ to limits in $\mathcal{V}$. Suppose that $\mathcal{C}$ admits all colimits indexed by the F_γ occurring in Φ. Then if $Q(B, -)\colon \mathcal{C} \longrightarrow \mathcal{V}$ is representable for all $B \in \mathcal{A}$, it is representable for all $B \in \mathcal{B}$.*

Proof. Let $\mathcal{D}$ be the full subcategory of all those B for which a representation $Q(B, C) \cong \mathcal{C}(KB, C)$ exists. Then $\mathcal{D}$ contains $\mathcal{A}$, and it remains to show that $\mathcal{D}$ contains $F_\gamma \star G_\gamma$ whenever G_γ lands in $\mathcal{D}$. But then the hypothesis on $Q(-, C)$ gives $Q(F_\gamma \star G_\gamma, C) \cong \{F_\gamma, Q(G_\gamma -, C)\}$, and this is $\{F_\gamma, \mathcal{C}(KG_\gamma -, C)\} \cong \mathcal{C}(F_\gamma \star KG_\gamma, C)$. ∎

Since the hypothesis on $Q(-, C)$ in the above proposition is automatically satisfied if $Q(B, C) = \mathcal{B}(B, TC)$, we have:

Proposition 3.37 *Let $T\colon \mathcal{C} \longrightarrow \mathcal{B}$ where $\mathcal{C}$ is cocomplete and where $\mathcal{B}$ is the closure of*

a full subcategory $\mathcal{A}$ with respect to a family of small colimits. Then T has a left adjoint if $\mathcal{B}(A, T-)$ is representable for each $A \in \mathcal{A}$. ∎

3.6 Strongly generating functors

In the next section we need, for ordinary categories, the notion of *strong generator*. Since the notion is useful for $\mathcal{V}$-categories too, we discuss it at the enriched level.

A $\mathcal{V}$-functor T is said to be *conservative* if the underlying ordinary functor T_0 is conservative. Clearly ST is conservative if S and T are, while if ST is conservative so is T. Any fully-faithful T is conservative, while by (2.27) a functor $[M, 1]\colon [\mathcal{C}, \mathcal{B}] \longrightarrow [\mathcal{A}, \mathcal{B}]$ is conservative if $M\colon \mathcal{A} \longrightarrow \mathcal{C}$ is essentially surjective on objects. *Clearly a conservative functor reflects such limits as it preserves*; in the sense that if $\{F, G\}$ exists and is preserved by T, a cylinder $F \longrightarrow \mathcal{B}(B, G-)$ is a limit-cylinder if the composite $F \longrightarrow \mathcal{B}(B, G-) \longrightarrow \mathcal{C}(TB, TG-)$ is.

If $\mathcal{A}$ is small and $Z\colon \mathcal{A} \longrightarrow \mathcal{B}$, we define functors

$$\widehat{Z}\colon \mathcal{B}^{\mathrm{op}} \longrightarrow [\mathcal{A}, \mathcal{V}], \qquad \widetilde{Z}\colon \mathcal{B} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \tag{3.38}$$

by

$$\widehat{Z}B = \mathcal{B}(B, Z-), \qquad \widetilde{Z}B = \mathcal{B}(Z-, B). \tag{3.39}$$

We say that Z is *strongly generating* if $\widetilde{Z}$ is conservative; and dually that Z is *strongly cogenerating* if $\widehat{Z}$ is conservative. In other words, Z is strongly generating if a map $f\colon C \longrightarrow D$ in $\mathcal{B}_0$ is an isomorphism whenever $\mathcal{B}(ZA, f)$ is an isomorphism in $\mathcal{V}_0$ for each $A \in \mathcal{A}$. This clearly depends only on the full image of $\mathcal{A}$ in $\mathcal{B}$; and is also expressed by saying that the small set $\{ZA\}_{A \in \mathcal{A}}$ of objects of $\mathcal{B}$ is a *strong generator* for $\mathcal{B}$. Note that *the $\mathcal{V}$-functor $Z\colon \mathcal{A} \longrightarrow \mathcal{B}$ is certainly strongly generating when the ordinary functor $Z_0\colon \mathcal{A}_0 \longrightarrow \mathcal{B}_0$ is so*; for if $\mathcal{B}(ZA, f)$ is an isomorphism so is $\mathcal{B}_0(ZA, f)$, this being $V\mathcal{B}(ZA, f)$ by (1.33). However the converse is false; the single object $\{1\}$ is a strong generator in the **Cat**-category **Cat** but not in the **Set**-category $\mathbf{Cat}_0$

In the case that $\mathcal{V} = \mathbf{Set}$ we further say that Z is *generating*, or that $\{ZA\}_{A \in \mathcal{A}}$ is a *generator* for $\mathcal{B}$, if $\widetilde{Z}$ is faithful; so that $f = g$ whenever $\mathcal{B}(ZA, f) = \mathcal{B}(ZA, g)$ for all $A \in \mathcal{A}$. By the remarks at the end of 3.4, a strong generator in the case $\mathcal{V} = \mathbf{Set}$ is *a fortiori* a generator, provided that $\mathcal{B}$ admits equalizers. In fact the above definition of a strong generator is far from ideal unless $\mathcal{B}_0$ admits finite limits; but we have chosen it for simplicity since this is usually the case in applications. For a general $\mathcal{V}$ it would take us into side-issues to discuss the various kinds of monomorphisms and epimorphisms in the context of $\mathcal{V}$-categories; so we refrain from defining faithfulness for a $\mathcal{V}$-functor, and hence from defining a generator for a $\mathcal{V}$-category.

We state the following proposition in the enriched case, although some remarks in 3.8 below are needed to justify its proof in that case; in the meantime we use it only for $\mathcal{V} = \mathbf{Set}$.

Proposition 3.40 *The inclusion $Z\colon \mathcal{A} \longrightarrow \mathcal{B}$ of a small full subcategory is strongly generating if $\mathcal{B}$ is the closure of $\mathcal{A}$ for some set Φ of colimits. Conversely, if Z is strongly*

generating, $\mathcal{B}$ is the closure of $\mathcal{A}$ for small colimits; provided that $\mathcal{B}$ is cocomplete, that $\mathcal{B}_0$ is finitely complete, and that each object of $\mathcal{B}_0$ has only a small set of extremal-epimorphic quotients.

Proof. For the direct part, let $f\colon C \longrightarrow D$ in $\mathcal{B}$ be such that $\mathcal{B}(A, f)$ is an isomorphism for all $A \in \mathcal{A}$. Then the full subcategory of $\mathcal{B}$ given by those B for which $\mathcal{B}(B, f)$ is an isomorphism contains $\mathcal{A}$, and it clearly contains $F_\gamma \star G_\gamma$ whenever G_γ lands in it; so it is all of $\mathcal{B}$, and consequently f is an isomorphism.

For the converse, let $\mathcal{C}$ be the closure of $\mathcal{A}$ in $\mathcal{B}$ under small colimits. Since $\mathcal{B}$ is cocomplete, $F \star Z$ exists for all $F\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{V}$; hence $\widetilde{Z}\colon \mathcal{B} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ has by (3.5) the left adjoint $- \star Z\colon [\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \longrightarrow \mathcal{B}$. Write RB for $\widetilde{Z}B \star Z$: clearly $RB \in \mathcal{C}$. Since $(\widetilde{Z})_0$ is conservative, the counit $\epsilon_B\colon RB \longrightarrow B$ of the adjunction is an extremal epimorphism in $\mathcal{B}_0$, by the remarks at the end of 3.4.

Given any $f\colon C \longrightarrow D$ with $C \in \mathcal{C}$, let $u, v\colon E \longrightarrow C$ be its kernel-pair. By 3.8 below, $\mathcal{B}_0$ is cocomplete since $\mathcal{B}$ is; hence u, v have a coequalizer $p_1\colon C \longrightarrow C_1$ in $\mathcal{B}_0$, and $f = f_1 p_1$ for some $f_1\colon C_1 \longrightarrow D$. Since ϵ_E is epimorphic, p_1 is also the coequalizer of the diagram $u\epsilon_E, v\epsilon_E\colon RE \longrightarrow C$ in $\mathcal{B}_0$; by 3.8 below again, a conical colimit in $\mathcal{B}_0$ is a colimit in $\mathcal{B}$ when $\mathcal{B}$ is cocomplete; hence $C_1 \in \mathcal{C}$.

We now apply the same process to factorize f_1 as $C_1 \longrightarrow C_2 \longrightarrow D$ with $C_2 \in \mathcal{C}$, so that f is $C \longrightarrow C_2 \longrightarrow D$; and continue by transfinite induction, defining C_α at a limit-ordinal α as the cointersection of the $C \longrightarrow C_\beta$ for $\beta < \alpha$. At each stage $C_\alpha \in \mathcal{C}$, and f is $C \longrightarrow C_\alpha \longrightarrow D$. By the results of [41], each $C \longrightarrow C_\alpha$ is an extremal epimorphism in $\mathcal{B}_0$; so the process terminates at a small α, since C has but a small set of extremal epimorphic quotients. Yet the process can terminate only when $C_\alpha \longrightarrow D$ is a monomorphism; so that if $F\colon C \longrightarrow D$ is an extremal epimorphism, it terminates only with $C_\alpha = D$. We conclude that $D \in \mathcal{C}$ in this case.

Taking now for f the extremal epimorphism $\epsilon_B\colon RB \longrightarrow B$ for any $B \in \mathcal{B}$, we conclude that $B \in \mathcal{C}$, as desired. ∎

Let us now apply this, in the case of ordinary categories, to the functor $V = \mathcal{V}_0(I, -)\colon \mathcal{V}_0 \longrightarrow \mathbf{Set}$. This functor is just the $\widetilde{Z}$ corresponding to the functor $Z\colon \mathbf{1} \longrightarrow \mathcal{V}_0$ sending the unique object of $\mathbf{1}$ to I; so $\{I\}$ is a generator of $\mathcal{V}_0$ when V is faithful, and a strong generator when V is conservative. Hence:

Proposition 3.41 *If $\mathcal{V}_0$ is the closure of $\{I\}$ under small colimits, $V\colon \mathcal{V}_0 \longrightarrow \mathbf{Set}$ is conservative; and the converse is true if each object of $\mathcal{V}_0$ has but a small set of extremal epimorphic quotients.* ∎

This last *weak cowellpoweredness* condition on $\mathcal{V}_0$ is certainly satisfied in the examples we have given with V conservative, namely **Set**, **Ab**, R-**Mod**, and **Ban**; in fact it seems very hard to think of any complete and cocomplete locally-small category in nature that is not weakly cowellpowered, although artificial examples exist: see Example 5.3 of [36].

3.7 Tensor and cotensor products

In this section and the next three, we consider for a general $\mathcal{V}$ certain important special cases of the indexed limit $\{F, G\}$, and the extent to which the general case may be reduced to these.

The first special case is that where the domain-category $\mathcal{K}$ of F and G is the unit $\mathcal{V}$-category $\mathcal{I}$, so that $F\colon \mathcal{I} \longrightarrow \mathcal{V}$ is in effect an object X of $\mathcal{V}$ and $G\colon \mathcal{I} \longrightarrow \mathcal{B}$ is an object C of $\mathcal{B}$. Then the object $\{F, G\}$ of $\mathcal{B}$ is written $X \pitchfork C$, or sometimes $[X, C]$, and is called the *cotensor product* in $\mathcal{B}$ of X and C. By (3.1), its defining equation is a $\mathcal{V}$-natural isomorphism

$$\mathcal{B}(B, X \pitchfork C) \cong [X, \mathcal{B}(B, C)] \tag{3.42}$$

with counit

$$X \longrightarrow \mathcal{B}(X \pitchfork C, C). \tag{3.43}$$

If $X \pitchfork C$ exists for all $X \in \mathcal{V}$ and all $C \in \mathcal{B}$, the $\mathcal{V}$-category $\mathcal{B}$ is said to *admit cotensor products*, or to be *cotensored*. Because $\mathcal{I}$ is small, any complete $\mathcal{B}$ is cotensored. Since by (1.27) we have the $\mathcal{V}$-natural isomorphism $[Y, [X, Z]] \cong [X, [Y, Z]]$ in $\mathcal{V}$, we conclude that *$\mathcal{V}$ is cotensored, the cotensor product $X \pitchfork Z$ being just the internal-hom $[X, Z]$*. When $\mathcal{V} = \mathbf{Set}$, the cotensor product $X \pitchfork C$ in $\mathcal{B}$ is clearly just the *power* C^X – the product of X copies of the object C of $\mathcal{B}$.

The dual notion is the *tensor product* $X \otimes C \in \mathcal{B}$ of $X \in \mathcal{V}$ and $C \in \mathcal{B}$, defined by

$$\mathcal{B}(X \otimes C, B) \cong [X, \mathcal{B}(C, B)]. \tag{3.44}$$

Using (1.27) again, we have that *$\mathcal{V}$ is tensored, $X \otimes Y$ being the ordinary tensor product in $\mathcal{V}$*. When $\mathcal{V} = \mathbf{Set}$, the tensor product $X \otimes C$ is just the *copower* $X \cdot C$ – the coproduct in $\mathcal{B}$ of X copies of the object C.

The natural bijection

$$\mathcal{B}_0(B, X \pitchfork C) \cong \mathcal{V}_0\big(X, \mathcal{B}(B, C)\big) \tag{3.45}$$

obtained by applying V to (3.42) corresponds to (3.4), and is in general insufficient to ensure that $X \pitchfork C$ is the cotensor product of X and C, unless this latter is known to exist. For instance, take $\mathcal{V} = \mathbf{Cat}$ and $X = \mathbf{2}$, and let $\mathcal{B}$ be the 2-category generated by the 2-graph

$$\begin{tikzcd} 0 \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow\alpha"] & 1 \arrow[r, bend left, "g"] & 2 \arrow[l, bend left, "h"] \end{tikzcd}$$

with the relations $g\alpha = 1$ and $gh = 1$. If we were to define $\mathbf{2} \pitchfork 2$ to be 1, with as counit (3.43) the functor from $\mathbf{2}$ corresponding to the map $1_g\colon g \longrightarrow g$, we should satisfy (3.45) but not (3.42).

When V is *conservative*, however, a map (3.43) that induces a bijection (3.45) also induces an isomorphism (3.42), as we saw in 1.11. In fact we have:

Proposition 3.46 *Let V be conservative, and let each object of $\mathcal{V}_0$ have but a small set of extremal-epimorphic quotients. Then a $\mathcal{V}$-category $\mathcal{B}$ is cotensored if the ordinary category $\mathcal{B}_0$ is complete.*

Proof. For a fixed $C \in \mathcal{B}$ we apply Proposition 3.37 to the ordinary functor $\mathcal{B}(-,C)_0 \colon \mathcal{B}_0^{\mathrm{op}} \longrightarrow \mathcal{V}_0$; observing that $\mathcal{V}_0$ is the closure of $\{I\}$ under small colimits by Proposition 3.41, while $\mathcal{V}_0(I, \mathcal{B}(-,C))$ is representable as $\mathcal{B}_0(-,C)$. This gives a natural bijection (3.45), the counit (3.43) of which then induces as above a natural isomorphism (3.42) because V is conservative. ∎

When $\mathcal{B}$ is cotensored, a third term can be added to the equation (3.22) expressing the continuity of a limit in its index; using a proof like that of (3.18), we easily get

$$\{F \star G, T\} \cong \{(G-)?, F- \pitchfork T?\} \cong \{F, \{G-, T\}\}, \tag{3.47}$$

where $F- \pitchfork T?$ is the functor sending (K, A) to $FK \pitchfork TA$; here the first isomorphism in (3.47) holds under the hypothesis that $F \star G$ exists, and the second under the hypothesis that each $\{GK, T\}$ exists. The corresponding extension of (3.23), now supposing $\mathcal{B}$ to be tensored, is

$$(F \star G) \star T \cong (G-)? \star (F \otimes T) \cong F \star (G- \star T). \tag{3.48}$$

3.8 Conical limits in a $\mathcal{V}$-category

For the next special case of a limit $\{F, G\}$, we begin with a $\mathcal{V}$-category $\mathcal{B}$, a locally-small ordinary category $\mathcal{L}$, and a functor $T \colon \mathcal{L} \longrightarrow \mathcal{B}_0$. We take for $\mathcal{K}$ the free $\mathcal{V}$-category $\mathcal{L}_{\mathcal{V}}$ on $\mathcal{L}$ as in 2.5, for $G \colon \mathcal{K} \longrightarrow \mathcal{B}$ the $\mathcal{V}$-functor corresponding in 2.5 to $T \colon \mathcal{L} \longrightarrow \mathcal{B}_0$, and for $F \colon \mathcal{K} \longrightarrow \mathcal{V}$ the $\mathcal{V}$-functor corresponding to the ordinary functor $\Delta I \colon \mathcal{L} \longrightarrow \mathcal{V}_0$ constant at $I \in \mathcal{V}_0$. Observe that, since the adjunction of 2.5 is of course 2-natural, $\mathcal{B}(B, G-) \colon \mathcal{K} \longrightarrow \mathcal{V}$ is the $\mathcal{V}$-functor corresponding to the ordinary functor

$$\mathcal{L} \xrightarrow[T]{} \mathcal{B}_0 \xrightarrow[\mathcal{B}(B,-)_0]{} \mathcal{V}_0, \tag{3.49}$$

which we may loosely call $\mathcal{B}(B, T-)$.

A family $\alpha_K \colon X \longrightarrow [FK, \mathcal{B}(B, GK)]$ is $\mathcal{V}$-natural by 1.8(m) if and only if the corresponding $\overline{\alpha}_K \colon FK \longrightarrow [X, \mathcal{B}(B, GK)]$ is so; by 2.5 this is equivalent to the mere naturality of $\overline{\alpha}_K \colon (\Delta I)K \longrightarrow [X, \mathcal{B}(B, TK)]$, and hence to the requirement that $\alpha_K \colon X \longrightarrow [I, \mathcal{B}(B, TK)] \cong \mathcal{B}(B, TK)$ be a cone over the ordinary functor $\mathcal{B}(B, T-)$ of (3.49). It follows from (2.10) that

$$[\mathcal{K}, \mathcal{V}](F, \mathcal{B}(B, G-)) \cong \lim \mathcal{B}(B, T-), \tag{3.50}$$

this being the ordinary conical limit in $\mathcal{V}_0$ of the ordinary functor (3.49) – the identification of which with the left side of (3.50) exhibits it as being $\mathcal{V}$-functorial in B.

In this situation we call $\{F, G\}$, if it exists, *the (conical) limit in* $\mathcal{B}$ *of* $T \colon \mathcal{L} \longrightarrow \mathcal{B}_0$, denoting it formally by $\lim_{\mathcal{B}} T$, but usually in practice simply by $\lim T$. By (3.1) and (3.50), its defining equation is the $\mathcal{V}$-natural isomorphism

$$\mathcal{B}(B, \lim T) \cong \lim \mathcal{B}(B, T-) \tag{3.51}$$

in $\mathcal{V}_0$, the counit μ of (3.2) here reducing to the *limit cone*

$$\mu_K \colon \lim T \longrightarrow TK. \tag{3.52}$$

Since the representable $V = \mathcal{V}_0(I,-)\colon \mathcal{V}_0 \longrightarrow \mathbf{Set}$ preserves limits, applying V to (3.51) gives the natural bijection

$$\mathcal{B}_0(B, \lim T) \cong \lim \mathcal{B}_0(B, T-), \tag{3.53}$$

showing by (3.29) that $\lim_{\mathcal{B}} T$, *if it exists, with the cone* (3.52), *is also the ordinary conical limit* $\lim_{\mathcal{B}_0} T$ *of the ordinary functor* $T\colon \mathcal{L} \longrightarrow \mathcal{B}_0$. It is for this reason that we write simply $\lim T$; but we distinguish verbally *the existence* (3.51) *of* $\lim T$ *in* $\mathcal{B}$ *from the (mere) existence* (3.53) *of* $\lim T$ *in* $\mathcal{B}_0$. The latter is in general strictly weaker, as we remarked in (3.4); for instance, if $\mathcal{V} = \mathbf{Cat}$ and $\mathcal{B}$ is the 2-category freely generated by the 2-graph

$$\begin{tikzcd} 0 \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow\alpha"] & 1, \end{tikzcd} \tag{3.54}$$

then the cone $1 \longleftarrow 1 \longrightarrow 1$ exhibits 1 as the product 1×1 in $\mathcal{B}_0$, but not in $\mathcal{B}$. Note that an equivalent way of expressing the existence in $\mathcal{B}$ of $\lim T$ is to say that $\lim T$ *exists in* $\mathcal{B}_0$ *and is preserved by all the functors* $\mathcal{B}(B,-)_0\colon \mathcal{B}_0 \longrightarrow \mathcal{V}_0$.

The difference between the existence in $\mathcal{B}$ *and that in* $\mathcal{B}_0$ *of such conical limits vanishes, of course, if* V *is conservative. It also vanishes for any* $\mathcal{V}$ *if* $\mathcal{B}$ *is tensored.* To see this we have only to replace B in (3.53) by $X \otimes B$ for any $X \in \mathcal{V}$, to apply the dual of (3.45), and to recall that $\mathcal{V}_0(X,-)$ preserves limits; we conclude that each $\mathcal{V}_0(X,-)$ carries the map $\mathcal{B}(B, \lim T) \longrightarrow \lim \mathcal{B}(B, T-)$ induced by μ into an isomorphism; whence this map is itself an isomorphism.

In particular the difference vanishes if $\mathcal{B} = \mathcal{V}$, so that $\mathcal{V}$ *admits all small conical limits.* Note that the right side of (3.51), which until now was the limit in $\mathcal{V}_0$, is in fact also the limit in $\mathcal{V}$. The difference also vanishes if $\mathcal{B} = \mathcal{V}^{\mathrm{op}}$, since $\mathcal{V}^{\mathrm{op}}$ is tensored because $\mathcal{V}$ is cotensored; hence $\mathcal{V}$ *admits all small conical colimits.* Here the notion of conical colimit is the evident dual to that of conical limit, defined for $T\colon \mathcal{L} \longrightarrow \mathcal{B}_0$ by

$$\mathcal{B}(\operatorname{colim} T, B) \cong \lim \mathcal{B}(T-, B). \tag{3.55}$$

Since $\mathcal{L}_{\mathcal{V}}$ is small when $\mathcal{L}$ is, *any complete* $\mathcal{B}$ *admits all small conical limits.* In consequence, by 3.4, $\mathcal{B}_0$ *is complete whenever* $\mathcal{B}$ *is.* However *the existence in* $\mathcal{B}$ *of all small conical limits does not suffice, in general, for completeness of* $\mathcal{B}$. Take $\mathcal{V} = \mathbf{Cat}$, and let $\mathcal{B}$ be the full sub-2-category of $\mathbf{Cat}$ determined by all categories with at most one object. Since $\mathcal{V}$ admits all small conical limits so does $\mathcal{B}$, for the conical limit in $\mathcal{V}$ of a diagram in $\mathcal{B}$ clearly lies in $\mathcal{B}$. Yet $\mathcal{B}$ is not complete, for $\mathbf{2} \pitchfork B$ does not exist in $\mathcal{B}$ if B has more than one map.

3.9 The inadequacy of conical limits

This inadequacy of conical limits deserves some further comment. It shows that, for a general $\mathcal{V}$, there can be no analogue of (3.32) allowing us to express, as in (3.33), a general limit as a conical one. In fact, as in 3.4, any $F\colon \mathcal{K} \longrightarrow \mathcal{V}$ is the vertex of a canonical inductive cone in $[\mathcal{K}, \mathcal{V}]_0$, whose generators are all the maps $\mathcal{K}(K,-) \longrightarrow F$ from representables

into F. Since such maps correspond by Yoneda to elements $x \in \mathcal{V}_0(I, FK) = VFK$, the base of this cone is a functor with domain $\mathrm{el}(VF_0)^{\mathrm{op}}$, the dual of the category of elements of the ordinary functor

$$\mathcal{K}_0 \xrightarrow[F_0]{} \mathcal{V}_0 \xrightarrow[V]{} \mathbf{Set}.$$

When $\mathcal{K}$ is small so is $\mathrm{el}(VF_0)$, so that this functor $\mathrm{el}(VF_0)^{\mathrm{op}} \longrightarrow [\mathcal{K}, \mathcal{V}]_0$ has a conical colimit in $[\mathcal{K}, \mathcal{V}]$; for $[\mathcal{K}, \mathcal{V}]$ admits small conical colimits pointwise since $\mathcal{V}$ admits them. We therefore have a comparison map

$$\{\mathrm{colim}(\mathrm{el}(VF_0)^{\mathrm{op}} \xrightarrow[d^{\mathrm{op}}]{} \mathcal{K}_0^{\mathrm{op}} \xrightarrow[Y_0]{} [\mathcal{K}, \mathcal{V}]_0)\} \longrightarrow F \tag{3.56}$$

in $[\mathcal{K}, \mathcal{V}]_0$. However, in contrast to the case $\mathcal{V} = \mathbf{Set}$, *the map* (3.56) *is not in general an isomorphism – even when V is conservative.*

To see this it suffices to take for $\mathcal{K}$ the unit $\mathcal{V}$-category $\mathcal{I}$, so that F is just an object X of $\mathcal{V}$. The canonical cone is now the cone in $\mathcal{V}_0$ given by all the maps $I \longrightarrow X$. To ask that this be a colimit-cone for each X is to ask that $\{I\}$ be *dense* in $\mathcal{V}_0$, in the sense of 5.1 below; this is stronger than asking V to be conservative, which by 3.6 is to ask that $\{I\}$ be a strong generator for $\mathcal{V}_0$; it is false even for $\mathcal{V} = \mathbf{Ab}$. It is of course true for $\mathcal{V} = \mathbf{Set}$, and for certain other $\mathcal{V}$: for instance, sets on which a commutative monoid acts, with a tensor product analogous to that for modules over a commutative ring.

What *is* true when V is conservative, by Proposition 3.41, is that any $X \in \mathcal{V}_0$, although not the canonical colimit of the maps from I, is in all reasonable cases in the closure of $\{I\}$ under small colimits; it is, so to speak, an *iterated colimit* of I. In this case an arbitrary $F\colon \mathcal{K} \longrightarrow \mathcal{V}$ is an iterated colimit of representables; and so the existence of all small conical limits in $\mathcal{B}$ *does* imply the existence of all indexed limits $\{F, G\}$, as we shall show formally in 3.10 below – even though $\{F, G\}$ cannot be expressed as a single conical limit.

In the study of enriched universal algebra, it is an important observation that, for many $\mathcal{V}$ including **Cat** and **Gpd**, (3.56) *is* an isomorphism whenever F is *left exact*; see [45].

Before ending this section, we look at the possible existence in certain cases of more general conical limits. The notion of a cone involves the notion of a constant functor; yet in general there is no such thing as "the $\mathcal{V}$-functor $\mathcal{K} \longrightarrow \mathcal{B}$ constant at $B \in \mathcal{B}$". The object $B \in \mathcal{B}$ may be seen as a $\mathcal{V}$-functor $\mathcal{I} \longrightarrow \mathcal{B}$, but there is no canonical $\mathcal{V}$-functor $\mathcal{K} \longrightarrow \mathcal{I}$; in fact there may well be none at all. There is certainly a unique V-functor $\mathcal{K} \longrightarrow 1$, where 1 is the terminal $\mathcal{V}$-category with one object 0 and with, for hom-object $1(0,0)$, the terminal object 1 of $\mathcal{V}_0$; but there is no $\mathcal{V}$-functor $1 \longrightarrow \mathcal{I}$ unless I is a retract of 1 in $\mathcal{V}_0$ – which is false for $\mathcal{V} = R$-**Mod**, **DG**-R-**Mod**, $\mathbf{CGTop}_*$, and so on. This is why we have defined conical limits as above, taking for $\mathcal{K}$ a free $\mathcal{V}$-category $\mathcal{L}_{\mathcal{V}}$; for there are always constant ordinary functors $\mathcal{L} \longrightarrow \mathcal{B}_0$.

Yet in the special case of *cartesian* closed $\mathcal{V}$ (and in some other cases as well), it happens that $I = 1$ in $\mathcal{V}_0$, so that the $\mathcal{V}$-categories $\mathcal{I}$ and 1 coincide; then there is for any $B \in \mathcal{B}$ a "constant $\mathcal{V}$-functor" $\Delta B\colon \mathcal{K} \longrightarrow 1 \longrightarrow \mathcal{B}$ with this value. In particular there is a constant $\mathcal{V}$-functor $\Delta 1\colon \mathcal{K} \longrightarrow \mathcal{V}$, and we can consider for any $G\colon \mathcal{K} \longrightarrow \mathcal{B}$ the indexed

limit $\{\Delta 1, G\}$. If this exists it represents $[\mathcal{K},\mathcal{V}](\Delta 1, \mathcal{B}(B, G-))$; which is easily seen to be $[\mathcal{K},\mathcal{B}](\Delta B, G)$, the isomorphism $[(\Delta 1)K, \mathcal{B}(B, GK')] \cong \mathcal{B}((\Delta B)K, GK')$ being clearly $\mathcal{V}$-natural in all three variables. Thus we are led, when $\mathcal{V}$ is cartesian closed, to define the *conical limit* $\lim G$ by

$$\lim G = \{\Delta 1, G\} \tag{3.57}$$

for any $\mathcal{V}$-functor $G\colon \mathcal{K} \longrightarrow \mathcal{B}$; so that $\lim G$ is the representing object in

$$\mathcal{B}(B, \lim G) \cong [\mathcal{K},\mathcal{B}](\Delta B, G). \tag{3.58}$$

This reduces to $\lim T$ when $G\colon \mathcal{L}_{\mathcal{V}} \longrightarrow \mathcal{B}$ is the $\mathcal{V}$-functor corresponding to the ordinary functor $T\colon \mathcal{L} \longrightarrow \mathcal{B}_0$; and thus it extends the notion of conical limit from functors to $\mathcal{V}$-functors, for cartesian closed $\mathcal{V}$.

By (3.57), $\lim G$ certainly exists for every $\mathcal{V}$-functor $G\colon \mathcal{K} \longrightarrow \mathcal{B}$ with $\mathcal{K}$ small, when $\mathcal{V}$ is cartesian closed, if the $\mathcal{V}$-category $\mathcal{B}$ is complete. Even the existence of these more general conical limits, however, does not conversely imply completeness of $\mathcal{B}$. The example in 3.8 above of a non-complete $\mathcal{B}$ when $\mathcal{V} = \mathbf{Cat}$, given by the full subcategory of $\mathbf{Cat}$ determined by the categories with at most one object, clearly admits all such conical limits indexed by a 2-category.

3.10 Ends and coends in a general $\mathcal{V}$-category; completeness

For the final special case of a limit $\{F, G\}$ we take the indexing-type $F\colon \mathcal{K} \longrightarrow \mathcal{V}$ to be $\mathrm{Hom}_{\mathcal{A}}\colon \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathcal{V}$ for some $\mathcal{V}$-category $\mathcal{A}$. Then a diagram G in $\mathcal{B}$ of this type is just a $\mathcal{V}$-functor $G\colon \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathcal{B}$. If $\{\mathrm{Hom}_{\mathcal{A}}, G\}$ exists we denote it by

$$\textstyle\int_{A\in\mathcal{A}} G(A, A) = \{\mathrm{Hom}_{\mathcal{A}}, G\}, \tag{3.59}$$

and call it the *end* of G. By the extra-variable Yoneda isomorphism (2.38), the right side of (3.1) may here be written as $\int_A \mathcal{B}(B, G(A, A))$, this being the end of a bifunctor $\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathcal{V}$ as defined in 2.1. Hence the defining equation of the end $\int_A G(A, A)$ in $\mathcal{B}$ is

$$\mathcal{B}(B, \textstyle\int_A G(A, A)) \cong \int_A \mathcal{B}(B, G(A, A)), \tag{3.60}$$

while the counit μ of (3.2) corresponds to a $\mathcal{V}$-natural family

$$\lambda_A\colon\ \textstyle\int_A G(A, A) \longrightarrow G(A, A). \tag{3.61}$$

It is immediate from (2.3) that, when $\mathcal{B} = \mathcal{V}$, the end $\int_A G(A, A)$ in the sense of 2.1 is also the end in the present sense, so that there is no ambiguity; indeed (3.60) is just one more illustration of the fact that limits in $\mathcal{B}$ are defined representably in terms of limits in $\mathcal{V}$.

Applying $V = \mathcal{V}_0(I, -)$ to (3.60), and observing that $\mathcal{V}_0(I, \int_A \mathcal{B}(B, G(A, A)))$ is by 2.1 the set of $\mathcal{V}$-natural families $I \longrightarrow \mathcal{B}(B, G(A, A))$, which is in effect the set $\mathcal{V}\text{-nat}(B, G(A, A))$ of $\mathcal{V}$-natural families $B \longrightarrow G(A, A)$, we have a bijection of sets

$$\mathcal{B}_0(B, \textstyle\int_A G(A, A)) \cong \mathcal{V}\text{-nat}(B, G(A, A)), \tag{3.62}$$

showing that the counit λ of (3.61) is the universal $\mathcal{V}$-natural family with codomain $G(A,A)$. When $\mathcal{B}=\mathcal{V}$, this universal property sufficed in 2.1 to *define* $\int_A G(A,A)$; but in general it does not. For instance, if $\mathcal{V}=\mathbf{Cat}$ and $\mathcal{A}$ is the discrete 2-category 2, then $\int_A G(A,A)$ is just the product $G(0,0)\times G(1,1)$, which for suitable G can be *any* product in $\mathcal{B}$; so that (3.54) again provides a counter-example. Of course this universal property does suffice to *detect* $\int_A G(A,A)$ if this is known to exist; and it *does* suffice to define it if V is conservative, or if $\mathcal{B}$ is tensored – the argument for this last being like the argument for the analogous fact in 3.8.

The general Fubini and interchange-of-limits theorems of 3.3 have as particular cases, under the hypothesis that the inner limits exist, such results as

$$\begin{aligned}
\textstyle\int_A\int_B G(A,A,B,B) &\cong \textstyle\int_{(A,B)} G(A,A,B,B)\cong\int_B\int_A G(A,A,B,B), && (3.63)\\
\textstyle\int_A \lim_\alpha G_\alpha(A,A) &\cong \lim_\alpha \textstyle\int_A G_\alpha(A,A), && (3.64)\\
\textstyle\int_A X\pitchfork G(A,A) &\cong X\pitchfork\textstyle\int_A G(A,A); && (3.65)
\end{aligned}$$

there is no point in specifically remembering any but the general results of 3.3.

The dual notion is that of the *coend* $\int^A G(A,A)$ of $G\colon \mathcal{A}^{\mathrm{op}}\otimes\mathcal{A}\longrightarrow\mathcal{B}$, defined by

$$\textstyle\int^A G(A,A)=\mathrm{Hom}_{\mathcal{A}}\star G, \tag{3.66}$$

and hence determined by

$$\mathcal{B}(\textstyle\int^A G(A,A),B)\cong\int_A\mathcal{B}(G(A,A),B). \tag{3.67}$$

We now observe that *any $\mathcal{B}$ which admits cotensor products and small conical limits admits every end $\int_{A\in\mathcal{A}} G(A,A)$ with small $\mathcal{A}$.* For let us temporarily *define* an object $\int_A G(A,A)$ as the equalizer of the evident pair ρ,σ of maps (cf. (2.2)) in

$$\textstyle\int_{A\in\mathcal{A}} G(A,A)\xrightarrow[\lambda]{}\prod_{A\in\mathcal{A}} G(A,A)\underset{\sigma}{\overset{\rho}{\rightrightarrows}}\prod_{A,C\in\mathcal{A}}\mathcal{A}(A,C)\pitchfork G(A,C). \tag{3.68}$$

Applying the limit-preserving $\mathcal{B}(B,-)$ to (3.68), using (3.42), and comparing the result with (2.2), we get precisely the desired (3.60).

We next observe that, for any $F\colon\mathcal{K}\longrightarrow\mathcal{V}$ and $G\colon\mathcal{K}\longrightarrow\mathcal{B}$, we have

$$\{F,G\}\cong\textstyle\int_K FK\pitchfork GK \qquad \textit{if } \mathcal{B} \textit{ is cotensored;} \tag{3.69}$$

or even if only these particular cotensor products $FK\pitchfork GK$ exist: either side of (3.69) existing if the other does. For the right side of (3.1) is $\int_K[FK,\mathcal{B}(B,GK)]$ by (2.10), which here is $\int_K\mathcal{B}(B,FK\pitchfork GK)$ by (3.42). The dual of this is

$$F\star G\cong\textstyle\int^K FK\otimes GK \qquad \textit{if } \mathcal{B} \textit{ is tensored.} \tag{3.70}$$

Note that the Yoneda isomorphisms (3.10) for cotensored [resp. tensored] $\mathcal{B}$ take the forms

$$\textstyle\int_L\mathcal{K}(K,L)\pitchfork GL\cong GK,\qquad \int^L\mathcal{K}(L,K)\otimes GL\cong GK; \tag{3.71}$$

while the expression (3.17) of an arbitrary $F\colon \mathcal{K} \longrightarrow \mathcal{V}$ as a colimit of representables takes the form

$$\int^L \mathcal{K}(L,-) \otimes FL \cong F. \tag{3.72}$$

The two observations above give at once:

Theorem 3.73 *The $\mathcal{V}$-category $\mathcal{B}$ is complete if and only if it admits cotensor products and all small conical limits. When this is so, the $\mathcal{V}$-functor $T\colon \mathcal{B} \longrightarrow \mathcal{C}$ is continuous if and only if it preserves cotensor products and small conical limits.* ∎

We have already seen in 3.2 that $\mathcal{V}$ is complete. Since $\mathcal{V}$ admits tensor products by 3.7 and small conical colimits by 3.8, we conclude that

$$\mathcal{V} \text{ is cocomplete.} \tag{3.74}$$

By 3.3 and 3.5, we have:

Proposition 3.75 *If $\mathcal{A}$ is small, every reflective full subcategory of $[\mathcal{A},\mathcal{V}]$ is complete and cocomplete.* ∎

It is a matter of common experience that, in the case $\mathcal{V} = \mathbf{Ab}$ of additive categories, it has never proved necessary to distinguish the completeness of $\mathcal{B}$ from that of $\mathcal{B}_0$. In fact Proposition 3.46, along with the observation in 3.8 that conical limits in $\mathcal{B}_0$ are also limits in $\mathcal{B}$ when V is conservative, gives:

Proposition 3.76 *Let V be conservative, and let each object of $\mathcal{V}_0$ have but a small set of extremal-epimorphic quotients. Then a $\mathcal{V}$-category $\mathcal{B}$ is complete whenever the ordinary category $\mathcal{B}_0$ is so.* ∎

We finally observe that (3.68) and (3.69) do not "reduce all indexed limits to conical limits and cotensor products"; the $\{F,G\}$ in (3.69) may well exist although the $FK \pitchfork GK$ do not, an example being provided by the Yoneda isomorphisms (3.10) for an arbitrary $\mathcal{B}$. It would be equally false for ordinary categories to say that "all conical limits reduce to products and equalizers"; there all conical limits exist if all products and equalizers do, but a particular pullback may well exist even though the product of which it would be a subobject does not. The moral is that it is proper to recognize the most general (indexed) limit concept, while seeing the results above as simple tests for completeness.

3.11 The existence of a limit-preserving universe-enlargement $\mathcal{V} \subset \mathcal{V}'$

We now justify our assertion of 2.6, that every $\mathcal{V}$ has a well-behaved extension to a higher-universe version $\mathcal{V}'$. The analysis below is a very special case of that in Day's consideration [12] of monoidal biclosed structures on enriched functor categories.

Given a set $\{\mathcal{A}\}$ of $\mathcal{V}$-categories $\mathcal{A}$ that we should like to be small when seen as $\mathcal{V}'$-categories, let $\mathbf{Set}'$ be the category of sets in some universe containing as elements $\mathrm{ob}(\mathbf{Set})$, $\mathrm{ob}\,\mathcal{V}$, and $\mathrm{ob}\,\mathcal{A}$ for each $\mathcal{A} \in \{\mathcal{A}\}$.

Take $\mathcal{V}'_0 = [\mathcal{V}_0^{\mathrm{op}}, \mathbf{Set}']$, so that $\mathcal{V}'_0$ is a $\mathbf{Set}'$-category, complete and cocomplete in the $\mathbf{Set}'$-sense; and denote by $y\colon \mathcal{V}_0 \longrightarrow \mathcal{V}'_0$ the Yoneda full embedding sending X to

$\mathcal{V}_0(-, X)$. Then *there is an essentially unique symmetric monoidal closed structure* $\mathcal{V}' = (\mathcal{V}'_0, \otimes', I', [\ ,\ ]')$ *on* $\mathcal{V}'_0$ *extending that on* $\mathcal{V}_0$.

First observe that, since the left adjoint functors $F \otimes' -$ and $- \otimes' G$ must preserve all colimits, and since F is by (3.17) the colimit $F \star y$ of representables, and similarly for G, we must have $F \otimes' G \cong (F \star y) \otimes' G \cong F \star (y \otimes' G) \cong F \star (G \star (y \otimes' y))$, which by (3.21) is $(F \times G) \star (y \otimes' y)$. Since $\otimes'$ is to extend $\otimes$, we must have

$$yX \otimes' yY = \mathcal{V}_0(-, X) \otimes' \mathcal{V}_0(-, Y) \cong \mathcal{V}_0(-, X \otimes Y) = y(X \otimes Y). \tag{3.77}$$

Thus we are forced to define $F \otimes' G$ as $(F- \times G?) \star y(- \otimes ?)$, which written in the more explicit coend form (3.70) is

$$F \otimes' G = \textstyle\int^{X,Y} FX \times GY \times \mathcal{V}_0(-, X \otimes Y). \tag{3.78}$$

With this definition, moreover, (3.77) is clearly satisfied, by Yoneda in the form (3.71).

The definition (3.5) of $\star$ applied to $F \otimes' G = (F- \times G?) \star y(- \otimes ?)$ gives $\mathcal{V}'_0(F \otimes' G, H) \cong [\mathcal{V}_0^{\mathrm{op}} \times \mathcal{V}_0^{\mathrm{op}}, \mathbf{Set}'](F- \times G?, \mathcal{V}'_0(y(- \otimes ?), H))$. But $\mathcal{V}'_0(y(- \otimes ?), H) \cong H(- \otimes ?)$ by Yoneda, so that $\mathcal{V}'_0(F \otimes' G, H) \cong [\mathcal{V}_0^{\mathrm{op}} \times \mathcal{V}_0^{\mathrm{op}}, \mathbf{Set}'](F- \times G?, H(- \otimes ?))$, which by (2.29) is also $\mathcal{V}'_0(F, \mathcal{V}'_0(G?, H(- \otimes ?)))$. In other words $- \otimes' G$ has as desired a right adjoint $[G,\ ]'$:

$$\mathcal{V}'_0(F \otimes' G, H) \cong \mathcal{V}'_0(F, [G, H]'), \tag{3.79}$$

where

$$[G, H]' = \mathcal{V}'_0(G?, H(- \otimes ?)) = \textstyle\int_Y [GY, H(- \otimes Y)], \tag{3.80}$$

$[\ ,\ ]$ here denoting the internal-hom in $\mathbf{Set}'$. Note that $[\mathcal{V}_0(-, X), \mathcal{V}_0(-, Z)]' = \int_Y [\mathcal{V}_0(Y, X), \mathcal{V}_0(- \otimes Y, Z)]$ is, by the form (3.71) of Yoneda, $\mathcal{V}_0(- \otimes X, Z) \cong \mathcal{V}_0(-, [X, Z])$. Thus

$$[yX, yZ]' \cong y[X, Z], \tag{3.81}$$

so that $[\ ,\ ]'$ is an extension of the $[\ ,\ ]$ of $\mathcal{V}$.

From the existence of this right adjoint and the evident symmetry of $\otimes'$, it follows that $(F \otimes' G) \otimes' H$ preserves colimits in each variable separately; so that from the colimit-expressions of the form $F \cong F \star y$ and from (3.77), we get

$$(F \otimes' G) \otimes' H \cong \textstyle\int^{X,Y,Z} FX \times GY \times HZ \times \mathcal{V}_0(-, (X \otimes Y) \otimes Z), \tag{3.82}$$

with a similar expression for $F \otimes' (G \otimes' H)$. Hence the associativity isomorphism for $\otimes$ induces one for $\otimes'$ as in (2.4); and similarly of course for the commutativity. Again, if we set

$$I' = yI = \mathcal{V}_0(-, I), \tag{3.83}$$

the cocontinuity in F of $F \otimes' I'$, along with (3.77), gives an evident isomorphism $F \otimes' I' \cong F$.

Verification of the coherence conditions in $\mathcal{V}'$ from those in $\mathcal{V}$ is straightforward: the essential logical point is the uniqueness of the induced map $\int_A \alpha_{AA}$ in (2.4). A careful discussion is given by Day in [12], and a more streamlined one in his thesis [13].

Note that the derivation of (3.79) made no use of the *closed* structure of the symmetric monoidal $\mathcal{V}$. This proves our assertion at the end of 1.4 above, that *any symmetric monoidal category can be embedded in a closed one.*

An essential property of the embedding $y\colon \mathcal{V}_0 \longrightarrow \mathcal{V}'_0$ is that it *preserves all limits that exist in* $\mathcal{V}_0$; in fact these become pointwise limits in $\mathcal{V}'_0$, for the composite with y of the evaluation $E_X\colon \mathcal{V}'_0 \longrightarrow \mathbf{Set}'$ is $\mathcal{V}_0(X,-)\colon \mathcal{V}_0 \longrightarrow \mathbf{Set} \longrightarrow \mathbf{Set}'$, which preserves all limits. It then follows from (2.1) that the end in $\mathcal{V}$ of $T\colon \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathcal{V}$, if it exists, coincides with the end in $\mathcal{V}'$ of $\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathcal{V} \longrightarrow \mathcal{V}'$; so that $\int_A T(A,A)$ is unambiguous for such a functor. In consequence, the definition (2.10) of $[\mathcal{A},\mathcal{B}](T,S)$ is also unambiguous: if it exists as an object of $\mathcal{V}$, it has this same value when $\mathcal{A}$ and $\mathcal{B}$ are thought of as $\mathcal{V}'$-categories.

This justifies the alternative view mentioned in 3.1 of the definition (3.1) of $\{F,G\}$ for $F\colon \mathcal{K} \longrightarrow \mathcal{V}$ and $G\colon \mathcal{K} \longrightarrow \mathcal{B}$: namely that $[\mathcal{K},\mathcal{V}]$ always exists, even if $\mathcal{K}$ is large, as a $\mathcal{V}'$-category for a suitable $\mathcal{V}'$, and that the only question is the representability of $B \mapsto [\mathcal{K},\mathcal{V}](F,\mathcal{B}(B,G-))\colon \mathcal{B}^{\mathrm{op}} \longrightarrow \mathcal{V}'$ by an object $\{F,G\}$ of $\mathcal{B}$.

However even more is true: it does not matter in these circumstances whether we perceive the *base* category as $\mathcal{V}$ or as $\mathcal{V}'$ – in the latter case seeing the indexing type as $yF\colon \mathcal{K} \longrightarrow \mathcal{V} \longrightarrow \mathcal{V}'$; for if $\{\ ,\ \}'$ denotes the indexed limit of a $\mathcal{V}'$-functor, we have

$$\{yF,G\}' \cong \{F,G\}, \tag{3.84}$$

either existing if the other does. (For once we are using y loosely to denote both the ordinary functor $\mathcal{V}_0 \longrightarrow \mathcal{V}'_0$ and the $\mathcal{V}'$-functor $\mathcal{V} \longrightarrow \mathcal{V}'$.) The point is that $[\mathcal{K},\mathcal{V}'](yF,y\mathcal{B}(B,G-))$, as the end in $\mathcal{V}'$ of $[FK,\mathcal{B}(B,GK)]'$, is by (3.81) the end in $\mathcal{V}'$ (and hence also in $\mathcal{V}$ if it exists there) of $[FK,\mathcal{B}(B,GK)]$. Of course it is equally true for colimits that $(yF) \star' G \cong F \star G$, for these are just limits in $\mathcal{B}^{\mathrm{op}}$.

3.12 The existence of a limit- and colimit-preserving universe-enlargement $\mathcal{V} \subset \mathcal{V}'$

On the other hand, $y\colon \mathcal{V}_0 \longrightarrow \mathcal{V}'_0$ does not preserve colimits; and we must ask how far this renders our formulae ambiguous under such a change of universe.

The first place where we used colimits in $\mathcal{V}_0$ was in the construction in 2.5 of the free $\mathcal{V}$-category $\mathcal{L}_{\mathcal{V}}$ on the locally-small ordinary category $\mathcal{L}$. This, seen as a $\mathcal{V}'$-category, is *not* the same as the free $\mathcal{V}'$-category $\mathcal{L}_{\mathcal{V}'}$ on $\mathcal{L}$. We applied this notion in 3.8 to exhibit a conical limit $\lim T$, where $T\colon \mathcal{L} \longrightarrow \mathcal{B}_0$, as an indexed limit $\{F,G\}$, with $\mathcal{L}_{\mathcal{V}}$ as the domain of F and G. When $\mathcal{L}$ is not $\mathbf{Set}$-locally-small, $\mathcal{L}_{\mathcal{V}}$ does not in general exist, but $\mathcal{L}_{\mathcal{V}'}$ does so for a suitable $\mathcal{V}'$; and this enables us to define $\lim T$ as the appropriate $\{F',G'\}'$, where F' and G' have domain $\mathcal{L}_{\mathcal{V}'}$ and F' has codomain $\mathcal{V}'$. The question arises whether the two definitions of $\lim T$ agree when $\mathcal{L}$ *is* locally small. The characterization (3.51) shows at once that they do: for the right side of (3.51) is an ordinary limit in $\mathcal{V}_0$ and hence in $\mathcal{V}'_0$.

The only other place where we have used colimits in $\mathcal{V}_0$ (except in the negative considerations of 3.9) is in the formula (3.22) expressing the "continuity" of a limit in its index – this meaning more precisely that *colimits* in $[\mathcal{A},\mathcal{V}]$ are sent to *limits* in $\mathcal{B}$. Note that the $\mathcal{V}'$-functor $y\colon \mathcal{V} \longrightarrow \mathcal{V}'$, although it preserves small tensor products by (3.77),

fails to preserve small conical colimits (since by (3.53) these are the ordinary colimits in $\mathcal{V}_0$ and $\mathcal{V}_0'$), and hence fails to preserve small indexed colimits; whence the same is true of $[\mathcal{A}, \mathcal{V}] \longrightarrow [\mathcal{A}, \mathcal{V}']$.

The argument given for (3.22) implicitly presupposes the existence of $[\mathcal{A}, \mathcal{V}]$ as a $\mathcal{V}$-category in which $F \star G$ is to live. However the result remains literally true even if $[\mathcal{K}^{\mathrm{op}}, \mathcal{V}]$ and $[\mathcal{A}, \mathcal{V}]$ exist only as $\mathcal{V}'$-categories, provided that we interpret the existence of $F \star G$ as meaning that we have an isomorphism $[\mathcal{A}, \mathcal{V}](F \star G, H) \cong [\mathcal{K}^{\mathrm{op}}, \mathcal{V}'](F, [\mathcal{A}, \mathcal{V}](G-, H))$ in $\mathcal{V}'$. For then we have

$$\begin{aligned}
\mathcal{B}(B, \{F, \{G-, T\}\}) &\cong [\mathcal{K}^{\mathrm{op}}, \mathcal{V}](F, \mathcal{B}(B, \{G-, T\})) \\
&\cong [\mathcal{K}^{\mathrm{op}}, \mathcal{V}](F, [\mathcal{A}, \mathcal{V}]((G-)?, \mathcal{B}(B, T?))) \\
&\cong [\mathcal{A}, \mathcal{V}]((F \star G)?, \mathcal{B}(B, T?)) \\
&\cong \mathcal{B}(B, \{F \star G, T\}).
\end{aligned}$$

We also have, however, the $\mathcal{V}'$-analogue of (3.22); *this is a different true statement.* For now F is replaced by the composite $F' = yF \colon \mathcal{K}^{\mathrm{op}} \longrightarrow \mathcal{V} \longrightarrow \mathcal{V}'$, and G by the composite $G' \colon \mathcal{K} \longrightarrow [\mathcal{A}, \mathcal{V}] \longrightarrow [\mathcal{A}, \mathcal{V}']$; so that $G'K$ is $y(GK)$. The right side of (3.22) in fact remains unchanged, by (3.84); but its new left side is $\{F' \star' G', T\}'$, while its old left side, by two applications of (3.84), is $\{F \star G, T\} \cong \{y(F \star G), T\}' \cong \{yF \star' G, T\}'$; and $yF \star' G = F' \star' G$ differs from $F' \star' G'$ because $[\mathcal{A}, \mathcal{V}] \longrightarrow [\mathcal{A}, \mathcal{V}']$ does not preserve colimits.

Thus, in a sense, the ambiguities that arise from the failure of $y \colon \mathcal{V} \longrightarrow \mathcal{V}'$ to preserve colimits are of but minor importance. Still, the notation would be that much more flexible if by modifying $\mathcal{V}'$ we could remove them; and more adapted to universal algebra, where colimits in $\mathcal{V}$ do play a central role. Indeed, in the *concrete* examples of 2.6, where $\mathcal{V}_0'$ is *not* $[\mathcal{V}_0^{\mathrm{op}}, \mathbf{Set}']$, but consists – in the case $\mathcal{V} = \mathbf{Cat}$ for example – of the category objects in $\mathbf{Set}'$, the inclusion $\mathcal{V}_0 \longrightarrow \mathcal{V}_0'$ does preserve all limits *and* all colimits that exist in $\mathcal{V}_0$. We now show that we can do nearly as well as this in the general case, replacing $\mathcal{V}_0' = [\mathcal{V}_0^{\mathrm{op}}, \mathbf{Set}']$ by a reflective full subcategory $\mathcal{V}_0''$ containing $\mathcal{V}_0$; it is then still the case, by 3.5, that $\mathcal{V}_0 \longrightarrow \mathcal{V}_0''$ preserves all the limits that exist in $\mathcal{V}_0$; and we can arrange for it to preserve, if not *all* colimits that exist in $\mathcal{V}_0$, at least those that are $\mathbf{Set}'$-small – and so *a fortiori* all small (= $\mathbf{Set}$-small) ones.

We just take $\mathcal{V}_0''$ to be the full subcategory of $\mathcal{V}_0' = [\mathcal{V}_0^{\mathrm{op}}, \mathbf{Set}']$ determined by those $F \colon \mathcal{V}_0^{\mathrm{op}} \longrightarrow \mathbf{Set}'$ which send all the $\mathbf{Set}'$-small colimits that exist in $\mathcal{V}_0$ – these being *limits* in $\mathcal{V}_0^{\mathrm{op}}$ – to limits in $\mathbf{Set}'$. By 3.2, the representables constituting $\mathcal{V}_0$ lie in $\mathcal{V}_0''$. That $\mathcal{V}_0 \longrightarrow \mathcal{V}_0''$ preserves the colimits in question is immediate; and that $\mathcal{V}_0''$ is reflective in $\mathcal{V}_0'$ follows from Kennison [50] or from Freyd-Kelly [30]; see also 6.3 below. Let the reflexion be $R \colon \mathcal{V}_0' \longrightarrow \mathcal{V}_0''$.

The first thing to observe is that

$$[G, H]' \in \mathcal{V}_0'' \textit{ if } G \in \mathcal{V}_0' \textit{ and } H \in \mathcal{V}_0''. \tag{3.85}$$

For $[G, H]' = \int_Y [GY, H(- \otimes Y)]$ by (3.80); but $- \otimes Y$ preserves all colimits, and in particular those in question; H sends these by hypothesis to limits in $\mathbf{Set}'$; the representable $[GY, -]$ preserves all limits; and so by (3.64) does $\int_Y$.

By the results of Day [14], this is enough to ensure that $\mathcal{V}_0''$ has a symmetric monoidal closed structure $\mathcal{V}'' = (\mathcal{V}_0'', \otimes'', I', [\ ,\ ]')$, where

$$F \otimes'' G = R(F \otimes' G). \tag{3.86}$$

This will give all that we want; the symmetric monoidal closed structure of $\mathcal{V}''$ will extend that of $\mathcal{V}$ by (3.81), (3.83), and (3.77) taken with the fact that $\mathcal{V}_0 \subset \mathcal{V}_0''$.

An adaptation of Day's argument to the present case is as follows; in fact we need (3.85) only for $G, H \in \mathcal{V}_0''$. Then for $F, G, H \in \mathcal{V}_0''$ we have from (3.86) that $\mathcal{V}_0''(F \otimes'' G, H) \cong \mathcal{V}_0'(F \otimes' G, H) \cong \mathcal{V}_0'(F, [G, H]')$, which by (3.85) is $\mathcal{V}_0''(F, [G, H]')$. Thus $- \otimes'' G \dashv [G, -]' \colon \mathcal{V}_0'' \longrightarrow \mathcal{V}_0''$.

There is an evident symmetry $F \otimes'' G \cong G \otimes'' F$; moreover $I' \otimes'' G = R(I' \otimes' G) \cong RG = G$ for $G \in \mathcal{V}_0''$. As for associativity, let $F, G, H \in \mathcal{V}_0''$. Then for any $K \in \mathcal{V}_0''$ we have $[H, K]' \in \mathcal{V}_0''$ by (3.85). Thus any map $F \otimes' G \longrightarrow [H, K]'$ factorizes uniquely through the reflexion $F \otimes' G \longrightarrow R(F \otimes' G) = F \otimes'' G$. It follows that any map $(F \otimes' G) \otimes' H \longrightarrow K$ factorizes uniquely through $(F \otimes' G) \otimes' H \longrightarrow (F \otimes'' G) \otimes' H$. Using (3.86) again, we conclude that $(F \otimes'' G) \otimes'' H \cong R((F \otimes' G) \otimes' H)$. So the associativity isomorphism for $\otimes'$ reflects onto one for $\otimes''$; and moreover all the coherence conditions reflect too.

Let us change notation now, and write $\mathcal{V}'$ for $\mathcal{V}''$; so that henceforth $\mathcal{V}'$ denotes an extension of $\mathcal{V}$ for which $\mathcal{V} \longrightarrow \mathcal{V}'$ preserves all limits that exist in $\mathcal{V}$ and also all colimits that exist and are $\mathbf{Set}'$-small. In future all functor categories too big to exist as $\mathcal{V}$-categories are to be interpreted as $\mathcal{V}'$-categories for a suitable such $\mathcal{V}'$; and similarly for limits and colimits in $\mathcal{V}$ or in some $[\mathcal{A}, \mathcal{V}]$.

Chapter 4

Kan extensions

4.1 The definition of Kan extensions; their expressibility by limits and colimits

Recall from (3.38) and (3.39) that a $\mathcal{V}$-functor $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ induces $\widehat{K}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow [\mathcal{A}, \mathcal{V}]$ and $\widetilde{K}\colon \mathcal{C} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$, given by $\widehat{K}C = \mathcal{C}(C, K-)$ and $\widetilde{K}C = \mathcal{C}(K-, C)$. Note that, for $F\colon \mathcal{C} \longrightarrow \mathcal{V}$ and $H\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{V}$, we have

$$F \star \widehat{K} \cong FK, \qquad H \star \widetilde{K} \cong HK^{\mathrm{op}}; \tag{4.1}$$

since by (3.16) we have $(F \star \widehat{K})A = F? \star \mathcal{C}(?, KA)$, which by the symmetry (3.9) and the Yoneda isomorphism (3.10) is FKA.

Given $\mathcal{V}$-functors and a $\mathcal{V}$-natural transformation

$$\begin{tikzcd} & \mathcal{C} \arrow[dr, "T"] \arrow[d, phantom, "\Swarrow \psi"] & \\ \mathcal{A} \arrow[ur, "K"] \arrow[rr, "G"'] & {} & \mathcal{B}, \end{tikzcd} \tag{4.2}$$

we get for each $F\colon \mathcal{C} \longrightarrow \mathcal{V}$ a map

$$[\mathcal{C}, \mathcal{V}]\big(F, \mathcal{B}(B, T-)\big) \longrightarrow [\mathcal{A}, \mathcal{V}]\big(FK, \mathcal{B}(B, TK-)\big) \longrightarrow [\mathcal{A}, \mathcal{V}]\big(FK, \mathcal{B}(B, G-)\big), \tag{4.3}$$

where the first arrow is $[K, 1]$ and the second is $[\mathcal{A}, \mathcal{V}]\big(1, \mathcal{B}(1, \psi-)\big)$. Hence by (3.1) we get an induced map

$$(K, \psi)_*\colon \{F, T\} \longrightarrow \{FK, G\} \tag{4.4}$$

whenever these limits exist. When $\mathcal{A} = \mathcal{C}$ and $K = 1$, this is just

$$\{1, \psi\}\colon \{F, T\} \longrightarrow \{F, G\};$$

we saw in (3.11) that $\{\ ,\ \}$ is a $\mathcal{V}$-functor in so far as it exists. When $G = TK$ and $\psi = 1$, we may write (4.4) as $K_*\colon \{F, T\} \longrightarrow \{FK, TK\}$. The general (4.4) is the composite

$$\{F, T\} \xrightarrow[K_*]{} \{FK, TK\} \xrightarrow[\{1,\psi\}]{} \{FK, G\} \tag{4.5}$$

if all these limits exist; but the outer ones may well exist when the central one does not.

Theorem 4.6 *For a diagram* (4.2)*, the following are equivalent:*

(i) *For each* $F\colon \mathcal{C} \longrightarrow \mathcal{V}$, $(K,\psi)_*\colon \{F,T\} \longrightarrow \{FK,G\}$ *is an isomorphism, either limit existing if the other does.*

(ii) *For each* $C \in \mathcal{C}$*, the limit* $\{\mathcal{C}(C,K-),G,\}$ *is given by* TC*, with as counit*

$$\mathcal{C}(C,K-) \xrightarrow[T]{} \mathcal{B}(TC,TK-) \xrightarrow[\mathcal{B}(1,\psi-)]{} \mathcal{B}(TC,G-). \tag{4.7}$$

(iii) *For each* $B \in \mathcal{B}$ *and* $C \in \mathcal{C}$*, the map*

$$\mathcal{B}(B,TC) \longrightarrow [\mathcal{A},\mathcal{V}]\big(\mathcal{C}(C,K-),\mathcal{B}(B,G-)\big) \tag{4.8}$$

induced by (4.7) *is an isomorphism.*

Proof. (i) gives (ii) on taking $F = \mathcal{C}(C,-)$ and using the Yoneda isomorphism (3.10); (ii) expresses (iii) by the definition (3.1); (iii) asserts by Yoneda that (4.3) is an isomorphism in the case $F = \mathcal{C}(C,-)$, whence (4.3) is an isomorphism for all F by (3.17), both sides of (4.3) sending pointwise colimits in the variable F to limits in $\mathcal{V}$; which then gives (i) by definition. ∎

When these conditions are satisfied, it follows from (ii) that T and ψ are uniquely determined to within isomorphism by K and G; we say that *the diagram* (4.2) (or *the map* ψ therein) *exhibits* T *as the right Kan extension of* G *along* K, and we write $T = \mathrm{Ran}_K G$, calling ψ the *counit* of this right Kan extension. Thus (ii) gives the formula

$$(\mathrm{Ran}_K G)C = \{\mathcal{C}(C,K-),G\} = \{\widehat{K}C,G\}, \tag{4.9}$$

and $\mathrm{Ran}_K G$ *exists*, by definition, when the limit on the right exists for each $C \in \mathcal{C}$: so that we may also write

$$\mathrm{Ran}_K G = \{\widehat{K}-,G\}. \tag{4.10}$$

(We may sometimes loosely express the existence of $\{\widehat{K}C,G\}$ for a *particular* C by saying that $(\mathrm{Ran}_K G)C$ exists.) In terms of the counit $\mu_C\colon \mathcal{C}(C,K-) \longrightarrow \mathcal{B}\big((\mathrm{Ran}_K G)C,G-\big)$ of the limit (4.9), the counit ψ of the right Kan extension is, by (4.7) and Yoneda in the form (1.48),

$$\psi_A = \mu_{KA,A}(1_{KA})\colon (\mathrm{Ran}_K G)KA \longrightarrow GA. \tag{4.11}$$

The equivalent defining property (i) of Theorem 4.6 may be recorded as

$$\{F,\mathrm{Ran}_K G\} \cong \{FK,G\} \quad \textit{for all } F\colon \mathcal{C} \longrightarrow \mathcal{V}; \tag{4.12}$$

this may also be seen as coming directly from (4.10), (3.22), and (4.1) – which gives an alternative proof that (iii) implies (i) above. Clearly

$$\mathrm{Ran}_K G, \textit{ where } G\colon \mathcal{A} \longrightarrow \mathcal{B}, \textit{ exists if } \mathcal{A} \textit{ is small and } \mathcal{B} \textit{ is complete.} \tag{4.13}$$

If (4.2) is a right Kan extension, a functor $P\colon \mathcal{B} \longrightarrow \mathcal{D}$ is said to *preserve* this right Kan extension if $P\psi$ exhibits PT as $\mathrm{Ran}_K PG$. Obviously:

Proposition 4.14 P *preserves the right Kan extension* $\mathrm{Ran}_K G$ *if and only if* P *preserves the limit* $\{\widehat{K}C,G\}$ *for all* C. ∎

In particular, therefore, *a right adjoint preserves any right Kan extension that exists*; and *the representables* $\mathcal{B}(B,-)\colon \mathcal{B}\longrightarrow\mathcal{V}$ *not only preserve* $\mathrm{Ran}_K G$ *but, in their totality, detect it* (which by (3.7) and (4.9) is a re-statement of Theorem 4.6(iii)).

For the dual notion we observe that a diagram

$$\begin{tikzcd} & \mathcal{C} \arrow[dr, "S"] \arrow[d, phantom, "\Nearrow\phi"] & \\ \mathcal{A} \arrow[ur, "K"] \arrow[rr, "G"'] & {} & \mathcal{B} \end{tikzcd} \tag{4.15}$$

induces, for any $H\colon \mathcal{C}^{\mathrm{op}}\longrightarrow\mathcal{V}$, a map

$$(K,\phi)^*\colon HK^{\mathrm{op}}\star G\longrightarrow H\star S; \tag{4.16}$$

we say that ϕ *exhibits* S *as the left Kan extension* $\mathrm{Lan}_K G$ *of* G *along* K if (4.16) is an isomorphism for each H whenever either side exists; equivalently, if ϕ induces an isomorphism of S with the functor

$$\mathrm{Lan}_K G = (\widetilde{K}-)\star G, \tag{4.17}$$

which exists when the colimit

$$(\mathrm{Lan}_K G)C = \widetilde{K}C\star G = \mathcal{C}(K-,C)\star G \tag{4.18}$$

does so for each C: as it surely does if $\mathcal{A}$ is small and $\mathcal{B}$ is cocomplete. Then (4.16) becomes

$$H\star\mathrm{Lan}_K G\cong HK^{\mathrm{op}}\star G \quad \textit{for all } H\colon\mathcal{C}^{\mathrm{op}}\longrightarrow\mathcal{V}. \tag{4.19}$$

Any left adjoint $Q\colon\mathcal{B}\longrightarrow\mathcal{D}$ preserves the left Kan extension $\mathrm{Lan}_K G$; and a representable $\mathcal{B}(-,B)\colon\mathcal{B}^{\mathrm{op}}\longrightarrow\mathcal{V}$ turns it into a right Kan extension in $\mathcal{V}$ – giving

$$\mathcal{B}\big((\mathrm{Lan}_K G)C,B\big)\cong[\mathcal{A}^{\mathrm{op}},\mathcal{V}]\big(\mathcal{C}(K-,C),\mathcal{B}(G-,B)\big) \tag{4.20}$$

as a further characterization of $\mathrm{Lan}_K G$. Since left Kan extensions occur more than right ones in our applications below, we shall give our further results in terms of these; of course the two notions are precisely dual, $(\ )^{\mathrm{op}}$ turning a left Kan extension (4.15) into a right one. For the genesis of these ideas, see Kan [38].

4.2 Elementary properties and examples

We continue with a collection of elementary results, formulae, and examples.

The accepted term "extension" is somewhat misleading here, for in the diagram

$$\begin{tikzcd} & \mathcal{C} \arrow[dr, "\mathrm{Lan}_K G"] \arrow[d, phantom, "\Nearrow\phi"] & \\ \mathcal{A} \arrow[ur, "K"] \arrow[rr, "G"'] & {} & \mathcal{B} \end{tikzcd} \tag{4.21}$$

it is not in general the case that $(\mathrm{Lan}_K G)K \cong G$, as (4.34) below shows. The fact is that $\widetilde{K}K\colon \mathcal{A} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ sends A to $\widetilde{K}KA = \mathcal{C}(K-, KA)$, and we have a canonical map $K_{-A}\colon \mathcal{A}(-, A) \longrightarrow \mathcal{C}(K-, KA)$, which is the A-component of a canonical map $\kappa\colon Y \longrightarrow \widetilde{K}K\colon \mathcal{A} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ where Y is the Yoneda embedding: and clearly

$$\kappa\colon Y \longrightarrow \widetilde{K}K \textit{ is an isomorphism if and only if } K \textit{ is fully faithful.} \tag{4.22}$$

We now have:

Proposition 4.23 *$\phi\colon G \longrightarrow (\mathrm{Lan}_K G)K$ is an isomorphism if K is fully faithful; while conversely K is fully faithful if ϕ is an isomorphism for all $\mathcal{B}$, G with $\mathrm{Lan}_K G$ existing.*

Proof. $(\mathrm{Lan}_K G)K = (\widetilde{K}K-) \star G$ by (4.17), while $G \cong (Y-) \star G$ by the Yoneda isomorphism (3.10); clearly $\phi\colon G \longrightarrow (\mathrm{Lan}_K G)K$ is $(\kappa-) \star G$; hence it is by (4.22) an isomorphism if K is fully faithful. For the converse take $\mathcal{B} = \mathcal{V}$ and $G = \mathcal{A}(A, -)$; then by (3.9) and (3.10) the isomorphism $\phi_A = (\kappa-) \star \mathcal{A}(A, -)$ is κ_A, so that κ is an isomorphism and K is fully faithful. ∎

When $\mathcal{B}$ has cotensor products [resp. tensor products] the formulae (4.10) and (4.17) have by (3.69) and (3.70) the more explicit forms

$$\mathrm{Ran}_K G \cong \textstyle\int_A \mathcal{C}(-, KA) \pitchfork GA, \tag{4.24}$$

$$\mathrm{Lan}_K G \cong \textstyle\int^A \mathcal{C}(KA, -) \otimes GA; \tag{4.25}$$

either side existing if the other does.

In the classical case $\mathcal{V} = \mathbf{Set}$ we can use (3.33) and (3.34) to express the Kan extensions by conical limits and colimits. Observe that $\mathrm{el}\,\mathcal{C}(C, K-)$, where $\mathcal{C}(C, K-)\colon \mathcal{A} \longrightarrow \mathbf{Set}$, is the comma-category C/K; an object is $(A \in \mathcal{A}, f\colon C \longrightarrow KA)$, while a map $(A, f) \longrightarrow (A', f')$ is some $g\colon A \longrightarrow A'$ with $Kg.f = f'$. Similarly $\mathcal{C}(K-, C)\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ has $\mathrm{el}\,\mathcal{C}(K-, C) = (K/C)^{\mathrm{op}}$. Hence (3.33) and (3.34) give

$$(\mathrm{Ran}_K G)C \cong \lim(C/K \xrightarrow[d]{} \mathcal{A} \xrightarrow[G]{} \mathcal{B}) \quad \textit{for } \mathcal{V} = \mathbf{Set}, \tag{4.26}$$

$$(\mathrm{Lan}_K G)C \cong \mathrm{colim}(K/C \xrightarrow[d]{} \mathcal{A} \xrightarrow[G]{} \mathcal{B}) \quad \textit{for } \mathcal{V} = \mathbf{Set}; \tag{4.27}$$

again either side existing if the other does.

Clearly (4.18) simplifies if $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ has a right adjoint L, for then $(\mathrm{Lan}_K G)C = \mathcal{C}(K-, C) \star G \cong \mathcal{A}(-, LC) \star G$, which is GLC by the Yoneda isomorphism (3.10). Thus

$$\mathrm{Lan}_K G \cong GL \quad \textit{if } K \dashv L. \tag{4.28}$$

Note that obviously

$$\textit{Any functor } Q\colon \mathcal{B} \longrightarrow \mathcal{D} \textit{ preserves the } \mathrm{Lan}_K G \textit{ of } (4.28). \tag{4.29}$$

We pass to some particular examples. Let $G\colon \mathcal{A} \longrightarrow \mathcal{B}$, and let $Y\colon \mathcal{A} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ be the Yoneda embedding. Since $\widetilde{Y}F = [\mathcal{A}^{\mathrm{op}}, \mathcal{V}](Y-, F)$ is isomorphic to F by the Yoneda isomorphism (2.31), we have

$$\widetilde{Y} \cong 1\colon [\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]. \tag{4.30}$$

Hence (4.17) gives

$$\mathrm{Lan}_Y G \cong - \star G \colon [\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \longrightarrow \mathcal{B}. \tag{4.31}$$

On the other hand, (4.17) also gives $\mathrm{Lan}_G Y = (\widetilde{G} - \star Y)$, so that by the Yoneda isomorphism (3.17), we have

$$\mathrm{Lan}_G Y \cong \widetilde{G} \colon \mathcal{B} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]. \tag{4.32}$$

We cannot quite say that (4.31) exhibits any colimit $F \star G$ as the value $(\mathrm{Lan}_Y G)F$ of a left Kan extension; for the left Kan extension $\mathrm{Lan}_Y G$ may not exist, even though $F \star G$ exists for a *particular* F. Yet we *can* always exhibit a colimit $F \star G$ as the value of a Kan extension. Identify $\mathcal{A}$ with its full image under $Y \colon \mathcal{A} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$, and take for $\mathcal{C}$ the full subcategory of $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ determined by all the $A \in \mathcal{A}$ and the given $F \colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{V}$; let $K \colon \mathcal{A} \longrightarrow \mathcal{C}$ be the inclusion. Then $\widetilde{K}K \cong Y$ by (4.22), while $\widetilde{K}F \cong F$ by Yoneda. Thus for any $G \colon \mathcal{A} \longrightarrow \mathcal{B}$ we have $(\mathrm{Lan}_K G)A \cong GA$ for $A \in \mathcal{A}$ (cf. Proposition 4.23), while $(\mathrm{Lan}_K G)F \cong F \star G$. This gives in particular a converse to (4.13):

Proposition 4.33 *$\mathcal{B}$ admits all left Kan extensions $\mathrm{Lan}_K G$, where $K \colon \mathcal{A} \longrightarrow \mathcal{C}$, $G \colon \mathcal{A} \longrightarrow \mathcal{B}$, and $\mathcal{A}$ is small, if and only if it is cocomplete.*

In theory, therefore, Kan extensions could replace indexed limits as the basic "limit" notion; we could use (4.8) to define Kan extensions without explicit mention of indexed limits, merely using ordinary limits in $\mathcal{V}_0$ to define the functor categories as in 2.2. In fact, as we see in 4.3 below, we can even define Kan extensions without first introducing functor categories: we could then recover these by defining indexed limits, and in particular ends, in terms of Kan extensions. However we find the order adopted in this account more natural.

A conical limit has a particularly simple expression as a Kan extension. If $\mathcal{V}$ is cartesian closed, for instance, $\operatorname{colim} G$ coincides with the left Kan extension

$$\begin{tikzcd} & 1 \arrow[dr, "\operatorname{colim} G"] \arrow[d, phantom, "\Nearrow \phi"] & \\ \mathcal{A} \arrow[ur, "K"] \arrow[rr, "G"'] & {} & \mathcal{B}; \end{tikzcd} \tag{4.34}$$

for clearly $\widetilde{K}0$ is here $\Delta 1 \colon \mathcal{A} \longrightarrow \mathcal{V}$, so that (4.18) gives $(\mathrm{Lan}_K G)0 = \Delta 1 \star G, \cong \operatorname{colim} G$ by (3.57). This illustrates our assertion preceding Proposition 4.23, that ϕ is not in general an isomorphism. (Note that (4.34) follows alternatively when $\mathcal{V} = \mathbf{Set}$ from (4.27), d then being an equivalence: but the argument we have used here generalizes to conical limits for any $\mathcal{V}$.)

For the purposes of a counter-example we shall use below, we look at a generalization of (4.34). Consider, in the case $\mathcal{V} = \mathbf{Set}$, the left Kan extension

$$\begin{tikzcd} & \mathcal{C} \arrow[dr, "\mathrm{Lan}_K G"] \arrow[d, phantom, "\Nearrow \phi"] & \\ \mathcal{L} \times \mathcal{C} \arrow[ur, "K"] \arrow[rr, "G"'] & {} & \mathcal{B}. \end{tikzcd} \tag{4.35}$$

where K is the projection. Here $\widetilde{K}C = \mathcal{C}(K-,C)\colon \mathcal{L}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is clearly $\Delta 1 \times \mathcal{C}(-,C)$; so that $(\mathrm{Lan}_K G)C = \big(\Delta 1 \times \mathcal{C}(-,C)\big) \star G$. By the Fubini theorem (3.21), this is $(\Delta 1)? \star \big(\mathcal{C}(-,C) \star G(?,-)\big)$, which is $(\Delta 1)? \star G(?,C)$ by the Yoneda (3.10); whence by (3.26)

$$(\mathrm{Lan}_K G)C \cong \operatorname{colim} G(-,C) \tag{4.36}$$

either side existing if the other does. Thus if G corresponds to $T\colon \mathcal{L} \longrightarrow [\mathcal{C},\mathcal{B}]$, we conclude that $\mathrm{Lan}_K G$ *exists in* (4.35) *exactly when* $\operatorname{colim} T$ *exists pointwise.*

4.3 A universal property of $\mathrm{Lan}_K G$; its inadequacy as a definition

The left Kan extension

$$\begin{tikzcd} & \mathcal{C} \arrow[dr, "\mathrm{Lan}_K G"] \arrow[d, phantom, "\Nearrow \phi"] & \\ \mathcal{A} \arrow[ur, "K"] \arrow[rr, "G"'] & {} & \mathcal{B} \end{tikzcd} \tag{4.37}$$

has an important universal property, different from the defining one in its three equivalent forms (4.18), (4.19), (4.20); namely:

Theorem 4.38 *For any $S\colon \mathcal{C} \longrightarrow \mathcal{B}$, we have a $\mathcal{V}$-natural isomorphism*

$$[\mathcal{C},\mathcal{B}](\mathrm{Lan}_K G, S) \cong [\mathcal{A},\mathcal{B}](G, SK); \tag{4.39}$$

the unit of this representation being the unit ϕ of (4.37)*. The dual result is*

$$[\mathcal{C},\mathcal{B}](S, \mathrm{Ran}_K G) \cong [\mathcal{A},\mathcal{B}](SK, G). \tag{4.40}$$

Proof.

$$\begin{aligned}
[\mathcal{C},\mathcal{B}](\mathrm{Lan}_K G, S) &= \textstyle\int_C \mathcal{B}\big((\mathrm{Lan}_K G)C, SC\big) && \text{by (2.10)}\\
&\cong \textstyle\int_C [\mathcal{A}^{\mathrm{op}},\mathcal{V}]\big(\mathcal{C}(K-,C), \mathcal{B}(G-,SC)\big) && \text{by (4.20)}\\
&\cong \textstyle\int_C \int_A [\mathcal{C}(KA,C), \mathcal{B}(GA,SC)] && \text{by (2.10)}\\
&\cong \textstyle\int_A \int_C [\mathcal{C}(KA,C), \mathcal{B}(GA,SC)] && \text{by (2.9)}\\
&\cong \textstyle\int_A \mathcal{B}(GA, SKA) && \text{by (2.33)}\\
&\cong [\mathcal{A},\mathcal{B}](G, SK) && \text{by (2.10).}
\end{aligned}$$

Putting $S = \mathrm{Lan}_K G$ on the left side and evaluating at 1 to determine the unit, we find this to be ϕ, using the duals of (4.7) and (4.8). ∎

Note that the left side of (4.39) exists in $\mathcal{V}$ (and not merely in some extension $\mathcal{V}'$) if the right side does; in particular

$$[\mathcal{C},\mathcal{B}](\mathrm{Lan}_K G, S) \in \mathcal{V} \textit{ if the domain } \mathcal{A} \textit{ of } K \textit{ is small.} \tag{4.41}$$

Applying $V = \mathcal{V}_0(I,-)\colon \mathcal{V}_0 \longrightarrow \mathbf{Set}$ to (4.39), we get a bijection of sets

$$[\mathcal{C},\mathcal{B}]_0(\mathrm{Lan}_K G, S) \cong [\mathcal{A},\mathcal{B}]_0(G, SK), \tag{4.42}$$

which in view of (2.16) may be expressed as:

Theorem 4.43 *If* $\phi\colon G \longrightarrow LK$ *exhibits* L *as* $\mathrm{Lan}_K G$*, then every* $\mathcal{V}$*-natural* $\alpha\colon G \longrightarrow SK$ *factorizes as*

$$\begin{tikzcd} & \mathcal{C} \arrow[dr, bend left, "S"] \arrow[dr, bend right, "L"'] \arrow[d, phantom, "\Swarrow\beta"] & \\ \mathcal{A} \arrow[ur, "K"] \arrow[rr, "G"'] \arrow[ur, phantom, shift right=3ex, "\Swarrow\phi"] & & \mathcal{B} \end{tikzcd} \tag{4.44}$$

for a unique $\beta\colon L \longrightarrow S$*.* ∎

The universal property of Theorem 4.43 can be satisfied by at most one (L,ϕ) to within isomorphism; for it is a question of a representation $[\mathcal{C},\mathcal{B}]_0(L,S) \cong [\mathcal{A},\mathcal{B}]_0(G,SK)$. It is moreover a very simple universal property, that makes sense not only in $\mathcal{V}$-**CAT** but in any 2-category. This has led many authors to use it as the *definition* of left Kan extension. Let us call an (L,ϕ) with this universal property a *weak left Kan extension of* G *along* K, writing $L = \mathrm{lan}_K G$ with a small "l" (and $\mathrm{ran}_K G$ for the dual). We now compare this with our definition. Theorem 4.43 asserts that $\mathrm{Lan}_K G = \mathrm{lan}_K G$ if the former exists. However $L = \mathrm{lan}_K G$ need not satisfy

$$[\mathcal{C},\mathcal{B}](L,S) \cong [\mathcal{A},\mathcal{B}](G,SK), \tag{4.45}$$

and hence by (4.39) need not be $\mathrm{Lan}_K G$. Take $\mathcal{V} = \mathbf{Cat}$, with $\mathcal{A}$ a mere category and $\mathcal{C} = 1$. The universal property of $L = \mathrm{lan}_K G$ clearly asserts that L is the colimit of G in $\mathcal{B}_0$, while (4.45) asserts that L is the colimit of G in $\mathcal{B}$; yet by (3.54) the first of these does not imply the second. (In this particular case, by (4.34), an L satisfying (4.45) is necessarily $\mathrm{Lan}_K G$.) Of course $L = \mathrm{lan}_K G$ *does* satisfy (4.45) if V is conservative; it also does so if $\mathcal{B}$ is cotensored, by an argument like that used in 3.8 when comparing conical limits in $\mathcal{B}_0$ with those in $\mathcal{B}$. Yet even an (L,ϕ) satisfying (4.45) *still* need not be $\mathrm{Lan}_K G$; in other words, the isomorphism (4.39) is strictly weaker than our definition of $\mathrm{Lan}_K G$. This is so even in the classical case $\mathcal{V} = \mathbf{Set}$; a counter-example is provided by (4.35), where $\mathrm{Lan}_K G$ exists by (4.36) precisely when $T\colon \mathcal{L} \longrightarrow [\mathcal{C},\mathcal{B}]$ has a *pointwise* colimit, while (4.45) merely asserts that $L \cong \operatorname{colim} T$, which by 3.3 does not imply pointwiseness.

For such reasons, various authors [22, 60, 9] who call $\mathrm{lan}_K G$ a Kan extension use the name *pointwise* Kan extension for $\mathrm{Lan}_K G$. Our present choice of nomenclature is based on our failure to find a single instance where a weak Kan extension plays any mathematical role whatsoever. (Street in [71] has given a definition of "pointwise Kan extension" in

any 2-category with finite limits; it agrees with our notion when $\mathcal{V} = \mathbf{Set}$, but is strictly stronger for a general $\mathcal{V}$, and is hence not suited to our context.)

When $\mathcal{B}$ is cotensored and $L = \mathrm{lan}_K G$ exists, we have observed that L satisfies (4.45). If in this we now take for S the functor $\mathcal{C}(-, C) \pitchfork B$ for fixed $C \in \mathcal{C}$ and $B \in \mathcal{B}$, we easily get $\mathcal{B}(LC, B) \cong [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]\big(\mathcal{C}(K-, C), \mathcal{B}(G-, B)\big)$, so that $L = \mathrm{Lan}_K G$ by (4.20). Thus for cotensored $\mathcal{B}$ there is no distinction between $\mathrm{lan}_K G$ and $\mathrm{Lan}_K G$. This is so in particular for $\mathcal{B} = \mathcal{V}$; moreover, since $\mathcal{V}$ is also tensored, $\mathrm{ran}_K G$ and $\mathrm{Ran}_K G$ too have the same meaning for $\mathcal{B} = \mathcal{V}$. This gives a way of defining $\mathrm{Ran}_K G$, for any $\mathcal{B}$, in terms of weak Kan extensions:

Proposition 4.46 *The diagram*

$$\begin{tikzcd} & \mathcal{C} \arrow[dr, "T"] \arrow[d, phantom, "\Swarrow \psi"] & \\ \mathcal{A} \arrow[ur, "K"] \arrow[rr, "G"'] & {} & \mathcal{B}, \end{tikzcd}$$

exhibits T as $\mathrm{Ran}_K G$ if and only if, for each $B \in \mathcal{B}$, the 2-cell $\mathcal{B}(B, \psi)$ exhibits $\mathcal{B}(B, T-)$ as $\mathrm{ran}_K \mathcal{B}(B, G-)$.

Proof. "Only if" is clear since representables preserve $\mathrm{Ran}_K G$. For the converse, $\mathcal{B}(B, T-)$ is actually $\mathrm{Ran}_K \mathcal{B}(B, G-)$ by the remark preceding the proposition; so that (4.8) is an isomorphism by (4.9) and (3.7), giving $T = \mathrm{Ran}_K G$. ∎

It is this way of defining $\mathrm{Ran}_K G$, in terms of the universal property of ran, that is envisaged in our remarks following Proposition 4.33.

Before leaving this comparison of Lan with lan, we should perhaps emphasize that, even when V is conservative, $\mathrm{Lan}_K G$ (or $\mathrm{lan}_K G$) has meaning heavily dependent on $\mathcal{V}$. Thus, when $\mathcal{V} = \mathbf{Ab}$, $\mathrm{Lan}_K G$ is quite different from $\mathrm{Lan}_{K_0} G_0$. The former is an *additive* functor, and satisfies the universal property of Theorem 4.43 among all *additive* functors; the latter is in general not additive at all.

4.4 Iterated Kan extensions; Kan adjoints; $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ as the free cocompletion of a small $\mathcal{A}$

We continue with further elementary properties of Kan extensions which, in one way or another, make use of Theorem 4.38 or Theorem 4.43.

Theorem 4.47 *If $\mathrm{Lan}_K G$ exists we have*

$$\mathrm{Lan}_Z \mathrm{Lan}_K G \cong \mathrm{Lan}_{ZK} G, \tag{4.48}$$

either side existing if the other does. Then if ϕ and θ are the units of $\mathrm{Lan}_K G$ and $\mathrm{Lan}_Z \mathrm{Lan}_K G$, the unit χ of $\mathrm{Lan}_{ZK} G$ is the composite natural transformation in the di-

agram

$$\begin{tikzcd}
 & & \mathcal{D} \arrow[dd, "M"] \\
 & \mathcal{C} \arrow[ur, "Z"] \arrow[dr, "L"] \arrow[u, phantom, "\Nearrow\theta"] & \\
\mathcal{A} \arrow[ur, "K"] \arrow[rr, "G"'] \arrow[u, phantom, "\Nearrow\phi"] & & \mathcal{B},
\end{tikzcd}$$

where L denotes $\mathrm{Lan}_K G$ and M the common value in (4.48)*. If Z is fully faithful, $\mathrm{Lan}_K G$ exists whenever $\mathrm{Lan}_{ZK} G$ exists, being $(\mathrm{Lan}_{ZK} G)Z$.*

Proof. (4.19) gives $\widetilde{Z}D \star \mathrm{Lan}_K G \cong (\widetilde{Z}D.K^{\mathrm{op}}) \star G$, either side existing if the other does. Since $\widetilde{Z}D.K^{\mathrm{op}} = \mathcal{D}(ZK-, D)$, this is by (4.18) the desired isomorphism (4.48). That the units compose as shown is immediate from Theorem 4.43. The last statement comes from the easy observation that

$$\widetilde{ZKZ} \cong \widetilde{K} \text{ if } Z \text{ is fully faithful.} \tag{4.49}$$

■

If $\mathcal{A}$ and $\mathcal{C}$ are small, both $[\mathcal{A}, \mathcal{B}]$ and $[\mathcal{C}, \mathcal{B}]$ exist as $\mathcal{V}$-categories; and any $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ induces a $\mathcal{V}$-functor $[K, 1]\colon [\mathcal{C}, \mathcal{B}] \longrightarrow [\mathcal{A}, \mathcal{B}]$ as in (2.26), sending S to SK. Then from Theorem 4.38 we get the *theorem of Kan adjoints:*

Theorem 4.50 *Let $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ be a $\mathcal{V}$-functor with $\mathcal{A}$ and $\mathcal{C}$ small. Then the functor $[K, 1]\colon [\mathcal{C}, \mathcal{B}] \longrightarrow [\mathcal{A}, \mathcal{B}]$ has the left adjoint Lan_K if $\mathrm{Lan}_K G$ exists for all $G\colon \mathcal{A} \longrightarrow \mathcal{B}$, as it surely does when $\mathcal{B}$ is cocomplete; and $[K, 1]$ has the right adjoint Ran_K if $\mathrm{Ran}_K G$ exists for each G, as it surely does when $\mathcal{B}$ is complete.* ■

We now consider the sense in which $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$, for a small $\mathcal{A}$, is the *free cocompletion of* $\mathcal{A}$ (embedded in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ by Yoneda).

Theorem 4.51 *Let $Y\colon \mathcal{A} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ be the Yoneda embedding, where $\mathcal{A}$ is small, and let $\mathcal{B}$ be arbitrary. Then for any cocontinuous $S\colon [\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \longrightarrow \mathcal{B}$ we have*

$$S \cong \mathrm{Lan}_Y G \cong - \star G\colon [\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \longrightarrow \mathcal{B}, \tag{4.52}$$

where

$$G = SY\colon \mathcal{A} \longrightarrow \mathcal{B}. \tag{4.53}$$

Such a cocontinuous S has a right adjoint T given by

$$T = \widetilde{G}\colon \mathcal{B} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]. \tag{4.54}$$

The full subcategory $\mathrm{Cocts}\big[[\mathcal{A}^{\mathrm{op}}, \mathcal{V}], \mathcal{B}\big]$ of $\big[[\mathcal{A}^{\mathrm{op}}, \mathcal{V}], \mathcal{B}\big]$ determined by the cocontinuous functors exists as a $\mathcal{V}$-category, and $S \mapsto SY$ is an equivalence of $\mathcal{V}$-categories

$$[Y, 1]\colon \mathrm{Cocts}\big[[\mathcal{A}^{\mathrm{op}}, \mathcal{V}], \mathcal{B}\big] \simeq [\mathcal{A}, \mathcal{B}]', \tag{4.55}$$

where $[\mathcal{A},\mathcal{B}]'$ *is the full subcategory of* $[\mathcal{A},\mathcal{B}]$ *determined by those* $G\colon \mathcal{A}\longrightarrow\mathcal{B}$ *for which* $\mathrm{Lan}_Y G$ *exists; the inverse to this equivalence sends* G *to* $\mathrm{Lan}_Y G$*. In particular, if* $\mathcal{B}$ *is cocomplete, this is an equivalence*

$$\mathrm{Cocts}\big[[\mathcal{A}^{\mathrm{op}},\mathcal{V}],\mathcal{B}\big]\simeq[\mathcal{A},\mathcal{B}]. \tag{4.56}$$

Proof. Expressing an arbitrary $F\colon\mathcal{A}^{\mathrm{op}}\longrightarrow\mathcal{V}$ as a small colimit $F\cong F\star Y$ of representables by (3.17), we see that a cocontinuous S must have $SF\cong F\star SY$, which is $F\star G$ by (4.53), or $(\mathrm{Lan}_Y G)F$ by (4.31). We then have $\mathcal{B}(SF,B)\cong\mathcal{B}(F\star G,B)\cong[\mathcal{A}^{\mathrm{op}},\mathcal{V}](F,\widetilde{G}B)$ by (3.5), so that $S\dashv\widetilde{G}$.

Since the cocontinuous S is $\mathrm{Lan}_Y G$, it follows from (4.41) that $\mathrm{Cocts}\big[[\mathcal{A}^{\mathrm{op}},\mathcal{V}],\mathcal{B}\big]$ exists as a $\mathcal{V}$-category, and from (4.39) that

$$\mathrm{Cocts}\big[[\mathcal{A}^{\mathrm{op}},\mathcal{V}],\mathcal{B}\big](S,S')\cong\big[[\mathcal{A}^{\mathrm{op}},\mathcal{V}],\mathcal{B}\big](\mathrm{Lan}_Y G,S')\cong[\mathcal{A},\mathcal{B}](G,S'Y)=[\mathcal{A},\mathcal{B}](SY,S'Y).$$

Thus $[Y,1]$ in (4.55) is fully faithful. It is moreover essentially surjective, since if $\mathrm{Lan}_Y G=-\star G$ exists, it is cocontinuous by (3.23), and $(\mathrm{Lan}_Y G)Y\cong G$ by Proposition 4.23 (or equally by (3.10)). ∎

4.5 Initial diagrams as the left Kan extensions into $\mathcal{V}$; initial and final functors when $\mathcal{V}=\mathbf{Set}$

As the motivation for our definition of right Kan extension, we chose in 4.1 the property (4.12) which allows us to write a limit $\{FK,G\}$ as $\{F,\mathrm{Ran}_K G\}$. We now examine the corresponding problem of writing a limit $\{H,TK\}$ in the form $\{F,T\}$.

First we need a result resembling (4.12) or (4.19), but where the Kan extension has codomain $\mathcal{V}$ and the other functor need not:

Proposition 4.57 *Let* $\mathrm{Lan}_K H$ *exist, where* $K\colon\mathcal{A}\longrightarrow\mathcal{C}$ *and* $H\colon\mathcal{A}\longrightarrow\mathcal{V}$*; then for any* $T\colon\mathcal{C}\longrightarrow\mathcal{B}$ *we have*

$$\{\mathrm{Lan}_K H,T\}\cong\{H,TK\}\ , \tag{4.58}$$

either side existing if the other does.

Proof. The formula (3.22) expressing the "continuity of a limit in its index" here gives $\{H?\star\mathcal{C}(K?,-),T-\}\cong\{H?,\{\mathcal{C}(K?,-),T-\}\}$; but $H?\star\mathcal{C}(K-,?)$ is $\mathrm{Lan}_K H$ by (4.18) and the symmetry (3.9), while $\{\mathcal{C}(K-,?),T-\}\cong TK$ by the Yoneda isomorphism (3.10). ∎

Given now a diagram

$$\begin{tikzcd} & \mathcal{C} \arrow[dr, "F"] \arrow[d, phantom, "\Nearrow\phi"] & \\ \mathcal{A} \arrow[ur, "K"] \arrow[rr, "H"'] & {} & \mathcal{V}, \end{tikzcd} \tag{4.59}$$

we get for each $T\colon \mathcal{C} \longrightarrow \mathcal{B}$ a map

$$[\mathcal{C},\mathcal{V}](F,\mathcal{B}(B,T-)) \longrightarrow [\mathcal{A},\mathcal{V}](FK,\mathcal{B}(B,TK-)) \longrightarrow [\mathcal{A},\mathcal{V}](H,\mathcal{B}(B,TK-)), \tag{4.60}$$

where the first arrow is $[K,1]$ and the second is $[\mathcal{A},\mathcal{V}](\phi,1)$. This induces by (3.1) a map

$$(K,\phi)^{\#}\colon \{F,T\} \longrightarrow \{H,TK\} \tag{4.61}$$

whenever these limits exist; it is the composite

$$\{F,T\} \xrightarrow[K_*]{} \{FK,TK\} \xrightarrow[\{\phi,1\}]{} \{H,TK\} \tag{4.62}$$

whenever the middle limit exists as well.

Theorem 4.63 *For a diagram* (4.59)*, the following are equivalent:*

(i) *For every $\mathcal{B}$ and every $T\colon \mathcal{C} \longrightarrow \mathcal{B}$, the $(K,\phi)^{\#}$ of* (4.61) *is an isomorphism, either side existing if the other does.*

(ii) *For every $\mathcal{B}$, every $T\colon \mathcal{C} \longrightarrow \mathcal{B}$, and every $B \in \mathcal{B}$, the map* (4.60) *is an isomorphism.*

(iii) *ϕ exhibits F in* (4.59) *as* $\mathrm{Lan}_K H$.

Proof. Clearly (ii) implies (i) by the definition (3.1) of the limits, while (ii) itself is, by (3.7), the special case of (i) obtained by replacing T in (4.61) by $\mathcal{B}(B,T-)$. Since (iii) implies (i) by Proposition 4.57, it remains only to prove that (i) implies (iii).

Let $C \in \mathcal{C}$ and $Z \in \mathcal{V}$. Then applying (i) with $\mathcal{B} = \mathcal{V}$ and with $T = [\mathcal{C}(-,C),Z]$ gives $\{F,[\mathcal{C}(-,C),Z]\} \cong \{H,[\mathcal{C}(K-,C),Z]\}$. Since the internal-hom $[X,Z]$ in $\mathcal{V}$ is the cotensor product $X \pitchfork Z$, the interchange-of-limits formula (3.20) lets us write the last isomorphism as $\{\mathcal{C}(-,C),[F-,Z]\} \cong \{\mathcal{C}(K-,C),[H-,Z]\}$. By the Yoneda isomorphism (3.10), the left side here is $[FC,Z]$; while by (3.7) the right side is $[\mathcal{A}^{\mathrm{op}},\mathcal{V}](\mathcal{C}(K-,C),\mathcal{V}(H-,Z))$. It now follows from (4.20) that F, with unit ϕ, is $\mathrm{Lan}_K H$. ∎

When these equivalent conditions are satisfied, we say that the pair (K,ϕ) in (4.59) is *initial*. The characterization (i) of Theorem 4.63 is the assertion that *the F-indexed limit of any T is the H-indexed limit of TK, when either exists*. The characterization (ii) is a $\mathcal{V}$-enriched version of the assertion that, whether the limits exist or not, *the (F,B)-cylinders over T coincide with the (H,B)-cylinders over TK*. From the proof, it is immaterial whether we require this only when the domain and codomain of (4.60) exist in $\mathcal{V}$, or when they exist merely in an extension $\mathcal{V}'$ as in 3.12.

For the dual notion we keep the diagram (4.59) as it is, but consider in place of T a functor $S\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{B}$. Then (4.60) is replaced by a map

$$[\mathcal{C},\mathcal{V}](F,\mathcal{B}(S-,B)) \longrightarrow [\mathcal{A},\mathcal{V}](H,\mathcal{B}(SK-,B)), \tag{4.64}$$

and (4.61) by a map

$$(K,\phi)_{\#}\colon H \star SK^{\mathrm{op}} \longrightarrow F \star S. \tag{4.65}$$

We call $(K^{\mathrm{op}}, \phi^{\mathrm{op}})$, in the diagram formed by applying $(\ )^{\mathrm{op}}$ to (4.59), *final* if (K, ϕ) is initial; that is, if (4.64) is always an isomorphism, or if (4.65) is an isomorphism whenever either side exists, or if $F \cong \mathrm{Lan}_K H$, or if $F^{\mathrm{op}} \cong \mathrm{Ran}_{K^{\mathrm{op}}} H^{\mathrm{op}}$.

For a cartesian closed $\mathcal{V}$, and in particular when $\mathcal{V} = \mathbf{Set}$, we can consider the special case

$$\begin{tikzcd} & \mathcal{C} \arrow[dr, "\Delta 1"] & \\ \mathcal{A} \arrow[ur, "K"] \arrow[rr, "\Delta 1"'] & & \mathcal{V}, \end{tikzcd} \tag{4.66}$$

of (4.59); here the unique 2-cell ϕ that can be inserted in the diagram is the identity *id*. We call the functor K *initial*, and the functor K^{op} *final*, if the pair (K, id) in (4.66) is initial.

Theorem 4.67 *For a cartesian closed $\mathcal{V}$ and a $\mathcal{V}$-functor $K\colon \mathcal{A} \longrightarrow \mathcal{C}$, the following are each equivalent to the initialness of K:*

(i) *For any $T\colon \mathcal{C} \longrightarrow \mathcal{B}$ we have $\lim T \cong \lim TK$, either side existing if the other does.*

(ii) *For such a T and any $B \in \mathcal{B}$ we have $[\mathcal{C}, \mathcal{B}](\Delta B, T) \cong [\mathcal{A}, \mathcal{B}](\Delta B, TK)$.*

(iii) *The diagram* (4.66) *exhibits $\Delta 1$ as $\mathrm{Lan}_K \Delta 1$.*

(iv) *For each $C \in \mathcal{C}$ we have $\mathrm{colim}\, \mathcal{C}(K-, C) = 1$.*

When $\mathcal{V} = \mathbf{Set}$, (ii) *here is just a bijection between the two sets of cones; and* (iv) *becomes*

(v) *For each C, the comma-category K/C is connected.*

Proof. (i) here is by (3.57) the (i) of Theorem 4.63; (ii) is the (ii) of that theorem since, for a cartesian closed $\mathcal{V}$, we have as in 3.9 that $[\mathcal{C}, \mathcal{V}](\Delta 1, \mathcal{B}(B, T-)) \cong [\mathcal{C}, \mathcal{B}](\Delta B, T)$; and (iii) is the (iii) of that theorem. Calculating $(\mathrm{Lan}_K \Delta 1)C$ by (4.18) and using the dual of (3.57) gives (iv); which when $\mathcal{V} = \mathbf{Set}$ becomes (v), since (3.35) then gives $\mathrm{colim}\, \mathcal{C}(K-, C) = \pi(\mathrm{el}\, \mathcal{C}(K-, C)) = \pi(K/C)$. ∎

If $\mathcal{A} = 1$ in the above theorem, in the case $\mathcal{V} = \mathbf{Set}$, K is just an object of $\mathcal{C}$; and the comma-category K/C is just the set $\mathcal{C}(K, C)$ regarded as a discrete category. Hence (v) gives:

Proposition 4.68 *When $\mathcal{V} = \mathbf{Set}$, the functor $K\colon 1 \longrightarrow \mathcal{C}$ is initial [resp. final] exactly when K is an initial [resp. final] object of $\mathcal{C}$.*

By (4.28), if $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ has a right adjoint L, we have $\mathrm{Lan}_K H \cong HL$ for any $H\colon \mathcal{A} \longrightarrow \mathcal{V}$. If $\mathcal{V}$ is cartesian closed and $H = \Delta 1$, then $HL = \Delta 1$; so that Theorem 4.67(iii) gives

Proposition 4.69 *For cartesian closed $\mathcal{V}$, any left adjoint $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ is initial.*

4.6 Filtered categories when $\mathcal{V} = \mathbf{Set}$; the commutativity in $\mathbf{Set}$ of filtered colimits with finite limits

Taking $\mathcal{V} = \mathbf{Set}$ for this section, we recall for later use the notion of *filtered* category. There is an interplay between *filtered* and *finite*: an ordinary category $\mathcal{P}$ is *finite* if its set of morphisms (and hence its set of objects) is finite. A category is *finitely complete* if it admits all (conical) limits indexed by finite categories; and a functor with a finitely-complete domain is *left exact* if it preserves all such finite limits: the duals are *finitely cocomplete* and *right exact*.

Theorem 4.70 *For an ordinary category $\mathcal{C}$, the following are equivalent:*

(i) *Every functor $T\colon \mathcal{P} \longrightarrow \mathcal{C}$ with $\mathcal{P}$ finite is the base of some inductive cone.*

(ii) *$\mathcal{C}$ is not empty; for any $A, B \in \mathcal{C}$ there is some $C \in \mathcal{C}$ and maps $A \longrightarrow C$ and $B \longrightarrow C$; and for any maps $f, g\colon A \longrightarrow B$ in $\mathcal{C}$ there is some map $h\colon B \longrightarrow D$ with $hf = hg$.*

(iii) *For any finite category $\mathcal{P}$ the diagonal $\Delta\colon \mathcal{C} \longrightarrow [\mathcal{P}, \mathcal{C}]$ has the property that T/Δ is non-empty for each $T\colon \mathcal{P} \longrightarrow \mathcal{C}$.*

(iv) *For any finite category $\mathcal{P}$ the diagonal $\Delta\colon \mathcal{C} \longrightarrow [\mathcal{P}, \mathcal{C}]$ is final.*

Proof. The properties in (ii) are special cases of (i), and in turn imply (i) by a trivial induction; while (iii) merely re-states (i). It remains to show that these imply (iv), which by Theorem 4.67(v) is the requirement that the T/Δ in (iii) be not merely non-empty but in fact connected. Given objects α and β of T/Δ – that is, cones $(\alpha_P\colon TP \longrightarrow A)$ and $(\beta_P\colon TP \longrightarrow B)$ – we can find by (ii) maps $f\colon A \longrightarrow C$ and $g\colon B \longrightarrow C$ such that $f\alpha_P = g\beta_P$ for each $P \in \mathcal{P}$. This common value is now a cone $(\gamma_P\colon TP \longrightarrow C)$; and we have maps $f\colon \alpha \longrightarrow \gamma$ and $g\colon \beta \longrightarrow \gamma$ in T/Δ. ∎

A category $\mathcal{C}$ satisfying these equivalent conditions is said to be *filtered*. It is clear that *a filtered category is connected*, and also that *$[\mathcal{Q}, \mathcal{C}]$ is filtered if $\mathcal{Q}$ is finite and $\mathcal{C}$ is filtered*. Observe that *a finitely cocomplete category is filtered*, since $T\colon \mathcal{P} \longrightarrow \mathcal{C}$ is then the base of its colimit-cone; in this case, of course, $\Delta\colon \mathcal{C} \longrightarrow [\mathcal{P}, \mathcal{C}]$ has a left adjoint, which by Proposition 4.69 is stronger than being final. The initialness condition of Theorem 4.67(v), or rather its dual, simplifies when $\mathcal{C}$ is filtered and K is fully faithful:

Proposition 4.71 *If $\mathcal{C}$ is filtered, the fully-faithful $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ is final if and only if each C/K is non-empty.*

Proof. Only the "if" clause needs proof. Consider objects $f\colon C \longrightarrow KA$ and $g\colon C \longrightarrow KB$ of C/K. Because $\mathcal{C}$ is filtered, there are maps $h\colon KA \longrightarrow D$ and $k\colon KB \longrightarrow D$ with $hf = kg$; because D/K is non-empty we may suppose D to be of the form KE; because K is fully faithful, h and k have the form Ku and Kv. Thus f and g are connected in C/K by the maps $u\colon f \longrightarrow hf$ and $v\colon g \longrightarrow kg = hf$. ∎

The classical example of this is that where $\mathcal{C}$ is a filtered preordered set (in which case *directed* is often used for *filtered*), and K is the inclusion of a subset $\mathcal{A}$. To say that each C/K is non-empty is then to say that each $C \in \mathcal{C}$ is less than or equal to some $A \in \mathcal{A}$. Such a subset $\mathcal{A}$, which we call *final*, was classically called *cofinal*.

A *filtered colimit* is a conical one indexed by a *small* filtered category. For example, suppose we have in any category a coproduct $A = \sum_{x \in X} A_x$. The finite subsets J of X, ordered by inclusion, form a filtered ordered set $\mathcal{C}$; and if the finite coproduct $A_J = \sum_{x \in J} A_x$ exists for each $J \in \mathcal{C}$, it is clear that A is the filtered colimit of the A_J.

Recall from (3.35) that $\operatorname{colim} F$ for $F\colon \mathcal{C} \longrightarrow \mathbf{Set}$ is $\pi(\operatorname{el} F)$.

Theorem 4.72 *For $F\colon \mathcal{C} \longrightarrow \mathbf{Set}$ where $\mathcal{C}$ is small and filtered, two objects (A, x) and (B, y) of $\operatorname{el} F$ have the same image in $\pi(\operatorname{el} F) = \operatorname{colim} F$ if and only if there are maps $f\colon A \longrightarrow C$ and $g\colon B \longrightarrow C$ with $(Ff)x = (Fg)y$.*

Proof. The relation between (A, x) and (B, y) expressed by the existence of such f and g is reflexive and symmetric, and clearly generates the equivalence relation of being connected. It therefore coincides with this latter if it is transitive; and that it is so follows easily from the filteredness. ∎

If $\mathcal{B}$ is an equational category of algebras whose operations all have finite arity, the conservative forgetful functor $U\colon \mathcal{B} \longrightarrow \mathbf{Set}$ preserves (and therefore reflects) filtered colimits. For if $F\colon \mathcal{C} \longrightarrow \mathcal{B}$ with $\mathcal{C}$ small and filtered, any finite number of elements of $X = \operatorname{colim} UF$ can by the filteredness be represented by elements $(C, x_i) \in \operatorname{el}(UF)$ with a common C, so that we can apply an n-ary operation ω to get $\omega(x_1, \ldots, x_n) \in UFC$. By Theorem 4.72, the image of this in X is independent of the choice of C; thus X becomes UA for an algebra A, which is clearly the colimit of F.

Theorem 4.73 *Filtered colimits commute with finite limits in* **Set**. *That is to say, if $F\colon \mathcal{P} \times \mathcal{C} \longrightarrow \mathbf{Set}$ with $\mathcal{P}$ finite and $\mathcal{C}$ small and filtered, the evident canonical map*

$$\rho\colon \operatorname{colim}_C \lim_P F(P, C) \longrightarrow \lim_P \operatorname{colim}_C F(P, C)$$

is an isomorphism.

Proof. By (3.28) and (3.35), an element of $\lim_P \operatorname{colim}_C F(P, C)$ is a cone $(\alpha_P\colon 1 \longrightarrow \pi(\operatorname{el} F(P, -)))$. By the filteredness of $\mathcal{C}$ we can find representatives $(C, x_P \in F(P, C))$ of the α_P, with a common C. For $f\colon P \longrightarrow Q$ in $\mathcal{P}$, the objects $(C, F(f, C)x_P)$ and (C, x_Q) represent the same element of $\pi(\operatorname{el} F(Q, -))$, because α is a cone. Since there are but a finite number of maps in $\mathcal{P}$, we can by the filteredness of $\mathcal{C}$ find a $g\colon C \longrightarrow D$ in $\mathcal{C}$ such that $(D, F(f, g)x_P) = (D, F(Q, g)x_Q)$ for each map $f\colon P \longrightarrow Q$ in $\mathcal{P}$. Now the $y_P = F(P, g)x_P$ constitute a cone with vertex 1 over $F(-, D)$, or an element y of $\lim_P F(P, D)$. If z is the image of this in $\operatorname{colim}_C \lim_P F(P, C)$, we clearly have $\rho(z) = \alpha$; thus ρ is surjective.

To see that ρ is injective, suppose that $y \in \lim_P F(P, D)$ and $y' \in \lim_P F(P, D')$ represent elements of $\operatorname{colim}_C \lim_P F(P, C)$ with the same image under ρ. Then $y_P \in F(P, D)$ and $y'_P \in F(P, D')$ represent for each P the same element of $\operatorname{colim}_C F(P, C)$. As in Theorem 4.72, but using the filteredness to do it simultaneously for each $P \in \mathcal{P}$, we can find $h\colon D \longrightarrow E$ and $h'\colon D' \longrightarrow E$ such that $F(P, h)y_P = F(P, h')y'_P = w_P$ say. Now w

is a cone in $\lim_P F(P,E)$, representing the same object of $\operatorname{colim}_C \lim_P F(P,C)$ as y and y'. ∎

By the remark before the theorem, we have the same commutativity if **Set** is replaced by an equational category of algebras. The theorem can also be stated as:

Proposition 4.74 *For small $\mathcal{C}$, the functor* $\operatorname{colim}\colon [\mathcal{C},\mathbf{Set}] \longrightarrow \mathbf{Set}$ *is left exact if $\mathcal{C}$ is filtered.*

In this form it has a converse, to be given in 5.10 below.

The ideas of this section have the following extension. For a regular cardinal α, a category $\mathcal{P}$ is *α-small* if its set of morphisms has cardinal less than α; and $\mathcal{C}$ is *α-filtered* if every $T\colon \mathcal{P} \longrightarrow \mathcal{C}$ with α-small $\mathcal{P}$ is the base of some inductive cone. Then α-filtered colimits commute with α-small limits in **Set**, by an argument like that for Theorem 4.73 – which is the special case $\alpha = \aleph_0$ of this.

4.7 The factorization of a functor, when $\mathcal{V} = \mathbf{Set}$, into an initial functor and a discrete op-fibration

In this section, again taking $\mathcal{V} = \mathbf{Set}$, we describe the *comprehensive factorization system* of Street and Walters [74] on the ordinary category $\mathbf{Cat}_0$ of small categories; for its logical significance, see their article.

A functor $J\colon \mathcal{C} \longrightarrow \mathcal{B}$ is called a *discrete op-fibration* (or *dof* for short) if there is a functor $F\colon \mathcal{B} \longrightarrow \mathbf{Set}$ such that J is the composite of an isomorphism $\mathcal{C} \cong \operatorname{el} F$ and the canonical projection $d\colon \operatorname{el} F \longrightarrow \mathcal{B}$.

Proposition 4.75 *$J\colon \mathcal{C} \longrightarrow \mathcal{B}$ is a dof if and only if every map $f\colon JC \longrightarrow B$ in $\mathcal{B}$ is of the form $f = Jg$ for a unique map $g\colon C \longrightarrow D$ in $\mathcal{C}$.*

Proof. "Only if" is immediate; "if" is easy when we define $F\colon \mathcal{B} \longrightarrow \mathbf{Set}$ on objects by $FB = \{D \in \mathcal{C} \mid JD = B\}$ and on maps by the property in the proposition. ∎

It is clear from this proposition that the composite of two dofs is a dof; and it is clear from Theorem 4.67(i) that the composite of two initial functors is initial. Of course an isomorphism in $\mathbf{Cat}_0$ is both initial and a dof. We shall show that every map in $\mathbf{Cat}_0$ factorizes, essentially uniquely and functorially, into an initial functor followed by a dof.

First observe that, if $F\colon \mathcal{B} \longrightarrow \mathbf{Set}$ gives rise to the dof $d\colon \operatorname{el} F \longrightarrow \mathcal{B}$, *to give a functor $T\colon \mathcal{A} \longrightarrow \operatorname{el} F$ is to give a functor $S\colon \mathcal{A} \longrightarrow \mathcal{B}$ and a cone $\alpha\colon \Delta 1 \longrightarrow FS$* (or, by (3.28), an element $\alpha \in \lim FS$). Then $TA = (SA, \alpha_A) \in \operatorname{el} F$.

Given now an arbitrary functor $S\colon \mathcal{A} \longrightarrow \mathcal{B}$ in $\mathbf{Cat}_0$, let (F,ϕ) be the left Kan extension

$$\begin{tikzcd} & \mathcal{B} \arrow[dr, "F=\mathrm{Lan}_S\Delta 1"] \arrow[d, phantom, "\Uparrow\phi"] & \\ \mathcal{A} \arrow[ur, "S"] \arrow[rr, "\Delta 1"'] & {} & \mathbf{Set}. \end{tikzcd} \tag{4.76}$$

By (4.18), (3.26), and (3.35) we have

$$FB = \mathcal{B}(S-, B) \star \Delta 1 \cong \operatorname{colim} \mathcal{B}(S-, B) \cong \pi(S/B). \tag{4.77}$$

The functor S, together with the cone $\phi\colon \Delta 1 \longrightarrow FS$, constitutes a functor $T\colon \mathcal{A} \longrightarrow \operatorname{el} F$ with $dT = S$. We shall have factorized S in the desired way if we show that T *is initial*.

This seems to be best done by a direct calculation using Theorem 4.67(v). (The purported proof in [74] is erroneous; it supposes that, in the situation of our Theorem 4.47, if X and θ are left Kan extensions, so is ϕ; that this is false is easily seen by taking $\mathcal{D} = 1$ and using (4.34).)

A typical object of $\operatorname{el} F$, by (4.77), is $(B, [f])$ where $[f]$ is the connected component of some $f\colon SA \longrightarrow B$. It is clear that $T\colon \mathcal{A} \longrightarrow \operatorname{el} F$ sends A to $(SA, [1_{SA}])$ and sends $h\colon A \longrightarrow A'$ to Sh. An object of the comma-category $T/(B, [f])$ is at once seen to be a pair $(A, g\colon SA \longrightarrow B)$ with $[g] = [f]$. A morphism $(A, g) \longrightarrow (A', g')$ is an $h\colon A \longrightarrow A'$ for which $g'Sh = g$. It follows that $T/(B, [f])$ is connected, being isomorphic to the connected component $[f]$ of S/B.

To see that this factorization is essentially unique and functorial – that is, that we have a *factorization system* in the sense of [30] – it remains to show the existence of a unique "diagonal" $R\colon \mathcal{D} \longrightarrow \mathcal{A}$ in any commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "K"] \arrow[d, "P"'] & \mathcal{D} \arrow[d, "Q"] \\ \mathcal{A} \arrow[r, "d"'] & \mathcal{B}, \end{tikzcd}$$

where K is initial and d is a dof. We may as well suppose at once that $\mathcal{A} = \operatorname{el} F$ for some $F\colon \mathcal{B} \longrightarrow \mathbf{Set}$. Then P corresponds as above to a functor $S\colon \mathcal{C} \longrightarrow \mathcal{B}$ with $dP = S$, and a cone $\alpha\colon \Delta 1 \longrightarrow FS = FdP = FQK$. Since K is initial, this cone is by Theorem 4.67(ii) of the form $\alpha = \beta K$ for a unique cone $\beta\colon 1 \longrightarrow FQ$. Now Q and β constitute a functor $R\colon \mathcal{D} \longrightarrow \mathcal{A}$ with $dR = Q$, and clearly with $RK = P$. It is evident that any $R'\colon \mathcal{D} \longrightarrow \mathcal{A}$ with $dR' = Q$ and $R'K = P$ must coincide with R. This completes the proof.

If we write $\operatorname{dof}/\mathcal{B}$ for the full subcategory of $\mathbf{Cat}_0/\mathcal{B}$ determined by those $J\colon \mathcal{A} \longrightarrow \mathcal{B}$ that are dofs, it is clear that this factorization gives a reflexion $\mathbf{Cat}_0/\mathcal{B} \longrightarrow \operatorname{dof}/\mathcal{B}$. On the other hand $\operatorname{dof}/\mathcal{B}$ is evidently equivalent to $[\mathcal{B}, \mathbf{Set}]$. Thus we get an adjunction

$$\Omega \dashv \Theta\colon [\mathcal{B}, \mathbf{Set}] \longrightarrow \mathbf{Cat}_0/\mathcal{B}\ , \tag{4.78}$$

with Θ fully faithful. Here Θ sends $F\colon \mathcal{B} \longrightarrow \mathbf{Set}$ to $d\colon \operatorname{el} F \longrightarrow \mathcal{B}$, and may be called *the discrete Grothendieck construction*; its left adjoint Ω sends S to the F of (4.77).

While this is special to the case $\mathcal{V} = \mathbf{Set}$, it has generalizations to the important cases $\mathcal{V} = \mathbf{Gpd}$ and $\mathcal{V} = \mathbf{Cat}$, the latter being the setting for the original Grothendieck construction [34]. We do not give these here, for they transcend the context of enriched

category theory as developed in this account, and involve "transformations" more general then the **Gpd**-natural or **Cat**-natural ones; we shall treat them in [46].

Note finally that $\Omega\Theta S \cong S$, since Θ is fully faithful. This means by (4.76) that, for any $F\colon \mathcal{B} \longrightarrow \mathbf{Set}$, we have a left Kan extension

$$\begin{tikzcd} & \mathcal{B} \arrow[dr, "F"] \arrow[d, phantom, "\Nearrow\phi"] & \\ \mathrm{el}\, F \arrow[ur, "d"] \arrow[rr, "\Delta 1"'] & {} & \mathbf{Set}; \end{tikzcd} \tag{4.79}$$

since this asserts by (4.77) that $FB \cong \operatorname{colim} \mathcal{B}(d-, B)$, it is precisely a restatement of (3.32).

4.8 The general representability and adjoint-functor theorems; the special case of a complete domain-category

In this section and the next we look at conditions ensuring the representability of a $\mathcal{V}$-functor $F\colon \mathcal{C} \longrightarrow \mathcal{V}$, especially in the simplest case of a complete $\mathcal{C}$. Since by 1.11 the $\mathcal{V}$-functor $T\colon \mathcal{C} \longrightarrow \mathcal{B}$ has a left adjoint precisely when each $\mathcal{B}(B, T-)$ is representable, the same considerations yield conditions for a left adjoint to exist.

Since a colimit in $\mathcal{C}$ is itself the representing object for a functor $\mathcal{C} \longrightarrow \mathcal{V}$, the representing object for $F\colon \mathcal{C} \longrightarrow \mathcal{V}$, if it exists, is in some sense a "generalized colimit". Representing objects are, in fact, very commonly constructed as colimits: consider some examples. First, in the case $\mathcal{V} = \mathbf{Set}$, a representation for F corresponds, as we observed in 1.10, to an initial object in $\mathrm{el}\, F$; that is, to the colimit of the empty diagram in $\mathrm{el}\, F$. Next, in Proposition 3.36, a functor $F\colon \mathcal{C} \longrightarrow \mathcal{V}$ was in effect given as an iterated limit of representables; and we constructed its representing object as the corresponding iterated colimit of the representing objects for these. Again, the left Kan adjoint Lan_K of Theorem 4.50 was constructed via the colimits $\widetilde{K}C \star G$. A free abelian group is constructed as a coproduct of infinite cyclic groups; the tensor product of two abelian groups is constructed as a cokernel of a free abelian group; and the free monoid on a set X is the coproduct $\sum_{n \geq 0} X^n$. More generally, numerous constructions of free algebras, free monoids, and free monads, as iterated colimits, under cocompleteness but no completeness hypotheses, are given in [44].

Yet the only general necessary condition that we so far have for the representability of F is expressed in terms of limits: by 3.2, F must preserve whatever limits exist. Hence if $\mathcal{C}$ is complete, F must be continuous (and similarly for a $T\colon \mathcal{C} \longrightarrow \mathcal{B}$ that is to have a left adjoint). We here consider what must be added to this continuity to ensure representability (or existence of an adjoint). Since the desired representing object is a generalized colimit, this is analogous to the question of what must be added to

completeness to ensure cocompleteness – or rather, to ensure the existence of a particular colimit.

A *small* complete **Set**-category $\mathcal{C}$ is of necessity a preordered set – for if $\mathcal{C}(A,B)$ had at least two elements, $\mathcal{C}(A,B^n)$ would have at least 2^n, making $\mathcal{C}$ large if B^n existed for all small cardinals n. Hence, by the classical argument on complete lattices, a small complete $\mathcal{C}$ is also cocomplete: the supremum of a subset is the infimum of its set of upper bounds. Not so, if $\mathcal{C}$ is large – even if $\mathcal{C}$ is a preordered set: the ordered set of small ordinals is cocomplete but not complete. What must be added is a condition ensuring that the colimit we need can be manufactured from *small* limits. Similarly in the case of representability of $F\colon \mathcal{C} \longrightarrow \mathcal{V}$: the extra condition needed is a "smallness" one on F.

We begin with a general result expressing a representing object as a limit: a *large* one, unless $\mathcal{C}$ happens to be small.

Theorem 4.80 *$F\colon \mathcal{C} \longrightarrow \mathcal{V}$ is representable if and only if the limit $\{F, 1_{\mathcal{C}}\}$ exists and is preserved by F. Then the representing object is $\{F,1\}$, and the counit $\mu\colon F \longrightarrow \mathcal{C}(\{F,1\},-)$ of this limit is an isomorphism giving the representation of F.*

Proof. One direction is immediate: if F is the representable $\mathcal{C}(D,-)$, then by (3.10) the limit $\{\mathcal{C}(D,-),1\}$ is D, with counit $\mu = 1\colon \mathcal{C}(D,-) \longrightarrow \mathcal{C}(D,-)$; similarly, with appropriate changes, when F is only *isomorphic* to $\mathcal{C}(D,-)$; and the representable F preserves this limit.

For the other direction we have by hypothesis that

$$F \xrightarrow[\mu]{} C(\{F,1\},-) \xrightarrow[F]{} [F\{F,1\}, F-]$$

is the counit of the limit $\{F,F\}$. Hence by (3.4) the (F,I)-cylinder over F given by the isomorphism $i\colon F \longrightarrow [I,F-]$ has the form $i = [g,1]F\mu$ for a unique $g\colon I \longrightarrow F\{F,1\}$. Writing $\lambda\colon \mathcal{C}(\{F,1\},-) \longrightarrow F$ for the composite $i^{-1}[g,1]F$, we therefore have $\lambda\mu = 1$; and to prove μ an isomorphism it remains to show that $\mu\lambda\colon \mathcal{C}(\{F,1\},-) \longrightarrow \mathcal{C}(\{F,1\},-)$ is 1.

But by Yoneda, this map $\mu\lambda$ is $\mathcal{C}(f,1)$ for a unique $f\colon \{F,1\} \longrightarrow \{F,1\}$. Since $\lambda\mu = 1$ we have $\mu\lambda\mu = \mu$, or $\mathcal{C}(f,1)\mu = \mu = \mathcal{C}(1,1)\mu$. However, as the counit of the limit $\{F,1\}$, the cylinder μ factorizes through itself as $\mathcal{C}(f,1)\mu$ for a *unique* f; whence $f = 1$ and $\mu\lambda = 1$. ∎

There is a corresponding adjunction theorem:

Theorem 4.81 *$T\colon \mathcal{C} \longrightarrow \mathcal{B}$ has a left adjoint if and only if $\mathrm{Ran}_T\, 1_{\mathcal{C}}$ exists and is preserved by T. Then the left adjoint S is $\mathrm{Ran}_T\, 1$, and the counit $ST \longrightarrow 1$ of the Kan extension is the counit of the adjunction.*

Proof. If $S \dashv T$ we have $\mathrm{Ran}_T\, 1 \cong S$ by (the dual of) (4.28); that the counit is as described is immediate; and the right adjoint T preserves the right Kan extension.

For the converse, observe that $(\mathrm{Ran}_T\, 1)B = \{\mathcal{B}(B,T-),1\}$ by (4.9). This limit is, by hypothesis, preserved by T: hence it is also preserved by $\mathcal{B}(B,T-)$. Thus $\mathcal{B}(B,T-)$ is representable by Theorem 4.80. This being for all B, we conclude that T has a left adjoint. ∎

Combining Theorem 4.63 with Theorem 4.80 gives the following, which replaces the large limit $\{F, 1\}$ by one that may be smaller:

Proposition 4.82 *Let $F\colon \mathcal{C} \longrightarrow \mathcal{V}$ be $\mathrm{Lan}_K H$, where $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ and $H\colon \mathcal{A} \longrightarrow \mathcal{V}$. Then F is representable if and only if the limit $\{H, K\}$ exists and is preserved by F; and the representing object is then $\{H, K\}$.* ∎

We consider the possibility of a small $\mathcal{A}$ here:

Proposition 4.83 *The following properties of $F\colon \mathcal{C} \longrightarrow \mathcal{V}$ are equivalent:*

(i) F is a small (indexed) colimit of representables.

(ii) F has the form $\mathrm{Lan}_K H$ where the domain $\mathcal{A}$ of K is small.

(iii) F is the left Kan extension of its restriction to a small full subcategory of $\mathcal{C}$.

Proof. To say that F is a small colimit of representables is to say that we have a small indexing-type $H\colon \mathcal{A} \longrightarrow \mathcal{V}$ and a functor $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ such that, if $Y\colon \mathcal{C}^{\mathrm{op}} \longrightarrow [\mathcal{C}, \mathcal{V}]$ is the Yoneda embedding ($[\mathcal{C}, \mathcal{V}]$ being a $\mathcal{V}'$-category for some suitable extension $\mathcal{V}'$ of $\mathcal{V}$), we have $H \star YK^{\mathrm{op}} \cong F$. Since small colimits in $[\mathcal{C}, \mathcal{V}]$ exist pointwise by 3.3, this is to say that $H \star \mathcal{C}(K-, C) \cong FC$ for each C; and hence by (4.18) and (3.9) to say that $F = \mathrm{Lan}_K H$. Thus (i) and (ii) are equivalent. It remains to show that, if we have (ii), we also have its special case (iii).

For this, let $\mathcal{A}'$ be the full image of $K\colon \mathcal{A} \longrightarrow \mathcal{C}$, so that K factorizes as say $W\colon \mathcal{A} \longrightarrow \mathcal{A}'$ followed by the inclusion $Z\colon \mathcal{A}' \longrightarrow \mathcal{C}$. Then $\mathcal{A}'$ is still small; and because Z is fully faithful, we have $F \cong \mathrm{Lan}_Z FZ$ by Theorem 4.47. ∎

We shall call a functor $F\colon \mathcal{C} \longrightarrow \mathcal{V}$ satisfying the equivalent properties of Proposition 4.83 *accessible*. In particular every representable $\mathcal{C}(D, -)\colon \mathcal{C} \longrightarrow \mathcal{V}$ is accessible; it is in fact $\mathrm{Lan}_D I$ where $D\colon \mathcal{I} \longrightarrow \mathcal{C}$ and $I\colon \mathcal{I} \longrightarrow \mathcal{V}$. So Proposition 4.82 now gives our desired representability theorem for complete $\mathcal{C}$:

Theorem 4.84 *If $\mathcal{C}$ is complete, $F\colon \mathcal{C} \longrightarrow \mathcal{V}$ is representable if and only if it is continuous and accessible.* ∎

For a general complete $\mathcal{C}$, the representability of F is not implied by its continuity alone, even in the case $\mathcal{V} = \mathbf{Set}$. The classical counter-example takes for $\mathcal{C}$ the category of (small) groups, and for $F\colon \mathcal{C} \longrightarrow \mathbf{Set}$ the functor given by $FC = \prod_\alpha \mathcal{C}(G_\alpha, C)$, where $\{G_\alpha\}$ is the large set of all small simple groups.

4.9 Representability and adjoint-functor theorems when $\mathcal{V} = \mathbf{Set}$

Although Theorem 4.84 gives a simple general criterion that we shall use below in 5.6, it is often advantageous, in considering the representability of a particular $\mathcal{V}$-functor, to reduce the problem to the more familiar case $\mathcal{V} = \mathbf{Set}$ by using:

Theorem 4.85 *Let $\mathcal{C}$ admit cotensor products. Then the $\mathcal{V}$-functor $F\colon \mathcal{C} \longrightarrow \mathcal{V}$ is representable if and only if the ordinary functor*

$$\mathcal{C}_0 \xrightarrow[F_0]{} \mathcal{V}_0 \xrightarrow[V]{} \mathbf{Set}$$

is representable and F preserves cotensor products. Moreover a $\mathcal{V}$-functor $T\colon \mathcal{C} \longrightarrow \mathcal{B}$ has a left adjoint if and only if the ordinary functor $T_0\colon \mathcal{C}_0 \longrightarrow \mathcal{B}_0$ does so and T preserves cotensor products.

Proof. "Only if" is clear both in the representability case and in the adjunction case – in the former, if $F \cong \mathcal{C}(D,-)$, then $VF_0 \cong \mathcal{C}_0(D,-)$ by (1.33).

For the converse, let $\beta\colon \mathcal{C}_0(D,-) \cong VF_0$ be a representation with unit $\eta \in VFD$; then $\eta\colon I \longrightarrow FD$ corresponds by Yoneda to a map $\alpha\colon \mathcal{C}(D,-) \longrightarrow F$, which we show to be an isomorphism. It suffices to verify that $\mathcal{V}_0(X,\alpha_C)\colon \mathcal{V}_0(X,\mathcal{C}(D,C)) \longrightarrow \mathcal{V}_0(X,FC)$ is an isomorphism for each $C \in \mathcal{C}$ and each $X \in \mathcal{V}$. The domain here is isomorphic by (3.45) to $\mathcal{C}_0(D, X \pitchfork C)$, and the codomain by (1.25) to $V[X,FC]$, which is then isomorphic to $VF(X \pitchfork C)$ since F preserves cotensor products. It is easily verified that, modulo these isomorphisms, $\mathcal{V}_0(X,\alpha_C)$ is in fact the isomorphism $\beta_{X\pitchfork C}$.

The case of adjoints follows from this, since $\mathcal{B}_0(B,T-) = V(\mathcal{B}(B,T-))_0$ by (1.33) and since, if T preserves cotensor products, so does $\mathcal{B}(B,T-)$. ∎

We accordingly take $\mathcal{V} = \mathbf{Set}$ for the rest of this section. Recall that, in taking $\mathcal{C}$ to be a $\mathbf{Set}$-category, we are automatically supposing it to be *locally small*. There are many classical results on the representability of $F\colon \mathcal{C} \longrightarrow \mathbf{Set}$; and to reproduce them here in detail, with examples of their use, would involve an unjustifiably long digression from our main path of development. Some briefer comments may however be useful. First, whether $\mathcal{C}$ is complete or not, applying Proposition 4.82 to the left Kan extension (4.79) gives:

Proposition 4.86 *$F\colon \mathcal{C} \longrightarrow \mathbf{Set}$ is representable if and only if $d\colon \operatorname{el} F \longrightarrow \mathcal{C}$ has a limit which is preserved by F; the representing object is then $\lim d$.* ∎

The following proposition is useful in various contexts; it follows at once from the observation in 4.6 that a functor $T\colon \mathcal{A} \longrightarrow \operatorname{el} F$ consists of a functor $S = dT\colon \mathcal{A} \longrightarrow \mathcal{C}$ and an $\alpha \in \lim FS$:

Proposition 4.87 *If $\mathcal{C}$ admits the conical limits of some class, and $F\colon \mathcal{C} \longrightarrow \mathbf{Set}$ preserves them, then $\operatorname{el} F$ also admits such limits, and $d\colon \operatorname{el} F \longrightarrow \mathcal{C}$ preserves and reflects them. In particular, $\operatorname{el} F$ is complete if $\mathcal{C}$ is complete and F is continuous.* ∎

By 3.4 and Proposition 4.83, $F\colon \mathcal{C} \longrightarrow \mathbf{Set}$ is accessible precisely when it is a small conical colimit of representables. There is a weaker condition – *strictly* weaker by [29] – related to this as finite-generation of an algebra is related to finite-presentability: namely that F be an epimorphic image of a small coproduct of representables; call this *weak accessibility*. Consider what it means. A map $\mathcal{C}(D_\alpha,-) \longrightarrow F$ corresponds by Yoneda to an object $(D_\alpha, y_\alpha \in FD)$ of $\operatorname{el} F$. Since epimorphisms in $[\mathcal{C},\mathbf{Set}]$ are pointwise, to say that $\sum_\alpha \mathcal{C}(D_\alpha,-) \longrightarrow F$ is epimorphic is to say that every $(C,x) \in \operatorname{el} F$ admits a map from some (D_α, y_α). If $\mathcal{S}$ is the small full subcategory of $\operatorname{el} F$ determined by the objects

(D_α, y_α), and $J\colon \mathcal{S} \longrightarrow \mathrm{el}\, F$ is the inclusion, this is to say that the comma-category $J/(C,x)$ is non-empty for each (C,x).

When $\mathrm{el}\, F$ is complete, however – or even finitely complete – so that $(\mathrm{el}\, F)^{\mathrm{op}}$ is filtered, the non-emptiness of each $J/(C,x)$ implies by Proposition 4.71 the initialness of $J\colon \mathcal{S} \longrightarrow \mathrm{el}\, F$. Thus from Theorem 4.84, Proposition 4.86, Proposition 4.87, and Theorem 4.67, we get Freyd's *General adjoint functor theorem* (cf. [28]):

Theorem 4.88 *For $F\colon \mathcal{C} \longrightarrow \mathbf{Set}$ where $\mathcal{C}$ is locally-small and complete, the following are equivalent:*

(i) F is representable.

(ii) F is continuous and accessible.

(iii) F is continuous and weakly accessible.

Then the representing object for F is $\lim dJ$. ∎

The weak accessibility of F is often called the *solution-set condition*. Since $\mathcal{C}$ is locally small, it can be expressed as follows: *there is a small set $\mathcal{T}$ of objects of $\mathcal{C}$ such that, for any $C \in \mathcal{C}$ and any $x \in FC$, there is some $E \in \mathcal{T}$, some $y \in FE$, and some $f\colon E \longrightarrow C$ with $(Ff)y = x$.*

Recall that we also have the quite elementary observation from 1.10:

Proposition 4.89 *For $F\colon \mathcal{C} \longrightarrow \mathbf{Set}$ the following are equivalent:*

(i) F admits a representation $\alpha\colon \mathcal{C}(D,-) \cong F$ with unit $\eta \in FD$.

(ii) $\mathrm{el}\, F$ *has an initial object (D,η).*

(iii) $\Delta 1\colon \mathrm{el}\, F \longrightarrow \mathbf{Set}$ *is representable by (D,η).* ∎

Applied to $\Delta 1\colon \mathcal{E} \longrightarrow \mathbf{Set}$ for any $\mathbf{Set}$-category $\mathcal{E}$, Theorem 4.88 gives that *$\mathcal{E}$ has an initial object if it is complete and has a small full subcategory $\mathcal{S}$* (the "solution set") *such that any $E \in \mathcal{E}$ admits a map from some $S \in \mathcal{S}$.* Conversely we could have proved Theorem 4.88 alternatively by using Proposition 4.89 and Proposition 4.87 to reduce it to this "initial-object theorem", and by giving a simple direct proof of the latter. In discussing variants of Theorem 4.88, we take this approach for brevity: we describe the corresponding initial-functor theorem, and leave the reader to translate it back to a representability theorem.

An unpublished result of Freyd, given in lectures in 1970, deals with a non-complete $\mathcal{E}$. It asserts that *$\mathcal{E}$ has an initial object if and only if* (i) *$\mathcal{E}$ satisfies the solution-set condition as above,* (ii) *every $A \in \mathcal{E}$ has a map from some $B \in \mathcal{E}$ all of whose idempotent endomorphisms split, and* (iii) *every small diagram in $\mathcal{E}$ admits some projective cone over it.* The corresponding representability theorem, with $\mathcal{C}$ the (non-complete) stable homotopy category of CW-complexes, gives Brown's result [11] on the representability of cohomology theories.

Very commonly, $\mathcal{E}$ is not only complete but admits arbitrary (even large) intersections of subobjects; by Proposition 4.87, $\mathrm{el}\,F$ is so if $\mathcal{C}$ is so and if F preserves, besides small limits, all intersections of subobjects. For such an $\mathcal{E}$, the intersection of all the subobjects of $E \in \mathcal{E}$ is an *atom* – an object with no proper subobject. So the set $\mathcal{S}$ of isomorphism-classes of atoms in $\mathcal{E}$ forms a solution-set if it is small. In many concrete cases, it is easy to find a small cardinal exceeding $\mathrm{card}\,\mathcal{S}$; this is probably the most common way of applying Theorem 4.88.

The adjoint-functor theorem of Day [15], in its initial-object-theorem form, requires only partial completeness of $\mathcal{E}$: the existence of equalizers, of pullbacks of monomorphisms, and of arbitrary intersections of subobjects; but it strengthens the solution-set condition to the requirement that there is some $S \in \mathcal{E}$ such that, for each $A \in \mathcal{E}$, there is a monomorphism $A \longrightarrow B$ and a map $S \longrightarrow B$. Then the atom in S is the initial object. Since this refers to no infinite limits except intersections of monomorphisms, it has an analogue [18] in an elementary topos, where such intersections exist in an internal sense.

Day's result also yields a proof of Freyd's *Special Adjoint Functor Theorem* (cf. [28]):

Theorem 4.90 *If the locally-small $\mathcal{C}$ is complete, admits all intersections of subobjects, and has a small cogenerating set, every continuous $F\colon \mathcal{C} \longrightarrow \mathbf{Set}$ is representable.*

Proof. It is clear that $\mathcal{E} = \mathrm{el}\,F$ has a small cogenerating set $\{G_\alpha\}$ in these conditions. The hypothesis of Day's initial-object theorem are clearly satisfied with $S = \prod_\alpha G_\alpha$ and with $A \longrightarrow B$ the canonical monomorphism $A \longrightarrow \prod_\alpha(\mathcal{E}(G_\alpha, A) \pitchfork G_\alpha)$. ∎

Applying this to the existence of a left adjoint to the diagonal $\Delta\colon \mathcal{C} \longrightarrow [\mathcal{A}, \mathcal{C}]$, we see that *$\mathcal{C}$ is cocomplete if it is complete, admits intersections of subobjects, and has a small cogenerating set.*

4.10 Existence and characterization of the left Kan extension along a fully-faithful $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ in terms of cylinders in $\mathcal{C}$ sent to colimits by the $\mathcal{C}(KA, -)$

For cocomplete $\mathcal{B}$, the left Kan extension of $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ along $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ exists by (4.13) if $\mathcal{A}$ is small. By Theorem 4.47, it still exists when $\mathcal{A}$ is not small, provided that K is itself a left Kan extension along some $\mathcal{A}' \longrightarrow \mathcal{A}$ with $\mathcal{A}'$ small. We now turn to existence results when $\mathcal{B}$ is no longer cocomplete, but admits colimits only of some specified class.

Theorem 4.91 *The left Kan extension of $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ along $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ exists if and only if, for each $C \in \mathcal{C}$, the functor $\widetilde{K}C\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{V}$ can be so expressed as a colimit $F \star YQ$ of representables that the colimit $F \star GQ$ exists in $\mathcal{B}$; then this latter colimit is $(\mathrm{Lan}_K G)C$.*

Proof. Let $\widetilde{K}C$ be so expressible, where $F\colon \mathcal{L}^{\mathrm{op}} \longrightarrow \mathcal{V}$ and $Q\colon \mathcal{L} \longrightarrow \mathcal{A}$. Since $(YQ-) \star G \cong GQ$ by (3.10), we have from (3.23) that $\widetilde{K}C \star G = (F \star YQ) \star G$ exists, being $F \star ((YQ-) \star G) \cong F \star GQ$. For the converse, we just take the canonical expression (3.17) of $\widetilde{K}C$ as a colimit of representables, corresponding to $F = \widetilde{K}C$ and $Q = 1$. ∎

As an example of this, take $\mathcal{V} = \mathbf{Set}$, let $\{G_x\}_{x\in X}$ be a small set of objects in the cocomplete category $\mathcal{C}$, and let $\mathcal{A}$ be the full subcategory of $\mathcal{C}$ determined by the small coproducts of the G_x, with $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ the inclusion. For any $C \in \mathcal{C}$, let $\epsilon\colon D = \sum_{x\in X} \mathcal{C}(G_x, C)\cdot G_x \longrightarrow C$ be the canonical map whose (x, f)-component, for $f \in \mathcal{C}(G_x, C)$, is f. Write M_x for the set of pairs of maps $g, h\colon G_x \longrightarrow D$ such that $\epsilon g = \epsilon h$, and let $\phi, \psi\colon E = \sum_x M_x \cdot G_x \longrightarrow D$ be the maps whose respective (x, g, h)-components are g and h. The commutative diagram

$$E \underset{\psi}{\overset{\phi}{\rightrightarrows}} D \xrightarrow[\epsilon]{} C \tag{4.92}$$

is not in general a coequalizer diagram; but for each x the diagram

$$\mathcal{C}(G_x, E) \underset{\mathcal{C}(1,\psi)}{\overset{\mathcal{C}(1,\phi)}{\rightrightarrows}} \mathcal{C}(G_x, D) \xrightarrow[\mathcal{C}(1,\epsilon)]{} \mathcal{C}(G_x, C) \tag{4.93}$$

is a *split coequalizer diagram*. That is to say, there are maps $t\colon \mathcal{C}(G_x, C) \longrightarrow \mathcal{C}(G_x, D)$ and $s\colon \mathcal{C}(G_x, D) \longrightarrow \mathcal{C}(G_x, E)$ for which $\mathcal{C}(1, \epsilon)t = 1$, $\mathcal{C}(1, \phi)s = 1$, and $\mathcal{C}(1, \psi)s = t\mathcal{C}(1, \epsilon)$; which easily implies that (4.93) is a coequalizer. To wit, $t(f)\colon G_x \longrightarrow D$ is the *coprojection* $i_{x,f}$ of the coproduct, while $s(u)\colon G_x \longrightarrow E$ is the coprojection corresponding to the index $(x, u, i_{x,\epsilon u})$. Any product of the diagrams (4.93) over a family (x_j) of the indices x is clearly again a split coequalizer; whence it follows that $\widetilde{K}$ sends (4.92) to a coequalizer diagram

$$\mathcal{A}(-, E) \rightrightarrows \mathcal{A}(-, D) \longrightarrow \widetilde{K}C$$

in $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$. We conclude that the left Kan extension of $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ along $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ exists whenever $\mathcal{B}$ admits coequalizers, and is given as the coequalizer of the two maps $G\phi, G\psi\colon GE \longrightarrow GD$. (We remark that, when (4.92) *is* a coequalizer diagram for each C, $\{G_x\}$ is said to be a *regular generator* of $\mathcal{C}$; it is easily verified to be *a fortiori* a strong generator and a generator.) Clearly Theorem 4.91 could be so modified as to cover the case where $\widetilde{K}C$ was expressed as an *iterated* colimit of representables. There is a particularly neat and useful result of this kind when K is *fully faithful*; which we now suppose.

Let $F\colon \mathcal{L}^{\mathrm{op}} \longrightarrow \mathcal{V}$, let $P\colon \mathcal{L} \longrightarrow \mathcal{C}$ be a diagram of type F, let $C \in \mathcal{C}$, and let $\alpha\colon F \longrightarrow \mathcal{C}(P-, C)$ be an (F, C)-cylinder over P, in the language of 3.1. Let us say that α is a K-cylinder if the composite cylinder

$$F \xrightarrow[\alpha]{} \mathcal{C}(P-, C) \xrightarrow[\widetilde{K}]{} [\mathcal{A}^{\mathrm{op}}, \mathcal{V}](\widetilde{K}P-, \widetilde{K}C)$$

is a colimit-cylinder, giving

$$\widetilde{K}C \cong F \star \widetilde{K}P. \tag{4.94}$$

For this it certainly suffices that, for each $A \in \mathcal{A}$, the cylinder

$$F \xrightarrow[\alpha]{} \mathcal{C}(P-, C) \xrightarrow[\mathcal{C}(KA,-)]{} [\mathcal{C}(KA, P-), \mathcal{C}(KA, C)]$$

be a colimit-cylinder, so that

$$\mathcal{C}(KA, C) \cong F \star \mathcal{C}(KA, P-); \tag{4.95}$$

this is just the *pointwise* existence of the colimit (4.94); and by 3.3 the pointwiseness is automatic if $\mathcal{L}$ is small.

An expression of $\widetilde{K}C$ as a colimit $F \star YQ$ of representables, as in Theorem 4.91, has as unit a cylinder $\beta\colon F \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}](YQ-, \widetilde{K}C)$. The codomain of β is isomorphic by Yoneda to $(\widetilde{K}C)Q^{\mathrm{op}} = \mathcal{C}(KQ-, C)$; moreover, since K is fully faithful, we have $Y \cong \widetilde{K}K$ by (4.22). It follows that β has the form

$$F \xrightarrow[\alpha]{} \mathcal{C}(KQ-, C) \xrightarrow[\widetilde{K}]{} [\mathcal{A}^{\mathrm{op}}, \mathcal{V}](\widetilde{K}KQ-, \widetilde{K}C), \tag{4.96}$$

in which the second map is in fact an isomorphism. Thus such an expression of $\widetilde{K}C$ as a colimit of representables is the same thing as a K-cylinder in which P factorizes through $\mathcal{A}$ as KQ.

In this fully-faithful case we have a characterization of $\mathrm{Lan}_K G$ as follows:

Theorem 4.97 *An extension $S\colon \mathcal{C} \longrightarrow \mathcal{B}$ of $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ along the fully-faithful $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ is $\mathrm{Lan}_K G$ (that is, S is canonically isomorphic to $\mathrm{Lan}_K SK$) if and only if it sends each K-cylinder to a colimit-cylinder. If $\mathcal{A}$ is small, it suffices to speak of* small *K-cylinders.*

Proof. The canonical expression of $\widetilde{K}C$ as a colimit $\widetilde{K}C \star Y$ of representables corresponds as in (4.96) to a K-cylinder $1\colon \widetilde{K}C \longrightarrow \mathcal{C}(K-, C)$. If S sends this to a colimit-cylinder we have $SC \cong \widetilde{K}C \star SK$, which is $\widetilde{K}C \star G$ since S is an extension of G; thus $S \cong \mathrm{Lan}_K G$. Conversely, if $\mathrm{Lan}_K G$ exists, any K-cylinder α is, by definition, sent to a colimit-cylinder by $\widetilde{K}$; by (3.23) the functor $- \star G$ preserves the colimit (4.94), since each $\widetilde{K}PL \star G$ exists; hence $\mathrm{Lan}_K G = \widetilde{K}- \star G$ sends α to a colimit-cylinder. ∎

Still supposing K fully faithful, consider a family $\Phi = (\alpha_\gamma\colon F_\gamma \longrightarrow \mathcal{C}(P_\gamma -, C_\gamma))_{\gamma \in \Gamma}$ of K-cylinders, where $F_\gamma\colon \mathcal{L}_\gamma^{\mathrm{op}} \longrightarrow \mathcal{V}$ and $P_\gamma\colon \mathcal{L}_\gamma \longrightarrow \mathcal{C}$. Generalizing 3.5, where it was a question of a family of colimits, we say that *$\mathcal{C}$ is the closure of $\mathcal{A}$ under the family Φ* if there is no proper replete full subcategory $\mathcal{D}$ of $\mathcal{C}$, containing $\mathcal{A}$, and such that $C_\gamma \in \mathcal{D}$ whenever P_γ takes its values in $\mathcal{D}$.

Theorem 4.98 *Let $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ be fully faithful, and let $\mathcal{C}$ be the closure of $\mathcal{A}$ for a family Φ of K-cylinders as above. Then a functor $S\colon \mathcal{C} \longrightarrow \mathcal{B}$ is exhibited by $1\colon SK \longrightarrow SK$ as $\mathrm{Lan}_K SK$ if and only if S sends each cylinder of Φ to a colimit-cylinder. Moreover $\mathrm{Lan}_K G$ exists for any $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ if $\mathcal{B}$ admits F_γ-indexed colimits for each $\gamma \in \Gamma$.*

Proof. Theorem 4.97 gives the necessity in the first assertion. For the sufficiency, let $\mathcal{D}$ be the replete full subcategory of $\mathcal{C}$ consisting of those C for which $S\colon \mathcal{C}(K-, C) \longrightarrow \mathcal{B}(SK-, SC)$ as unit exhibits SC as $\widetilde{K}C \star SK$. Then $\mathcal{A} \subset \mathcal{D}$ since we have canonical isomorphisms $SKA \cong YA \star SK \cong \widetilde{K}KA \star SK$. If P_γ takes its values in $\mathcal{D}$, we have a canonical isomorphism $\widetilde{K}P_\gamma - \star SK \cong SP_\gamma$ by the definition of $\mathcal{D}$; and $\widetilde{K}C_\gamma \cong F_\gamma \star \widetilde{K}P_\gamma$ by (4.94). Hence (3.23) gives $F_\gamma \star SP_\gamma \cong \widetilde{K}C_\gamma \star SK$. Since $F_\gamma \star SP_\gamma \cong SC_\gamma$ by hypothesis, we have $C_\gamma \in \mathcal{D}$; thus $\mathcal{D}$ is all of $\mathcal{C}$, as desired. The second assertion is proved by a similar argument, taking $\mathcal{D}$ to consist of those C for which $\widetilde{K}C \star G$ exists. ∎

For an example of this, let us modify the example following Theorem 4.91. With the notation as there, write $\mathcal{A}'$ for the full subcategory of $\mathcal{C}$ determined by the *finite* coproducts of the G_x. Since every coproduct is the *filtered* colimit of the finite coproducts of its summands, every object of $\mathcal{A}$ is a filtered colimit of objects of $\mathcal{A}'$. Suppose now that each G_x is *finitely presentable* – meaning that each $\mathcal{C}(G_x, -)\colon \mathcal{C} \longrightarrow \mathbf{Set}$ preserves filtered colimits. Then, since filtered colimits commute with finite limits in **Set** by Theorem 4.73, each object of $\mathcal{A}'$ is finitely presentable; and thus the filtered colimits in question are preserved by $\widetilde{Z}\colon \mathcal{C} \longrightarrow [\mathcal{A}'^{\mathrm{op}}, \mathbf{Set}]$, where Z is the inclusion $\mathcal{A}' \longrightarrow \mathcal{A} \longrightarrow \mathcal{C}$; that is, they correspond to Z-cylinders. It now follows from Theorem 4.98 that any $G\colon \mathcal{A}' \longrightarrow \mathcal{B}$ admits a left Kan extension along $Z\colon \mathcal{A}' \longrightarrow \mathcal{C}$ if $\mathcal{B}$ admits coequalizers and filtered colimits.

We end with a result, a special case of which we have already used in Theorem 4.51:

Theorem 4.99 *Let $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ be fully faithful, and denote by $[\mathcal{A}, \mathcal{B}]'$ the full subcategory of $[\mathcal{A}, \mathcal{B}]$ determined by those G for which $\mathrm{Lan}_K G$ exists. Then $\mathrm{Lan}_K\colon [\mathcal{A}, \mathcal{B}]' \longrightarrow [\mathcal{C}, \mathcal{B}]$ is fully faithful; so that we have an equivalence $[\mathcal{A}, \mathcal{B}]' \simeq [\mathcal{C}, \mathcal{B}]^\ell$, where the latter is the full subcategory of $[\mathcal{C}, \mathcal{B}]$ determined by the functors isomorphic to some $\mathrm{Lan}_K G$; the inverse to this equivalence is composition with K. If $\mathcal{A}$ is small, it follows that $[\mathcal{C}, \mathcal{B}]^\ell$ is a $\mathcal{V}$-category.*

Proof. We have by Theorem 4.38 that $[\mathcal{C}, \mathcal{B}](\mathrm{Lan}_K G, \mathrm{Lan}_K G') \cong [\mathcal{A}, \mathcal{B}](G, (\mathrm{Lan}_K G')K)$, which by Proposition 4.23 is isomorphic to $[\mathcal{A}, \mathcal{B}](G, G')$. ∎

Of course this fully-faithfulness is just a special case of the observation in 1.11 following (1.53), to which we could have appealed: for we are dealing with an adjunction $\mathrm{Lan}_K \dashv [K, 1]$ whose unit ϕ is an isomorphism. In practice we use the equivalence $[\mathcal{A}, \mathcal{B}]' \simeq [\mathcal{C}, \mathcal{B}]^\ell$ chiefly when $[\mathcal{A}, \mathcal{B}]'$ is all of $[\mathcal{A}, \mathcal{B}]$, as for instance under the hypotheses of Theorem 4.98 or when $\mathcal{A}$ is small and $\mathcal{B}$ is cocomplete.

Chapter 5

Density

5.1 Definition of density, and equivalent formulations

A function $k\colon A \longrightarrow C$, not necessarily continuous, between hausdorff spaces is *dense* (in the sense that its image is a dense subset) precisely when any continuous map from C into another hausdorff space B is uniquely determined by its composite with k; that is, when $\mathbf{Set}(k,1)\colon \mathbf{Haus}(C,B) \longrightarrow \mathbf{Set}(A,B)$ is injective. We give an analogous definition of density for a $\mathcal{V}$-functor $K\colon \mathcal{A} \longrightarrow \mathcal{C}$.

For a $\mathcal{V}$-category $\mathcal{C}$, the functor category $[\mathcal{C},\mathcal{B}]$ exists at least as a $\mathcal{V}'$-category for some suitable extension $\mathcal{V}'$ of $\mathcal{V}$ as in 3.12. Consider those $S\colon \mathcal{C} \longrightarrow \mathcal{B}$ which preserve every colimit existing in $\mathcal{C}$ whose indexing-type $F\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{V}$ has domain $\mathcal{A}^{\mathrm{op}}$; these constitute a full subcategory $\mathcal{A}$-$\mathrm{Cocts}[\mathcal{C},\mathcal{B}]$, which if $\mathcal{A}$ is small contains the full subcategory $\mathrm{Cocts}[\mathcal{C},\mathcal{B}]$ of those S preserving all small colimits that exist.

Theorem 5.1 *For a $\mathcal{V}$-functor $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ the following are equivalent:*

(i) For any $\mathcal{B}$, the restriction to $\mathcal{A}$-$\mathrm{Cocts}[\mathcal{C},\mathcal{B}]$ of the functor $[K,1]\colon [\mathcal{C},\mathcal{B}] \longrightarrow [\mathcal{A},\mathcal{B}]$ is fully faithful.

(ii) The functor $\widetilde{K}\colon \mathcal{C} \to [\mathcal{A}^{\mathrm{op}},\mathcal{V}]$ is fully faithful.

(iii) For each $C,D \in \mathcal{C}$ the map $\widetilde{K}\colon \mathcal{C}(C,D) \longrightarrow [\mathcal{A}^{\mathrm{op}},\mathcal{V}](\mathcal{C}(K-,C),\mathcal{C}(K-,D))$ is an isomorphism.

(iv) For each $C \in \mathcal{C}$, the identity cylinder $1\colon \mathcal{C}(K-,C) \longrightarrow \mathcal{C}(K-,C)$ as unit exhibits C as $\widetilde{K}C \star K$.

(v) The identity $1\colon K \longrightarrow 1_{\mathcal{C}}K$ exhibits $1_{\mathcal{C}}$ as $\mathrm{Lan}_K K$.

(vi) Some *isomorphism $\phi\colon K \longrightarrow 1_{\mathcal{C}}K$ exhibits $1_{\mathcal{C}}$ as $\mathrm{Lan}_K K$.*

Proof. Because a representable $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{V}$ preserves *all* limits that exist, its dual $\mathcal{C} \longrightarrow \mathcal{V}^{\mathrm{op}}$ preserves *all* colimits that exist in $\mathcal{C}$. Since the Yoneda embedding is fully faithful, and since $\widetilde{K}$ is the composite

$$\mathcal{C} \xrightarrow[Y]{} [\mathcal{C}^{\mathrm{op}},\mathcal{V}] \xrightarrow[{[K^{\mathrm{op}},1]}]{} [\mathcal{A}^{\mathrm{op}},\mathcal{V}],$$

it follows that (i) implies (ii). The direct translation (iii) of (ii) is equivalent by (3.5) to (iv), which is in turn equivalent by (4.18) to (v), itself a special case of (vi); so that it remains only to show that (vi) implies (i). Suppose then that $S\colon \mathcal{C} \longrightarrow \mathcal{B}$ is $\mathcal{A}$-cocontinuous; it thus preserves the colimit $\widetilde{K}C \star K$, and hence the left Kan extension in (vi), so that $S\phi\colon SK \longrightarrow SK$ exhibits S as $\mathrm{Lan}_K(SK)$. By Theorem 4.38 we therefore have for any $S'\colon \mathcal{C} \longrightarrow \mathcal{B}$ a representation $[\mathcal{C},\mathcal{B}](S,S') \cong [\mathcal{A},\mathcal{B}](SK,S'K)$, whose unit is $S\phi$. By (1.48), this isomorphism is the composite of $[K,1]\colon [\mathcal{C},\mathcal{B}](S,S') \longrightarrow [\mathcal{A},\mathcal{B}](SK,S'K)$ and $[\mathcal{A},\mathcal{B}](S\phi,1)\colon [\mathcal{A},\mathcal{B}](SK,S'K) \longrightarrow [\mathcal{A},\mathcal{B}](SK,S'K)$; the latter being an isomorphism since ϕ is, so is the former – which proves (i). ∎

A $\mathcal{V}$-functor $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ satisfying these equivalent conditions is said to be *dense*. (The original term used by Isbell, who introduced the concept for $\mathcal{V} = \mathbf{Set}$ in [35], was *left adequate*.) For the dual notion (Isbell's *right adequate*) we use *codense*; $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ is codense if $K^{\mathrm{op}}\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{C}^{\mathrm{op}}$ is dense. In the important special case where the dense $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ is the inclusion of a full subcategory, we merely say that $\mathcal{A}$ *is dense in* $\mathcal{C}$, leaving K to be understood; and if $\mathcal{A}$ is the one-object full subcategory $\{A\}$, we say that A *is dense in* $\mathcal{C}$.

It is clear that the density of K depends only on "the equivalence class of K in the 2-category of $\mathcal{V}$-functors"; that is, if K is dense, so is the composite

$$\overline{\mathcal{A}} \simeq \mathcal{A} \xrightarrow[K']{} \mathcal{C} \simeq \overline{\mathcal{C}} \tag{5.2}$$

where the outer functors are equivalences and where K' is isomorphic to K.

When C admits tensor products, the criterion (v) for the density of $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ becomes by (4.25) the requirement that the canonical map $\mathcal{C}(KA,C) \otimes KA \longrightarrow C$ express C as the coend

$$C \cong \int^{A} \mathcal{C}(KA,C) \otimes KA; \tag{5.3}$$

while when $\mathcal{V} = \mathbf{Set}$ it becomes by (4.27) the requirement that the canonical cone express C as the colimit

$$C \cong \mathrm{colim}(K/C \xrightarrow[d]{} \mathcal{A} \xrightarrow[K]{} \mathcal{C}). \tag{5.4}$$

If $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ is dense, applying $V\colon \mathcal{V}_0 \longrightarrow \mathbf{Set}$ to the isomorphism in Theorem 5.1(iii) above gives a bijection of sets

$$\mathcal{C}_0(C,D) \cong [A^{\mathrm{op}},\mathcal{V}]_0(\mathcal{C}(K-,C),\mathcal{C}(K-,D)), \tag{5.5}$$

whose meaning is that every $\mathcal{V}$-natural transformation $\alpha\colon \mathcal{C}(K-,C) \longrightarrow \mathcal{C}(K-,D)$ is of the form $\mathcal{C}(K-,f)$ for a unique $f\colon C \longrightarrow D$. It is clear that (5.5) is in turn equivalent to the density of K if V is conservative, and in particular in the classical case $\mathcal{V} = \mathbf{Set}$. It is also equivalent to density if $\mathcal{C}$ is cotensored; or even if $\mathcal{C}$ admits cotensor products $X_\beta \pitchfork C$ for some set $\{X_\beta\}$ constituting a strong generator of $\mathcal{V}_0$: for if we write (5.5) with $X \pitchfork D$ in place of D, we easily get $\mathcal{V}_0(X,\mathcal{C}(C,D)) \cong \mathcal{V}_0(X,[\mathcal{A}^{\mathrm{op}},\mathcal{V}](\mathcal{C}(K-,C),\mathcal{C}(K-,D)))$. Yet in general (5.5) is strictly weaker than density: take $\mathcal{V} = \mathbf{Cat}$, let $\mathcal{C}$ be the 2-category (3.54), and let K be the inclusion of the full subcategory $\{1\}$; we have the bijection (5.5) but not the isomorphism in $\mathcal{V}$ of Theorem 5.1(iii).

After first surveying various examples and elementary properties of density, we shall return in 5.11 below to the study of the important map $[K, 1]\colon \mathrm{Cocts}[\mathcal{C}, \mathcal{B}] \longrightarrow [\mathcal{A}, \mathcal{B}]$ of Theorem 5.1(i).

5.2 Composability and cancellability properties of dense functors

We begin by considering the composability and cancellation properties of dense functors. The positive results below are for any $\mathcal{V}$, while the counter-examples are all for $\mathcal{V} = \mathbf{Set}$; the assertions about the density or non-density of particular functors in the counter-examples are easily verified using (5.5). We consider a diagram

$$\begin{tikzcd} & \mathcal{D} \arrow[dr, "J"] \arrow[d, phantom, "\Uparrow\alpha"] & \\ \mathcal{A} \arrow[ur, "P"] \arrow[rr, "K"'] & & \mathcal{C} \end{tikzcd} \tag{5.6}$$

in which α is an isomorphism, so that $K \cong JP$. First, the density of P and of J, even if both are fully faithful, does not imply the density of JP. For an example, let $\mathcal{A} = 1$, let $\mathcal{D}$ be the category of finite sets, and let $\mathcal{C}$ be the functor category $[\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]$; let $P\colon 1 \longrightarrow \mathcal{D}$ correspond to the object $1 \in \mathcal{D}$, and let $J\colon \mathcal{D} \longrightarrow [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]$ be the Yoneda embedding.

Again, the density of P and of J does not imply the density of JP when P is essentially surjective and J is fully faithful; that is, when (5.6) is the factorization of K through its full image $\mathcal{D}$ in $\mathcal{C}$. For an example, let $\mathcal{A}$ be the discrete category identified with the set 2, let $\mathcal{D}$ be the unit category 1, let $\mathcal{C} = \mathbf{Set}$, and let J be the functor $1 \to \mathbf{Set}$ corresponding to the object $1 \in \mathbf{Set}$.

The following observation leads to some positive results:

Proposition 5.7 *If the isomorphism α in* (5.6) *exhibits J as* $\mathrm{Lan}_P K$*, then J is dense if and only if K is dense.*

Proof. Applying Theorem 4.7 to the diagram

$$\begin{tikzcd} & & \mathcal{C} \arrow[dd, "1_{\mathcal{C}}"] \\ & \mathcal{D} \arrow[ur, "J"] \arrow[u, phantom, "\Uparrow 1"] \arrow[dr, "J"] \arrow[d, phantom, "\Uparrow\alpha"] & \\ \mathcal{A} \arrow[ur, "P"] \arrow[rr, "K"'] & & \mathcal{C}, \end{tikzcd} \tag{5.8}$$

we conclude that 1 exhibits $1_{\mathcal{C}}$ as $\mathrm{Lan}_J J$ if and only if α exhibits $1_{\mathcal{C}}$ as $\mathrm{Lan}_{JP} K$. Pasting to the top edge of (5.8) the isomorphism $\alpha^{-1}\colon JP \longrightarrow K$, we see that α exhibits $1_{\mathcal{C}}$ as $\mathrm{Lan}_{JP} K$ if and only if 1 exhibits $1_{\mathcal{C}}$ as $\mathrm{Lan}_K K$. The conclusion now follows from Theorem 5.1(v). ∎

To say that α exhibits J as $\mathrm{Lan}_P K$ is clearly, since α is an isomorphism, to say that 1 exhibits J as $\mathrm{Lan}_P JP$. Now if P is dense, so that 1 exhibits $1_{\mathcal{D}}$ as $\mathrm{Lan}_P P$, this is by 4.1 to say that J preserves the Kan extension $\mathrm{Lan}_P P$. Thus Proposition 5.7 gives:

Proposition 5.9 *If P is dense and if J preserves $\mathrm{Lan}_P P$, then J is dense if and only if $K \cong JP$ is dense.* ∎

If P is fully faithful and if, for *any* $K\colon \mathcal{A} \longrightarrow \mathcal{C}$, we *define* $J\colon \mathcal{D} \longrightarrow \mathcal{C}$ as $\mathrm{Lan}_P K$, supposing this to exist, the unit α of this Kan extension is necessarily an isomorphism by Proposition 4.23. So Proposition 5.7 gives:

Proposition 5.10 *If $P\colon \mathcal{A} \longrightarrow \mathcal{D}$ is fully faithful and if $J\colon \mathcal{D} \longrightarrow \mathcal{C}$ is $\mathrm{Lan}_P K$ for some $K\colon \mathcal{A} \longrightarrow \mathcal{C}$, then J is dense if and only if K is dense.* ∎

A simple cancellability result is given by:

Proposition 5.11 *If P is essentially surjective and $K \cong JP$ is dense, then J is dense. In particular, if $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ is dense, the full image of K is a dense full subcategory of $\mathcal{C}$.*

Proof. $\widetilde{K}$ being fully faithful, so is $\widetilde{JP}$, which is clearly the composite

$$\mathcal{C} \xrightarrow[\widetilde{J}]{} [\mathcal{D}^{\mathrm{op}}, \mathcal{V}] \xrightarrow[{[P^{\mathrm{op}},1]}]{} [\mathcal{A}^{\mathrm{op}}, \mathcal{V}].$$

Thus, for $C, D \in \mathcal{C}$, the composite $[P^{\mathrm{op}}, 1]_{\widetilde{J}C, \widetilde{J}D} \cdot \widetilde{J}_{CD}$ is an isomorphism. The desired result that $\widetilde{J}_{CD}$ is an isomorphism will follow if $[P^{\mathrm{op}}, 1]_{TS}$ is shown to be a monomorphism in $\mathcal{V}_0$ (or in $\mathcal{V}_0'$ if $\mathcal{D}$ is large) for all $T, S\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathcal{V}$.

We have by (2.26) the commutative diagram

$$\begin{tikzcd} {[\mathcal{D}^{\mathrm{op}}, \mathcal{V}](T,S)} \arrow[rr, "{[P^{\mathrm{op}},1]_{TS}}"] \arrow[dr, "E_{PA}"'] & & {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}](TP,SP)} \arrow[dl, "E'_A"] \\ & {[TPA, SPA]}, & \end{tikzcd} \tag{5.12}$$

where E and E' are the respective evaluations. By the definition in 2.1 of the end in V, the family of maps $(E_D\colon [\mathcal{D}^{\mathrm{op}}, \mathcal{V}](T,S) \longrightarrow [TD, SD])_{D \in \mathcal{D}}$ is jointly monomorphic. Since P is essentially surjective, the same is true of the family $(E_{PA})_{A \in \mathcal{A}}$. It now follows from (5.12) that $[P^{\mathrm{op}}, 1]_{TS}$ is monomorphic, as desired. ∎

Note that, in contrast, the essential surjectivity of P, combined with the density of $K = JP$, does not imply the density of P. For an example, let $\mathcal{D}$ be any category that is not a preordered set, let $\mathcal{A}$ be the coproduct $\mathcal{D} + \mathcal{D}$, let $P\colon \mathcal{A} \longrightarrow \mathcal{D}$ be the codiagonal functor, and let $\mathcal{C} = 1$. As for the second assertion of the proposition above, we have seen in the example preceding Proposition 5.7 that the converse is false; *$K\colon \mathcal{A} \longrightarrow \mathcal{C}$ need not be dense when its full image $\mathcal{D}$ is dense in $\mathcal{C}$, even if $P\colon \mathcal{A} \longrightarrow \mathcal{D}$ is also dense.*

There is however a strong cancellability result when J is fully faithful:

Theorem 5.13 *Let J be fully faithful and let $K \cong JP$ be dense. Then both P and J are dense. Moreover the isomorphism $K \cong JP$ exhibits J as $\mathrm{Lan}_P K$.*

Proof. Using (5.2), suppose for simplicity that $K = JP$. Because J is fully faithful, we have $\widetilde{P} \cong \widetilde{K}J$ by (4.49). Since $\widetilde{K}$ and J are each fully faithful by hypothesis, so is $\widetilde{P}$; so that P is dense by Theorem 5.1(ii).

Again, because $\widetilde{K}$ and J are both fully faithful, we have the composite $\mathcal{V}$-natural isomorphism

$$\begin{tikzcd} \mathcal{C}(JD, C) \arrow[r, "\widetilde{K}"] & {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}](\mathcal{C}(JP-, JD), \mathcal{C}(JP-, C))} \arrow[d, "{[\mathcal{A}^{\mathrm{op}},\mathcal{V}](J_{P-,D},1)}"] \\ & {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}](\mathcal{D}(P-, D), \mathcal{C}(JP-, C)).} \end{tikzcd}$$

By (3.5), this isomorphism exhibits JD as $\widetilde{P}D \star JP$, with unit

$$J\colon \mathcal{D}(P-, D) \longrightarrow \mathcal{C}(JP-, JD).$$

By the dual of Theorem 4.6(ii), this is to say that $1\colon K \longrightarrow JP$ exhibits J as $\mathrm{Lan}_P K$. That J is dense now follows from Proposition 5.7. ∎

There are no similar cancellability results when $K = JP$ is fully faithful and dense, and P too is fully faithful. Here, even if P is dense, J need not be: for an example, let $\mathcal{A} = 1$, let $\mathcal{D}$ be the ordered set $\mathbf{2} = (0 \longrightarrow 1)$, let $P\colon 1 \longrightarrow \mathbf{2}$ have image 1, let $\mathcal{C} = \mathbf{Set}$, and let $J\colon \mathbf{2} \longrightarrow \mathbf{Set}$ send $(0 \longrightarrow 1)$ to the unique map $(\mathbf{2} \longrightarrow 1)$. Furthermore, even if J is dense, P need not be. Let $\mathcal{A} = 1$, let $\mathcal{D}$ be the category of the finite ordinals $\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots$ and increasing functions, let $P\colon 1 \longrightarrow \mathcal{D}$ have image 1, let $\mathcal{C} = \mathbf{Set}$, and let $J\colon \mathcal{D} \longrightarrow \mathbf{Set}$ be the (non-full) inclusion.

5.3 Examples of density; comparison with strong generation

Since a fully-faithful $\mathcal{V}$-functor is *a fortiori* conservative, comparison of Theorem 5.1 with 3.6 shows that a dense $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ is strongly generating; or rather, since we defined "strongly generating" only when $\mathcal{A}$ was small, that this is the case for small $\mathcal{A}$. While we saw in 3.6, however, that $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ is strongly generating precisely when its full image in $\mathcal{C}$ is so, we have seen in Proposition 5.11 and in the remark preceding Theorem 5.13 that the density of K is strictly stronger than the density of its full image. At any rate, it is clear that *$\mathcal{C}$ admits some dense $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ with $\mathcal{A}$ small if and only if it admits a small dense full subcategory; which latter is then* a fortiori *a strong generator.* (We shall abbreviate "small dense full subcategory" to "small dense subcategory" in appropriate contexts.)

When $\mathcal{V} = \mathbf{Set}$, if $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ is dense with $\mathcal{C}$ cocomplete and $\mathcal{A}$ small, (5.3) and the dual of (3.68) show the canonical map $\sum_A \mathcal{C}(KA, C) \cdot KA \longrightarrow C$ to be a coequalizer; whence it easily follows that the objects $\{KA\}_{A \in \mathcal{A}}$ constitute a regular generator for $\mathcal{C}$ in the sense of 4.10. However density is much stronger than being a strong generator, or even a regular one; and this even for $\mathcal{V} = \mathbf{Set}$.

In the category **Comp** of compact hausdorff spaces, for instance, the single space 1 is a regular generator – for it is clearly a generator, and every epimorphism in **Comp**

is a coequalizer. Yet *no* small subcategory $\mathcal{A}$ of **Comp** is dense. For let θ be a regular cardinal exceeding card A for each $A \in \mathcal{A}$, and let Ω be the set of ordinals $\leq \theta$, made into a compact hausdorff space by giving it the order topology. Let $2 = \{0, 1\}$ be the discrete two-element space, and define $f\colon \Omega \longrightarrow 2$ by $f(\theta) = 1$, $f(\alpha) = 0$ for $\alpha < \theta$. Although f is not continuous, composition with f defines a map $\phi_A\colon \mathbf{Comp}(A, \Omega) \longrightarrow \mathbf{Comp}(A, 2)$ for each $A \in \mathcal{A}$; because for any continuous $h\colon A \longrightarrow \Omega$, the subset $h^{-1}(\theta)$ is closed since $\{\theta\}$ is closed, while $h^{-1}(\Omega - \{\theta\})$ is closed by the choice of θ. Because ϕ_A is clearly natural in $A \in \mathcal{A}$, it would if $\mathcal{A}$ were dense be induced by some continuous $g\colon \Omega \longrightarrow 2$. Consideration of constant maps $A \longrightarrow \Omega$ gives the contradiction $g = f$ if some A is non-empty; while the empty set 0 is clearly not dense.

By Theorem 5.1(ii), each dense $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ exhibits $\mathcal{C}$, to within equivalence, as a full subcategory of $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$. We can characterize the subcategories $\mathcal{C}$ of $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ that arise in this way, and recapture K from $\mathcal{C}$:

Proposition 5.14 *The inclusion $J\colon \mathcal{C} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ of a full subcategory is isomorphic to $\widetilde{K}$ for some (necessarily dense) $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ if and only if each representable $\mathcal{A}(-, A) \in [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ admits a reflexion in $\mathcal{C}$. Then KA is the reflexion of $\mathcal{A}(-, A)$, and K is unique to within isomorphism.*

Proof. We are asking for a $\mathcal{V}$-natural isomorphism $JC \cong \mathcal{C}(K-, C)$, or $(JC)A \cong \mathcal{C}(KA, C)$. Since $(JC)A \cong [\mathcal{A}^{\mathrm{op}}, \mathcal{V}](\mathcal{A}(-, A), JC)$ by Yoneda, this is precisely to ask that each $\mathcal{A}(-, A)$ admit a reflexion KA in $\mathcal{C}$; whereupon K has by 1.10 a unique $\mathcal{V}$-functor structure. ∎

This has a simple consequence for cocomplete $\mathcal{C}$ and small $\mathcal{A}$:

Proposition 5.15 *Let $\mathcal{A}$ be small. Then $\mathcal{C}$ is equivalent to a reflective full subcategory of $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ if and only if $\mathcal{C}$ is cocomplete and there is a dense functor $K\colon \mathcal{A} \longrightarrow \mathcal{C}$. To within isomorphism, the "inclusion" $J\colon \mathcal{C} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is $\widetilde{K}$, while K is the composite of the Yoneda embedding $Y\colon \mathcal{A} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ and the left adjoint $S\colon [\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \longrightarrow \mathcal{C}$ of J. Such a $\mathcal{C}$ is complete.*

Proof. If $\mathcal{C}$ is reflective in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$, with reflexion $S \dashv J$, then certainly each $\mathcal{A}(-, A)$ admits a reflexion $KA = SYA$ in $\mathcal{C}$; whence by Proposition 5.14 we have $J \cong \widetilde{K}$ for the dense $K = SY\colon \mathcal{A} \longrightarrow \mathcal{C}$. Moreover $\mathcal{C}$ is cocomplete by 3.5, $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ being cocomplete by 3.3 and 3.74. Conversely, if $\mathcal{C}$ is cocomplete and $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ dense, the fully-faithful $\widetilde{K}\colon \mathcal{C} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ has by Theorem 4.51 the left adjoint $- \star K$; and $\mathcal{C}$ is complete by 3.5. ∎

Another simple case is that of a fully-faithful K:

Proposition 5.16 *$\mathcal{C}$ is equivalent to a full subcategory of $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ containing the representables if and only if there is a dense fully-faithful $K\colon \mathcal{A} \longrightarrow \mathcal{C}$. Then, to within isomorphism, the "inclusion" $J\colon \mathcal{C} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is $\widetilde{K}$, while the inclusion of $\mathcal{A}$ (identified with the representables) into $\mathcal{C}$ is K.*

Proof. If $\mathcal{C}$ contains the representables, each $\mathcal{A}(-, A)$ is its own reflexion in $\mathcal{C}$; whence by Proposition 5.14, the inclusion $J\colon \mathcal{C} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is isomorphic to $\widetilde{K}$ where $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ is the inclusion. Conversely, if $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ is fully faithful and dense, (4.22) gives $\widetilde{K}K \cong$

$Y\colon \mathcal{A} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$, so that $\mathcal{C}$ is equivalent to a full subcategory of $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ containing the representables. ∎

As extreme cases of Proposition 5.16, we have the density of $1\colon \mathcal{A} \longrightarrow \mathcal{A}$ ($\widetilde{1}$ here being the fully-faithful $Y\colon \mathcal{A} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$), and the density of $Y\colon \mathcal{A} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ ($\widetilde{Y}$ here by (4.30) being $1\colon [\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$). If in this latter we take $\mathcal{A} = \mathcal{I}$, which we may identify with the full subcategory $\{I\}$ of $\mathcal{V}$, we observe that:

$$\textit{The single object } I \textit{ is always dense in } \mathcal{V}. \tag{5.17}$$

For instance, 1 is dense in **Set**, the infinite cyclic group **Z** is dense in **Ab**, and **1** is dense in **Cat**.

The assertion (5.17) and these examples of it must be interpreted strictly in accordance with our conventions; there is no suggestion that the single object I is necessarily dense in the ordinary category $\mathcal{V}_0$, when $\mathcal{V} \neq \mathbf{Set}$: cf. 3.9 above. For instance, although **Z** is dense in the **Ab**-category **Ab**, it is not dense in the ordinary category $\mathbf{Ab}_0$, although it is a regular generator there. In fact, the single object $\mathbf{Z} \oplus \mathbf{Z}$ is dense in $\mathbf{Ab}_0$; this is a special case of the fact (cf. Isbell [35]) that, if $\mathcal{C}$ is an equational category of algebras whose basic operations all have arity $\leq n$, the free algebra Fn on n generators is dense in $\mathcal{C}$. (Once again, these assertions of density and of non-density for particular functors in the case $\mathcal{V} = \mathbf{Set}$ are easily verified using (5.5).)

Again, although **1** is dense in the **Cat**-category **Cat**, it is not even a generator in the ordinary category $\mathbf{Cat}_0$. The ordered-set category **2** is clearly a strong generator in $\mathbf{Cat}_0$; but it is not a regular generator, since $\mathbf{Cat}_0(\mathbf{2}, A) \cdot \mathbf{2} \longrightarrow A$ is not a coequalizer if A has one object and has a single non-identity map e with $e^2 = e$. Hence *a fortiori* **2** is not dense in $\mathbf{Cat}_0$. It is easily verified, however, that the ordered set $\mathbf{3} = \{0, 1, 2\}$ is dense in $\mathbf{Cat}_0$.

It is in fact possible, in contrast to (5.17), that $\mathcal{V}_0$ admit no small dense subcategory $\mathcal{A}$ whatsoever. An example is the cartesian closed category $\mathcal{V} = \mathbf{CGTop}$ of compactly-generated topological spaces. If $\mathcal{A}$ is a small full subcategory (which we may assume to contain some nonempty space), let θ be a regular cardinal exceeding $\operatorname{card} A$ for each $A \in \mathcal{A}$, and form the compact hausdorff space Ω as in the example preceding Proposition 5.14. By the argument used there, the non-closed set $\Omega - \{\theta\}$ has a closed inverse image under each continuous $h\colon A \longrightarrow \Omega$ with $A \in \mathcal{A}$; hence the canonical surjective map $\sum_{A \in \mathcal{A}} \mathcal{A}(A, \Omega) \cdot A \longrightarrow \Omega$ is not a topological quotient map. It factorizes therefore through the set Ω with the strictly-finer quotient topology (easily seen to be compactly generated), and thus through a proper subobject of A. Hence $\mathcal{A}$ is not even a strong generator, much less dense.

We observed in 3.6 that a $\mathcal{V}$-functor $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ is certainly strongly generating if the ordinary functor $K_0\colon \mathcal{A}_0 \longrightarrow \mathcal{C}_0$ is so. The analogue for density is false; if $\mathcal{V}$ is **Cat** and $\mathcal{C}$ is the 2-category of (3.54), the object 1 is dense in $\mathcal{C}_0$ but not in $\mathcal{C}$. Even more is true: a $\mathcal{V}$-category $\mathcal{C}$ may admit no small dense subcategory at all, although $\mathcal{C}_0$ admits one. For an example of this, take $\mathcal{V} = \mathbf{Cat}$ again, and let $\mathcal{C}_0 = \mathbf{Set}$, which has the small dense subcategory 1. Let the category $\mathcal{C}(X, Y)$ be discrete unless X is the empty set and Y is not.

The category $\mathcal{C}(0,Y)$ has only one object, t_Y say, and is therefore to be a monoid: let it be the free monoid on the symbols α_Y, where α runs through all cardinals less than $\operatorname{card} Y$. It remains to define the composite of the 2-cell $\alpha_Y\colon t_Y \longrightarrow t_Y\colon 0 \longrightarrow Y$ with $f\colon Y \longrightarrow Z$; we take it to be $\alpha_Z\colon t_Z \longrightarrow t_Z\colon 0 \longrightarrow Z$ if $\alpha < \operatorname{card} Z$, but to be $1\colon t_Z \longrightarrow t_Z\colon 0 \longrightarrow Z$ otherwise. Clearly $\mathcal{C}$ is now a 2-category. Let $\mathcal{A}$ be a small subcategory of $\mathcal{C}$ that is a candidate for density; by Theorem 5.13 we may suppose without loss of generality that $0 \in \mathcal{A}$. Let $\beta = \sup\{\operatorname{card} A \mid A \in \mathcal{A}\}$, and choose an $X \in \mathcal{C}$ with $\beta < \operatorname{card} X$. We exhibit a 2-natural transformation $\phi = (\phi_A\colon \mathcal{C}(A,X) \longrightarrow \mathcal{C}(A,1))_{A\in\mathcal{A}}$ not induced by the unique map $X \longrightarrow 1$, showing by (5.5) that $\mathcal{A}$ is not dense in $\mathcal{C}$.

For $A \neq 0$, we are forced to take for ϕ_A the unique functor into $\mathcal{C}(A,1) = 1$; while ϕ_0 is to be some monoid-map $\psi\colon \mathcal{C}(0,X) \longrightarrow \mathcal{C}(0,1)$. For any choice of ψ, the naturality of ϕ (as distinct from its 2-naturality) is automatic. The only non-trivial 2-naturality conditions involve the 2-cells $\alpha_A\colon t_A \longrightarrow t_A\colon 0 \longrightarrow A$ for $A \in \mathcal{A}$; such a condition evaluated at $f \in \mathcal{C}(A,X)$ gives $\psi(\alpha_X) = 1$ if $\alpha > 0$ and $\psi(0_X) = 0_1$; and hence ϕ is 2-natural if $\psi(\alpha_X)$ is so given for each $\alpha < \beta$. We are at liberty to choose ψ arbitrarily on the other generators of $\mathcal{C}(0,X)$, and in particular on β_X. If we choose $\psi(\beta_X)$ to be not 1 but 0_1, the 2-natural ϕ is not induced by $X \longrightarrow 1$.

5.4 Presentations of the density of a fully-faithful K in terms of K-absolute colimits

In this section we use the results of 4.10 above to discuss, in terms of colimits in $\mathcal{C}$, the density of a *fully faithful* $K\colon \mathcal{A} \longrightarrow \mathcal{C}$.

Let $F\colon \mathcal{L}^{\mathrm{op}} \longrightarrow \mathcal{V}$ and $P\colon \mathcal{L} \longrightarrow \mathcal{C}$. If the colimit $F \star P$ exists in $\mathcal{C}$ and is preserved by $\widetilde{K}\colon \mathcal{C} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$, we shall call it a *$K$-absolute colimit* in $\mathcal{C}$. If $\mathcal{L}$ is small, it follows from 3.3 that K-absoluteness of the colimit $F \star P$ is equivalent to its preservation by $\mathcal{C}(KA,-)\colon \mathcal{C} \longrightarrow \mathcal{V}$ for each $A \in \mathcal{A}$; and in any case preservation by these representables implies K-absoluteness. It is immediate from the definition in 4.10 of *K-cylinder* that:

Lemma 5.18 *The unit $\alpha\colon F \longrightarrow \mathcal{C}(P-, F \star P)$ of any K-absolute colimit in $\mathcal{C}$ is a K-cylinder.*

In proving Theorem 4.93 we observed that $1\colon \widetilde{K}C \longrightarrow \mathcal{C}(K-,C)$ is always a K-cylinder. Indeed, since the colimit $\widetilde{K}C \cong \widetilde{K}C \star Y$ is pointwise by (3.10), this cylinder is sent to a colimit-cylinder not only by $\widetilde{K}$ but by each $\mathcal{C}(KA,-)$: which for large $\mathcal{A}$ is in general a stronger assertion.

Theorem 5.19 *For a fully-faithful $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ the following are equivalent:*

(i) *K is dense.*

(ii) *Every K-cylinder $\alpha\colon F \longrightarrow \mathcal{C}(P-,C)$ is the unit of a K-absolute colimit $F \star P \cong C$.*

(iii) *For each $C \in \mathcal{C}$, the identity $1\colon \widetilde{K}C \longrightarrow \mathcal{C}(K-,C)$ as unit exhibits C as $\widetilde{K}C \star K$, and this colimit is K-absolute, preserved in fact by the representables $\mathcal{C}(KA,-)\colon \mathcal{C} \longrightarrow \mathcal{V}$.*

(iv) *Each $C \in \mathcal{C}$ is a K-absolute colimit of objects of $\mathcal{A}$, in the sense that C is the value of a K-absolute colimit of the form $F \star KQ$.*

(v) *There is a family $\Phi = (F_\gamma \colon \mathcal{L}_\gamma^{\mathrm{op}} \longrightarrow \mathcal{V}, P_\gamma \colon \mathcal{L}_\gamma \longrightarrow \mathcal{C})_{\gamma \in \Gamma}$ of diagrams in $\mathcal{C}$ such that each colimit $F_\gamma \star P_\gamma$ exists and is K-absolute, and such that $\mathcal{C}$ is the closure of $\mathcal{A}$ under this family of colimits.*

If $\mathcal{A}$ is small, the colimits in (iv) *and* (v) *may be taken to be small.*

Proof. (i) implies (ii) since, when $\widetilde{K}$ is fully faithful, it reflects colimits by 3.5. By the remark preceding the theorem, (ii) gives (iii) as a special case. Then (iii) gives (iv) by taking $F = \widetilde{K}C$ and $Q = 1$; and the domain of F here is $\mathcal{A}$, so that the colimit is small if $\mathcal{A}$ is. From (iv) we get (v) by taking for Φ the one-object family (F, KQ). Finally, by Theorem 4.98 and Lemma 5.4, it follows from (v) that $1 \colon K \longrightarrow K$ exhibits $1_{\mathcal{C}}$ as $\mathrm{Lan}_K K$, giving (i) by Theorem 5.1(v). ∎

The chief purpose of this theorem is to exhibit (v) as a practical criterion for the density of K. We shall say that the density of $\mathcal{A}$ in $\mathcal{C}$ is *presented* (or *exhibited*) by the family $\Phi = (F_\gamma, P_\gamma)_{\gamma \in \Gamma}$; or, more loosely, by the family $F_\gamma \star P_\gamma$ of K-absolute colimits. We may call such a family a *presentation* of the density of $\mathcal{A}$ in $\mathcal{C}$, or simple a *density presentation*. We give a few examples of density exhibited by such a presentation.

Proposition 5.20 *Given full subcategories $\mathcal{A} \subset \mathcal{D} \subset \mathcal{C}$, if every $D \in \mathcal{D}$ is a retract in $\mathcal{D}_0$ of some $A \in \mathcal{A}$, and if $\mathcal{D}$ is dense in $\mathcal{C}$, then $\mathcal{A}$ is dense in $\mathcal{C}$.*

Proof. Given $D \in \mathcal{D}$ there is an $A \in \mathcal{A}$ with $i \colon D \longrightarrow A$ and $r \colon A \longrightarrow D$ such that $ri = 1$. In these circumstances

$$A \underset{1}{\overset{ir}{\rightrightarrows}} A \xrightarrow[r]{} D \tag{5.21}$$

is a coequalizer in the ordinary category $\mathcal{C}_0$, and is moreover one preserved by *any* functor whatsoever. Being in particular preserved by the representables $\mathcal{C}(C, -)_0 \colon \mathcal{C}_0 \longrightarrow \mathcal{V}_0$, it is by 3.8 in fact a coequalizer in $\mathcal{C}$; and then, being preserved by the representables $\mathcal{C}(KA, -) \colon \mathcal{C} \longrightarrow \mathcal{V}$, where $K \colon \mathcal{A} \longrightarrow \mathcal{C}$ is the inclusion, it is a K-absolute colimit. On the other hand, if $J \colon \mathcal{D} \longrightarrow \mathcal{C}$ is the inclusion, every $C \in \mathcal{C}$ is the colimit $\widetilde{J}C \star J$ of objects of $\mathcal{D}$; and by Theorem 5.19(iii) this colimit is preserved by each representable $\mathcal{C}(JD, -)$, and so *a fortiori* by each $\mathcal{C}(KA, -)$; whence it, too, is K-absolute. Since $\mathcal{C}$ is therefore the closure of $\mathcal{A}$ under the K-absolute colimits of the two types (5.21) and $\widetilde{J}C \star J$, these constitute a presentation exhibiting $\mathcal{A}$ as dense in $\mathcal{C}$. ∎

A category with a small dense subcategory does not in general admit a one-object full subcategory that is dense; for instance, in $\mathbf{Set} \times \mathbf{Set}$, no single object can even be a generator. However:

Proposition 5.22 *Let $\mathcal{C}$ have the small dense subcategory $\mathcal{G}$, and let $H = \sum_{G \in \mathcal{G}} G$ exist in $\mathcal{C}$. Then H is dense in $\mathcal{C}$ provided that no $\mathcal{G}_0(G, G')$ is empty.*

Proof. Because of the last proviso, every $G \in \mathcal{G}$ is a retract in $\mathcal{C}_0$ of H. Since G is dense in $\mathcal{C}$, so too by Theorem 5.13 is $\mathcal{G} \cup \{H\}$. The density of H now follows from Proposition 5.20. ∎

Proposition 5.23 *Let $\mathcal{C}$ have a full subcategory $\mathcal{A}$, and an initial object 0 such that $\mathcal{C}(A, 0) = 0$ for each $A \in \mathcal{A}$. Then if $\mathcal{A} \cup \{0\}$ is dense in $\mathcal{C}$, so is $\mathcal{A}$.*

Proof. $0 \in \mathcal{C}$ is the colimit of the empty diagram, and by hypothesis it is K-absolute where $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ is the inclusion. The proof concludes like that of Proposition 5.20. ∎

Another example of a density presentation comes at once from the examples following Theorem 4.91 and Theorem 4.98:

Proposition 5.24 *In the case $\mathcal{V} = \mathbf{Set}$, let the small set $\{G_x\}_{x \in X}$ of objects constitute a regular generator of the cocomplete category $\mathcal{C}$. Then the full subcategory given by the small coproducts of the G_x is dense in $\mathcal{C}$; and if the G_x are all finitely presentable, we can replace "small" here by "finite".* ∎

For example, if $\mathcal{C}$ is an equational category of algebras (with finitary operations), the free algebra $F1$ on one generator is easily seen to be a regular generator and to be finitely presentable. We conclude by Proposition 5.24 that the free algebras Fm with m finite constitute a small dense subcategory. Here, however, we have already noted in 5.3 Isbell's result that, if all the basic operations have arity $\leq n$, then Fn alone is dense in $\mathcal{C}$.

Further examples of density presentations will be given in 5.7–5.10 below.

5.5 The characterization of functor categories $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$; small projectives and Cauchy completion; Morita equivalence

We now apply some of the results above to the characterization of $\mathcal{V}$-categories of the form $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ for some small $\mathcal{A}$.

We shall say that an object A in a cocomplete $\mathcal{V}$-category $\mathcal{C}$ is *small-projective* if the representable $\mathcal{C}(A, -)\colon \mathcal{C} \longrightarrow \mathcal{V}$ preserves all small colimits. We record for future use:

Proposition 5.25 *A retract D of a small-projective A in a cocomplete $\mathcal{C}$ is again small-projective.*

Proof. Let $F \star G$ be a small colimit in $\mathcal{C}$. We have the coequalizer (5.21) which is preserved by any functor (and hence sent to an equalizer by a contravariant functor). So we have a commutative diagram

$$\begin{tikzcd}
\mathcal{C}(D, F \star G) \arrow[r] \arrow[d] & \mathcal{C}(A, F \star G) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"'] & \mathcal{C}(A, F \star G) \arrow[d, "\cong"'] \\
F \star \mathcal{C}(D, G) \arrow[r] & F \star \mathcal{C}(A, G) \arrow[r, shift left] \arrow[r, shift right] & F \star \mathcal{C}(A, G)
\end{tikzcd}$$

where the rows are equalizer diagrams and the marked comparison maps are isomorphisms; whence the remaining comparison map is an isomorphism too. ∎

Theorem 5.26 *In order that $\mathcal{C}$ be equivalent to $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ for some small $\mathcal{A}$, it is necessary and sufficient that $\mathcal{C}$ be cocomplete and that there be a small set of small-projective objects in $\mathcal{C}$ constituting a strong generator for $\mathcal{C}$.*

Proof. We have seen that $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is cocomplete, and that the small full subcategory $\mathcal{A}$ of the representables is dense in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$, and thus *a fortiori* strongly generating. Moreover the representable $\mathcal{A}(-, A)$ is small-projective; for $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}](\mathcal{A}(-, A), ?)$ is isomorphic by Yoneda to the evaluation $E_A \colon [\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \longrightarrow \mathcal{V}$, which preserves all small colimits by 3.3. This proves the necessity.

For the sufficiency, let $K \colon \mathcal{G} \longrightarrow \mathcal{C}$ be the inclusion of the full subcategory determined by the small set of objects in question. Since K is strongly generating, $\widetilde{K} \colon \mathcal{C} \longrightarrow [\mathcal{G}^{\mathrm{op}}, \mathcal{V}]$ is conservative. Moreover $\widetilde{K}$ preserves all small colimits, since each $\mathcal{C}(KG, -)$ does so. In particular it preserves for each $C \in \mathcal{C}$ the colimit $\widetilde{K}C \star K$. Since, by the remarks following Lemma 5.4, $\widetilde{K}$ sends $1 \colon \widetilde{K}C \longrightarrow \mathcal{C}(K-, C)$ to a colimit cylinder, we conclude by the observations at the beginning of 3.6 that 1 is already a colimit cylinder, exhibiting C as $\widetilde{K}C \star K$. Thus K is dense by Theorem 5.1(iv); and consequently $\widetilde{K}$ is fully faithful.

To prove $\widetilde{K}$ an equivalence, it remains by 1.11 to prove it essentially surjective. Given $F \in [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$, the colimit $F \star K$ exists because $\mathcal{A}$ is small and $\mathcal{C}$ is cocomplete; and since $\widetilde{K}$ preserves small colimits we have $\widetilde{K}(F \star K) \cong F \star \widetilde{K}K \cong F \star Y \cong F$. ∎

There may well be various inequivalent $\mathcal{A}$ for which $\mathcal{C}$ is equivalent to $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$. If $\overline{\mathcal{A}}$ denotes the full subcategory of $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ determined by the small-projectives, then $\mathcal{A} \subset \overline{\mathcal{A}}$, and the proof above gives $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \simeq [\overline{\mathcal{A}}^{\mathrm{op}}, \mathcal{V}]$ if $\overline{\mathcal{A}}$ is small. In general $\overline{\mathcal{A}}$ is strictly bigger than $\mathcal{A}$; and for many $\mathcal{V}$ it is small whenever $\mathcal{A}$ is. We shall show that in 5.8 that, for $\mathcal{V} = \mathbf{Set}$, $\overline{\mathcal{A}}$ consists of the retracts of the representables. The same is true for $\mathcal{V} = \mathbf{Cat}$ or $\mathbf{CGTop}$; when $\mathcal{V} = \mathbf{Ab}$, $\overline{\mathcal{A}}$ consists of the retracts of finite coproducts of representables; when $\mathcal{V} = \overline{\mathbb{R}}_+$, Lawvere [54] has shown $\overline{\mathcal{A}}$ to be the Cauchy completion of the (generalized) metric space $\mathcal{A}$. In all these cases $\overline{\mathcal{A}}$ is small when $\mathcal{A}$ is.

If $\mathcal{V}$ is such that this last is so, we may call $\overline{\mathcal{A}}$ the *Cauchy completion* of $\mathcal{A}$. The passage from $\mathcal{A}$ to $\overline{\mathcal{A}}$ is a closure operation; for $\overline{\overline{\mathcal{A}}}$, as the small-projectives in $[\overline{\mathcal{A}}^{\mathrm{op}}, \mathcal{V}] \simeq [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$, is equivalent to $\overline{\mathcal{A}}$. Then $\mathcal{A}$ is *Cauchy complete* if the inclusion $\mathcal{A} \longrightarrow \overline{\mathcal{A}}$ is an equivalence. In these circumstances it follows at once from the facts above that, for small $\mathcal{A}$ and $\mathcal{B}$, we have $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \simeq [B^{\mathrm{op}}, \mathcal{V}]$ if and only if $\overline{\mathcal{A}} \simeq \overline{\mathcal{B}}$. This relation is called *Morita equivalence* between $\mathcal{A}$ and $\mathcal{B}$; it was first studied by Morita [65] (see also [2]) in the case $\mathcal{V} = \mathbf{Ab}$ with $\mathcal{A}$ and $\mathcal{B}$ one-object $\mathbf{Ab}$-categories – that is, rings R and S; in which case $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ and $[\mathcal{B}^{\mathrm{op}}, \mathcal{V}]$ are the categories of right R-modules and right S-modules. The general situation has been studied by various authors ([66], [68], [27], [33]), and may be expressed in terms of the *profunctors* or *distributors* introduced independently by Lawvere [53] and Bénabou [8]; a profunctor from $\mathcal{A}$ to $\mathcal{B}$ is a functor $T \colon \mathcal{B} \otimes \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{V}$, and these compose by the operation $S(C, -) \star T(-, A)$ to form a biclosed bicategory.

For some $\mathcal{V}$, however, $\overline{\mathcal{A}}$ may be large when $\mathcal{A}$ is small. An example is the category $\mathcal{V}_0$ of complete lattices and sup-preserving maps, with the tensor product such that the maps $A \otimes B \longrightarrow C$ are the functions $A \times B \longrightarrow C$ which are sup-preserving in each variable separately. Here the Cauchy completion of the unit $\mathcal{V}$-category $\mathcal{I}$ is not small, containing

for each set X the lattice $\mathcal{P}X$ of subsets of X. For such $\mathcal{V}$ the relation between $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ and $[\overline{\mathcal{A}}^{\mathrm{op}}, \mathcal{V}]$ is more complicated, and has been given by Lindner [55]:

Theorem 5.27 *For any $\mathcal{V}$ and any small $\mathcal{V}$-category $\mathcal{A}$, let $\overline{\mathcal{A}}$ consist of the small-projectives in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$. Then if $K\colon \overline{\mathcal{A}} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is the inclusion,*

$$\widetilde{K}\colon [\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \longrightarrow [\overline{\mathcal{A}}^{\mathrm{op}}, \mathcal{V}]$$

induces an equivalence of $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ with the full subcategory $\mathrm{Acc}[\overline{\mathcal{A}}^{\mathrm{op}}, \mathcal{V}]$ determined by the accessible functors; and the inverse to this equivalence is composition with the inclusion $Z\colon \mathcal{A} \longrightarrow \overline{\mathcal{A}}$.

Proof. Since $KZ = Y\colon \mathcal{A} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is dense, the fully-faithful K is dense by Theorem 5.13. Hence $\widetilde{K}\colon [\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \longrightarrow [\overline{\mathcal{A}}^{\mathrm{op}}, \mathcal{V}]$ is fully faithful, and it remains to determine its full image. For any $F\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{V}$ we have $\widetilde{K}F = [\mathcal{A}^{\mathrm{op}}, \mathcal{V}](K-, F)$, which by the Yoneda (3.17) is $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}](K-, F \star Y)$. Since each KB is small-projective, this is isomorphic to the colimit $F? \star [\mathcal{A}^{\mathrm{op}}, \mathcal{V}](K-, Y?) = F? \star [\mathcal{A}^{\mathrm{op}}, \mathcal{V}](K-, KZ?) \cong F? \star \overline{\mathcal{A}}(-, Z?)$. The last term here is a small colimit of representables $\overline{\mathcal{A}}^{\mathrm{op}} \longrightarrow \mathcal{V}$, and hence is accessible by Proposition 4.83. Moreover composing it with Z gives $F? \star \overline{\mathcal{A}}(Z-, Z?) \cong F? \star \mathcal{A}(-, ?) \cong F$, verifying that composition with Z is the inverse. To show finally that every accessible is in the image of $\widetilde{K}$, it suffices to show that every representable $\overline{\mathcal{A}}(-, B)$ is in this image, since $\widetilde{K}$ preserves small colimits. Composing this representable with Z gives $\overline{\mathcal{A}}(Z-, B)\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{V}$; but $\overline{\mathcal{A}}(Z-, B) = [\mathcal{A}^{\mathrm{op}}, \mathcal{V}](Y-, KB) \cong KB$. Now we have $\widetilde{K}KB = [\mathcal{A}^{\mathrm{op}}, \mathcal{V}](K-, KB) \cong \overline{\mathcal{A}}(-, B)$, as desired. ∎

This gives at once:

Proposition 5.28 *For any $\mathcal{V}$ and any small $\mathcal{A}$, $\mathcal{B}$, we have an equivalence $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \simeq [\mathcal{B}^{\mathrm{op}}, \mathcal{V}]$ if and only if we have $\overline{\mathcal{A}} \simeq \overline{\mathcal{B}}$.* ∎

Of course, since $\mathrm{Acc}[\overline{\mathcal{A}}^{\mathrm{op}}, \mathcal{V}] = [\overline{\mathcal{A}}^{\mathrm{op}}, \mathcal{V}]$ when $\overline{\mathcal{A}}$ is small, these results include those derived more simply above for small $\overline{\mathcal{A}}$. (Another approach to Proposition 5.28 uses the observation that, if $\mathcal{B}$ is cocomplete in Theorem 4.21, then $\widetilde{G}$ is cocontinuous if and only if each GA is small-projective; whereupon SF is small-projective for $F \in \overline{\mathcal{A}}$.)

5.6 Existence and characterization of the left Kan extension along a fully-faithful dense functor in terms of a density presentation; adjoint-functor theorems involving density

We return to the consideration of a fully-faithful $K\colon \mathcal{A} \longrightarrow \mathcal{C}$, now supposing it to be dense; and we use the results of 4.10 again, this time to discuss left Kan extensions along K. When we say of a functor $G\colon \mathcal{C} \longrightarrow \mathcal{B}$ that it preserves K-absolute colimits in $\mathcal{C}$, we mean only that it sends them to colimits in $\mathcal{B}$; the epithet "K-absolute" has in general no meaning when applied to colimits in $\mathcal{B}$. Since K is fully faithful, we have by Proposition 4.23 that $S\colon \mathcal{C} \longrightarrow \mathcal{B}$ is of the form $\mathrm{Lan}_K G$ for some $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ precisely when S is canonically isomorphic to $\mathrm{Lan}_K SK$.

Theorem 5.29 *If $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ is fully faithful and dense, the following properties of $S\colon \mathcal{C} \longrightarrow \mathcal{B}$ are equivalent:*

(i) $1\colon SK \longrightarrow SK$ *exhibits* S *as* $\mathrm{Lan}_K SK$.

(ii) S *preserves all K-absolute colimits.*

(iii) *For any density presentation $\Phi\colon (F_\gamma, P_\gamma)_{\gamma\in\Gamma}$ of $K\colon \mathcal{A} \longrightarrow \mathcal{C}$, each of the colimits $F_\gamma \star P_\gamma$ is preserved by S.*

(iv) S *preserves the colimit $\widetilde{K}C \star K$ for each $C \in \mathcal{C}$.*

(v) *There is some density presentation $\Phi = (F_\gamma, P_\gamma)_{\gamma\in\Gamma}$ of $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ for which S preserves each of the colimits $F_\gamma \star P_\gamma$.*

If $\mathcal{A}$ is small, it suffices in (ii) *that S preserve all small K-absolute colimits; and moreover it may then be supposed that the colimits in* (v) *are small.*

Proof. (i) implies (ii) (including the modification when $\mathcal{A}$ is small) by Theorem 4.97 and Theorem 5.19(ii). Trivially (ii) implies (iii); and since $(\widetilde{K}C, K)$ is, by Theorem 5.19, itself a density presentation, (iii) implies (iv) and (iv) implies (v). Finally (v) implies (i) by Theorem 4.98 and Lemma 5.4. ∎

As for the existence of $\mathrm{Lan}_K G$, a direct application of Theorem 4.98 gives:

Theorem 5.30 *Let $\Phi = (F_\gamma, P_\gamma)_{\gamma\in\Gamma}$ be a density presentation of the fully-faithful and dense $K\colon \mathcal{A} \longrightarrow \mathcal{C}$, and let $\mathcal{B}$ admit all F_γ-indexed colimits for all $\gamma \in \Gamma$. Then every $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ admits a left Kan extension $\mathrm{Lan}_K G\colon \mathcal{C} \longrightarrow \mathcal{B}$.* ∎

Recall the equivalence $[\mathcal{A}, \mathcal{B}]' \simeq [\mathcal{C}, \mathcal{B}]^l$ of Theorem 4.99, where $[\mathcal{A}, \mathcal{B}]'$ is the full subcategory given by those $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ admitting left Kan extensions along K, and where $[\mathcal{C}, \mathcal{B}]^l$ is the full subcategory given by those $S\colon \mathcal{C} \longrightarrow \mathcal{B}$ of the form $\mathrm{Lan}_K G$ for some G (necessarily isomorphic to SK). Where K is dense, Theorem 5.29 now identifies such S as those preserving all K-absolute colimits. We extract from this the most important practical case, writing $\mathrm{Cocts}^K[\mathcal{C}, \mathcal{B}]$ for the full subcategory of $[\mathcal{C}, \mathcal{B}]$ given by those $S\colon \mathcal{C} \longrightarrow \mathcal{B}$ which preserve all *small* K-absolute colimits:

Theorem 5.31 *Let $\Phi = (F_\gamma, P_\gamma)_{\gamma\in\Gamma}$ be a density presentation of the fully-faithful and dense $K\colon \mathcal{A} \longrightarrow \mathcal{C}$, for which the domain L_γ of each F_γ is small; and let $\mathcal{B}$ admit all F_γ-indexed colimits for all $\gamma \in \Gamma$. Then we have an equivalence*

$$\mathrm{Lan}_K\colon [\mathcal{A}, \mathcal{B}] \simeq \mathrm{Cocts}^K[\mathcal{C}, \mathcal{B}]$$

whose inverse is the restriction of $[K, 1]\colon [\mathcal{C}, \mathcal{B}] \longrightarrow [\mathcal{A}, \mathcal{B}]$. ∎

Note that, since the fully-faithful and dense $Y\colon \mathcal{A} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ has $\widetilde{Y} = 1$, *every* colimit in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is Y-absolute. Thus Theorem 5.31 generalizes the assertion (4.56) of Theorem 4.51.

We end this section with two adjoint-functor theorems involving small dense subcategories.

Theorem 5.32 *Let $\mathcal{C}$ be complete and cocomplete, and let $\mathcal{A}$ be a small dense full subcategory whose inclusion $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ has a density presentation $\Phi = (F_\gamma, P_\gamma)_{\gamma\in\Gamma}$. Let $\mathcal{B}$ be the closure under small colimits of a full subcategory $\mathcal{D}$. Let $T\colon \mathcal{C} \longrightarrow \mathcal{B}$ preserve the colimits $F_\gamma \star P_\gamma$, and let each $F_\gamma \star TP_\gamma$ be further preserved by the representables $\mathcal{B}(D, -)$ with $D \in \mathcal{D}$. Then T has a left adjoint if it is continuous.*

Proof. For each $D \in \mathcal{D}$, the functor $\mathcal{B}(D, T-)\colon \mathcal{C} \longrightarrow \mathcal{V}$ preserves the colimits $F_\gamma \star P_\gamma$, and hence by Theorem 5.29 is the left Kan extension of its restriction to $\mathcal{A}$. It is therefore accessible by Proposition 4.83; and since it is continuous, it is by Theorem 4.84 representable. That T has a left adjoint now follows from Proposition 3.37. ∎

A well-known special case of this theorem (see Satz 14.6 of [31]) is that where $\mathcal{V} = \mathbf{Set}$, where $\mathcal{C}$ and $\mathcal{B}$ are categories of finitary essentially-algebraic structures, and where $\mathcal{A}$ and $\mathcal{D}$ are the finitely-presentable objects in $\mathcal{C}$ and $\mathcal{B}$ respectively; the conclusion is then that a continuous T has a left adjoint if it preserves filtered colimits.

The next theorem, although we place it here for comparison, is quite different in nature, and could have been proved immediately after Theorem 5.1. It is a variant of Theorem 4.90 (the Special Adjoint Functor Theorem) – or rather, of its dual. A small dense subcategory replaces the generator, and then the hypothesis on cointersections may be dropped; moreover the present theorem is valid for any $\mathcal{V}$.

Theorem 5.33 *If the cocomplete $\mathcal{C}$ has a small dense subcategory $\mathcal{A}$, every cocontinuous $S\colon \mathcal{C} \longrightarrow \mathcal{B}$ has a right adjoint.*

Proof. If $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ is the inclusion, $\mathrm{Lan}_{SK}K$ exists because $\mathcal{A}$ is small and $\mathcal{C}$ is cocomplete. Since $\mathrm{Lan}_K K \cong 1_{\mathcal{C}}$ by Theorem 5.1, it follows from Theorem 4.47 that $\mathrm{Lan}_S 1_{\mathcal{C}}$ exists. Again, since $\mathcal{A}$ is small, the cocontinuous S preserves the Kan extensions $\mathrm{Lan}_K K$ and $\mathrm{Lan}_{SK} K$; whence, by Theorem 4.47 again, it preserves $\mathrm{Lan}_S 1_{\mathcal{C}}$. That S has a right adjoint now follows from Theorem 4.81. ∎

Note that a special case of this theorem appeared in Theorem 4.51.

5.7 The free completion of $\mathcal{A}$ with respect to all colimits with indexing-type in $\mathcal{F}$

We now describe the free cocompletion of a $\mathcal{V}$-category with respect to colimits of some given class.

First observe that:

Proposition 5.34 *For any $\mathcal{V}$-category $\mathcal{A}$, the set of accessible functors $\mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{V}$ is closed under small colimits.*

Proof. Let $H\colon \mathcal{L}^{\mathrm{op}} \longrightarrow \mathcal{V}$ and $P\colon \mathcal{L} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$, where $\mathcal{L}$ is small and each PL is accessible. By Proposition 4.83, each PL is the left Kan extension of its restriction to some small full subcategory $\mathcal{C}_L$ of $\mathcal{A}^{\mathrm{op}}$. The union $\mathcal{C}$ of the $\mathcal{C}_L$ for $L \in \mathcal{L}$ is still small, because $\mathcal{L}$ is small; write $K\colon \mathcal{C} \longrightarrow \mathcal{A}^{\mathrm{op}}$ for the inclusion. Since K is fully faithful, it follows from Theorem 4.47 that each PL is the left Kan extension of its restriction $(PL)K$ to $\mathcal{C}$. Write $Q\colon \mathcal{L} \longrightarrow [\mathcal{C}, \mathcal{V}]$ for the composite of P with $[K, 1]\colon [\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \longrightarrow [\mathcal{C}, \mathcal{V}]$, so that $QL = (PL)K$; then we have $PL \cong \mathrm{Lan}_K QL$. Now (3.21), expressing the cocontinuity of $\star$ in its second variable, along with (3.16), justifies the calculation $(H \star P)A = H? \star (P?)A \cong H? \star (\widetilde{K}A \star Q?) \cong \widetilde{K}A \star (H \star Q) = (\mathrm{Lan}_K(H \star Q))A$. So $H \star P \cong \mathrm{Lan}_K(H \star Q)$ is accessible by Theorem 4.83. ∎

In other words, the full subcategory $\mathrm{Acc}[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ of $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is closed under small colimits. Note from (4.41) that $\mathrm{Acc}[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ *is a $\mathcal{V}$-category, even if $\mathcal{A}$ is large.*

Now let $\mathcal{F}$ be any set of functors with codomain $\mathcal{V}$, for each element $F\colon \mathcal{L} \longrightarrow \mathcal{V}$ of which the domain $\mathcal{L}$ is *small*; that is, *$\mathcal{F}$ is a set of small indexing types.* We say that $\mathcal{C}$ *is $\mathcal{F}$-cocomplete* if it admits all colimits indexed by the elements of $\mathcal{F}$, and that $S\colon \mathcal{C} \longrightarrow \mathcal{B}$ is *$\mathcal{F}$-cocontinuous* if it preserves all such colimits – which we may call *$\mathcal{F}$-colimits*. We write $\mathcal{F}$-$\mathrm{Cocts}[\mathcal{C}, \mathcal{B}]$ for the full subcategory of $[\mathcal{C}, \mathcal{B}]$ determined by the $\mathcal{F}$-cocontinuous functors. In the following we regard $\mathcal{A}$ as embedded in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ by Yoneda.

Theorem 5.35 *For any set $\mathcal{F}$ of small indexing types and any $\mathcal{V}$-category $\mathcal{A}$, let $\overline{\mathcal{A}}$ be the closure of $\mathcal{A}$ in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ under $\mathcal{F}$-colimits, and let $K\colon \mathcal{A} \longrightarrow \overline{\mathcal{A}}$ be the inclusion. Then $\overline{\mathcal{A}} \subset \mathrm{Acc}[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$, and is therefore a $\mathcal{V}$-category. When $\mathcal{A}$ is small and the set $\mathcal{F}$ is small, $\overline{\mathcal{A}}$ is small. $\overline{\mathcal{A}}$ is $\mathcal{F}$-cocomplete, and every $\mathcal{F}$-colimit in $\overline{\mathcal{A}}$ is K-absolute; the totality of these $\mathcal{F}$-colimits constitutes a density presentation of K. A functor $S\colon \overline{\mathcal{A}} \longrightarrow \mathcal{B}$ is of the form $\mathrm{Lan}_K SK$ precisely when it is $\mathcal{F}$-cocontinuous; and we have an equivalence*

$$\mathrm{Lan}_K\colon [\mathcal{A}, \mathcal{B}]' \simeq \mathcal{F}\text{-}\mathrm{Cocts}[\overline{\mathcal{A}}, \mathcal{B}],$$

whose inverse is the restriction of $[K, 1]$, where $[\mathcal{A}, \mathcal{B}]'$ is the full subcategory of $[\mathcal{A}, \mathcal{B}]$ consisting of those G for which $\mathrm{Lan}_K G$ exists. If $\mathcal{B}$ is $\mathcal{F}$-cocomplete, this is all of $[\mathcal{A}, \mathcal{B}]$, and the equivalence becomes

$$\mathrm{Lan}_K\colon [\mathcal{A}, \mathcal{B}] \simeq \mathcal{F}\text{-}\mathrm{Cocts}[\overline{\mathcal{A}}, \mathcal{B}],$$

exhibiting $\overline{\mathcal{A}}$ as "the free $\mathcal{F}$-cocompletion of $\mathcal{A}$".

Proof. That $\overline{\mathcal{A}} \subset \mathrm{Acc}[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ follows from the transfinite construction of $\overline{\mathcal{A}}$ in 3.5, along with Proposition 5.34. That $\overline{\mathcal{A}}$ is small when $\mathcal{A}$ and $\mathcal{F}$ are small was observed in 3.5. $\overline{\mathcal{A}}$ is $\mathcal{F}$-cocomplete by its construction, since $\mathcal{F}$-colimits certainly exist in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$, each element of $\mathcal{F}$ being small; and these $\mathcal{F}$-colimits in $\overline{\mathcal{A}}$ are K-absolute, being preserved by construction under the inclusion $\overline{\mathcal{A}} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$, which is $\widetilde{K}$ by Proposition 5.16. The $\mathcal{F}$-colimits in $\overline{\mathcal{A}}$ form a density presentation for K by the definition of this following Theorem 5.19; and then the characterisation of the S of the form $\mathrm{Lan}_K SK$ follows from Theorem 5.29. The first equivalence displayed comes from Theorem 4.99, and the identification of $[\mathcal{A}, \mathcal{B}]'$ with $[\mathcal{A}, \mathcal{B}]$ when $\mathcal{B}$ is $\mathcal{F}$-cocomplete comes from Theorem 5.30. ∎

Note that the second equivalence of Theorem 5.35 is a special case of that of Theorem 5.31; what is new here is the *freeness* of the $\mathcal{F}$-cocompletion, which allows us to replace $\mathrm{Cocts}^K[\mathcal{C},\mathcal{B}]$ by $\mathcal{F}$-$\mathrm{Cocts}[\mathcal{C},\mathcal{B}]$. (Often, using the word "completion" in a wider sense, we may refer to the $\mathcal{F}$-cocompletion of $\mathcal{A}$ as the *completion of* $\mathcal{A}$ *under* $\mathcal{F}$*-colimits.*)

As the first example of such an $\mathcal{F}$-cocompletion, take $\mathcal{F}$ to consist of *all* small indexing types, so that $\overline{\mathcal{A}}$ is simply *the cocompletion of* $\mathcal{A}$. From Proposition 4.83 and Proposition 5.34, this cocompletion is precisely the $\mathcal{V}$-category $\mathrm{Acc}[\mathcal{A}^{\mathrm{op}},\mathcal{V}]$. When $\mathcal{A}$ is small, this reduces to $[\mathcal{A}^{\mathrm{op}},\mathcal{V}]$, and we re-find (part of) Theorem 4.51.

Other examples with $\mathcal{V}\neq\mathbf{Set}$ will be found in the article [45] on enriched essentially-algebraic structures. For the remaining examples discussed below we take $\mathcal{V}=\mathbf{Set}$. The colimits we are interested in will then be conical ones, so that $\mathcal{F}$ is in effect a set of small categories.

Before we proceed, however, we anticipate a possible misunderstanding that may arise from the very special nature of our main examples. In the transfinite construction of 3.5 for the closure $\overline{\mathcal{A}}$ of $\mathcal{A}$ under $\mathcal{F}$-colimits in $[\mathcal{A}^{\mathrm{op}},\mathcal{V}]$, the first step consists in taking all the $\mathcal{F}$-colimits in $[\mathcal{A}^{\mathrm{op}},\mathcal{V}]$ of the objects of $\mathcal{A}$. In some important cases the full subcategory given by these $\mathcal{F}$-colimits is itself closed under $\mathcal{F}$-colimits and hence constitutes $\overline{\mathcal{A}}$; the construction then terminates after one step. This was so in the example above, where $\mathcal{F}$ consisted of *all* small indexing types; and it will also be so in the three important examples of 5.8–5.10 below. However, it is far from true in general. For instance, taking $\mathcal{V}=\mathbf{Set}$ henceforth, if $\mathcal{F}$ consists of two discrete categories 0 and 2, so that $\overline{\mathcal{A}}$ is the closure of $\mathcal{A}$ under finite coproducts, we reach $\overline{\mathcal{A}}$ only after ω steps of the iteration. Less trivially, if $\mathcal{F}$ consists of the single category $0\rightrightarrows 1$, so that $\overline{\mathcal{A}}$ is the closure of $\mathcal{A}$ under coequalizers, consider the case where $\mathcal{A}$ has two objects A,B and three non identity maps $f,g,h\colon A\longrightarrow B$. In the first step we add in the coequalizers P,Q,R of the respective pairs (f,g), (f,h), and (g,h); but now there are two maps $A\longrightarrow P$, and we must next add in the coequalizer of these, and so on.

5.8 Cauchy completion as the idempotent-splitting completion when $\mathcal{V}=\mathbf{Set}$

For our first main example with $\mathcal{V}=\mathbf{Set}$, let $\mathcal{F}$ consist of the single category with one object and with one non-identity endomorphism e satisfying $e^2=e$. Then a diagram of type $\mathcal{F}$ in $\mathcal{A}$ is an object A with an idempotent endomorphism e. This diagram has a colimit $r\colon A\longrightarrow B$ in $\mathcal{A}$ if and only if the idempotent e *splits*; that is, e has the form $e=ir$ for maps $r\colon A\longrightarrow B$ and $i\colon B\longrightarrow A$ with $ri=1$. Such a splitting is unique (to within isomorphism) if it exists; so that the split idempotents correspond bijectively to the retracts of A in $\mathcal{A}$. In this case $\overline{\mathcal{A}}$, which we may call the *idempotent-splitting completion of* $\mathcal{A}$, clearly consists of all the retracts in $[\mathcal{A}^{\mathrm{op}},\mathbf{Set}]$ of the representables $\mathcal{A}(-,A)\in\mathcal{A}$; it is evidently small if $\mathcal{A}$ is. Moreover $\mathcal{B}$ is $\mathcal{F}$-complete precisely when all idempotents split in $\mathcal{B}$. In fact we have the result promised in 5.5 above:

Theorem 5.36 *For a small* $\mathcal{A}$*, the small projectives in* $[\mathcal{A}^{\mathrm{op}},\mathbf{Set}]$ *are precisely the re-*

tracts of the representables; so that the idempotent-splitting completion of $\mathcal{A}$ is its Cauchy completion. Moreover $\mathcal{A}$ is Cauchy complete exactly when all the idempotents split in $\mathcal{A}$.

Proof. We saw in the proof of Theorem 5.26 that the representables in $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ are small-projective, whence so too are their retracts by Proposition 5.25. Suppose conversely that $F\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is small-projective. Then $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}](F, -)$ preserves all small colimits, and in particular the canonical expression (3.32) of F as a colimit of representables. Thus $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}](F, F)$ is the canonical colimit, indexed by $(\mathrm{el}\, F)^{\mathrm{op}}$, of the sets $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}](F, \mathcal{A}(-, A))$. Since colimits in $\mathbf{Set}$ are calculated by (3.35), the element 1_F of $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}](F, F)$ is the composite of some $r\colon \mathcal{A}(-, A) \longrightarrow F$ in $\mathrm{el}\, F$ with some $i\colon F \longrightarrow \mathcal{A}(-, A)$; that is, F is a retract of some $\mathcal{A}(-, A)$.

To say that $\mathcal{A}$ is Cauchy complete is, by 5.5, to say that the inclusion $\mathcal{A} \longrightarrow \overline{\mathcal{A}}$ is an equivalence, or that every object of $\overline{\mathcal{A}}$ is isomorphic to one in $\mathcal{A}$. Since idempotents split uniquely if at all, this is exactly to say that all idempotents split in $\mathcal{A}$. ∎

Note that, in this example, the inclusion $\mathcal{A} \longrightarrow \overline{\mathcal{A}}$ preserves such $\mathcal{F}$-colimits as exist in $\mathcal{A}$. In fact $\mathcal{F}$-colimits for this $\mathcal{F}$ are *absolute* – they are preserved by any functor whatsoever. This example is quite atypical, however: for a general $\mathcal{F}$ the inclusion $\mathcal{A} \longrightarrow \overline{\mathcal{A}}$ by no means preserves the $\mathcal{F}$-colimits that exist in $\mathcal{A}$. For instance, consider the completion of $\mathcal{A}$ under finite coproducts: $\mathcal{A}(C, A+B)$ is in general quite different from $\mathcal{A}(C, A) + \mathcal{A}(C, B)$.

5.9 The finite-colimit completion when $\mathcal{V} = \mathbf{Set}$

Our next example with $\mathcal{V} = \mathbf{Set}$ is the completion of $\mathcal{A}$ under finite colimits; since the number of finite categories is small, *the finite completion of $\mathcal{A}$ is small if $\mathcal{A}$ is.*

Theorem 5.37 *The completion of $\mathcal{A}$ under finite colimits is the full subcategory of $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ determined by all finite colimits of the representables.*

Proof. Let $\overline{\mathcal{A}} \subset [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ consist of all finite colimits of the representables; we are to show that $\overline{\mathcal{A}}$ is closed under finite colimits. Clearly $\overline{\mathcal{A}}$ contains the initial object of $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$. Consider objects $F = \mathrm{colim}\, YT$ and $G = \mathrm{colim}\, YS$ of $\overline{\mathcal{A}}$, where $T\colon \mathcal{P} \longrightarrow \mathcal{A}$ and $F\colon \mathcal{Q} \longrightarrow \mathcal{A}$ with $\mathcal{P}$ and $\mathcal{Q}$ finite. Let $R\colon \mathcal{P} + \mathcal{Q} \longrightarrow \mathcal{A}$ be the functor with components T and S. Since an inductive cone over YR with any vertex H clearly consists of independent cones over YT and YS, it follows that $\mathrm{colim}\, YR = F + G$. Hence $\overline{\mathcal{A}}$ is closed under finite coproducts, and it remains to prove it closed under finite coequalizers.

With F and G as above, let the colimit cones be $(\rho_P\colon \mathcal{A}(-, TP) \longrightarrow F)$ and $(\sigma_Q\colon \mathcal{A}(-, SQ) \longrightarrow G)$, and consider $\alpha, \beta\colon F \longrightarrow G$. Since the ρ_P are jointly epimorphic, a map $\gamma\colon G \longrightarrow H$ satisfies $\gamma\alpha = \gamma\beta$ if and only if it satisfies $\gamma\alpha\rho_P = \gamma\beta\rho_P$ for each $P \in \mathcal{P}$. The map $\alpha\rho_P\colon \mathcal{A}(-, TP) \longrightarrow G$ corresponds by Yoneda to an element of GTP. Since the $\sigma_{Q,TP}\colon \mathcal{A}(TP, SQ) \longrightarrow GTP$ are jointly surjective in $\mathbf{Set}$, this element is the image under $\sigma_{P',TP}$ of some $u_P \in \mathcal{A}(TP, SP')$ for some $P' \in \mathcal{Q}$. By the naturality of Yoneda we then

have commutativity in the left diagram of

$$\begin{tikzcd} \mathcal{A}(-,TP) \arrow[r, "\mathcal{A}(-,u_P)"] \arrow[d, "\rho_P"'] & \mathcal{A}(-,SP') \arrow[d, "\sigma_{P'}"] \\ F \arrow[r, "\alpha"'] & G, \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}(-,TP) \arrow[r, "\mathcal{A}(-,v_P)"] \arrow[d, "\rho_P"'] & \mathcal{A}(-,SP'') \arrow[d, "\sigma_{P''}"] \\ F \arrow[r, "\beta"'] & G; \end{tikzcd}$$

while the corresponding right diagram comes from a similar argument applied to β. The condition $\gamma\alpha\rho_P = \gamma\beta\rho_P$ now becomes $\gamma\sigma_{P'}\mathcal{A}(-,u_P) = \gamma\sigma_{P''}\mathcal{A}(-,v_P)$.

Define a finite graph $\mathcal{G}$ whose object-set is $(\operatorname{ob}\mathcal{P}) + (\operatorname{ob}\mathcal{Q})$, and whose arrows are all those of $\mathcal{Q}$ together with arrows $x_P\colon P \longrightarrow P'$ and $y_P\colon P \longrightarrow P''$ for each $P \in \mathcal{P}$. Let $\mathcal{S}$ be the category generated by $\mathcal{G}$ subject to the composition and identity relations that already obtain in $\mathcal{Q}$; clearly $\mathcal{S}$ is finite. Define $N\colon \mathcal{S} \longrightarrow \mathcal{A}$ to agree with S on $\mathcal{Q}$, to send $P \in \mathcal{P}$ to TP, and to send x_P and y_P to u_P and v_P respectively. Then clearly $\operatorname{colim} YN$ is the coequalizer of α and β. ∎

(It may not be inferred from the truth of Proposition 5.34 and Theorem 5.37 that, for a *general* cocomplete $\mathcal{C}$, the construction of the colimit-closure or of the finite-colimit-closure of a full subcategory $\mathcal{A}$ terminates after one step. The ordinary category $\mathbf{Cat}_0$ is by Proposition 3.40 the colimit-closure of the full subcategory $\{\mathbf{2}\}$, since $\mathbf{2}$ is a strong generator, as we observed in 5.3. The category A with one object and one non-identity idempotent map is in fact easily seen to lie in the finite-colimit-closure of $\{\mathbf{2}\}$. Yet A is not the colimit in $\mathbf{Cat}_0$ of any functor factorizing through $\{\mathbf{2}\}$; for it is soon verified that no map from a copower of $\mathbf{2}$ to A is a coequalizer.)

5.10 The filtered-colimit completion when $\mathcal{V} = \mathbf{Set}$; flat and left-exact functors

For our final example with $\mathcal{V} = \mathbf{Set}$ we consider the completion of a small $\mathcal{A}$ under small filtered colimits. Let us write this as $\widehat{\mathcal{A}} \subset [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$, and for the rest of this section keep $\overline{\mathcal{A}}$ for the completion of $\mathcal{A}$ under finite colimits.

A functor $F\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is called *flat* if the functor $F\star -\colon [\mathcal{A}, \mathbf{Set}] \longrightarrow \mathbf{Set}$ is left exact – that is, finite-limit-preserving. Every representable $\mathcal{A}(-,A)$ is flat, for $\mathcal{A}(-,A)\star ? = (?)A$ by Yoneda, which is the evaluation $E_A\colon [\mathcal{A}, \mathbf{Set}] \longrightarrow \mathbf{Set}$, preserving all small limits. Moreover, since $\star$ is cocontinuous in both variables, it follows from Theorem 4.73 that a small filtered colimit of flat functors is flat.

Theorem 5.38 *For $F\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ with $\mathcal{A}$ small, the following are equivalent:*

(i) $(\operatorname{el} F)^{\mathrm{op}}$ *is filtered.*

(ii) *The canonical expression* (3.32) *of F as a colimit of representables is a filtered colimit.*

(iii) *F is some small filtered colimit of representables.*

(iv) *F lies in the completion $\widehat{\mathcal{A}}$ of $\mathcal{A}$ under small filtered colimits.*

(v) *F is flat.*

Proof. (i) implies (ii) since the colimit (3.32) is indexed by $(\operatorname{el} F)^{\mathrm{op}}$; that (ii) implies (iii) and (iii) implies (iv) are trivial. Since $\widehat{\mathcal{A}}$ is by Theorem 5.35 the closure of $\mathcal{A}$ in $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ under filtered colimits, (iv) implies (v) by the remarks preceding the theorem. Thus we have only to show that (v) implies (i).

Let $T\colon \mathcal{P} \longrightarrow \operatorname{el} F$ where $\mathcal{P}$ is finite. As in 4.7, T corresponds to a functor $S\colon \mathcal{P} \longrightarrow \mathcal{A}^{\mathrm{op}}$ and an element $\alpha \in \lim FS$. By the flatness of F we have $F \star \lim_P \mathcal{A}(SP, -) \cong \lim_P (F \star \mathcal{A}(SP, -))$; and by Yoneda the latter is $\lim_P FSP = \lim FS$. By the formula (3.70) and the dual of (3.68), we have therefore a canonical surjection

$$\sum_{A \in \mathcal{A}} FA \times \lim_P \mathcal{A}(SP, A) \longrightarrow \lim FS.$$

Let some inverse image of α under this surjection be given by $(x \in FA,\ \beta \in \lim_P \mathcal{A}(SP, A))$. By (3.28), we can regard β as an inductive cone in $\mathcal{A}$ over S with vertex A. Since $F\beta_P\colon FA \longrightarrow FSP$ maps x to α_P by construction, β is equally an inductive cone in $(\operatorname{el} F)^{\mathrm{op}}$ over T with vertex (A, x). Thus $(\operatorname{el} F)^{\mathrm{op}}$ is filtered. ∎

Proposition 5.39 *A flat functor $F\colon \mathcal{A} \longrightarrow \mathbf{Set}$ with $\mathcal{A}$ small preserves any finite limit that exists in $\mathcal{A}$. If $\mathcal{A}$ is finitely complete, the flat functors $F\colon \mathcal{A} \longrightarrow \mathbf{Set}$ are precisely the left-exact ones.*

Proof. If $S\colon \mathcal{P} \longrightarrow \mathcal{A}$ with $\mathcal{P}$ finite has a limit in $\mathcal{A}$, the flatness of F gives, as in the proof above, that $F \star \lim_P \mathcal{A}(-, SP) \cong \lim FS$. However the left side is now isomorphic to $F \star \mathcal{A}(-, \lim S) \cong F(\lim S)$. So a flat functor F is certainly left exact if $\mathcal{A}$ is finitely complete. Conversely, if $\mathcal{A}$ is finitely complete and F is left exact, $\operatorname{el} F$ is finitely complete by Proposition 4.87; hence $(\operatorname{el} F)^{\mathrm{op}}$ is finitely cocomplete and therefore filtered, so that F is flat by Theorem 5.38. ∎

Proposition 5.40 *The finitely-presentable objects in the filtered-colimit-completion $\widehat{\mathcal{A}}$ of a small $\mathcal{A}$ are the retracts of the representables; so that they reduce to $\mathcal{A}$ if idempotents split in $\mathcal{A}$.*

Proof. Since filtered colimits in $\widehat{\mathcal{A}}$ are formed as in $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$, the proof of Theorem 5.36 applies here virtually unchanged. The new point is that, for a finitely presentable F, the functor $\widehat{\mathcal{A}}(F, -)$ is only known to preserve *filtered* colimits; but now the canonical colimit of representables giving F *is* filtered. ∎

Proposition 5.41 *If $\overline{\mathcal{A}}$ is the finite-colimit-completion of the small $\mathcal{A}$, then the inclusion $\overline{\mathcal{A}} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ is, to within equivalence, the filtered-colimit-completion $\widehat{\overline{\mathcal{A}}}$ of $\overline{\mathcal{A}}$. In consequence, every $F\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is a filtered colimit of objects of $\overline{\mathcal{A}}$, and $\overline{\mathcal{A}}$ consists of the finitely-presentable objects of $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$.*

Proof. Let $K\colon \mathcal{A} \longrightarrow \overline{\mathcal{A}}$ be the inclusion, so that by Proposition 5.16 the inclusion $\overline{\mathcal{A}} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ is $\widetilde{K}$. Using Lex for "left exact", we have by Theorem 5.38 and Proposition 5.39 that $\widehat{\overline{\mathcal{A}}}$ is the full subcategory $\mathrm{Lex}[\overline{\mathcal{A}}^{\mathrm{op}}, \mathbf{Set}]$ of $[\overline{\mathcal{A}}^{\mathrm{op}}, \mathbf{Set}]$. By the dual of Theorem 5.35 we have an equivalence $\mathrm{Ran}_{K^{\mathrm{op}}}\colon [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}] \simeq \mathrm{Lex}[\overline{\mathcal{A}}^{\mathrm{op}}, \mathbf{Set}]$, whose inverse is the restriction of $[K^{\mathrm{op}}, 1]$. Since the composite of $[K^{\mathrm{op}}, 1]$ with the Yoneda embedding $\overline{\mathcal{A}} \longrightarrow [\overline{\mathcal{A}}, \mathbf{Set}]$ is $\widetilde{K}\colon \overline{\mathcal{A}} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$, this gives the first assertion. The second one comes from Theorem 5.38 and the third from Proposition 5.40. ∎

Finally we give the converse to Proposition 4.74 as promised in 4.6:

Proposition 5.42 *For a small $\mathcal{A}$ with finite-colimit-completion $\overline{\mathcal{A}}$, the following are equivalent:*

(i) *$\mathcal{A}$ is filtered.*

(ii) $\mathrm{colim}\colon [\mathcal{A}, \mathbf{Set}] \longrightarrow \mathbf{Set}$ *is left exact.*

(iii) *The inclusion $K\colon \mathcal{A} \longrightarrow \overline{\mathcal{A}}$ is final.*

Proof. Since by (3.26) we have $\mathrm{colim}\, F \cong \Delta 1 \star F$, what (ii) asserts is the flatness of $\Delta 1\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Set}$. By Theorem 5.38, this is equivalent to the filteredness of $(\mathrm{el}\,\Delta 1)^{\mathrm{op}}$, which is $\mathcal{A}$; thus (ii) is equivalent to (i).

Since $\overline{\mathcal{A}}$ is finitely complete and hence filtered, what (iii) asserts by Proposition 4.71 is the non-emptiness of each F/K for each $F \in \overline{\mathcal{A}}$. Any finite diagram in $\mathcal{A}$ has a colimit F in $\overline{\mathcal{A}}$, and every F in $\overline{\mathcal{A}}$ so arises. To say that the diagram admits an inductive cone with vertex $A \in \mathcal{A}$ is exactly to say that there is a map $F \longrightarrow KA$ for some A; thus (iii) is equivalent to (i). ∎

5.11 The image under $[K, 1]\colon [\mathcal{C}, \mathcal{B}] \longrightarrow [\mathcal{A}, \mathcal{B}]$, for a general dense K, of the left adjoint functors; examples and counter-examples

From the beginning of 5.4 until now we have dealt only with *fully-faithful* dense functors – exploiting the notion of a density presentation, which does not seem to generalize usefully to the non-fully-faithful case. Now we drop this restriction, and consider an *arbitrary* dense $K\colon \mathcal{A} \longrightarrow \mathcal{C}$. For any $\mathcal{B}$, write $\mathrm{Ladj}[\mathcal{C}, \mathcal{B}]$ for the full subcategory of $[\mathcal{C}, \mathcal{B}]$ given by those $S\colon \mathcal{C} \longrightarrow \mathcal{B}$ which *are* left adjoints (or which *have* right adjoints); since left adjoints are certainly $\mathcal{A}$-cocontinuous, it follows from Theorem 5.1 that the restriction of $[K, 1]\colon [\mathcal{C}, \mathcal{B}] \longrightarrow [\mathcal{A}, \mathcal{B}]$ to $\mathrm{Ladj}[\mathcal{C}, \mathcal{B}]$ is fully-faithful, so that $\mathrm{Ladj}[\mathcal{C}, \mathcal{B}]$ is a $\mathcal{V}$-category if $\mathcal{A}$ is small, being equivalent to its replete image in $[\mathcal{A}, \mathcal{B}]$ under $[K, 1]$; the main goal of this section and the next is to determine this replete image.

In this section, devoted mostly to examples, we suppose for simplicity that $\mathcal{A}$ is small; if it were not, the expressions $\mathrm{Cocts}[\mathcal{C}, \mathcal{B}]$ and $\mathrm{Cocts}^K[\mathcal{C}, \mathcal{B}]$ would have to be modified. Then Proposition 5.11 and Theorem 5.33 give

$$\mathrm{Ladj}[\mathcal{C}, \mathcal{B}] = \mathrm{Cocts}[\mathcal{C}, \mathcal{B}] \qquad \textit{if } \mathcal{C} \textit{ is cocomplete.} \tag{5.43}$$

We also abbreviate *full subcategory* to *subcategory*, since we consider no others. Finally, we loosely write $[K,1]$ and Lan_K for various restrictions, specified by the context, of the functors properly so named.

The following remarks, to the end of this paragraph, apply to any $K\colon \mathcal{A} \longrightarrow \mathcal{C}$, dense or not, and do not need $\mathcal{A}$ to be small. In Theorem 4.51, and in its generalizations Theorem 4.99 and Theorem 5.35, we have used the notation $[\mathcal{A},\mathcal{B}]'$ for the subcategory given by those $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ for which $\mathrm{Lan}_K G$ exists. Since we have soon to deal with more than one functor K, we replace this notation by $[\mathcal{A},\mathcal{B}]^{(K)}$. Similarly we have used, in Theorem 4.99 and elsewhere, $[\mathcal{C},\mathcal{B}]^{\ell}$ for the subcategory given by those $S\colon \mathcal{C} \longrightarrow \mathcal{B}$ of the form $\mathrm{Lan}_K G$ for some G; we now change this to $[\mathcal{C},\mathcal{B}]^{\ell K}$. Write $[\mathcal{A},\mathcal{B}]_*^{(K)}$ for the subcategory of $[\mathcal{A},\mathcal{B}]^{(K)}$ given by those G for which the unit $G \longrightarrow (\mathrm{Lan}_K G)K$ is an isomorphism; and write $[\mathcal{C},\mathcal{B}]_*^{\ell K}$ for the subcategory of $[\mathcal{C},\mathcal{B}]^{\ell K}$ given by those S for which the identity $1\colon SK \longrightarrow SK$ exhibits S as $\mathrm{Lan}_K SK$. It is immediate from Theorem 4.38 that we have an equivalence (with the evident unit and counit)

$$\mathrm{Lan}_K \dashv [K,1]\colon [\mathcal{C},\mathcal{B}]_*^{\ell K} \simeq [\mathcal{A},\mathcal{B}]_*^{(K)}, \tag{5.44}$$

where (as we warned) we are using $[K,1]$ and Lan_K loosely for their restrictions. By Proposition 4.23,

$$[\mathcal{A},\mathcal{B}]_*^{(K)} = [\mathcal{A},\mathcal{B}]^{(K)} \quad \textit{and} \quad [\mathcal{C},\mathcal{B}]_*^{\ell K} = [\mathcal{C},\mathcal{B}]^{\ell K} \quad \textit{for fully-faithful } K; \tag{5.45}$$

and in this special case, (5.44) becomes the equivalence

$$\mathrm{Lan}_K \dashv [K,1]\colon [\mathcal{C},\mathcal{B}]^{\ell K} \simeq [\mathcal{A},\mathcal{B}]^{(K)} \qquad \textit{for fully-faithful } K \tag{5.46}$$

of Theorem 4.99: but by the same Proposition 4.23, the first equality in (5.45) fails for a general $\mathcal{B}$ (and in particular for $\mathcal{B} = \mathcal{V}$) whenever K is *not* fully faithful, so that (5.46) then fails too.

Henceforth we suppose once again that K is dense. As in the proof of Theorem 5.1, a cocontinuous $S\colon \mathcal{C} \longrightarrow \mathcal{B}$ preserves the colimit $\widetilde{K}C * K$; so that by Theorem 5.1(iv) we have $S \in [\mathcal{C},\mathcal{B}]_*^{\ell K}$ (from which, using (5.44), we re-find the conclusion of Theorem 5.1(i), that the restriction of $[K,1]$ to $\mathrm{Cocts}[\mathcal{C},\mathcal{B}]$ is fully faithful). We therefore have inclusions

$$\mathrm{Ladj}[\mathcal{C},\mathcal{B}] \subset \mathrm{Cocts}[\mathcal{C},\mathcal{B}] \subset [\mathcal{C},\mathcal{B}]_*^{\ell K} \subset [\mathcal{C},\mathcal{B}]^{\ell K} \subset [\mathcal{C},\mathcal{B}]. \tag{5.47}$$

The first of these inclusions is by (5.43) an equality when $\mathcal{C}$ is cocomplete, but is proper in general – the inclusion into **Set** of the small full subcategory of finite sets is continuous but has no left adjoint. The third inclusion is by (5.45) an equality when K is fully faithful, and in fact Theorem 5.29 then gives the common value explicitly as

$$[\mathcal{C},\mathcal{B}]_*^{\ell K} = [\mathcal{C},\mathcal{B}]^{\ell K} = \mathrm{Cocts}^K[\mathcal{C},\mathcal{B}] \qquad \textit{for fully-faithful } K. \tag{5.48}$$

However the second and fourth inclusions are in general proper even when $\mathcal{V} = \mathbf{Set}$, $\mathcal{C}$ and $\mathcal{B}$ are complete and cocomplete, and K is fully faithful.

For an example of this, let $K\colon 1 \longrightarrow \mathbf{2}$ correspond to $1 \in \mathbf{2} = (0 \longrightarrow 1)$; it is clearly dense. Then $[\mathbf{2}, \mathcal{B}]$ consists of all maps $A \longrightarrow B$ in $\mathcal{B}$; while $[\mathbf{2}, \mathcal{B}]^{\ell K} = [\mathbf{2}, \mathcal{B}]_*^{\ell K}$ consists of those with $A = 0$, the initial object of $\mathcal{B}$; and $\mathrm{Cocts}[\mathbf{2}, \mathcal{B}] = \mathrm{Ladj}[\mathbf{2}, \mathcal{B}]$ consists of those $0 \longrightarrow B$ which are epimorphisms in $\mathcal{B}$. Here of course Lan_K is fully faithful on $[\mathcal{A}, \mathcal{B}]_*^{(K)} = [\mathcal{A}, \mathcal{B}]^{(K)} = [\mathcal{A}, \mathcal{B}] = \mathcal{B}$; but $[K, 1]$, although necessarily fully faithful on $[\mathbf{2}, \mathcal{B}]^{\ell K}$, is not so on $[\mathbf{2}, \mathcal{B}]$: it sends $A \longrightarrow B$ to $B \in \mathcal{B}$. The replete image under $[K, 1]$ of $\mathrm{Ladj}[\mathbf{2}, \mathcal{B}]$ is the proper subcategory of $\mathcal{B}$ given by those B for which $0 \longrightarrow B$ is epimorphic.

When K is not fully faithful, the second, third and fourth inclusions in (5.47) may all be proper, even when $\mathcal{V} = \mathbf{Set}$, $\mathcal{C}$ and $\mathcal{B}$ are complete and cocomplete, and K is surjective on objects. For an example, let $K\colon 2 \longrightarrow \mathbf{2}$ be the inclusion, where 2 is the discrete category $\{0, 1\}$, and let $\mathcal{B} = \mathbf{Set}^{\mathrm{op}}$; again K is clearly dense. An object of $[\mathbf{2}, \mathcal{B}]$ may be identified with a map $B \longrightarrow A$ in $\mathbf{Set}$; this is sent by $[K, 1]$ to $(B, A) \in [2, \mathbf{Set}] = \mathbf{Set} \times \mathbf{Set}$; and Lan_K sends (D, C) to the projection $D \times C \longrightarrow C$. Clearly $[\mathbf{2}, \mathcal{B}]^{\ell K}$ consists of those $B \longrightarrow A$ isomorphic to such projections, while $[\mathbf{2}, \mathcal{B}]_*^{\ell K}$ consists of such projections for which the other projection $D \times C \longrightarrow D$ is an isomorphism – that is, the maps $D \longrightarrow 1$ and the maps $0 \longrightarrow C$. Finally (as in the last example) $\mathrm{Ladj}[\mathbf{2}, \mathcal{B}]$ consists of the maps $1 \longrightarrow 1$ and $0 \longrightarrow 1$. Observe how badly (5.46) fails: although $[K, 1]$ is conservative on $[\mathbf{2}, \mathcal{B}]$ (as it must be for an essentially surjective K), it is not fully faithful on $[\mathbf{2}, \mathcal{B}]^{\ell K}$; and Lan_K is not fully faithful on $[2, \mathcal{B}] = [2, \mathcal{B}]^{(K)}$. Clearly $[2, \mathcal{B}]_*^{(K)}$ consists of the pairs of the forms $(D, 1)$ and $(0, C)$, while the replete image under $[K, 1]$ of $\mathrm{Ladj}[\mathbf{2}, \mathcal{B}]$ consists of the pairs $(1, 1)$ and $(0, 1)$.

For a non-fully-faithful K, not only does Proposition 4.23 fail, but even a *left adjoint* $S\colon \mathcal{C} \longrightarrow \mathcal{B}$ may be $\mathrm{Lan}_K G$ for some G other than SK; in which case, by (5.44) and (5.47), G must lie outside $[\mathcal{A}, \mathcal{B}]_*^{(K)}$. The example above does not illustrate this possibility, but the following one does so. With $\mathcal{V} = \mathbf{Set}$ again, let $K = \Delta 1\colon \mathbf{2} \longrightarrow \mathbf{2}$. Since the unique map $\mathbf{2} \longrightarrow 1$ is obviously initial, the left Kan extension of K along $\mathbf{2} \longrightarrow 1$ is $\Delta 1\colon 1 \longrightarrow \mathbf{2}$ by Theorem 4.67; since the latter functor is clearly dense, so is K by Proposition 5.7. Taking $\mathcal{B}$ cocomplete, we see that Lan_K sends $C \longrightarrow D$ to $0 \longrightarrow D$, while $[K, 1]$ sends $A \longrightarrow B$ to $1\colon B \longrightarrow B$. Here $[\mathcal{C}, \mathcal{B}]_*^{\ell K} = [\mathcal{C}, \mathcal{B}]^{\ell K}$, even though K is not fully faithful – each consisting of the maps $0 \longrightarrow B$; while $[\mathcal{A}, \mathcal{B}]_*^{(K)}$ consists of the maps $1\colon D \longrightarrow D$. As in the penultimate example, $\mathrm{Ladj}[\mathcal{C}, \mathcal{B}]$ consists of those $0 \longrightarrow B$ which are epimorphic; and such an object is the image under Lan_K of *any* $\mathcal{C} \longrightarrow \mathcal{B}$, which need not in general – for instance, if $\mathcal{B} = \mathbf{Set}^{\mathrm{op}}$ – lie in $[\mathcal{A}, \mathcal{B}]_*^{(K)}$.

5.12 Description of the image above in terms of K-comodels; generalization to an equivalence $\mathrm{Lan}_Z \dashv [Z, 1]$ between categories of comodels

We now answer the question raised in 5.11 for a dense $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ by identifying the replete image of $[K, 1]\colon \mathrm{Ladj}[\mathcal{C}, \mathcal{B}] \longrightarrow [\mathcal{A}, \mathcal{B}]$ as a certain subcategory $K\text{-}\mathrm{Com}[\mathcal{A}, \mathcal{B}]^{(K)}$ of $[\mathcal{A}, \mathcal{B}]_*^{(K)} \subset [\mathcal{A}, \mathcal{B}]$; in this form, the result is independent of the smallness of $\mathcal{A}$.

We call a functor $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ a *K-comodel* if $\widetilde{G}\colon \mathcal{B} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ factorizes through $\widetilde{K}\colon \mathcal{C} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ as

$$\widetilde{G} \cong \widetilde{K}T \tag{5.49}$$

for some $T\colon \mathcal{B}\longrightarrow\mathcal{C}$. Since $\widetilde{K}$ is fully faithful by Theorem 5.1, such a T is unique to within isomorphism if it exists. We can write (5.49) more explicitly as a $\mathcal{V}$-natural isomorphism

$$\mathcal{B}(GA,B)\cong\mathcal{C}(KA,TB). \tag{5.50}$$

Since $\widetilde{K}$ trivially factorizes through itself, we have:

$$\textit{The dense } K\colon\mathcal{A}\longrightarrow\mathcal{C} \textit{ is always itself a } K\textit{-comodel.} \tag{5.51}$$

We write K-Com$[\mathcal{A},\mathcal{B}]$ for the subcategory of $[\mathcal{A},\mathcal{B}]$ given by the K-comodels, and K-Com$[\mathcal{A},\mathcal{B}]^{(K)}$ for the intersection of this with $[\mathcal{A},\mathcal{B}]^{(K)}$ – that is, the category of those K-comodels G for which $\mathrm{Lan}_K G$ exists. More generally, for any $Z\colon\mathcal{A}\longrightarrow\mathcal{D}$, we write K-Com$[\mathcal{A},\mathcal{B}]^{(Z)}$ for the intersection K-Com$[\mathcal{A},\mathcal{B}]\cap[\mathcal{A},\mathcal{B}]^{(Z)}$.

The name "K-comodel" for a $G\colon\mathcal{A}\longrightarrow\mathcal{B}$ satisfying (5.49) is inspired by the fact (see 5.13 below) that a dense $K\colon\mathcal{A}\longrightarrow\mathcal{C}$, at least for small $\mathcal{A}$ and cocomplete $\mathcal{C}$, may be thought of as a very general kind of *essentially-algebraic theory*, a model of which in $\mathcal{P}$ is a functor $H\colon\mathcal{A}^{\mathrm{op}}\longrightarrow\mathcal{P}$ such that $G=H^{\mathrm{op}}\colon\mathcal{A}\longrightarrow\mathcal{P}^{\mathrm{op}}$ is a comodel in the sense above. These models constitute a full subcategory K-Mod$[\mathcal{A}^{\mathrm{op}},\mathcal{P}]$ of $[\mathcal{A}^{\mathrm{op}},\mathcal{P}]$, isomorphic to $(K\text{-Com}[\mathcal{A},\mathcal{P}^{\mathrm{op}}])^{\mathrm{op}}$. If we talk here of comodels rather than models, it is because we have elected to emphasize left Kan extensions and colimits, rather than their duals. Particular examples of such theories are given in 6.2 and 6.3 below.

Rather than establish directly the promised equivalence $\mathrm{Ladj}[\mathcal{A},\mathcal{B}]\simeq K\text{-Com}[\mathcal{A},\mathcal{B}]^{(K)}$, we first prove a more general result, itself of value in practice:

Theorem 5.52 *Let the dense $K\colon\mathcal{A}\longrightarrow\mathcal{C}$ be (isomorphic to) the composite of $Z\colon\mathcal{A}\longrightarrow\mathcal{D}$ and a fully-faithful $J\colon\mathcal{D}\longrightarrow\mathcal{C}$ (so that, by Theorem 5.13, both Z and J are also dense). Then we have an equivalence*

$$\mathrm{Lan}_Z\dashv[Z,1]\colon J\text{-Com}[\mathcal{D},\mathcal{B}]\simeq K\text{-Com}[\mathcal{A},\mathcal{B}]^{(Z)}, \tag{5.53}$$

whose unit and counit are the canonical maps.

Proof. Let $H\in J$-Com$[\mathcal{D},\mathcal{B}]$, so that, corresponding to (5.50), we have

$$\mathcal{B}(HD,B)\cong\mathcal{C}(JD,TB) \tag{5.54}$$

for some $T\colon\mathcal{B}\longrightarrow\mathcal{C}$. Putting $D=ZA$ in this gives $\mathcal{B}(HZA,B)\cong\mathcal{C}(JZA,TB)\cong\mathcal{C}(KA,TB)$; showing that HZ is a K-comodel, and giving $(\widetilde{HZ})$ explicitly as $\widetilde{K}T$. Since (4.49) gives $\widetilde{Z}\cong\widetilde{K}J$, we have ($\widetilde{K}$ being fully faithful)

$$[\mathcal{A}^{\mathrm{op}},\mathcal{V}](\widetilde{Z}D,(\widetilde{HZ})B)\cong[\mathcal{A}^{\mathrm{op}},\mathcal{V}](\widetilde{K}JD,\widetilde{K}TB)\cong\mathcal{C}(JD,TB)\cong\mathcal{B}(HD,B),$$

which by (4.20) shows that $H\cong\mathrm{Lan}_Z(HZ)$; moreover the unit of this Kan extension is easily calculated by Theorem 4.6 to be $1\colon HZ\longrightarrow HZ$. Since we have seen that HZ is a K-comodel, it does indeed lie in K-Com$[\mathcal{A},\mathcal{B}]^{(Z)}$.

Now let $G\in K$-Com$[\mathcal{A},\mathcal{B}]^{(Z)}$, so that $\widetilde{G}\cong\widetilde{K}T$ for some $T\colon\mathcal{B}\longrightarrow\mathcal{C}$. Writing H for $\mathrm{Lan}_Z G$, using (4.20) for the first step, and recalling that $\widetilde{Z}\cong\widetilde{K}J$, we have

$$\mathcal{B}(HD,B)\cong[\mathcal{A}^{\mathrm{op}},\mathcal{V}](\widetilde{Z}D,\widetilde{G}B)\cong[\mathcal{A}^{\mathrm{op}},\mathcal{V}](\widetilde{K}JD,\widetilde{K}TB)\cong\mathcal{C}(JD,TB),$$

which is (5.54), showing that $H = \mathrm{Lan}_Z G$ belongs to J-$\mathrm{Com}[\mathcal{D},\mathcal{B}]$. Putting $D = ZA$ in (5.54) gives as before $\mathcal{B}(HZA,B) \cong \mathcal{C}(KA,TB)$; the latter now being $\mathcal{B}(GA,B)$ by (5.50), Yoneda gives an isomorphism $G \cong HZ$, which is easily verified to be the unit $G \longrightarrow (\mathrm{Lan}_Z G)Z$ of the Kan extension. ∎

Since the unit and counit of the equivalence (5.53) are the canonical maps, it follows that

$$J\text{-}\mathrm{Com}[\mathcal{D},\mathcal{B}] \subset [\mathcal{D},\mathcal{B}]_*^{\ell Z} \quad \text{and} \quad K\text{-}\mathrm{Com}[\mathcal{A},\mathcal{B}]^{(Z)} \subset [\mathcal{A},\mathcal{B}]_*^{(Z)}, \tag{5.55}$$

and that the equivalence (5.53) is just the restriction of the equivalence $[\mathcal{D},\mathcal{B}]_*^{\ell Z} \cong [\mathcal{A},\mathcal{B}]_*^{(Z)}$ of (5.44). Now consider the special case of Theorem 5.52 in which $\mathcal{D} = \mathcal{C}$, $J = 1_{\mathcal{C}}$, and $Z = K$. A $1_{\mathcal{C}}$-comodel is by (5.50) a functor $S\colon \mathcal{C} \longrightarrow \mathcal{B}$ for which $\mathcal{B}(SC,B) \cong \mathcal{C}(C,TB)$ for some T; that is, it is a left-adjoint functor. Thus Theorem 5.52 (together with (5.43)) gives the following answer to the problem proposed at the beginning of 5.11:

Theorem 5.56 *For a dense $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ and any $\mathcal{B}$, the equivalence* $\mathrm{Lan}_K \dashv [K,1]\colon [\mathcal{C},\mathcal{B}]_*^{\ell K} \simeq [\mathcal{A},\mathcal{B}]_*^{(K)}$ *restricts to an equivalence*

$$\mathrm{Lan}_K \dashv [K,1]\colon \mathrm{Ladj}[\mathcal{C},\mathcal{B}] \simeq K\text{-}\mathrm{Com}[\mathcal{A},\mathcal{B}]^{(K)}. \tag{5.57}$$

Here, if $\mathcal{C}$ is cocomplete and $\mathcal{A}$ is small, $\mathrm{Ladj}[\mathcal{C},\mathcal{B}]$ *may be replaced by* $\mathrm{Cocts}[\mathcal{C},\mathcal{B}]$*; and if every $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ admits a left Kan extension along K, as when $\mathcal{A}$ is small and $\mathcal{B}$ is cocomplete, K-$\mathrm{Com}[\mathcal{A},\mathcal{B}]^{(K)}$ may be replaced by K-$\mathrm{Com}[\mathcal{A},\mathcal{B}]$.* ∎

Note that, when $\mathcal{A}$ is small and K is the Yoneda embedding $Y\colon \mathcal{A} \longrightarrow [\mathcal{A}^{\mathrm{op}},\mathcal{V}]$, a Y-comodel is *any* functor $\mathcal{A} \longrightarrow \mathcal{B}$, since $\widetilde{Y} = 1$ by (4.30). Thus (5.57) is another generalization of the equivalence (4.55), *different from* the generalization given by Theorem 5.31. In the situation of Theorem 5.52, combining that theorem with Theorem 5.56 gives a diagram of multiple equivalences

$$\begin{array}{lcl} \mathrm{Ladj}[\mathcal{C},\mathcal{B}] \quad \simeq J\text{-}\mathrm{Com}[\mathcal{D},\mathcal{B}]^{(J)} & \simeq & K\text{-}\mathrm{Com}[\mathcal{A},\mathcal{D}]^{(K)} \\ \qquad\qquad\qquad\qquad \cap & & \cap \\ \qquad\qquad J\text{-}\mathrm{Com}[\mathcal{D},\mathcal{B}] & \simeq & K\text{-}\mathrm{Com}[\mathcal{A},\mathcal{D}]^{(Z)} \end{array} \tag{5.58}$$

in which the horizontal maps from left to right are restrictions, and those from right to left are left Kan extensions. The vertical inclusions here, although equalities when $\mathcal{A}$ is small and $\mathcal{B}$ is cocomplete, are in general proper, even if $\mathcal{C}$ is cocomplete and $\mathcal{A}$ and $\mathcal{D}$ are small. For let $J\colon \mathcal{D} \longrightarrow \mathcal{C}$ be fully faithful and dense, take $\mathcal{A} = \mathcal{D}$ and $Z = 1$, and take $\mathcal{B} = \mathcal{D}$; then $1\colon \mathcal{D} \longrightarrow \mathcal{D}$ is a J-comodel, since $\widetilde{1} = Y$ is isomorphic by (4.22) to $\widetilde{J}J$; yet $\mathrm{Lan}_J 1_{\mathcal{D}}$ need not exist. To see this, let $\mathcal{C} = \mathcal{V} = \mathbf{Set}$, and let J be the inclusion of the full subcategory $\{2\}$ determined by the single two-element set 2; since $\{1\}$ is dense in $\mathbf{Set}$ by (5.17), so is $\{1,2\}$ by Theorem 5.13, whence $\{2\}$ is dense by Proposition 5.20, as 1 is a retract of 2. Clearly $\widetilde{J}0$ is $\Delta 0$, the initial object of $[\{2\}^{\mathrm{op}},\mathbf{Set}]$; so that $\widetilde{J}0 \star 1_{\mathcal{D}}$ would, if it existed, be the initial object of $\{2\}$. Since $\{2\}$ has no initial object, $\mathrm{Lan}_J 1_{\mathcal{D}}$ fails to exist.

5.13 A dense $K\colon \mathcal{A} \longrightarrow \mathcal{C}$, with $\mathcal{A}$ small and $\mathcal{C}$ cocomplete, as an essentially-algebraic theory with $\mathcal{C}$ as the category of algebras

We now settle on the precise definition of an *essentially-algebraic theory*: namely, a dense $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ with $\mathcal{A}$ small and $\mathcal{C}$ cocomplete. The models of the theory in $\mathcal{V}$ itself are usually called its *algebras*. We have:

Theorem 5.59 *For any dense $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ with $\mathcal{A}$ small, there is an equivalence K-Mod$[A^{\mathrm{op}}, \mathcal{V}] \simeq \mathcal{C}'$, where $\mathcal{C}'$ is the subcategory of $\mathcal{C}$ given by those $\mathcal{C}$ such that $X \pitchfork \mathcal{C}$ exists for all $X \in \mathcal{V}$. Thus if $\mathcal{C}$ is cotensored, K-Mod$[A^{\mathrm{op}}, \mathcal{V}] \simeq \mathcal{C}$. This is so in particular if $\mathcal{C}$ is cocomplete.*

Proof. Since $\mathcal{V}^{\mathrm{op}}$ is cocomplete, Theorem 5.56 gives K-Com$[\mathcal{A}, \mathcal{V}^{\mathrm{op}}] \simeq$ Ladj$[\mathcal{C}, \mathcal{V}^{\mathrm{op}}]$, or equivalently K-Mod$[\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \simeq$ (Ladj$[\mathcal{C}, \mathcal{V}^{\mathrm{op}}])^{\mathrm{op}}$; and this last category is equivalent to Radj$[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$. But all right-adjoint functors $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{V}$ lie in the subcategory $\mathcal{C} \subset [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ given by the representables, and constitute in fact the category $\mathcal{C}'$ of the theorem. The last assertion follows from Proposition 5.15. ∎

This leads to the following point of view. If $\mathcal{C}$ is a cocomplete category admitting a small dense subcategory, any choice of a dense $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ with small $\mathcal{A}$ gives an essentially-algebraic theory, for which $\mathcal{C}$ is the category of algebras. As long as we consider models of the theory only in complete categories $\mathcal{P}$, the concept of a model is independent of the choice of K; a model of the theory in $\mathcal{P}$, also called a *$\mathcal{C}$-object in $\mathcal{P}$*, is by Theorem 5.56 just a continuous (and hence right-adjoint) functor $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{P}$. At this level, any continuous $\mathcal{P} \longrightarrow \mathcal{P}'$ carries models to models; and any model in $\mathcal{P}$ is so obtained by applying the continuous functor $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{P}$ to the *generic model* in $\mathcal{C}^{\mathrm{op}}$ given by $1\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{C}^{\mathrm{op}}$. In terms of $K\colon \mathcal{A} \longrightarrow \mathcal{C}$, this generic model in $\mathcal{C}^{\mathrm{op}}$ corresponds to $K^{\mathrm{op}}\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{C}^{\mathrm{op}}$, which is a model by (5.51).

The choice of $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ affects to some extent the notion of a model in $\mathcal{P}$ when $\mathcal{P}$ is not complete. If $K'\colon \mathcal{A}' \longrightarrow \mathcal{C}$ is another dense functor with $\mathcal{A}'$ small, let $\mathcal{D}$ be the union of the full images in $\mathcal{C}$ of K and of K'; then the inclusion $J\colon \mathcal{D} \longrightarrow \mathcal{C}$ is dense by Theorem 5.13. Thus Theorem 5.52 gives, in an evident notation, K-Mod$[\mathcal{A}^{\mathrm{op}}, \mathcal{P}]^{(Z)} \simeq K'$-Mod$[\mathcal{A}'^{\mathrm{op}}, \mathcal{P}]^{(Z')}$. It follows that K-Mod$[\mathcal{A}^{\mathrm{op}}, \mathcal{P}]$ and K'-Mod$[A'^{\mathrm{op}}, \mathcal{P}]$ coincide when $\mathcal{P}$ is "complete enough" to admit all right Kan extensions along $Z\colon \mathcal{A} \longrightarrow \mathcal{D}$ and $Z'\colon \mathcal{A}' \longrightarrow \mathcal{D}$. A concrete example of such "sufficient completeness" will occur in 6.4 below.

We will end with three remarks. First, it is easily seen that, for any $\mathcal{P}$ and $\mathcal{P}'$, a right-adjoint functor $\mathcal{P} \longrightarrow \mathcal{P}'$ takes K-models to K-models. Secondly, the dense $I\colon \mathcal{I} \longrightarrow \mathcal{V}$ of (5.17) is "the theory of an object"; for, $\widetilde{I}$ being 1, we have I-Mod$[\mathcal{I}^{\mathrm{op}}, \mathcal{P}] \cong \mathcal{P}$ for any $\mathcal{P}$. Thirdly, if $\mathcal{P}$ too is cocomplete with a small dense subcategory (and hence complete by Proposition 5.15), the notion of a $\mathcal{C}$-object in $\mathcal{P}$ coincides with that of a $\mathcal{P}$-object in $\mathcal{C}$: for a right-adjoint $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{P}$ has as its left adjoint $\mathcal{P} \longrightarrow \mathcal{C}^{\mathrm{op}}$ what is in effect a right adjoint $\mathcal{P}^{\mathrm{op}} \longrightarrow \mathcal{C}$.

5.14 The image under $[K,1]\colon [\mathcal{C},\mathcal{B}]\longrightarrow[\mathcal{A},\mathcal{B}]$, for a dense K, of the equivalences; characterization theorems for $\mathcal{C}$

We now determine the image, under the equivalence $\mathrm{Ladj}[\mathcal{C},\mathcal{B}]\simeq K\text{-}\mathrm{Com}[\mathcal{A},\mathcal{B}]^{(K)}$ of (5.57), of the subcategory of *equivalences* $S\colon\mathcal{C}\simeq\mathcal{B}$. We do not suppose $\mathcal{A}$ small nor $\mathcal{C}$ cocomplete; but we do restrict ourselves to the simpler case of a *fully-faithful* dense $K\colon\mathcal{A}\longrightarrow\mathcal{C}$.

We thus get a criterion for a given category $\mathcal{B}$ to be equivalent to $\mathcal{C}$, in terms of there being a K-comodel $G\colon\mathcal{A}\longrightarrow\mathcal{B}$ satisfying our characterization of the image in question; this contains as a special case the criterion of Theorem 5.26 above for $\mathcal{B}$ to be equivalent to $[\mathcal{A}^{\mathrm{op}},\mathcal{V}]$. We give only a few applications below; numerous others can be found in the article [21] of Diers, to whom the results (in the case $\mathcal{V}=\mathbf{Set}$) are due.

If there is an equivalence $S\dashv T\colon\mathcal{B}\simeq\mathcal{C}$, the image of S under (5.57) is the K-comodel $G=SK\colon\mathcal{A}\longrightarrow\mathcal{B}$. Since we have $\mathcal{B}(GA,B)=\mathcal{B}(SKA,B)\cong\mathcal{C}(KA,TB)$, the unique T which, as in (5.49) or (5.50), exhibits G as being a K-comodel, is precisely the equivalence-inverse of S. We therefore start with any K-comodel $G\colon\mathcal{A}\longrightarrow\mathcal{B}$, exhibited as such by $\widetilde{G}\cong\widetilde{K}T$ as in (5.49), and seek conditions for this T to be an equivalence.

This isomorphism $\sigma\colon\widetilde{G}\cong\widetilde{K}T$, written as a $\mathcal{V}$-natural isomorphism $\mathcal{B}(GA,B)\cong\mathcal{C}(KA,TB)$ as in (5.50), is (by Yoneda in the form (1.48)) of the form $\sigma=\mathcal{C}(\rho,1)T$ for some $\mathcal{V}$-natural $\rho\colon K\longrightarrow TG$. The diagram (1.39) expressing the $\mathcal{V}$-naturality of ρ may be written as

$$\begin{tikzcd} Y \arrow[r, "\cong", "\kappa_K"'] \arrow[d, "\kappa_G"'] & \widetilde{K}K \arrow[d, "\widetilde{K}\rho"] \\ \widetilde{G}G \arrow[r, "\cong", "\sigma G"'] & \widetilde{K}TG, \end{tikzcd}$$

where κ is the map of (4.22), according to which κ_G is an isomorphism exactly when G is fully faithful. Since K is fully faithful, both the horizontal maps here are isomorphisms; whence, since $\widetilde{K}$ is fully faithful, *ρ is an isomorphism if and only if G is fully faithful.* By Theorem 5.56, T admits a left adjoint S precisely when G admits a left Kan extension S along K. Suppose for the moment that this is so, and let $\eta,\epsilon\colon S\dashv T$ be the adjunction. Then T is an equivalence precisely when η and ϵ are isomorphisms. It is clear that the isomorphism $G\cong SK$ identifies $\rho\colon K\longrightarrow TG$ with $\eta K\colon K\longrightarrow TSK$. Let $\Phi=(F_\gamma,P_\gamma)_{\gamma\in\Gamma}$ be a density presentation of K as in Theorem 5.19, so that $\mathcal{C}$ is the closure of $\mathcal{A}$ under the K-absolute colimits $F_\gamma\star P_\gamma$ of Φ. It now follows that *$\eta\colon 1\longrightarrow TS$ is an isomorphism if and only if ρ is an isomorphism and TS preserves the colimits of Φ.* Because the fully-faithful K both preserves and reflects these colimits, and because $\widetilde{G}\cong\widetilde{K}T$, we conclude that TS preserves these colimits if and only if $\widetilde{G}S$ does so. Since the left-adjoint S preserves all colimits, we have finally that *η is an isomorphism if and only if G is fully faithful and the colimits $F_\gamma\star SP_\gamma$ are G-absolute.*

Once η is known to be an isomorphism, $T\epsilon$ is an isomorphism by the triangular equation $T\epsilon.\eta T=1$ of the adjunction. So *ϵ too is an isomorphism if T is conservative*; which of course T must be if it is an equivalence. Since $\widetilde{G}\cong\widetilde{K}T$ where $\widetilde{K}$ is fully faithful, *T is*

conservative if and only if $\widetilde{G}$ is conservative; that is, if and only if $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ is strongly generating. (In contrast to 3.6, we are using "strongly generating" here without thereby implying that $\mathcal{A}$ is small.)

It remains to ensure the existence of the left adjoint S. Propositions 3.36 and 3.37 give this if $\mathcal{B}$ admits all F_γ-indexed colimits for all $\gamma \in \Gamma$, but this is more than we need. The proof of Proposition 3.36 shows that we only need the (inductive) existence of the particular colimits $F_\gamma \star SP_\gamma$ in $\mathcal{B}$. Since we can run simultaneously the inductive proof of the existence of S and the inductive proof that $\eta\colon 1 \longrightarrow TS$ is an isomorphism, we have finally:

Theorem 5.60 *Let $K\colon \mathcal{A} \longrightarrow \mathcal{C}$ be fully faithful and dense, with density presentation $(F_\gamma, P_\gamma)_{\gamma\in\Gamma}$, and let $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a K-comodel with $\widetilde{G} \cong \widetilde{K}T$. Then T is an equivalence if and only if the following conditions are satisfied (whereupon $K \cong TG$):*

(i) *G is fully faithful.*

(ii) *G is strongly generating (or, equivalently, T is conservative).*

(iii) *Whenever $P_\gamma \cong TQ_\gamma$ for some Q_γ, the colimit $F_\gamma \star Q_\gamma$ exists in $\mathcal{B}$ and is preserved by T (or, equivalently, is G-absolute).* ∎

This form of the theorem can be used to derive Beck's monadicity criterion; see ([21], Corollaire 5.91). The theorem takes a simpler form, sufficient for many applications, when the density presentation Φ of K consists of *all* colimits with indexing-types in some set $\mathcal{F}$; as was the case, for instance, in Theorem 5.35 above. We then have:

Theorem 5.61 *Let $\mathcal{C}$ admit all colimits with indexing types in $\mathcal{F}$, and let these colimits constitute a density presentation of the fully-faithful dense $K\colon \mathcal{A} \longrightarrow \mathcal{C}$. Let $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a K-comodel with $\widetilde{G} \cong \widetilde{K}T$. Then T is an equivalence if and only if the following conditions are satisfied (whereupon $K \cong TG$):*

(i) *G is fully faithful and strongly generating.*

(ii) *$\mathcal{B}$ admits all $\mathcal{F}$-colimits and they are $\mathcal{G}$-absolute.* ∎

We can apply this to the $K\colon \mathcal{A} \longrightarrow \overline{\mathcal{A}}$ of Theorem 5.35, where $\overline{\mathcal{A}}$ is the free $\mathcal{F}$-cocompletion of $\mathcal{A}$ for a set $\mathcal{F}$ of small indexing types. If $\mathcal{B}$ is to be equivalent to $\overline{\mathcal{A}}$, certainly $\mathcal{B}$ must be the closure under $\mathcal{F}$-colimits of $\mathcal{A}$, embedded in $\mathcal{B}$ by G. By Proposition 3.40, whose first assertion does not really need the smallness of $\mathcal{A}$, it is then automatic that G is strongly generating. If, moreover, $\mathcal{F}$-colimits are G-absolute in $\mathcal{B}$, each $\widetilde{G}B$ in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is an iterated $\mathcal{F}$-colimit of objects of $\mathcal{A}$, and hence lies in $\mathcal{C}$; so that G is automatically a K-comodel. This gives:

Proposition 5.62 *In order that $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ be equivalent to the the free $\mathcal{F}$-cocompletion of $\mathcal{A}$ for a given set $\mathcal{F}$ of small indexing types, it is necessary and sufficient that G be fully faithful, that $\mathcal{B}$ admit all $\mathcal{F}$-colimits and be the closure of $\mathcal{A}$ under these, and that all $\mathcal{F}$-colimits in $\mathcal{B}$ be G-absolute.* ∎

Taking $\mathcal{F}$ here to consist of *all* small indexing types, with $\mathcal{A}$ small, does not in fact given Theorem 5.26 as stated, but a variant. Yet Theorem 5.26 does follow from Theorem 5.61; for, since $\widetilde{Y} \cong 1$, every $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ is automatically a Y-comodel.

Chapter 6

Essentially-algebraic theories defined by reguli and by sketches

6.1 Locally-bounded categories; the local boundedness of $\mathcal{V}$ in all our examples

All that we have so far required of the symmetric monoidal closed $\mathcal{V}$ is that $\mathcal{V}_0$ be locally small, complete, and cocomplete. To ensure the reflectivity of certain naturally-occurring subcategories of $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$, we now impose upon $\mathcal{V}$ extra conditions – which are however satisfied by all the examples of symmetric monoidal closed categories given in 1.1 above.

We call upon some reflectivity results (for ordinary categories) of [44], which improve those of [30]. The reader will rightly suspect, as we proceed, that similar results could be formulated and proved directly in the context of enriched categories. The length of such an undertaking, however, would not be justified by our limited applications; we accordingly arrange, at the cost of some elegance, to use the results as they stand – appealing to Theorem 4.85 to pass from an ordinary reflexion to a $\mathcal{V}$-reflexion.

We refer to [30] for the notion of a *proper factorization system* $(\mathcal{E}, \mathcal{M})$ on an ordinary category $\mathcal{K}$, which for simplicity we take to be locally small, complete, and cocomplete. We recall that $\mathcal{E}$ here is a set of epimorphisms, and $\mathcal{M}$ a set of monomorphisms, each containing the isomorphisms and closed under composition, such that every map factorizes as ip with $i \in \mathcal{M}$ and $p \in \mathcal{E}$, this factorization being essentially unique and functorial, in virtue of a "diagonal fill-in" property. We further recall that the $\mathcal{M}$*-subobjects* of an object $A \in \mathcal{K}$ form a lattice admitting small intersections and unions: the intersection $\cap B_i$ being the limit of the diagram $B_i \longrightarrow A$, while the union $\cup B_i$ is obtained by forming the $(\mathcal{E}, \mathcal{M})$-factorization $\sum B_i \longrightarrow \cup B_i \longrightarrow A$ of the map $\sum B_i \longrightarrow A$; and that there is a further factorization $\sum B_i \longrightarrow \operatorname{colim} B_i \longrightarrow \cup B_i$, both factors here being in $\mathcal{E}$. Similarly we speak of the *cointersection* $A \longrightarrow C$ of a set $A \longrightarrow C_i$ of maps in $\mathcal{E}$: it being the colimit of the diagram, and itself in $\mathcal{E}$. Finally we recall that a small set $\mathcal{G}$ of objects of $\mathcal{K}$ is an *($\mathcal{E}$,$\mathcal{M}$)-generator* if the canonical map $\sum_{G \in \mathcal{G}} \mathcal{K}(G, A) \cdot G \longrightarrow A$ is in $\mathcal{E}$ for each $A \in \mathcal{K}$; which is equally to say that, for every proper $\mathcal{M}$-subobject B of A, there is some $G \in \mathcal{G}$ for which some $g \colon G \longrightarrow A$ fails to factorize through B. The existence of such a generator implies that the lattice of $\mathcal{M}$-subobjects of each $A \in \mathcal{K}$ is small; that is, that

$\mathcal{K}$ is $\mathcal{M}$*-wellpowered.*

We shall call a locally small, complete, and cocomplete ordinary category $\mathcal{K}$ *locally bounded* if there is a proper factorization system $(\mathcal{E}, \mathcal{M})$ on $\mathcal{K}$ such that

(i) *every family* $A \longrightarrow C_i$ *of maps in* $\mathcal{E}$*, however large, admits a cointersection;*

(ii) $\mathcal{K}$ *has an* $(\mathcal{E}, \mathcal{M})$*-generator* $\mathcal{G}$*;*

(iii) *there is a small infinite regular cardinal* α *such that* $\mathcal{K}(G, -) : \mathcal{K} \longrightarrow \mathbf{Set}$ *preserves* α*-filtered unions of* $\mathcal{M}$*-subobjects for each* $G \in \mathcal{G}$*.*

Here an α-filtered union $\cup B_i$ is one such that, for any subset J of the indices with $\operatorname{card} J < \alpha$, there is an index i such that $B_j \subset B_i$ for each $j \in J$. To say that $\mathcal{K}(G, -)$ preserves the union is to say that $\mathcal{K}(G, \cup B_i)$ is the set-theoretic union of the $\mathcal{K}(G, B_i)$; and hence to say that any map $G \longrightarrow \cup B_i$ factorizes through one of the B_i.

If A is any object in a locally bounded $\mathcal{K}$, there is a regular cardinal $\beta = \beta(A)$ such that $\mathcal{K}(A, -)$ preserves β-filtered unions (of $\mathcal{M}$-subobjects). We have only to choose β at least equal to α and greater than $\sum_{G \in \mathcal{G}} \operatorname{card} \mathcal{K}(G, A)$. Then, if $f\colon A \longrightarrow \cup B_i$ is a map into a β-filtered union, there is for each $G \in \mathcal{G}$ and each $g\colon G \longrightarrow A$ some B_i through which fg factorizes, and i can be chosen to be independent of G and g; so that f factorizes through B_i by the diagonal fill-in property applied to the diagram

$$\begin{tikzcd}
\sum_G \mathcal{K}(G, A) \cdot G \arrow[r, "\mathcal{E}"] \arrow[d] & A \arrow[d, "f"] \\
B_i \arrow[r, "\mathcal{M}"'] & \bigcup B_i.
\end{tikzcd}$$

We now say that the symmetric monoidal closed $\mathcal{V}$ is *locally bounded* if the ordinary category $\mathcal{V}_0$ is locally bounded and if, moreover,

$$(Z \otimes p) \in \mathcal{E} \text{ whenever } Z \in \mathcal{V} \text{ and } p \in \mathcal{E}. \tag{6.1}$$

This is clearly equivalent to

$$[Z, i] \in \mathcal{M} \text{ whenever } Z \in \mathcal{V} \text{ and } i \in \mathcal{M}; \tag{6.2}$$

and combined with the diagonal fill-in property for $\mathcal{V}_0$, it easily implies that *the diagram*

$$\begin{tikzcd}
{[N, X]} \arrow[r, "{[p,1]}"] \arrow[d, "{[1,i]}"'] & {[M, X]} \arrow[d, "{[1,i]}"] \\
{[N, Y]} \arrow[r, "{[p,1]}"'] & {[M, Y]}
\end{tikzcd} \tag{6.3}$$

is a pullback whenever $p\colon M \longrightarrow N$ *is in* $\mathcal{E}$ *and* $i\colon X \longrightarrow Y$ *is in* $\mathcal{M}$. This "internal" version of the diagonal fill-in property is stronger than the original one, which we regain by applying the limit-preserving $V\colon \mathcal{V}_0 \longrightarrow \mathbf{Set}$ to the pullback (6.3).

We now verify that all the examples of symmetric monoidal closed categories in 1.1 are locally bounded. For the small preorders such as $\mathbf{2}$ and $\overline{\mathbb{R}}_+$, all the conditions are trivially satisfied with $\mathcal{E}$ = the epimorphisms, $\mathcal{M}$ = the extremal monomorphisms = the isomorphisms.

In the examples **Set**, **Cat**, **Gpd**, **Ord**, **Shv** S, **Ab**, R-**Mod**, **G**-R-**Mod** and **DG**-R-**Mod**, the category $\mathcal{V}_0$ is *locally presentable* in the sense of [31]; and any such category $\mathcal{K}$ is locally bounded with $\mathcal{E}$ = the extremal epimorphisms, $\mathcal{M}$ = the monomorphisms. For it is certainly $\mathcal{E}$-cowellpowered, by Satz 7.14 of [31]; it has by definition a strong generator $\mathcal{G}$, and this is an $(\mathcal{E}, \mathcal{M})$-generator by the remarks at the end of 3.4 above; and there is by definition a regular α such that each $\mathcal{K}(G, -)\colon \mathcal{K} \longrightarrow \mathbf{Set}$ for $G \in \mathcal{G}$ preserves α-filtered colimits – that is, each $G \in \mathcal{G}$ is *α-presentable*. Now, in the presence of such a generator $\mathcal{G}$, when $\cup B_i$ is an α-filtered union of subobjects, the map $\operatorname{colim} B_i \longrightarrow \cup B_i$ is a monomorphism, as pointed out in Lemma 3.2.1 of [30]; and hence in the present case is an isomorphism, for (lying in $\mathcal{E}$) it is an extremal epimorphism. Thus in fact each $\mathcal{K}(G, -)$ preserves α-filtered unions. Finally, (6.2) is satisfied in each of these examples, since the right-adjoint functor $[Z, -]$ preserves monomorphisms. (In all of these examples except **Shv** S, the category $\mathcal{V}_0$ is *locally finitely presentable*, in the sense that, for a suitable choice of $\mathcal{G}$, we may take $\alpha = \aleph_0$. Another account of locally-presentable categories can be found in [45].)

In the topological examples **CGTop**, **CGTop**$_*$, **HCGTop**, and **QTop**, we take for $\mathcal{M}$ the subspace-inclusions and for $\mathcal{E}$ the surjections. The first three of these are easily seen to be $\mathcal{E}$-cowellpowered. The fourth is not – the two-point set admits a large set of quasitopologies – yet it does admit all cointersections of surjections; we merely form the cointersection in **Set** and give it the appropriate quasitopology. The one-point space is in each case an $(\mathcal{E}, \mathcal{M})$-generator, which satisfies (iii) with $\alpha = \aleph_0$. Moreover, (6.1) is clearly satisfied, since the tensor product is the cartesian product – not only in $\mathcal{V}_0$, but also at the level of the underlying sets.

For the final example **Ban**, we take $\mathcal{E}$ = the epimorphisms = the dense maps, $\mathcal{M}$ = the extremal monomorphisms = the inclusions of closed subspaces with the induced norm. Cowellpoweredness is easy: if $X \longrightarrow Y$ is dense, Y can have no more points than there are filters on the image of X. An $(\mathcal{E}, \mathcal{M})$-generator is $\{G\}$, where G is the base-field $\mathbb{R}$ or $\mathbb{C}$. It satisfies (iii) with $\alpha = \aleph_1$; for then an α-filtered union $\cup B_i$ is the set-theoretical union. (In general $\cup B_i$ is the *closure* of the set-theoretical union, so that every point of it is the limit of some sequence of the latter; when the union is $\aleph_1$-filtered, the sequence lies entirely in some B_i.) Finally (6.2) is clearly satisfied, $[Z, Y]$ being the usual function-space.

6.2 The reflectivity in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ of the category of algebras for a regulus

The first step in making available the results of [44] is the proof of:

Proposition 6.4 *If* $\mathcal{V}$ *is locally bounded, so is the ordinary category* $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]_0$ *for any*

small $\mathcal{A}$.

Proof. Let $(\mathcal{E},\mathcal{M})$ be the proper factorization system on $\mathcal{V}_0$, and $\mathcal{G}$ the corresponding $(\mathcal{E},\mathcal{M})$-generator. We define sets of maps in $[\mathcal{A}^{\mathrm{op}},\mathcal{V}]_0$, again called $\mathcal{E}$ and $\mathcal{M}$, by decreeing that a $\mathcal{V}$-natural $\rho\colon T\longrightarrow S$ is to be in $\mathcal{E}$ or in $\mathcal{M}$ precisely when all of its components $\rho_A\colon TA\longrightarrow SA$ are in the $\mathcal{E}$ or the $\mathcal{M}$ of $\mathcal{V}_0$. Clearly the new $\mathcal{E}$ and $\mathcal{M}$ contain the isomorphisms and are closed under composition. Since we test for a monomorphism as in 3.3 by pulling it back along itself, and since small limits in $[\mathcal{A}^{\mathrm{op}},\mathcal{V}]$ are formed pointwise, the $\mathcal{M}$ of $[\mathcal{A}^{\mathrm{op}},\mathcal{V}]_0$ consists of monomorphisms; similarly the $\mathcal{E}$ consists of epimorphisms.

To see that every $\rho\colon T\longrightarrow S$ factorizes into an $\mathcal{E}$ and an $\mathcal{M}$, we factorize each component ρ_A as

$$TA\xrightarrow[\sigma_A]{}RA\xrightarrow[\tau_A]{}SA$$

with $\sigma_A\in\mathcal{E}$ and $\tau_A\in\mathcal{M}$. To make R into a $\mathcal{V}$-functor $\mathcal{A}^{\mathrm{op}}\longrightarrow\mathcal{V}$, such that σ and τ are $\mathcal{V}$-natural, we must so define R_{AB} as to render commutative the top and left regions of

$$\begin{tikzcd}
\mathcal{A}(B,A) \arrow[rr, "T_{AB}"] \arrow[dr, "R_{AB}"] \arrow[dd, "S_{AB}"'] & & {[TA,TB]} \arrow[d, "{[1,\sigma_B]}"] \\
& {[RA,RB]} \arrow[r, "{[\sigma_A,1]}"'] \arrow[d, "{[1,\tau_B]}"] & {[TA,RB]} \arrow[d, "{[1,\tau_B]}"] \\
{[SA,SB]} \arrow[r, "{[\tau_A,1]}"'] & {[RA,SB]} \arrow[r, "{[\sigma_A,1]}"'] & {[TA,SB]};
\end{tikzcd}$$

since the bottom right region is a pullback by (6.3), there is exactly one such map R_{AB} – which is then easily seen to satisfy the axioms (1.5) and (1.6) for a $\mathcal{V}$-functor.

To verify the diagonal fill-in property, let $\sigma\colon T\longrightarrow S$ be in $\mathcal{E}$ and let $\tau\colon P\longrightarrow Q$ be in $\mathcal{M}$. By (6.3) we have for each $A\in\mathcal{A}$ a pullback

$$\begin{tikzcd}
{[SA,PA]} \arrow[r, "{[\sigma_A,1]}"] \arrow[d, "{[1,\tau_A]}"'] & {[TA,PA]} \arrow[d, "{[1,\tau_A]}"] \\
{[SA,QA]} \arrow[r, "{[\sigma_A,1]}"'] & {[TA,QA]};
\end{tikzcd}$$

applying $\int_A$ to this in accordance with (2.10), and recalling from (3.20) that limits commute with limits, we get a corresponding pullback in which the top left corner is now $[\mathcal{A}^{\mathrm{op}},\mathcal{V}](S,P)$; applying $V\colon\mathcal{V}_0\longrightarrow\mathbf{Set}$ to this, we get a pullback in $\mathbf{Set}$ expressing the desired diagonal fill-in property. Hence $(\mathcal{E},\mathcal{M})$ is a proper factorization system on $[\mathcal{A}^{\mathrm{op}},\mathcal{V}]_0$.

Write $\mathcal{H}$ for the small set $\{G\otimes\mathcal{A}(-,A)\}_{G\in\mathcal{G},A\in\mathcal{A}}$ of objects of $[\mathcal{A}^{\mathrm{op}},\mathcal{V}]_0$. If R is a proper $\mathcal{M}$-subobject of S, there is some A for which RA is a proper $\mathcal{M}$-subobject of SA: and then there is some G for which some $g\colon G\longrightarrow SA$ factorizes through RA. This gives

a map $G \otimes \mathcal{A}(-, A) \longrightarrow S$ that factorizes through R; whence $\mathcal{H}$ is an $(\mathcal{E}, \mathcal{M})$-generator for $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]_0$.

Unions of $\mathcal{M}$-subobjects in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]_0$ are clearly formed pointwise. Since $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]_0(G \otimes \mathcal{A}(-, A), S) \cong \mathcal{V}_0(G, SA)$ by Yoneda, it follows that each $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]_0(G \otimes \mathcal{A}(-, A), -)$ preserves α-filtered unions if each $\mathcal{V}_0(G, -)$ does so. ∎

An object B of a $\mathcal{V}$-category $\mathcal{B}$ is said to be *orthogonal in* $\mathcal{B}$ to a map $\theta\colon M \longrightarrow N$ of $\mathcal{B}_0$ if the map $\mathcal{B}(\theta, 1)\colon \mathcal{B}(N, B) \longrightarrow \mathcal{B}(M, B)$ is an isomorphism in $\mathcal{V}_0$. When this is so, applying $V\colon \mathcal{V}_0 \longrightarrow \mathbf{Set}$ to $\mathcal{B}(\theta, 1)$ gives an isomorphism $\mathcal{B}_0(\theta, 1)\colon \mathcal{B}_0(N, B) \longrightarrow \mathcal{B}_0(M, B)$ in $\mathbf{Set}$, showing that B is *a fortiori* orthogonal to θ in the ordinary category $\mathcal{B}_0$. In general, of course, the latter condition is strictly weaker. To ask that $\mathcal{B}(\theta, 1)$ be an isomorphism is to ask that $\mathcal{V}_0(X, \mathcal{B}(\theta, 1))$ be an isomorphism for each $X \in \mathcal{V}$ (or equivalently for each X in a strong generator of $\mathcal{V}_0$, if it has one). If $\mathcal{B}$ is tensored, this is to ask that B be orthogonal in $\mathcal{B}_0$ to $X \otimes \theta\colon X \otimes M \longrightarrow X \otimes N$ for each $X \in \mathcal{V}$.

Orthogonality sets up a Galois connexion between sets of maps and sets of objects; in the case of ordinary categories, simple closure properties of the sets so arising are listed in §2 of [30]; such of these properties as we use in the proof below are easy to verify directly.

The reflectivity result for ordinary categories that we call upon is the following case of Theorem 10.2 of [44]: *Let the locally-small, complete, and cocomplete $\mathcal{K}$ be locally bounded, and let Θ be a (possibly large) set of maps $\theta\colon M_\theta \longrightarrow N_\theta$ in $\mathcal{K}$, such that those N_θ for which θ is not in $\mathcal{E}$ form (to within isomorphism) only a small set. Then the objects orthogonal in $\mathcal{K}$ to each $\theta \in \Theta$ determine a reflective full subcategory of $\mathcal{K}$.* (In getting this precise form from the theorem as stated, one must recall that $\mathcal{K}$ is $\mathcal{M}$-wellpowered in virtue of the existence of the $(\mathcal{E}, \mathcal{M})$-generator.)

Our goal is the following $\mathcal{V}$-based version, in the special case where the ambient category is $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$; its statement makes no explicit reference to $(\mathcal{E}, \mathcal{M})$:

Theorem 6.5 *Let $\mathcal{A}$ be a small $\mathcal{V}$-category, where $\mathcal{V}$ is locally bounded. Let Θ be a set of maps $\theta\colon M_\theta \longrightarrow N_\theta$ in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]_0$, such that the N_θ form (to within isomorphism) only a small set. Let $\mathcal{C}$ be the full subcategory of $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ determined by those objects orthogonal in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ to each $\theta \in \Theta$. Then $\mathcal{C}$ is reflective in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$.*

Proof. It is immediate that $\mathcal{C}$ is closed under cotensor products in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$; hence it suffices by Theorem 4.85 to prove the reflectivity of $\mathcal{C}_0$ in the ordinary category $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]_0$, which we now denote by $\mathcal{K}$. The objects of $\mathcal{C}_0$ are those orthogonal in $\mathcal{K}$ to $X \otimes \theta\colon X \otimes M_\theta \longrightarrow X \otimes N_\theta$ for each $X \in \mathcal{V}$. If $\mathcal{G}$ is the $(\mathcal{E}, \mathcal{M})$-generator in $\mathcal{V}_0$, write $\epsilon_X\colon RX \longrightarrow X$ for the canonical map $\sum_{G \in \mathcal{G}} \mathcal{V}_0(G, X) \cdot G \longrightarrow X$, which is in $\mathcal{E}$ by hypothesis. For each θ

and each X, form the commutative diagram

$$\begin{tikzcd}
RX \otimes M_\theta \arrow[r, "RX\otimes\theta"] \arrow[d, "\epsilon_X\otimes M_\theta"'] & RX \otimes N_\theta \arrow[d, "\sigma_{X,\theta}"'] \arrow[ddr, "\epsilon_X\otimes N_\theta"] & \\
X \otimes M_\theta \arrow[r, "\rho_{X,\theta}"] \arrow[drr, "X\otimes\theta"'] & P(X,\theta) \arrow[dr, "\tau_{X,\theta}"] & \\
& & X \otimes N_\theta,
\end{tikzcd} \tag{6.6}$$

in which the top left region is a pushout. Since tensor products in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ are formed pointwise, it follows from (6.1) that $\epsilon_X \otimes N_\theta$ is in $\mathcal{E}$, whence its factor $\tau_{X,\theta}$ is in $\mathcal{E}$. Every object orthogonal in $\mathcal{K}$ to $RX \otimes \theta$ is also orthogonal to its pushout $\rho_{X,\theta}$; it is then orthogonal to the composite $X \otimes \theta = \tau_{X,\theta}\rho_{X,\theta}$ if and only if it is orthogonal to $\tau_{X,\theta}$. Thus $\mathcal{C}_0$ consists of the objects in $\mathcal{K}$ orthogonal to each $RX \otimes \theta$ and to each $\tau_{X,\theta}$. Since $RX \otimes \theta = (\sum_G \mathcal{V}_0(G, X) \cdot G) \otimes \theta \cong \sum_G \mathcal{V}_0(G, X) \cdot (G \otimes \theta)$, any object orthogonal in $\mathcal{K}$ to each $G \otimes \theta$ is also orthogonal to the coproduct $RX \otimes \theta$ of these; and thus $\mathcal{C}_0$ consists of the objects in $\mathcal{K}$ orthogonal to each $G \otimes \theta$ for $G \in \mathcal{G}$ and to each $\tau_{X,\theta}$. Since the number of different $G \otimes N_\theta$ is small, and since each $\tau_{X,\theta}$ is in $\mathcal{E}$, the desired reflectivity of $\mathcal{C}_0$ in $\mathcal{K}$ follows from Proposition 6.4 and the result quoted before the theorem. ∎

By Proposition 5.15, the cocomplete $\mathcal{V}$-category $\mathcal{C}$ of Theorem 6.5 is the category of algebras for an essentially-algebraic theory $K\colon \mathcal{A} \longrightarrow \mathcal{C}$, where K is the composite of the Yoneda embedding $\mathcal{A} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ and the reflexion $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \longrightarrow \mathcal{C}$. Clearly $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ is a comodel for the theory precisely when $\mathcal{B}(G-, B)\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{V}$ is for each $B \in \mathcal{B}$ orthogonal in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ to each $\theta \in \Theta$. By a very free adaptation of the terminology of Isbell [37], we shall in future call such a set Θ as occurs in Theorem 6.5 a *regulus*. We shall use such terms as Θ-comodel and K-comodel interchangeably, and shall write Θ-**Alg** for the category $\mathcal{C}$ of K-algebras.

6.3 The category of algebras for a sketch, and in particular for an $\mathcal{F}$-theory; algebraic functors and their adjoints

We suppose henceforth that $\mathcal{V}$ is *locally bounded*. Let $\mathcal{A}$ be a small $\mathcal{V}$-category, and consider a set (not necessarily small)

$$\Phi = \{\phi_\gamma\colon F_\gamma \longrightarrow \mathcal{A}(P_\gamma -, A_\gamma)\}_{\gamma\in\Gamma} \tag{6.7}$$

of cylinders in $\mathcal{A}$, where $F_\gamma\colon \mathcal{L}_\gamma^{\mathrm{op}} \longrightarrow \mathcal{V}$ and $P_\gamma\colon \mathcal{L}_\gamma \longrightarrow \mathcal{A}$, with each $\mathcal{L}_\gamma$ small. Call a functor $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ a Φ-*comodel* if, for each γ, the cylinder

$$F_\gamma \xrightarrow[\phi_\gamma]{} \mathcal{A}(P_\gamma -, A_\gamma) \xrightarrow[G]{} \mathcal{B}(GP_\gamma -, GA_\gamma) \tag{6.8}$$

is a colimit-cylinder; and write Φ-Com$[\mathcal{A},\mathcal{B}]$ for the full subcategory of $[\mathcal{A},\mathcal{B}]$ given by these comodels. Similarly, write Φ-Mod$[\mathcal{A}^{\mathrm{op}},\mathcal{P}]$ for $(\Phi\text{-Com}[\mathcal{A},\mathcal{P}^{\mathrm{op}}])^{\mathrm{op}}$, and write Φ-Alg for Φ-Mod$[\mathcal{A}^{\mathrm{op}},\mathcal{V}]$. Thus the Φ-algebras are those $\mathcal{A}^{\mathrm{op}}\longrightarrow\mathcal{V}$ that send each ϕ_γ to a limit-cone. By very freely adapting the terminology of Ehresmann [24] (see also [3]), we may call the pair $(\mathcal{A}^{\mathrm{op}},\Phi)$, or just Φ for short, a *sketch*.

Given such a sketch, each composite

$$F_\gamma \xrightarrow[\phi_\gamma]{} \mathcal{A}(P_\gamma -, A_\gamma) \xrightarrow[Y]{} [\mathcal{A}^{\mathrm{op}},\mathcal{V}](YP_\gamma -, YA_\gamma) \tag{6.9}$$

corresponds by (3.5) to a map

$$\theta_\gamma\colon F_\gamma \star YP_\gamma \longrightarrow YA_\gamma \tag{6.10}$$

in the cocomplete $[\mathcal{A}^{\mathrm{op}},\mathcal{V}]$. Write $\Theta=\{\theta_\gamma\}_{\gamma\in\Gamma}$ for the set of these maps; because $\mathcal{A}$ is small, the set of codomains in (6.10) is small, so that Θ is indeed a regulus in the sense of 6.2.

Theorem 6.11 *If the regulus Θ corresponds as above to the sketch Φ, then*

$$\Phi\text{-Com}[\mathcal{A},\mathcal{B}] = \Theta\text{-Com}[\mathcal{A},\mathcal{B}]$$

for any category $\mathcal{B}$. In particular Φ-Alg *is the reflective subcategory* Θ-Alg *of $[\mathcal{A}^{\mathrm{op}},\mathcal{V}]$. The corresponding dense functor $K\colon\mathcal{A}\longrightarrow\Phi$-Alg *is fully faithful precisely when the cylinders ϕ_γ are already colimit-cylinders in $\mathcal{A}$.*

Proof. By the remarks at the end of 6.2, $G\colon\mathcal{A}\longrightarrow\mathcal{B}$ is a Θ-comodel precisely when each $\mathcal{B}(G-,B)$ is orthogonal in $[\mathcal{A}^{\mathrm{op}},\mathcal{V}]$ to each θ_γ; that is, when each

$$[\mathcal{A}^{\mathrm{op}},\mathcal{V}](\theta_\gamma,1)\colon [\mathcal{A}^{\mathrm{op}},\mathcal{V}]\big(YA_\gamma,\mathcal{B}(G-,B)\big)\longrightarrow[\mathcal{A}^{\mathrm{op}},\mathcal{V}]\big(F_\gamma\star YP_\gamma,\mathcal{B}(G-,B)\big)$$

is an isomorphism. By (3.5) and Yoneda, this map is isomorphic to a map

$$\mathcal{B}(GA_\gamma,B)\longrightarrow[\mathcal{L}_\gamma^{\mathrm{op}},\mathcal{V}]\big(F_\gamma,\mathcal{B}(GP_\gamma-,B)\big), \tag{6.12}$$

whose unit is easily verified to be (6.8); to say that it is an isomorphism is therefore by (3.5) to say that (6.8) is a colimit-cylinder, or that G is a Φ-comodel. Applying this to the comodels in $\mathcal{V}^{\mathrm{op}}$ gives by Theorem 5.59 the equality Φ-Alg $=\Theta$-Alg.

By Proposition 5.16, K is fully faithful if and only if Φ-Alg contains the representables $\mathcal{A}(-,A)$. To ask each $\mathcal{A}(-,A)$ to be an algebra is by (6.12) to ask that each

$$\mathcal{A}(A_\gamma,A)\longrightarrow[\mathcal{L}_\gamma^{\mathrm{op}},\mathcal{V}]\big(F_\gamma,\mathcal{A}(P_\gamma-,A)\big)$$

be an isomorphism, and hence by (3.5) that each ϕ_γ be a colimit-cylinder. ∎

In particular, Φ-Alg is reflective in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ when Φ consists of *all* small colimit-cylinders in $\mathcal{A}$; the reflectivity of $\mathcal{V}_0''$ in $\mathcal{V}_0' = [\mathcal{V}_0^{\mathrm{op}}, \mathbf{Set}']$ of 3.12 above is an example of this.

A very important special case of Theorem 6.11 is that where $\mathcal{A}$ is $\mathcal{F}$-cocomplete for some set $\mathcal{F}$ of small indexing-types, and where Φ consists of (the units of) all the $\mathcal{F}$-colimits in $\mathcal{A}$. In this case $\Phi\text{-Com}[\mathcal{A}, \mathcal{B}] = \mathcal{F}\text{-Cocts}[\mathcal{A}, \mathcal{B}]$ and $\Phi\text{-Alg} = \mathcal{F}\text{-Cts}[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$. The full name $(\mathcal{A}^{\mathrm{op}}, \Phi)$ of the sketch Φ might here be replaced by $(\mathcal{A}^{\mathrm{op}}, \mathcal{F})$. It is also suggestive to refer to a small $\mathcal{F}$-complete category $\mathcal{A}^{\mathrm{op}}$ as an $\mathcal{F}$-theory. For instance, with $\mathcal{V} = \mathbf{Set}$, a Lawvere-Bénabou theory ([52], [7]) is a small category $\mathcal{A}^{\mathrm{op}}$ with finite products; while a small $\mathcal{A}^{\mathrm{op}}$ with finite limits is sometimes called a *finitary essentially-algebraic theory*. (Joyal suggests calling the latter a *cartesian theory*; in that finite limits may be said to go back to Descartes, who introduced both the product $\mathbb{R} \times \mathbb{R}$ and the equalizer $\{(x, y) \mid f(x, y) = g(x, y)\}$.) This terminology moreover allows $\mathcal{A}^{\mathrm{op}}$-Alg ($\mathcal{F}$ being understood) as a useful abbreviation for $\mathcal{F}\text{-Cts}[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$.

Consider now a sketch $(\mathcal{A}^{\mathrm{op}}, \Phi)$ where Φ is given by (6.7); let $\mathcal{F}$ be a set of small indexing-types containing each F_γ occurring in Φ; and let $\mathcal{B}$ be a small $\mathcal{F}$-cocomplete category. Write $K\colon \mathcal{A} \longrightarrow \Phi\text{-Alg}$ and $L\colon \mathcal{B} \longrightarrow \mathcal{B}^{\mathrm{op}}\text{-Alg} = \mathcal{F}\text{-Cts}[\mathcal{B}^{\mathrm{op}}, \mathcal{V}]$ for the respective dense functors, observing by Theorem 6.11 that L is fully faithful. Note that, since L is also an L-comodel by (5.51), we have:

Proposition 6.13 *For a small $\mathcal{F}$-cocomplete $\mathcal{B}$, the dense inclusion $L\colon \mathcal{B} \longrightarrow \mathcal{B}^{\mathrm{op}}\text{-Alg} = \mathcal{F}\text{-Cts}[\mathcal{B}^{\mathrm{op}}, \mathcal{V}]$ preserves and reflects $\mathcal{F}$-colimits.*

Consider a Φ-comodel $G\colon \mathcal{A} \longrightarrow \mathcal{B}$. If $Q\colon \mathcal{B} \longrightarrow \mathcal{D}$ is any $\mathcal{F}$-continuous functor, the composite QG is clearly another Φ-comodel. Taking $\mathcal{D} = \mathcal{V}^{\mathrm{op}}$ here, we see that $[G^{\mathrm{op}}, 1]\colon [\mathcal{B}^{\mathrm{op}}, \mathcal{V}] \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ restricts to a functor

$$G^*\colon \mathcal{B}^{\mathrm{op}}\text{-Alg} \longrightarrow \Phi\text{-Alg}. \tag{6.14}$$

Such a functor between the categories of algebras, induced by a *realization* (that is, a model) G^{op} of one theory in the other, is often said to be *algebraic*: especially in the case where $\mathcal{A}$ too is $\mathcal{F}$-cocomplete and $\Phi = (\mathcal{A}^{\mathrm{op}}, \mathcal{F})$.

Since $\widetilde{L}\colon \mathcal{B}^{\mathrm{op}}\text{-Alg} \longrightarrow [\mathcal{B}^{\mathrm{op}}, \mathcal{V}]$ and $\widetilde{K}\colon \Phi\text{-Alg} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ are fully faithful, since

$$[G^{\mathrm{op}}, 1]\colon [\mathcal{B}^{\mathrm{op}}, \mathcal{V}] \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$$

has by Theorem 4.50 the left adjoint $\mathrm{Lan}_{G^{\mathrm{op}}}$, and since $\widetilde{L}$ has by Theorem 4.51 the left adjoint $- \star L$, it follows that the algebraic functor $\widehat{G}^*$ has the left adjoint

$$G_* = (- \star L).\mathrm{Lan}_{G^{\mathrm{op}}}.\widetilde{K}\colon \mathcal{B}^{\mathrm{op}}\text{-Alg} \longrightarrow \Phi\text{-Alg}.$$

The value of this at $P \in \mathcal{B}^{\mathrm{op}}$-Alg is $\big(\mathrm{Lan}_{G^{\mathrm{op}}}(\widetilde{K}P)\big) \star L$, which by (4.65) is also $(\widetilde{K}P) \star (LG)$; so that by (4.17) we have

$$G_* \cong \mathrm{Lan}_K(LG). \tag{6.15}$$

In fact G is recoverable from G_*. First observe that Proposition 6.13 gives:

Proposition 6.16 *In the circumstances above, $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ is a Φ-comodel if and only if $LG\colon \mathcal{A} \longrightarrow \mathcal{B}^{\mathrm{op}}$-Alg is a Φ-comodel. Thus Φ-Com$[\mathcal{A}, \mathcal{B}]$ is the full subcategory of Φ-Com$[\mathcal{A}, \mathcal{B}^{\mathrm{op}}$-Alg$]$ given by the functors factorizing through L.*

Using this and (6.15), we have:

Theorem 6.17 *Let the small $\mathcal{B}$ be $\mathcal{F}$-cocomplete, where the set $\mathcal{F}$ of small indexing-types contains all the F_γ occurring in the sketch $(\mathcal{A}^{\mathrm{op}}, \Phi)$; and let $K\colon \mathcal{A} \longrightarrow \Phi$-Alg and $L\colon \mathcal{B} \longrightarrow \mathcal{B}^{\mathrm{op}}$-Alg be the respective dense functors. Then the composite of the full inclusion Φ-Com$[\mathcal{A}, \mathcal{B}] \longrightarrow \Phi$-Com$[\mathcal{A}, \mathcal{B}^{\mathrm{op}}$-Alg$]$ sending G to LG, and the equivalence*

$$[K, 1] \dashv \mathrm{Lan}_K\colon \Phi\text{-Com}[\mathcal{A}, \mathcal{B}^{\mathrm{op}}\text{-Alg}] \simeq \mathrm{Ladj}[\Phi\text{-Alg}, \mathcal{B}^{\mathrm{op}}\text{-Alg}]$$

of Theorem 5.56, is isomorphic to a functor

$$(\ )_*\colon \Phi\text{-Com}[\mathcal{A}, \mathcal{B}] \longrightarrow \mathrm{Ladj}[\Phi\text{-Alg}, \mathcal{B}^{\mathrm{op}}\text{-Alg}] \tag{6.18}$$

sending G to the left adjoint G_* of G^*. Accordingly we have

$$G_* K \cong LG; \tag{6.19}$$

and the functor $(\ )_*$ of (6.18) is fully faithful, its image consisting of those left-adjoint $S\colon \Phi$-Alg $\longrightarrow \mathcal{B}^{\mathrm{op}}$-Alg for which SK factors through the fully-faithful L as $SK \cong LG$ for some (essentially unique) G. ∎

We may consider this theorem in the case where $\mathcal{A}$ too is $\mathcal{F}$-cocomplete, and $\Phi = (\mathcal{A}^{\mathrm{op}}, \mathcal{F})$. The $\mathcal{F}$-theories $\mathcal{A}^{\mathrm{op}}$, the $\mathcal{F}$-continuous functors between these, and the $\mathcal{V}$-natural transformations between the latter, form a 2-category – that is, a **Cat**-category. In fact they form more; for $\mathcal{F}$-Cocts$[\mathcal{A}, \mathcal{B}]$ is not just a category but a $\mathcal{V}$-category. We saw in 2.3 that $\mathcal{V}$-**Cat** is symmetric monoidal closed; and what the $\mathcal{F}$-theories actually constitute is a ($\mathcal{V}$-**Cat**)-category. There is a ($\mathcal{V}$-**Cat**)-functor into $\mathcal{V}$-**CAT** sending the $\mathcal{F}$-theory $\mathcal{A}^{\mathrm{op}}$ to $\mathcal{A}^{\mathrm{op}}$-Alg and defined on hom-objects by (6.18). It is *locally fully faithful*, in the sense that (6.18) is fully faithful. Theorem 6.17 above determines, in a certain sense, its local image; not as absolutely as we should like, since the criterion for $S\colon \mathcal{A}^{\mathrm{op}}$-Alg $\longrightarrow \mathcal{B}^{\mathrm{op}}$-Alg to be in this local image refers back to the inclusions of $\mathcal{A}$ and $\mathcal{B}$. The remaining step in determining globally the image of this ($\mathcal{V}$-**Cat**)-functor is that of characterizing the $\mathcal{V}$-categories of the form $\mathcal{A}^{\mathrm{op}}$-Alg.

For this we can get something from Theorem 5.60; in the absence of any particularly simple density presentation of $K\colon \mathcal{A} \longrightarrow \mathcal{A}^{\mathrm{op}}$-Alg, we can just use the canonical one $\widetilde{K}C \star K \cong C$ from Theorem 5.19. We have:

Proposition 6.20 *If the small $\mathcal{A}$ is $\mathcal{F}$-cocomplete and $K\colon \mathcal{A} \longrightarrow \mathcal{A}^{\mathrm{op}}$-Alg is the fully-faithful dense functor as above, let $G\colon \mathcal{A} \longrightarrow \mathcal{D}$ be an $\mathcal{F}$-cocontinuous functor, and hence a K-comodel; and define $T\colon \mathcal{D} \longrightarrow \mathcal{A}^{\mathrm{op}}$-Alg by $\widetilde{G} \cong \widetilde{K}T$. In order that T be an equivalence, the following conditions are necessary and sufficient:*

(i) *$\mathcal{D}$ is cocomplete.*

(ii) *G is fully faithful and strongly generating.*

(iii) *The canonical map* $\mathcal{A}^{\mathrm{op}}\text{-Alg}(KA, C) \longrightarrow \mathcal{D}\big(GA, \mathcal{A}^{\mathrm{op}}\text{-Alg}(K-, C) \star G\big)$ *is an isomorphism.*

Proof. The conditions are clearly necessary. For their sufficiency we refer not to Theorem 5.60 but to its proof. Since $\mathcal{D}$ is cocomplete, G admits a left Kan extension S along K, which by Theorem 5.56 is a left adjoint to T. For η and ϵ to be isomorphisms, we need that G is fully faithful and strongly generating (which is (ii)) and that the colimit $\widetilde{K}C \star SK$ is G-absolute. Since $SK \cong G$, this last requirement is that $\mathcal{D}(GA, \widetilde{K}C \star G)$ be isomorphic to $\widetilde{K}C \star \mathcal{D}(GA, G-)$; and as G is fully faithful the latter is $\widetilde{K}C \star \mathcal{A}(A, -)$ or $(\widetilde{K}C)A$, giving (iii). ∎

Again this recognition theorem is not as absolute as we should like, in that condition (iii) refers in detail to $\mathcal{A}^{\mathrm{op}}$-Alg. A far better description of the image of $\mathcal{A}^{\mathrm{op}} \mapsto \mathcal{A}^{\mathrm{op}}$-Alg is often available when both $\mathcal{V}$ and $\mathcal{F}$ are suitably restricted – see for example [45]. The fact is that we are coming to the end of what can usefully be said in the present generality.

6.4 The $\mathcal{F}$-theory generated by a small sketch

One last series of important observations can however be made.

Call a sketch $(\mathcal{A}^{\mathrm{op}}, \Phi)$ *small* if the set Φ of cylinders is small. If all the F_γ occurring in Φ lie in a *small* set $\mathcal{F}$ of indexing-types, the smallness of Φ is automatic; for the small $\mathcal{A}$ admits but a small set of cylinders with any given indexing-type.

Proposition 6.21 *Let* $(\mathcal{A}^{\mathrm{op}}, \Phi)$ *be a small sketch which involves indexing-types only from the small set* $\mathcal{F}$*; and let* $K\colon \mathcal{A} \longrightarrow \mathcal{C} = \Phi\text{-Alg}$ *be the corresponding dense functor. Let* $\mathcal{D}$ *be the closure under* $\mathcal{F}$*-colimits in* $\mathcal{C}$ *of the full image of* K*; it is small by 3.5. Let the factorization of* K *through* $\mathcal{D}$ *be* $Z\colon \mathcal{A} \longrightarrow \mathcal{D}$ *followed by the full inclusion* $J\colon \mathcal{D} \longrightarrow \mathcal{C}$*; by Theorem 5.13 both* Z *and* J *are dense, so that in particular* $\widetilde{J}\colon \mathcal{C} \longrightarrow [\mathcal{D}^{\mathrm{op}}, \mathcal{V}]$ *is fully faithful. We assert that the replete image of* $\widetilde{J}$ *is* $\mathcal{F}\text{-Cts}[\mathcal{D}^{\mathrm{op}}, \mathcal{V}]$.

Proof. One direction is immediate. J preserves $\mathcal{F}$-colimits by the construction of $\mathcal{D}$, so that $\mathcal{C}(J-, C)$ sends them to limits in $\mathcal{V}$, and hence $\widetilde{J}C$ is $\mathcal{F}$-continuous.

Suppose now that $P^{\mathrm{op}}\colon \mathcal{D}^{\mathrm{op}} \longrightarrow \mathcal{V}$ is $\mathcal{F}$-continuous. Since $K = JZ$ is a Φ-comodel by (5.51), and since the fully-faithful $J\colon \mathcal{D} \longrightarrow \mathcal{C}$ not only preserves but also reflects $\mathcal{F}$-colimits, $Z\colon \mathcal{A} \longrightarrow \mathcal{D}$ is itself a Φ-comodel. Since P is $\mathcal{F}$-cocontinuous, $PZ\colon \mathcal{A} \longrightarrow \mathcal{V}^{\mathrm{op}}$ too is a Φ-comodel. By Theorem 5.56, we have $PZ \cong SK$ where $S\colon \mathcal{C} \longrightarrow \mathcal{V}^{\mathrm{op}}$ is a left-adjoint functor given by $S = \mathrm{Lan}_K PZ$, the unit $PZ \longrightarrow SK$ of this left Kan extension being an isomorphism. By Theorem 4.47, since $K = JZ$ with J fully faithful, we have $\mathrm{Lan}_Z PZ \cong SJ$, with unit λ an isomorphism. Applying Theorem 4.43 to $1\colon PZ \longrightarrow PZ$ as in

$$\begin{tikzcd}
 & \mathcal{D} \arrow[dr, bend left, "P"] \arrow[dr, bend right, "SJ"'] \arrow[d, phantom, "\mu\Uparrow"] & \\
\mathcal{A} \arrow[ur, "Z"] \arrow[rr, "PZ"'] \arrow[ur, phantom, shift right=3ex, "\lambda\Uparrow"] & {} & \mathcal{V}^{\mathrm{op}}
\end{tikzcd}$$

gives a map $\mu\colon SJ \longrightarrow P$ with μZ an isomorphism. Since both P and SJ preserve $\mathcal{F}$-colimits, and since $\mathcal{D}$ is the closure under $\mathcal{F}$-colimits of the full image of Z (which is the full image of K), we conclude that $\mu\colon SJ \longrightarrow P$ is an isomorphism.

The right-adjoint $S^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{V}$ being necessarily a representable $\mathcal{C}(-, C)$, we have $P^{\mathrm{op}} \cong S^{\mathrm{op}} J^{\mathrm{op}} \cong \mathcal{C}(J-, C) = \widetilde{J}C$, as required. ∎

Now Theorem 6.11 gives:

Corollary 6.22 *In these circumstances the functor J is isomorphic to the inclusion $\mathcal{D} \longrightarrow \mathcal{D}^{\mathrm{op}}$-Alg of the $\mathcal{F}$-cocomplete small $\mathcal{D}$, so that for any $\mathcal{B}$ we have J-Com$[\mathcal{D}, \mathcal{B}] = \mathcal{F}$-Cocts$[\mathcal{D}, \mathcal{B}]$.* ∎

Since the size of $\mathcal{F}$ here is bounded only below by the knowledge of the sketch Φ, we cannot expect the comodels of the sketches $(\mathcal{A}^{\mathrm{op}}, \Phi)$ and $(\mathcal{D}^{\mathrm{op}}, \mathcal{F})$ to "coincide" in *every* $\mathcal{B}$; that they do so when $\mathcal{B}$ is $\mathcal{F}$-cocomplete is an example of the "sufficient completeness" referred to in the penultimate paragraph of 5.13. The precise result is:

Theorem 6.23 *In the circumstances of Proposition 6.21, every Φ-comodel $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ admits a left Kan extension along $Z\colon \mathcal{A} \longrightarrow \mathcal{D}$ whenever $\mathcal{B}$ is $\mathcal{F}$-cocomplete; and we have an equivalence*

$$\mathrm{Lan}_Z \dashv [Z, 1]\colon \mathcal{F}\text{-Cocts}[\mathcal{D}, \mathcal{B}] \simeq \Phi\text{-Com}[\mathcal{A}, \mathcal{B}]$$

whose unit and counit are the canonical maps.

Proof. By Corollary 6.22 and Theorem 5.52, we have only to show that *every* Φ-comodel $G\colon \mathcal{A} \longrightarrow \mathcal{B}$ has the form HZ for some $\mathcal{F}$-continuous $H\colon \mathcal{D} \longrightarrow \mathcal{B}$.

We suppose first that $\mathcal{B}$ is small, and consider the inclusion $L\colon \mathcal{B} \longrightarrow \mathcal{B}^{\mathrm{op}}$-Alg of Proposition 6.13. Then LG, being itself a Φ-comodel as in Proposition 6.16, is by Theorem 5.56 isomorphic to SK for some left-adjoint $S\colon \Phi\text{-Alg} \longrightarrow \mathcal{B}^{\mathrm{op}}$-Alg; in fact S is the G_* of (6.19). The set $\mathcal{H}$ of those objects $C \in \mathcal{C}$ for which SC lies in $\mathcal{B} \subset \mathcal{B}^{\mathrm{op}}$-Alg certainly contains all the objects KA, since $SK \cong LG$; and it is closed under $\mathcal{F}$-colimits by Proposition 6.13, since the left-adjoint S preserves colimits; hence $\mathcal{H}$ contains $\mathcal{D}$ by the definition of this latter. Thus $SJ \cong LH$ for some $H\colon \mathcal{D} \longrightarrow \mathcal{B}$, which is clearly $\mathcal{F}$-cocontinuous since SJ is so. Now $LG \cong SK = SJZ \cong LHZ$; and ($L$ being fully faithful) we have the desired result $G \cong HZ$.

Passing to the case where the $\mathcal{F}$-cocomplete $\mathcal{B}$ need not be small, let $\mathcal{B}'$ be the closure under $\mathcal{F}$-colimits in $\mathcal{B}$ of the full image of the Φ-comodel $G\colon \mathcal{A} \longrightarrow \mathcal{B}$; this latter factorizing as $G'\colon \mathcal{A} \longrightarrow \mathcal{B}'$ followed by the full inclusion $N\colon \mathcal{B}' \longrightarrow \mathcal{B}$. Since N preserves and reflects $\mathcal{F}$-colimits, G' is itself a Φ-comodel; and since $\mathcal{B}'$ is small by 3.5, we have as above $G' \cong H'Z$ for some $\mathcal{F}$-cocontinuous $H'\colon \mathcal{D} \longrightarrow \mathcal{B}'$. Now $G \cong HZ$, where $H = NH'$ is $\mathcal{F}$-cocontinuous. ∎

We may say that $\mathcal{D}^{\mathrm{op}}$ is the *$\mathcal{F}$-theory generated by the sketch* $(\mathcal{A}^{\mathrm{op}}, \Phi)$, or that $(\mathcal{A}^{\mathrm{op}}, \Phi)$ is a *sketch of the $\mathcal{F}$-theory* $\mathcal{D}^{\mathrm{op}}$. The latter form of words comes closer to explicating Ehresmann's choice of the word "sketch". As he used the word (in the case $\mathcal{V} = \mathbf{Set}$), the small category $\mathcal{A}$ was not in general explicitly given, but presented by generators and relations; often this presentation was finite, and Φ was a finite set of finite cones (the colimits involved in $\mathcal{F}$ classically being conical ones). In such cases, one can literally see

(the presentation of) $(\mathcal{A}^{\mathrm{op}}, \Phi)$ as a finitary way of "sketching" the generally infinite and complicated structure of $\mathcal{D}^{\mathrm{op}}$, or the still more complicated structure of $\mathcal{D}^{\mathrm{op}}$-Alg.

When $\mathcal{V} = \mathbf{Set}$ and $\mathcal{F}$ consists of the functors $\Delta 1\colon \mathcal{L} \longrightarrow \mathbf{Set}$ for finite $\mathcal{L}$, the $\mathcal{F}$-theories are the cartesian theories of 6.3 above; the sketch $(\mathcal{A}^{\mathrm{op}}, \Phi)$ gives rise to such a theory if Φ is a small set of finite cones. It is exactly the categories Φ-Alg for such a Φ that are the *locally finitely presentable* categories; $\mathcal{D} \subset \Phi$-Alg is a small dense subcategory composed of objects which (since finite limits commute with filtered colimits in **Set**) are finitely presentable. In all the examples of categories $\mathcal{V}_0$ that we declared locally finitely presentable in 6.1, it is easy to see that $\mathcal{V}_0 = \Phi$-Alg for such a sketch Φ. For further information on the relation of locally presentable categories to reguli and sketches, see the classical [31], or the newer [45] which extends the ideas to certain important $\mathcal{V}$ other than **Set**. These $\mathcal{V}$ include **Cat** and **Gpd**, which are the appropriate base categories for the study of *categories* with essentially-algebraic structure. At this level, as is well known, equivalence becomes more central than isomorphism, and the naturally-occurring "functors" and "transformations" are more general than the $\mathcal{V}$-functors and the $\mathcal{V}$-natural transformations; which leads to various problems of *coherence*. The study of essentially-algebraic theories in this context must take account of these more general notions; this is not touched on in [45], but a beginning is made in [46].

6.5 The symmetric monoidal closed structure on the category of $\mathcal{F}$-cocomplete categories and $\mathcal{F}$-cocontinuous functors

Before ending, we give a final application of Theorem 6.23. We consider the ($\mathcal{V}$-**Cat**)-category of small $\mathcal{F}$-cocomplete categories $\mathcal{A}$, where $\mathcal{F}$ is a small set of indexing-types. Since $\mathcal{F}$-Cocts$[\mathcal{A}, \mathcal{B}]$ denotes the ($\mathcal{V}$-**Cat**)-valued hom here, an appropriate name for this category is $\mathcal{F}$-Cocts.

If $\mathcal{B}$ and $\mathcal{C}$ are $\mathcal{F}$-cocomplete, so of course is $[\mathcal{B}, \mathcal{C}]$, with $\mathcal{F}$-colimits formed pointwise. The full subcategory $\mathcal{F}$-Cocts$[\mathcal{B}, \mathcal{C}]$ of this is again $\mathcal{F}$-cocomplete, since colimits commute with colimits. The functors $Q\colon \mathcal{A} \longrightarrow \mathcal{F}$-Cocts$[\mathcal{B}, \mathcal{C}]$ correspond to those functors $P\colon \mathcal{A} \otimes \mathcal{B} \longrightarrow \mathcal{C}$ for which each partial functor $P(A, -)\colon \mathcal{B} \longrightarrow \mathcal{C}$ is $\mathcal{F}$-cocontinuous. If $\mathcal{A}$ too is $\mathcal{F}$-cocomplete, the functor Q is *itself* $\mathcal{F}$-cocontinuous precisely when the *other* partial functors $P(-, B)\colon \mathcal{A} \longrightarrow \mathcal{C}$ are also $\mathcal{F}$-cocontinuous. We are thus led to the notion of a $P\colon \mathcal{A} \otimes \mathcal{B} \longrightarrow \mathcal{C}$ which is $\mathcal{F}$-cocontinuous in each variable separately.

Such a P is a comodel for a sketch $\big((\mathcal{A} \otimes \mathcal{B})^{\mathrm{op}}, \Phi\big)$. If $\lambda\colon F \longrightarrow \mathcal{A}(T-, M)$ is a cylinder in $\mathcal{A}$, where $F\colon \mathcal{L}^{\mathrm{op}} \longrightarrow \mathcal{V}$ and $T\colon \mathcal{L} \longrightarrow \mathcal{A}$, we can consider for $B \in \mathcal{B}$ the functor

$$\mathcal{L} \cong \mathcal{L} \otimes I \xrightarrow[T \otimes B]{} \mathcal{A} \otimes \mathcal{B}$$

sending L to (TL, B), and called $T \otimes B$. Then we can consider the cylinder

$$F \xrightarrow[\lambda]{} \mathcal{A}(T-, M) \cong \mathcal{A}(T-, M) \otimes I \xrightarrow[1 \otimes j]{} \mathcal{A}(T-, M) \otimes \mathcal{B}(B, B) = (\mathcal{A} \otimes \mathcal{B})\big((T-, B), (M, B)\big),$$

and call this $\lambda \otimes B$. Note that this is a colimit-cylinder in $\mathcal{A} \otimes \mathcal{B}$ if λ is one in $\mathcal{A}$; essentially because $- \otimes Y\colon \mathcal{V} \longrightarrow \mathcal{V}$ preserves colimits. If we take Φ to consist of all the $\lambda \otimes B$ and

all the $A \otimes \mu$, where λ is an $\mathcal{F}$-colimit-cylinder in $\mathcal{A}$ and μ is an $\mathcal{F}$-colimit-cylinder in $\mathcal{B}$, the Φ-comodels $P\colon \mathcal{A} \otimes \mathcal{B} \longrightarrow \mathcal{C}$ are precisely those P that are $\mathcal{F}$-cocontinuous in each variable. In fact more than this is true; it is easy to see that

$$\Phi\text{-Com}[\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] \cong \mathcal{F}\text{-Cocts}\big[\mathcal{A}, \mathcal{F}\text{-Cocts}[\mathcal{B}, \mathcal{C}]\big].$$

Write $(\mathcal{A} \otimes_{\mathcal{F}} \mathcal{B})^{\mathrm{op}}$ for the $\mathcal{F}$-theory generated by the sketch $\big((\mathcal{A} \otimes \mathcal{B})^{\mathrm{op}}, \Phi\big)$. In view of Theorem 6.23 we get

$$\mathcal{F}\text{-Cocts}[\mathcal{A} \otimes_{\mathcal{F}} \mathcal{B}, \mathcal{C}] \simeq \mathcal{F}\text{-Cocts}\big[\mathcal{A}, \mathcal{F}\text{-Cocts}[\mathcal{B}, \mathcal{C}]\big]. \tag{6.24}$$

The symmetry in $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ of $\mathcal{F}\text{-Cocts}\big[\mathcal{A}, \mathcal{F}\text{-Cocts}[\mathcal{B}, \mathcal{F}\text{-Cocts}[\mathcal{C}, \mathcal{D}]]\big]$ now easily gives

$$(\mathcal{A} \otimes_{\mathcal{F}} \mathcal{B}) \otimes_{\mathcal{F}} \mathcal{C} \simeq \mathcal{A} \otimes_{\mathcal{F}} (\mathcal{B} \otimes_{F} \mathcal{C}) \tag{6.25}$$

and

$$\mathcal{A} \otimes_{\mathcal{F}} \mathcal{B} \simeq \mathcal{B} \otimes_{\mathcal{F}} \mathcal{A}. \tag{6.26}$$

If $\overline{\mathcal{A}}$ is the free $\mathcal{F}$-cocompletion of $\mathcal{A}$ as in Theorem 5.35, it is clear that (for any small $\mathcal{A}$ and $\mathcal{B}$)

$$\overline{\mathcal{A}} \otimes_{\mathcal{F}} \overline{\mathcal{B}} \simeq \overline{\mathcal{A} \otimes \mathcal{B}}\ ; \tag{6.27}$$

it is also clear that, for any small $\mathcal{A}$ and for $\mathcal{F}$-cocomplete $\mathcal{B}$, the $\mathcal{F}$-cocontinuous functors $\overline{\mathcal{A}} \otimes_{\mathcal{F}} \mathcal{B} \longrightarrow \mathcal{C}$ correspond to the functors $P\colon \mathcal{A} \otimes \mathcal{B} \longrightarrow \mathcal{C}$ for which $P(A, -)$ is $\mathcal{F}$-cocontinuous. It follows that

$$\overline{I} \otimes_{\mathcal{F}} \mathcal{B} \simeq \mathcal{B}\ , \tag{6.28}$$

where $\overline{I}$ is the free $\mathcal{F}$-cocompletion of I; which by Theorem 5.35 is the closure of I in $\mathcal{V}$ under $\mathcal{F}$-colimits.

The coherence, in the appropriate sense, of the maps (6.25), (6.26), and (6.28), follows from the universal property (6.24). Thus the ($\mathcal{V}$-**Cat**)-category $\mathcal{F}$-Cocts is endowed with something like a symmetric monoidal closed structure – the difference being that we have equivalences in place of isomorphisms.

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Index

Absolute colimit, 101
  K-absolute colimit, 92
Accessible functor, 77
  weakly accessible functor, 78
Adequate, 86
adjoint
  partial, 23
Adjoint functor theorem
  general, 79
  special, 80
Adjoint, left, 23
Adjoint, right, 23
Adjunction, 24
Algebra, 109, 119
  for a regulus, 118
  for a sketch, 119
Algebraic functor, 120
Atom, 80

Bicategory, 4
Biclosed
  bicategory, 14
  monoidal category, 14
Boolean algebra, 15

$\mathcal{C}$-object in $\mathcal{P}$, 109
C-object in P, 109
Cartesian theory, 120
Category
  2-category, 8
  cartesian closed, 13
  cartesian monoidal, 7
  closed, 13
  enriched over $\mathcal{V}$, 8
  filtered, 71
  finite, 71
  functor category, 30
  monoidal, 7
  monoidal biclosed, 14
  monoidal closed, 13
  of elements of F, 22
  underlying, 10
Cauchy complete, 95
Cauchy completion, 95
Closed, 7
Closed monoidal category, 13
Closure
  under a family of colimits, 45, 82
  under a family of cylinders, 82
Cocomplete, 27
  $\mathcal{F}$-cocomplete, 99
  finitely, 71
Cocompletion, free, 67
Cocontinuous, 39, 99
  $\mathcal{F}$-cocontinuous, 99
Codense, 86
Coend, 53
Cogenerator, strong, 46
Coherence, 124
Coherence axioms, 7, 11
Cointersection, 113
Colimit
  absolute, 101
  classical, 42
  conical, 42, 50, 52
  $\mathcal{F}$-colimits, 99
  filtered, 72
  indexed, 38
  iterated, 51
  K-absolute, 92
  pointwise, 40
Comodel, 106
  for a regulus, 118

for a sketch, 119
Complete, 27, 50
finitely, 71
Completion
Cauchy, 95
idempotent-splitting, 100
under $\mathcal{F}$-colimits, 100
under finite colimits, 101
Component
connected, 44
Components
of a $\mathcal{V}$-natural transformation, 9
Composition law, 8
Comprehensive factorization system, 73
Cone, 42
canonical inductive, 43
projective, 42
Conical limit or colimit
classical, 42
in a $\mathcal{V}$-category, 49, 50, 52
Connected component, 44
Conservative, 8, 46
Continuity of a limit in its index, 42
Continuous, 39
Contravariant functor, 12
representable, 15
Copower, 48
Cotensor product, 48
Cotensored, 48
Counit
of a general end, 52
of a limit, 37
of a representation, 22
of a right Kan extension, 60
of an adjunction, 23
of an end in $\mathcal{V}$, 27
Cowellpoweredness, weak, 47
Cylinder, 38, 118–120, 122, 124
K-cylinder, 81

Dense, 86
Dense functors, 119–122
composition of, 87
Density presentation, 121
Density, formulations for, 85
Diagonal fill-in, 113
Diagram, 37
Directed, 72
Discrete Grothendieck construction, 74
Discrete op-fibration, 73
Distributor, 95
Dual $\mathcal{V}$-category, 12

$(\mathcal{E},\mathcal{M})$-generator, 113
Element
of $X \in \mathcal{V}$, 8
of a set-valued functor, 29
End
in $\mathcal{V}$, 27
in any $\mathcal{V}$-category, 52
Epimorphism, extremal, 44
Equivalence, 24
Essentially surjective, 24
Essentially-algebraic
structure, 124
theory, 107, 109, 124
finitary, 120
Extraordinary $\mathcal{V}$-naturality, 17
Extremal epimorphism, 44

$\mathcal{F}$-cocomplete, 99
$\mathcal{F}$-Cocontinuous, 99
$\mathcal{F}$-colimits, 99
$\mathcal{F}$-theory, 120
generated by a sketch, 123, 125
Factorization system, 74
comprehensive, 73
proper, 113
Filtered category, 71
α-filtered, 73
Filtered colimit, 72
Filtered-colimit completion, 102
Final diagram, 70
Final functor, 70
Finitary essentially-algebraic theory, 120
Finite-colimit completion, 101
Finitely cocomplete, 71
Finitely complete, 71
Finitely presentable, 83

Flat, 102
Free cocompletion, 67
Free $\mathcal{V}$-category, 35
Fubini theorem
  for ends in $\mathcal{V}$, 29
  general, 41
Full image, 25
Full subcategory, 9
Fully faithful
  locally, 121
Fully faithful functor, 9
Functor
  2-functor, 10
  algebraic, 120
  conservative, 8
  contravariant, 12
  contravariant representable, 15
  enriched over $\mathcal{V}$, 9
  flat, 102
  fully faithful, 9
  generating, 46
  of two variables, 12
  partial, 12
  representable, 15, 22
  underlying, 10
Functor category, 30

General adjoint functor theorem, 79
Generating functor, 46
Generator, 46
  $(\mathcal{E},\mathcal{M})$-generator, 113
  regular, 81
  strong, 46
Generic model, 109
Grothendieck construction
  discrete, 74

Heyting algebra, 15
Hom, internal, 14

Idempotent monad, 25
Idempotent, split, 100
Idempotent-splitting completion, 100
Identity element, 8
Image
  full, 25
  replete, 25
Indexed colimit, 38
Indexed limit, 37
Indexing type, 37
Inductive cone, canonical, 43
Initial diagram, 69
Initial functor, 70
Interchange
  of ends in $\mathcal{V}$, 29
  of limits in general, 41
Internal hom, 14
Intersection of subobjects, 80
  of $\mathcal{M}$-subobjects, 113
Iterated colimit, 51

K-absolute colimit, 92
K-comodel, 106
K-cylinder, 81
K-model, 107
Kan adjoint, 67
Kan extension, 60, 61
  pointwise, 65
  weak, 65

Left adequate, 86
Left adjoint, 23
Left exact, 71
Left Kan extension, 61
  weak, 65
Limit
  classical, 42
  conical, 42, 49, 52
  indexed, 37
  pointwise, 40
Limit cone, 42, 49
Locally bounded, 114, 118
Locally finitely presentable, 115, 124
Locally fully faithful, 121
Locally presentable, 115
Locally small, 8, 10

$\mathcal{M}$-subobject, 113
$\mathcal{M}$-wellpowered, 114
Mates, 25

Model, 107
  for a regulus, 118
  for a sketch, 119
  generic, 109
Monad, idempotent, 25
Monoidal category, 7
  biclosed, 14
  cartesian, 7
  closed, 13
  strict, 8
  symmetric, 12
Morita equivalence, 95

Opposite $\mathcal{V}$-category, 12
Orthogonal, 117, 119

Partial adjoint, 23
Partial functor, 12
Pointwise Kan extension, 65
Pointwise limit or colimit, 40
Pointwise limits, preservation of, 42
Power, 48
Presentation of density, 93
Preservation
  of a limit, 39
  of a right Kan extension, 60
Profunctors, 95
Projective cone, 42
Proper factorization system, 113

Realization, 120
Reflect limits, 44
Reflective full subcategory, 25
Regular generator, 81
Regulus, 118, 119
Replete full subcategory, 25
Replete image, 25
Repletion, 25
Representable $\mathcal{V}$-functor, 15, 22
Representation, 22
Right adequate, 86
Right adjoint, 23
Right exact, 71
Right Kan extension, 60

Sketch, 119, 123
  of an $\mathcal{F}$-theory, 123
  small, 122
Small, 10, 33
Small limit
  α-small, 73
Small-projective, 94
Small sketch, 122
Small $\mathcal{V}$-category, 9, 27
Solution-set condition, 79
Special Adjoint Functor Theorem, 80
Split coequalizer diagram, 81
Split idempotent, 100
Strict monoidal category, 8
Strong cogenerator, strongly cogenerating, 46
Strong generator, strongly generating, 46, 121
Subobject, 80
  $\mathcal{M}$-subobject, 113
Surjective, essentially, (on objects), 24
Symmetric, 7
Symmetric monoidal category, 12
Symmetry, 11

Tensor product, 8, 48
  of $\mathcal{F}$-theories, 124
  of $\mathcal{V}$-theories, 12
Tensored, 48
Theory, cartesian, 120
Theory, essentially-algebraic, 107, 109
  $\mathcal{F}$-theory, 120
  finitary, 120
Transformation, $\mathcal{V}$-natural, 9
Triangular equations, 23

Underlying category, 10
Underlying functor, 10
Union, of $\mathcal{M}$-subobjects, 113
Unit object, 8
Unit of a(n)
  adjunction, 23
  colimit, 38
  left Kan extension, 61
  representation, 22
Unit $\mathcal{V}$-category, 10

Universal $\mathcal{V}$-natural family, 27

$\mathcal{V}$-category, 8
    cocomplete, 39
    complete, 39
    dual, 12
    opposite, 12
    small, 9, 27
    unit, 10
$\mathcal{V}$-category
    free, 35
$\mathcal{V}$-functor, 9
    accessible, 77
    cocontinuous, 39
    conservative, 46
    continuous, 39
    dense, 86
    representable, 15, 22
    strongly cogenerating, 46
    strongly generating, 46
$\mathcal{V}$-natural family
    extraordinary, 17
    universal, 27
$\mathcal{V}$-natural transformation, 9

Weak Kan extension, 65
Weakly accessible, 78
Weakly cowellpowered, 47
Wellpowered: $\mathcal{M}$-wellpowered, 114

Yoneda embedding, 34
Yoneda isomorphism, 33, 39
Yoneda lemma
    strong, 33
    weak, 21

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Inventiones math. 4, 159–164 (1967)

On the Scale of a Uniform Space

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Introduction

Starting with an arbitrary uniform space $(S, \mathscr{U})$, there is an associated complete lattice $(P, \leqq)$ called the scale[1] of $(S, \mathscr{U})$. By means of a very natural construction, a uniformity $\mathscr{V}$ can be defined on the scale so as to make $(P, \mathscr{V}, \leqq)$ a uniform lattice (in the sense that the lattice operations are uniformly continuous). Of even more interest is the retracted scale $(P_R, \leqq)$ with its associated uniformity $\mathscr{V}_R$; the retracted scale uniform space is a Hausdorff quotient uniform space of the scale uniform space.

The scale and retracted scale were introduced by BUSHAW [1] for the purpose of generalizing the Liapuniv direct method (see [2]) for describing stability in dynamical systems. BUSHAW defines stability in subsets of a quasi-ordered uniform space $(S, \mathscr{U})$, and goes on to prove that a subset A of S is stable if and only if there exists a generalized Liapunov function for A into the retracted scale of $(S, \mathscr{U})$.

Although the retracted scale uniform space was invented for a very special application, it turns out to have at least one property of general interest in uniform space theory: it gives a new representation for the completion of a Hausdorff uniform space. The retracted scale uniform space of any uniform space is Hausdorff and complete, and any Hausdorff uniform space can be uniformly embedded in its retracted scale uniform space.

1.

A non-empty collection α of subsets of a set S is called a *prefilter* on S if $A \in \alpha$ and $B \supset A$ implies $B \in \alpha$. A prefilter which is closed under finite intersections is a filter.

Let $(S, \mathscr{U})$ be a uniform space; i.e., $\mathscr{U}$ is a filter on $S \times S$ with the properties: (1) each member of $\mathscr{U}$ contains the diagonal; (2) for each $U \in \mathscr{U}$, there is $V \in \mathscr{U}$ such that $V^2 = VV \subset U$; (3) $U \in \mathscr{U}$ implies $U^{-1} \in \mathscr{U}$. Let P be the set of all prefilters on $S \times S$ which are subsets of $\mathscr{U}$. For any two elements α and β in P, define $\alpha \leqq \beta$ to mean $\alpha \supset \beta$. The order structure $(P, \leqq)$ is called the *scale* of $(S, \mathscr{U})$. The simplicity of the order relation yields some desirable lattice properties.

[1] The origin of this term is found in [1].

Proposition 1. *For any uniform space $(S,\mathscr{U})$, the scale $(P,\leqq)$ is a complete distributive lattice whose suprema and infima are set intersections and set unions, respectively, and with least element $\mathscr{U}$ and greatest element $\{S\times S\}$.*

For each $U\in\mathscr{U}$, let $U'=\{(\alpha,\beta)\in P\times P\colon A\in\alpha$ implies $UA\in\beta$, and $B\in\beta$ implies $UB\in\alpha\}$.

Proposition 2. *For each $U\in\mathscr{U}$, $U'^2=U^{2\prime}$.*

Proof. Let $(\alpha,\beta)\in U'$ and $(\beta,\gamma)\in U'$. Then $T\in\alpha$ implies $UT\in\beta$ implies $U^2T\in\gamma$; $T\in\gamma$ implies $UT\in\beta$ implies $U^2T\in\alpha$. Thus $U'^2\subset U^{2\prime}$. Conversely, let (α,β) be in $U^{2\prime}$, and let γ be the prefilter which contains all oversets of sets of the form UT, for some T in α, or UV, for some V in β. It follows easily that (α,γ) and (γ,β) are both in U'. ‖

It is clear that each set of the form U' is symmetric and contains the diagonal of $P\times P$; thus $\{U'\colon U\in\mathscr{U}\}$ is the base for a uniformity on P, which we call $\mathscr{V}$. $(P,\mathscr{V})$ is the *scale uniform space* associated with $(S,\mathscr{U})$. If $(S,\mathscr{U})$ is a pseudo-metrizable uniform space, then its scale $(P,\mathscr{V})$ is also pseudo-metrizable. The Hausdorff property, however, does not extend from a uniform space to its scale except under pathological conditions. This situation can be remedied by introducing the retracted scale uniform space, which is obtained from the scale by means of a quotient construction and is always T_2.

2.

Starting with the scale uniform space $(P,\mathscr{V})$ of a uniform space $(S,\mathscr{U})$, we introduce an equivalence relation R on P as follows: $(\alpha,\beta)\in R$ if and only if $(\alpha,\beta)\in U'$ for all $U\in\mathscr{U}$. (The notation "$(\alpha,\beta)\in R$" will be shortened to $\alpha R\beta$.) Let α_R denote the R-equivalence class containing a prefilter α in P, and let $P_R=\{\alpha_R\colon\alpha\in P\}$. With each prefilter α in P, we associate prefilters α_0 and α^0 defined by: $\alpha_0=\{U\in\mathscr{U}\colon VU\in\alpha$, all $V\in\mathscr{U}\}$; $\alpha^0=\{U\in\mathscr{U}\colon VA\subset U$, for some $V\in\mathscr{U}$, and some $A\in\alpha\}$.

Proposition 3. (a) *For each $\alpha\in P$, $\alpha_R=\{\beta\in P\colon\alpha_0\leqq\beta\leqq\alpha^0\}$.* (b) *$\alpha R\beta$ implies $\alpha_0=\beta_0$ and $\alpha^0=\beta^0$.* (c) *$\alpha_0\subset\beta_0$ if and only if $\alpha^0\subset\beta^0$.*

Given $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ in P_R, we define $\boldsymbol{\alpha}\leqq\boldsymbol{\beta}$ to mean $\alpha_0\supset\beta_0$. The order structure $(P_R,\leqq)$ is called the *retracted scale* of $(S,\mathscr{U})$. Let φ be the canonical mapping from P to P_R defined by $\varphi(\alpha)=\alpha_R$. It is easy to see that φ is order-preserving.

Proposition 4. *Let $\{\alpha_i\colon i\in I\}$ and $\{\beta_i\colon i\in I\}$ be sets of prefilters with $\alpha_iR\beta_i$, all $i\in I$. Then $(\bigcup\{\alpha_i\colon i\in I\})R(\bigcup\{\beta_i\colon i\in I\})$ and $(\bigcap\{\alpha_i\colon i\in I\})R(\bigcap\{\beta_i\colon i\in I\})$.*

Proposition 5. *For any pair of elements α, β in P, the following equalities hold*: (1) $\alpha_0 \wedge \beta_0 = \inf(\alpha_R \wedge \beta_R)$; (2) $\alpha^0 \wedge \beta^0 = \sup(\alpha_R \wedge \beta_R)$; (3) $\alpha_0 \vee \beta_0 = \inf(\alpha_R \vee \beta_R)$; (4) $\alpha^0 \vee \beta^0 = \sup(\alpha_R \vee \beta_R)$.

Proposition 6. *The retracted scale of any uniform space is a complete distributive lattice. The operations* sup *and* inf *in* $(P, \leqq)$ *are preserved under the canonical mapping* φ.

Proof. To see that the sup operation is invariant, let $A \subset P$, $B = \varphi(A)$, and $\alpha = \sup A$; we shall show that $\varphi(\alpha) = \sup B$. Note that $\varphi(\alpha) \geqq \gamma$, all γ in B, since φ is order preserving. If $\boldsymbol{\delta} \geqq \gamma$ for all $\gamma \in B$, $\delta \in \boldsymbol{\delta}$, and $\gamma \in A$, then $\delta_0 \subset \gamma_0$, and hence $\delta_0 \subset \bigcap \{\gamma_0 : \gamma \in A\}$. By Proposition 4, the latter prefilter is R-equivalent to α, and so $\boldsymbol{\delta} = \varphi(\delta_0) \geqq \varphi(\alpha)$. A similar argument establishes that $\varphi(\inf A) = \inf B$. Since φ is an onto mapping, $(P_R, \leqq)$ is a complete lattice. The fact that the retracted scale is distributive follows from Proposition 5 and the distributive properties of union and intersection. ‖

Our retracted scale differs somewhat from that used by BUSHAW [*1*]; his retracted scale is, in our notation, $\{\alpha_0 : \alpha \in P\}$. In other words, we are using equivalence classes, whereas he used representatives of equivalence classes. The two retracted scales are essentially identical since they are order isomorphic, and, as far as the retracted scale uniformities are concerned, uniformly isomorphic.

We next define the retracted scale uniformity $\mathscr{V}_R$. Recall that with each $U \in \mathscr{U}$ we have an associated entourage U' in the scale uniformity $\mathscr{V}$. Let $U'' = \varphi(U') = \{(\alpha_R, \beta_R) : (\alpha, \beta) \in U'\}$.

Proposition 7. (a) *For each* $U \in \mathscr{U}$, $U^{2\prime\prime} \subset U''^2 \subset U^{3\prime\prime}$. (b) *For each* $U \in \mathscr{U}$, $\varphi^{-1}(U'') \subset U^{3\prime}$.

Proof. (a) Let $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and $(\boldsymbol{\beta}, \boldsymbol{\gamma})$ belong to U''. Then there is $\alpha \in \boldsymbol{\alpha}$, $\beta \in \boldsymbol{\beta}$ such that $(\alpha, \beta) \in U'$, and there is $\delta \in \boldsymbol{\beta}$, $\gamma \in \boldsymbol{\gamma}$ such that $(\delta, \gamma) \in U'$. $V \in \alpha$ implies $UV \in \beta$ implies $U^2 V \in \delta$ implies $U^3 V \in \gamma$; similarly $V \in \gamma$ implies $U^3 V \in \alpha$. Thus $U''^2 \subset U^{3\prime\prime}$.

If $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in U^{2\prime\prime}$, then there is α in $\boldsymbol{\alpha}$, β in $\boldsymbol{\beta}$ such that $(\alpha, \beta) \in U^{2\prime}$. Since $U^{2\prime} = U'^2$ by Proposition 2, there is γ in P such that $(\alpha, \gamma) \in U'$ and $(\gamma, \beta) \in U'$. Thus $(\boldsymbol{\alpha}, \boldsymbol{\gamma}) \in U''$, $(\boldsymbol{\gamma}, \boldsymbol{\beta}) \in U''$, and $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in U''^2$.

(b) Assume that $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in U''$, with $\alpha \in \boldsymbol{\alpha}$, $\beta \in \boldsymbol{\beta}$. By definition of U'', there is a pair $(\gamma, \delta) \in U'$ such that $\varphi(\gamma) = \boldsymbol{\alpha}$ and $\varphi(\delta) = \boldsymbol{\beta}$. $A \in \alpha$ implies $UA \in \gamma$ implies $U^2 A \in \delta$ implies $U^3 A \in \beta$; this, along with the reciprocal argument, gives $(\alpha, \beta) \in U^{3\prime}$. ‖

The filter $\mathscr{V}_R$ generated by $\{U'' : U \in \mathscr{U}\}$ is a T_2 uniformity by Proposition 7; $(P_R, \mathscr{V}_R)$ is called the *retracted scale uniform space* associated with $(S, \mathscr{U})$.

Theorem 1. *The retracted scale uniform space $(P_R, \mathscr{V}_R)$ associated with an arbitrary uniform space $(S, \mathscr{U})$ is Hausdorff and is the quotient uniform structure derived from $(P, \mathscr{V})$ under the mapping φ. The lattice operations for the retracted scale are uniformly continuous with respect to $\mathscr{V}_R$.*

Proof. It remains only to verify the last statement. Given $U \in \mathscr{U}$, let $U'' \vee U'' = \{(\boldsymbol{\alpha} \vee \boldsymbol{\beta}, \boldsymbol{\gamma} \vee \boldsymbol{\delta}) : (\boldsymbol{\alpha}, \boldsymbol{\gamma}) \in U'', (\boldsymbol{\beta}, \boldsymbol{\delta}) \in U''\}$; we shall verify that $U'' \vee U'' = U''$, proving uniform continuity for the "join" operation. Let $(\boldsymbol{\alpha}, \boldsymbol{\gamma})$, $(\boldsymbol{\beta}, \boldsymbol{\delta})$ belong to U'', and choose (α, γ) and (β, δ) in U' such that $\alpha \in \boldsymbol{\alpha}$, $\beta \in \boldsymbol{\beta}$, $\gamma \in \boldsymbol{\gamma}$, and $\delta \in \boldsymbol{\delta}$. If $V \in \alpha \vee \beta$, then V belongs both to α and β, and so UV is in both γ and δ. This, along with the reciprocal argument, establishes that $(\alpha \vee \beta, \gamma \vee \delta)$ belongs to U'. The desired equality holds, since φ is a lattice homomorphism by Proposition 6. A similar argument shows that the "meet" operation is uniformly continuous. ||

Let τ denote the uniform topology on P_R compatible with $\mathscr{V}_R$. A few additional properties of the retracted scale uniform space are listed below without proof.

(a) The graph of the order relation for P_R is closed in the product topology.

(b) For each $\boldsymbol{\alpha}$, the sets $L(\boldsymbol{\alpha})$ of all lower bounds of $\boldsymbol{\alpha}$ and $U(\boldsymbol{\alpha})$ of all upper bounds of $\boldsymbol{\alpha}$ are τ-closed.

(c) The interior of each uniform neighborhood $U''(\boldsymbol{\alpha})$ is a sublattice of P_R.

(d) The retracted scale topology τ has an open subbase consisting of ideals and dual ideals in the lattice $(P_R, \leqq)$.

(e) $(P_R, \mathscr{V}_R)$ is uniformly isomorphic to a subspace of $(P, \mathscr{V})$ (see [1]).

3.

If $(L, \leqq)$ is an arbitrary lattice and $\mathscr{F}$ a filter on L, then let $L(\mathscr{F}) = \{y \in L$: there is $F \in \mathscr{F}$ such that y is a lower bound of $F\}$, and let $U(\mathscr{F}) = \{y \in L$: there is $F \in \mathscr{F}$ such that y is an upper bound of $F\}$. A filter $\mathscr{F}$ *order-converges* to x in L if and only if $x = \sup L(\mathscr{F}) = \inf U(\mathscr{F})$.

We shall now consider order convergence in the scale $(P, \leqq)$ of an arbitrary uniform space $(S, \mathscr{U})$.

Lemma 1. *Each ultrafilter $\mathscr{G}$ on P order-converges to some element of P.*

Proof. Let $\mathscr{G}$ be an ultrafilter on P. For each G in $\mathscr{G}$, let $\rho_G = \bigcup\{\alpha : \alpha \in G\}$ and let $\sigma_G = \bigcap\{\alpha : \alpha \in G\}$. If $\rho = \bigcap\{\rho_G : G \in \mathscr{G}\}$ and $\sigma = \bigcup\{\sigma_G : G \in \mathscr{G}\}$, then $\sigma \subset \rho$ is clear. If $A \in \rho$, then for each $G \in \mathscr{G}$, there is α_G in G such that $A \in \alpha_G$. Let $K_G = \{\alpha_H : H \subset G$ and $H \in \mathscr{G}\}$. Then the sets $\{K_G : G \in \mathscr{G}\}$ generate a filter on P finer than, and hence equal to, $\mathscr{G}$.

But since $A \in \bigcap\{\alpha : \alpha \in K_G\}$ for each $G \in \mathscr{G}$, it follows that $A \in \sigma$. Thus $\rho = \sigma = \sup L(\mathscr{G}) = \inf U(\mathscr{G})$, and $\mathscr{G}$ order-converges to σ. ||

Lemma 2. *Let $U \in \mathscr{U}$, $F \subset P$, and $F \times F \subset U'$. If α is an element of P such that* $\inf F \leqq \alpha \leqq \sup F$, *then $(\alpha, \gamma) \in U'$ for all $\gamma \in F$.*

Proof. Let $\gamma \in F$ and assume $V \in \gamma$. Then $F \times F \subset U'$ implies that $UV \in \beta$, all $\beta \in F$. Thus UV belongs to $\sup F$, and hence to α. If $W \in \alpha$, then $W \in \inf F$, and W belongs to some β in F. But $(\beta, \gamma) \in U'$ implies $UW \in \gamma$. ||

Theorem 2. *For any uniform space $(S, \mathscr{U})$, $(P, \mathscr{V})$ and $(P_R, \mathscr{V}_R)$ are both complete uniform spaces.*

Proof. First we will show that $(P, \mathscr{V})$ is complete. Let $\mathscr{F}$ be any Cauchy filter on P. Given $U \in \mathscr{U}$, choose $F \in \mathscr{F}$ such that $F \times F \subset U'$. If $\mathscr{G}$ is an ultrafilter finer than $\mathscr{F}$, then $\mathscr{G}$ order-converges to some point α in P by Lemma 1. Since $F \in \mathscr{G}$, $\inf F \leqq \alpha \leqq \sup F$, and the desired result now follows from Lemma 2.

Next, let $\mathscr{H}$ be a Cauchy filter on P_R. It follows from Proposition 7b that $\varphi^{-1}(\mathscr{H})$ is a Cauchy filter on P. If $\varphi^{-1}(\mathscr{H})$ converges to α in the topology compatible with $\mathscr{V}$, then $\mathscr{H} = \varphi(\varphi^{-1}(\mathscr{H}))$ τ-converges to α by Theorem 1. ||

Lemma 3. *Let $(S, \mathscr{U})$ be a uniform space, and let a be any fixed element of S. Let $v_a : S \to P$ be defined at each x in S by $v_a(x) = \{V \in \mathscr{U} : (a, x) \in V\}$. Then v_a is a uniformly continuous function with respect to the scale uniformity.*

Proof. Choose U a symmetric entourage in $\mathscr{U}$, and let $(x, y) \in U$. We will show that $(v_a(x), v_a(y)) \in U'$. Indeed, $V \in v_a(x)$ implies $(a, x) \in V$. Since $(x, y) \in U$, we have $(a, y) \in UV$, and hence $UV \in v_a(y)$. The reciprocal argument establishes that $V \in v_a(y)$ implies $UV \in v_a(x)$, and the proof is complete. ||

Lemma 4. *If $(S, \mathscr{U})$ is a Hausdorff uniform space and v_a is defined as in Lemma 3, then the mapping $\varphi\, v_a$ is a uniform embedding of $(S, \mathscr{U})$ onto a subspace of $(P_R, \mathscr{V}_R)$.*

Proof. (1) That $\varphi\, v_a$ is one to one. Given $x \neq y$ in S, choose symmetric entourages W_1, W_2, W_3 in $\mathscr{U}$ such that $(a, x) \notin W_1$, $(a, y) \notin W_2$, $(x, y) \notin W_3$; let $W = W_1 \cap W_2 \cap W_3$, and choose $V \in \mathscr{U}$ such that $V^2 \subset W$. Then clearly $V \cup \{(a, x)\}$ is in $v_a(x)$. Furthermore, if $(a, y) \in V \circ (V \cup \{(a, x)\})$, then there is z in S such that $(a, z) \in V \cup \{(a, x)\}$ and $(z, y) \in V$. If $(a, z) \in V$, then $(a, y) \in V^2 \subset W_2$, a contradiction. If $x = z$, then $(x, y) \in V$, again a contradiction. It follows that $v_a(x)$ and $v_a(y)$ are not members of the same equivalence class. The special case where $y = a$ can be disposed of by a similar argument.

(2) That $(\varphi v_a)^{-1}$ is uniformly continuous. Given $U \in \mathscr{U}$, choose symmetric $W \in \mathscr{U}$ such that $W^4 \subset U$, and assume that $v_a(x) = \alpha$, $v_a(y) = \beta$, and $(\alpha, \beta) \in W'$. Then $(a,y) \in W \circ (W \cup \{(a,x)\})$, and there is z_1 in S such that $(z_1, y) \in W$ and $(a, z_1) \in W \cup \{(a,x)\}$. If $z_1 = x$, then $(x,y) \in U$ is clear; suppose rather that $(z_1, y) \in W$ and $(a, z_1) \in W$. We also have $(a,x) \in W \circ (W \cup \{(a,y)\})$, and thus there is z_2 in S such that $(z_2, x) \in W$ and $(a, z_2) \in W \cup \{(a,y)\}$. Using the symmetry of W, we have $(z_2, z_1) \in W^2$, $(z_1, x) \in W^3$, and $(x,y) \in W^4 \subset U$. ‖

Theorem 3. *The completion of a Hausdorff uniform space is a subspace of its retracted scale uniform space.*

Indeed, the completion of $(S, \mathscr{U})$ is simply the closure in P of the range of φv_a, for any a in S.

Concluding Remarks

It can be shown that the scale (or retracted scale) of a uniform space is compact if and only if the original space is either finite or indiscrete. However there are many unanswered questions which suggest directions for future research; a few such questions are listed below.

(1) Under what conditions on $(S, \mathscr{U})$ are $(P, \mathscr{V})$ and $(P_R, \mathscr{V}_R)$ connected, locally connected, second countable, separable, locally compact, etc.?

(2) Are there simple conditions which characterize those (ordered) uniform spaces which are uniformly (order) isomorphic to the scale or retracted scale uniform space of some uniform space?

(3) Can the retracted scale (or some closely related construction) be used to obtain completions for various generalized uniform spaces, such as quasi-uniform spaces, uniform convergence structures, etc.?

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(Received April 3/July 31, 1967)

An Introduction to the Theory of Quasi-uniform Spaces

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7 May, 2005

1 Introduction

If the symmetry condition in the definition of a pseudometric is deleted, the notion of a quasi-pseudometric is obtained. Asymmetric distance functions already occur in the work of Hausdorff in the beginning of the twentieth century when in his book on set-theory [27] he discusses what is now called the Hausdorff metric of a metric space.

A family of pseudo-metrics on a set generates a uniformity. Similarly, a family of quasi-pseudometrics on a set generates a quasi-uniformity [60].

In 1937 Weil published his booklet on (entourage) uniformities, which is now usually considered as the beginning of the modern theory of uniformities. Three years later Tukey suggested an approach to uniformities via uniform coverings. The study of quasi-uniformities started in 1948 with Nachbin's investigations (recorded in [57]) on uniform preordered spaces, that is, those topological preordered spaces whose preorder is given by the intersection of the entourages of a (filter) quasi-uniformity $\mathcal{U}$ and whose topology is induced by the associated sup-uniformity $\mathcal{U} \vee \mathcal{U}^{-1}$. He proved that the topological ordered spaces of this kind can be characterized by the property that they admit T_2-ordered compactifications.

The filter $\mathcal{U}^{-1}$ of inverse relations of a quasi-uniformity $\mathcal{U}$ is also a quasi-uniformity. Similarly, each quasi-pseudometric has an obvious conjugate by interchanging the order of points. Hence quasi-uniformities and quasi-metrics naturally yield bitopological spaces (in the sense of Kelly [35]), that is, sets endowed with two topologies.

The term quasi-proximity first appeared in the articles of Pervin and Steiner [59, 66]. The connection between totally bounded quasi-uniformities and quasi-proximities generalizes the well-known correspondence between totally bounded uniformities and proximities.

The work of Fox, Junnila and Kofner showed that in the class of T_1-spaces the concepts of a γ-space (= a topological space admitting a local quasi-uniformity with a countable base), a quasi-pseudometrizable space and a non-archimedeanly quasi-pseudometrizable space are all distinct [22, 36] and that the fine quasi-uniformity of metrizable and suborderable (= generalized ordered) spaces has a base consisting of transitive entourages [33, 38].

Császár [4] developed the theory of the bicompletion for quasi-uniform spaces that was later popularized by Fletcher, Lindgren and Salbany [17, 63]. It satisfactorily generalizes the theory of the completion from the metric and uniform setting to the asymmetric context. Since its underlying idea is that of a complete uniformity however, many further attempts have been made to find other theories of asymmetric completions (see for instance the work of Deák and Doitchinov [6, 11]).

Brümmer (see e.g. [3]) was first to consider explicitly the class of all functorial quasi-uniformities on topological spaces, although some basic work on canonical quasi-uniformities was done at about the same time by Fletcher and Lindgren.

In the last decade the interest in the study of quasi-uniform function spaces and hyperspaces increased considerably. Partially this fact is explained by their applications in theoretical computer science (see the work of Smyth and Sünderhauf [65, 67]).

In the spirit of Nachbin several authors tried to develop with the help of quasi-pseudometrics a common generalization of the two well-established theories of metric spaces and partially ordered sets that would unify common classical results like fixed point theorems and completions.

In connection with such investigations also the idea to replace the reals in the definition of a quasi-pseudometric by some more general structure became popular (e.g. in the studies on Kopperman's continuity spaces, e.g. [39]). In these theories a quasi-uniformity is understood as a kind of generalized quasi-pseudometric.

Quasi-uniform structures were also investigated in various kinds of topological algebraic structures (see for instance the recent work of Romaguera and his collaborators [55, 25]). In particular the study of paratopological groups with the help of quasi-uniformities is well known. Furthermore the study of asymmetric norms naturally leads to a theory of asymmetric functional analysis.

Various researchers attempted more or less successfully to extend important classical results about quasi-uniformities to fuzzy mathematics. Based on Lowen's theory of approach spaces [53], the concept of an approach quasi-uniformity was introduced and investigated. In recent years more and more results about quasi-uniformities were also generalized to a pointfree setting (for instance in the work of Picado and his colleagues [15]).

Over the years, numerous tools from various mathematical theories like lattice theory, category theory, nonstandard analysis and descriptive set-theory were employed with great success in the study of quasi-uniform spaces.

In this chapter we wish to introduce the reader to many of the aforementioned basic ideas. Obviously, for the more sophisticated topics and results he/she will often be referred to the literature because of lack of space. Besides two survey articles dealing mainly with historic aspects of our topic [1, 45] we would like to mention the two monographs [56, 19] and the survey articles [3, 5, 6, 37, 41, 42, 43, 46]. Of course, also well-known texts (that contain sections) on uniformities [31, 12, 28, 32] and the short articles on (quasi-)uniformities in the recent Encyclopedia of General Topology [26, 47] should be quite useful to the interested reader.

Throughout we shall make use of the entourage approach to the theory of (quasi-)uniformities. While the theory of covering uniformities [31] is often easier to use than the equivalent theory of entourage uniformities, a theory of quasi-uniformities based on (pair)covers is possible [24], but often rather cumbersome. Nevertheless it should be stressed that the latter approach has its merits in pointfree topology and bitopology.

2 Basic Concepts and Results

2.1 Quasi-uniform and Quasi-pseudometric Spaces

It is well known that for topological spaces there is no obvious way to compare the sizes of neighborhoods of distinct points. Hence important concepts from the theory of metric spaces like total boundedness and uniform continuity cannot be formulated for topological spaces.

Of course, one natural way to overcome this problem is to choose for a given topological space X some sufficiently large index set I and to consider for each point $x \in X$ an indexed neighborhood base $\{N_i(x) : i \in I\}$. However this is not yet what we want. For instance certainly there should still be some order $\leq$ on the index set I so that if $i_1, i_2 \in I$ with $i_1 \leq i_2$ then we have $N_{i_1}(x) \subseteq N_{i_2}(x)$ whenever $x \in X$.

In fact, it has turned out to be useful to put even more structure on the index set I in order to be able to formulate a halving condition for the chosen families of neighborhoods, which is somewhat reminiscent of the triangle inequality. We are then naturally led to the fairly simple, but nevertheless very useful concept of a quasi-uniform space.

Definition 2.1.1 *A* quasi-uniformity $\mathcal{U}$ *on a set X is a filter on $X \times X$ such that*
(1) Each member U of $\mathcal{U}$ contains the diagonal $\Delta = \{(x, x) : x \in X\}$ of X;
(2) For each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V^2 \subseteq U$.
(Here $V^2 = V \circ V = \{(x, z) \in X \times X :$ There is $y \in X$ such that $(x, y) \in V$ and $(y, z) \in V\}$. Hence $\circ$ is the usual composition of binary relations.)

The members $U \in \mathcal{U}$ are called the *entourages* of $\mathcal{U}$. The elements of X are called *points*. The tuple $(X,\mathcal{U})$ is called a *quasi-uniform space*.

Example 2.1.1 *Let T be a reflexive and transitive relation, that is, a* preorder*, on a set X. Then the filter $X \times X$ generated by the base $\{T\}$ is a quasi-uniformity on X. For any quasi-uniformity $\mathcal{U}$ on X, $\bigcap \mathcal{U}$ is a preorder on X.*

Definition 2.1.2 *Let $[0, \infty)$ be the set of nonnegative reals. A quasi-pseudometric d on a set X is a function $d : X \times X \to [0, \infty)$ such that $d(x, x) = 0$ whenever $x \in X$, and $d(x, z) \leq d(x, y) + d(y, z)$ whenever $x, y, z \in X$.*

The quasi-pseudometric $d^{-1} : X \times X \to [0, \infty)$ defined by $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$ is a called the quasi-pseudometric conjugate *to d.*

A quasi-pseudometric d on X is called non-archimedean *if d satisfies a strong form of the triangle inequality, namely $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ whenever $x, y, z \in X$.*

A quasi-pseudometric is called a pseudo-metric *provided that $d(x, y) = d(y, x)$ whenever $x, y \in X$.*

Let d be a quasi-pseudometric on a set X. For each $\epsilon > 0$ set $U_\epsilon = \{(x, y) \in X \times X : d(x, y) < \epsilon\}$. The quasi-uniformity on $X \times X$ generated by the base $\{U_\epsilon : \epsilon > 0\}$ is called the *quasi-pseudometric quasi-uniformity* $\mathcal{U}_d$ induced by d on X.

Definition 2.1.3 *Each quasi-uniformity $\mathcal{U}$ on a set X* induces a topology $\tau(\mathcal{U})$ *as follows: For each $x \in X$ and $U \in \mathcal{U}$ set $U(x) = \{y \in X : (x, y) \in U\}$. A subset G belongs to $\tau(\mathcal{U})$ if and only if it satisfies the following condition: For each $x \in G$ there exists $U \in \mathcal{U}$ such that $U(x) \subseteq G$.*

It turns out that the neighborhood filter at $x \in X$ with respect to the topology $\tau(\mathcal{U})$ is given by $\mathcal{U}(x) = \{U(x) : U \in \mathcal{U}\}$.

Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $\tau(\mathcal{U})$ is a T_0-topology if and only if $\bigcap \mathcal{U}$ is a partial order. Furthermore $\tau(\mathcal{U})$ is a T_1-topology if and only if $\bigcap \mathcal{U}$ is equal to the diagonal of X.

Note also that given a quasi-pseudometric d on a set X, $\tau(\mathcal{U}_d)$ is the usual quasi-pseudometric topology $\tau(d)$ on X having all the open balls $B_d(x,\epsilon) = \{y \in X : d(x,y) < \epsilon\}$ $(x \in X, \epsilon \in (0,\infty))$ as a base.

Observe that $\tau(d)$ is a T_0-topology if and only if for all $x, y \in X$, $d(x,y) = d(y,x) = 0$ implies that $x = y$. Furthermore $\tau(d)$ is a T_1-topology if and only if for all $x, y \in X$, $d(x,y) = 0$ implies that $x = y$.

Exercise 2.1.1 *Let d be a quasi-pseudometric on a set X. Show that the weights of the topologies $\tau(d)$ and $\tau(d^{-1})$ are equal.*

Definition 2.1.4 *A map $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ between two quasi-uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ is called* uniformly continuous *provided that for each $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $(f \times f)(U) \subseteq V$. Here $f \times f$ is the product map from $X \times X$ into $Y \times Y$ defined by $(f \times f)(x_1, x_2) = (f(x_1), f(x_2))$ $(x_1, x_2 \in X)$.*

Of course the identity map on any quasi-uniform space into itself is always uniformly continuous. Furthermore the composition of two uniformly continuous maps between quasi-uniform spaces is readily seen to be uniformly continuous.

Two quasi-uniform spaces are called *isomorphic* provided that there is a uniformly continuous bijection between them whose inverse map is also uniformly continuous.

It is obvious that each uniformly continuous map between quasi-uniform spaces is continuous with respect to the induced topologies.

Somewhat surprisingly, each topological space (X,τ) is *quasi-uniformizable*, that is, there is a quasi-uniformity $\mathcal{U}$ on X such that $\tau(\mathcal{U}) = \tau$. (If the latter condition is satisfied, we shall say that $\mathcal{U}$ is *compatible* with the topology τ or that (X,τ) *admits* $\mathcal{U}$.)

Indeed for each subset A of X set

$$S_A = [(X \setminus A) \times X] \cup [X \times A].$$

Note that S_A is a preorder. We conclude that the collection $\{S_G : G \in \tau\}$ yields a subbase of a quasi-uniformity on X.

This quasi-uniformity is called the *Pervin quasi-uniformity* $\mathcal{P}(\tau)$ of the topological space (X,τ). One checks that indeed it induces the topology τ on X.

Recall that an open collection $\mathcal{C}$ of a topological space X is called *interior-preserving* provided that $\bigcap \mathcal{E}$ is open whenever $\mathcal{E} \subseteq \mathcal{C}$. Observe that if T is a preorder on a topological space X such that $T(x)$ is open whenever $x \in X$, then $\{T(x) : x \in X\}$ is interior preserving.

We note that for the following construction some authors assume that $\mathcal{C}$ is also a cover of X. Obviously this is merely a matter of taste, since it can always be achieved by just adding the set X to $\mathcal{C}$.

(The Fletcher construction) Let $\mathcal{D}$ be a collection of interior-preserving open collections $\mathcal{C}$ of a given topological space X such that $\bigcup_{\mathcal{C}\in\mathcal{D}} \mathcal{C}$ is a subbase of X. For each $\mathcal{C} \in \mathcal{D}$ set $T_{\mathcal{C}} = \bigcap_{C\in\mathcal{C}} S_C$. Note that for each $x \in X$, $T_{\mathcal{C}}(x) = \bigcap\{C : x \in C \in \mathcal{C}\}$. Hence it is readily checked that $\{T_{\mathcal{C}} : \mathcal{C} \in \mathcal{D}\}$ yields a subbase for a compatible quasi-uniformity $\mathcal{U}_{\mathcal{D}}$ of X.

In fact if $\mathcal{D}$ consists of the collection of all finite open collections of X, then $\mathcal{U}_{\mathcal{D}}$ is the Pervin quasi-uniformity of X.

We recall that a family $\mathcal{L}$ of subsets of a topological space (X,τ) is called *well-monotone* provided that the partial order $\subseteq$ of set inclusion is a well-order on $\mathcal{L}$. The quasi-uniformity $\mathcal{U}_{\mathcal{D}}$ where $\mathcal{D}$ is the collection of all well-monotone open collections of X is called the *well-monotone quasi-uniformity* $\mathcal{M}(\tau)$ of X.

If $\mathcal{U}$ is a quasi-uniformity on a set X, then the filter $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$ on $X \times X$ is also a quasi-uniformity on X. (Here and in the following, $R^{-1} = \{(x,y) \in X \times X : (y,x) \in R\}$ is the inverse of the binary relation R on X.)

The quasi-uniformity $\mathcal{U}^{-1}$ is called the *conjugate* of $\mathcal{U}$. A quasi-uniformity that is equal to its conjugate is called a *uniformity*.

If $\mathcal{U}_1$ and $\mathcal{U}_2$ are two quasi-uniformities on a set X, then we say that $\mathcal{U}_1$ is *coarser* than $\mathcal{U}_2$ (or $\mathcal{U}_2$ is *finer* than $\mathcal{U}_1$) provided that $\mathcal{U}_1 \subseteq \mathcal{U}_2$.

The set of all quasi-uniformities on a set X ordered by set-theoretic inclusion turns out to be a complete lattice.

The smallest element of this lattice is the *indiscrete uniformity* $\mathcal{I} = \{X \times X\}$. The largest element is the *discrete uniformity* $\mathcal{D}$ which consists of all reflexive binary relations on X.

Exercise 2.1.2 *[7] Let A be a nonempty subset of X and let $\mathcal{U}$ be the quasi-uniformity generated by the base $\{S_A\}$ on $X \times X$. Show that there is no quasi-uniformity $\mathcal{V}$ on X such that $\mathcal{I} \subset \mathcal{V} \subset \mathcal{U}$, that is, $\mathcal{U}$ is an* atom *in the lattice of quasi-uniformities on X.*

Obviously the supremum of a family $(\mathcal{U}_i)_{i\in I}$ of quasi-uniformities on X is generated by the subbase $\bigcup_{i\in I}\mathcal{U}_i$ on $X \times X$.

It is easy to see that $\tau(\bigvee_{i\in I}\mathcal{U}_i) = \bigvee_{i\in I}\tau(\mathcal{U}_i)$. It follows that each topological space (X,τ) admits a finest compatible quasi-uniformity, which will be called the *fine quasi-uniformity* and is denoted by $\mathcal{F}N(\tau)$ in the following.

In particular the union of a quasi-uniformity $\mathcal{U}$ and its conjugate $\mathcal{U}^{-1}$ yields a subbase of the coarsest uniformity finer than $\mathcal{U}$. It will be denoted by $\mathcal{U}^s$ in the following.

We also note that if a function $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is uniformly continuous, then $f : (X,\mathcal{U}^{-1}) \to (Y,\mathcal{V}^{-1})$ and $f : (X,\mathcal{U}^s) \to (Y,\mathcal{V}^s)$ are also uniformly continuous.

Definition 2.1.5 *A quasi-uniform space $(X,\mathcal{U})$ is said to have the* Lebesgue property *provided that for each open cover $\mathcal{C}$ of $(X,\tau(\mathcal{U}))$ there is $U \in \mathcal{U}$ such that $\{U(x) : x \in X\}$ is a refinement of $\mathcal{C}$.*

Exercise 2.1.3 *[19] Let $(X,\mathcal{U})$ be a quasi-uniform space such that $\tau(\mathcal{U})$ is compact. Prove that $(X,\mathcal{U})$ has the Lebesgue property.*

Definition 2.1.6 *A quasi-uniform space $(X,\mathcal{U})$ is called* small-set symmetric *provided that $\tau(\mathcal{U}^{-1}) \subseteq \tau(\mathcal{U})$. (The conjugate quasi-uniformity $\mathcal{U}^{-1}$ is then called* point-symmetric.*)*

Exercise 2.1.4 *[19] Let $(X,\mathcal{U})$ be a quasi-uniform space such that $\tau(\mathcal{U})$ is a compact T_1-topology. Show that $\mathcal{U}$ is point-symmetric.*

Exercise 2.1.5 *Show that a countably compact quasi-pseudometric T_1-space is compact.*

Exercise 2.1.6 *Equip $X = \omega_1$ with the quasi-pseudometric $d(x,y) = 0$ if $x \geq y$ and $d(x,y) = 1$ otherwise. Note that $\tau(d)$ is countably compact, but not compact.*

Exercise 2.1.7 *A quasi-uniform space $(X,\mathcal{U})$ is called* locally symmetric *provided that for all $U \in \mathcal{U}$ and $x \in X$, there is $V \in \mathcal{U}$ such that $V^{-1}(V(x)) \subseteq U(x)$. Observe that each point-symmetric quasi-uniform space is locally symmetric. Show that a topological space X admits a locally symmetric quasi-uniformity if and only if X is regular.*

Proposition 2.1.1 *[54] Let $(X,\mathcal{U})$ be a quasi-uniform space satisfying the Lebesgue property, let $(Y,\mathcal{V})$ be a small-set symmetric quasi-uniform space and let $f : (X,\tau(\mathcal{U})) \to (Y,\tau(\mathcal{V}))$ be a continuous map. Then $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is uniformly continuous. (Indeed, $f : (X,\mathcal{U}) \to (Y,\mathcal{V}^s)$ is uniformly continuous.)*

Proof. Let $V \in \mathcal{V}$. Choose $W \in \mathcal{V}$ such that $W^2 \subseteq V$. Then for each $x \in X$ there is $W_x \in \mathcal{V}$ such that $W_x \subseteq W$ and $W_x(f(x)) \subseteq W^{-1}(f(x))$ by small-set symmetry of $(Y,\mathcal{V})$. By continuity of f, $\mathcal{D} = \{\mathrm{int} f^{-1}[W_x(f(x))] : x \in X\}$ is an open cover of X. There is $U \in \mathcal{U}$ such that $\{U(x) : x \in X\}$ refines $\mathcal{D}$ by the Lebesgue property of $(X,\mathcal{U})$. Fix $x \in X$. Then there exists $y \in X$ such that $U(x) \subseteq f^{-1}W_y(f(y))$. Thus $f(U(x)) \subseteq W_y(f(y)) \subseteq W^{-1}(f(y))$ and hence $f(y) \in W(f(x))$. Therefore $f(U(x)) \subseteq W_y(f(y)) \subseteq W^2(f(x)) \subseteq V(f(x))$. Hence $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is uniformly continuous. The remark in parenthesis immediately follows from this result, since by small-set symmetry $f : (X,\tau(\mathcal{U})) \to (Y,\tau(\mathcal{V}^s))$ is also continuous and $\mathcal{V}^s$, as any uniformity, is small-set symmetric.

Corollary 2.1.1 *A small-set symmetric quasi-uniformity $\mathcal{U}$ on a set X that induces a compact topology $\tau(\mathcal{U})$ is a uniformity.*

Proof. Consider the map $\mathrm{id}_X : (X,\mathcal{U}) \to (X,\mathcal{U}^s)$. By the preceding result it is uniformly continuous. Therefore $\mathcal{U}^s \subseteq \mathcal{U}$ and thus $\mathcal{U}$ is a uniformity.

There seems to be no way to explicitly describe a subbase of the infimum of a family $(\mathcal{U}_i)_{i\in I}$ of quasi-uniformities in terms of the entourages of the quasi-uniformities $\mathcal{U}_i$. Also in general topological spaces do not admit a coarsest compatible quasi-uniformity.

Nevertheless it is easy to see that conjugation commutes with arbitrary infima and suprema in the lattice of quasi-uniformities. In particular for any quasi-uniformity $\mathcal{U}$, $\mathcal{U} \wedge \mathcal{U}^{-1}$ is a uniformity.

Exercise 2.1.8 *Let R and S be preorders on X. Consider the quasi-uniformities $\mathcal{U}$ resp. $\mathcal{V}$ on X generated by the bases $\{R\}$ resp. $\{S\}$ on $X \times X$. Show that $\mathcal{U} \wedge \mathcal{V}$ is generated by the base $\{C\}$ where C is the transitive closure of $R \cup S$. Observe that $\{R \cap S\}$ is a base of the quasi-uniformity $\mathcal{U} \vee \mathcal{V}$.*

A map $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ between quasi-uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ is called *uniformly open* if for each $U \in \mathcal{U}$ there is $V \in \mathcal{V}$ such that $V(f(x)) \subseteq f(U(x))$ whenever $x \in X$.

Proposition 2.1.2 *Let $(X,\mathcal{U})$ be a compact* uniform *space and $(Y,\mathcal{V})$ a quasi-uniform space. Then any map $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ that is open and continuous is uniformly open.*

Proof. Let $U \in \mathcal{U}$. There is $P \in \mathcal{U}$ such that $P^2 \subseteq U$. Since f is open, for each $a \in X$ we find $W_a \in \mathcal{V}$ such that $W_a^2(f(a)) \subseteq f(P(a))$. By continuity of f and since $\mathcal{U}$ is a uniformity, we can consider the open cover $\{\mathrm{int}(P^{-1}(a) \cap f^{-1}W_a(f(a))) : a \in X\}$ of X. Since X is compact, there is a finite subset F of X such that $\bigcup_{a\in F} \mathrm{int}(P^{-1}(a) \cap f^{-1}W_a(f(a))) = X$. Set $W = \bigcap_{a\in F} W_a$ and note that $W \in \mathcal{V}$. Consider $x \in X$. There is $b \in F$ such that $x \in P^{-1}(b) \cap f^{-1}W_b(f(b))$. Therefore $f(x) \in W_b(f(b))$ and $W(f(x)) \subseteq W_b^2(f(b)) \subseteq f(P(b)) \subseteq f(P^2(x)) \subseteq f(U(x))$. We have shown that f is uniformly open.

Definition 2.1.7 *Let X be a set and suppose that for each $i \in I$ there is given a map $f_i : X \to Y_i$ into a quasi-uniform space $(Y_i, \mathcal{V}_i)$. Then there is a coarsest quasi-uniformity $\mathcal{V}$ on X such that all maps f_i are uniformly continuous, the so-called* initial quasi-uniformity*. It is generated by the subbase $\{(f_i \times f_i)^{-1}(V) : i \in I, V \in \mathcal{V}_i\}$ on $X \times X$.*

One checks that the initial quasi-uniformity $\mathcal{V}$ induces the coarsest topology on X such that all maps $f_i : X \to (Y_i, \tau(\mathcal{V}_i))$ $(i \in I)$ are continuous, that is, the corresponding initial topology. Note also that the initial quasi-uniformity $\mathcal{V}$ is a uniformity if all quasi-uniformities $\mathcal{V}_i$ are indeed uniformities. We give two applications for the concept of the initial quasi-uniformity.

Let $((X_i,\mathcal{U}_i))_{i\in I}$ be a family of quasi-uniform spaces. Let $\Pi_{i\in I}X_i$ be the set-theoretic product of the family $(X_i)_{i\in I}$ and let $\pi_j : \Pi_{i\in I}X_i \to X_j$ $(j \in I)$ be the projection onto $(X_j,\mathcal{U}_j)$. Then the coarsest quasi-uniformity on $\Pi_{i\in I}X_i$ that makes all projections uniformly continuous is called the *product quasi-uniformity.* It induces the product topology of the topological spaces $((X_i, \tau(\mathcal{U}_i))_{i\in I}$.

Let $(X,\mathcal{U})$ be a quasi-uniform space and let A be a subset of X. Then the coarsest quasi-uniformity on X such that the set-theoretic embedding $e : A \to X$ is uniformly continuous is called the *subspace quasi-uniformity* $\mathcal{U}|A$: It consists of the entourages $U|A := U \cap (A \times A)$ with $U \in \mathcal{U}$ and induces the subspace topology $(\tau(\mathcal{U}))|A$ of $(X, \tau(\mathcal{U}))$ on A.

Exercise 2.1.9 *Let (X, τ) be a topological space and let A be a subset of X. Show that $(\mathcal{P}(\tau))|A = \mathcal{P}(\tau|A)$ and $(\mathcal{M}(\tau))|A = \mathcal{M}(\tau|A)$.*

Let Y be a set and suppose that for each $i \in I$ there is given a map $f_i : X_i \to Y$ from a quasi-uniform space $(X_i,\mathcal{U}_i)$ to Y. Then there is a finest quasi-uniformity $\mathcal{V}$ on Y such that all maps f_i are uniformly continuous, the so-called *final quasi-uniformity.* Observe that $\mathcal{V}$ is a uniformity if all quasi-uniformities $\mathcal{U}_i$ are uniformities.

The final quasi-uniformity $\mathcal{V}$ need not be compatible with the final topology on Y determined by the family of maps $f_i : (X_i, \tau(\mathcal{U}_i)) \to Y$ $(i \in I)$.

Exercise 2.1.10 *Let $\mathbb{R}$ be the set of the reals equipped with its usual metric uniformity. Consider the partition $\mathbb{R}/\sim = \{(-\infty, 0], (0, \infty)\}$ of $\mathbb{R}$ and the corresponding quotient map $q : \mathbb{R} \to \mathbb{R}/\sim$. Determine the quotient (that is, final) quasi-uniformity on $\mathbb{R}/\sim$ and compare its induced topology with the quotient topology.*

2.2 Transitive Quasi-uniform and Topological Spaces

A quasi-uniform space $(X,\mathcal{U})$ is called *transitive* provided that it has a base consisting of transitive entourages (that is, preorders). Such a base is called a *transitive* base. Note that all quasi-uniformities obtained by the Fletcher construction are transitive. Furthermore observe that if d is a non-archimedean quasi-pseudometric on a set X, then the quasi-pseudometric quasi-uniformity $\mathcal{U}_d$ has a countable transitive base.

Exercise 2.2.1 *Let $\mathcal{U}$ be a quasi-uniformity having a countable base of transitive entourages. Then $\mathcal{U}$ is* non-archimedeanly quasi-pseudometrizable*, that is, there is a non-archimedean quasi-pseudometric d on X such that $\mathcal{U} = \mathcal{U}_d$.*

Corollary 2.2.1 *The topology of a topological space can be induced by a non-archimedean quasi-pseudometric if and only if it has a σ-interior preserving base (that is, a base of open sets that can be written as the union of countably many interior-preserving collections).*

Proof. The statement is a consequence of the preceding exercise and a straightforward application of the Fletcher construction.

It is easy to see that the supremum of any family of transitive quasi-uniformities is transitive. On the other hand the infimum of a transitive quasi-uniformity with its conjugate need not be transitive, as we shall see later (see **??**).

The proof of the following result is crucial, but well known and will not be included here.

Lemma 2.2.1 *Let $(U_n)_{n\in\omega}$ be a sequence of binary reflexive relations on a set X such that for each $n \in \omega$, $U_{n+1} \circ U_{n+1} \circ U_{n+1} \subseteq U_n$. Then there is a quasi-pseudometric d on X such that $U_{n+1} \subseteq \{(x,y) : d(x,y) < 2^{-n}\} \subseteq U_n$ whenever $n \in \omega$. If each U_n is a symmetric relation, then d can be constructed to be a pseudo-metric.*

Corollary 2.2.2 *Each (quasi-)uniformity $\mathcal{U}$ on a set X with a countable base is (quasi-)pseudometrizable (that is, there is a (quasi-)pseudometric d on X such that $\mathcal{U}_d = \mathcal{U}$).*

Corollary 2.2.3 *Each (quasi-)uniformity $\mathcal{U}$ can be written as the supremum of a family of (quasi-)pseudometric quasi-uniformities.*

Corollary 2.2.4 *A topological space is completely regular if and only if it is* uniformizable*, that is, can be induced by a uniformity.*

Proof. Let (X,τ) be a topological space that is uniformizable by some uniformity $\mathcal{U}$. Then by the preceding corollary $\mathcal{U}$ can be written as the supremum of pseudometric uniformities. Hence τ is the supremum of pseudometric topologies and thus is completely regular.

In order to prove the converse, assume that X is completely regular. Then X carries the initial topology with respect to the family of all continuous maps from X into the real unit interval $([0,1],\tau(m))$ where m denotes the usual metric on $[0,1]$. Consider the initial (quasi-)uniformity $\mathcal{C}^*(\tau)$ on X determined by the family of these maps into the uniform space $([0,1],\mathcal{U}_m)$. Since $\mathcal{C}^*(\tau)$ induces τ, we have shown that (X,τ) is uniformizable.

Corollary 2.2.5 *Let $\mathcal{V}$ be a compatible quasi-uniformity on a completely regular space X. If $\mathcal{V}$ has the Lebesgue property, then it is finer than the finest compatible uniformity $\mathcal{U}$ on X.*

Proof. The continuous map $\mathrm{id}_X : (X,\mathcal{V}) \to (X,\mathcal{U})$ is uniformly continuous by Proposition 2.1.1.

Corollary 2.2.6 *Let (X,τ) be a compact regular space and let $\mathcal{U}$ be a compatible uniformity and $\mathcal{V}$ a compatible quasi-uniformity on X. Then $\mathcal{U} \subseteq \mathcal{V}$.*

In particular $\mathcal{C}^*(\tau)$ is the unique compatible uniformity on X and the coarsest compatible quasi-uniformity on X.

Proof. Consider the identity map $\mathrm{id}_X : (X,\tau(\mathcal{U})) \to (X,\tau(\mathcal{V}))$. It is continuous and open, so the map $\mathrm{id}_X : (X,\mathcal{U}) \to (X,\mathcal{V})$ is uniformly open. Hence $\mathcal{U} \subseteq \mathcal{V}$.

Definition 2.2.1 *Let X be a set. A filter $\mathcal{U}$ on $X \times X$ consisting of reflexive relations is called a* local quasi-uniformity *provided that for each $U \in \mathcal{U}$ and each $x \in X$ there is $V \in \mathcal{U}$ such that $V^2(x) \subseteq U(x)$. A local quasi-uniformity $\mathcal{U}$ on a set X such that $\mathcal{U} = \mathcal{U}^{-1}$ is called a* local uniformity*. A local quasi-uniformity such that $\mathcal{U}^{-1}$ is a local quasi-uniformity will be called a* pairwise-local quasi-uniformity*.*

*In the same way as for a quasi-uniformity, a local quasi-uniformity $\mathcal{U}$ induces a topology $\tau(\mathcal{U})$. A topological space is called a γ-*space *provided that its topology can be induced by a local quasi-uniformity having a countable base.*

Exercise 2.2.2 *Let $(X,\mathcal{U})$ be a local quasi-uniform space and $x \in X$. Show that $\mathcal{U}(x) = \{U(x) : U \in \mathcal{U}\}$ is the neighborhood filter at x with respect to the topology $\tau(\mathcal{U})$.*

Exercise 2.2.3 *[70] Prove that a topological space (X,τ) admits a local uniformity if and only if its topology τ is regular.*

Example 2.2.1 *A method to build γ-spaces that are not quasi-pseudometrizable was developed by Fox [22], answering a question that originated in work of Ribeiro [62]. Fox and Kofner [23] constructed a γ-space with a Tychonoff topology that is not quasi-pseudometrizable. The problem whether a γ-space satisfying some additional property is quasi-pseudometrizable often turns out to be quite difficult.*

The following positive result is not deep, but the method is often useful.

Proposition 2.2.1 *A topological space X is quasi-pseudometrizable if and only if its topology can be induced by a pairwise-local quasi-uniformity $\mathcal{U}$ with a countable base.*

Proof. Necessity is obvious. In order to prove sufficiency let $\mathcal{U}$ be a local quasi-uniformity with a countable base $\{U_n : n \in \omega\}$ for X such that $\mathcal{U}^{-1}$ is a local quasi-uniformity. Without loss of generality we may assume that $U_{n+1} \subseteq U_n$ whenever $n \in \omega$. Inductively we define a sequence $(V_n)_{n\in\omega}$ of $\tau(\mathcal{U}^{-1}) \times \tau(\mathcal{U})$-neighborhoods of Δ as follows. If $n = 0$, choose for each $x \in X$ an $n(x) \in \omega$ such that $U^3_{n(x)}(x) \subseteq U_0(x)$ and $(U^{-1}_{n(x)})^3(x) \subseteq U_0^{-1}(x)$ and set

$$V_0 = \bigcup\{U^{-1}_{n(x)}(x) \times U_{n(x)}(x) : x \in X\}.$$

If $n > 0$ and V_{n-1} is already defined, choose for each $x \in X$ an $n(x) \in \omega$ such that $U^7_{n(x)}(x) \subseteq (V_{n-1} \cap U_n)(x)$ and $(U^{-1}_{n(x)})^7(x) \subseteq (V_{n-1} \cap U_n)^{-1}(x)$ and set

$$V_n = \bigcup\{U^{-1}_{n(x)}(x) \times U_{n(x)}(x) : x \in X\}.$$

An easy computation shows that $V_0^2 \subseteq U_0^2$ and $V_n^4 \subseteq V_{n-1}^2 \cap U_n^2$ for each $n > 0$. Hence the quasi-uniformity $\mathcal{V}$ on X generated by $\{V_n^2 : n \in \omega\}$ satisfies $\tau(\mathcal{V}) = \tau(\mathcal{U})$ and $\tau(\mathcal{V}^{-1}) = \tau(\mathcal{U}^{-1})$.

Corollary 2.2.7 *[70] A topological space is pseudometrizable if and only its topology can be induced by a local uniformity possessing a countable base.*

Problem 2.2.1 *(for research) A lot of work has gone into the problem of characterizing topologically those topological spaces that admit a compatible quasi-pseudometric. Numerous solutions to this problem have been suggested in the literature (see e.g. [40, 29]). While in special cases the obtained criteria turn out to be quite useful, it seems fair to say that no known general characterization of quasi-pseudometrizability is as useful as for instance the Nagata-Smirnov Theorem in the theory of metrizable topological spaces.*

Each quasi-uniformity contains a finest transitive quasi-uniformity coarser than $\mathcal{U}$, often denoted by $\mathcal{U}_t$. Of course, $(\mathcal{U}_t)^{-1} = (\mathcal{U}^{-1})_t$. Note that $\mathcal{U}_t$ and $\mathcal{U}$ may induce distinct topologies, for instance on any uniform space $(X, \mathcal{U})$ with a connected topology $\tau(\mathcal{U})$, the indiscrete uniformity on X is the only transitive quasi-uniformity coarser than $\mathcal{U}$.

For any topological space (X, τ) we shall call $(\mathcal{F}N(\tau))_t$ the *fine transitive quasi-uniformity* and denote it by $\mathcal{F}T(\tau)$. Since each topological space admits a transitive quasi-uniformity, namely its Pervin quasi-uniformity, $\mathcal{F}T(\tau)$ is compatible with τ.

Exercise 2.2.4 *Observe that a topological space is zero-dimensional if and only if it admits a transitive uniformity.*

A topological space is called *transitive* if its fine quasi-uniformity is transitive.

In the following example each $n \in \omega$ is identified with its set of predecessors, that is $n = \{0, \dots, n-1\}$.

Example 2.2.2 *[51] Let X be the set $\mathbf{Q}^\omega$ (of sequences from ω to the rationals $\mathbf{Q}$) equipped with the topology induced by the quasi-uniformity $\mathcal{U}$ having the following countable base $\{U_n : n \in \omega\}$.*

For each $n \in \omega$ and $x \in X$ the set $U_n(x)$ is the set of all $x' \in X$ which (1) agree with x on the initial segment n, and (2) for which the first coordinate $\Delta := \Delta(x, x')$ in which x and x' differ satisfies $x(\Delta) \le x'(\Delta) \le x(\Delta) + 2^{-n}$.

In order to see that $\mathcal{U}$ is a quasi-uniformity, it suffices to show that for any $n \in \omega$ and any $x, x', x'' \in X$, we have that $x' \in U_{n+1}(x)$ and $x'' \in U_{n+1}(x')$ imply $x'' \in U_n(x)$.

Indeed, note first that under our hypothesis x, x', x'' agree on $n+1$. Set $\Delta_0 := \Delta(x', x)$ and $\Delta_1 := \Delta(x'', x')$. We now distinguish three cases.

Case 1: If $\Delta_1 < \Delta_0$, then $\Delta_1 = \Delta(x'', x)$, $x'(\Delta_1) = x(\Delta_1)$, $x'(\Delta_1) \le x''(\Delta_1) \le x'(\Delta_1) + 2^{-(n+1)}$, and thus $x'' \in U_{n+1}(x)$.

Case 2: If $\Delta_1 > \Delta_0$, then $\Delta_0 = \Delta(x'', x)$, $x''(\Delta_0) = x'(\Delta_0)$, $x(\Delta_0) \le x'(\Delta_0) \le x(\Delta_0) + 2^{-(n+1)}$, and thus $x'' \in U_{n+1}(x)$.

Case 3: If $\Delta_1 = \Delta_0$, then $\Delta_0 = \Delta(x'', x)$, $x(\Delta_0) \le x'(\Delta_0) \le x(\Delta_0) + 2^{-(n+1)}$ and $x'(\Delta_0) \le x''(\Delta_0) \le x'(\Delta_0) + 2^{-(n+1)}$, and thus $x'' \in U_n(x)$.

Let us note that $\tau(\mathcal{U})$ is a T_1-topology, since $\cap_{n\in\omega} U_n(x) = \{x\}$ whenever $x \in X$.

We want to show that $(X, \tau(\mathcal{U}))$ is not transitive. Indeed we shall show that $\tau(\mathcal{U})$ is not non-archimedeanly quasi-pseudometrizable. In order to reach a contradiction, we assume that it admits a quasi-uniformity with a countable (decreasing) base $\{T_n : n \in \omega\}$ consisting of transitive entourages.

Then for each $x \in X$, choose $k(x) \in \omega$ such that $T_{k(x)}(x) \subseteq U_1(x)$; furthermore fix $n_x \in \omega$ such that $U_{n_x}(x) \subseteq T_{k(x)}(x)$.

Viewing the rationals as having the discrete topology and working in the resulting space ω^ω, for each $m, k \in \omega$ we set $A_{m,k} = \{x \in \omega^\omega : n_x = m \text{ and } k(x) = k\}$. With the help of the Baire Category Theorem we find $m, k \in \omega$ such that $\text{int}\overline{A_{m,k}} \neq \emptyset$ in the space ω^ω. Hence there are $n \in \omega$, $\sigma \in \mathbf{Q}^n$ and a dense $D \subseteq [\sigma]$ such that, for all $x \in D$, $k(x) = k$ and $n_x \leq n$. (Here, $[\sigma] = \{x' \in \mathbf{Q}^\omega : x'|n = \sigma\}$.)

Construct $x_1, \ldots, x_{2^n+1} \in D$ such that for each $i = 1, \ldots, 2^{n+1}$, x_i restricted to n equals σ and such that for each $i = 1, \ldots, 2^n$, $x_i(n) + 2^{-n} = x_{i+1}(n)$. We see that $x_{i+1} \in T_k(x_i)$, because $x_{i+1} \in U_n(x_i) \subseteq U_{n_{x_i}}(x_i)$. By transitivity we conclude that $x_{2^n+1} \in T_k(x_1) \subseteq U_1(x_1)$. But now $x_{2^n+1} \in U_1(x_1)$ requires $x_{2^n+1}(\Delta) \leq x_1(\Delta) + 2^{-1}$ where $\Delta := \Delta(x_1, x_{2^n+1}) = n$. However $x_1(\Delta) + 1 = x_1(\Delta) + 2^{-n}(2^n) = x_{2^n+1}(\Delta)$. This contradiction completes the proof.

Remark 2.2.1 *The first example of a topological space that is quasi-pseudometrizable, but not non-archimedeanly quasi-pseudometrizable was constructed by to Kofner and is now called the* Kofner plane *[36]:*

(The Kofner plane) Let $X = \mathbb{R}^2$. For each $x \in X$ and $\epsilon > 0$ let $C(x, \epsilon)$ be the closed disk of radius ϵ lying above the horizontal line through x and tangent to this line at x. For $x, y \in X$ define $d(x, y)$ as follows: Set $d(x, y) = 1$ if $y \notin C(x, 1)$, set $d(x, y) = r$ if $r \leq 1$, $y \in C(x, r)$ and $y \notin C(x, s)$ for all $s < r$, and set $d(x, y) = 0$ if $x = y$. An application of the Baire Category Theorem similarly to the one given above shows that the quasi-pseudometric space (X, d) is not non-archimedeanly quasi-pseudometrizable. Hence it cannot be transitive, since this would imply that it admits a quasi-uniformity with a countable base of transitive relations. We omit the proof that can be found in [19].

Problem 2.2.2 *(for research) It is not known whether each (regular) quasi-pseudometric space (X, d) such that $\tau(d) \subseteq \tau(d^{-1})$ is non-archimedeanly quasi-pseudometrizable.*

Problem 2.2.3 *(for research) No non-archimedeanly quasi-pseudometrizable (Tychonoff) space seems to be known that is not transitive.*

2.3 Bitopological Spaces

Definition 2.3.1 *A bitopological space (X, τ_1, τ_2) consists of a set X equipped with two topologies τ_1 and τ_2. A map $f : X \to Y$ between two bitopological spaces $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called* bicontinuous *provided that $f : (X, \tau_1) \to (Y, \sigma_1)$ and $f : (X, \tau_2) \to (Y, \sigma_2)$ are both continuous.*

Of course, it is natural to consider a topological space (X, τ) as a special case of a bitopological space, namely the bitopological space (X, τ, τ). A lot of work has been done in extending concepts and results about topological spaces to bitopological spaces. As expected, while in some cases such generalizations are more or less straightforward, in other cases they are quite demanding and often highly controversial. Sometimes they also lead to completely unsatisfactory results, since concepts that in the case of topological spaces are equivalent may extend naturally to concepts that are no longer equivalent in the bitopological setting.

Given a quasi-pseudometric d on a set X, it is natural to study the bitopological space $(X, \tau(d), \tau(d^{-1}))$.

Let X be a set and let A, B subsets of X. We let $T_{A,B}$ denote the binary relation

$$(X \times X) \setminus (A \times B) = [(X \setminus A) \times X] \cup [X \times (X \setminus B)].$$

In particular note that $S_A = T_{A, X \setminus A}$.

In the following suppose that ξ is a collection of pairs $< A, B >$ of subsets of X such that $A \subseteq B$.

If for a topology τ on X, for each $\zeta \subseteq \xi$, we have $\text{cl}_\tau \bigcup\{A :< A, B > \in \zeta\} \subseteq \bigcup\{B :< A, B > \in \zeta\}$, then ξ is called *(τ-)cushioned.*

If for a topology τ on X, for each $\zeta \subseteq \xi$, we have $\bigcap\{A :< A, B >\in \zeta\} \subseteq \text{int}_\tau \bigcap\{B :< A, B >\in \zeta\}$ then ξ is called *(τ-)cocushioned.*

If $\xi = \bigcup_{n \in \omega} \xi_n$ where each ξ_n is τ-cushioned (resp. τ-cocushioned), then ξ is called *σ-τ-cushioned* (resp. *σ-τ-cocushioned*).

Exercise 2.3.1 *Let X be a topological space. Prove that a collection of pairs $< A, B >$ of subsets of X is cushioned if and only if the collection of the corresponding pairs $< X \setminus B, X \setminus A >$ is cocushioned.*

As above, let ξ be a collection of pairs $< A, B >$ of subsets of X such that if $< A, B >\in \xi$ then $A \subseteq B$. A *pairbase* for a topological space X is such a collection ξ satisfying

(i) if $< A, B >\in \xi$, then A is open, (ii) if C is a neighborhood of $x \in X$, then there exists $< A, B > \in \xi$ with $x \in A \subseteq B \subseteq C$.

Proposition 2.3.1 *(Fox [21]) A bitopological space (X, τ_1, τ_2) possesses a quasi-pseudometric inducing its bitopology (that is, (X, τ_1, τ_2) is* quasi-pseudometrizable*) if and only if τ_1 has both a σ-τ_2-cushioned pairbase and a σ-τ_1-cocushioned pairbase, and τ_2 has both a σ-τ_1-cushioned pairbase and a σ-τ_2-cocushioned pairbase.*

Proof. Let (X, d) be a quasi-pseudometric space. Set $B_d(x, 2^{-n}) = \{y \in X : d(x, y) < 2^{-n}\}$ whenever $x \in X$ and $n \in \omega$. Let $\beta_n = \{< B_d(x, 2^{-(n+1)}), B_d(x, 2^{-n}) >: x \in X\}$ and $\delta_n = \{< B_{d^{-1}}(x, 2^{-(n+1)}), B_{d^{-1}}(x, 2^{-n}) >: x \in X\}$. Then β_n is both τ_2-cushioned and τ_1-cocushioned, and $\beta = \bigcup_{n\in\omega} \beta_n$ is a pairbase for τ_1. Also δ_n is both τ_1-cushioned and τ_2-cocushioned, and $\delta = \bigcup_{n\in\omega} \delta_n$ is a pairbase for τ_2.

In order to prove sufficiency we shall make use of the following lemmas due to Fox:

Lemma 2.3.1 *If (X, τ_1, τ_2) is a bitopological space where τ_1 has a σ-τ_2-cushioned pairbase μ and a σ-τ_1-cocushioned pairbase ξ, and where τ_2 has a σ-τ_1-cushioned pairbase λ and a σ-τ_2-cocushioned pairbase ζ, then the hypothesis of the next lemma is satisfied.*

Proof. If $\beta = \{< A \cap C, B \cup D >:< A, B >\in \xi$ and $< C, D >\in \mu\}$ and $\delta = \{< A \cap C, B \cup D >:< A, B >\in \zeta$ and $< C, D >\in \lambda\}$, then β is a σ-τ_2-cushioned, σ-τ_1-cocushioned pairbase for τ_1, and δ is a σ-τ_1-cushioned, σ-τ_2-cocushioned pairbase for τ_2.

Lemma 2.3.2 *Let (X, τ_1, τ_2) be a bitopological space, where τ_1 has a σ-τ_2-cushioned, σ-τ_1-cocushioned pairbase β and where τ_2 has a σ-τ_1-cushioned and σ-τ_2-cocushioned pairbase δ. Then (X, τ_1, τ_2) has a compatible pairwise-local quasi-uniformity with a countable base.*

Proof. Without loss of generality we may assume $\beta = \bigcup_{n\in\omega} \beta_n$ and $\delta = \bigcup_{n\in\omega} \delta_n$, where $\beta_n \subseteq \beta_{n+1}$ and $\delta_n \subseteq \delta_{n+1}$, and where each β_n is both τ_2-cushioned and τ_1-cocushioned and each δ_n is both τ_1-cushioned and τ_2-cocushioned. We define for any $n \in \omega$,

$$V_n = \bigcap\{T_{A,X\setminus B} :< A, B >\in \beta_n\} \cap \bigcap\{(T_{A,X\setminus B})^{-1} :< A, B >\in \delta_n\}.$$

Then $V_n(x)$ is a τ_1-neighborhood of x as β_n is τ_1-cocushioned and δ_n is τ_1-cushioned. Also $V^{-1}(x)$ is a σ_2-neighborhood of x as β_n is τ_2-cushioned and δ_n is τ_2-cocushioned. Given any τ_1-neighborhood S of a point $x \in X$, as β is a pairbase for τ_1 we may find $n \in \omega$ and $< A, B >, < C, D >\in \beta_n$ such that $x \in A \subseteq B \subseteq C \subseteq D \subseteq S$. Then $x \in A$ implies $V_n(x) \subseteq B \subseteq C$, which implies $V_n^2(x) \subseteq D \subseteq S$. Similarly, given any τ_2-neighborhood R of x, we may find $n \in \omega$ such that $V_n^{-2}(x) \subseteq R$. It follows that $\{V_n : n \in \omega\}$ is a countable base for a compatible pairwise-local quasi-uniformity on (X, τ_1, τ_2).

The assertion now follows by the proof of Proposition 2.2.1.

Corollary 2.3.1 *[35] A bitopological space (X, τ_1, τ_2) is quasi-pseudometrizable if X is pairwise regular (that is, each point has a τ_i neighborhood base consisting of τ_j-closed sets $(i, j \in \{1, 2\}, i \neq j)$) and both topologies have a countable base.*

Exercise 2.3.2 *Show that a topological space (X, τ) admits a local quasi-uniformity with a countable base if and only if it possesses a σ-τ-cocushioned pairbase.*

Corollary 2.3.2 *A topological space is pseudometrizable if and only if it a γ-space that has a σ-cushioned pairbase. (Note that a topological T_1-space that has a σ-cushioned pairbase is called* stratifiable.*)*

Exercise 2.3.3 *(compare [19, Theorem 2.32]) Let $(X, \mathcal{U})$ be a locally symmetric quasi-uniform space with a countable base. Show that $\tau(\mathcal{U})$ is pseudometrizable. (Hint: Observe that $\tau(\mathcal{U})$ has a σ-cushioned pairbase.)*

Exercise 2.3.4 *Let d be a quasi-pseudometric on a set X. For fixed $x \in X$ consider the function $f_x : X \to \mathbb{R}$ defined by $f_x(y) = d(x, y)$ whenever $y \in X$. Then f_x is (in the usual sense, see e.g. [12, p. 61]) $\tau(d)$-upper semi-continuous and $\tau(d^{-1})$-lower semi-continuous.*

Evidently, each quasi-uniform space $(X, \mathcal{U})$ induces the bitopological space $(X, \tau(\mathcal{U}), \tau(\mathcal{U}^{-1}))$. The obvious question which bitopological spaces can be induced by a quasi-uniformity has the expected answer.

Example 2.3.1 *Let $[0, 1]$ be the real unit interval. For all $x, y \in [0, 1]$ set $r(x, y) = \max\{0, x - y\}$. Then r is a quasi-pseudometric on $[0, 1]$ and $\tau(r)$ is the so-called* upper topology *$\{(a, 1] : a \in [0, 1]\} \cup \{\emptyset, [0, 1]\}$ on $[0, 1]$. Similarly $\tau(r^{-1})$ is the* lower topology *$\{[0, a) : a \in [0, 1]\} \cup \{\emptyset, [0, 1]\}$ on $[0, 1]$.*

A bitopological space (X, τ_1, τ_2) is called *pairwise completely regular* if it carries the coarsest bitopology that makes all the bicontinuous maps $f : (X, \tau_1, \tau_2) \to ([0, 1], \tau(r), \tau(r^{-1}))$ bicontinuous (that is, τ_1 and τ_2 are both initial with respect to this collection of maps and the topologies $\tau(r)$ and $\tau(r^{-1})$, respectively).

Proposition 2.3.2 *A bitopological space (X, τ_1, τ_2) is quasi-uniformizable if and only if it is pairwise completely regular.*

Proof. The proof is completely analogous to the proof of Corollary 2.2.4. Of course, we have to equip $[0, 1]$ with the quasi-uniformity $\mathcal{U}_r$ and its induced bitopology, and consider bicontinuous maps instead of continuous maps.

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The scale of a quasi-uniform space

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Thesis prepared under the supervision of
Prof. H.-P. A. Künzi
in fulfilment of the requirements for the degree of
Masters of Sciences in Mathematics

Cape Town, March 2009

University of Cape Town

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Acknowledgements

I would like to thank my supervisor, Professor H.-P. Künzi for his infinite support, patience, proper guidance, stimulating discussions on the subject and encouragement he has given me throughout my studies.

I would like to thank the Topology and Category Theory Research Group for all kinds of support and a marvellous research environment.

I express my gratitude to the Department of Mathematics and Applied Mathematics for opportunities offered during my studies.

I acknowledge financial support from the National Research Foundation (NRF) during 2008 and 2009.

To my family and my friends thank you for support and encouragement.

Contents

Acknowledgements i
Abstract iii

1 Introduction **1**

2 Basic definitions **5**
2.1 Definitions 5
2.2 The Hausdorff hyperspace quasi-uniformity 10
2.3 The quasi-uniformity of quasi-uniform convergence on a multifunction space 11

3 Summary of some results about the scale of a uniform space **12**
3.1 The scale of a filter 12
3.2 Definition of the scale of a uniform space 13
3.3 Construction of the retracted scale of a uniform space 16
3.4 Stability in quasi-ordered uniform spaces 21
3.5 Pseudometrizability, separability, the Lindelöf condition, and the first and second axioms of countability of the uniform scale 23
3.6 The order scale 27

4 Topological properties of the scale of a uniform space **30**
4.1 Connectedness in the scale of a uniform space 30
4.2 Connectedness conditions in the scale of a uniform space . . . 34
4.3 Other topological properties and the cardinality of the scale of a uniform space 35

5 The scale of a quasi-uniform space **40**
5.1 Definition of the scale of a quasi-uniform space 41

5.2 The retracted scale of a quasi-uniform space 46
5.3 Properties of the scale of a quasi-uniform space 50
5.4 Quasi-pseudometrizability of the scale 57
5.5 The prefilter space of a quasi-uniform space 62
5.6 The left-sided scale of a quasi-uniform space 67
5.7 The two-sided scale of a quasi-uniform space 70

6 Conclusion **74**

Bibliography **75**

Abstract

Over the last forty years much progress has been made in the investigation of the scale of a uniform space. In particular, Bushaw, Kent, Ramsey and Richardson published several articles concerning the scale of a uniform space. The aim of this dissertation is to begin a similar investigation into the scale of a quasi-uniform space. It starts off with a summary of results obtained for the scale of a uniform space, which, has been investigated in the past. We conclude by commencing an investigation into the scale of a quasi-uniform space. Here several results obtained for the scale of a uniform space are generalized, and some original results for the scale of a quasi-uniform space are presented.

Chapter 1

Introduction

The concept of a "scale of a uniform space" was first introduced by D. Bushaw in 1967 [6] to support the generalization of Liapunov's "direct method" in stability theory to abstract systems. He investigated a stability concept analogous to the classical uniform stability as a relationship between a quasi-order and a uniformity on the same set. He showed that stability in this sense occurs if and only if there exists a Liapunov function taking values in a certain partially ordered uniform space associated with the given uniformity and called its retracted scale.

In 1967 D. C. Kent published a paper [16] defining an partial order in the retracted scale which makes it a complete distributive lattice and the canonical mapping from the scale to the retracted scale is order preserving. The lattice operations in the retracted scale are uniformly continuous. His main result is that the completion of a Hausdorff uniform space is a subspace of its retracted scale uniform space. Both the scale and retracted scale are complete.

In 1967 O. C. Ramsey [29] studied some properties of the scale of a uniform space. He defined a function called "écart" ρ on $X \times X$ into the scale P where P is the collection of all nonempty subsets α of $\mathcal{U}$ satisfying $U \in \alpha$ and $V \supseteq U$ implying $V \in \alpha$ and $(X,\mathcal{U})$ is a uniform space, and he showed that the uniform properties of $(X,\mathcal{U})$ can be written in terms of ρ and the neighborhoods of 0 in P without explicitly mentioning $\mathcal{U}$. These properties then resemble metric properties. Then ρ is used to prove that the following are equivalent: $(X,\mathcal{U})$ is pseudometrizable, $(P,\mathcal{U}')$ is pseudometrizable, and

$(P,\mathcal{U}')$ satisfies the first axiom of countability, where $\mathcal{U}'$ is a uniformity defined on the scale P. The écart ρ is also used to investigate the scales of two spaces which have the same topology. Necessary conditions are obtained for a scale to be separable or Lindelöf or to satisfy the second axiom of countability.

In 1972 G. C. Leslie and D. G. Kent [19] show that the scale of a uniform space is uniformly connected if and only if the original uniform space is bounded. D. C. Richardson [33] characterizes uniform connectedness for uniform subspaces of the scale of the real line and he also shows that the scale of any uniform space is locally compact if and only if the original uniformity $\mathcal{U}$ has a last element, see [32]. The scale is compact if and only if the original space X is finite or $\mathcal{U} = \{X \times X\}$. This statement remains true if compact is replaced by countably compact, totally bounded, Lindelöf, second countable, or separable.

In 1983 D. C. Kent introduced the notion of an order scale, see [19], and showed that the order topology is compact and T_2 in both scale and retracted scale of any uniform space $(X,\mathcal{U})$. If $(X,\mathcal{U})$ is T_2 and totally bounded, the Samuel compactification associated with $(X,\mathcal{U})$ can be obtained by uniformly embedding $(X.\mathcal{U})$ in its order retracted scale. This implies that every compact T_2 space is a closed subspace of a complete, infinitely distributive lattice in its order topology, and also a continuous, closed image of a closed subspace of a complete atomic Boolean algebra in its order topology [19].

In the light of the above, it is natural to start an investigation into the scale of a quasi-uniform space. This is our focus in this dissertation.
We define our scale of a quasi-uniform space motivated by the definition of the Hausdorff hyperspace quasi-uniformity of a quasi-uniform space as introduced in [3]. In Proposition 5.1.3 we show that if the original space is a uniform space then our scale is the same as the scale investigated by Bushaw and Kent, see [6] and [16]. In Corollary 5.6.1 we also show that the quasi-uniform space given by a Hausdorff hyperspace quasi-uniformity is quasi-uniformly embedded into the left-sided scale of a quasi-uniform space.

It is interesting to note the various resemblances between the scale of a uniform space and the scale of a quasi-uniform space such as that the scale and retracted scale of a uniform space are both complete and the scale and retracted scale of a quasi-uniform space are both bicomplete.

The scale of a uniform space and its retracted scale are pseudometrizable if and only if the original uniform space is pseudometrizable. Nearly the same applies to the quasi-uniform space (Theorem 3.5.2 and Theorem 5.4.2), where "pseudometrizable" must be replaced in the statement by "quasi-pseudometrizable". In addition to these coincidences, it is interesting to note that the same condition that is both necessary and sufficient for the scale of a uniform space to be totally bounded is also necessary and sufficient for the scale of a quasi-uniform space to be totally bounded (Proposition 4.3.3 and Theorem 5.3.4).

We then introduce two slight generalizations of the scale of a quasi-uniform space, which we call the prefilter space and the left-sided scale of a quasi-uniform space. We show that the prefilter space on $X \times X$, where $(X,\mathcal{U})$ is a quasi-uniform space and the left-sided scale of the quasi-uniform space $(X,\mathcal{U})$ are bicomplete (Proposition 5.6.2).

We also show that total boundedness is preserved by another modified scale called the two-sided scale of a quasi-uniform space (Proposition 5.7.3).

We bring it to the reader's attention that some of the most interesting new results on the scale of a quasi-uniform space obtained during this investigation are collected in [28] for possible publication. The proofs given in this dissertation and in [28] respectively may sometimes differ.

This dissertation starts with some preliminary definitions which are given in the next chapter. That chapter contains two separate sections on the Hausdorff hyperspace quasi-uniformity and the quasi-uniformity of quasi-uniform convergence on a multifunction space respectively, each of which lists some often used basic definitions and results needed to understand this dissertation.

Chapter 3 consists of a short summary of some uniform results about the scale of a uniform space.

Chapter 4 deals with the topological properties of the scale of a uniform space. It mentions some known results about connectedness in the scale of a uniform space and about the cardinality of the scale of a uniform space.

Chapter 5 starts our investigations into the scale of a quasi-uniform space. We show that total boundedness is preserved by the two-sided scale of a quasi-uniform space. The prefilter space of a quasi-uniform space is quasi-uniformly embedded into the left-sided scale of a quasi-uniform space. We also show that the scale of a quasi-uniform space, the prefilter space of a quasi-uniform space and the left-sided scale of a quasi-uniform space are bi-complete.

We conclude this dissertation by listing a few unsolved problems in Chapter 6.

Chapter 2

Basic definitions

In this chapter. firstly we recall the definition and some properties of quasi-uniform spaces and we summarize some results about order convergence. Secondly we also summarize facts about the Hausdorff hyperspace quasi-uniformity discussed by many authors (see e.g. [3], [23]) and thirdly we recall the notion of the quasi-uniform multifunction space discussed in [8]. We also introduce some notations used throughout this dissertation.

2.1 Definitions

In this section we recall the concept of a quasi-uniform space.

The following definitions can be found in the book [12].

Definition 2.1.1 *([12]) A* quasi-uniformity $\mathcal{U}$ *on the set* X *is a filter on* $X \times X$ *such that*
(1) *Each member* U *of* $\mathcal{U}$ *contains the diagonal* $\triangle_X = \{(x,x) : x \in X\}$ *of* X*;*
(2) *For each* $U \in \mathcal{U}$ *there is* $V \in \mathcal{U}$ *such that* $V^2 \subseteq U$.
(Here $V^2 = V \circ V = \{(x,z) \in X \times X :$ *there is* $y \in X$ *such that* $(x,y) \in V$ *and* $(y,z) \in V\}$*. Hence* $\circ$ *is the usual composition of binary relations.)*
The pair $(X,\mathcal{U})$ *is called a quasi-uniform space.*

The members of $U \in \mathcal{U}$ are called the *entourages* of $\mathcal{U}$. The elements of X are called *points*.

If $\mathcal{U}$ is a quasi-uniformity on a set X, then $\mathcal{U}^{-1}$ is also a quasi-uniformity on X called the *conjugate* of $\mathcal{U}$.

Definition 2.1.2 *([12]) Given a quasi-uniform space $(X,\mathcal{U})$ we shall denote by $\mathcal{U}^s$ the coarsest uniformity finer than $\mathcal{U}$ and its conjugate $\mathcal{U}^{-1}$ (i.e. $\mathcal{U}^s = \mathcal{U} \vee \mathcal{U}^{-1}$). If $U \in \mathcal{U}$ we denote by U^s the entourage $U \cap U^{-1}$ in $\mathcal{U}^s$.*

It is easily seen that a quasi-uniformity $\mathcal{U}$ is a uniformity if and only if $\mathcal{U} = \mathcal{U}^{-1}$.

Definition 2.1.3 *If $\mathcal{U}$ is a quasi-uniformity on X, $\mathcal{B}$ is called a* base *for $\mathcal{U}$ if $\mathcal{B} \subseteq \mathcal{U}$ and for every $U \in \mathcal{U}$ there is a $B \in \mathcal{B}$ such that $B \subseteq U$. Furthermore $\mathcal{B}$ is called a subbase for $\mathcal{U}$ if $\{\bigcap \mathcal{H} \mid \mathcal{H} \subseteq \mathcal{B}, \mathcal{H}$ is non-empty and finite$\}$ is a base for $\mathcal{U}$.*

Topologies induced by quasi-uniformities

Definition 2.1.4 *([12]) Each quasi-uniformity $\mathcal{U}$ on a set X induces a topology $\tau(\mathcal{U})$ as follows: For each $x \in X$ and $U \in \mathcal{U}$ set $U(x) = \{y : (x,y) \in U\}$. A subset G of X belongs to $\tau(\mathcal{U})$ if and only if for each $x \in G$ there exists $U \in \mathcal{U}$ such that $U(x) \subseteq G$.*

If τ is a topology on X, then $\mathcal{U}$ is said to be compatible with τ if $\tau(\mathcal{U}) = \tau$, and (X,τ) is said to admit $\mathcal{U}$.

Proposition 2.1.1 *([12]) If $\mathcal{U}$ and $\mathcal{V}$ are quasi-uniformities on a set X such that $\mathcal{U} \subseteq \mathcal{V}$, then $\tau(\mathcal{U}) \subseteq \tau(\mathcal{V})$.*

Proof. See [12, Proposition 1.29]. □

Quasi-uniform continuity

In the theory of quasi-uniform spaces the structure-preserving maps are the quasi-uniformly continuous maps.

Definition 2.1.5 *([12]) A map $f : (X,\mathcal{U}) \longrightarrow (Y,\mathcal{V})$ between two quasi-uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ is called* quasi-uniformly continuous *provided that for each $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $(f \times f)(U) \subseteq V$. Here $f \times f$ is the product map from $X \times X$ to $Y \times Y$ defined by $(f \times f)(x_1, x_2) = (f(x_1), f(x_2))$ $(x_1, x_2 \in X)$. Moreover f is called a* quasi-uniform isomorphism *if it is one-to-one, onto, and f as well as its inverse are quasi-uniformly continuous.*

Proposition 2.1.2 *([12]) Suppose $f : (X,\mathcal{U}) \longrightarrow (Y,\mathcal{V})$ is quasi-uniformly continuous. Then $f : (X,\tau(\mathcal{U})) \longrightarrow (Y,\tau(\mathcal{V}))$ is continuous.*

Proof. See [12, Proposition 1.14]. □

Cauchy filter

Definition 2.1.6 *([23]) A filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is called a* $\mathcal{U}^s$-Cauchy filter *if for each $U \in \mathcal{U}$ there is an $F \in \mathcal{F}$ such that $F \times F \subseteq U$.*

Definition 2.1.7 *([34]) A quasi-uniform space $(X,\mathcal{U})$ is said to be* bicomplete *if each $\mathcal{U}^s$-Cauchy filter on X converges with respect to the topology $\tau(\mathcal{U}^s)$, i.e., if the uniform space $(X,\mathcal{U}^s)$ is complete.*

A bicompletion of a quasi-uniform space $(X,\mathcal{U})$ is a bicomplete quasi-uniform space $(Y,\mathcal{V})$ that has a $T(\mathcal{V}^s)$-dense subspace quasi-uniformly isomorphic to $(X,\mathcal{U})$.

We have the following uniqueness property for T_0 bicompletions.

Proposition 2.1.3 *([21]) Let $(X,\mathcal{U})$ be a quasi-uniform space, let $(Y,\mathcal{V})$ be a bicomplete quasi-uniform space T_0-space, let D be a dense subset of $(X,\tau(\mathcal{U}^s))$, and let $f : (D,\mathcal{U}\,|D) \to (Y,\mathcal{V})$ be a quasi-uniformly continuous map. Then there is a unique continuous extension $g : (X,\tau(\mathcal{U}^s)) \to (Y,\tau(\mathcal{V}^s))$ of f, and $g : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is quasi-uniformly continuous.*

Proof. See [21, Theorem 2.7.3]. □

Remark 2.1.1 *([21]) Let $(X,\mathcal{U})$ be a quasi-uniform space that is a $\tau(\mathcal{V}^s)$-dense subspace of a bicomplete quasi-uniform T_0-space $(Y,\mathcal{V})$. Then $(Y,\mathcal{V})$ is isomorphic to the bicompletion of $(X,\mathcal{U})$ under an isomorphism that keeps the points of X pointwise fixed. When this identification is made, the minimal $\mathcal{U}^s$-Cauchy filters on X are the traces of the $\tau(\mathcal{V}^s)$-neighborhood filters of the points of Y.*

Totally bounded quasi-uniformity

Definition 2.1.8 *Let $\mathcal{U}$ be a quasi-uniformity on a set X. Then $\mathcal{U}$ is said to be* totally bounded *if for every $U \in \mathcal{U}$ there is a finite cover $\{A_i \mid 1 \leq i \leq n\}$ (for some integer n) of X such that for each $1 \leq i \leq n$, $A_i \times A_i \subseteq U$.*

Definition 2.1.9 *([15]) A quasi-uniform space $(X,\mathcal{U})$ is called* precompact *provided that for each $U \in \mathcal{U}$ there is a finite subset A of X such that $U(A) = X$.*

Definition 2.1.10 *([31]) Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $(X,\mathcal{U})$ is* totally bounded *if and only if $(X,\mathcal{U}^s)$ is totally bounded.*

Remark 2.1.2 *It is not too difficult to see that every totally bounded quasi-uniform space is precompact, and that a uniformity is totally bounded if and only if it is precompact.*

Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $\bigcap_{U\in\mathcal{U}}(U \cap U^{-1}) = \triangle$ if and only if $\tau(\mathcal{U}^s)$ is Hausdorff if and only if $(\bigcap_{U\in\mathcal{U}}U)\cap (\bigcap_{U\in\mathcal{U}^{-1}}U) = \triangle$ if and only if $\bigcap_{U\in\mathcal{U}}U(x)\cap \bigcap_{U\in\mathcal{U}}U^{-1}(x) = \{x\}$ whenever $x \in X$ if and only if $cl_{\tau(\mathcal{U}^{-1})}\{x\}\cap cl_{\tau(\mathcal{U})}\{x\} = \{x\}$ whenever $x \in X$ (see [21]).

Order convergence

We define the relevant lattice concepts, and summarize some known results about order convergence.

Definition 2.1.11 *([17]) Let L be a complete lattice. If $x \in L$, $A \subseteq L$, and $\mathcal{F}$ is a filter on L, then let:*

$$\uparrow x = \{y \in L : y \geq x\},\ \uparrow A = \bigcap\{\uparrow x : x \in A\},\ \uparrow \mathcal{F} = \bigcup\{\uparrow F : F \in \mathcal{F}\}.$$

The symbols $\downarrow x$, $\downarrow A$, $\downarrow \mathcal{F}$ designate the corresponding sets of lower bounds.

Definition 2.1.12 *([17]) Let L be a complete lattice. Let x, y be two elements on L. If $x \leq y$, then $[x, y] = \uparrow x \cap \downarrow y$ denotes the closed interval spanned by x and y.*

Definition 2.1.13 *([17]) A filter $\mathcal{F}$ on L order-converges to x if*

$$x = sup \downarrow \mathcal{F} = \inf \uparrow \mathcal{F}.$$

The order topology on L has for its closed sets those sets A which contain all of their order-convergence limit points. Order convergence does not always coincide with convergence in the order topology; when they do coincide, order convergence is said to be *topological*, and the resulting order topology is regular and T_1 (see [17]).

Definition 2.1.14 *([17]) An element x of L is* compact *if $A \subseteq L$ and $x \leq supA$ implies that there is a finite subset B of A such that $x \leq supB$.*

An element with the dual property is said to be *cocompact.*

Definition 2.1.15 *([17]) The complete lattice L is* compactly (cocompactly) *generated if each element of L is the supremum (infimum) of a set of compact (cocompact) elements; if L is both compactly and cocompactly generated, then L is said to be* bicompactly *generated.*

Definition 2.1.16 *([17]) A subset A of L with the inherited order is called a* subcomplete lattice *of L if, for every nonempty subset B of A,* $\sup_A B = \sup_L B$ *and* $\inf_A B = \inf_L B$.

Proposition 2.1.4 *([17]) (a) If L is bicompactly generated, then order convergence in L is topological, and if $x \in L$, then the neighborhood filter with respect to the order topology at x has an open base consisting of sets of the form $[a, b]$, where a is a compact lower bound of x and b is a cocompact upper bound of x. The order topology is totally disconnected.*
(b) If L is a subcomplete lattice of an atomic Boolean algebra, then L is bicompactly generated and the order topology is, in addition, compact (see [17]).

Proof. See [17, Proposition 1.3]. □

2.2 The Hausdorff hyperspace quasi-uniformity

The Hausdorff hyperspace quasi-uniformity of a quasi-uniform space was investigated by many authors (see e.g. [3], [22]). This section contains the basic definitions.

Definition 2.2.1 *([23]) Given a quasi-uniform space $(X,\mathcal{U})$ we shall consider the $\mathcal{U}$-equivalence relation $\sim$ on X that underlies its T_0-reflection: For $x, y \in X$ we have $x \sim y$ if and only if $(x, y) \in \bigcap \mathcal{U} \cap (\bigcap \mathcal{U}^{-1})$.*

In [22] Künzi and Ryser have used the following definition.

Definition 2.2.2 *([22]) Let $(X,\mathcal{U})$ be a quasi-uniform space and let $\mathcal{P}_0(X)$ be the set of nonempty subsets of X. For any $U \in \mathcal{U}$ let*

$$U_+ = \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : B \subseteq U(A)\}$$

and

$$U_- = \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : A \subseteq U^{-1}(B)\}.$$

Furthermore set $U_H = (U_-) \cap (U_+)$ whenever $U \in \mathcal{U}$. Then $\{U_- : U \in \mathcal{U}\}$ is a base for the *lower quasi-uniformity* $\mathcal{U}_-$ on $\mathcal{P}_0(X)$ and $\{U_+ : U \in \mathcal{U}\}$ is a base for the *upper quasi-uniformity* $\mathcal{U}_+$ on $\mathcal{P}_0(X)$. Moreover $\mathcal{U}_H = \mathcal{U}_+ \vee \mathcal{U}_-$ is the so-called *Hausdorff hyperspace quasi-uniformity or Bourbaki quasi-uniformity* of $(X,\mathcal{U})$ (see [21]).

Lemma 2.2.1 *([22, Lemma 1]) (a) Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $x \mapsto \{x\}$ is a quasi-uniform embedding of $(X,\mathcal{U})$ into $(\mathcal{P}_0(X),\mathcal{U}_H)$.*
(b) Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be quasi-uniform spaces and let $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ be a quasi-uniformly continuous map. Then the map $f : (\mathcal{P}_0(X),\mathcal{U}_H) \to (\mathcal{P}_0(Y), \mathcal{V}_H)$ defined by $f(A) := \{f(a) : a \in A\}$ is quasi-uniformly continuous, too.

Proof. See [22, Lemma 1]. □

2.3 The quasi-uniformity of quasi-uniform convergence on a multifunction space

In this section we mention some basic definitions on the quasi-uniformity of quasi-uniform convergence on a function space used in this dissertation. The notation and conventions employed mostly correspond to those used in [8].

Definition 2.3.1 *([8]) A* multifunction (or multi-valued function) $F : X \to Y$ *is a point-to-set correspondence from* X *to* Y *such that* $F(x) \neq \emptyset$ *for each* $x \in X$.

Definition 2.3.2 *([8]) Let* Y^{mX} *be the set of all multifunctions from a nonempty set* X *to a quasi-uniform space* $(Y, \mathcal{U})$. *For each* $U \in \mathcal{U}$, *define*

$W(U) = \{(F, H) : H(x) \subseteq U(F(x))$ *and* $F(x) \subseteq U^{-1}(H(x))$ *for all* $x \in X\}$.

Moreover $\{W(U) : U \in \mathcal{U}\}$ *is a base for a quasi-uniformity* $\mathcal{U}_{mX}$ *on* Y^{mX} *called the* quasi-uniformity of quasi-uniform convergence *on* Y^{mX}. *The topology induced by* $\mathcal{U}_{mX}$ *on* Y^{mX} *is called the* topology of quasi-uniform convergence.

We shall use the multifunction space in Section 5.7.

Chapter 3

Summary of some results about the scale of a uniform space

In this chapter we summarize the most often used fundamental results pertaining to the scale of a uniform space. We will first recall the concept of the scale of a filter. Secondly, we will recall the scale of a uniform space. In the third section we shall recall the construction of the retracted scale of a uniform space. The fourth section will recall the notion of stability in a quasi-ordered uniform space. In the last section, we shall summarize pseudometrizabilty, separability, the Lindelöf condition, and the first and second axioms of countability of the scale of a uniform space. A large portion of this chapter was obtained from [6] and [16]. In order to avoid repetitions it will sometimes happen that we state a result in this chapter but delay its proof until Chapter 5. There it is either proved for the more general case of quasi-uniformities, or derived from other results obtained in that chapter.

3.1 The scale of a filter

In this section, we recall the notions of a prefilter and the scale of a filter introduced by D. Bushaw [6].

Our prefilters are the "nonempty stacks" of some other authors (see e.g. Száz in [38]).

Definition 3.1.1 *([6]) By a* prefilter *on a nonempty set X we shall*

mean a nonempty collection α of nonempty subsets of X such that $A \in \alpha$ and $A \subseteq B \subseteq X$ imply $B \in \alpha$.

For any nonempty subset A_0 of X, the collection $\{A \subseteq X : A \supseteq A_0\}$ is a prefilter; it will be denoted by $\langle A_0 \rangle$ and called the *principal* prefilter generated by A_0. We shall say that a prefilter α is proper if $\alpha \neq \{X\}$ or equivalently (since $X \in \alpha$ in any case) if α has at last two elements. A prefilter α is a filter if it is multiplicative: $\{A, B\} \subseteq \alpha$ implies $A \cap B \in \alpha$.

Definition 3.1.2 *([6]) Let $\mathcal{F}$ be a filter on a set X. We shall denote by $P(\mathcal{F})$, or simply by P where there is no risk of ambiguity, the set of all prefilters on the set X which are contained in $\mathcal{F}$.*

The set P may be partially ordered by the inverse of inclusion.

Definition 3.1.3 *([6]) For any $\{\alpha, \beta\} \subseteq P$ we shall say that $\alpha \leq \beta$ if and only if $\beta \subseteq \alpha$. The partially ordered set $(P, \leq)$ will be called the* scale *of a filter $\mathcal{F}$.*

Some significant information about the order-theoretic structure of the scale of a filter is provided by the following proposition.

Proposition 3.1.1 *([6]) The scale $(P, \leq)$ of a filter $\mathcal{F}$ has a last element 0, where $0 = \{X\}$. The subset $P^+ = P - \{0\}$ is directed downward (" filtrant à gauche") and has a last element if and only if $\mathcal{F}$ is a proper principal filter.*

Proof. See [6, Proposition 2.1]. □

3.2 Definition of the scale of a uniform space

We shall recall in this section the definition of the scale of a uniform space and we will introduce an example of the scale of a uniform space.

Definition 3.2.1 *([16]) Let $(X, \mathcal{U})$ be a uniform space, and let P be the set of all prefilters on $X \times X$ which are subsets of $\mathcal{U}$.*

Definition 3.2.2 *For any two elements α and β in P, $\alpha \leq \beta$ means that $\beta \subseteq \alpha$.*

The following definition is according to Kent (see [16]).

Definition 3.2.3 *([16]) The order structure of $(P, \leq)$ is called the* order of the scale *of $(X, \mathcal{U})$.*

Proposition 3.2.1 *For any uniform space $(X, \mathcal{U})$, the order scale $(P, \leq)$ is a complete distributive lattice whose suprema and infima are set intersections and set unions, respectively, and $\mathcal{U}$ is the last element and $\{X \times X\}$ is the greatest element.*

Proof. See [16, Proposition 1]. □

In the following, we are going to define a uniformity on the scale of a uniform space.

Definition 3.2.4 *([16]) Let $(X, \mathcal{U})$ be a uniform space. For each $U \in \mathcal{U}$,*

$U' = \{(\alpha, \beta) \in P \times P :$ *if $A \in \alpha$ then $U \circ A \in \beta$ and if $B \in \beta$ then $U \circ B \in \alpha$*$\}$.

Note that $\{U' : U \in \mathcal{U}\}$ is a base for a uniformity $\mathcal{U}'$ on P. Indeed the following proposition shows that the scale of a uniform space is a uniform space.

Proposition 3.2.2 *Let $(X, \mathcal{U})$ be a uniform space. Then $(P, \mathcal{U}')$ is a uniform space.*

Proof. (1) We show that $\mathcal{U}'$ is a filter on $P \times P$.

For each $U \in \mathcal{U}$, $U' \subseteq P \times P$ by definition of U'.

If U, V belong to $\mathcal{U}$ then it suffices to show that $(U \cap V)' \subseteq U' \cap V'$. Let $(\alpha, \beta) \in (U \cap V)'$, if $A \in \alpha$, $(U \cap V) \circ A \in \beta$ and if $B \in \beta$, $(U \cap V) \circ B \in \alpha$. Then $(U \cap V) \circ A \subseteq U \circ A \in \beta$ and $(U \cap V) \circ B \subseteq U \circ B \in \alpha$, where we use that α, β are prefilters. If $A \in \alpha$ then $U \circ A \in \beta$ and if $B \in \beta$ then $U \circ B \in \alpha$

implies $(\alpha, \beta) \in U'$. Similarly $(\alpha, \beta) \in V'$ so $(\alpha, \beta) \in U' \cap V'$.

(2) Each member of $\mathcal{U}'$ contains the diagonal.

(3) Let $U' \in \mathcal{U}'$, $U'^{-1} \in \mathcal{U}'$ because U' is symmetric for each $U \in \mathcal{U}$.

(4) Let $U \in \mathcal{U}$ then there is $V \in \mathcal{U}$ such that $V^2 \subseteq U$. We show that $V'^2 \subseteq U'$.
Let $(\alpha, \beta) \in V'^2$. This implies there is a $\gamma \in P$ such that $(\alpha, \gamma) \in V'$ and $(\gamma, \beta) \in V'$. If $A \in \alpha$, $V \circ V \circ A = V^2 \circ A \subseteq U \circ A \in \beta$ and if $B \in \beta$, $V \circ V \circ B = V^2 \circ B \subseteq U \circ B \in \alpha$ thus $(\alpha, \beta) \in U'$. So $\{U' : U \in \mathcal{U}\}$ is a filter base generating the uniformity $\mathcal{U}'$. □

Proposition 3.2.3 *([16]) Let $(X, \mathcal{U})$ be a uniform space. For each $U \in \mathcal{U}$,*
$U'^2 = U^{2'}$.

Proof. By the previous argument, $U'^2 \subseteq U^{2'}$.

Conversely, let (α, β) be in $U^{2'}$, and let γ be the prefilter which contains all oversets of sets of the form $U \circ T$, for some T in α, or $U \circ V$, for some V in β. It follows easily that (α, γ) and (γ, β) are both in U'. □

The following definition is according to Kent in [16].

Definition 3.2.5 *([16]) The pair $(P, \mathcal{U}')$ is called the* scale *of the uniform space $(X, \mathcal{U})$.*

The next paragraph discusses the example of the discrete uniform space.

Example 3.2.1 *If $(X, \mathcal{U})$ is a discrete uniform space, then its scale $(P, \mathcal{U}')$ is a discrete uniform space.*

Indeed, $\mathcal{U} = \langle \triangle \rangle$ and $\alpha \in P$ if and only if α is a nonempty family of reflexive relations on X such that if $A \in \alpha$ and $A \subseteq B$ then $B \in \alpha$.

Let $(\alpha, \beta) \in \triangle'$, if $A \in \alpha$ implies $\triangle \circ A = A \in \beta$ and if $B \in \beta$ implies $\triangle \circ B = B \in \alpha$ thus $\alpha = \beta$.

3.3 Construction of the retracted scale of a uniform space

This is the section where we investigate the construction of the retracted scale of a uniform space. It will be constructed with a simple method that has been used by Kent in [16] by introducing an equivalence relation R on P.

Definition 3.3.1 *([16]) Let α, β in P. We define*

$$\alpha \ R \ \beta \ (or \ (\alpha, \beta) \in R) \ if \ (\alpha, \beta) \in U' \ for \ each \ U \in \mathcal{U}.$$

Lemma 3.3.1 *R is an equivalence relation on P.*

Proof. Let $\alpha \in P$, α R α because each U' contains the diagonal of $P \times P$.

If α R β implies for each $U \in \mathcal{U}$, $(\alpha, \beta) \in U'$, and U' is symmetric so $(\beta, \alpha) \in U'$, which implies β R α.

If α R β and β R γ, implies $(\alpha, \beta) \in U'$ and $(\beta, \gamma) \in U'$ for each $U \in \mathcal{U}$, if $A \in \alpha$, $U \circ U \circ A \in \gamma$ and if $C \in \gamma$, $U \circ U \circ C \in \alpha$ implies $(\alpha, \beta) \in U'$ whenever $U \in \mathcal{U}$, thus α R γ . □

The following notations are used by Kent in [16].

Notations. Let

$$\alpha_R = \{\beta \in P : \alpha \text{ R } \beta\}$$

and

$$P_R = \{\alpha_R : \alpha \in P\} \ .$$

Kent gave the following definition (see [16]).

Definition 3.3.2 *For each prefilter $\alpha \in P$, we associate prefilters α_0 and α^0 defined by :*

$$\alpha_0 = \{U \in \mathcal{U} : V \circ U \in \alpha \ for \ all \ V \in \mathcal{U}\}$$

and

$$\alpha^0 = \{U \in \mathcal{U}\colon V \circ A \subseteq U \textit{ for some } V \in \mathcal{U} \textit{ and some } A \in \alpha\}.$$

Before defining an equivalence relation in the retracted scale, we need the following proposition.

Proposition 3.3.1 (*a*) *For each* $\alpha \in P$, $\alpha_R = \{\beta \in P : \alpha_0 \leq \beta \leq \alpha^0\}$.

(*b*) $\alpha\ R\ \beta$ *implies* $\alpha_0 = \beta_0$ *and* $\alpha^0 = \beta^0$.

(*c*) $\alpha_0 \subseteq \beta_0$ *if and only if* $\alpha^0 \subseteq \beta^0$.

Proof. (*a*) Let $\beta \in \alpha_R$ which implies α R β. Let $B \in \alpha^0$. Then $U \circ A \subseteq B$ for some $U \in \mathcal{U}$ and $A \in \alpha$, since $U \circ A \in \beta$ we have $B \in \beta$, so $\alpha^0 \subseteq \beta$. Let $C \in \beta$. Then $U \circ C \in \alpha$ for each $U \in \mathcal{U}$ which implies $C \in \alpha_0$, so $\beta \subseteq \alpha_0$. Thus $\alpha_0 \leq \beta \leq \alpha^0$.

(*b*) Let $B \in \beta_0$. For each $U \in \mathcal{U}$, $B \subseteq U \circ B \in \beta$ implies that $\beta_0 \subseteq \beta$ since α R β by (*a*) $\alpha^0 \subseteq \beta \subseteq \alpha_0$, so $\beta_0 \subseteq \alpha_0$. By the symmetry of R and by a similar argument $\alpha_0 \subseteq \beta_0$ thus $\alpha_0 = \beta_0$.

Let $T \in \alpha^0$. Then $U \circ A \subseteq T$ for some $U \in \mathcal{U}$ and $A \in \alpha$ since α R β, $U \circ A \in \beta$ implies that $T \in \beta^0$ so $\alpha^0 \subseteq \beta^0$. Similarly $\beta^0 \subseteq \alpha^0$, thus $\alpha^0 = \beta^0$.

(*c*) Suppose $\alpha_0 \subseteq \beta_0$. We show that $\alpha^0 \subseteq \beta^0$. Indeed, $T \in \alpha^0$, for some $U \in \mathcal{U}$ and $A \in \alpha$, $U \circ A \subseteq T$, $A \subseteq U \circ A \in \beta_0$ implies $U \circ A \in \beta$. So $T \in \beta^0$. We obtain the reciprocal inclusion by a similar argument. □

In the following we define an equivalence relation in the retracted scale.

Definition 3.3.3 *([16]) Let* α_R, β_R *in* P_R. *We define*

$$\alpha_R \leq \beta_R \textit{ to mean } \beta_0 \subseteq \alpha_0.$$

Lemma 3.3.2 *The relation* $\leq$ *defined above is an partial order on* P_R.

Proof. We show that $\leq$ is well defined: If α R γ and β R δ such that $\beta_0 \subseteq \alpha_0$ then $\delta_0 \subseteq \gamma_0$. Let $B' \in \delta_0$. Since β R δ implies that $\beta_0 = \delta_0$ so $B' \in \alpha_0$ thus $B' \in \gamma_0$.

The relation $\leq$ is reflexive, antisymmetric and transitive by dual inclusion of prefilters. □

In [16] Kent used the following definition:

Definition 3.3.4 *([16]) The order structure* $(P_R, \leq)$ *is called* order retracted scale *of* $(X,\mathcal{U})$.

In the following, we are going to show that the canonical map from the scale to its retracted scale is order-preserving.

Lemma 3.3.3 *Let* φ *be the canonical map from* P *to* P_R *defined by for* $\alpha \in P$, $\varphi(\alpha) = \alpha_R$. *Then* φ *is order-preserving.*

Proof. Let α, β in P be such that $\alpha \leq \beta$. We show that $\alpha_R \leq \beta_R$. Indeed, $\alpha \leq \beta$ means that $\beta \subseteq \alpha$. Let $B \in \beta_0$. For each $U \in \mathcal{U}$, $U \circ B \in \beta$ implies that $U \in \mathcal{U}$, $U \circ B \in \alpha$ which means that $B \in \alpha_0$. So $\beta_0 \subseteq \alpha_0$ thus $\alpha_R \leq \beta_R$. □

Proposition 3.3.2 *([16]) Let* $\{\alpha_i : i \in I\}$ *and* $\{\beta_i : i \in I\}$ *be sets of prefilters with* α_i R β_i, *for all* $i \in I$. *Then* $(\bigcup\{\alpha_i : i \in I\})$ R $(\bigcup\{\beta_i : i \in I\})$ *and* $(\bigcap\{\alpha_i : i \in I\})$ R $(\bigcap\{\beta_i : i \in I\})$.

Proof. See [16, Proposition 4]. □

Proposition 3.3.3 *([16]) The order retracted scale of any uniform space is a complete distributive lattice. The operations sup and* inf *are preserved under the canonical map* $\varphi : P \to P_R$ *defined in Lemma 3.3.3.*

Proof. See [16, Proposition 6]. □

We next define the retracted scale uniformity $\mathcal{U}''$ on P_R.

Definition 3.3.5 *For each $U \in \mathcal{U}$,*
$U" = \varphi(U') = \{(\alpha_R, \beta_R) \in P_R \times P_R : (\alpha, \beta) \in U'\}$.

Definition 3.3.6 *([16]) The pair $(P_R, \mathcal{U}")$ is called the* retracted scale uniform space *associated with $(X, \mathcal{U})$, where $\mathcal{U}"$ is defined in the next line.*

One can show that the filter $\mathcal{U}"$ generated by $\{U" : U \in \mathcal{U}\}$ is a T_2 uniformity.

Theorem 3.3.1 *([16]) The retracted scale uniform space $(P_R, \mathcal{U}")$ associated with an arbitrary uniform space $(X, \mathcal{U})$ is Hausdorff and is the quotient uniform structure derived from $(P, \mathcal{U}')$ under the map φ. The lattice operations for the retracted scale are uniformly continuous with respect to $\mathcal{U}"$.*

Proof. For the first part we refer the reader to [16, Theorem 1].

We show that the lattice operations for the retracted scale are uniformly continuous.

(a) Given $U \in \mathcal{U}$, let $U" \vee U" = \{(\alpha_R \vee \beta_R, \delta_R \vee \gamma_R)$: $(\alpha_R, \delta_R) \in U", (\beta_R, \gamma_R) \in U"\}$, we shall verify that $U" \vee U" = U"$. It suffices to show that $U" \vee U" \subseteq U"$.
Let (α_R, δ_R), (β_R, γ_R) belong to $U"$ and choose αR δ, β R γ such that $\alpha \in \alpha_R$, $\delta \in \delta_R$, $\beta \in \beta_R$ and $\gamma \in \gamma_R$. If $A \in \alpha \vee \beta$ implies A belongs to α or A belongs to β, so $U \circ A \in \delta$ or $U \circ A \in \gamma$, and if $B \in \delta \vee \gamma$ implies B belongs to δ or B belongs to γ, and so $U \circ B \in \alpha$ or $U \circ B \in \beta$. We have : if $A \in \alpha \vee \beta$ then $U \circ A \in \delta \vee \gamma$ and if $B \in \delta \vee \gamma$ then $U \circ B \in \alpha \vee \beta$ means $(\alpha \vee \beta, \delta \vee \gamma) \in U'$.

(b) Given $U \in \mathcal{U}$, we can prove that $U" = U" \wedge U"$. Similarly to the "join ", where it follows that $U" \subseteq U" \wedge U"$. □

We shall now consider order convergence in the scale $(P, \leq)$ of an arbitrary uniform space $(X, \mathcal{U})$.

Lemma 3.3.4 *([16]) Each ultrafilter ξ on P order-converges to some element of P.*

Proof. See [16, Lemma 1]. A similar proof is given in Lemma 5.5.2. □

Kent needs the following lemma for his proof that the scale of a uniform space and its retracted scale are both complete uniform spaces.

Lemma 3.3.5 *([16]) Let $U \in \mathcal{U}$, $F \subseteq P$, and $F \times F \subseteq U'$. If α is an element of P such that $\inf F \leq \alpha \leq \sup F$, then $(\alpha, \delta) \in U'$ for all $\delta \in F$.*

Proof. See [16, Lemma 2]. □

Theorem 3.3.2 *([16]) For any uniform space $(X, \mathcal{U})$, $(P, \mathcal{U}')$ and $(P_R, \mathcal{U}")$ are both complete uniform spaces.*

Proof. See [16, Theorem 2]. The analogous results hold for quasi-uniformities (see Theorem 5.3.1). □

Lemma 3.3.6 *([16]) Let $(X, \mathcal{U})$ be a uniform space, and let a be any fixed element of X. Let $v_a : X \longrightarrow P$ be defined at each $x \in X$ by $v_a(x) = \{V \in \mathcal{U}\colon (a, x) \in V\}$. Then v_a is a uniformly continuous map with respect to the scale uniformity on P.*

Proof. Choose U a symmetric entourage in $\mathcal{U}$, and let $(x, y) \in U$. We will show that $(v_a(x), v_a(y)) \in U'$. Indeed, $V \in v_a(x)$ implies $(a, x) \in V$. Since $(x, y) \in U$, we have $(a, y) \in U \circ V$, and hence $U \circ V \in v_a(y)$. The reciprocal argument establishes that $V \in v_a(y)$ implies $U \circ V \in v_a(x)$, and the proof is complete. □

Lemma 3.3.7 *([16]) If $(X, \mathcal{U})$ is a Hausdorff uniform space and v_a as in the lemma above, then the map $\varphi \circ v_a$ is a uniform embedding of $(X, \mathcal{U})$ onto a subspace of $(P_R, \mathcal{U}")$.*

Proof. See [16, Lemma 4]. □

Theorem 3.3.3 *([16]) The completion of a Hausdorff uniform space is a subspace of its retracted scale uniform space.*

Proof. Indeed, the completion of $(X, \mathcal{U})$ is simply the closure in P of the range of $\varphi \circ v_a$. for any chosen a in X. □

3.4 Stability in quasi-ordered uniform spaces

Many proposed abstract models for the study of control theory, differential equations with or without uniqueness of solutions, finite automata, etc. ([1], [7], [13], [26], [35] and [36] constitute a sample of this literature), although differing considerably from one another, have at last the following elements in common: a set E of "events", often interpreted as pairs (t, x), where t represents a time variable and x a "state" of some system; furthermore quasi-order (reflexive and transitive binary relation on E) ρ, so interpreted that $(e_1, e_2) \in \rho$ means that the event e_2 could follow e_1 in some feasible history of the system. In other words, for any $e \in E$, $\rho(e)$ is the set of all events which may follow, or are attainable from e.
We shall consider a concept of stability in this very general setting due to Bushaw. To formulate it we assume that there is given also a uniformity $\mathcal{U}$ on E. A set $A \subseteq E$ will be called *stable* relative to ρ and $\mathcal{U}$ if, for every $V \in \mathcal{U}$, there exists a $W \in \mathcal{U}$ such that $\rho W(A) \subseteq V(A)$, which in terms of the interpretation sketched above means that any event attainable from an event sufficiently close to A is itself arbitrarily close to A. As stability concepts go, it is rather simple.

Stability is usually defined only for *invariant* sets, i.e., sets A satisfying $\rho(A) \subseteq A$. The following observation shows that this discrepancy is not severe.

Proposition 3.4.1 *([6]) Let E be a uniform space. A set $A \subseteq E$ is stable relative to ρ and $\mathcal{U}$ if and only if its closure $\overline{A}$ in the uniform topology is stable; and any closed stable set is invariant.*

Proof. See [6, Proposition 4.1]. □

Liapunov functions and the stability criterion

Let $(M, \leq)$ be a partially ordered set with first element 0. The set $M - \{0\}$, assumed nonempty, will be denoted by M^+.

Definition 3.4.1 *([6]) Generalizing an existing definition (see [2]), Bushaw said that a map $v : N \longrightarrow M$, where N is some invariant uniform neighborhood of A, is a* Liapunov function *for A if it satisfies:*

(L_1) *for any* $e_1 \in N$, $(e_1, e_2) \in \rho$ *implies* $v(e_1) \geq v(e_2)$;

(L_2) *for any* $\lambda \in M^+$ *there exists a* $V \in \mathcal{U}$ *such that* $v(e)$ *is not* $\geq \lambda$ *if* $e \in V(A) \cap N$;

(L_3) *for any* $V \in \mathcal{U}$, *there exists a* $\lambda \in M^+$ *such that* $v(e) \geq \lambda$ *if* $e \in N - V(A)$.

Thus, informally, a Liapunov function is one which (L_1) decreases relative to ρ, (L_2) "approaches 0" uniformly as e approaches A, and (L_3) is bounded away from 0 outside any uniform neighborhood of A.

Proposition 3.4.2 *([6]) If* v *is a Liapunov function for* A, *then* $v(e) = 0$ *if and only if* $e \in \overline{A}$.

Proof. See [6, Proposition 5.1]. □

Liapunov's direct method in stability theory characterizes stability properties by the existence of Liapunov functions of corresponding types. It seems that before Bushaw started his investigations the partially ordered set $(M, \leq)$ had always been taken to be the set of non-negative reals with the standard order, or a very mild extension thereof, and the types of systems $(E, \rho, \mathcal{U})$ for which Liapunov stability criteria had been found had been restricted accordingly. The introduction of the (retracted) scale of $\mathcal{U}$ at this point wiped away the restrictions (see [6]).

Theorem 3.4.1 *([6]) A set* A *of a uniform space* $(X, \mathcal{U})$ *is stable if and only if there exists a Liapunov function for* A *into the retracted scale of* $\mathcal{U}$.

Proof. See [6, Theorem 1]. □

3.5 Pseudometrizability, separability, the Lindelöf condition, and the first and second axioms of countability of the uniform scale

Ramsey introduced in [29] a function which acts like a pseudometric on a uniform space, but which has as its range a subset of the uniform scale of that space. This function will then be used to investigate pseudometrizability, separability, the Lindelöf condition, and the first and second axioms of countability.

Definition 3.5.1 *([29]) Let X and M be nonempty sets, κ a function from $X \times X$ into M, t_0 a fixed point in M, and h_{t_0} a family of sets $\{X_\lambda : \lambda \in \wedge\}$ in M such that $t_0 \in X_\lambda$ for every $\lambda \in \wedge$. Then κ will be called an* écart *if the following conditions are satisfied.*

1. $\kappa(a, a) = t_0$ for every $a \in X$.

2. $\kappa(a, b) = \kappa(b, a)$ for every a and b in X.

3. h_{t_0} is submultiplicative, i.e., for every λ and λ' in $\wedge$, there exists a $\mu \in \wedge$, such that $X_\mu \subseteq X_\lambda \cap X_{\lambda'}$.

4. For every $\lambda \in \wedge$, there exists a $\mu \in \wedge$ such that $\kappa(a, b) \in X_\mu$ and $\kappa(b, c) \in X_\mu$ imply that $\kappa(a, c) \in X_\lambda$.

If κ is an écart on X, then a uniformity can be defined on X by using the following base:

$$W_\lambda = \{(a, b) \in X \times X : \kappa(a, b) \in X_\lambda\},\ \lambda \in \wedge.$$

The uniformity will be Hausdorff if and only if the following condition is satisfied.

5. $\kappa(x, a) \in \bigcap\{X_\lambda : \lambda \in \wedge\}$ implies that $a = x$.

Now, let $(X, \mathcal{U})$ be an arbitrary uniform space and $(P, \mathcal{U}')$ its uniform scale.

Definition 3.5.2 *([29]) Let 0 be the last element of P, and let $h_0 = \{U'(0) : U \in \mathcal{U}\}$. Define $\kappa : X \times X \longrightarrow P$ by*

$$\kappa(x, y) = \{U \in \mathcal{U} : (x, y) \in U \text{ and } (y, x) \in U\}$$

for every x and y in X.

Theorem 3.5.1 *([29]) Then κ is an écart, and the uniformity defined on X by κ is $\mathcal{U}$. Moreover, if $\mathcal{B}$ is a closed, symmetric base for $\mathcal{U}$, then for any $V \in \mathcal{B}$, $(a, b) \in V$ if and only if $\kappa(a, b) \in \mathcal{B}$.*

Proof. See [29, Theorem 2.1]. A similar proof is given in Theorem 5.4.1.□

Let $\varphi : P \longrightarrow P_R$ be the retraction map that is the (canonical map) as defined in Lemma 3.3.3. It is clear that if κ is an écart with values in P, then $\varphi\kappa$ is an écart with values in P_R. The following proposition implies that $\varphi\kappa$ also induces the uniformity $\mathcal{U}$ on X.

Theorem 3.5.2 *([29]) Let $(X,\mathcal{U})$ be a uniform space, $(P,\mathcal{U}')$ its scale and $(P_R,\mathcal{U}'')$ its retracted scale. Then the following statements are equivalent.*

1. $(X,\mathcal{U})$ is pseudometrizable.

2. $(P,\mathcal{U}')$ is pseudometrizable.

3. $(P,\mathcal{U}')$ satisfies the first axiom for countability.

4. $(P,\mathcal{U}')$ has a countable base for the neighborhood filter at 0.

5. There exists a map ϕ from P into the nonnegative real numbers such that $\phi(0) = 0$ and $\{\phi^{-1}([0, \varepsilon)) : \varepsilon > 0\}$ is a base for the neighborhood filter at 0.

Furthermore $(P,\mathcal{U}')$ can be replaced by $(P_R,\mathcal{U}'')$ in any of the statements (2) through (5).

Proof. See [29, Theorem 2.3]. A similar proof is given in Theorem 5.4.2.□

Questions (compare [29]): The écart κ maps $X \times X$ into the scale P. That suggests the following questions: Does there exist a uniformly continuous, order preserving map ϕ from P into the nonnegative real numbers and,

if so, what can be said about the map $\phi\kappa : X \times X \longrightarrow \mathbb{R}^+$. The next series of propositions will be concerned with this problem.

First let $(X, d, \mathcal{U})$ be a pseudometric space, where $\mathcal{U}$ is the pseudometric uniformity. For $\varepsilon > 0$, let $U_\varepsilon = \{(x, y) \in X \times X : d(x, y) \leq \varepsilon\}$; note that this is not the usual meaning.

Lemma 3.5.1 *([29]) For any $\varepsilon, \delta \geq 0$, $U_\varepsilon{}' \circ U_\delta{}' \subseteq U_{\varepsilon+\delta}{}'$.*

Proof. Here the proof is completed by using the composition of entourages.□

Now suppose that d is a bounded pseudometric.

Definition 3.5.3 *([29]) Define $\widetilde{d} : P \times P \longrightarrow \mathbb{R}^+$ by*

$$\widetilde{d}(\alpha, \beta) = \inf\{\varepsilon > 0 : (\alpha, \beta) \in U_\varepsilon{}'\}.$$

(The fact that d is bounded ensures that $\widetilde{d}$ is well defined.)

Proposition 3.5.1 *([29]) Then $\widetilde{d}$ is a pseudometric on P, and the scale uniformity $\mathcal{U}'$ is the pseudometric uniformity induced by $\widetilde{d}$.*

Proof. First we show that $\widetilde{d}$ is pseudometric.

i. $\widetilde{d}(\alpha, \alpha) = 0$ for every $\alpha \in P$ since $(\alpha, \alpha) \in U_\varepsilon{}'$ for every $\varepsilon > 0$.

ii. $\widetilde{d}(\alpha, \beta) = \widetilde{d}(\beta, \alpha)$ for every α and β in P by the symmetry of the $U_\varepsilon{}'$'s.

iii. Let $\varepsilon > 0$, $\widetilde{d}(\alpha, \beta) = a$, and $\widetilde{d}(\beta, \gamma) = b$. Then $(\alpha, \beta) \in U_{a+\varepsilon}{}'$ and $(\beta, \gamma) \in U_{b+\varepsilon}{}'$ imply that $(\alpha, \gamma) \in U_{a+\varepsilon}{}' \circ U_{b+\varepsilon}{}' \subseteq U_{a+b+2\varepsilon}{}'$. Therefore for any $\varepsilon > 0$, $\widetilde{d}(\alpha, \gamma) \leq \widetilde{d}(\alpha, \beta) + \widetilde{d}(\beta, \gamma) + 2\varepsilon$; hence $\widetilde{d}(\alpha, \gamma) \leq \widetilde{d}(\alpha, \beta) + \widetilde{d}(\beta, \gamma)$.

Thus $\widetilde{d}$ is a pseudometric. It is clear that $U_\varepsilon{}' = \{(\alpha, \beta) : \widetilde{d}(\alpha, \beta) \leq \varepsilon\}$, so $\mathcal{U}'$ is the pseudometric uniformity. □

Remark 3.5.1 *Note that $\widetilde{d}(\alpha, \beta) = 0$ if and only if $(\alpha, \beta) \in U_\varepsilon{}'$ for every $\varepsilon > 0$, and the latter is true if and only if α is equivalent to β. Thus the pseudometric becomes a metric on the retracted scale $(P_R, \mathcal{U}")$.*

Lemma 3.5.2 *Let $(X,\mathcal{U})$ be a uniform space and $\alpha, \beta \in P$. If $\alpha \leq \beta$ and $\beta \in U'(0)$ for some $U \in \mathcal{U}$, then $\alpha \in U'(0)$.*

Proof. Let $\beta \in U'(0)$ for some $U \in \mathcal{U}$, which implies that for every $V \in \mathcal{U}$, $U \circ V \in \beta$. If $\alpha \leq \beta$, then $\beta \subseteq \alpha$, so $U \circ V \in \alpha$ for every $V \in \mathcal{U}$. Thus $\alpha \in U'(0)$. □

Proposition 3.5.2 *([29]) If $(X,\mathcal{U})$ is a pseudometrizable space and $(P,\mathcal{U}')$ its scale, then there exists a uniformly continuous, order preserving map ϕ from $(P,\mathcal{U}')$ into the nonnegative real numbers. (The latter set is assumed to be equipped with its standard metric uniformity.)*

Proof. Let d be a bounded pseudometric on X such that $\mathcal{U}$ is the pseudometric uniformity for d. Let $\widetilde{d}$ be the pseudometric on P defined by

$$\widetilde{d}(\alpha, \beta) = \inf\{\varepsilon > 0 : (\alpha, \beta) \in U_\varepsilon{}'\},$$

for all $\alpha, \beta \in P$. Then $\widetilde{d}$ is a pseudometric on P, and $\mathcal{U}'$ is its pseudometric uniformity by Proposition 3.5.1. Define $\phi : P \longrightarrow \mathbb{R}^+$ by $\phi(\alpha) = \widetilde{d}(\alpha, 0)$ for every $\alpha \in P$.

i. ϕ is order preserving: If $\alpha \leq \beta$, then by Lemma 3.5.2, for any $\varepsilon > 0$, $\beta \in U_\varepsilon{}'(0)$ implies that $\alpha \in U_\varepsilon{}'(0)$. Then $\widetilde{d}(\alpha, 0) \leq \widetilde{d}(\beta, 0)$, and so $\phi(\alpha) \leq \phi(\beta)$.

ii. ϕ is uniformly continuous:
Let $\varepsilon > 0$. If $(\alpha, \beta) \in U_\varepsilon{}'$, then $\widetilde{d}(\alpha, \beta) \leq \varepsilon$, so $\mid \phi(\alpha) - \phi(\beta) \mid = \mid \widetilde{d}(\alpha, 0) - \widetilde{d}(\beta, 0) \mid \leq \widetilde{d}(\alpha, \beta) \leq \varepsilon$. Therefore ϕ is uniformly continuous, and the proof is complete. □

If $(X,\mathcal{U})$ is not pseudometrizable, then a similar map ϕ can still be constructed. The next proposition gives the necessary tools to prove this statement.

Proposition 3.5.3 *([29]) Let d be a bounded pseudometric on X such that $\mathcal{U}_d \subseteq \mathcal{U}$, where $\mathcal{U}_d$ is the pseudometric uniformity defined by*

$$\widetilde{d}(\alpha, \beta) = \inf\{\varepsilon > 0 : (\alpha, \beta) \in U_\varepsilon{}'\}.$$

where U_ε is defined with respect to d. Then $\widetilde{d}$ is a pseudometric on P,

$$U_\varepsilon{}' = \{(\alpha, \beta) : \widetilde{d}(\alpha, \beta) \leq \varepsilon\},$$

and $\mathcal{U}_{\widetilde{d}} \subseteq \mathcal{U}'$, where $\mathcal{U}_{\widetilde{d}}$ is the uniformity generated on P by $\widetilde{d}$.

Proof. This is just like the proof of Proposition 3.5.1. We have $\{U_\varepsilon{}' : \varepsilon > 0\} \subseteq \mathcal{U}'$ since $\mathcal{U}_d \subseteq \mathcal{U}$, and it is clear that these sets form a base for a uniformity on P. □

Theorem 3.5.3 *([29]) If $(X,\mathcal{U})$ is any uniform space and $(P,\mathcal{U}')$ its scale, then there exists a uniformly continuous, order preserving map ϕ from $(P,\mathcal{U}')$ into the nonnegative real numbers.*

Proof. This is a consequence of Proposition 3.5.2 and Proposition 3.5.3. Then ϕ is order preserving and uniformly continuous with respect to $\mathcal{U}_{\widetilde{d}}$. Since $\mathcal{U}_{\widetilde{d}} \subseteq \mathcal{U}'$, ϕ is also uniformly continuous with respect to $\mathcal{U}'$. □

If $(X,\mathcal{U})$ is not pseudometrizable, then the sets $\{(x, y) \in X \times X : \phi\kappa(x, y) < \varepsilon\}$, $\epsilon > 0$, do not form a base for $\mathcal{U}$.

Proposition 3.5.4 *([29]) If $(X, d, \mathcal{U})$ is a bounded pseudometric space, $(P, \widetilde{d}, \mathcal{U}')$ its scale, κ the écart of Theorem 3.5.1 , and ϕ the map in Proposition 3.5.2, then $d = \phi\kappa$.*

Proof. See [29, Proposition 2.10]. A similar proof is given in Proposition 5.4.5. □

3.6 The order scale

We start this section with some basic results on the order scale of a uniform space. This structure was introduced in [17].

Let $(X,\mathcal{U})$ be a uniform space and $(P,\mathcal{U}')$ its scale uniform space. From Proposition 3.2.1 P is a complete distributive lattice with last element $\mathcal{U}$ (which we denote by 0) and greatest element $\{X \times X\}$(denoted by 1); infima

and suprema in P are set unions and intersections, respectively, and from this it follows that P is infinitely distributive. Indeed, P is a subcomplete lattice of the power set of the power set of $X \times X$, with the dual of its usual ordering, which means that P is a subcomplete lattice of an atomic Boolean algebra, and thus bicompactly generated (see [17]).

Definition 3.6.1 *([17]) Let $(X,\mathcal{U})$ be a uniform space. For each $U \in \mathcal{U}$, we associate two prefilters σ_U and ρ_U as follows:*

$$\sigma_U = \{V \in \mathcal{U} : U \subseteq V\},$$

$$\rho_U = \{V \in \mathcal{U} : U \textit{ is not a subset of } V\}.$$

Note that $\sigma_U = \langle U \rangle$ as defined in Definition 3.1.1.

Proposition 3.6.1 *([17]) Let $(X,\mathcal{U})$ be a uniform space and $(P,\mathcal{U}')$ its scale.*
(a) Then $\alpha \in P$ is compact if and only if there are entourages $U_1, ..., U_n$ in $\mathcal{U}$ such that $\alpha = \rho_{U_1} \vee ... \vee \rho_{U_n}$.
(b) Then $\alpha \in P$ is cocompact if and only if there are entourages $U_1, ..., U_n$ in $\mathcal{U}$ such that $\alpha = \sigma_{U_1} \wedge ... \wedge \sigma_{U_n}$.

Proof. See [17, Proposition 2.1]. □

By Proposition 2.1.4, order convergence in P is topological; we denote the order topology by θ.

The following theorem summarizes properties of the order scale topology.

Theorem 3.6.1 *For any uniform space $(X,\mathcal{U})$, (P,θ) is a compact, T_2, totally disconnected topological space. If $\alpha \neq 0$, the θ-neighborhood filter $\mathcal{U}_\theta(\alpha)$ at α has an open subbase of sets of the form $[\rho_U, \sigma_V]$, where U does not belong to α and $V \in \alpha$. The θ-neighborhood filter of 0 has an open subbase of the form $[0, \sigma_V]$, $V \in \mathcal{U}$.*

Proof. This follows from Proposition 2.1.4 and Proposition 3.6.1. □

Since (P,θ) is compact and T_2, there is a unique uniformity ϑ for P which induces θ. We shall call ϑ the *order scale uniformity.*

Definition 3.6.2 *([17]) Let $(X,\mathcal{U})$ be a uniform space and $(P,\mathcal{U}')$ its uniform scale. The term* order scale *will be used ambiguously to mean either the topological space (P,θ) or the uniform space (P,ϑ).*

Theorem 3.6.2 *([17]) For any uniform space $(X,\mathcal{U})$, (P,ϑ) is a uniform lattice.*

Proof. See [17, Theorem 2.3]. □

Theorem 3.6.3 *([17]) The order scale (P,θ) is first countable (second countable, metrizable) in its order topology if and only if $(X,\mathcal{U})$ is finite or discrete.*

Proof. See [17, Theorem 2.5]. □

Kent showed that a uniformly continuous map between two uniform spaces can be "lifted" to a uniformly continuous map between their order scales (see [17]).

Theorem 3.6.4 *Let $(X_1,\mathcal{U}_1)$, $(X_2,\mathcal{U}_2)$ be two uniform spaces. If $f : (X_1,\mathcal{U}_1) \to (X_2,\mathcal{U}_2)$ is uniformly continuous, then so is $\widehat{f} : (P_1,\vartheta_1) \to (P_2,\vartheta_2)$. Let $\alpha \in P_1$. We define $\widehat{f}(\alpha) = [(f \times f)(\alpha)] \cap \mathcal{U}_2$, where $[(f \times f)(\alpha)]$ is the smallest prefilter on $X_2 \times X_2$ containing $\{(f \times f)(A) : A \in \alpha\}$. Then $\widehat{f}$ is also uniformly continuous. (As we shall see later, a similar result holds for the (quasi)-uniform scale (see Proposition 5.3.1.).)*

Proof. See [17, Theorem 2.6]. □

The relationship between the scale and order scale uniformities is summarized in the next proposition.

Proposition 3.6.2 *([17]) Let $(X,\mathcal{U})$ be a uniform space, $(P,\mathcal{U}')$ its uniform scale and (P,ϑ) its order scale.*
(a) $\mathcal{U}' \subseteq \vartheta$ if and only if $(X,\mathcal{U})$ is discrete.
(b) $\vartheta \subseteq \mathcal{U}'$ if and only if $(X,\mathcal{U})$ is finite or discrete.

Proof. See [17, Proposition 3.4]. □

Chapter 4

Topological properties of the scale of a uniform space

In this chapter, we shall summarise known results about the topological properties of the uniform scale of a uniform space. We shall first discuss the property of connectedness in the scale of a uniform space. These results were obtained by Leslie and Kent [19]. We also mention conditions due to Leslie and Kent that are necessary for local connectedness, connectedness and arcwise connectedness in the scale of a uniform space. In the last section of Chapter 4, we shall investigate other topological properties of the scale of a uniform space dealt with by Richardson [32].

4.1 Connectedness in the scale of a uniform space

In this section, we show that the scale of a uniform space is uniformly connected if and only if the uniform space is bounded [19]. Sufficient conditions are found on the uniform space which make the uniform scale connected and arcwise-connected, and the relationship between connectedness and local connectedness is also considered.

Let $(X,\mathcal{U})$ be a uniform space and $(P,\mathcal{U}')$ its scale uniform space. The last element of P is $\mathcal{U}$, denoted by 0; the greatest element is $\{X \times X\}$, denoted by 1, as mentioned before.

Definition 4.1.1 *([19]) Let $(X,\mathcal{U})$ be a uniform space; $(X,\mathcal{U})$ is said to be bounded if, for each $U \in \mathcal{U}$, there is a positive integer n such that $U^n = X \times X$.*

Definition 4.1.2 *([19]) Let $(X,\mathcal{U})$ be a uniform space; $(X,\mathcal{U})$ is said to be uniformly connected if every uniformly continuous map from the space into a discrete uniform space is a constant map.*

The following lemma shows that boundedness implies uniform connectedness.

Lemma 4.1.1 *(Mrówka and Pervin)([19]) A uniform space $(X,\mathcal{U})$ is uniformly connected if and only if, for each pair x, y of elements in X and each $U \in \mathcal{U}$, there is an integer n such that $(x, y) \in U^n$.*

Proof. See [19, Lemma 1]. □

Theorem 4.1.1 *([19]) The following statements are equivalent.*

(1) *$(X,\mathcal{U})$ is bounded.*

(2) *$(P,\mathcal{U}')$ is bounded.*

(3) *$(P,\mathcal{U}')$ is uniformly connected.*

Proof. That (1) implies (2) follows from Proposition 3.2.3. That (2) implies (3) is a consequence of Lemma 4.1.1. To show that (3) implies (1), assume that $(P,\mathcal{U}')$ is uniformly connected and let $U \in \mathcal{U}$. By Lemma 4.1.1. there is an integer n such that $(0, 1) \in U'^n$; furthermore we can assume by Proposition 3.2.3 that $U'^n = U^{n'}$. Thus for each $V \in \mathcal{U}$, $U^n \circ V = X \times X$, and in particular $U^{n+1} = X \times X$. So $(X,\mathcal{U})$ is bounded. □

A topological space is connected if and only if the only continuous map from the space into a discrete topological space is a constant map. Thus connectedness implies uniform connectedness in a uniform space, and boundedness in $(X,\mathcal{U})$ is necessary for connectedness in $(P,\mathcal{U}')$.

Definition 4.1.3 *([19]) For each $U \in \mathcal{U}$ and $\alpha \in P$, let*

$$\alpha_U = \{V \in \mathcal{U} : U \circ V \in \alpha\}$$

and

$$\alpha^U = \{V \in \mathcal{U} : \textit{for some } A \in \alpha,\ U \circ A \subseteq V\}.$$

Lemma 4.1.2 *([19]) For each $U \in \mathcal{U}$ and for each $\alpha \in P$, $U'(\alpha) = [\alpha_U, \alpha^U]$.*

Proof. Let $\beta \in U'(\alpha)$. Let $V \in \alpha^U$, for some $A \in \alpha$, then $U \circ A \subseteq V \in \beta$ so $\alpha^U \subseteq \beta$. Let $B \in \beta$ which implies $U \circ B \in \alpha$ so $B \in \alpha_U$.

Conversely, let β be such that $\alpha^U \subseteq \beta \subseteq \alpha_U$. Let $V \in \alpha^U$. For some $A \in \alpha$, $U \circ A \subseteq V \in \beta$ and let $B \in \beta$ which implies $U \circ B \in \alpha$, so $(\alpha, \beta) \in U'$. Thus $B \in U'(\alpha)$. □

Proposition 4.1.1 *([19]) The collection $\{[\alpha_U, \alpha^U] : U \in \mathcal{U}\}$ forms a basic system of neighborhoods at α for the uniform topology on P generated by $\mathcal{U}'$.*

Proof. See [19, Proposition 2]. □

We next define the composition of prefilters.

Definition 4.1.4 *([19]) Let M be a subset of P. Given $\alpha, \beta \in P$, we define their product as follows:*

$$\alpha \circ \beta = \{W \in \mathcal{U} : \textit{for some } U \in \alpha \textit{ and } V \in \beta, W \supseteq U \circ V\}.$$

Let

$$M \bullet \alpha = \{\beta \circ \alpha : \beta \in M\}. \textit{ Let } M \vee \alpha = \{\beta \vee \alpha : \beta \in M\}$$

and

$$M \wedge \alpha = \{\beta \wedge \alpha : \beta \in M\}.$$

The lattice operations $\vee$ and $\wedge$ in P are uniformly continuous by Theorem 3.3.1. Also for each α the map $\beta \mapsto \beta \circ \alpha$ is uniformly continuous [19]. Then we obtain the following results about connectedness and local connectedness in the uniform scale space. They are immediate (see [19]).

Proposition 4.1.2 *([19]) If M is a connected (arcwise connected) subset of P, then the sets $M \bullet \alpha$, $M \vee \alpha$ and $M \wedge \alpha$ are likewise connected (arcwise connected) for any choice of α.*

Proposition 4.1.3 *([19]) If there is a connected (arcwise connected) subset C of P containing α and β, then $[\alpha \wedge \beta, \alpha \vee \beta]$ is connected (arcwise connected).*

Corollary 4.1.1 *([19]) The scale of a uniform space is connected (arcwise connected) if 0 and 1 belong to the same connected (arcwise connected) component.*

Corollary 4.1.2 *([19]) If the closed interval $[\alpha, \beta]$ in P is a subset of a connected (arcwise connected) set M, then $[\alpha, \beta]$ is connected (arcwise connected).*

Corollary 4.1.3 *([19]) The scale of a uniform space is locally connected (locally arcwise connected) if and only if there is a connected (arcwise connected) neighborhood for each point.*

The following theorem characterizes the local connectedness condition.

Theorem 4.1.2 *([19]) The scale $(P, \mathcal{U}')$ of a uniform space $(X, \mathcal{U})$ is locally connected (locally arcwise connected) if and only if there is a connected (arcwise connected) neighborhood of 0.*

We next study connectedness of the scale of a uniform space.

Theorem 4.1.3 *([19]) The scale $(P, \mathcal{U}')$ of a bounded uniform space $(X, \mathcal{U})$ is connected (arcwise connected) if and only if it is locally connected (arcwise connected).*

Proof. See [19, Theorem 3]. □

Boundedness in $(X, \mathcal{U})$ is not a necessary condition for local connectedness in the scale of a uniform space $(P, \mathcal{U}')$. In the next proposition, we cite a weaker condition found by Leslie and Kent which is necessary for local connectedness in the scale of a uniform space.

Proposition 4.1.4 *([19]) If $(P,\mathcal{U}')$ is locally connected, then there is an entourage $U \in \mathcal{U}$ such that, for every $V \in \mathcal{U}$, $U \subseteq V^n$ for some integer n.*

Proof. See [19, Proposition 5]. □

4.2 Connectedness conditions in the scale of a uniform space

In this section, we will now give conditions on $(X,\mathcal{U})$ which are sufficient for connectedness, local connectedness, and arcwise connectedness in $(P,\mathcal{U}')$.

In [19] Leslie and Kent used the following definitions.

Definition 4.2.1 *([19]) A uniform space $(X,\mathcal{U})$ satisfies Condition A if there is a base $\mathcal{B}$ for the uniform space which is closed under unions and finite compositions such that, given a chain $\{V_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{B}$ and $W \in \mathcal{U}$, there is an index $\lambda_0 \in \Lambda$ such that $W \circ V_{\lambda_0} \supseteq \cup_{\lambda\in\Lambda} V_\lambda$.*

Definition 4.2.2 *([19]) Let (X,d) be a pseudo-metric space, and for each $\epsilon > 0$ let O_ϵ be the entourage $\{(x,y) : d(x,y) < \epsilon\}$. If $O_\epsilon \circ O_\delta = O_{\epsilon+\delta}$ for all positive real numbers ϵ, δ then (X,d) is said to satisfy Condition B.*

The uniformity derived from a pseudo-metric which satisfies Condition B will surely satisfy Condition A, since $\mathcal{B}$ can be chosen to consist of $\{O_\epsilon : \epsilon > 0\}$.

Theorem 4.2.1 *([19]) If $(X,\mathcal{U})$ is a bounded uniform space which satisfies Condition A, then $(P,\mathcal{U}')$ is connected.*

Proof. Assume that there is a subset M of P which is both closed and open, distinct from P, and contains an element α. We shall show that 1 is in M. If N is the complement of M, then precisely the same argument establishes that $1 \in N$, a contradiction.

Let $\mathcal{C} = \{V \in \mathcal{U} : \alpha^V \in M, V \in \mathcal{B}\}$, where $\mathcal{B}$ is the base for $\mathcal{U}$ specified in Condition A. Then $\mathcal{C}$ is non-empty since M is open. We shall partially order $\mathcal{C}$ by set inclusion and apply Zorn's Lemma to establish the existence of a

maximal element U in $\mathcal{C}$. Indeed, if $\{V_i\}$ is a chain in $\mathcal{C}$, and $V = \cup V_i$, then Condition A is precisely what is needed to show that the net $\{\alpha^{V_i}\}$ converges to α^V. Since M is closed, α^V is in M, and $V \in \mathcal{C}$.

Since $\alpha^U \in M$ and M is open, there is $W \in \mathcal{B}$ such that $(\alpha^U)^W = \alpha^{WU} \in M$. Thus $WU \in \mathcal{C}$ (recall that $\mathcal{C}$ is closed under finite compositions) and so $WU = U$. The proof will be completed by showing that, necessarily, $U = X \times X$. Indeed, $W^2U = WWU = U$, and by induction $W^kU = U$ for all integers k. But $(X,\mathcal{U})$ is bounded, and so $W^n = X \times X$ for some integer n. Thus $U = X \times X$, and $\alpha^U = 1 \in M$, which completes the proof. □

If (X, d) satisfies Condition B, then boundedness in (X, d) is equivalent to boundedness in the associated uniform space, and hence this term as used in the next theorem can be interpreted in either sense.

Theorem 4.2.2 *([19]) If a pseudo-metric space (X, d) satisfies Condition B, then the scale $(P,\mathcal{U}')$ of the associated uniform space is locally arcwise connected. If, in addition, (X, d) is bounded, then $(P,\mathcal{U}')$ is arcwise connected.*

Proof. We can assume without loss of generality that d assumes values less than or equal to 1. Let f be the function mapping the unit interval $[0, 1]$ into $(P,\mathcal{U}')$ defined by $f(r) = \langle O_r \rangle$, where $\langle O_r \rangle$ denotes the prefilter consisting of all supersets of O_r; in particular let $f(0) = 0$. Condition B guarantees that f is uniformly continuous. Thus by Proposition 4.1.3, the interval $[0, \langle O_r \rangle]$ is arcwise connected, and it follows from Theorem 4.1.2 that $(P,\mathcal{U}')$ is locally arcwise connected. □

4.3 Other topological properties and the cardinality of the scale of a uniform space

In the following section we describe under which condition the scale of a uniform space is compact and we shall discuss the cardinality of the scale of a uniform space and its retracted scale.

The next proposition will be useful in proving that a necessary and sufficient condition for the scale to have a totally bounded neighborhood of 0 is that $\mathcal{U}$ has a last element.

Proposition 4.3.1 *([32]) Suppose that $(X,\mathcal{U})$ is a uniform space such that there exists a $U \in \mathcal{U}$ with the property that the complement of U is an infinite set. Then either $\mathcal{U}$ has a last element or there exists a countably infinite discrete subspace of $[0, \langle U \rangle]$.*

Proof. See [32, Proposition 3]. □

Proposition 4.3.2 *([32]) Let $(X,\mathcal{U})$ be a uniform space. Then $(P,\mathcal{U}')$ has a totally bounded neighborhood of 0 if and only if $\mathcal{U}$ has a last element.*

Proof. See [32, Proposition 4]. □

Theorem 4.3.1 *([32]) Let $(X,\mathcal{U})$ be a uniform space. Then the following are equivalent conditions:*

(*a*) *$(P,\mathcal{U}')$ has a totally bounded neighborhood of 0.*

(*b*) *$(P,\mathcal{U}')$ has a compact neighborhood of 0.*

(*c*) *$(P,\mathcal{U}')$ is locally compact.*

(*d*) *$\mathcal{U}$ has a last element.*

Proof. $(a) \Longrightarrow (b)$: Proposition 4.3.1 implies that $\mathcal{U}$ has a last element U. Then $[0, \langle U' \rangle] = \{0\}$ is a compact neighborhood of 0. $(b) \Longrightarrow (c)$: Let $\alpha \in P$; then if U is the last element of $\mathcal{U}$, $U'(\alpha) \subseteq V'(\alpha)$ for each $V \in \mathcal{U}$, and so $U'(\alpha)$ is a compact neighborhood of α. That $(c) \Longrightarrow (d)$ and $(d) \Longrightarrow (a)$ follows from Proposition 4.3.1. □

Example 4.3.1 *([32]) It is possible for a uniformity to have a last element which is not the diagonal $\triangle$. Let $X = \mathbb{R}$ and let $\mathcal{U}$ be the set of all subsets of $\mathbb{R}^2$ that contain $\triangle \cup \{(-1,1),(1,-1)\}$. Then the set $\triangle \cup \{(-1,1),(1,-1)\}$ is the last element of $\mathcal{U}$.*

We now turn from local properties of the uniform scale space to its global properties.

Proposition 4.3.3 *([32]) Let $(X,\mathcal{U})$ be a uniform space. Then $(P,\mathcal{U}')$ is totally bounded if and only if either X is finite or $\mathcal{U} = \{X \times X\}$.*

Proof. See [32, Proposition 5]. We shall see later that the same results hold for quasi-uniformities (Theorem 5.3.4). □

Note that in Example 4.3.1 $(P,\mathcal{U}')$ has a totally bounded neighborhood of 0, but $(P,\mathcal{U}')$ is not totally bounded.

The following lemmas are immediate ([32]). They will be used to show that the statement "the scale of a uniform space is compact if and only if X is finite or $\mathcal{U} = \{X \times X\}$" remains true if compact is replaced by countably compact, totally bounded, Lindelöf, second countable, or separable.

Lemma 4.3.1 *([32]) Let $(X,\mathcal{U})$ be a uniform space and $V \in \mathcal{U}$ be such that $X - V(x) =: A$ for some $x \in X$, where the cardinality of A is infinite. Then P_R contains a closed discrete subset of cardinality 2^{cardA}.*

Lemma 4.3.2 *([32]) Suppose that $(X,\mathcal{U})$ is a uniform space where $card(X) = \kappa$, and V is a symmetric entourage such that for all $x \in X$, $card(X - V(x)) = \lambda < \kappa$ where κ and λ are infinite cardinal numbers. Then $V^2 = X \times X$.*

Remark 4.3.1 *([32]) The conclusion of the above lemma remains true if κ is infinite and $X - V(x)$ has finitely many elements for all $x \in X$.*

Theorem 4.3.2 *([32]) Let $(X,\mathcal{U})$ be a uniform space. Then the following conditions are equivalent:*

(a) $(P,\mathcal{U}')$ *is countably compact.*

(b) $(P,\mathcal{U}')$ *is totally bounded.*

(c) $(P,\mathcal{U}')$ *is Lindelöf.*

(d) $(P,\mathcal{U}')$ *is compact.*

(e) $(P,\mathcal{U}')$ *is second countable.*

(f) $(P,\mathcal{U}')$ *is separable.*

(g) *Either* X *is finite or* $\mathcal{U} = \{X \times X\}$.

Proof. If X is finite or if $\mathcal{U} = \{X \times X\}$, all the remaining conditions easily follow.
Suppose that $(P,\mathcal{U}')$ is countably compact, totally bounded, Lindelöf, compact, second countable, or separable. Suppose that X is not finite. We wish to show that $\mathcal{U} = \{X \times X\}$.
Assume that $\mathcal{U} \neq \{X \times X\}$. Then by Lemma 4.3.2 and Remark 4.3.1, there exist $V \in \mathcal{U}$ and $x \in X$ such that $X - V(x)$ is infinite. By Lemma 4.3.1, P_R contains a closed, discrete subset of cardinality $2^{card(X-V(x))}$. This fact is contrary to $(P,\mathcal{U}')$ being countably compact, totally bounded, Lindelöf, compact, or second countable.
Moreover, we claim that it is contrary to $(P,\mathcal{U}')$ being separable. Suppose that there is a countable dense subset E of P. Choose a symmetric $W \in \mathcal{U}$ such that $W^3 \subseteq V$. Let B be the uncountable subset of P which is defined in the proof of Lemma 4.3.1. By that proof for each $\alpha \in P$, $W'(\alpha)$ contains at most one element of B. Since E is a countable dense subset of P and B is an uncountable subset of P, there must exist distinct elements $\alpha, \beta \in B$ such that $W'(\alpha) \cap W'(\beta) \cap E \neq \emptyset$. This implies that $\beta \in W'(\alpha)$, a contradiction. We conclude that $\mathcal{U} = \{X \times X\}$, and the theorem is proved. □

Since the retracted scale P_R is made up of equivalence classes of elements of the scale P, the cardinality of P_R must be less than or equal to the cardinality of P. For certain uniform spaces the cardinalities are equal. Let T be a set. In the proof of the following theorem $\mathcal{P}(T)$ will denote the power set of T.

Theorem 4.3.3 *([32]) Let* X *be an infinite set and* $\mathcal{U}$ *a uniformity on* X *such that* $\mathcal{U} \neq \{X \times X\}$. *If* $card(X) = \kappa$, *then* $card(P_R) = 2^{2^\kappa} = card(P)$.

Proof. Since $P \subseteq \mathcal{P}(\mathcal{P}(X \times X))$, $card(P_R) \leq card(P) \leq 2^{2^\kappa}$. By Lemma 4.3.2 there exists $V \in \mathcal{U}$ such that $card(X - V(x)) = \kappa$ for some $x \in X$. Let $W \in \mathcal{U}$ be such that $W^3 \subseteq V$ and let $B = (X - V(x)) \times \{x\}$ where $x \in X$, and W and V are symmetric. Let $F(B)$ denote the collection of all filters on $X \times X$ which contain B. Then $card(F(B)) = 2^{2^\kappa}$. For each $\mathcal{F} \in F(B)$, let

$\alpha_{\mathcal{F}} = \langle W \rangle \cap \mathcal{F}$, and $A = \{\alpha_{\mathcal{F}} | \mathcal{F} \in F(B)\}$. Notice that if $\mathcal{F}, \mathcal{G} \in F(B)$ such that $\mathcal{F} \neq \mathcal{G}$, then $\alpha_{\mathcal{F}} \neq \alpha_{\mathcal{G}}$, which implies that $card(A) = 2^{2^{\kappa}}$. Also, note that $A \subseteq P$.

We claim that A is discrete. If $\alpha \in P$, and $\alpha_{\mathcal{F}}$, $\alpha_{\mathcal{G}} \in W'(\alpha)$, then $(\alpha_{\mathcal{F}}, \alpha_{\mathcal{G}}) \in (W^2)'$ since W is symmetric. If $\mathcal{F} \neq \mathcal{G}$, we may assume without loss of generality that there exists $F \in \mathcal{F} - \mathcal{G}$. For all $G \in \mathcal{G}$ the set $(X - F) \cap G$ is nonempty. Since $(\alpha_{\mathcal{F}}, \alpha_{\mathcal{G}}) \in (W^2)'$ and $W \cup F \in \alpha_{\mathcal{F}}$, $W^2 \circ (W \cup F) \in \alpha_{\mathcal{G}}$. Hence $W^2 \circ (W \cup F) \supseteq W \cup G$ for some $G \in \mathcal{G}$. Let (s, x) be a point in $G - F$. Then $(s, z) \in W \cup F$, and $(z, x) \in W^2$ for some $z \in X$. Now, (s, x) does not belong to $W^3 \subseteq V$ because $(s, x) \in G$ and G is contained in the complement of V. Therefore, $(s, z) \in F \subseteq B$, which implies that $z = x$ and $(s, x) \in F$. We have reached a contradiction to our choice of (s, x); hence, $W'(\alpha)$ contains at most one element of A. Therefore, A is a discrete subset of P, the closure of A in P is $\cup\{(\alpha_{\mathcal{F}})_R | \mathcal{F} \in F(B)\}$, and the image of the closure of A under φ is $\varphi(A)$.

Now P and P_R both contain discrete subsets of cardinality $2^{2^{\kappa}}$, so $card(P)$ and $card(P_R)$ are at last $2^{2^{\kappa}}$. Recalling that $card(P) \leq 2^{2^{\kappa}}$, we conclude that $card(P) = 2^{2^{\kappa}} = card(P_R)$. $\square$

Chapter 5

The scale of a quasi-uniform space

In this main chapter we introduce the scale of a quasi-uniform space. In [16] and [17], Kent studied the scale of a uniform space and its retracted scale. We summarized some of the results of his investigations in the first part of this thesis.

We will first define the scale of a quasi-uniform space and the retracted scale of a quasi-uniform space associated with it and we will establish the connection and similarities between our scale of a quasi-uniform space and the scale of a uniform space investigated by Kent [16] and Bushaw [6]. In particular we shall succeed in generalizing several results from the uniform to the quasi-uniform setting.

We will then define the prefilter space of a quasi-uniform space. That space is closely related to our (left-sided) scale of a quasi-uniform space. We will also show that the Hausdorff hyperspace quasi-uniform space is embedded into the prefilter space of a quasi-uniform space.

In the last section we will define the two-sided scale of a quasi-uniform space and we will show that total boundedness is preserved by this modified scale. When the results in this chapter are new, we shall give complete proofs.

5.1 Definition of the scale of a quasi-uniform space

In the following we shall start by defining the scale of a quasi-uniform space. First it is not clear which set of prefilters we should choose for a quasi-uniform space $(X,\mathcal{U})$: Prefilters of $\mathcal{U}$, of $\mathcal{U} \cup \mathcal{U}^{-1}$ or $\mathcal{U}^s$. We decided to make the following choice. Let $(X,\mathcal{U})$ be a quasi-uniform space and let $P_{\mathcal{U}^s} = \{\alpha : \alpha$ is prefilter on $X \times X$ and $\alpha \subseteq \mathcal{U}^s\}$. But see Section 5.6.

Definition 5.1.1 *Let $\alpha \in P_{\mathcal{U}^s}$. We define α^{-1} by*

$\alpha^{-1} = \{A^{-1} : A \in \alpha\}$. *Of course $\alpha^{-1} \in P_{\mathcal{U}^s}$, too.*

Definition 5.1.2 *For any two elements α and β in $P_{\mathcal{U}^s}$, $\alpha \leq \beta$ if $\alpha \supseteq \beta$.*

We next define the order of the scale.

Definition 5.1.3 *The order structure $(P_{\mathcal{U}^s}, \leq)$ is called the* order of the scale *of the quasi-uniform space $(X,\mathcal{U})$.*

Proposition 5.1.1 *(Compare Proposition 3.2.1) For any quasi-uniform space $(X,\mathcal{U})$, the order of the scale $(P_{\mathcal{U}^s}, \leq)$ is a complete distributive lattice whose suprema and infima are set intersections and set unions, respectively, and $\mathcal{U}^s$ is the last element and $\{X \times X\}$ is the greatest element.*

Proof. For the proof we refer to Proposition 3.2.1. □

Again it is not obvious how to define the quasi-uniformity of the scale of a quasi-uniform space. Our definition is motivated by the definition of the Hausdorff hyperspace quasi-uniformity of a quasi-uniform space.

We introduce the following notations:

Let X be a set. For any $R \subseteq X \times X$ and some prefilters α and β on $X \times X$.

We write $R \circ \alpha := \{R \circ A : A \in \alpha\}$,

and

$$\alpha \circ R := \{A \circ R : A \in \alpha\}$$

Proposition 5.1.2 *Let $(X,\mathcal{U})$ be a quasi-uniform space and let $P_{\mathcal{U}^s} = \{\alpha : \alpha$ is prefilter on $X \times X$ and $\alpha \subseteq \mathcal{U}^s\}$. For each $U \in \mathcal{U}$ we set*

$$S(U)_+ = \{(\alpha,\beta) \in P_{\mathcal{U}^s} \times P_{\mathcal{U}^s} : U \circ \alpha \subseteq \beta\}$$

and

$$S(U)_- = \{(\alpha,\beta) \in P_{\mathcal{U}^s} \times P_{\mathcal{U}^s} : \beta^{-1} \circ U \subseteq \alpha^{-1}\}.$$

Furthermore set $S(U) = S(U)_- \cap S(U)_+$ whenever $U \in \mathcal{U}$.
Then $\{S(U)_- : U \in \mathcal{U}\}$ is a base for the negative *quasi-uniformity $S(\mathcal{U})_-$ on $P_{\mathcal{U}^s}$ and $\{S(U)_+ : U \in \mathcal{U}\}$ is a base for the* positive *quasi-uniformity $S(\mathcal{U})_+$ on $P_{\mathcal{U}^s}$. Moreover $\{S(U) : U \in \mathcal{U}\}$ is a base for the* scale quasi-uniformity *$S(\mathcal{U})$.*

Proof. For each $U \in \mathcal{U}$ and any $\alpha \in P_{\mathcal{U}^s}$, we have $(\alpha,\alpha) \in S(U)_+$ and similarly $(\alpha,\alpha) \in S(U)_-$ and $(\alpha,\alpha) \in S(U)$. Observe also that $U, V \in \mathcal{U}$ with $U \subseteq V$ implies that $S(V)_+ \subseteq S(U)_+$ and similarly $S(V)_- \subseteq S(U)_-$, and $S(V) \subseteq S(U)$. Hence $\{S(U)_+ : U \in \mathcal{U}\}$, $\{S(U)_- : U \in \mathcal{U}\}$ and $\{S(U) : U \in \mathcal{U}\}$ are filter bases on $P_{\mathcal{U}^s} \times P_{\mathcal{U}^s}$.

Let $U \in \mathcal{U}$ and $V \in \mathcal{U}$ be such that $V^2 \subseteq U$. Let $(\alpha,\gamma) \in S(V)_+{}^2$. Then there is $\beta \in P_{\mathcal{U}^s}$ such that $(\alpha,\beta) \in S(V)_+$ and $(\beta,\gamma) \in S(V)_+$. Let $A \in \alpha$. Then $V \circ A \in \beta$. We have $V^2 \circ A \in \gamma$ implies $U \circ A \in \gamma$. We have shown that $(\alpha,\gamma) \in S(U)_+$. Similarly $S(V)_-{}^2 \subseteq S(U)_-$ and $S(V)^2 \subseteq S(U)$. We deduce that $S(\mathcal{U})_+$, $S(\mathcal{U})_-$ and $S(\mathcal{U})$ are quasi-uniformities on $P_{\mathcal{U}^s}$. □

The following lemma is similar to Proposition 3.2.3.

Lemma 5.1.1 *Let $(X,\mathcal{U})$ be a quasi-uniform space. For each $U \in \mathcal{U}$, $S(U^2) = S(U)^2$.*

Proof. Let $(\alpha,\gamma) \in S(U)^2$ which implies that there exists $\beta \in P_{\mathcal{U}^s}$ such that $(\alpha,\beta) \in S(U)$ and $(\beta,\gamma) \in S(U)$. Then $U \circ \alpha \subseteq \beta$ implies $U^2 \circ \alpha \subseteq \gamma$

and $\gamma^{-1} \circ U \subseteq \beta^{-1}$ implies $\gamma^{-1} \circ U^2 \subseteq \alpha^{-1}$. Thus $(\alpha, \gamma) \in S(U^2)$.

Conversely, let (α, γ) be in $S(U^2)$ and let β be the prefilter which contains all oversets of sets of the form $U \circ T$, for some $T \in \alpha$, or $V^{-1} \circ U$, for some $V \in \gamma$. So (α, β) and (β, γ) are both in $S(U)$, thus $(\alpha, \gamma) \in S(U)^2$. □

We next define the scale of a quasi-uniform space.

Definition 5.1.4 *Let $(X, \mathcal{U})$ be a quasi-uniform space. Then $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ is called the* quasi-uniform scale *of the quasi-uniform space $(X, \mathcal{U})$.*

The following definition is similar to Definition 4.1.4.

Definition 5.1.5 *Let M be a subset of $P_{\mathcal{U}^s}$. Given $\alpha, \beta \in P_{\mathcal{U}^s}$, we define their product as follows:*

$$\alpha \circ \beta = \{W \in \mathcal{U}^s : \textit{for some } U \in \alpha \textit{ and } V \in \beta, W \supseteq U \circ V\}.$$

Let

$$M \bullet \alpha = \{\beta \circ \alpha : \beta \in M\}. \textit{ Let } M \vee \alpha = \{\beta \vee \alpha : \beta \in M\}$$

and

$$M \wedge \alpha = \{\beta \wedge \alpha : \beta \in M\}.$$

The following proposition makes a connection between the scale of a quasi-uniform space and the scale of a uniform space investigated by Bushaw [6] and Kent [16] (see Chapter 3 of this thesis).

Proposition 5.1.3 *If $(X, \mathcal{U})$ is a uniform space then $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ gives the Kent-Bushaw scale.*

Proof. Suppose that $(X, \mathcal{U})$ is a uniform space. We show that $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ is the scale of the uniform space $(X, \mathcal{U})$ of Kent-Bushaw. Note that for $U \in \mathcal{U}$, we have $S(U \cap U^{-1}) \subseteq S(U)$ and $U \cap U^{-1} \in \mathcal{U}$ so that $\{S(U) : U \in \mathcal{U}, U$ symmetric $\}$ is a base of the scale. Indeed $P = P_{\mathcal{U}^s}$. Let $(\alpha, \beta) \in S(U)$. Then $A \in \alpha$ implies $U \circ A \in \beta$ and if $B \in \beta$ then $B^{-1} \circ U \in \alpha^{-1}$, so $(B^{-1} \circ U)^{-1} = U^{-1} \circ B \in \alpha$. Indeed $\mathcal{U}$ is a uniformity so that for each $U \in \mathcal{U}$ with $U^{-1} = U$ thus $(B^{-1} \circ U)^{-1} = U \circ B \in \alpha$; therefore $(\alpha, \beta) \in U'$. □

Proposition 5.1.4 *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $(P_{\mathcal{U}^s}, S(\mathcal{U})^{-1}) = (P_{\mathcal{U}^s}, S(\mathcal{U}^{-1}))$, where the latter space is the scale of $(X,\mathcal{U}^{-1})$.*

Proof. Let $U \in \mathcal{U}$. Let $(\alpha,\beta) \in S(U)_+{}^{-1}$ which implies $(\beta,\alpha) \in S(U)_+$. We have $U \circ \beta \subseteq \alpha$ implies $\beta^{-1} \circ U^{-1} \subseteq \alpha^{-1}$, so $(\alpha,\beta) \in S(U^{-1})_-$. Hence $S(U)_+{}^{-1} = S(U^{-1})_-$.

Let $(\alpha,\beta) \in S(U)_-{}^{-1}$ which implies $(\beta,\alpha) \in S(U)_-$. We have $\alpha^{-1} \circ U \subseteq \beta^{-1}$ implies $U^{-1} \circ \alpha \subseteq \beta$ so $(\alpha,\beta) \in S(U^{-1})_+$. Hence $S(U)_-{}^{-1} = S(U^{-1})_+$. Thus for each $U \in \mathcal{U}$, $S(U)^{-1} = S(U^{-1})$. □

Call two prefilters α and β on $X \times X$ equivalent if $(\alpha,\beta) \in \bigcap_{U\in\mathcal{U}} S(U) \cap S(U^{-1})$.

The following lemma shows when two prefilters are equivalent.

Lemma 5.1.2 *Let $(X,\mathcal{U})$ be a quasi-uniform space and $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ its quasi-uniform scale. Then $(\alpha,\beta) \in \bigcap_{U\in\mathcal{U}} S(U) \cap S(U^{-1})$ if and only if $\mathcal{U} \circ \alpha = \mathcal{U} \circ \beta$ and $\mathcal{U}^{-1} \circ \alpha = \mathcal{U}^{-1} \circ \beta$. We also note that for each $\alpha \in P_{\mathcal{U}^s}$, α is equivalent to $\mathcal{U} \circ \alpha \cup \mathcal{U}^{-1} \circ \alpha$.*

Proof. Indeed, suppose that $(\alpha,\beta) \in \bigcap_{U\in\mathcal{U}} S(U) \cap S(U^{-1})$. Let $U \in \mathcal{U}$, we have $(\alpha,\beta) \in S(U)$ and $(\alpha,\beta) \in S(U^{-1})$. Then $U \circ \alpha \subseteq \beta$, $\beta^{-1} \circ U \subseteq \alpha^{-1}$ and $U^{-1} \circ \alpha \subseteq \beta$, $U \circ \beta \subseteq \alpha$. By conjugation $\alpha^{-1} \circ U \subseteq \beta^{-1}$, $U \circ \beta \subseteq \alpha$. We have for each $U \in \mathcal{U}$, $U^2 \circ \alpha \subseteq U \circ \beta$ and $U^2 \circ \beta \subseteq U \circ \alpha$. Hence $\mathcal{U} \circ \alpha = \mathcal{U} \circ \beta$. Similarly $\mathcal{U}^{-1} \circ \alpha = \mathcal{U}^{-1} \circ \beta$. The converse is trivial. Since for instance $\mathcal{U} \circ \beta \subseteq \beta$ and $\mathcal{U}^{-1} \circ \beta \subseteq \beta$. □

The following proposition introduces a family of quasi-uniformly continuous functions from the original quasi-uniform space to its scale quasi-uniform space. It should be compared with Lemma 3.3.6.

Proposition 5.1.5 *Let $(X,\mathcal{U})$ be a quasi-uniform space, and let a be any fixed element of X. Let $\eta_a : X \to P_{\mathcal{U}^s}$ be defined at each x in X by $\eta_a(x) = \{V \in \mathcal{U}^s : (a,x) \in V\}$. Then η_a is a quasi-uniformly continuous map with respect to $\mathcal{U}$ on X and the scale quasi-uniformity $S(\mathcal{U})$ on $P_{\mathcal{U}^s}$.*

Proof. Choose an entourage U in $\mathcal{U}$, and let $(x, y) \in U$. We will show that $(\eta_a(x), \eta_a(y)) \in S(U)$.

Indeed, $V \in \eta_a(x)$ implies $(a, x) \in V$. Since $(x, y) \in U$, we have $(a, y) \in U \circ V$, and hence $U \circ V \in \eta_a(y)$. Let $W \in \eta_a(y)$ with $(a, y) \in W$, and thus $(y, a) \in W^{-1}$.

Since $(x, y) \in U$, we have $(x, a) \in W^{-1} \circ U$, and hence $W^{-1} \circ U \in \eta_a(x)^{-1}$.□

In the next lemma, we show that any T_0-quasi-uniform space can be quasi-uniformly embedded in its quasi-uniform scale space.

Lemma 5.1.3 *If $(X, \mathcal{U})$ is a T_0-quasi-uniform space and η_a is defined as in Proposition 5.1.5, then the map η_a is a quasi-uniform embedding.*

Proof. Indeed, from Proposition 5.1.5 η_a is quasi-uniformly continuous.

Let $x, y \in X$ be such that $\eta_a(x) = \eta_a(y)$. Let $U \in \mathcal{U}$. Since $U \circ \{(a, x)\} \in \eta_a(x) = \eta_a(y)$ we have $(a, y) \in U \circ \{(a, x)\}$. Thus $(x, y) \in U$. Similarly $(y, x) \in U$. Thus $(x, y) \in \bigcap_{U \in \mathcal{U}} U \cap \bigcap_{U \in \mathcal{U}} U^{-1} = \triangle$ and $x = y$.

Given $U \in \mathcal{U}$ there is $H \in \mathcal{U}$ such that $H^4 \subseteq U$. Suppose that for $x, y \in X$, $(\eta_a(x), \eta_a(y)) \in S(H)$. Then $H \circ \eta_a(x) \subseteq \eta_a(y)$ and $(\eta_a(y))^{-1} \circ H \subseteq (\eta_a(x))^{-1}$.

Consequently $(a, y) \in H \circ (H \cup \{(a, x)\})$ and $(a, x) \in (H \cup \{(y, a)\}) \circ H$. Then $(a, y) \in H^2$ and $(x, a) \in H^2$, or $(x, y) \in H$. In any case $(x, y) \in H^4 \subseteq U$. Hence ${\eta_a}^{-1} \mid \eta_a(X)$ is quasi-uniformly continuous as a map from $(\eta_a(X), S(\mathcal{U}) \mid \eta_a(X)) \to (X, \mathcal{U})$. □

We next give an example of a quasi-uniform scale space.

Example 5.1.1 *Let X be a set. If ρ is a reflexive and transitive relation on X, then $\mathcal{U} = \langle \{\rho\} \rangle$ is a quasi-uniformity on X. The scale $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ is the quasi-uniform space on $P_{\mathcal{U}^s}$ generated by the preorder $S(\rho)$.*

Indeed $\alpha \in P_{\mathcal{U}^s}$ if and only if α is a family of relations that contain ρ^s and if $A \in \alpha$ and $A \subseteq B$ then $B \in \alpha$.

$(\alpha, \beta) \in S(\rho) \Longleftrightarrow$ *(if* $A \in \alpha$ *then* $\rho \circ A \in \beta$ *and if* $B \in \beta$ *then* $\rho^{-1} \circ B \in \alpha$*).*

$S(\rho)$ *is reflexive because it contains the diagonal.*

Let $(\alpha, \beta) \in S(\rho)^2$*. This implies there exists* $\gamma \in P_{\mathcal{U}^s}$ *such that* $(\alpha, \gamma) \in S(\rho)$ *and* $(\gamma, \beta) \in S(\rho)$*. If* $A \in \alpha$ *which implies* $\rho^2 \circ A \in \beta$ *by* $\rho^2 \subseteq \rho$ *then* $\rho \circ A \in \beta$*. Reciprocally if* $B \in \beta$ *which implies* $\rho^{-2} \circ B \in \alpha$ *then* $\rho^{-1} \circ B \in \alpha$*. Thus* $S(\rho)$ *is transitive, as expected.*

5.2 The retracted scale of a quasi-uniform space

Starting with the scale quasi-uniform space $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ of a quasi-uniform space $(X, \mathcal{U})$ in this section, we will define the retracted scale quasi-uniform space.
We recall that $\alpha \sim \beta$ if $\mathcal{U} \circ \alpha = \mathcal{U} \circ \beta$ and $\mathcal{U}^{-1} \circ \alpha = \mathcal{U}^{-1} \circ \beta$ by Lemma 5.1.2.

Definition 5.2.1 *Let* $\alpha \in P_{\mathcal{U}^s}$*. We set:*

$$\alpha_R = \{\beta \in P_{\mathcal{U}^s} : \alpha \sim \beta\}$$

and

$$(P_{\mathcal{U}^s})_R = \{\alpha_R : \alpha \in P_{\mathcal{U}^s}\}$$

We introduce an partial order on $(P_{\mathcal{U}^s})_R$ as follows:

Definition 5.2.2 *For each prefilter* $\alpha \in P_{\mathcal{U}^s}$*, we associate the prefilter* $\alpha^{\mathcal{U}}$ *defined by*

$\alpha^{\mathcal{U}} = \mathcal{U} \circ \alpha$.

We introduce an partial order on $(P_{\mathcal{U}^s})_R$ as follows:

Definition 5.2.3 *Let* α_R *and* β_R *in* $(P_{\mathcal{U}^s})_R$*,* $\alpha_R \leq \beta_R$ *means* $\beta^{\mathcal{U}} \cup \beta^{\mathcal{U}^{-1}} \subseteq \alpha^{\mathcal{U}} \cup \alpha^{\mathcal{U}^{-1}}$.

Lemma 5.2.1 *The relation* $\leq$ *defined above is an partial order on* $(P_{\mathcal{U}^s})_R$.

Proof. We show that $\leq$ is well defined. If $\alpha_1 \sim \alpha_2$ and $\beta_1 \sim \beta_2$ such that ${\beta_1}^{\mathcal{U}} \cup {\beta_1}^{\mathcal{U}^{-1}} \subseteq {\alpha_1}^{\mathcal{U}} \cup {\alpha_1}^{\mathcal{U}^{-1}}$ then ${\beta_2}^{\mathcal{U}} \cup {\beta_2}^{\mathcal{U}^{-1}} \subseteq {\alpha_2}^{\mathcal{U}} \cup {\alpha_2}^{\mathcal{U}^{-1}}$.

The relation $\leq$ is reflexive, transitive by dual inclusion of prefilters and antisymmetric, since $\mathcal{U} \circ \alpha \cup \mathcal{U}^{-1} \circ \alpha = \mathcal{U} \circ \beta \cup \mathcal{U}^{-1} \circ \beta$ implies $\mathcal{U} \circ \alpha = \mathcal{U} \circ \beta$ and $\mathcal{U}^{-1} \circ \alpha = \mathcal{U}^{-1} \circ \beta$. □

Definition 5.2.4 *The order structure* $((P_{\mathcal{U}^s})_R, \leq)$ *is called the* order retracted scale *of the quasi-uniform space* $(X, \mathcal{U})$.

The next lemma shows that the canonical map is order-preserving.

Lemma 5.2.2 *Let* ψ *be the canonical map from* $P_{\mathcal{U}^s}$ *into* $(P_{\mathcal{U}^s})_R$ *defined by* $\psi(\alpha) = \alpha_R$, *for* $\alpha \in P_{\mathcal{U}^s}$. *Then* ψ *is order-preserving.*

Proof. Let α and β in $P_{\mathcal{U}^s}$ such that $\alpha \leq \beta$. We show that $\beta^{\mathcal{U}} \cup \beta^{\mathcal{U}^{-1}} \subseteq \alpha^{\mathcal{U}} \cup \alpha^{\mathcal{U}^{-1}}$.

Indeed, $\alpha \leq \beta$, which implies $\beta \subseteq \alpha$. Hence $\mathcal{U} \circ \beta \subseteq \mathcal{U} \circ \alpha$ and $\mathcal{U}^{-1} \circ \beta \subseteq \mathcal{U}^{-1} \circ \alpha$. Then we have $\mathcal{U} \circ \beta \cup \mathcal{U}^{-1} \circ \beta \subseteq \mathcal{U} \circ \alpha \cup \mathcal{U}^{-1} \circ \alpha$. □

Proposition 5.2.1 *For any pair of elements* α_R, β_R *in* $(P_{\mathcal{U}^s})_R$, *we have:*

$$\alpha_R \vee \beta_R = (\alpha \vee \beta)_R$$

and

$$\alpha_R \wedge \beta_R = (\alpha \wedge \beta)_R.$$

Proof. We leave it to the reader to check that these equalities make sense. □

The following proposition is similar to Proposition 3.3.3.

Proposition 5.2.2 *The order retracted scale of any quasi-uniform space is a complete distributive lattice. The operations* sup *and* inf *in* $(P_{\mathcal{U}^s}, \leq)$ *are preserved under the canonical map* ψ.

Proof. To see that the sup operation is invariant, let $A \subseteq P_{\mathcal{U}^s}$, $B = \psi(A)$, and $\alpha = \sup(A)$; we shall show that $\psi(\alpha) = \sup(B)$. Note $\psi(\alpha) \geq \gamma_R$, for all γ_R, since ψ is order-preserving. If $\beta_R \geq \gamma_R$ for all $\gamma_R \in B$, $\beta \in \beta_R$ and $\gamma \in A$ then $\beta \geq \gamma$ so $\beta \subseteq \gamma \subseteq \alpha$ thus $\beta_R = \psi(\beta) \geq \psi(\alpha)$. A similar argument establishes that $\psi(\inf(A)) = \inf(B)$. Since ψ is an onto map, $(P_{\mathcal{U}^s}, \leq)$ is a complete lattice. The fact that the retracted scale is distributive follows from Proposition 5.2.1. □

We next define the retracted scale quasi-uniformity $S(\mathcal{U})_R$. We recall that with each $U \in \mathcal{U}$ we have an associated entourage $S(U)$ in the scale quasi-uniformity $S(\mathcal{U})$.

Definition 5.2.5 *Let $(X,\mathcal{U})$ be a quasi-uniform space. For each $U \in \mathcal{U}$,*

$$S(U)_R = \psi(S(U)) = \{(\alpha_R, \beta_R) : (\alpha, \beta) \in S(U)\}.$$

We next define the retracted scale quasi-uniform space.

Definition 5.2.6 *The quotient space $((P_{\mathcal{U}^s})_R, S(\mathcal{U})_R)$ of $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ under the equivalence relation $\sim$ is called the* retracted scale quasi-uniform space *associated with the quasi-uniform space $(X,\mathcal{U})$.*

The next proposition observes that the topology $\tau(S(\mathcal{U})_R)$ of the retracted scale of a quasi uniform space $(X,\mathcal{U})$ is T_0.

Proposition 5.2.3 *The retracted scale quasi-uniform space $((P_{\mathcal{U}^s})_R, S(\mathcal{U})_R)$ associated with an arbitrary quasi-uniform space $(X,\mathcal{U})$ is T_0 for the topology associated with $\tau(S(\mathcal{U})_R)$ and is the quotient quasi-uniform space structure derived from $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ under the map ψ.*

Proof. This is a consequence of Proposition 5.2.2. □

The next definition should be compared with Section 3.4 of Chapter 3.

Definition 5.2.7 *Let $(X,\mathcal{U})$ be a quasi-uniform space and ρ a quasi-order (reflexive and transitive binary relation) on X. A set $A \subseteq X$ will be called* stable *relative to ρ and $\mathcal{U}$ if, for every $V \in \mathcal{U}$, there exists a $W \in \mathcal{U}$ such that $\rho(W(A)) \subseteq V(A)$.*

The next proposition shows that the composition of the canonical mapping ψ from the scale of a quasi-uniform space into the retracted scale and the quasi-uniformly continuous map v_a (defined below) from the original quasi-uniform space into its scale is a Liapunov function. A Liapunov function on a quasi-uniform space is defined analogously to the Definition 3.4.1.

Proposition 5.2.4 *Let $(X,\mathcal{U})$ be a quasi-uniform space and $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ its scale. If A is a stable subset of X then $\psi \circ v_a$ is a Liapunov function, where a is a fixed element of A, ψ is the canonical map of $P_{\mathcal{U}^s}$ into $(P_{\mathcal{U}^s})_R$, $v_a(x) = \{V \in \mathcal{U}^s : \rho(x) \subseteq V(a)\}$, whenever $x \in X$ and ρ is a quasi-order relation on X.*

Proof. We suppose that A is stable and N an invariant neighborhood of A. We take $N = X$ and for each $x \in X$, set $v(x) = \psi(\{V \in \mathcal{U}^s : \rho(x) \subseteq V(a)\})$ to simplify the notation.

We shall show that v is a Liapunov function for A.
Let $(x, y) \in \rho$ and $\rho(x) \subseteq V(a)$, by transitivity of ρ, $\rho(y) \subseteq \rho^2(x) = \rho(x)$, so $\rho(y) \subseteq V(a)$. Thus $\{V \in \mathcal{U}^s : \rho(x) \subseteq V(a)\} \subseteq \{V \in \mathcal{U}^s : \rho(y) \subseteq V(a)\}$, and the property of order-preserving of ψ gives $v(x) \subseteq v(y)$ so $v(y) \leq v(x)$.

Let $\gamma_R \in (P_{\mathcal{U}^s})_R - \{\mathcal{U}\}$ and let $V \in \mathcal{U} - \bigcup\gamma_R$. Choose $W \in \mathcal{U}$ so that $\rho W(a) \subseteq V(a)$. We suppose that $x \in W(a)$. If we had $v(e) \geq \gamma_R$, then we would have $V \in \{V \in \mathcal{U}^s : \rho(x) \subseteq V(a)\} \subseteq v(x) \subseteq \gamma_R$, in contradiction to the choice of V.

Let $V \in \mathcal{U}$. If $V(a) = X$, there exists a $\gamma_R \in (P_{\mathcal{U}^s})_R$ such that $v(x) \geq \gamma_R$ where $x \in N - V(a)$. So suppose that $V(a) \neq X$ and let $W^2 \subseteq V$. Put $\alpha_R = \bigcup\{v(x) : x \notin V(a)\}$. We shall show that $W \notin \alpha_R$. In the contrary case we would have $W \in v(x)$ for some $x \notin V(a)$. If $W \in v(x)$, then by definition of v, $\rho(x) \subseteq W(a) \subseteq W^2(a) \subseteq V(a)$. Since ρ is reflexive, this implies $x \in V(a)$, a contradiction. Since $W \notin \alpha_R$, we have $\alpha_R \neq 0$ and $\gamma_R = \psi(\alpha) \neq 0$. If $x \notin V(a)$, then $v(x) \subseteq \alpha \subseteq \gamma$, which means that $v(x) \geq \gamma_R$, and γ_R meets the requirements. □

5.3 Properties of the scale of a quasi-uniform space

In this section we shall investigate some properties of the scale of a quasi-uniform space.

Bicompletion of the scale of a quasi-uniform space

We next state a lemma that turns out to be useful in the proof that any quasi-uniform scale space is indeed bicomplete (see Lemma 3.3.4).

Lemma 5.3.1 *Let Υ be an ultrafilter on $P_{\mathcal{U}^s}$. Then $\cup_{Z\in\Upsilon}\cap_{\zeta\in Z}\zeta = \cap_{Z\in\Upsilon}\cup_{\zeta\in Z}\zeta$.*

Proof. The proof is similar to the one of Lemma 3.3.4 (see also Lemma 5.5.2). □

The following theorem generalizes Theorem 3.3.2 from the theory dealing with uniform spaces.

Theorem 5.3.1 *For any quasi-uniform space $(X,\mathcal{U})$, $(P_{\mathcal{U}^s},S(\mathcal{U}))$ and $((P_{\mathcal{U}^s})_R,S(\mathcal{U})_R)$ are both bicomplete quasi-uniform spaces.*

Proof. (1) Let Ξ be a filter on $P_{\mathcal{U}^s}$ that is $S(\mathcal{U})^s$-Cauchy. We show that Ξ $\tau(S(\mathcal{U})^s)$-converges to $\gamma := \cup_{Z\in\Upsilon}\cap_{\zeta\in Z}\zeta = \cap_{Z\in\Upsilon}\cup_{\zeta\in Z}\zeta$ where Ξ is an ultrafilter on $P_{\mathcal{U}^s}$ containing Ξ. Let $U \in \mathcal{U}$. By assumption there is $G \in \Xi$ such that $G \times G \subseteq S(U)$. Let $\xi' \in G$ be arbitrary. Then $U \circ \xi' \subseteq \xi$ whenever $\xi \in G$ by definition of $S(U)$.

Thus $U \circ \xi' \subseteq \cap_{\xi\in G}\xi \subseteq \gamma$. So $U \circ \xi' \subseteq \gamma$. Let $W \in \gamma^{-1}$ be arbitrary. Then $W \in \cup_{\xi\in G}\xi^{-1}$, since $G \in \Upsilon$.

Thus there is $\xi \in G$ such that $W \in \xi^{-1}$. Furthermore $(\xi',\xi) \in S(U)$ and thus $W \circ U \in \xi^{-1} \circ U \subseteq (\xi')^{-1}$. We conclude that $\gamma^{-1} \circ U \subseteq (\xi')^{-1}$. Therefore $G \subseteq (S(U))^{-1}(\gamma)$ and Ξ $\tau(S(\mathcal{U}^{-1}))$-converges to γ.

Let $\xi' \in G$ be arbitrary. Then $U \circ \xi \subseteq \xi'$ whenever $\xi \in G$ by definition of $S(U)$. Thus $U \in \gamma \subseteq U\circ \cup_{\xi\in G}\xi \subseteq \xi'$ and so $U \circ \gamma \subseteq \xi'$. Let $W \in (\xi')^{-1}$ be

arbitrary. We have that $(\xi, \xi') \in S(U)$ and thus $W \circ U \in (\xi')^{-1} \circ U \subseteq \xi^{-1}$.

Hence $W \circ U \in \cap_{\xi \in G} \xi^{-1} \subseteq \gamma^{-1}$. So $W \circ U \in (\xi')^{-1} \circ U \subseteq \gamma^{-1}$. Consequently $G \subseteq (S(U))(\gamma)$ and Ξ $\tau(S(\mathcal{U}))$-converges to γ. Thus $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ is bicomplete.

Next, let Γ be a Cauchy filter on $(P_{\mathcal{U}^s})_R$. It follows from Proposition 5.2.2 that $\psi^{-1}(\Gamma)$ is a Cauchy filter on $P_{\mathcal{U}^s}$. If $\psi^{-1}(\Gamma)$ converge to α in the topology compatible with $S(\mathcal{U})$, then $\Gamma = \psi(\psi^{-1}(\Gamma))$ τ-converges to α_R by Proposition 5.2.2. □

The next proposition describes an induced map between two scales of quasi-uniform spaces when the original quasi-uniform spaces are connected by a quasi-uniformly continuous map.

Proposition 5.3.1 *Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be quasi-uniform spaces and $f : (X, \mathcal{U}) \longrightarrow (Y, \mathcal{V})$ a quasi-uniformly continuous map.*
If $\alpha \in P_{\mathcal{U}^s}$, then $(f \times f)(\alpha) \cap \mathcal{V}^s \in P_{\mathcal{V}^s}$. Furthermore the map $f_S : (P_{\mathcal{U}^s}, S(\mathcal{U})) \longrightarrow (P_{\mathcal{V}^s}, S(\mathcal{V}))$ defined by $f_S(\alpha) = (f \times f)(\alpha) \cap \mathcal{V}^s$ is quasi-uniformly continuous.

Proof. Let $V \in \mathcal{V}$. By quasi-uniform continuity of f there is $U \in \mathcal{U}$ such that $(f \times f)U \subseteq V$. Since $\alpha \in P_{\mathcal{U}^s}$, $\alpha \subseteq \mathcal{U}^s$ and then $(f \times f)(\alpha) \cap \mathcal{V}^s \in P_{\mathcal{V}^s}$.

It remains to be shown that f_S is quasi-uniformly continuous. Let $V \in \mathcal{V}$, as above, there is $U \in \mathcal{U}$ such that $(f \times f)U \subseteq V$. Consider $(\alpha, \beta) \in S(U)$, then $U \circ \alpha \subseteq \beta$ and $\beta^{-1} \circ U \subseteq \alpha^{-1}$. Consequently $V \circ ((f \times f)(\alpha) \cap \mathcal{V}^s) \subseteq (f \times f)(\beta) \cap \mathcal{V}^s$ and $(((f \times f)(\beta))^{-1} \cap \mathcal{V}^s) \circ V \subseteq ((f \times f)(\alpha))^{-1} \cap \mathcal{V}^s$. Hence the map f_S is quasi-uniformly continuous. □

The next theorem introduces a functor from the category of quasi-uniform spaces into itself.

Theorem 5.3.2 *Let QU be the category of quasi-uniform spaces where the quasi-uniformly continuous maps are the morphisms. If $(X, \mathcal{U})$ is a quasi-uniform space, we set $S((X, \mathcal{U})) := P_{\mathcal{U}^s}$ and $S(f) := f_S$ where f is a quasi-uniformly continuous map from $(X, \mathcal{U})$ into $(Y, \mathcal{V})$. Then S is a functor from QU into itself.*

Proof. Let $f : (X,\mathcal{U}) \longrightarrow (Y,\mathcal{V})$ and $g : (Y,\mathcal{V}) \longrightarrow (Z,\mathcal{W})$ quasi-uniformly continuous maps. We must show that $S(g \circ f) = S(g) \circ S(f)$ and $S(id_{(X,\mathcal{U})}) = id_{S((X,\mathcal{U}))}$.

Indeed, let $\alpha \in P_{\mathcal{U}^s}$ then $(g_S \circ f_S)(\alpha) = g_S(f_S(\alpha)) = ((g \times g) \circ (f \times f))(\alpha) \cap \mathcal{W}^s = (g \circ f \times g \circ f)(\alpha) \cap \mathcal{W}^s = (g \circ f)_S(\alpha)$.

Let $\alpha \in P_{\mathcal{U}^s}$. Then $id_S(\alpha) = (id \times id)(\alpha) \cap \mathcal{U}^s = id_{P_{\mathcal{U}^s}}(\alpha)$. □

Lemma 5.3.2 *Let X and Y be sets and let $f : X \longrightarrow Y$ be a surjective map. Let A, B and U be subsets of $Y \times Y$. If $B \subseteq U \circ A$ then $(f \times f)^{-1}B \subseteq ((f \times f)^{-1}U) \circ ((f \times f)^{-1}A)$.*

Proof. Let $(x, y) \in (f \times f)^{-1}B$. We have $(f(x), f(y)) \in B$ which implies there exists $z \in Y$ such that $(f(x), z) \in A$ and $(z, f(y)) \in U$. By surjectivity of f there exists $c \in X$ such that $f(c) = z$. Hence $(x, c) \in (f \times f)^{-1}A$ and $(c, y) \in (f \times f)^{-1}U$ which implies $(x, y) \in ((f \times f)^{-1}U) \circ ((f \times f)^{-1}A)$. □

Proposition 5.3.2 *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be quasi-uniform spaces and $f : (X,\mathcal{U}) \longrightarrow (Y,\mathcal{V})$ be a quasi-uniformly continuous surjective map.*
If $\alpha \in P_{\mathcal{V}^s}$, we set $(f \times f)^{-1}(\alpha) :=$ prefilter coarser than $\mathcal{U}^s$ generated by $\{(f \times f)^{-1}(A) : A \in \alpha\}$. Then $(f \times f)^{-1}(\alpha) \in P_{\mathcal{U}^s}$. Furthermore the map $f_T : (P_{\mathcal{V}^s}, S(\mathcal{V})) \longrightarrow (P_{\mathcal{U}^s}, S(\mathcal{U}))$ defined by $f_T(\alpha) = (f \times f)^{-1}(\alpha)$ is quasi-uniformly continuous.

Proof. Let $V \in \mathcal{V}$. By quasi-uniform continuity of f there is $U \in \mathcal{U}$ such that $(f \times f)U \subseteq V$. Since $\alpha \in P_{\mathcal{V}^s}$ implies $\alpha \subseteq \mathcal{V}^s$, then $(f \times f)^{-1}(\alpha) \subseteq \mathcal{U}^s$.

It remains to be shown that f_T is quasi-uniformly continuous. Let $V \in \mathcal{V}$, as above. There exists $U \in \mathcal{U}$ such that $U \subseteq (f \times f)^{-1}V$. Consider $(\alpha, \beta) \in S(U)$. Then $U \circ \alpha \subseteq \beta$ and $\beta^{-1} \circ U \subseteq \alpha^{-1}$. Consequently $V \circ (f \times f)^{-1}(\alpha) \subseteq (f \times f)^{-1}(\beta)$ and $(f \times f)^{-1}(\beta^{-1}) \circ V \subseteq (f \times f)^{-1}(\alpha^{-1})$.□

Connectedness of the scale of a quasi-uniform space

In the next paragraphs we study connectedness of the quasi-uniform scale of a quasi-uniform space.

Let $(X,\mathcal{U})$ be a quasi-uniform space and $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ its quasi-uniform scale. It will be convenient to introduce the following "closed interval" notation:

For $\alpha \leq \beta$, let $[\alpha, \beta] = \{\gamma \in P_{\mathcal{U}^s} : \alpha \leq \gamma \leq \beta\}$. For each $U \in \mathcal{U}$ and $\alpha \in P_{\mathcal{U}^s}$, let

$$\alpha^U = U \circ \alpha,$$

$$\alpha_{U^{-1}} = \{V \in \mathcal{U}^s : U^{-1} \circ V \in \alpha\}.$$

Lemma 5.3.3 *Let $U \in \mathcal{U}$. Then $S(U)(\alpha) = [\alpha_{U^{-1}}, \alpha^U]$ for each $\alpha \in P_{\mathcal{U}^s}$.*

Proof. Suppose that $\beta \in S(U)(\alpha)$. We show that $\alpha^U \subseteq \beta \subseteq \alpha_{U^{-1}}$.

Let $V \in \alpha^U$. Then for some $F \in \alpha$, $U \circ F \subseteq V \in \beta$. Let $G \in \beta$. Then $U^{-1} \circ G \in \alpha$ implies $G \in \alpha_{U^{-1}}$.

Conversely, if $V \in \alpha^U$ then for some $F \in \alpha$, $U \circ F \subseteq V \in \beta$. If $G \in \beta$ then $G \in \alpha_{U^{-1}}$ implies $U^{-1} \circ G \in \beta$. Thus $(\alpha, \beta) \in S(U)$. □

The following proposition is similar to Proposition 4.1.1.

Proposition 5.3.3 *The collection $\{[\alpha_{U^{-1}}, \alpha^U] : U \in \mathcal{U}\}$ forms a basic system of neighborhoods at α for the quasi-uniform topology on $P_{\mathcal{U}^s}$ generated by $S(\mathcal{U})$.* □

Consider the case $\alpha = 0$. Here $\mathcal{U}^s$ is the last element of $P_{\mathcal{U}^s}$. Note that $0_U = 0$. Recall that $\langle U \rangle$ denotes the prefilter consisting of all oversets of U. Since $0^V \leq \langle U \rangle \leq 0^U$ for $V^2 \subseteq U$, it follows from Proposition 5.3.3 that the collection $\{[0, \langle U \rangle] : U \in \mathcal{U}\}$ is a basic neighborhood system for 0.

One can show that the lattice operations $\vee$ and $\wedge$ in $P_{\mathcal{U}^s}$ are quasi-uniformly continuous (compare Theorem 3.3.1) and that for any $\alpha \in P_{\mathcal{U}^s}$ $\beta \mapsto \beta \circ \alpha$ is quasi-uniformly continuous.

Proposition 5.3.4 *If M is a connected (arcwise connected) subset of $P_{\mathcal{U}^s}$, then the sets $M \bullet \alpha$, $M \vee \alpha$ and $M \wedge \alpha$ are likewise connected (arcwise connected) for any choice of α.*

Proof. See [19, Proposition 3]. □

Proposition 5.3.5 *If there is a connected (arcwise connected) subset C of $P_{\mathcal{U}^s}$ containing α and β, then $[\alpha \wedge \beta, \alpha \vee \beta]$ is connected (arcwise connected).*

Proof. The sets $C \vee \alpha$ and $C \wedge \beta$ are connected sets which contain α, and $A = (C \vee \alpha) \cup (C \wedge \alpha)$ is a connected set which contains both $\alpha \vee \beta$ and $\alpha \wedge \beta$. If $\gamma \in [\alpha \wedge \beta, \alpha \vee \beta]$, $A \vee \gamma$ is a connected set which contains γ and $\alpha \vee \beta$. Thus

$$B = \cup\{A \vee \gamma : \gamma \in [\alpha \wedge \beta, \alpha \vee \beta]\}$$

is a connected set which includes $[\alpha \wedge \beta, \alpha \vee \beta]$ as a subset. But

$$[\alpha \wedge \beta, \alpha \vee \beta] = (B \vee (\alpha \wedge \beta)) \wedge (\alpha \vee \beta),$$

and the proof is complete. □

Corollary 5.3.1 *The scale of a quasi-uniform space is connected (arcwise connected) if 0 and 1 belong to the same connected (arcwise connected) component.*

Proof. This follows from Proposition 5.3.5. □

Corollary 5.3.2 *If the closed interval $[\alpha, \beta]$ in $P_{\mathcal{U}^s}$ is a subset of a connected (arcwise connected) set M, then $[\alpha, \beta]$ is connected (arcwise connected).*

Proof. This is a consequence of Corollary 5.3.1. □

Corollary 5.3.3 *The quasi-uniform scale of a quasi-uniform space $(X, \mathcal{U})$ is locally connected (locally arcwise connected) if and only if there is a connected (arcwise connected) neighborhood for each point of $(P_{\mathcal{U}^s}, \tau(S(\mathcal{U})))$.*

Proof. This follows from Corollary 5.3.2 and Proposition 5.3.3. □

Total boundedness of the quasi-uniform scale of a quasi-uniform space

We will need the following proposition to show under which condition the scale of a quasi-uniform space is totally bounded.

Proposition 5.3.6 *Let $(X, \mathcal{U})$ be a quasi-uniform space. If X is infinite and $\mathcal{U}$ distinct from $\{X \times X\}$, then there is $U \in \mathcal{U}$ such that $(X \times X) \backslash U^2$ is infinite.*

Proof. Indeed, otherwise for any $V \in \mathcal{U}$ and any $x, y \in X$, we have that both $X \backslash V^2(x)$ and $X \backslash V^2(y)$ are finite and thus there is $z \in V^2(x) \cap V^{-2}(y)$, since X is infinite. Hence $(x, y) \in V^4$. We conclude that $\mathcal{U} = \{X \times X\}$, a contradiction. □

Remark 5.3.1 *Let $\mathcal{U}$ be distinct from $\{X \times X\}$. By the assertion just proved there exists $U \in \mathcal{U}$ such that for each $n \in \omega$ we can choose $(a_n, b_n) \in (X \times X) \backslash U^2$ such that Case 1 $(a_n)_{n \in \omega}$ is injective or Case 2 $(b_n)_{n \in \omega}$ is injective.*

In Case 1 for each $n \in \omega$ set $\alpha_n = \langle U^s \cup \{(a_n, b_n)\}\rangle$, in case 2 for each $n \in \omega$ set $\beta_n = \langle U^s \cup \{(b_n, a_n)\}\rangle$. Observe that for each $n \in \omega$, $\alpha_n, \beta_n \in P_{\mathcal{U}^s}$.

In Case 1 we have for all $n, m \in \omega$ with $n \neq m$ that $U \circ \alpha_n$ is not a subset of α_m, since $(a_m, b_m) \notin U \circ (U^s \cup \{(a_n, b_n)\}$, because $a_m \neq a_n$ and $(a_m, b_m) \notin U^2$. In Case 2 for all $n, m \in \omega$ with $n \neq m$ we have that $U^{-1} \circ \beta_n$ is not a subset of β_m, since $(b_m, a_m) \notin U^{-1} \circ (U^s \cup \{(b_n, a_n)\})$, because $b_m \neq b_n$ and $(a_m, b_m) \notin U^2$. This remark will be used in the proof of Theorem 5.3.4.

The following theorem generalizes Proposition 4.3.3 from the theory of uniform spaces.

Theorem 5.3.3 *Let $(X, \mathcal{U})$ be a quasi-uniform space. Then $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ is totally bounded if and only if X is finite or $\mathcal{U} = \{X \times X\}$.*

Proof. Let X be finite. Then $P_{\mathcal{U}^s}$ is finite and thus $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ is totally bounded. If $\mathcal{U} = \{X \times X\}$, then $S(\mathcal{U}) = \{P_{\mathcal{U}^s} \times P_{\mathcal{U}^s}\}$, hence $S(\mathcal{U})$ is totally bounded, too.

Conversely, suppose that X is infinite and $\mathcal{U}$ distinct from $\{X \times X\}$. By Proposition 5.3.6 there is $U \in \mathcal{U}$ such that $(X \times X)\backslash U$ is an infinite set of (a_n, b_n)'s where $a_n \neq a_m$ whenever $n \neq m$ or $b_n \neq b_m$ whenever $n \neq m$. For each $n \in \omega$ set (Case 1) $\alpha_n = \langle U^s \cup \{(a_n, b_n)\}\rangle$ and (Case 2) $\beta_n = \langle U^s \cup \{(b_n, a_n)\}\rangle$. (See Remark 5.3.1).

Hence in Case 1 for all $m, n \in \omega$ with $n \neq m$ we have $(\alpha_n, \alpha_m) \notin S(U)$. Similarly in Case 2 for all $m, n \in \omega$ with $n \neq m$ we have $(\beta_n, \beta_m) \notin S(U)$. Thus in either case we have found an infinite discrete subspace of $(P_{\mathcal{U}^s}, S(\mathcal{U}))$. Hence $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ is not totally bounded. □

Our next example shows that given a quasi-uniform space $(X, \mathcal{U})$ the quasi-uniformity $S(\mathcal{U})_+$ on $P_{\mathcal{U}^s}$ can be totally bounded under weaker conditions than those given in the theorem above.

Example 5.3.1 *Let $X = \omega$ and let F be a finite subset of ω. Set $V_F = [(X\backslash F) \times X] \cup \triangle_\omega$. Consider the quasi-uniformity $\mathcal{U}$ on X generated by the base $\{V_F : F \subseteq \omega$ and F is finite$\}$. We are going to show that $(P_{\mathcal{U}^s}, S(\mathcal{U})_+)$ is totally bounded.*

Proof. Fix $F \subseteq X$ finite. Let B be any reflexive binary relation on X. Set $B_F = \{x \in F : B(x) \subseteq F\}$. Then $V_F \circ B = [(X\backslash B_F) \times X] \cup (\bigcup_{x \in B_F} \{x\} \times B(x))$.

Indeed: Case 1, let $x \in X\backslash F$. Then $V_F = X$, so $(x, x) \in B$ yields the result $X = (V_F \circ B)(x)$.

Case 2, let $x \in F\backslash B_F$. Then there exists $b \in B(x)\backslash F$. Therefore $(x, b) \in B$ and $V_F(b) = X$. The assertion $X = (V_F \circ B)(x)$ follows.

Case 3, let $x \in B_F$. Then $B_F(x) \subseteq F$. Thus $(x, c) \in B$ and $(c, d) \in V_F$ implies that $c = d$. Therefore $(V_F \circ B)(x) = B(x)$.

Altogether for each $x \in X$ such that $B(x) \subseteq F$ we get that $(V_F \circ B)(x) = B(x)$ and $(V_F \circ B)(x) = X$ otherwise. We conclude that $\mathcal{M}_F = \{V_F \circ B : B$

is a reflexive relation on $X\}$ is finite.

For any nonempty collection $\mathcal{M}'$ of $\mathcal{M}_F$ we consider the prefilter $\gamma_{\mathcal{M}'} = \bigcup\{\langle M'\rangle : M' \in \mathcal{M}'\}$.

Let $\beta \in P_{\mathcal{U}^s}$ be arbitrary. Then $\mathcal{N} = \{V_F \circ B : B \in \beta\}$ is a nonempty finite subcollection of $\mathcal{M}_F$. Hence $\gamma_{\mathcal{N}} = \bigcup\{\langle V_F \circ B\rangle : B \in \beta\}$. Obviously $V_F \circ B \subseteq \gamma_{\mathcal{N}}$, since $\mathcal{N} \subseteq \gamma_{\mathcal{N}}$.

Furthermore $V_F \circ \gamma_{\mathcal{N}} \subseteq \beta$, since $C \in \gamma_{\mathcal{N}}$ implies that there is $B \in \beta$ such that $V_F \circ B \subseteq C$, and so $B \subseteq V_F \circ (V_F \circ B) \subseteq V_F \circ C$, because V_F is reflexive. We conclude that $S(\mathcal{U})_+$ is totally bounded. □

5.4 Quasi-pseudometrizability of the scale

We use the notion of an écart in a quasi-uniform space that was introduced in Chapter 1 for a uniform space. The definition of an écart is the same as in Definition 3.5.1, except that the condition (2) introducing the symmetry of the écart will be omitted.

Let $(X,\mathcal{U})$ be an arbitrary quasi-uniform space, and $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ its quasi-uniform scale.

Definition 5.4.1 *Let 0 be the last element of $P_{\mathcal{U}^s}$, and let $h_0 = \{S(U)(0) : U \in \mathcal{U}\}$. Define $\sigma : X \times X \longrightarrow P_{\mathcal{U}^s}$ by*

$$\sigma(x,y) = \{U \in \mathcal{U}^s : (x,y) \in U\}$$

for every x and y in X.

Theorem 5.4.1 *Let σ be defined as above. Then σ is an écart, and the quasi-uniformity defined on X by σ is $\mathcal{U}$. Moreover, if $\mathcal{B}$ is a $\tau(\mathcal{U}^s) \times \tau(\mathcal{U}^s)$-closed base of $\mathcal{U}$ then for any $V \in \mathcal{B}$, $(a,b) \in V$ if and only if $\sigma(a,b) \in S(V)(0)$.*

Proof. We start by proving the second part. Let $\mathcal{B}$ be a $\tau(\mathcal{U}^s) \times \tau(\mathcal{U}^s)$-closed base of $\mathcal{U}$. Then for any $V \in \mathcal{B}$, we have the following: Let $\sigma(a,b) \in$

$S(V)(0)$ if and only if $(0, \sigma(a,b)) \in S(V)$ if and only if $V \circ U \in \sigma(a,b)$ if $U \in \mathcal{U}^s$ and $A^{-1} \circ V \in \mathcal{U}^s$ if $A \in \sigma(a,b)$ if and only if $(a,b) \in V \circ U$ for every $U \in \mathcal{U}^s$ if and only if for every $U \in \mathcal{U}^s$, $(a,c) \in U$, $(c,b) \in V$ for some $c \in X$ if and only if for every $U \in \mathcal{U}^s$, $V(a) \cap U^{-1}(b) \neq \emptyset$ if and only if $b \in cl_{\tau(\mathcal{U}^s)} V(a) = V(a)$ if and only $(a,b) \in V$. Thus the quasi-uniformity induced on X by σ is $\mathcal{U}$.

It will now be shown that conditions (1), (3) and (4) of the definition of an écart are satisfied.

1. That $\sigma(a,a) = 0$ for every $a \in X$ is clear.

2. Let $S(U)(0)$ and $S(V)(0)$ be in h_0. There exists a $W \in \mathcal{U}$ such that $W \subseteq U \cap V$. Then $S(W)(0) \subseteq S(U)(0) \cap S(V)(0)$, and $S(W)(0) \in h_0$.

3. For every $U \in \mathcal{U}$, there exists a $V \in \mathcal{U}$ such that $V^2 \subseteq U$. Then $\sigma(a,b) \in S(V)(0)$ and $\sigma(b,c) \in S(V)(0)$ which imply (a,b) and (b,c) are in V; hence (a,c) is in V^2. It follows that $\sigma(a,c) \in S(V^2)(0) \subseteq S(U)(0)$. This completes the proof. □

We recall the definition of the map $\psi : P_{\mathcal{U}^s} \longrightarrow (P_{\mathcal{U}^s})_R$ (see Lemma 5.2.2). It is clear that if σ is an écart with values in $P_{\mathcal{U}^s}$, then $\psi\sigma$ is an écart with values in $(P_{\mathcal{U}^s})_R$. The following proposition implies that $\psi\sigma$ also induces the quasi-uniformity $\mathcal{U}$ in X.

Proposition 5.4.1 *Let $V \in \mathcal{U}$ be $\tau(\mathcal{U}^s) \times \tau(\mathcal{U}^s)$-closed. Then $\alpha \in S(V)(0)$ if and only if $\psi(\alpha) \in S(V)(0)$.*

Proof. Let $\alpha \in S(V)(0)$, which implies $(0,\alpha) \in S(V)$. Let $\beta \in \psi(\alpha)$. Since $\alpha \sim \beta$ we have $V \circ U \in \psi(\alpha)$ for every $U \in \mathcal{U}^s$. Then $V \circ \mathcal{U}^s \subseteq \psi(\alpha)$. Furthermore $V^{-1} \circ \psi(\alpha) \subseteq \mathcal{U}^s$. Thus $(0, \psi(\alpha)) \in S(V)$. Similarly, if $\psi(\alpha) \in S(V)(0)$ then $(0,\alpha) \in S(V)$. □

The écart σ will now be used to establish some topological properties of the scale. The next theorem is similar to Theorem 3.5.2.

Theorem 5.4.2 *Let $(X,\mathcal{U})$ be a quasi-uniform space, $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ its quasi-uniform scale and $((P_{\mathcal{U}^s})_R, S(\mathcal{U})_R)$ its retracted scale. Then the follow-*

ing statements are equivalent:

(1). *$(X,\mathcal{U})$ is quasi-pseudometrizable.*

(2). *$(P_{\mathcal{U}^s},S(\mathcal{U}))$ is quasi-pseudometrizable.*

(3). *$(P_{\mathcal{U}^s},S(\mathcal{U}))$ satisfies the first axiom for countability.*

(4). *$(P_{\mathcal{U}^s},S(\mathcal{U}))$ has a countable base for the neighborhood filter at 0.*

(5). *There exists a map ϕ from $P_{\mathcal{U}^s}$ into the nonnegative real numbers such that $\phi(0) = 0$ and $\{\phi^{-1}([0,\varepsilon)) : \varepsilon > 0\}$ is a base for the neighborhood filter at 0.*

Moreover $(P_{\mathcal{U}^s},S(\mathcal{U}))$ can be replaced by $((P_{\mathcal{U}^s})_R,S(\mathcal{U})_R)$ in any of the statements (2) through (5).

Proof. (1) $\Longrightarrow$ (2), (2) $\Longrightarrow$ (3), (3) $\Longrightarrow$ (4) are trivial.

If (4) is satisfied, then a countable, nested neighborhood base at $0 = M_1 \supseteq M_2 \supseteq \ldots$ can be constructed.
Define $\phi : P_{\mathcal{U}^s} \to \mathbb{R}^+$ by $\phi(\alpha) = 1/n$ if $\alpha \in M_n - M_{n+1}$ and 0 otherwise.
For each $\alpha \in P_{\mathcal{U}^s}$, ϕ is well defined since the M_i's are nested.
Furthermore $0 \in \bigcap_{i=1}^{\infty} M_i$, so $\phi(0) = 0$ and $\phi^{-1}([0,\epsilon)) = M_n$, where n is the last integer greater than $1/\epsilon$. Thus ϕ satisfies (5).

We show that (5) $\Longrightarrow$ (1). Let $f = \phi\sigma$, where σ is the écart of Theorem 5.4.1 and ϕ is the given map. Consider the map $f : X \times X \to \mathbb{R}^+$. Let $B_r = \{\alpha : \phi(\alpha) < r\}$ for each $r > 0$. Finally let $\mathcal{B}$ be a $\tau(\mathcal{U}^s) \times \tau(\mathcal{U}^s)$-closed base of $\mathcal{U}$.

i. Given $U \in \mathcal{B}$, there exists an $r > 0$ such that $B_r \subseteq S(U)(0)$ that is, $f(x,y) < r$ implies $\sigma(x,y) \in B_r \subseteq S(U)(0)$, and hence $(x,y) \in U$. Therefore there exists an $r > 0$ such that $\{(x,y) \in X \times X : f(x,y) < r\} \subseteq U$.

ii. Let $\epsilon > 0$. There exists a $U \in \mathcal{B}$ such that $S(U)(0) \subseteq B_\epsilon$, and so $(x,y) \in U$ implies that $f(x,y) < \epsilon$. Therefore there exists a $U \in \mathcal{B}$ such that $U \subseteq \{(x,y) \in X \times X : f(x,y) < \epsilon\}$.

It follows that $\{f^{-1}([0,\epsilon)) : \epsilon > 0\}$ is a base for $\mathcal{U}$. Then $\{f^{-1}([0,1/n)) : n$ is a positive integer$\}$ is a countable base for $\mathcal{U}$, and $(X,\mathcal{U})$ is quasi-pseudometrizable.

We suppose that $((P_{\mathcal{U}^s})_R, S(\mathcal{U})_R)$ is substituted for $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ in statements (2) through (5). Again it is clear that (1) $\Longrightarrow$ (2), (2) $\Longrightarrow$ (3), (3) $\Longrightarrow$ (4), (4) $\Longrightarrow$ (5) are true. To prove that (5) implies (1), one must use the écart $\psi\sigma$ instead of σ. Then with the aid of Proposition 5.4.1, everything works as before. □

Definition 5.4.2 *Let $(X,d,\mathcal{U}_d)$ be a quasi-pseudometric space. For $\epsilon > 0$, let $U_\epsilon = \{(x,y) \in X \times X : d(x,y) \leq \epsilon\}$.*

Lemma 5.4.1 *For any ϵ, $\delta \geq 0$, $S(U_\epsilon) \circ S(U_\delta) \subseteq S(U_{\epsilon+\delta})$.*

Proof. Let $(\alpha,\beta) \in S(U_\epsilon) \circ S(U_\delta)$. Then $(\alpha,\gamma) \in S(U_\delta)$ and $(\gamma,\beta) \in S(U_\epsilon)$ for some $\gamma \in P_{\mathcal{U}^s}$. Therefore $U_\delta \circ F \in \gamma$ for every $F \in \alpha$, $U_\varepsilon \circ H \in \beta$, and $U_\delta^{-1} \circ H \in \alpha$ for every $H \in \gamma$, $U_\epsilon^{-1} \circ G \in \gamma$ for every $G \in \beta$. Then $U_\epsilon \circ U_\delta \circ F \in \beta$ for every $F \in \alpha$ and $U_\delta^{-1} \circ U_\epsilon^{-1} \circ G \in \alpha$ for every $G \in \beta$. So $U_{\epsilon+\delta} \circ F \in \beta$ for every $F \in \alpha$ and $U_{\varepsilon+\delta}^{-1} \circ G \in \alpha$. Thus $(\alpha,\beta) \in S(U_{\epsilon+\delta})$.□

Definition 5.4.3 *We suppose that d is a bounded quasi-pseudometric. Define $d_s : P_{\mathcal{U}^s} \times P_{\mathcal{U}^s} \to \mathbb{R}^+$ by*

$$d_s(\alpha,\beta) = \inf\{\epsilon > 0 : (\alpha,\beta) \in S(U_\epsilon)\}.$$

The fact that d is bounded ensures that d_s is well defined.

The following proposition is similar to Proposition 3.5.3.

Proposition 5.4.2 *Let $(X,d,\mathcal{U}_d)$ be a bounded quasi-pseudometric space. Then d_s is a quasi-pseudometric on $P_{\mathcal{U}^s}$, and the scale quasi-uniformity $S(\mathcal{U})$ is the quasi-pseudometric quasi-uniformity of d_s.*

Proof. We shall show that d_s is a quasi-pseudometric.

i. $d_s(\alpha,\alpha) = 0$ for every $\alpha \in P_{\mathcal{U}^s}$, since $(\alpha,\alpha) \in S(U_\epsilon)$ for every $\epsilon > 0$.

ii. Let $\epsilon > 0$, $d_s(\alpha, \beta) = a$, and $d_s(\beta, \gamma) = b$. Then $(\alpha, \beta) \in S(U_{a+\epsilon})$ and $(\beta, \gamma) \in S(U_{b+\epsilon})$ imply that $(\alpha, \gamma) \in S(U_{b+\epsilon}) \circ S(U_{a+\epsilon}) \subseteq S(U_{a+b+2\epsilon})$. Therefore for any $\epsilon > 0$, $d_s(\alpha, \gamma) \leq d_s(\alpha, \beta) + d_s(\beta, \gamma) + 2\epsilon$; hence $d_s(\alpha, \gamma) \leq d_s(\alpha, \beta) + d_s(\beta, \gamma)$.

Thus d_s is a quasi-pseudometric. It is clear that $S(U_\epsilon) = \{(\alpha, \beta) : d_s(\alpha, \beta) \leq \epsilon\}$, so $S(\mathcal{U})$ is the quasi-pseudometric quasi-uniformity induced by d_s. □

Remark 5.4.1 *$d_s(\alpha, \beta) = 0$ if and only if $(\alpha, \beta) \in S(U_\epsilon)$ for every $\epsilon > 0$.*

Definition 5.4.4 *As before let $X = \mathbb{R}^+$ be the set of nonnegative real numbers. We define a quasi-pseudo-metric F on X by*
$F(x, y) = y - x$ if $y \geq x$ and $F(x, y) = 0$ if $y < x$.
For each $r > 0$, set $O_r = \{(x, y) \in X \times X : F(x, y) < r\}$. The filter on $X \times X$ generated by the base $\{O_r : r > 0\}$ is a quasi-uniformity and is called the standard quasi-pseudometric quasi-uniformity $\mathcal{U}_r$ induced by F on X.

The following proposition is similar to Theorem 3.5.3.

Proposition 5.4.3 *If $(X, \mathcal{U})$ is quasi-pseudometrizable space and $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ its scale, then there exists a quasi-uniformly continuous, order-preserving map ϕ from $P_{\mathcal{U}^s}$ into the nonnegative real numbers.*

Proof. Let d be a bounded quasi-pseudometric on X such that $\mathcal{U}$ is the quasi-pseudometric quasi-uniformity for d. Let d_s be the quasi-pseudometric on $P_{\mathcal{U}^s}$ defined by

$$d_s(\alpha, \beta) = \inf\{\epsilon > 0 : (\alpha, \beta) \in S(U_\epsilon)\}$$

for all $\alpha, \beta \in P_{\mathcal{U}^s}$. Then d_s is a quasi-pseudometric on $P_{\mathcal{U}^s}$, and $S(\mathcal{U})$ is the quasi-pseudometric quasi-uniformity by Proposition 5.4.2. Define
$\phi : P_{\mathcal{U}^s} \to \mathbb{R}^+$ by $\phi(\alpha) = d_s(0, \alpha)$ for every $\alpha \in P_{\mathcal{U}^s}$.

For the proof that ϕ is quasi-uniformly continuous and order preserving we use the same argument as in the proof of Proposition 3.5.3(ii). □

Proposition 5.4.4 *Let d be a bounded quasi-pseudometric on X such that $\mathcal{U}_d \subseteq \mathcal{U}$, where $\mathcal{U}_d$ is the quasi-pseudometric quasi-uniformity defined by d. Let $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ be the scale of $(X, \mathcal{U})$, and define $d_s : P_{\mathcal{U}^s} \times P_{\mathcal{U}^s} \to \mathbb{R}^+$ by*

$$d_s(\alpha, \beta) = \inf\{\epsilon > 0 : (\alpha, \beta) \in S(U_\epsilon)\},$$

where, U_ϵ is defined with respect to d. Then d_s is a quasi-pseudometric on $P_{\mathcal{U}^s}$, $S(U_\epsilon) = \{(\alpha, \beta) : d_s(\alpha, \beta) \leq \epsilon\}$, and $\mathcal{U}_{d_s} \subseteq S(\mathcal{U})$, where $\mathcal{U}_{d_s}$ is the quasi-uniformity generated on $P_{\mathcal{U}^s}$ by d_s.

Proof. It is shown in Proposition 5.4.2 that d_s is a quasi-pseudometric. $\{S(U_\epsilon) : \epsilon > 0\} \subseteq S(\mathcal{U})$ since $\mathcal{U}_d \subseteq \mathcal{U}$, and it is clear that these sets $S(U_\epsilon)$ form a base for a quasi-uniformity on $P_{\mathcal{U}^s}$. □

Theorem 5.4.3 *If $(X, \mathcal{U})$ is any quasi-uniform space and $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ its scale, then there exists a quasi-uniformly continuous, order-preserving map ϕ from $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ into the nonnegative real numbers equipped with $\mathcal{U}_r$.*

Proof. This is a consequence of Proposition 5.4.3 and Proposition 5.4.4.□

Proposition 5.4.5 *If $(X, d, \mathcal{U})$ is a bounded quasi-pseudometric space, $(P_{\mathcal{U}^s}, d_s, S(\mathcal{U}))$ its quasi-uniform scale, σ the écart of Theorem 5.4.1, and ϕ the map in Proposition 5.4.3, then $d = \phi\sigma$.*

Proof. For any $x, y \in X$,
$d(x, y) = \inf\ \{\epsilon > 0 : (x, y) \in U_\epsilon\}$ and therefore
$d(x, y) = \inf\ \{\epsilon > 0 : \sigma(x, y) \in S(U_\epsilon)(0)\}$. Thus
$d(x, y) = d_s(0, \sigma(x, y))$ and hence
$d(x, y) = \phi(\sigma(x, y))$. □

5.5 The prefilter space of a quasi-uniform space

In this section, we will first introduce the prefilter space of a quasi-uniform space. Secondly we will establish a connection between the prefilter quasi-uniform space and the Hausdorff hyperspace quasi-uniform space. Furthermore we will show that total boundedness is preserved by the prefilter space

of a quasi-uniform space. The prefilter space is a kind of generalization of the quasi-uniform scale with similar properties.

The following defines the set of all prefilters of a quasi-uniform space.

Definition 5.5.1 *Let $(X,\mathcal{U})$ be a quasi-uniform space. We define $\mathcal{PF}(X)$ to be the set of all prefilters on X.*

Definition 5.5.2 *(Compare Definition 5.1.2) We equip the set $\mathcal{PF}(X)$ with the following partial order: For $\alpha, \beta \in \mathcal{PF}(X)$ we set $\alpha \leq \beta$ if $\alpha \supseteq \beta$.*

The next proposition is similar to Proposition 5.1.1.

Proposition 5.5.1 *For any set X, the ordered prefilter space $(\mathcal{PF}(X),\leq)$ is a complete lattice with $\inf A = \bigcup_{\alpha\in A}\alpha$ and $\sup A = \bigcap_{\alpha\in A}\alpha$ whenever $A \subseteq \mathcal{PF}(X)$. For a nonempty set X, observe that $\{X\}$ is the largest element and $\mathcal{P}_0(X)$ is the smallest element of $\mathcal{PF}(X)$ (if $X = \emptyset$, then $\mathcal{PF}(X) = \emptyset$).*

Proof. This is obvious from the definition of the partial order $\leq$ in the prefilter space. □

Let $(X,\mathcal{U})$ be a quasi-uniform space. For each $U \in \mathcal{U}$ and $A \subseteq X$, set $U(A) = \bigcup_{a\in A}U(a) = \{y \in X\colon$ there is $a \in A$ such that $(a,y) \in U\}$.
Similarly as before, given a subset $\mathcal{S}$ of $\mathcal{P}_0(X)$, $\langle\mathcal{S}\rangle$ will denote the smallest prefilter containing $\mathcal{S}$ on X.

The following defines the $\mathcal{U}$-envelope and $\mathcal{U}$-roundness.

Definition 5.5.3 *Let $(X,\mathcal{U})$ be a quasi-uniform space. For each $U \in \mathcal{U}$ and each $\alpha \in \mathcal{PF}(X)$ we set $U(\alpha)$ equal to $\langle\{U(A) : A \in \alpha\}\rangle$, and call it the* U-hull *of α.*
*Similarly we define $\mathcal{U}(\alpha)$ as the prefilter $\langle\{U(A) : A \in \alpha, U \in \mathcal{U}\}\rangle$. Let us note that $\mathcal{U}(\alpha)$ is usually called the $\mathcal{U}$-*envelope *of α. A prefilter will be called $\mathcal{U}$-*round *if it is equal to its $\mathcal{U}$-envelope. Note that for any prefilter on X its $\mathcal{U}$-envelope is $\mathcal{U}$-round.*

Proposition 5.5.2 *Let $(X,\mathcal{U})$ be a quasi-uniform space and let $\mathcal{PF}(X)$ be the set of all prefilters on X. For each $U \in \mathcal{U}$ we set*

$$U_{\oplus} = \{(\alpha,\beta) \in \mathcal{PF}(X) \times \mathcal{PF}(X) : U(\alpha) \subseteq \beta\}.$$

Then $\{U_{\oplus} : U \in \mathcal{U}\}$ is a base for a quasi-uniformity $\mathcal{U}_{\oplus}$ on $\mathcal{PF}(X)$ which will be called the positive quasi-uniformity. *For each $U \in \mathcal{U}$ we set*

$$U_{\ominus} = \{(\alpha,\beta) \in \mathcal{PF}(X) \times \mathcal{PF}(X) : U^{-1}(\beta) \subseteq \alpha\}.$$

Then $\{U_{\ominus} : U \in \mathcal{U}\}$ is a base for a quasi-uniformity $\mathcal{U}_{\ominus}$ on $\mathcal{PF}(X)$ which will be called the negative quasi-uniformity.

Furthermore for each $U \in \mathcal{U}$ set $U_{PF} = U_{\oplus} \cap U_{\ominus}$. Then $\{U_{PF} : U \in \mathcal{U}\}$ is a base for a quasi-uniformity $\mathcal{U}_{PF}$ on $\mathcal{PF}(X)$ which will be called the prefilter quasi-uniformity.

Proof. Note first that for each $U \in \mathcal{U}$ and any $\alpha \in \mathcal{PF}(X)$, we have $(\alpha,\alpha) \in U_{\oplus}$, and similarly $(\alpha,\alpha) \in U_{\ominus}$ and $(\alpha,\alpha) \in U_{PF}$. Observe also that $U, V \in \mathcal{U}$ with $U \subseteq V$ implies that $U_{\oplus} \subseteq V_{\oplus}$, $U_{\ominus} \subseteq V_{\ominus}$, and $U_{PF} \subseteq V_{PF}$. Hence $\{U_{\oplus} : U \in \mathcal{U}\}$, $\{U_{\ominus} : U \in \mathcal{U}\}$ and $\{U_{PF} : U \in \mathcal{U}\}$ are filter bases on $\mathcal{PF}(X) \times \mathcal{PF}(X)$.

Let $U \in \mathcal{U}$ and $V \in \mathcal{U}$ be such that $V^2 \subseteq U$. Let $(\alpha,\gamma) \in {V_{\oplus}}^2$ then there is $\beta \in \mathcal{PF}(X)$ such that $(\alpha,\beta) \in V_{\oplus}$ and $(\beta,\gamma) \in V_{\oplus}$. Let $A \in \alpha$, then $V(A) \in \beta$. We have $V^2(A) \in \gamma$ which implies $U(A) \in \gamma$. We have shown that $(\alpha,\gamma) \in U_{\oplus}$. Thus ${V_{\oplus}}^2 \subseteq U_{\oplus}$. Similarly ${V_{\ominus}}^2 \subseteq U_{\ominus}$, and thus $(V_{PF})^2 \subseteq U_{PF}$. We deduce that $\mathcal{U}_{\oplus}$, $\mathcal{U}_{\ominus}$ and $\mathcal{U}_{PF}$ are quasi-uniformities on $\mathcal{PF}(X)$. □

We next define the prefilter space of a quasi-uniform space.

Definition 5.5.4 *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $(\mathcal{PF}(X),\mathcal{U}_{PF})$ is called the* prefilter space *of the quasi-uniform space $(X,\mathcal{U})$.*

The next proposition shows that the lattice operations in the prefilter space of a quasi-uniform space are quasi-uniformly continuous.

Proposition 5.5.3 *The lattice operations on $(\mathcal{PF}(X),\mathcal{U}_{PF})$ are quasi-uniformly continuous (that is, $(\mathcal{PF}(X),\mathcal{U}_{PF})$ is a quasi-uniform lattice).*

Proof. Let $U \in \mathcal{U}$. Consider $(\alpha, \beta), (\alpha', \beta') \in U_\oplus \cap U_\ominus$. Then $U(\alpha) \subseteq \beta$, $U^{-1}(\beta) \subseteq \alpha$, $U(\alpha') \subseteq \beta'$ and $U^{-1}(\beta') \subseteq \alpha'$. Thus $U(\alpha \cup \alpha') \subseteq \beta \cup \beta'$ and $U^{-1}(\beta \cup \beta') \subseteq \alpha \cup \alpha'$. Hence $(\alpha \cup \alpha', \beta \cup \beta') \in U_\oplus \cap U_\ominus$ and similarly $(\alpha \cap \alpha', \beta \cap \beta') \in U_\oplus \cup U_\ominus$. Therefore $\cup : (\mathcal{PF}(X) \times \mathcal{PF}(X), \mathcal{U}_{PF} \times \mathcal{U}_{PF}) \to (\mathcal{PF}(X), \mathcal{U}_{PF})$ and $\cap : (\mathcal{PF}(X) \times \mathcal{PF}(X), \mathcal{U}_{PF} \times \mathcal{U}_{PF})$ are quasi-uniformly continuous. □

The following lemma makes a connection between the prefilter space and the Hausdorff hyperspace quasi-uniform space of a quasi-uniform space.

Lemma 5.5.1 *Let $(X, \mathcal{U})$ be a quasi-uniform space.*
(a) Then the Hausdorff hyperspace $(\mathcal{P}_0(X), \mathcal{U}_H)$ embeds quasi-uniformly via the map $A \longmapsto \langle A \rangle$ into $(\mathcal{PF}(X), \mathcal{U}_{PF})$.
(b) Let $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ be a quasi-uniformly continuous map. Then $\mathcal{PF}(f)(\alpha) = \langle f(\alpha) \rangle$ is a quasi-uniformly continuous map from $(\mathcal{PF}(X), \mathcal{U}_{PF})$ into $(\mathcal{PF}(Y), \mathcal{V}_{PF})$.

Proof. We leave the easy proof to the reader. □

The next proposition makes another connection between the prefilter space and the Hausdorff hyperspace quasi-uniform space.

Proposition 5.5.4 *Let $(X, \mathcal{U})$ be a quasi-uniform space. Then $\mathcal{U}_{PF}$ is equal to $(\mathcal{U}_+)_- \bigvee (\mathcal{U}_-)_+ \mid \mathcal{PF}(X)$.*

Proof. First note the following fact: Let $\alpha \in \mathcal{PF}(X)$. Then α is a nonempty collection of nonempty subsets of X. Therefore $\alpha \subseteq \mathcal{P}_0(X)$, hence $\alpha \in \mathcal{P}_0(\mathcal{P}_0(X))$. So $\mathcal{PF}(X) \subseteq \mathcal{P}_0(\mathcal{P}_0(X))$.

Let $U \in \mathcal{U}$ and $\alpha, \beta \in \mathcal{PF}(X)$. Suppose $(\alpha, \beta) \in U_\oplus$. We have $U(\alpha) \subseteq \beta$ means for each $A \in \alpha$ that there exists $B \in \beta$ such that $B \subseteq U(A)$, hence for each $A \in \alpha$ there exists $B \in \beta$ such that $(A, B) \in U_+$. We have $\alpha \subseteq (U_+)^{-1}(\beta)$ means $(\beta, \alpha) \in ((U_+)_-)^{-1}$ so $(\alpha, \beta) \in (U_+)_-$. Similarly $(\alpha, \beta) \in U_\ominus$ is equivalent to $(\alpha, \beta) \in (U_-)_+$.
Therefore $U_\oplus \cap U_\ominus$ is equivalent to $(U_+)_- \cap (U_-)_+$ on $\mathcal{PF}(X)$, thus $\mathcal{U}_{PF}$ is equal to $(\mathcal{U}_+)_- \bigvee (\mathcal{U}_-)_+ \mid \mathcal{PF}(X)$. □

We next show that total boundedness is preserved by the prefilter space of a quasi-uniform space.

Proposition 5.5.5 *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $\mathcal{U}$ is totally bounded if and only if $(\mathcal{PF}(X),\mathcal{U}_{PF})$ is totally bounded.*

Proof. This follows from Proposition 5.5.4 and Lemma 5.5.1 above, since $\mathcal{U}_H$ is known to be totally bounded if and only if $\mathcal{U}$ is totally bounded [22], and since total boundedness is preserved by quasi-uniform subspaces, as well as quasi-uniformly continuous images. □

Lemma 5.5.2 *Let X be a set and let Υ be an ultrafilter on $\mathcal{PF}(X)$. Then $\bigcup_{Z\in\Upsilon}(\bigcap_{\zeta\in Z}\zeta) = \bigcap_{Z\in\Upsilon}(\bigcup_{\zeta\in Z}\zeta)$.*

Proof. The inequality $\bigcup_{Z\in\Upsilon}(\bigcap_{\zeta\in Z}\zeta) \subseteq \bigcap_{Z\in\Upsilon}(\bigcup_{\zeta\in Z}\zeta)$ is obvious: For any $Z_1, Z_2 \in \Upsilon$ we have that $\bigcap_{\zeta\in Z_1}\zeta \subseteq \bigcup_{\zeta\in Z_2}\zeta$, since Z_1 and Z_2 intersect. Therefore $\bigcup_{Z_1\in\Upsilon}(\bigcap_{\zeta\in Z_1}\zeta) \subseteq \bigcup_{\zeta\in Z_2}\zeta$ and the assertion follows.

Let $A \in \bigcap_{Z\in\Upsilon}(\bigcup_{\zeta\in Z}\zeta)$. Then for each $Z \in \Upsilon$ there is $\zeta_Z \in Z$ such that $A \in \zeta_Z$. Let $E = \{\zeta_Z : Z \in \Upsilon\}$. Then $\zeta_Z \in E \cap Z$ whenever $Z \in \Upsilon$. Thus $E \in \Upsilon$, since Υ is an ultrafilter on $\mathcal{PF}(X)$. Furthermore $A \in \bigcap_{\zeta\in E}\zeta$ and $A \in \bigcup_{Z\in\Upsilon}\bigcap_{\zeta\in Z}\zeta$. Hence $\bigcap_{Z\in\Upsilon}(\bigcup_{\zeta\in Z}\zeta) \subseteq \bigcup_{Z\in\Upsilon}(\bigcap_{\zeta\in Z}\zeta)$ and the assertion is proved. □

We next show that the prefilter space of a quasi-uniform space is bicomplete (compare with Theorem 3.3.2).

Proposition 5.5.6 *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $(\mathcal{PF}(X),\mathcal{U}_{PF})$ is bicomplete. (Note that the proof also shows that $(\mathcal{PF}(X),\mathcal{U}_{\oplus})$ and $(\mathcal{PF}(X),\mathcal{U}_{\ominus})$ are bicomplete.)*

Proof. Let Ξ be a filter on $\mathcal{PF}(X)$ that is $(\mathcal{U}_{PF})^s$-Cauchy. We show that Ξ $\tau((\mathcal{U}_{PF})^s)$-converges to $\gamma := \bigcap_{Y\in\Upsilon}(\bigcup_{\zeta\in Y}\zeta) = \bigcup_{Y\in\Upsilon}(\bigcap_{\zeta\in Y}\zeta)$ where Υ is an ultrafilter on $\mathcal{PF}(X)$ containing Ξ by Lemma 5.5.2.

Let $U \in \mathcal{U}$. By assumption there is $C \in \Xi$ such that $(C\times C) \subseteq (U_{\oplus}\cap U_{\ominus})$. Let $\xi, \xi' \in C$ be arbitrary. Then $U(\xi') \subseteq \xi$ by definition of $U_{\oplus}$ and similarly $U^{-1}(\xi) \subseteq \xi'$ by definition of $U_{\ominus}$.

Thus for each $\xi' \in C$, $U(\xi') \subseteq \bigcap_{\xi\in C}\xi \subseteq \gamma$ and therefore $C \times \{\gamma\} \subseteq U_{\oplus}$. Also for each $\xi \in C$ we have $U(\gamma) \subseteq U(\bigcup_{\xi'\in C}\xi') \subseteq \xi$ and consequently

$\{\gamma\} \times C \subseteq U_{\ominus}$.

Analogously for each $\xi \in C$, $U^{-1}(\xi) \subseteq \bigcap_{\xi' \in C} \xi' \subseteq \gamma$ and therefore $\{\gamma\} \times C \subseteq U_{\ominus}$. Furthermore for each $\xi' \in C$, $U^{-1}(\gamma) \subseteq U^{-1}(\bigcup_{\xi \in C} \xi) \subseteq \xi'$. Therefore $C \times \{\gamma\} \subseteq U_{\ominus}$. We conclude that Ξ converges to γ in $(\mathcal{PF}(X),(\mathcal{U}_{PF})^s)$ and thus $(\mathcal{PF}(X),\mathcal{U}_{PF})$ is bicomplete. □

It follows from Proposition 5.5.7 and Lemma 5.5.1 (a) that the bicompletion of the T_0-reflection of $(\mathcal{P}_0(X),\mathcal{U}_H)$ is a subspace of the T_0-reflection of the prefilter space $(\mathcal{PF}(X),\mathcal{U}_{PF})$. From the results of [23] we conclude that the bicompletion can be identified with the subspace consisting of all 2-round doubly stable filters on $(X,\mathcal{U})$, compare [23] for more details.

5.6 The left-sided scale of a quasi-uniform space

In this section we revisit the quasi-uniform scale of a quasi-uniform space. In particular we shall extend the ground set of the quasi-uniform scale of a quasi-uniform space.

Definition 5.6.1 *Let $(X,\mathcal{U})$ be a quasi-uniform space. We define $\mathcal{R}(X)$ as the set of all reflexive binary relations on X.*

We next define the left-sided scale quasi-uniformity.

Proposition 5.6.1 *Let $(X,\mathcal{U})$ be a quasi-uniform space.*
We define $\mathcal{PF}_r(X^2) = \{\alpha : \alpha$ prefilter on $X \times X$ such that $\alpha \subseteq \mathcal{R}(X)\}$.
For any $U \in \mathcal{U}$ we set

$$U_{\uparrow} = \{(\alpha,\beta) \in \mathcal{PF}_r(X^2) \times \mathcal{PF}_r(X^2) \colon U \circ \alpha \subseteq \beta\}$$

and

$$U_{\downarrow} = \{(\alpha,\beta) \in \mathcal{PF}_r(X^2) \times \mathcal{PF}_r(X^2) : U^{-1} \circ \beta \subseteq \alpha\}.$$

Furthermore set $U_{\updownarrow} = U_{\uparrow} \cap U_{\downarrow}$ *whenever* $U \in \mathcal{U}$. *Then* $\{U_{\downarrow} : U \in \mathcal{U}\}$ *is a base for the* down-quasi-uniformity $\mathcal{U}_{\downarrow}$ *on* $\mathcal{PF}_r(X^2)$ *and* $\{U_{\uparrow} : U \in \mathcal{U}\}$ *is a base for the* up-quasi-uniformity $\mathcal{U}_{\uparrow}$ *on* $\mathcal{PF}_r(X^2)$. *Moreover* $\{U_{\updownarrow} : U \in \mathcal{U}\}$ *is the base for the* left-sided scale quasi-uniformity $\mathcal{U}_{\updownarrow}$.

Proof. This is similar to the proof of Proposition 5.1.2. □

Note that contrary to Bushaw and Kent we have defined the ground set of our left-sided scale as the set of all prefilters on $X \times X$ coarser than the filter $\mathcal{R}(X)$ and not as the set of all prefilters on $X \times X$ coarser than $\mathcal{U}^s$. This difference however seems of minor importance in the following investigations. Indeed in our context it is more convenient to work on the chosen larger set, since any quasi-uniformity $\mathcal{U}$ comes with closely related quasi-uniformities like $\mathcal{U}^{-1}$ and $\mathcal{U}^s$, and it is often inconvenient to change the ground set repeatedly according to varying needs.

We next define the left-sided scale of a quasi-uniform space.

Definition 5.6.2 *Let* $(X, \mathcal{U})$ *be a quasi-uniform space. Then* $(\mathcal{PF}_r(X^2), \mathcal{U}_{\updownarrow})$ *is called the* left-sided scale *of the quasi-uniform space* $(X, \mathcal{U})$.

We will need the next lemma in the following.

Lemma 5.6.1 *Let* M, N, A *be binary relations on a set* X. *Then* $M \circ A \circ N = \bigcup_{(a_1,a_2)\in A} N^{-1}(a_1) \times M(a_2)$.

Proof. Suppose $(x, y) \in M \circ A \circ N$. Then for some $(a_1, a_2) \in A$ we have $(x, a_1) \in N$, $(a_2, y) \in M$ and therefore $(a_1, x) \in N^{-1}$, $(a_2, y) \in M$. Hence with $(a_1, a_2) \in A$ we have $(x, y) \in N^{-1}(a_1) \times M(a_2)$. Thus $(x, y) \in \bigcup_{(a_1,a_2)\in A} N^{-1}(a_1) \times M(a_2)$. Conversely we use a similar argument. □

Our next results discuss connections between the left-sided scale quasi-uniform space and the prefilter space.

Proposition 5.6.2 *Let* $(X, \mathcal{U})$ *be a quasi-uniform space. Then on* $\mathcal{PF}_r(X^2)$ *we have:*

$\mathcal{U}_{\uparrow} = (\mathcal{D} \times \mathcal{U})_{\oplus} \mid \mathcal{PF}_r(X^2)$,

$\mathcal{U}_{\downarrow} = (\mathcal{D} \times \mathcal{U})_{\ominus} \mid \mathcal{PF}_r(X^2)$ *and*

$\mathcal{U}_{\updownarrow} = (\mathcal{D} \times \mathcal{U})_{PF} \mid \mathcal{PF}_r(X^2)$.

Here $\mathcal{D}$ denotes the discrete uniformity on X.

Furthermore $\mathcal{U}_{\uparrow}, \mathcal{U}_{\downarrow}, \mathcal{U}_{\updownarrow}$ are all bicomplete on $\mathcal{PF}_r(X^2)$.

Proof. The first statement follows from Lemma 5.6.1. For the second statement note that if in Lemma 5.5.2 Υ is an ultrafilter on $\mathcal{PF}_r(X^2)$, then $\gamma \in \mathcal{R}(X)$. The result follows from Proposition 5.5.7. □

The following corollary makes another connection between the prefilter space and the left-sided scale of a quasi-uniform space. It should be compared with Lemma 5.5.1. In particular we conclude that for any quasi-uniform space $(X,\mathcal{U})$, $(\mathcal{P}_0(X),\mathcal{U}_H)$ quasi-uniformly embeds into $(\mathcal{PF}_r(X^2),(\mathcal{D} \times \mathcal{U})_{PF} \mid \mathcal{PF}_r(X^2))$ via $A \mapsto \langle \triangle \cup (X \times A) \rangle$.

Corollary 5.6.1 *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then we can embed $(\mathcal{PF}(X),\mathcal{U}_{PF})$ into $(\mathcal{PF}_r(X^2),(\mathcal{D} \times \mathcal{U})_{PF} \mid \mathcal{PF}_r(X^2))$.*

Proof. For each $\alpha \in \mathcal{PF}(X)$ set $k(\alpha) = \langle \{\triangle \cup (X \times A) : A \in \alpha\} \rangle$.

We show that $k : (\mathcal{PF}(X),\mathcal{U}_{PF}) \to (\mathcal{PF}_r(X^2),(\mathcal{D} \times \mathcal{U})_{PF} \mid \mathcal{PF}_r(X^2))$ is a quasi-uniformly continuous map:

Let $(\alpha, \beta) \in U_{\oplus}$. Thus $U \circ \alpha \subseteq \beta$ and therefore for each $A \in \alpha$ there is $B \in \beta$ such that $B \subseteq U(A)$. Then for each $A \in \alpha$ there is $B \in \beta$ such that $\triangle \cup (X \times B) \subseteq U \circ (\triangle \cup (X \times A))$ if and only if $U \circ k(\alpha) \subseteq k(\beta)$. Similarly $(\alpha, \beta) \in U_{\ominus}$ implies that $U^{-1} \circ k(\beta) \subseteq k(\alpha)$. Thus $(k(\alpha), k(\beta)) \in U_{\updownarrow}$ and k is quasi-uniformly continuous. Note next that k is injective. Let $\alpha \neq \beta$, say α is not subset of β. Then there is $A_0 \in \alpha$ such for all $B \in \beta$, $B \setminus A_0 \neq \emptyset$. Then $\triangle \cup (X \times B)$ is not subset of $\triangle \cup (X \times A_0)$ whenever $B \in \beta$. Thus $k(\alpha) \neq k(\beta)$.

On the other hand suppose that $H \circ k(\alpha) \subseteq k(\beta)$. Then for each $A \in \alpha$ there exists $B \in \beta$ such that $B \subseteq H(A)$: Indeed let $B \in B$. Choose any

$a_0 \in A$. Then $(a_0, b) \in H \circ (X \times A)$ or $(a_0, b) \in H \circ \triangle = H$. We conclude that for some $a \in A$, we have $b \in H(a)$, or that $b \in H(a_0)$. Thus $B \subseteq H(A)$ in either case. Hence $H(\alpha) \subseteq \beta$. Similarly $H^{-1} \circ k(\beta) \subseteq k(\alpha)$ implies that $H^{-1}(\beta) \subseteq \alpha$. Therefore k is a quasi-uniform embedding. □

The following proposition discusses a connection between the left-sided scale of a quasi-uniform space and the scale of a quasi-uniform space defined in Definition 5.1.4.

Proposition 5.6.3 *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then the left-sided scale of a quasi-uniform space $(\mathcal{PF}_r(X^2),(\mathcal{D} \times \mathcal{U})_{PF} \mid \mathcal{PF}_r(X^2))$ coincides with the scale of a quasi-uniform space defined in Definition 5.1.4 on the subset $P_{\mathcal{U}^s}$ of $\mathcal{R}(X)$.*

Proof. Let $U \in \mathcal{U}$ and consider $(\alpha, \beta) \in (\triangle \times U)_{\mathcal{PF}}$, then $(\triangle \times U)(\alpha) \subseteq \beta$ and $(\triangle \times U)^{-1}(\beta) \subseteq \alpha$.
Suppose $A \in \alpha$ and let $(c_1, c_2) \in \bigcup_{(a_1,a_2)\in A} \triangle(a_1) \times U(a_2)$. By Lemma 5.6.1 we have $(c_1, c_2) \in U \circ A \circ \triangle = U \circ A$, hence $U \circ A \in \beta$. Similarly suppose $B \in \beta$ then $U^{-1} \circ B \in \alpha$. Thus $(\alpha, \beta) \in S(U)$. The converse is similar. □

Let us note that Corollary 5.6.1 also holds if we replace $(\mathcal{PF}_r(X^2),(\mathcal{D} \times \mathcal{U})_{PF} \mid \mathcal{PF}_r(X^2))$ by $(\mathcal{PF}_r(X^2),(\mathcal{U}^{-1} \times \mathcal{U})_{PF} \mid \mathcal{PF}_r(X^2))$, which is of interest in the study below which deals with the two-sided scale.

5.7 The two-sided scale of a quasi-uniform space

Remark 5.7.1 *Let $(X,\mathcal{U})$ be a quasi-uniform space and let $U, V \in \mathcal{R}(X)$, and $Q \in \mathcal{U}$. Note that $(\langle U \rangle, \langle V \rangle) \in Q_{\updownarrow}$ if and only if $V \subseteq Q \circ U$ and $U \subseteq Q^{-1} \circ V$ if and only if $V(x) \subseteq Q(U(x))$ and $U(x) \subseteq Q^{-1}(V(x))$ whenever $x \in X$. The latter condition exactly means that $(U(x), V(x)) \in Q_H$ whenever $x \in X$ where here Q_H is the standard entourage related to Q of the Hausdorff quasi-uniformity $\mathcal{U}_H$ on $\mathcal{P}_0(X)$. Hence the scale quasi-uniformity of a quasi-uniform space yields on the set of prefilters generated by reflexive relations (understood as multifunctions) the quasi-uniformity of quasi-uniform convergence with respect to the Hausdorff quasi-uniformity $\mathcal{U}_H$ on $\mathcal{P}_0(X)$.*

We recall that the quasi-uniform space of multifunctions has been summarized in Chapter 2. Observe that the multifunction space can be embedded into $(\mathcal{PF}(X \times Y), (\mathcal{D}_X \times \mathcal{U})_{PF})$, *where* $\mathcal{D}_X$ *denotes the discrete uniformity on* X.

It is somewhat surprising that the definition by Bushaw is admittedly left-sided [6, p.105], although as we see above this definition becomes less unnatural if one considers the scale as a generalization of the multifunction space of a quasi-uniform space instead of considering it as a generalization of the prefilter space of a quasi-uniform space.

One possibly unwanted consequence of the surprising appearance of the discrete uniformity in the formula in Proposition 5.6.2 is that total boundedness is not preserved by the scale quasi-uniformity.

Below we modify the definition of the left-sided scale given in Proposition 5.6.2 by suggesting to work on $\mathcal{PF}_r(X^2)$ with the restrictions of the quasi-uniformities $(\mathcal{U}^{-1} \times \mathcal{U})_{\oplus}$, $(\mathcal{U}^{-1} \times \mathcal{U})_{\ominus}$ and $(\mathcal{U}^{-1} \times \mathcal{U})_{PF}$ instead.

Proposition 5.7.1 *Let* $(X, \mathcal{U})$ *be a quasi-uniform space. On* $\mathcal{PF}_r(X^2)$ *for any* $U \in \mathcal{U}$ *we set*

$$U_{\Uparrow} = \{(\alpha, \beta) \in \mathcal{PF}_r(X^2) \times \mathcal{PF}_r(X^2) : U \circ \alpha \circ U \subseteq \beta\} \text{ and}$$

$U_{\Downarrow} = \{(\alpha, \beta) \in \mathcal{PF}_r(X^2) \times \mathcal{PF}_r(X^2) : U^{-1} \circ \beta \circ U^{-1} \subseteq \alpha\}$. *Furthermore for each* $U \in \mathcal{U}$ *set* $U_{\Updownarrow} = U_{\Uparrow} \cap U_{\Downarrow}$.

Then the two-sided scale *of the quasi-uniform space* $(X, \mathcal{U})$ *will be the quasi-uniform space* $(\mathcal{PF}_r(X^2), \mathcal{U}_{\Updownarrow})$ *where* $\{U_{\Updownarrow} : U \in \mathcal{U}\}$ *is the base of the quasi-uniformity* $\mathcal{U}_{\Updownarrow}$. *Similarly* $\mathcal{U}_{\Uparrow}$ *will be the quasi-uniformity generated by the base* $\{U_{\Uparrow} : U \in \mathcal{U}\}$ *and* $\mathcal{U}_{\Downarrow}$ *will be the quasi-uniformity generated by the base* $\{U_{\Downarrow} : U \in \mathcal{U}\}$.

Proof. Note first that for each $U \in \mathcal{U}$ and any $\alpha \in \mathcal{PF}_r(X^2)$, we have $(\alpha, \alpha) \in U_{\Uparrow}$, and similarly $(\alpha, \alpha) \in U_{\Downarrow}$ and $(\alpha, \alpha) \in U_{\Updownarrow}$. Observe also that $U, V \in \mathcal{U}$ with $U \subseteq V$ implies that $U_{\Uparrow} \subseteq V_{\Uparrow}$ and $U_{\Downarrow} \subseteq V_{\Downarrow}$, and $U_{\Updownarrow} \subseteq V_{\Updownarrow}$. Hence $\{U_{\Uparrow} : U \in \mathcal{U}\}$, $\{U_{\Downarrow} : U \in \mathcal{U}\}$ and $\{U_{\Updownarrow} : U \in \mathcal{U}\}$ are filter bases on

$\mathcal{PF}_r(X^2) \times \mathcal{PF}_r(X^2)$.

Let $U \in \mathcal{U}$ and $V \in \mathcal{U}$ be such that $V^2 \subseteq U$. Let $(\alpha, \gamma) \in {V_{\Uparrow}}^2$. Then there is $\beta \in \mathcal{PF}_r(X^2)$ such that $(\alpha, \beta) \in V_{\Uparrow}$ and $(\beta, \gamma) \in V_{\Uparrow}$. Let $A \in \alpha$. Then $V \circ A \circ U \in \beta$. We have $V^2 \circ A \circ V^2 \in \gamma$ implies $U \circ A \circ U \in \gamma$. We have shown that $(\alpha, \gamma) \in U_{\Uparrow}$. Thus ${V_{\Uparrow}}^2 \subseteq U_{\Uparrow}$. Similarly ${V_{\Downarrow}}^2 \subseteq U_{\Downarrow}$, and thus $(V_{\Updownarrow})^2 \subseteq U_{\Updownarrow}$. We deduce that $\mathcal{U}_{\Uparrow}$, $\mathcal{U}_{\Downarrow}$ and $\mathcal{U}_{\Updownarrow}$ are quasi-uniformities on $\mathcal{PF}_r(X^2)$. □

Our next proposition makes a connection between the prefilter quasi-uniformity and the two-sided quasi-uniformity.

Proposition 5.7.2 *Let $(X, \mathcal{U})$ be a quasi-uniform space and let $\alpha, \beta \in \mathcal{PF}_r(X^2)$. Then for each $U \in \mathcal{U}$, $(\alpha, \beta) \in (U^{-1} \times U)_{PF} \mid \mathcal{PF}_r(X^2)$ if and only if $U \circ \alpha \circ U \subseteq \beta$ and $U^{-1} \circ \beta \circ U^{-1} \subseteq \alpha$.*
Therefore $\mathcal{U}_{\Updownarrow} = (\mathcal{U}^{-1} \times \mathcal{U})_{PF} \mid \mathcal{PF}_r(X^2)$.

Proof. Let $U \in \mathcal{U}$ and suppose $(\alpha, \beta) \in (U^{-1} \times U)_{PF}$. We have $(U^{-1} \times U)(\alpha) \subseteq \beta$ and $(U^{-1} \times U)^{-1}(\beta) \subseteq \alpha$.
Let $A \in \alpha$. Then $(U^{-1} \times U)(A) \in \beta$ by Lemma 5.6.1. Thus $U \circ A \circ U \in \beta$. Hence $U \circ \alpha \circ U \subseteq \beta$, and similarly $U^{-1} \circ \beta \circ U^{-1} \subseteq \alpha$.

Conversely for each $U \in \mathcal{U}$, we suppose $U \circ \alpha \circ U \subseteq \beta$ and $U^{-1} \circ \beta \circ U^{-1} \subseteq \alpha$. By Lemma 5.6.1 we have $(U^{-1} \times U)(\alpha) \subseteq \beta$ and $(U^{-1} \times U)^{-1}(\beta) \subseteq \alpha$. □

Our next result shows that total boundedness in the two-sided scale is preserved, which does not hold for the Bushaw-Kent version of a scale, even in the context of a uniform space.

Proposition 5.7.3 *The two-sided scale $(\mathcal{PF}_r(X^2), \mathcal{U}_{\Updownarrow})$ of a totally bounded quasi-uniform space $(X, \mathcal{U})$ is joincompact, that is, the topology $\tau((\mathcal{U}_{\Updownarrow})^s)$ is compact.*

Proof. For any quasi-uniform space $(X, \mathcal{U})$, $(\mathcal{PF}_r(X^2), (\mathcal{U}^{-1} \times \mathcal{U})_{PF} \mid \mathcal{PF}_r(X^2))$ is bicomplete. Hence $(\mathcal{PF}_r(X^2), \mathcal{U}_{\Updownarrow})$ is bicomplete. It is totally bounded, since $\mathcal{U}^{-1} \times \mathcal{U}$ is totally bounded and the prefilter space preserves total boundedness. □

Corollary 5.7.1 *For any quasi-uniform space* $(X,\mathcal{U})$ *the two-sided scale* $(\mathcal{PF}_r(X^2),\mathcal{U}_{\Updownarrow})$ *is bicomplete.* $\square$

Corollary 5.7.2 *The two-sided scale* $(\mathcal{PF}_r(X^2),\mathcal{U}_{\Updownarrow})$ *of a quasi-uniform space* $(X,\mathcal{U})$ *is totally bounded if and only if* $(X,\mathcal{U})$ *is totally bounded.* $\square$

Chapter 6

Conclusion

In this dissertation, we have discussed the scale of a quasi-uniform space. We defined our scale of a quasi-uniform space and we made a connection between the scale of a quasi-uniform space and the scale of a uniform space introduced by Bushaw [6] and Kent [16]. We showed that the scale of a quasi-uniform space and its retracted scale are both bicomplete and that for any quasi-uniform space $(X,\mathcal{U})$ the associated hyperspace given by the Hausdorff quasi-uniformity is quasi-uniformly embedded into the left-sided scale of $(X,\mathcal{U})$.

We defined the prefilter space of a quasi-uniform space, and we made a connection between the Hausdorff hyperspace quasi-uniform space and our prefilter space. We also defined the two-sided scale of a quasi-uniform space and showed that total boundedness is preserved by our two-sided scale.

Our conclusion leads us to list some open problems encountered throughout the present investigation. We hope to study these questions in future work.

Problem 6.0.1 *Which quasi-uniform lattices are left-sided scales of a quasi-uniform space?*

Problem 6.0.2 *Under what conditions on the quasi-uniform space $(X,\mathcal{U})$ are the scale quasi-uniform space $(P_{\mathcal{U}^s},S(\mathcal{U}))$ and the retracted scale quasi-uniform space $((P_{\mathcal{U}^s})_R, S(\mathcal{U})_R)$ connected, locally connected, second countable, separable, locally compact, etc.?*

Problem 6.0.3 *Investigate the order scale of a quasi-uniform space (see Section 3.6)?*

Problem 6.0.4 *Are there simple conditions which characterize those quasi-uniform spaces which are quasi-uniformly isomorphic to the prefilter space or the two-sided scale quasi-uniform space of some quasi-uniform space?*

We will next list all those articles that we have consulted during the completion of this dissertation.

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Reprints in Theory and Applications of Categories, No. 1, 2002, pp. 1–37.

METRIC SPACES, GENERALIZED LOGIC, AND CLOSED CATEGORIES

F. WILLIAM LAWVERE

Author Commentary:
Enriched Categories in the Logic of Geometry and Analysis

Because parts of the following 1973 article have been suggestive to workers in several areas, the editors of TAC have kindly proposed to make it available in the present form. The idea on which it is based can be developed considerably further, as initiated in the 1986 article [1]. In the second part of this brief introduction I will summarize, for those familiar with the theory of enriched categories, some of the more promising of these further developments and possibilities, including suggestions coming from the modern theory of metric spaces which have not yet been elaborated categorically. (The 1973 and 1986 articles had also a didactic purpose, and so include a detailed introduction to the theory of enriched categories itself.)

While listening to a 1967 lecture of Richard Swan, which included a discussion of the relative codimension of pairs of subvarieties, I noticed the analogy between the triangle inequality and a categorical composition law. Later I saw that Hausdorff had mentioned the analogy between metric spaces and posets. The poset analogy is by itself perhaps not sufficient to suggest a whole system of constructions and theorems appropriate for metric spaces, but the categorical connection is! This connection is more fruitful than a mere analogy, because it provides a sequence of mathematical theorems, so that enriched category theory can suggest new directions of research in metric space theory and conversely, unusual for two subjects so old (1966 and 1906 respectively).

The closed interval $[0, \infty]$ of real numbers as objects, $\geq$ as maps, $+$ as "tensor" and truncated subtraction as adjoint "hom", constitute a bona fide example of a complete, symmetric, monoidal closed category V. For any such V there is the rich system of constructions and theorems (worked out by Eilenberg and Kelly, Day, and others) involving

- V-valued categories;

- V-strong functors;

Originally published as: Metric spaces, generalized logic, and closed categories, *Rendiconti del seminario matématico e fisico di Milano, XLIII (1973), 135-166*
Received by the editors 2002-04-01 and, in revised form, 2002-06-24.
Transmitted by Michael Barr. Reprint published on 2002-09-1.
2000 Mathematics Subject Classification: 18D20.
Key words and phrases: Metric spaces, enriched categories, logic.
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- V-natural transformations as an object of V and hence
- V-functor categories, λ-transformation and double dualization;
- the Cayley-Hausdorff-Yoneda lemma;
- free V-categories (generated by V-graphs) whose adjointness expresses Dedekind-Peano recursion via an objective geometric series;
- V-valued "relations" or bimodules and their convolution;
- Kan quantifiers which give extensions in particular well-defined situations.

All of these turn out to specialize, for the stated example V, to important results and constructions for metric spaces:

- V-functors are Lip_1 maps;
- the V-natural hom of two such turns out to be their sup-distance;
- some embedding and extension theorems of the Polish school and of MacShane follow from the general Yoneda-Kan lore;
- profunctor composition is Bellman-Fenchel convolution.

It is important that, in general, metric spaces satisfy only the two axioms for a V-category; the evidence is compelling that the usually-given more restrictive definition was too hastily fixed. Note that our V itself is quite non-symmetric (from now on we use "symmetry" of A to mean that $A^{\mathrm{op}} = A$ in an object-preserving way, rather than to mean that the tensor is commutative). A metric space can always be symmetrized (by one of two methods, + and max), but it is often better to delay that until the last stage of a calculation, because the natural asymmetry carries considerable information and also because the main rules for passing from one stage of a calculation to the next are adjointness relations. Even though examples from pure geometry are symmetric, many constructions arising in dynamics as well as many constructions in analysis lead naturally to non-symmetric metric spaces; for example, the Hausdorff metric on subsets of a metric space, or the usual distance between subsets of a probability space (usually discussed only in their symmetrized form) yield in particular an "approximate inclusion" partial order upon applying the standard monoidal functor (represented by the unit 0) from V to the cartesian-closed poset V_0 of truth-values.

Likewise, metric spaces need not have all distances finite, but one can (when appropriate) restrict consideration to those points which have finite distance to a given part. The coproducts in V-cat naturally have infinite distance between points in different summands; infinite distance corresponds to a vacuous hom-set in the case of ordinary categories.

The relation between truth-values V_0 and distances V may be understood, informally, in terms of the cost or work required to transform or move one point to another, and

formally in terms of three adjoint monoidal functors. The inclusion of V_0 into V interprets "true" as zero distance or "already achieved", but interprets "false" as infinite distance or "unattainably expensive". This inclusion has a right adjoint which transforms any V-category into the underlying V_0-category (or poset) as mentioned above; but it also has a left-adjoint π_0 which is also monoidal and hence transforms a V-category into a different V_0-category ordered this time by finiteness of cost.

The symbol π_0 for the "finiteness" monoidal truth functor was chosen by strict analogy with the relation between simplicial sets and abstract sets, where the connected components concept is indeed the left adjoint of a left adjoint and moreover monoidal (with respect to cartesian product). Following the Hurewicz tradition we can define for any V-category A a corresponding homotopy category $\pi_0 A$ and in particular for V-functors f_1 and f_2 from A to B a corresponding homotopy value $\pi_0(B^A)(f_1, f_2)$.

The content of the resulting "homotopy theory" is largely about rotations: Defining translations to mean automorphisms at finite distance from the identity, one sees that these form a normal sub-group with a recognizable quotient group in the case of Euclidian space, where the "search light effect" shows that these are indeed only the translations.

Closed subsets of a metric space have been identified with certain Lip_1 functions on the whole space in both constructive analysis and variational calculus. More precisely, every V presheaf is a V colimit of representables, but among those are the mere V_0 colimits (infima) and indeed between those the sup metric is the same as the (non-symmetric) Hausdorff metric. Every presheaf has in particular its zero set (and more generally sub-level sets).

The interpretation of presheaves as refined subsets suggests the following further construction: By definition, representables A are V-adequate in presheaves $V^{A^{\mathrm{op}}}$, but how co-adequate are they? That is measured by the monad which is the composite of the Isbell conjugacies to and from $(V^A)^{\mathrm{op}}$, i.e. double dualization into the identity bi-module. The action of this monad on subsets is the formation of the closed convex hull (at least in case A itself is a closed convex subset of a suitably reflexive Banach space).

Although habitually the diameter is used as a measure of the size of a subset, for many purposes a more appropriate (because more functorial) quantity is the radius, defined as follows: The direct limit functor from V^A to V exists and in fact is just inf; given any presheaf F on A

$$\mathrm{rad}(F) = \inf{}^{\mathrm{op}}(F^*)$$

where $(\ )^*$ is Isbell conjugation.

V-cat is itself a monoidal closed category and moreover the monoidal endo-functors of V act on it, giving rise to a fibered category whose maps include Lip_λ functions for various λ. But these monoidal functors are considerably more general than multiplication by a constant so that Lipschitz continuity, as well as Hausdorff dimension, admit much more refined measurements. Note that the square root, but not squaring, is monoidal. This suggests a whole family of monoidal structures on V-cat interpolating between the standard one given by V and the cartesian product at the other extreme (probably it

already occurred to analysts that an equation like

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$$

indicates that the parameterization chosen may not have been the most natural.) Thus, contrary to the apology in the introduction of the 1973 paper, it appears that the unique role of the Pythagorean tensor does indeed have expression strictly in terms of the enriched category structure.

The geodesic re-metrization G is the co-monad on V-cat, resulting from a general categorical idea: Namely, it measures the adequacy of a particular family of objects, in this case a family of intervals parameterized by V itself.

Recent work of Gromov and others suggests that V-cat itself has a useful structure as a V-category. Presumably the Gromov distance between two metric spaces A and B is the symmetrization of a more refined invariant obtained as their Hausdorff distance in an extremal metric on $A + B$; but the latter metrics are determined by bimodules, which is a standard V notion!

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METRIC SPACES, GENERALIZED LOGIC, AND CLOSED CATEGORIES

(Conferenza tenuta il 30 marzo 1973) *

SUNTO. — In questo articolo viene rigorosamente sviluppata l'analogia fra $\mathrm{dist}(a, b) + \mathrm{dist}(b, c) \geq \mathrm{dist}(a, c)$ e $\hom(A, B) \otimes \hom(B, C) \to \hom(A, C)$, giungendo a numerosi risultati generali sugli spazi metrici, come conseguenza di una « logica pura generalizzata » i cui « valori di verità » sono scelti in una arbitraria categoria chiusa.

INTRODUCTION.

It is a banality that all mathematical structures of a given kind constitute the objects of a category; the sequence: elements/structures/categories thus has led some people to attempt to characterize the philosophical significance of the theory of categories as that of a « third level of abstraction ». But the theory of categories actually penetrates much more deeply than that attempted characterizaion would suggest toward summing up the essence of mathematics. The kinds of structures which actually arise in the practice of geometry and analysis are far from being « arbitrary », and indeed in this paper we will investigate a particular case of the way in which logic should be specialized to take account of this experience of non-arbitrariness, as concentrated in the thesis that *fundamental* structures are themselves categories. Two cases of this thesis have been known for 30 years; an ordered set (often called poset) is a category in which for any ordered pair of objects there is at most one morphism from the first to the second, while a group is a category in which there is just one object and in which every morphism is a isomorphism. That the thesis has non-vacuous implications for these two cases follows from the facts that when the general idea

* Pervenuta in Redazione il 16 novembre 1973.

of functor between categories is restricted to the special categories (that is to the posets, respectively to the groups) it agrees with the correct idea of morphism between these fundamental structures (i.e. with order-preserving map, respectively group homomorphism) and that a functor from one of the special categories to some category (e.g. that of vector spaces) is itself an important structure (a direct system of vector spaces, respectively a linear group-representation; moreover the correct morphism between *these* structures are then just the natural transformations). Two further cases of the thesis have been developed in the past 10 years: functorial semantics, in which categories with special properties are identified as theories and special functors as interpretations or models, and the theory of topoi, in which certain categories correspond to (usefully generalized) topological spaces.

By taking account of a certain natural generalization of category theory within itself, namely the consideration of strong categories whose hom-functors take their values in a given « closed category » $\mathcal{V}$ (not necessarily in the category $\mathcal{S}$ of abstract sets), we will show below that it is possible to regard a metric space as a (strong) category and that moreover by specializing the constructions and theorems of general category theory we can deduce a large part of *general* metric space theory. The theory of closed categories (and of strong categories valued in them) was originally developed to deal with more complicated examples such as compactly generated topological spaces, Banach spaces, differential graded modules, etc., and moreover some of the publications on the subject seem forbiddingly technical to the beginner. I hope that this article can also be read as an introduction to closed categories on the basis of the guiding example of metric spaces considered as strong categories valued in the closed category of nonnegative real quantities.

Since closed categories are just what is suficient to have a reasonable theory of strong categories, we consider some examples of the latter first in order to bring out the elementary nature of the analogy on which the present work is based. In a *metric space* X, we will denote by $X(a, b)$ the non-negative real quantity (we will allow the value ∞) of X-distance from the point a to the point b. Then the laws satisfied by X are greater-than relations

$$\begin{aligned} X(a, b) + X(b, c) &\geq X(a, c) \\ 0 &\geq X(a, a) . \end{aligned}$$

(see below for remarks on the possible non-symmetry of the metric).

In a *category* X, we will denote by $X(a, b)$ the abstract set of X-morphisms from the object a to the object b. Then the composition law and specification of identity morphisms for X are mappings

$$\begin{array}{rcl} X(a, b) \times X(b, c) & \rightarrow & X(a, c) \\ 1 & \rightarrow & X(a, a) \end{array}$$

(which are subject to associativity and unity axioms which may be expressed as commutative diagrams of mappings betwen abstract sets, using elementary properties of the cartesian product $\times$ of abstract sets and of the one-element set 1). In a *poset* X, we will denote by $X(a, b)$ the truth-value of the X-dominance of the element a over the element b. Then the transitivity and reflexivity laws for X are entailments

$$\begin{array}{rcl} X(a, b) \wedge X(b, c) & \vdash & X(a, c) \\ \text{true} & \vdash & X(a, a)\,. \end{array}$$

If K is a commutative ring, then in a K-*additive category* X we will denote by $X(a, b)$ the K-module of X-morphisms from the object a to the object b. Then the composition and identity laws for X are K-linear mappings

$$\begin{array}{rcl} X(a, b) \otimes X(b, c) & \rightarrow & X(a, c) \\ K & \rightarrow & X(a, a) \end{array}$$

(again subject to associativity and unity axioms which may be expressed by commutative diagrams of K-linear mappings of K-modules, using elementary properties of the K-tensor product $\otimes$ of K-modules and of the K-module K). Thus we are led to consider that a greater-than-or-equal-to relation between nonnegative real quantities is analogous to a K-linear mapping between K-modules, since both are morphisms of possible hom-values for categories X, but in two different closed categories $\mathcal{V}$. Similarly, the sum of quantities is analogous to the tensor product of modules, both because they play the same role in the structure of a $\mathcal{V}$-valued category and also because they satisfy the same « elementary properties » (within $\mathcal{V}$ itself) of functoriality (i.e. monotonicity in the case of quantities), of associativity and commutativity up to $\mathcal{V}$-isomorphism (i.e. up to equality in the case of quantities), and of having a unit object K (the zero quantity). We have the table

X	hom-values for X	composition law and identity law for X	domain of composition law for X	domain of identity law for X
metric space	nonnegative real quantities	$\geqq$	sum	zero
category	abstract sets	mapping	cartesian product	one element set
poset	truth values	entailment	conjunction	true
$\mathcal{V}$-valued category	objects in $\mathcal{V}$	morphism in $\mathcal{V}$	« tensor » product in $\mathcal{V}$	unit object K for tensor product in $\mathcal{V}$

The associativity and unity axioms which the composition law and identity law of a $\mathcal{V}$-based category X must satisfy are automatic in the case of metric spaces or posets since *all* diagrams in $\mathcal{V}$ commute if $\mathcal{V} =$ reals or $\mathcal{V} =$ truth-values.

None of our results in this paper will depend on the additional Frechet axioms:

$$\begin{array}{llll} \text{if} & X(a, b) & = & 0 \text{ then } a = b \\ & X(a, b) & < & \infty \\ & X(a, b) & = & X(b, a)\,. \end{array}$$

The first of these is not very natural from the categorical viewpoint since it corresponds to requiring that isomorphic objects are equal; passage to the quotient can be avoided (as it *must* be for ordinary categories) by employing *equivalence* (broader than isomorphism) of strong categories. Allowing ∞ among the quantities is precisely analogous to including the empty set among abstract sets, and it is done for similar reasons of completeness; a metric space can be analyzed as a structured system of metric spaces with finite distances by considering the equivalence relation defined by « $X(a, b) < \infty$ and $X(b, a) < \infty$ ». The non symmetry is the more serious generalization, and moreover occurs in many naturally arising examples, such as $X(a, b) =$ work required to get from a to b in mountainous region X. Also within analysis itself a naturally arising metric is often

non-symmetric but traditionally symmetrized by one of the two procedures

$$X(a, b) + X(b, a)$$
$$\max(X(a, b) \quad , \quad X(b, a))$$

which in fact could be applied to any metric; we mention three common examples where this nonsymmetry exists. If M is any Boolean algebra equipped with an outer measure, then

$$M(a, b) \underset{\text{def}}{=} M(b - a)$$

(where $b - a = b \cap a'$ in the Boolean algebra and $M(c)$ denotes the measure of c) defines a metric space in which

$$0 \geq M(a, b) \quad \text{iff} \quad a \supseteq b$$

almost everywhere.

(For more about this example see the section below on closed functors).

If K is any convex set we may define a metric on it by

$$K(a, b) = \inf_{f: a \to b} \{ - \log(\alpha_f) \}$$

where $f: a \to b$ means that $f \in K$ with a on the open segment from f to b, with α_f then denoting the $0 < \alpha < 1$ with $a = (1 - \alpha) f + \alpha b$. (The proof of the triangle inequality follows from the fact that K is actually a « normed category », since if $g: b \to c$ we can define $fg: a \to c$ by

$$fg = \frac{1 - \alpha_f}{1 - \alpha_f \alpha_g} f + \frac{\alpha_f (1 - \alpha_g)}{1 - \alpha_f \alpha_g} g,$$

note that this is associative and that $\alpha_{fg} = \alpha_f \alpha_g$, and adjoin identities formally). The notion of « normed category » can also be related to the (nonsymmetric) Hausdorff metric

$$2^X(A, B) = \sup_{a \in A} \inf_{b \in B} X(a, b)$$

for subsets A, B of a metric space X: let $f: A \to B$ mean that f is any mapping from A to B and define

$$|f| = \sup_{a \in A} X(a, af);$$

then by the axiom of choice

$$2^X(A, B) = \inf_{f: A \to B} |f|$$

and the fundamental property of a normed category

$$|f| + |g| \geq |fg|$$

leads to a proof of the triangle inequality. We will leave as an exercise for the reader to define a closed category $\mathcal{S}(\mathbf{R})$ such that « normed categories » are just $\mathcal{S}(\mathbf{R})$-valued categories and a « closed functor » $\inf: \mathcal{S}(\mathbf{R}) \to \mathbf{R}$ which induces the passage from any « normed category » to a metric space with the same objects. Another approach to the Hausdorff metric will be evident from the discussion below of the comprehension scheme. The canonical symmetrization procedure actually applies to categories valued in any given closed category $\mathcal{V}$; in general, symmetry of a $\mathcal{V}$-valued category X has to mean (not a property but) a given structure consisting of $\mathcal{V}$-isomorphisms

$$\sigma_{ab}: X(a, b) \longrightarrow X(b, a)$$

subject to suitable coherence axioms, and the first canonical procedure (specializing to the sum in the case of a metric space) is

$$\text{sym}\ (X)(a, b) = X(a, b) \otimes X(b, a).$$

We still have not, in this introduction, touched on the property of closed categories for which they are called « closed ». Basically, closed categories are closed with respect to the operation of forming the hom of two objects, so that $\mathcal{V}$ itself is a fundamental example of a $\mathcal{V}$-category. The internal Hom (a, c) in $\mathcal{V}$ is related to the internal tensor product in $\mathcal{V}$ by adjointness so that there is a natural one-to-one correspondence between the $\mathcal{V}$-morphisms

$$b \longrightarrow \text{Hom}\ (a, c)$$

and the $\mathcal{V}$-morphisms

$$a \otimes b \longrightarrow c.$$

Thus in the closed category $\mathbf{R} = [0; \infty]$ in which metric spaces are valued

$$\text{Hom}\ (a, c) = \begin{cases} c - a & \text{if} \quad c \geq a \\ 0 & \text{if} \quad a \geq c \end{cases}$$

so that the Hom-$\otimes$ adjointness reduces (if we denote by « minus » this truncated subtraction) to the if-and-only-if

$$\frac{b \geq c - a}{a + b \geq c},$$

whereas in the closed category $\mathcal{S}$ in which ordinary categories are valued, it reduces to the rule of lambda conversion

$$\frac{B \to C^A}{A \times B \to C}$$

and in the closed category **2** in which posets are valued, it reduces essentially to modus ponens and the « deduction theorem »

$$\frac{\beta \vdash \alpha \Rightarrow \gamma}{\alpha \wedge \beta \vdash \gamma}$$

i.e. the internal Hom for truth-values is implication. We will also assume that our closed categories $\mathcal{V}$ have cartesian products and coproducts over arbitrary index sets as well as equalizers and coequalizers, i.e. that they are complete and cocomplete. From general properties of adjoint functors it follows that $\otimes$ preserves direct limits in each variable separately, while

$$\mathrm{Hom}\,(\varinjlim a_i, c) \cong \varprojlim \mathrm{Hom}\,(a_i, c)$$

$$\mathrm{Hom}\,(a, \varprojlim c_j) \cong \varprojlim \mathrm{Hom}\,(a, c_j)$$

For example in $\mathbf{R}$, $\varprojlim$ means sup and $\varinjlim$ means inf; in particular 0 is the empty $\varprojlim$ and ∞ is the empty $\varinjlim$ so that

$$\begin{aligned} a + \infty &= \infty \\ c - \infty &= 0 \\ 0 - a &= 0\,. \end{aligned}$$

This completeness of $\mathcal{V}$ itself will be necessary for most of our general constructions. We will not consider (categorical) completeness of $\mathcal{V}$-valued categories; but on the other hand we will see that completeness in the *Cauchy* sense does have a meaning for categories valued in any closed $\mathcal{V}$.

The general constructions of functor categories, free categories, left and right Kan extensions, and « discrete » fibrations reduce in the very special case $\mathcal{V} = \mathbf{2}$ to higher types, transitive closure of a relation, existential and universal quantification, and the principle of set abstraction. Like the operation of implication mentioned above, the position of all these constructions in the general scheme as well as their fundamental properties of transformation are uniquely determined by adjointness. Since logic signifies formal relationships which are general in character, we may more precisely identify logic with this scheme of interlocking adjoints and then observe that all of logic applies *directly* to structures valued in an arbitrary closed category $\mathcal{V}$ (not only to structures valued in truth-values). For example, in quantitative logic (the case $\mathcal{V} = \mathbf{R}$ with which we will be mainly concerned in this paper) the isometric embedding of a metric space into a space of functions with sup metric is the application of the exactly same principle of logic which in the case $\mathcal{V} = \mathbf{2}$ gives Dedekind's representation of a poset by order ideals, and the upper and lower integrals of a real function are precisely cases of the generalized universal and existential quantification. This vast generalization is quite compatible with the specialization of logic called for in the first paragraph: although for any given $\mathcal{V}$ we could consider « arbitrary » $\mathcal{V}$ -valued structures, there is one type of such structure which is of first importance, namely for $\mathcal{V}$ respectively truth-values, quantities, abstract sets, abelian groups, the structure of respectively poset, metric space, category, additive category (a very natural generalization of ring) is the generally useful first approximation possible with $\mathcal{V}$-valued logic for analyzing various problems; it even seems that there is a natural second approximation, namely the structure of a « rigidly $\mathcal{V}$-closed $\mathcal{V}$-category » which in the four cases mentioned specializes *roughly* to partially ordered abelian group, normed abelian group, rigidly closed category, and (in the additive case) to a common generalization of the category of locally free modules on an algebraic space and the category of finite-dimensional representations of an algebraic group. Detailed discussion of this second approximation awaits further investigation, as does the extension of the results of this paper to an arbitrary base topos, i.e. the extension from the *constant* abstract sets which we consider here to continuously *variable* sets as «sets» of points for metric spaces, as index «sets» for products, etc.

Of course the really deep results in a subject depend very much on the particularity of that subject and the results we offer here in the field of metric spaces, taken individually, will justly appear

shallow to those with any experience. Indeed for me the surprising aspect was that methods originally devised to deal with quite different fields of algebra and geometry could yield any significant known theorems at all (for example the known theorem on extension of Lipschitz maps in section three). But there are many particularities, for example the special role of quadratic metrics, which I do not see how could be a result of « generalized logic ».

1. - Closed categories, strong categories, strong functors, closed functors.

In this paper we use the term « closed category » as short for « bicomplete symmetric monoidal closed category ». That is, a closed category $\mathcal{V}$ has equalizers, coequalizers, set-indexed products and coproducts plus a « monoidal » structure, which is symmetric and closed. A monoidal structure is a given functor

$$\mathcal{V} \times \mathcal{V} \xrightarrow{\otimes} \mathcal{V}$$

which is symmetric

$$u \otimes v \cong v \otimes u$$

and associative

$$u \otimes (v \otimes w) \cong (u \otimes v) \otimes w$$

and has a unit object k satisfying

$$k \otimes v \cong v \cong v \otimes k\,;$$

more precisely, the symmetry, associativity, and unitary isomorphisms are in general required to be given, natural, and «coherent», but since these conditions are automatic in the particular examples we consider, we do not emphasize them (MacLane). That the monoidal structure is closed means that we are further given a functor

$$\mathcal{V}^{op} \times \mathcal{V} \xrightarrow{\mathrm{Hom}} \mathcal{V}$$

and two natural transformations

$$u \xrightarrow{\lambda} \mathrm{Hom}\,(a, a \otimes u)$$

$$a \otimes \mathrm{Hom}\,(a, v) \xrightarrow{\epsilon} v$$

such that the two processses

$$a \otimes u \xrightarrow{f} v \rightsquigarrow u \xrightarrow{\lambda} \mathrm{Hom}\,(a, a \otimes u) \xrightarrow{\mathrm{Hom}\,(a,f)} \mathrm{Hom}\,(a, v)$$

$$u \xrightarrow{g} \mathrm{Hom}\,(a, v) \rightsquigarrow a \otimes u \xrightarrow{a \otimes g} a \otimes \mathrm{Hom}\,(a, v) \xrightarrow{\in} v$$

are inverse bijections,

$$\downarrow\uparrow \quad \frac{a \otimes u \to v}{u \to \mathrm{Hom}\,(a, v)}\,.$$

Thus in particular there is a natural bijection

$$\frac{a \to v}{k \to \mathrm{Hom}\,(a, v)}$$

between $\mathcal{V}$-morphism $a \to v$ and $\mathcal{V}$-morphisms from the unit object to the object $\mathrm{Hom}(a, v)$ for any two objects a, v of $\mathcal{V}$, in terms of which the natural transformation $\in$ behaves as an « evaluation » morphism, and two successive applications of appropriate instances of evaluation corresponds to a $\mathcal{V}$-morphism

$$\mathrm{Hom}\,(a, b) \otimes \mathrm{Hom}\,(b, c) \to \mathrm{Hom}\,(a, c)$$

which represents a « strong » form of composition. There is moreover a natural double-dualization morphism

$$u \to \mathrm{Hom}\,(\mathrm{Hom}\,(u, b), b)$$

for any fixed object b.

The simplest non-trivial example **2** of a closed category, which serves as the values for classical logic, has two objects false and $k =$ true and three morphisms

$$\begin{aligned} \text{false} &\vdash \text{false} \\ \text{false} &\vdash \text{true} \\ \text{true} &\vdash \text{true} \end{aligned}$$

with conjunction and implication as the tensor and Hom. Conjunction is of course symmetric and associative and since

$$u \vdash a \Longrightarrow (a \wedge u)$$

$$a \wedge (a \Longrightarrow v) \vdash v$$

always, we have the bijection

$$\frac{a \wedge u \vdash v}{u \vdash (a \Longrightarrow v)}$$

i.e. for any assignments of false, true to the three variables a, u, v either both or neither of these entailments exist. In particular

$$\frac{a \vdash v}{\text{true} \vdash a \Longrightarrow v}$$

and

$$(a \Longrightarrow b) \wedge (b \Longrightarrow c) \vdash a \Longrightarrow c$$

and

$$u \vdash (u \Longrightarrow b) \Longrightarrow b \,.$$

A somewhat more general and « dual » example is as follows: let $\mathcal{M}$ be any system of subsets of a given finite set which contains the empty subset $k = \varnothing$ and which is closed with respect to finite unions and with respect to set-theoretic difference; for $a, v \in \mathcal{M}$ let $a \to v$ mean that $a \supseteq v$, and let $a \otimes u$ mean $a \cup u$. Then $\mathcal{M}$ is a closed category since

$$\frac{a \cup u \supseteq v}{u \supseteq v - a}$$

(Even if $\mathcal{M}$ is not finite, our condition of bicompleteness can often be achieved by considering e.g. measurable sets *modulo* null sets).

Our central example $\mathbf{R}$ has as objects all non-negative real quantities (including ∞), as morphisms $a \to v$ the greater-than-or-equal-to relations $a \geq v$, and as « tensor » $a \otimes u = a + u$ the sum of quantities. Then Hom is forced to be truncated subtraction

$$\mathrm{Hom}\,(a, v) = \begin{cases} v - a & v \geq a \\ 0 & a \geq v \end{cases}$$

(denoted simply as subtraction) where in particular

$$\begin{aligned} \infty - \infty &= 0 \\ \infty - a &= \infty \quad \textit{if} \quad a \neq \infty \\ v - \infty &= 0 \end{aligned}$$

(bicompleteness holds since

$$\prod_{i \in I} v_i = \sup_{i \in I} v_i$$

$$\sum_{i \in I} a_i = \inf_{i \in I} a_i$$

and equalizers and coequalizers are trivial). We have in particular

$$\frac{a \geq v}{0 \geq v - a}$$

$$(b - a) + (c - b) \geq c - a$$

$$u \geq b - (b - u) .$$

The « original » example of a closed category is the category $\mathcal{S}$ of abstract sets and mappings with the (unique) closed structure in which tensor means cartesian product, $k = 1$ is a one-element set, and $\mathrm{Hom}(a, v)$ is the (abstract) set of (indices for) all the mappings $a \to v$.

Given a closed category $\mathcal{V}$, a strong category valued in $\mathcal{V}$, or simply a $\mathcal{V}$-category X is any structure consisting of a specified set of X-objects $a, b, c \ldots$ together with the assignment of an object $X(a, b)$ of $\mathcal{V}$ to every ordered pair of X-objects $\langle a, b \rangle$, the assignment of a $\mathcal{V}$-morphism

$$X(a, b) \otimes X(b, c) \xrightarrow{\mu_{abc}} X(a, c)$$

to every ordered triple $\langle a, b, c \rangle$ of X-objects, and the assignment of a $\mathcal{V}$-morphism

$$k \xrightarrow{\eta_a} X(a, a)$$

to every X-object a, subject to the conditions that the following diagrams in $\mathcal{V}$ always commute

$$\begin{tikzcd}
X(a,b) \otimes (X(b,c) \otimes X(c,d)) \arrow[r, "X(a,b) \otimes \mu_{bcd}"] \arrow[d, "\mathcal{V}\text{ — assoc } \cong"'] & X(a,b) \otimes X(b,d) \arrow[dd, "\mu_{abd}"] \\
(X(a,b) \otimes X(b,c)) \otimes X(c,d) \arrow[d, "\mu_{abc} \otimes X(c,d)"'] & \\
X(b,c) \otimes X(c,d) \arrow[r, "\mu_{bcd}"'] & X(a,d)
\end{tikzcd}$$

$$\begin{tikzcd}
X(a,b) \arrow[r, "\mathcal{V}\text{—unitary}\cong"] \arrow[d, "\mathcal{V}\text{—unitary}\cong"'] & k \otimes X(a,b) \arrow[r, "\eta_a \otimes X(a,b)"] & X(a,a) \otimes X(a,b) \arrow[dd, "\mu_{abd}"] \\
X(a,b) \otimes k \arrow[d, "X(a,b) \otimes \eta_b"'] \arrow[rr, "\mathrm{id}"] & & \\
X(a,b) \otimes X(b,b) \arrow[rr, "\mu_{abb}"'] & & X(a,b)
\end{tikzcd}$$

Thus an $\mathcal{S}$-category is just an ordinary (small) category-whereas if $\mathcal{V} = Ab$ the category of abelian groups then an Ab-category with one object is just an ordinary ring (and more general Ab-categories arise in linear algebra just as naturally as do rings, e.g. *all* finite rectangular matrices over a given field form an Ab-category with the natural numbers as objects). The associativity and identity axioms (i.e. the above commutative diagrams) are (like coherence) automatic in case $\mathcal{V}$ itself is a poset, and hence, as claimed in the introduction, a **2** -category is an arbitrary poset while an **R**-category is an arbitrary (generalized) metric space.

Every $\mathcal{V}$-category X has an opposite X^{op} with the same objects and units but with

$$X^{op}(a, b) = X(b, a)$$

and

$$\begin{tikzcd}
X^{op}(a,b) \otimes X^{op}(b,c) \arrow[rr, "\mu^{op}_{abc}"] \arrow[d, equal] & & X^{op}(a,c) \arrow[d, equal] \\
X(b,a) \otimes X(c,b) \cong X(c,b) \otimes X(b,a) \arrow[rr, "\mu_{cba}"'] & & X(c,a)
\end{tikzcd}$$

If we define also

$$\mathrm{sym}\,(X)\,(a, b) = X(a, b) \otimes X(b, a)$$

with

$$\begin{tikzcd}
\mathrm{sym}\,(X)(a,b) \otimes \mathrm{sym}\,(X)(b,c) \arrow[rr, "\mu^{\mathrm{sym}}_{abc}"] \arrow[d, equal] & & \mathrm{sym}\,(X)(a,c) \arrow[dd, equal] \\
(X(a,b) \otimes X(b,a)) \otimes (X(b,c) \otimes X(c,b)) \arrow[d, "\cong"'] & & \\
(X(a,b) \otimes X(b,c)) \otimes (X(c,b) \otimes X(b,a)) \arrow[rr, "\mu_{abc} \otimes \mu_{cba}"'] & & X(a,c) \otimes X(c,a)
\end{tikzcd}$$

and

$$\begin{tikzcd} k \arrow[r, "\eta_a^{\mathrm{sym}}"] \arrow[d, "\cong"'] & \mathrm{sym}\,(X)\,(a,\,a) \arrow[d, equal] \\ k \otimes k \arrow[r, "\eta_a \otimes \eta_a"] & X(a,\,a) \otimes X(a,\,a) \end{tikzcd}$$

then sym(X) is a $\mathcal{V}$-category isomorphic with its opposite in an object-preserving manner.

In case the objects of $\mathcal{V}$ can be indexed by a set (which by bicompletness actually forces $\mathcal{V}$ to have a poset as its underlying category) then $\mathcal{V}$ itself is an example of a $\mathcal{V}$-category (more generally any small part of $\mathcal{V}$ becomes a $\mathcal{V}$-category) by setting

$$\mathcal{V}\,(a,\,b) \;=\; \mathrm{Hom}\,(a,\,b)$$

Thus **2** is an example of a poset, while **R** is a (highly non-symmetric) example of a metric space. But sym(**R**) is the usual metric on the reals.

If X and Y are two $\mathcal{V}$-categories, then by a $\mathcal{V}$-functor $X \xrightarrow{f} Y$ is meant any structure consisting of mapping of the objects of X into the objects of Y together with an assignment of a $\mathcal{V}$-morphism

$$X(a,\,b) \xrightarrow{f_{ab}} Y(fa\,,fb)$$

to every ordered pair $\langle a, b\rangle$ of X-objects, subject to the commutativity of the following diagrams in $\mathcal{V}$.

$$\begin{tikzcd} X(a,\,b) \otimes X(b\,,\,c) \arrow[r, "\mu^X_{abc}"] \arrow[d, "f_{ab} \otimes f_{bc}"'] & X(a\,,\,c) \arrow[d, "f_{ac}"] \\ Y(fa,fb) \otimes Y(fb,fc) \arrow[r, "\mu^Y_{fa\,fb\,fc}"'] & Y(fa,fc) \end{tikzcd}$$

$$\begin{tikzcd} & X(a,\,a) \arrow[dd] \\ k \arrow[ur, "\eta^X_a"] \arrow[dr, "\eta^Y_{fa}"'] & \\ & Y(fa,fa) \end{tikzcd}$$

Thus an $\mathcal{S}$-functor is just an ordinary functor between (small) categories, an Ab-functor is just an additive functor (e.g. a ring

homomorphism in case X has only one object) while a **2** functor (satisfying

$$X(a, b) \vdash Y(fa, fb))$$

is an arbitrary order-preserving mapping from the poset X to the poset Y, and an **R**-functor (satisfying

$$X(a, b) \geq Y(fa, fb))$$

is an arbitrary distance-decreasing map (Lipschitz map of Lipschitz constant ≤ 1) from the metric space X to the metric space Y.

If $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are $\mathcal{V}$-functors we can define $(gf)_{ab}$ as the composition

$$X(a, b) \xrightarrow{f_{ab}} Y(fa, fb) \xrightarrow{g_{fa, fb}} Z(gfa, gfb)$$

in $\mathcal{V}$ to obtain a $\mathcal{V}$-functor $X \xrightarrow{gf} Z$ and thus a category $\mathcal{V}$-Cat whose objects and morphisms are all the $\mathcal{V}$-categories and $\mathcal{V}$-functors. In the next section we will see that $\mathcal{V}$-Cat (e.g. the category of metric spaces and distance-decreasing maps) is itself a closed category.

A morphism between closed categories $\mathcal{V} \xrightarrow{\Phi} \mathcal{W}$ is basically the concentrated expression of a process which takes every $\mathcal{V}$-category X into a $\mathcal{W}$-category ΦX with the same objects, every $\mathcal{V}$-functor f into a $\mathcal{W}$-functor Φf and in general interprets $\mathcal{V}$-valued category-theory as $\mathcal{W}$-valued category-theory; in view of many examples it is too restrictive to require that Φ « strictly » preserves the tensor product and Hom, so we adopt the following definition. A closed functor is a triple consisting of a functor $\mathcal{V} \xrightarrow{\Phi} \mathcal{W}$, a $\mathcal{W}$-morphism

$$k_{\mathcal{W}} \xrightarrow{\varphi_0} \Phi(k_{\mathcal{V}})$$

and a natural transformation

$$\Phi(u) \underset{\mathcal{W}}{\otimes} \Phi(v) \xrightarrow{\varphi_{uv}} \Phi(u \underset{\mathcal{V}}{\otimes} v)$$

subject to compatibility between themselves and with the given symmetry, associativity and unitary isomorphisms in $\mathcal{V}$ and $\mathcal{W}$ (Eilenberg-Kelly).

For example, any locally small closed category $\mathcal{V}$ has a canonical closed functor

$$\mathcal{V} \xrightarrow{V} \mathcal{S}$$

to the category of abstract sets defined by

$$V(u) = \text{set of all } \mathcal{V}\text{-morphisms } k \to u$$
$$1 \xrightarrow{\varphi_0} V(k) = \text{the identity } \mathcal{V}\text{-morphism } k \to k$$

with

$$V(u_1) \times V(u_2) \xrightarrow{\varphi_{u_1 u_2}} V(u_1 \otimes u_2)$$

the mapping taking any ordered pair $k \xrightarrow{f_1} u_1$, $k \xrightarrow{f_2} u_2$ into

$$k \xrightarrow{\cong} k \otimes k \xrightarrow{f_1 \otimes f_2} u_1 \otimes u_2 .$$

If $\mathcal{M}$ is a Boolean ring of sets made into a closed category with union as tensor as above then a closed functor

$$\mathcal{M} \xrightarrow{M} \mathbf{R}$$

is any order-preserving real-valued function on $\mathcal{M}$ satisfying

$$0 = M(\emptyset)$$
$$M(a) + M(b) \geq M(a \cup b)$$

i.e. an « outer measure ». The closed functors $\mathbf{R} \xrightarrow{\lambda} \mathbf{R}$ are just the « subadditive » order-preserving functions, i.e.

$$u_1 \geq u_2 \implies \lambda u_1 \geq \lambda u_2$$
$$0 = \lambda 0$$
$$\lambda u_1 + \lambda u_2 \geq \lambda (u_1 + u_2) .$$

If $\mathcal{V} \xrightarrow{\Phi} \mathcal{W}$ is any closed functor and X any $\mathcal{V}$-category, then

$$(\Phi X)(a, b) = \Phi(X(a, b))$$

$$\begin{tikzcd}
(\Phi X)(a,b) \underset{\mathcal{W}}{\otimes} (\Phi X)(b,c) \arrow[r, "(\Phi\mu)_{abc}"] \arrow[d, "\varphi"] & (\Phi X)(a,c) \arrow[d, equal] \\
\Phi(X(a,b) \underset{\mathcal{V}}{\otimes} X(b,c)) \arrow[r, "\Phi(\mu_{abc})"'] & \Phi(X(a,c))
\end{tikzcd}$$

$$\begin{tikzcd}
k\mathcal{W} \arrow[rr, dashed, "(\Phi\eta)_a"] \arrow[dr, "\varphi_0"'] & & (\Phi X)(a,a) \\
& \Phi(k\mathcal{V}) \arrow[ur, "\Phi(\eta_a)"'] &
\end{tikzcd}$$

defines a $\mathcal{W}$-category ΦX with the same set of objects as X. Moreover if $X \xrightarrow{f} Y$ is any $\mathcal{V}$-functor by defining Φf to be the same mapping as f on objects and $(\Phi f)_{ab} = \Phi(f_{ab})$ we obtain a $\mathcal{W}$ — functor $\Phi X \to \Phi Y$, and moreover this process is compatible with composition of $\mathcal{V}$-functors (and more). For example, any $\mathcal{V}$-category X has an underlying $\mathcal{S}$-category VX because of the closed functor $\mathcal{V} \xrightarrow{V} \mathcal{S}$.

In particular if $\mathcal{V}$ is small and $\mathcal{V} \xrightarrow{\Phi} \mathcal{W}$ is a closed functor, then $\Phi\ \mathcal{V}$ is a $\mathcal{W}$-category. As an example, any Boolean ring of sets $\mathcal{M}$ equipped with an outer measure $\mathcal{M} \xrightarrow{M} \mathbf{R}$ becomes a metric space.

If $\mathbf{R} \xrightarrow{\lambda} \mathbf{R}$ is a closed functor X and Y are metric spaces, then an $\mathbf{R}$-functor

$$\lambda X \xrightarrow{f} Y$$

is just a function satisfying

$$\lambda(X(a,b)) \geq Y(fa, fb).$$

Since multiplication by a given constant is subadditive and thus a closed functor, we see that the study of (a very natural generalization of) arbitrary Lipschitz mappings $X \to Y$ is naturally incorporated into the functorial set-up.

A special class of closed categories are those in which the tensor product is actually the categorical product, which amounts to

$$\frac{u \to v_1 \otimes v_2}{u \to v_1,\ u \to v_2}$$

further « rule of inference of propositional Logic »; such closed categories are usually called cartesian closed. $\mathcal{S}$ and **2** are cartesian closed, while Ab and **R** are not. If a category admits a cartesian closed structure, then that structure is essentially unique. Ab does not admit a cartesian closed structure, but the underlying category of **R** does admit the cartesian closed structure $\mathbf{R}_{cart}$ in which

$$u \otimes v = \max(u, v)$$

$$\mathrm{Hom}(a, v) = \begin{cases} v & v > a \\ 0 & a \geq v \end{cases}$$

An $\mathbf{R}_{cart}$-category is just an *ultrametric space*, so that all of our general results may be reinterpreted as applying to ultrametric spaces. Moreover, the identity mapping $\mathbf{R}_{cart} \to \mathbf{R}$ is a closed functor (i.e. $a + v \geq \max(a, v)$) which induces the inclusion

$$\mathbf{R}_{cart}\text{-Cat} \to \mathbf{R}\text{-Cat}$$

of ultrametric spaces into all metric spaces.

2. - Functor Categories, Yoneda embedding, adequacy, comprehension scheme.

If A and X are two $\mathcal{V}$-categories then there is a $\mathcal{V}$-category $A \times X$ whose objects are the ordered pairs $\langle a, x \rangle$ of objects and whose hom-values are

$$(A \times X)(\langle a, x \rangle, \langle a', x' \rangle) = A(a, a') \times X(x, x')$$

the last being the cartesian product (e.g. max. in the case of **R**) in $\mathcal{V}$. But if $\mathcal{V}$ is not cartesian closed, there is another more important $\mathcal{V}$-category structure on the same objects, defined by

$$(A \otimes X)(\langle a, x \rangle, \langle a', x' \rangle) = A(a, a') \otimes X(x, x').$$

This gives, for example, the l_1-style metric on the product of two metric spaces. The unit k for this tensor product has one object $*$ with $k(*, *) = k$.

This tensor product of $\mathcal{V}$-categories always has a « Hom » adjoint to it, making $\mathcal{V}$-cat itself into a closed category. This Hom always has a concrete interpretation in terms of « strong natural

transformations »; we are going to denote it by Y^A, and its objects are just all the $\mathcal{V}$-functors $A \to Y$. Recall that in the usual $\mathcal{S}$-valued case, a natural transformation $f_1 \to f_2$ (where $A \xrightarrow{f_1, f_2} Y$) is a family of Y-morphisms indexed by the objects of A, more exactly, an element of the set $\prod_{a \in A} Y(f_1 a, f_2 a)$ subject to equations indexed by the elements of the $A(a, b)$. In the $\mathcal{V}$-valued case, we define directly (not the set of but) the $\mathcal{V}$-object of natural transformations $Y^A(f_1, f_2)$ by the equalizer in $\mathcal{V}$ of the two morphisms

$$\prod_a \rightrightarrows \prod_{a,b}$$

in

$$Y^A(f_1, f_2) \to \prod_{a \in A} Y(f_1 a, f_2 a) \rightrightarrows \prod_{a,b \in A} \operatorname{Hom}(A(a, b), Y(f_1 a, f_2 b))$$

whose constructions we leave to the reader. A $\mathcal{V}$-morphism $k \to Y^A(f_1, f_2)$ may then be identified with a strong natural transformation $f_1 \to f_2$. In case $\mathcal{V}$ is itself a poset, as with $\mathcal{V} = \mathbf{R}$, the two morphisms are already equal so that $Y^A(f_1, f_2) = \prod_{a \in A} Y(f_1 a, f_2 a)$ in such a case. Since Π in $\mathbf{R}$ means sup, we deduce that

$$Y^A(f_1, f_2) = \sup_{a \in A} Y(f_1 a, f_2 a)$$

in the case of metric spaces, i.e. that the sup metric on the space of 1-Lipschitz maps is a special case of the general notion of $\mathcal{V}$-natural transformations. The reader should be able to verify that there is a bijection

$$\frac{A \otimes X \to Y}{X \to Y^A}$$

between the two indicated sets of $\mathcal{V}$-functors at least in the case $\mathcal{V} = \mathbf{R}$.

Of special interest is the case $\mathcal{V}^{A^{op}}$, because of the existence of the basic Yoneda embedding

$$A \to \mathcal{V}^{A^{op}}$$

which is the $\mathcal{V}$-functor whose value at an object a of A is the $\mathcal{V}$-functor $A^{op} \to \mathcal{V}$ defined by

$$a' \rightsquigarrow A(a', a)$$

and which works on A-morphism objects precisely by use of A-composition. Saying that a $\mathcal{V}$-functor $A \xrightarrow{f} Y$ is $\mathcal{V}$-full-and-faithful iff each of the $\mathcal{V}$-morphisms

$$A(a, b) \xrightarrow{f_{ab}} Y(fa, fb)$$

is actually an isomorphism, we have the important *Yoneda Lemma* (Eilenberg-Kelley): The Yoneda embedding is, for any closed $\mathcal{V}$ and any $\mathcal{V}$-category A, a $\mathcal{V}$-full-and-faithful $\mathcal{V}$-functor.

As a simple example, note that for $\mathcal{V} = \mathbf{2}$, an order-preserving map $A^{op} \to \mathbf{2}$ is equivalent to an order-ideal in A, and the Yoneda embedding is simply Dedekind's representation of a poset by its principal ideals. In the case $\mathcal{V} = \mathcal{S}$ or $\mathcal{V} = k$-Modules, the Yoneda embedding is often (especially when A has one object) called the regular representation of A, and Yoneda's Lemma includes Cayley's theorem on representing an abstract group by transformations. For a metric space A and for each point a, the function assigning to any point a' its distance to a is a distance decreasing function, and Yoneda's Lemma states that assigning to each a the just-described function is an *isometric* embedding of A into the space $\mathbf{R}\, A^{op}$ where the last is equipped with the sup metric.

More generally, given any $\mathcal{V}$-functor $A \xrightarrow{i} X$, we can consider the Yoneda representation of X restricted to A, i.e. the composite $\mathcal{V}$-functor

$$X \to \mathcal{V}^{X^{op}} \xrightarrow{\mathcal{V}^{i^{op}}} \mathcal{V}^{A^{op}}$$

which assigns to each x the functor $A^{op} \to \mathcal{V}$ defined by $a' \rightsquigarrow X(ia', x)$. In case this restricted representation is still $\mathcal{V}$-full--and-faithful on all X, we say after Isbell that i is *$\mathcal{V}$-adequate*, or in case i is an inclusion that A is a $\mathcal{V}$-adequate subcategory of X. This concept is the basis of much representation theory of categories, especially for algebraic categories and topoi, since it often happens that a quite small category A is adequate in a quite large one X. (We hasten to point out that this paragraph is meaningful even when $\mathcal{V}$ is not small; for example $\mathcal{V}\, A^{op}$ can be interpreted in terms of « modules » as in the following section). For example, if $\mathcal{V}$ = abelian groups and A is any ring, then A (considered as a $\mathcal{V}$-category with only one object) is adequate in any category X of A-modules, no matter how large.

In particular, to say that a subspace A of a metric space X is adequate is to say that for any two points x_1, x_2 of X, the inequality

$$X(x_1, x_2) \geq \sup_{a \in A} [X(a, x_2) - X(a, x_1)]$$

is actually an equality, i.e. that for any $d > 0$ there exists $a \in A$ with

$$X(a, x_1) + X(x_1, x_2) \leq X(a, x_2) + d .$$

For example, the unit circle is adequate in the unit disc (this simple example was pointed out to me by Prof. Isbell). A more restricted notion is that of $\mathcal{V}$-density, by which we mean (here differing in terminology with some authors) that $i_* oi^* \cong 1_X$ in the sense of the next section on bimodules; this reduces in the case of metric spaces to the requirement that

$$X(x_1, x_2) = \inf_{a \in A} [X(x_1, ia) + X(ia, x_2)] .$$

Proposition. - If a distance-decreasing map $A \xrightarrow{i} X$ of metric spaces is $\mathbf{R}$-dense, then it is $\mathbf{R}$-adequate.

Proof. - We always have

$$X(a, x_1) + X(x_1, x_2) \leq X(a, x_1) + X(x_1, a) + X(a, x_2) .$$

But by taking $x_2 = x_1$ in the definition of density, we see that $X(a, x_1) + X(x_1, a) \leq d$ for suitable $a \in A$, as required.

Define a particular metric space A whose points are the natural numbers and for which

$$A(n, m) = \begin{cases} \infty & n \neq m \\ 0 & n = m \end{cases} .$$

Then $\mathbf{R}^{A^{op}}$ is just the space of all sequences of nonnegative reals, with the (nonsymmetric) sup metric. Say that a metric space X is *separable* iff there exists an $\mathbf{R}$-dense map $A \to X$. Then

Corollary. - Any separable metric space X can be isometrically embedded in the space of all sequences of non-negative reals with the sup metric.

To discuss the « comprehension scheme » we will limit ourselves to those closed categories $\mathcal{V}$ in which $K = 1$, i.e. in which the unit object for the tensor product is also the terminal object of $\mathcal{V}$; thus for this form of the comprehension scheme, $\mathcal{V}$ may be car-

tesian closed, e.g. $\mathcal{S}$ or $\mathbf{2}$, but more generally e.g. $\mathbf{R}$ satisfies this condition, as does the category of so-called « affine modules » over a given commutative ring (but not the usual category of modules). This has the effect that for any $\mathcal{V}$-category E, there is a canonical « augmentation » $E(e, e') \to K$. Thus we may simply consider the category $\mathcal{V}$-Cat$/B$ of all $\mathcal{V}$-categories equipped with with a $\mathcal{V}$--functor with codomain B, and compare it with the category $\mathcal{V}^B$ of all $\mathcal{V}$-functors with domain B and the fixed codomain $\mathcal{V}$. Namely, given any $E \xrightarrow{p} B$, we define $\varphi_p : B \to \mathcal{V}$ by the coequalizer diagram in $\mathcal{V}$

$$\sum_{e,e'} E(e, e') \otimes B(p(e'), b) \rightrightarrows \sum_{e \in E} B(p(e), b) \to \varphi_p(b)$$

where one of the two morphisms in induced by the functor p followed by composition in B, while the other is induced by the augmentation. In the case $\mathcal{V} = \mathbf{R}$, this simply means that for any distance-decreasing map p from a metric space E to the fixed metric space B, we define a real-valued function on B by

$$\varphi_p(b) = \inf_e B(p(e), b)$$

i.e. the distance from the image of p to the variable point b.

In the case $\mathcal{V} = \mathbf{2}$, φ_p is the order-preserving map $B \to \mathbf{2}$ defined by

$$\varphi_p(b) = \text{true} \quad \text{iff } \exists\, e\, [p(e) \text{ dominates } b \text{ in } B]$$

In the case $\mathcal{V} = \mathcal{S}$, $\varphi_p(b)$ may be interpreted as the set of components of the category p/b whose objects are pairs e, β with $p(e) \xrightarrow{\beta} b$ in B and whose morphisms are morphisms $e \xrightarrow{\xi} e'$ in E such that $\beta = p(\xi)\, \beta'$ in B.

The « comprehension scheme » then refers to the right adjoint of the functor

$$\mathcal{V}\text{—}Cat/B \to \mathcal{V}^B$$

defined by $p \rightsquigarrow \varphi_p$. Namely, given any $B\varphi \to \mathcal{V}$, define a category $\{B|\varphi\}$ whose objects are pairs $\langle b, x \rangle$ such that $k \xrightarrow{x} \varphi(b)$ in $\mathcal{V}$

and for which $\{B|\varphi\}$ $(\langle b, x\rangle, \langle b', x'\rangle)$ is defined by the following pullback diagram in $\mathcal{V}$

$$\begin{array}{ccccc}
\{B \mid \varphi\}\ (\langle b, x\rangle, \langle b', x'\rangle) & & \xrightarrow{\quad n \quad} & & k \\
\pi\ xx' \downarrow & & & & \downarrow x' \\
B(b, b') \cong k \otimes B(b, b') & \xrightarrow[x \otimes id]{} & \varphi(b) \otimes B(b, b') & \longrightarrow & \varphi(b')
\end{array}$$

where $\varphi(b) \otimes B(b, b') \to \varphi(b')$ is the « action » of B on φ adjoint to the functoriality of φ [note that $\mathcal{V}$-functors with codomain $\mathcal{V}$ always have an equivalent interpretation as « right modules » over the domain category]. The morphism n is unique since we have assumed $k = 1$, but without that assumption it would provide the $\mathcal{V}$-category $\{B|\varphi\}$ with an « augmentation » structure in addition to the structural functor π. In the case $\mathcal{V} = \mathcal{S}$, $\{B|\varphi\}$ is sometimes called the category of all elements of φ, and the functor $\{B|\varphi\} \xrightarrow{\pi} B$ is the discrete fibration corresponding to the set-valued functor φ. In case $\mathcal{V} = \mathbf{2}$

$$\{B \mid \varphi\} = \{\ b \in B \mid \varphi(b) = \text{true}\ \}$$

(accounting for the terminology « comprehension scheme ») and we have of course

$$\frac{Im(p) \subseteq \{B \mid \varphi\}}{\varphi_p \vdash \varphi}$$

But in the case $\mathcal{V} = \mathbf{R}$, we have for any « quantity-valued propositional function » φ on the metric space B that

$$\{B \mid \varphi\} = \{\ b \in B \mid 0 = \varphi(b)\}$$

and for any $E \xrightarrow{p} B$,

$$\frac{Im(p) \subseteq \{B \mid \varphi\}}{\inf B(p(e), b) \leq \varphi(b),\ \text{all } b}\ .$$

To what extent are objects in one of the categories $\mathcal{V}$-Cat/B, $\mathcal{V}^B$ « equivalent » to objects in the other via this adjoint pair? For $\mathcal{V} = \mathcal{S}$, $\mathbf{2}$, $\mathbf{R}$ respectively, $E \xrightarrow{p} B$ must be respectively a discrete fibration, the inclusion of an order-ideal, the inclusion of a *closed* subset in order to have $p \cong \{|\varphi_p\}$. On the other side, for $\mathcal{V} = \mathbf{R}$ there are distance decreasing maps $B \to \mathbf{R}$ *not* of the form « di-

stance from a certain closed set », but for $\mathcal{V} = \mathcal{S}$ or $\mathcal{V} = \mathbf{2}$ *every* $\varphi \in \mathcal{V}^B$ is of the form φ_p.

3. - Bimodules, Kan quantification, Cauchy completeness.

For any $\mathcal{V}$-category Y, we have the canonical Yoneda embedding $Y \to \mathcal{V}^{Y^{op}}$ which means in particular that we can consider that the concept of a $\mathcal{V}$-functor $X \to \mathcal{V}^{Y^{op}}$ is a generalization of the concept of a $\mathcal{V}$-functor $X \to Y$. Such a generalized $\mathcal{V}$-functor $X \mapsto Y$ is equivalent to a $\mathcal{V}$-functor $Y^{op} \otimes X \xrightarrow{\varphi} \mathcal{V}$, and may be considered as a « $\mathcal{V}$-valued relation » from X to Y, $\varphi(y, x)$ being the « truth-value of the φ-relatedness of y to x ». In particular, every $\mathcal{V}$-functor $X \xrightarrow{f} Y$ thus yields a $X \overset{f_*}{\mapsto} Y$ defined by

$$f_*(y, x) = Y(y, f(x))$$

but also yields $Y \overset{f^*}{\mapsto} X$ defined by

$$f^*(x, y) = Y(f(x), y) .$$

We first give an alternate description, without recourse to the notion of $\mathcal{V}$-functor, of such $\mathcal{V}$-valued relations as *bimodules*. We are using, by the way, the notational convention that inside a $\mathcal{V}$-category composition is written from left to right, while composition of $\mathcal{V}$-functors and of bimodules (= « relations ») is written from right to left.

If X, Y are $\mathcal{V}$-categories, a *bimodule* $X \overset{\varphi}{\mapsto} Y$ (also called a right X, left Y-module) consists of a family $\varphi(y, x)$ of objects of $\mathcal{V}$ indexed by the objects of Y and X together with morphisms

$$Y(y', y) \otimes \varphi(y, x) \longrightarrow \varphi(y', x)$$

$$\varphi(y, x) \otimes X(x, x') \longrightarrow \varphi(y, x')$$

in $\mathcal{V}$ which behave as « actions » in the sense that the five axioms (commutative diagrams in $\mathcal{V}$) of X-unity, X-associativity, Y-unity, Y-associativity, and mixed associativity (= commutativity of X-action with Y-action) hold. Thus in case $\mathcal{V} = Ab$, this is just the usual notion of bimodule suitably extended to « rings » X and Y with more than one object. In case $\mathcal{V} = \mathcal{S}$, bimodules are someti-

mes called « profunctors » or « distributeurs ». In case $\mathcal{V} = \mathbf{R}$, a bimodule $X \overset{\varphi}{\mapsto} Y$ between two metric spaces is just a real-valued function on the product $Y \times X$ satisfying

$$\inf_y \, [Y(y', y) + \varphi(y, x)] \geq \varphi(y', x)$$

$$\inf_x \, [\varphi(y, x) + X(x, x')] \geq \varphi(y, x') \, .$$

The latter is clearly equivalent to putting a metric on the sum $X + Y$ extending the given metrics and having $d(x, y) = \infty$ for $x \in X$, $y \in Y$, and indeed this alternate mode of description works generally, recalling that the infinite quantity corresponds to the empty coproduct in $\mathcal{V}$.

If $X \overset{\varphi}{\mapsto} Y \overset{\psi}{\mapsto} Z$ are bimodules, then composition $X \overset{\psi \circ \varphi}{\mapsto} Z$ is defined by the following coequalizer diagram in $\mathcal{V}$

$$\sum_{y_1, y_2} \psi(z, y_1) \otimes Y(y_1, y_2) \otimes \varphi(y_2, x) \rightrightarrows \sum_{y \in Y} \psi(z, y) \otimes \varphi(y, x) \longrightarrow (\psi \circ \varphi)(z, x)$$

Thus in case $\mathcal{V}$ is itself a poset, $\psi \circ \varphi$ reduces to a « matrix product »

$$\sum_{y \in Y} \psi(z, y) \otimes \varphi(y, x)$$

where the sigma denotes coproduct in $\mathcal{V}$, but in general we take the quotient of the matrix product modulo the discrepancy between the two actions of Y in the middle, just as in the familiar case $\mathcal{V} = Ab$ where one often writes more explicitly

$$\psi \circ \varphi = \psi \underset{Y}{\otimes} \varphi \, .$$

In particular for $\mathcal{V} = \mathbf{2}$

$$(\psi \circ \varphi)(z, x) = \exists \, y \, [\psi(z, y) \wedge \varphi(y, x)]$$

is the usual relational product, while for $\mathcal{V} = \mathbf{R}$ we have

$$(\psi \circ \varphi)(z, x) = \inf [\psi(z, y) + \varphi(y, x)] \, .$$

It can be verified in general that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ are $\mathcal{V}$-functors, then

$$(gf)_* \cong g_* \circ f_*$$

where isomorphism of bimodules has an obvious sense; also 1_X defined by

$$1_X(x, x') = X(x, x')$$

plays the role of identity with respect to composition with any bimodules (not only those of the form f_*). Further, the composition of bimodules is associative up to (coherent) isomorphism, but in general not commutative. Note that the endobimodules of the one-object category k are in correspondence with the objects of $\mathcal{V}$, with the composition of these reducing to the tensor product in $\mathcal{V}$.
The concept Hom, right adjoint to $\otimes$ in $\mathcal{V}$ has two extensions to mixed Hom of bimodules, right adjoint on opposite sides to the composition of bimodules. That is, given any three $\mathcal{V}$-categories X, Y, Z and a bimodule $X \overset{\varphi}{\mapsto} Z$, there are two universal problems: given any $X \overset{\alpha}{\mapsto} Y$, there is a $Y \mapsto Z$ denoted by $\mathrm{Hom}^X(\alpha, \varphi)$ which is « best » in the sense that for any bimodule $Y \overset{\beta}{\mapsto} Z$ there is a natural bijection of morphisms of bimodules

$$\frac{\beta \longrightarrow \mathrm{Hom}^X(\alpha, \varphi)}{\beta \circ \alpha \longrightarrow \varphi}$$

likewise for given β there is a « best α », denoted by $\mathrm{Hom}_Z(\beta, \varphi)$ satisfying

$$\frac{\alpha \longrightarrow \mathrm{Hom}_Z(\beta, \varphi)}{\beta \circ \alpha \longrightarrow \varphi}$$

for any α.

Explicitly, $\mathrm{Hom}^X(\alpha, \varphi)$ is defined by an equalizer diagram in $\mathcal{V}$

$$\mathrm{Hom}^X(\alpha, \varphi)(z, y) \longrightarrow \prod_{x \in X} \mathrm{Hom}(\alpha(y, x)\varphi, (z, x) \rightrightarrows \prod_{x_1, x_2} \mathrm{Hom}(X(x_1, x_2), \mathrm{Hom}(\alpha(y, x_1), \varphi(z, x_2)))$$

and similarly $\mathrm{Hom}_Z(\beta, \varphi)(y, x)$ is defined by an equalizer condition as a subobject of

$$\prod_{z \in Z} \mathrm{Hom}(\beta(z, y), \varphi(z, x)) .$$

For *some* bimodules $X \overset{\alpha}{\mapsto} Y$, the operation of composing with α, $\beta \rightsquigarrow \beta \circ \alpha$, has not only the right adjoint $\varphi \rightsquigarrow \mathrm{Hom}^X(\alpha, \varphi)$ but has also a *left* adjoint. The typical such bimodule α is one of the form

$\alpha = f_*$, where $X \xrightarrow{f} Y$ is a $\mathcal{V}$-functor; presently we will also discuss the extent to which « all » such bimodules α are induced by functors f.

Lemma. - For any $\mathcal{V}$-functor $X \xrightarrow{f} Y$ and any $\mathcal{V}$-bimodule $Y \overset{\beta}{\mapsto} Z$, there is a natural isomorphism

$$\beta \circ f_* \;=\; \mathrm{Hom}^Y(f^*, \beta)$$

of bimodules $X \mapsto Z$.

Proof. - One verifies that both sides are naturally isomorphic to the bimodule whose typical component is

$$\beta(z, f(x)) .$$

Theorem. - For $\alpha = f_*$, composition $\beta \rightsquigarrow \beta \circ \alpha$ has as left adjoint the operation of composing with f^*

$$\varphi \rightsquigarrow \varphi \circ f^*$$

Proof. -

$$\begin{array}{c} \varphi \circ f^* \longrightarrow \beta \\ \hline \varphi \longrightarrow \mathrm{Hom}^Y(f^*, \beta) \\ \hline \varphi \longrightarrow \beta \circ f_* \end{array}$$

Corollary. - (Kan quantification). For any $\mathcal{V}$-functor $X \xrightarrow{f} Y$, the functor « composition with f »

$$\mathcal{V}^Y \xrightarrow{\mathcal{V}^f} \mathcal{V}^X$$

has both left and right adjoints. On $\varphi \in \mathcal{V}^X$, the left adjoint gives as a result the functor $\varphi \circ f^*$, whose value at y is the quotient of

$$\sum_X \varphi(x) \otimes Y(f(x), y)$$

« modulo the distinction between the two actions of X in the middle », while the right adjoint gives as a result the functor $\mathrm{Hom}^X(f_*, \varphi)$ whose value at y is that subobject of

$$\prod_X \mathrm{Hom}\,(Y(y, f(x)), \varphi(x))$$

« on which the two actions of X agree ». In the special case that f

is $\mathcal{V}$-full-and-faithful, $\mathcal{V}^f$ is also full and faithful, or equivalently the two Kan quantifications are actually *extensions*, in the sense that

$$(\varphi \circ f^*) \circ f_* \cong \varphi$$

$$\mathrm{Hom}^X(f_*, \varphi) \circ f_* \cong \varphi$$

for any φ.

Proof. - The corollary is for the most part just a summary of the preceding discussion for the special case $Z = k$ (Indeed the corollary remains valid if $\mathcal{V}$ is replaced by $\mathcal{V}^{Z^{op}}$ for any (small) $\mathcal{V}$-category Z).

The assertion that, under the assumption that f is $\mathcal{V}$-full-and-faithful, first « extending » φ along f (in either of the two adjoint ways) and then « restricting » along f gives again φ, is proved in very much the same way as the Lemma above.

Corollary. - Let $X \xrightarrow{f} Y$ be an isometric embedding of a metric space X into a metric space Y. Then any Lipschitz function $X \xrightarrow{\varphi} \mathbf{R}$ can be extended to Y, with the same Lipschitz constant λ. In fact, among all extensions there is a largest one and a smallest one, given respectively by

$$\overline{\varphi}(y) = \inf_X [\varphi(x) + \lambda\, Y(f(x), y)]$$

$$\underline{\varphi}(y) = \sup_X [\varphi(x) - \lambda\, Y(y, f(x))].$$

Proof. - Apply the preceding discussion to the diagram

$$\begin{tikzcd} \lambda X \arrow[rr, "\lambda f"] \arrow[dr, "\varphi"'] & & \lambda Y \arrow[dl, dashed, shift right] \\ & \mathcal{V} & \end{tikzcd}$$

noting that λf as a function is the same as the inclusion f.

As an example of the last corollary, we could take for X the space of nonnegative step functions on a probability space S and for Y the space of all nonnegative functions with the natural sup metric, (so that $Y(y_2, y_1) = 0$ iff $y_1 \geq y_2$), and consider as φ the

elementary integral; thus in general we might call y φ-integrable if $\overline{\varphi}(y) \cong \underline{\varphi}(y)$.

The (non standard) name Kan quantification was suggested by the case $\mathcal{V} = \mathbf{2}$, in which the adjointness rules reduce to the usual rules of inference for quantification, φ being thought of as property or relation, i.e.

$$(\varphi \circ f^*)(z, y) \equiv \exists\, x\, [\varphi(z, x) \wedge f(x) \geq y]$$

$$\mathrm{Hom}^X(f_*, \varphi)(z, y) \equiv \forall\, x\, [y \geq f(x) \Longrightarrow \varphi(z, x)]$$

But for other choices of $\mathcal{V}$, induced representations and relatively free universal algebras can be shown to arise as special cases of these two constructions.

The essential property of a bimodule of the form f_* is that there exists another bimodule f^* which is right adjoint to it, in the sense that there are morphisms

$$1_X \longrightarrow f^* \circ f_*$$

$$f_* \circ f^* \longrightarrow 1_Y$$

satisfying the usual two adjunction equations.

Proposition. - In order that a metric space Y be Cauchy-complete, it is necessary and sufficient that every adjoint pair of bimodules

$$X \underset{f^*}{\overset{f^*}{\rightleftarrows}} Y$$

be induced by a $\mathcal{V}$-functor $X \xrightarrow{f} Y$.

Proof. - It suffices to consider $X = 1$ and show that an adjoint pair of $\mathbf{R}$-valued bimodules $1 \leftrightarrows Y$ is essentially just a point in the completion of Y. But adjointness just means that

$$X(x, x') \geq \inf_y\, [f^*(x, y) + f^*(y, x')]$$

$$\inf_X\, [f_*(y, x) + f^*(x, y')] \geq Y(y, y')$$

in addition to the bimodule property for each of f_*, f^*. In case $X = 1$, we have then

$$0 = \inf_y\, [f^*(y) + f_*(y)]$$

$$f_*(y) + f^*(y') \geq Y(y, y')\,.$$

Thus for each n we can choose y_n satisfying, for example

$$f^*(y_n) + f_*(y_n) \leq \frac{1}{n}$$

and then

$$Y(y_n, y_m) \leq \frac{1}{n} + \frac{1}{m}$$

so that we have a Cauchy sequence, and any other choice y'_n satisfying the same condition would have

$$Y(y_n\, y'_n) \leq \frac{2}{n}$$

i.e. would be an equivalent Cauchy sequence. Conversely, any equivalence class of Cauchy sequences yields an adjoint pair of bimodules by the definition

$$f^*(y) = \lim_{n \to \infty} Y(y_n, y)$$

$$f_*(y) = \lim_{n \to \infty} Y(y, y_n) .$$

It can be shown that the suggested definition of « $\mathcal{V}$-Cauchy-completeness » means in case $\mathcal{V} = Ab$ that a « point of the completion » of an additive category Y is simply any finitely generated projective module over Y, while in case $\mathcal{V} = \mathcal{S}$, it means that Y is Cauchy-complete iff all idempotents in Y split in Y.

4. - Free $\mathcal{V}$-categories.

Defining a $\mathcal{V}$-graph to be a pair consisting of any set X and any $X \times X$-indexed family of objects of $\mathcal{V}$, and a morphism $\langle X, \gamma \rangle \xrightarrow{f} \langle Y, \delta \rangle$ to be any pair consisting of a mapping $X \to Y$ and any family

$$\gamma(x, x') \xrightarrow{f_{x,x'}} \delta(f(x), f(x'))$$

of morphisms in $\mathcal{V}$, we have an obvious forgetful functor

$$\mathcal{V}\text{-Cat} \to \mathcal{V}\text{-Graph}.$$

This forgetful functor has a left-adjoint called taking the free $\mathcal{V}$-category generated by a $\mathcal{V}$-graph. Explicitly, the free category has as objects the vertices of the generating graph, and as hom from x to x' has the coproduct in $\mathcal{V}$

$$\sum_{x_1 \dots, x_n} \gamma(x, x_1) \otimes \gamma(x_1, x_2) \otimes \dots \otimes (x_n, x')$$

over all finite sequences $x_1 \dots x_n$ of vertices. For example with $\mathcal{V} = k$-modules, this formula contains the construction of the tensor algebra over a vector space, or with $\mathcal{V} = \mathcal{S}$, we have as a special case the word algebra (= free monoid) over a given set. For the case $\mathcal{V} = \mathbf{R}$, an $\mathbf{R}$-graph is just an arbitrary assignment γ of quantities to pairs of points, and in the « free metric space » over such, the distance from x to x' is

$$\inf_{x_1 \dots, x_n} [\gamma(x, x_1) + \gamma(x_1, x_2) + \dots\dots + \gamma(x_n, x')]$$

the well-known « least-cost » distance.

The adjointness of the free $\mathcal{V}$-category construction contains the essence of the notion of recursion, especially when one considers it in relation with bimodules, where it leads for example to the iteration of endobimodules.

5. - Further remarks.

We already remarked that the composition of bimodules is similar to matrix multiplication, and indeed the analogy with linear algebra goes further. For example, if bimodules rather than $\mathcal{V}$ functors are considered as the « morphisms » between $\mathcal{V}$-categories, then $A^{op} \otimes Y$ plays the role of « Hom ». Either further developing that remark in the case $A = Y$, or proceding directly, it is natural to define, for any endobimodule $A \xrightarrow{\alpha} A$, $Tr(\alpha)$ to be the object of $\mathcal{V}$ defined by coequalizer

$$\sum_{a,b \in A} A(b, a) \otimes \alpha(a, b) \rightrightarrows \sum_{a \in A} \alpha(a, a) \longrightarrow Tr(\alpha) .$$

The injections $\alpha(a, a) \to Tr(\alpha)$, denoted by tr_a, are a natural generalization of many examples of classical constructions such as trace of endomorphisms or Lefschitz numbers (when $\alpha = 1_A$) as has been

verified by A. Kock. In case A is a metric space and $\alpha = f_*$ where f is a distance decreasing endomap,

$$Tr(\alpha) = \inf_a A(a, fa)$$

showing that the vanishing of the trace is related to the existence of fixed points. It seems likely that there may be theorems holding for more general $\mathcal{V}$, relating the trace of an endobimodule to an infinite iteration of it, which would extend the Banach fixed-point theorem.

SUMMARY. — The analogy between $\mathrm{dist}(a, b) + \mathrm{dist}(b, c) \geq \mathrm{dist}(a, c)$ and $\hom(A, B) \otimes \hom(B, C) \to \hom(A, C)$ is rigorously developed to display many general results about metric spaces as consequences of a « generalized pure logic » whose « truth-values » are taken in an arbitrary closed category.

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Commentationes Mathematicae Universitatis Carolinae

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The completion monad and its algebra

Commentationes Mathematicae Universitatis Carolinae, Vol. 23 (1982), No. 2, 301--311

Persistent URL: http://dml.cz/dmlcz/106152

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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
23,2 (1982)

THE COMPLETION MONAD AND ITS ALGEBRA

Sergio SALBANY

Abstract: Let C represent the completion functor discussed by O. Wyler and S. Salbany. There is a monad associated with C and it is natural to ask for a characterization of the C-algebras. In this paper we show that the C-algebras are the complete spaces.

Key words: Quasi Uniform spaces, completion triple C, C-algebras.

Classification: Primary 54E15
Secondary 18C15

Introduction. As shown in [4] and [5], the completion functor on the category of Quasi-Uniform spaces is associated with a monad (C, η, μ). Keith Hardie asked us for a characterization of the C-algebras and persuaded us, over the years, that an answer should be given.

We shall follow the terminology of [3] and [1] concerning quasi-uniform spaces and that of [2] for the category theory.

1. Definitions, constructions and notations

A. Cauchy filters, convergence and completeness. Let QU denote the category of quasi-uniform spaces $(X, \mathcal{U})$ and quasi-uniformly continuous maps.

Definition 1. A filter $\mathcal{F}$ on $(X, \mathcal{U})$ is said to be a

Cauchy-filter if, for every U in the uniformity $\mathcal{U} \vee \mathcal{U}^{-1}$ there is F in $\mathcal{F}$ such that $F \times F \subset \mathcal{U}$.

Definition 2. A filter $\mathcal{F}$ on $(X, \mathcal{U})$ is said to converge to x if it converges to x in the topology induced by the uniformity $\mathcal{U} \vee \mathcal{U}^{-1}$.

Definition 3. $(X, \mathcal{U})$ is said to be complete if $\mathcal{U} \vee \mathcal{U}^{-1}$ is complete.

Definition 4. If A is a subset of $(X, \mathcal{U})$, denote by $\overline{A}$ the closure of A in the topology induced by the uniformity $\mathcal{U} \vee \mathcal{U}^{-1}$. x is called an adherence point of a filter $\mathcal{F}$ if $x \in \overline{F}$ for every F in $\mathcal{F}$.

Note. As for uniform spaces, if $\mathcal{F}$ is a Cauchy filter on $(X, \mathcal{U})$ and x is an adherence point of $\mathcal{F}$, then $\mathcal{F}$ converges to x, and conversely. Thus, if $x \in \overline{F}$ for all $F \in \mathcal{F}$, then $\mathcal{F}$ converges to x.

B. Description of the completion monad. Given $(X, \mathcal{U})$, let CX denote the set of all Cauchy filters on X and let $\mathcal{U}^*$ denote the quasi-uniformity on CX with basis elements U^*, where $U \in \mathcal{U}$ and $(\alpha, \beta) \in U^*$ if and only if there are sets A in α, B in β such that $A \times B \subset U$. The sets U^* do form a basis for $\mathcal{U}^*$ since $(U \cap V)^* = U^* \cap V^*$. We now describe the multiplication μ and the unity η of the triple C:

(i) Let $\eta_X : X \longrightarrow CX$ be given by $\eta_X(x) = \{F \mid F \subset X$ and $x \in F\}$. Then $\eta_X : (X, \mathcal{U}) \longrightarrow (CX, \mathcal{U}^*)$ is quasi-uniformly continuous.

(ii) Let $\mu_X : C^2X \longrightarrow CX$ be given by $\mu_X(\alpha) = \{H \mid H \subset X, H^* \in \alpha\}$, where H^* is such that $\beta \in H^*$ if and only if $H \in \beta$. Then $\mu_X : (C^2X, \mathcal{U}^{**}) \longrightarrow (CX, \mathcal{U}^*)$ is quasi-uniformly continuous.

Moreover,

(a) η_X is an initial and injective map onto a $\mathcal{U}^* \vee (\mathcal{U}^*)^{-1}$-dense subspace of CX.

(b) μ_X and η_X induce natural transformations $\mu : C^2 \to C$; $\eta : 1\!\!1 \longrightarrow C$ such that the following diagrams commute

$$\begin{tikzcd} C^3 \arrow[r, "C\mu"] \arrow[d, "\mu C"'] & C^2 \arrow[d, "\mu"] \\ C^2 \arrow[r, "\mu"'] & C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\eta C"] \arrow[dr, equal] & C^2 \arrow[d, "\mu"] & C \arrow[l, "C\eta"'] \arrow[dl, equal] \\ & C & \end{tikzcd}$$

Thus, every space can be densely embedded in a complete space in a "regular" way, which is expressed by the functoriality of C. Moreover, even though the completion process always enlarges a space, the existence of μ shows that the completion of a complete space is not "much larger" than the complete space itself.

C. The separated completion

Definition 5. A quasi-uniform space $(X, \mathcal{U})$ is separated if the uniformity $\mathcal{U} \vee \mathcal{U}^{-1}$ is separated, that is, the intersection of all members of $\mathcal{U} \vee \mathcal{U}^{-1}$ is the diagonal of $X \times X$.

Construction of the separated reflection. Given a quasi-uniform space $(X, \mathcal{U})$, let R denote the equivalence relation $x R y$ if and only if $\overline{\{x\}} = \overline{\{y\}}$. Denote by $[x]$, the R-equivalence class of x. Let X^s denote the set of R-equivalence classes on $(X, \mathcal{U})$. Let $s : X \longrightarrow X^s$ denote the map $s(x) = [x]$. For $U \in \mathcal{U}$, let $U^s = \{([x], [y]) \mid (x, y) \in U\}$, then the U^s form a basis for a quasi-uniformity $\mathcal{U}^s$ (since $(U \cap V)^s \subset U^s \cap V^s$). The map $s_X : (X, \mathcal{U}) \longrightarrow (X^s, \mathcal{U}^s)$ is an initial quasi-uniformly continuous map onto a separated quasi-uniform space

and the assignment $(X,\mathcal{U}) \longrightarrow (X^s,\mathcal{U}^s)$ is the separated reflection in QU. Moreover, $(X,\mathcal{U})$ is complete if and only if $(X^s,\mathcal{U}^s)$ is complete. The composite $s \circ C = C^s$ is the separated-completion-functor. The natural transformations $\eta^s = s\eta$ and $\mu^s = s\mu$ provides the separated-completion monad (C^s, η^s, μ^s).

The importance of separated completions lies in the fact that the embedding map $\eta^s_X : (X,\mathcal{U}) \longrightarrow (X^s,\mathcal{U}^s)$ is a map onto a $\mathcal{U}^s \vee (\mathcal{U}^s)^{-1}$-dense subspace, hence an epimorphism in the category of separated quasi-uniform spaces (see [1],[3]).

D. The algebra of a monad

Definition 6. Let (C, η, μ) be a monad on a category $\underline{A}$. An object A of $\underline{A}$ is a C-algebra if there is a morphism h in $\underline{A}$, called a structure map, $h: CA \longrightarrow A$ such that the following diagrams commute:

$$\begin{tikzcd} CA \arrow[r, "Ch"] \arrow[d, "\mu_A"'] & CA \arrow[d, "h"] \\ CA \arrow[r, "h"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, equal] & CA \arrow[d, "h"] \\ & A \end{tikzcd}$$

Examples of C-algebras

Example 1. Let $X = \{0,1\}$ and $\mathcal{U} = X \times X$. It is straightforward to verify that $(X,\mathcal{U})$ is a C-algebra for every map $h: (CX, \mathcal{U}^*) \longrightarrow (X,\mathcal{U})$.

Example 2. Let $(X,\mathcal{U})$ be a separated complete space and let $h: (CX, \mathcal{U}^*) \longrightarrow (X,\mathcal{U})$ be the limit map, $h(\mathcal{F})$ = limit of $\mathcal{F}$ (convergence in the topology of $\mathcal{U} \vee \mathcal{U}^{-1}$). Note that h is

well defined since limits are unique in separated quasi-uniform spaces. It will be shown in Section 3 that such a separated complete space is a C-algebra.

2. C-algebras. As expected, C-algebras are the complete spaces. Although expected, we have found the proof elusive. The result is surprising in that there is an arbitrariness in the structure map $h:CX \longrightarrow X$ (see Proposition 2) that suggests that not all complete spaces would be C-algebras.

Proposition 1. If $(X, \mathcal{U})$ is a C-algebra, then $(X, \mathcal{U})$ is complete.

This is an immediate consequence of the following two lemmas.

Lemma 1. If $S \subset X$, then $S^* \subset \overline{\eta_X[S]}$.

Proof. Let $\alpha \in S^*$. Given a symmetric U in $\mathcal{U} \vee \mathcal{U}^{-1}$, we show that $U[\alpha]$ intersects $\eta_X[S]$. From $\alpha \in S^*$ it follows that $S \in \alpha$. Now α is a Cauchy-filter, so there is F in α such that $F \times F \subset U$. But S and F are in α so there is x in $F \cap S$. By definition of U^* it follows that $(\alpha, \eta_X(x)) \in U^*$ since $F \in \alpha$ and $F \in \eta_X(x)$ and $F \times F \subset U$. Thus $U^*[\alpha]$ intersects $\eta_X[S]$, as required.

Lemma 2. Let $h:(CX, \mathcal{U}^*) \longrightarrow (X, \mathcal{U})$ be a quasi-uniformly continuous map such that $h \circ \eta_X(x) = x$ for all x in X. Then a Cauchy filter $\mathcal{F}$ converges to $h(\mathcal{F})$.

Proof. We show that $h(\mathcal{F})$ is an adherence point of $\mathcal{F}$, from which it follows that $\mathcal{F}$ converges to $h(\mathcal{F})$. Let $S \in \mathcal{F}$, then $F \in S^*$, so that $h(\mathcal{F}) \in h[S^*]$. By Lemma 1 we have $h[S^*] \subset$ $\subset h[\eta_X[S]] \subset h\,\overline{[\eta_X[S]]} = \overline{S}$. Thus $h(\mathcal{F}) \in \overline{S}$ for all S in $\mathcal{F}$,

showing that $h(\mathcal{F})$ is an adherence point of $\mathcal{F}$.

To establish the converse of Proposition 1 we require the following lemmas.

<u>Lemma 3</u>. Let $\alpha \in C^2X$, then $\bigcap\{\overline{H} \mid H \subset X$ and $h \leftarrow [H] \in \alpha\} = \bigcap\{\overline{h[k]} \mid k \in \alpha\}$.

<u>Proof</u>. Suppose $H \subset X$ and $h \leftarrow [H] \in \alpha$. $h[h \leftarrow [H]] = H$ (since h is surjective) shows that $\overline{H} = \overline{h[k]}$ for some k in α, since $k \in \alpha$ and $K \subset h \leftarrow [h[k]]$. The proof is complete.

<u>Lemma 4</u>. Let $\alpha \in C^2X$. If α converges to $\mathcal{F}$, then $ch(\alpha)$ converges to $h(\mathcal{F})$.

<u>Proof</u>. Suppose α converges to $\mathcal{F}$, then $\mathcal{F}$ is an adherence point of α so that $\mathcal{F} \in \overline{k}$ for all $k \in \alpha$. Hence $h(\mathcal{F}) \in h[\overline{k}] \subset \overline{h[k]}$ for all $k \in \alpha$. Thus, by Lemma 3, $h(\mathcal{F}) \in \bigcap\{\overline{H} \mid h \leftarrow [H] \in \alpha\}$, so that $h(\mathcal{F})$ is an adherence point of $Ch(\alpha)$, as required.

<u>Lemma 5</u>. Let $V \in \mathcal{U} \vee \mathcal{U}^{-1}$. Let U be symmetric and such that $U \circ U \subset V$. If $\mathcal{F}$ is a Cauchy filter which converges to x, then $U^*[\mathcal{F}] \subset (V[x])^*$.

<u>Proof</u>. Because $\mathcal{F}$ is Cauchy and converges to x, there is F in $\mathcal{F}$ such that $F_1 \times F_1 \subset U$ and $F_1 \subset U[x]$. Suppose $\chi \in U^*[\mathcal{F}]$, we show that $V[x] \in \chi$. By definition of U^*, if $\chi \in U^*[\mathcal{F}]$, there is $F_2 \in \mathcal{F}$ and $G \in \mathcal{F}$ such that $F_2 \times G \subset U$.

Let $F = F_1 \cap F_2$ so that (i) $F \in \mathcal{F}$, (ii) $F \times G \subset U$, and (iii) $F \subset U[x]$.

From (ii) we have $G \subset U[F]$ so that $U[F] \in \chi$ since $G \in \chi$.

But $U[F] \subset U \circ U[x] \subset V[x]$, from (iii), so that $V[x] \in \chi$,

as required.

<u>Lemma 6</u>. Let $\alpha \in C^2X$. If α converges to $\mathcal{F}$ and $\mathcal{F}$ converges to x, then $\mu_X(\alpha)$ converges to x.

<u>Proof</u>. Let $V \in \mathcal{U} \vee \mathcal{U}^{-1}$. We show that $V[x] \in \mu_X(\alpha)$. Choose a symmetric entourage U such that $U \circ U \subset V$. Because $\mathcal{F}$ converges to x, by Lemma 5, we have $U^*[\mathcal{F}] \subset (V[x])^*$. Since α converges to $\mathcal{F}$ we also have $U^*[\mathcal{F}]$ in α, hence $(V[x])^*$ is in α and, consequently, $V[x] \in \mu_X(\alpha)$, as required.

<u>Lemma 7</u>. Let $\alpha \in C^2X$, if there is $x \in X$ such that $\mu_X(\alpha) = \{H \mid x \in H\}$, then $Ch(\alpha) = \mu_X(\alpha)$.

<u>Proof</u>. Observe that, for any filter $\mathcal{F}$, $\mathcal{F} = \{H \mid x \in H\} \Longleftrightarrow \{x\} \in \mathcal{F}$. For convenience, let $\{H \mid x \in H\}$ be denoted by $\langle x \rangle$. To show that $Ch(\alpha) = \langle x \rangle$ it suffices to prove that $\{x\} \in Ch(\alpha)$, that is, $h \leftarrow [\{x\}] \in \alpha$.
Now, $\{\langle x \rangle\} \in \alpha$ since $\{x\} \in \mu(\alpha)$ implies $(\{x\})^* \in \alpha$ and $\mathcal{F} \in (\{x\})^*$ is equivalent to $\{x\} \in \mathcal{F}$ which states that $\mathcal{F} = \langle x \rangle$, so that $\{\langle x \rangle\} \in \alpha$. Now $\langle x \rangle \in h \leftarrow [\{x\}]$ since $h(\langle x \rangle) = x$. It then follows that $h \leftarrow [\{x\}] \in \alpha$, as required.

Note: Consideration of example 1 shows that if $Ch(\alpha) = \{H \mid x \in H\}$, then it does not follow that $\mu_X(\alpha) = Ch(\alpha)$.

<u>Lemma 8</u>. Let A_x consist of all Cauchy filters on X which converge to x. Let $\alpha \in C^2X$. If $A_x \in \alpha$, then $\mu_X(\alpha)$ converges to x.

<u>Proof</u>. It is straightforward to check $A_x = \overline{A}_x$. Also CX is complete so there is $\mathcal{F}$ in CX such that α converges to $\mathcal{F}$. Then $\mathcal{F}$ is an adherence point to α, so that $\mathcal{F} \in \overline{A}_x$. But

then $\mathcal{F} \in A_x$, so that $\mathcal{F}$ converges to x . By Lemma 6, we have that $\mu_X(\alpha)$ converges to x .

<u>Proposition 2</u>. If $(X, \mathcal{U})$ is complete, then it is a C-algebra.

<u>Proof</u>. Define $h: CX \longrightarrow X$ as follows: For each x , let $[x]$ denote the R equivalence class of x ($x R y \Longleftrightarrow \bar{x} = \bar{y}$). Let c be a choice function on X^s, so $c([x]) \in [x]$. Observe that if $\mathcal{F}$ is a Cauchy filter on X which converges to x and y, then $[x] = [y]$; now let

$$h(\mathcal{F}) = \begin{cases} x, \text{ if } \mathcal{F} = \eta_X(x) \text{ for some } x \cdot \\ c([x]), \text{ if } \mathcal{F} \notin \eta_X[X] \text{ and } \mathcal{F} \text{ converges to } x . \end{cases}$$

By the remarks above, h is well defined. Observe that $\mathcal{F}$ converges to $h[\mathcal{F}]$. It is readily checked that $h: (CX, \mathcal{U}^*) \longrightarrow (X, \mathcal{U})$ is quasi-uniformly continuous. We verify that the diagram corresponding to (1) in the definition of the algebra of a monad is commutative: Let $\alpha \in C^2X$. Because CX is complete, α converges to some $\mathcal{F}$ in CX.

By Lemma 4, $Ch(\alpha)$ converges to $h(\mathcal{F})$. By Lemma 6, $\mu_X(\alpha)$ converges to $h(\mathcal{F})$ and α converges to $\mathcal{F}$. Thus $\mu_X(\alpha)$ and $Ch(\alpha)$ both converge to $h(\mathcal{F})$. To show that $h[\mu_X(\alpha)] = h[Ch(\alpha)]$ we consider two cases.

<u>Case 1</u>. $Ch(\alpha) \notin \eta_X[X]$. Then $\mu_X(\alpha) \notin \eta_X[X]$, by Lemma 7. By definition of h, it follows that $h[\mu_X(\alpha)] = c([h(\mathcal{F})])$ and $h[Ch(\alpha)] = c([h(\mathcal{F})])$, as required.

<u>Case 2</u>. $Ch(\alpha) \in \eta_X[X]$. Then $Ch(\alpha) = \langle x \rangle$ (for a unique x in X) so that $h \leftarrow [\{x\}] \in \alpha$. We again distinguish

two cases:

Case 2 a. $x \neq c([x])$. In this case, $h \leftarrow [\{x\}] = \{\langle x \rangle\}$ since for any $\mathcal{F}$ not in $\eta_X[X]$, $h(\mathcal{F}) = x$ implies $\mathcal{F}$ converges to x so that $h(\mathcal{F}) = c([x]) \neq x$. Now $Ch(\alpha) = \langle x \rangle$ implies that $h \leftarrow [\{x\}] \in \alpha$, so that $\{\langle x \rangle\} \in \alpha$. But then $\mu_X(\alpha) = \langle x \rangle$. Thus $Ch(\alpha) = \mu_X(\alpha) = \langle x \rangle$ so that $h(Ch(\alpha)) = h(\mu_X(\alpha))$.

Case 2 b. $x = c([x])$. In this case $h \leftarrow [\{x\}]$ consists of all Cauchy filters $\mathcal{F}$ which converge to a point in $[x]$. Thus $h \leftarrow [\{x\}] = A_x$. From Lemma 8, it follows that $\mu_X(\alpha)$ converges to x. From the definition of h (whether or not $\mu_X(\alpha)$ is in $\eta_X[X]$) and $x = c([x])$, it follows that $h[\mu_X(\alpha)] = x$. Hence $h[\mu_X(\alpha)] = h[Ch(\alpha)]$.

It is remarkable that the regularity expressed in diagram 1 of the definition of a monad could be achieved with the arbitrariness involved in the function h of Proposition 2.

3. The separated-completion C^s. The results in Section 2 readily identify the C^s-algebras in $\underline{QU}_s$:

Proposition 3. The C^s-algebras are the separated and complete spaces.

However, the delicate comparison of filters and limits can be avoided and a simple proof of Proposition 3 will be given in two parts, where we rely on the fact that η_X is epic in $\underline{QU}_s$.

Part 1. Every C^s-algebra is complete and separated.

Proof. Let $h: C^s X \longrightarrow X$ be the structure map. Then $h \circ \eta_X = 1\!\!1_X$, so that $\eta_X \circ h \circ \eta_X = \eta_X \circ 1\!\!1_X = 1\!\!1_X \circ \eta_X$. But η_X is an

epimorphism, hence $\eta_X \circ h = 1_{C^S X}$. Thus X and $C^S X$ are isomorphic.

Part 2. Let X be a complete and separated space, then every Cauchy filter $\mathcal{F}$ converges to a unique point x. It is straightforward to verify that $h:(C^S X, (\mathcal{U}^*)^S) \longrightarrow (X, \mathcal{U})$ is quasi-uniformly continuous. Moreover $h \circ \eta_X = 1_X$, since the filter $\eta_X(x)$ converges to x.

To prove that $h \circ \mu_X = h \circ Ch$ we show that $h \circ \mu_X \circ \eta_{CX} = h \circ Ch \circ \eta_{CX} (= h)$ and use the fact that η_{CX} is epic.

Now $\mu_X \circ \eta_{CX} = 1_{CX}$, so that $h \circ \mu_X \circ \eta_{CX} = h$. Also, $Ch \circ \eta_{CX} = \eta_X \circ h$ (by naturality of η), so that

$$h \circ Ch \circ \eta_{CX} = h \circ \eta_X \circ h = 1_X \circ h = h.$$

The proof is complete.

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(Oblatum 9.11. 1981)

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Electronic Notes in Theoretical Computer Science

ELSEVIER Electronic Notes in Theoretical Computer Science 73 (2004) 165–205
www.elsevier.com/locate/entcs

Enriched Categories and Quasi-uniform Spaces

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Abstract

The metric space representation in terms of enriched categories of [14] is extended to quasi-uniform spaces and uniformly continuous maps. The bicompletion of the quasi-uniform spaces [6] is related to the Cauchy-completion. On one hand *abstract quasi-uniformities* and uniformly continuous maps are defined. They generalize quasi-uniform spaces and their morphisms and form a locally preordered 2-category $AQUnif$. On the other hand a 2-category $Enr(H)$ of enrichments over different bases is defined. H stands here for a parameter 2-functor with domain a locally filtered 2-category and codomain the 2-category with objects monoidal partial orders and arrows the so-called *super* monoidal functors. Also the bases considered are partial orders (or "quantales"). In $Enr(H)$ the Cauchy-complete objects defines a fully reflective sub-2-category. Eventually we show a 2-equivalence $AQUnif \cong Enr(H)$ for a particular H. Thus one gets a completion for abstract quasi-uniformities, that reduces to the bicompletion for quasi-uniformities.

Keywords: enriched categories, quasi uniform spaces

1 Introduction

In [14] it is shown that enrichments over a particular monoidal closed category, say $[0, +\infty]$, may capture the notion of generalized metric spaces (also called pseudo quasi-metrics). The Cauchy-completion of $\mathcal{V}$-enrichments for a general monoidal closed $\mathcal{V}$ is defined. It reduces to the metric space completion when $\mathcal{V} = [0, +\infty]$. Also a general theory of (co)completions for enrichments is de-

[1] Email: vs27mcs.le.ac.uk
[2] Support from grant GR/R63004/01 of EPSRC is gratefully acknowledged.

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doi:10.1016/j.entcs.2004.08.009

veloped in [9]. It is possible to parameterize the various completions according to classes of chosen colimits.

Since these fundamental works, the notion of completion of enrichments has been successfully applied to various areas of mathematics and computer science. Let us cite as a non-exhaustive lists of applications [23] for sheaves, [1] for fibrations, [22] for domain theory, [8], [2] for domain theory and metric spaces, also more recently [12] for metric spaces, [15] for ultra-metric spaces. Eventually we refer to [7] for many other mathematical examples of enrichments over quantales.

Nevertheless in some cases it seems necessary to develop further the theory of enrichments to render it more applicable. Let us mention for example the work on the change of base bicategories [10], that yielded the two-sided enrichments as bicategory morphisms. The original motivation for the change of base was to encode geometric morphisms using Walters' representation of sheaves [13].

Extending the original idea from [14], this paper treats quasi-uniformities and uniformly continuous maps. Its aim is to clarify the connection between quasi-uniform spaces together with uniformly continuous maps and enriched categories together with enriched functors. Again it will appear that the change of base techniques are required to capture the notion of uniformly continuous maps.

Quasi-uniformities arise from uniformities, as quasi-pseudo-metrics from pseudo metrics, by dropping the symmetry condition. These spaces were first studied in relation with topological groups (see [11] for a recent survey). They have also become a center of interest for computer science since the pioneering works of Smyth and Sunderhauf [16], [17], [18], [19], [20], [21]. From a technical point of view, a huge challenge is to define a correct theory of non symmetric space, that is, together with a well defined notion of non symmetric convergence. From this perspective, there is a hope that the analogy with enriched categories - which are essentially non-symmetric objects - should be fruitful. In this spirit, the connection established in the present paper enables the transposition of the categorical Cauchy-completion to the topological bicompletion.

Starting with the idea of [14] that enrichments are general metric spaces, and considering quasi-uniform spaces, a few remarks are in order. One may observe the following two points:

- The $[0, +\infty]$-functors correspond to non-increasing maps that are less general than the traditional uniformly continuous maps.
- Quasi-uniform spaces are more general than the metric ones and also admit a completion à la Cauchy, the so-called bicompletion [6]. After a little

thought, it seems reasonable that a quasi-uniform space (X, U) occurs as a $\mathcal{V}$-enrichment where the base category $\mathcal{V}$ is related to the quasi-uniformity U (indeed this is the case, as shown in this paper). But then it seems unrealistic to code the whole category of quasi-uniform spaces as a category of enrichments over a single a base category.

Therefore we investigated a theory of enrichments *over different bases* with in mind that:

($pb1$) Such a theory should capture the notion of quasi-uniform spaces with the right morphisms: the uniformly continuous maps;

($pb2$) There should exist a Cauchy-completion for enrichments over different bases that should capture the quasi-uniform space bicompletion.

It is fair to cite at this stage the recent work [3] that treated nicely ($pb1$). Nevertheless it is not obvious (at the moment for the author) that the general framework developed in the later paper may describe naturally the Cauchy completion. On that point our works may actually differ.

To present now an overview of the results obtained, it seems necessary to point out in an informal way the problems occurring with Lawvere's approach for quasi-uniform spaces. This should also provide some intuition for later technical developments.

Let us recall briefly some results of [14] regarding the (general) metric spaces.

$[0, +\infty]$ is the monoidal category with:

- objects: positive reals and $+\infty$;
- arrows: the reverse ordering, $x \to y$ if and only if $x \geq y$;
- tensor: the addition (with $+\infty + x = x + +\infty = +\infty$);
- unit: 0.

$[0, +\infty]$ is obviously symmetric. It is also closed as for any pair x, y of objects in $[0, +\infty]$, the exponential object $[x, y]$ is $max\{y - x, 0\}$.

A simple translation of the categorical definitions shows that $[0, +\infty]$-categories and $[0, +\infty]$-functors correspond respectively to the so-called general metric spaces and the non-increasing maps.

A *general metric space* or *gms* consists in a set of *objects* or *elements* denoted $Obj(A)$ or sometimes just A, together with a map $A(-,-) : Obj(A) \times Obj(A) \to [0, +\infty]$ that satisfies:

- for all $x, y, z \in A$, $A(y, z) + A(x, y) \geq A(x, z)$;
- for all $x \in A$, $A(x, x) = 0$.

For two gms a *map* $F : A \to B$ is non-increasing when

$$(*) \text{ for all } x, y \in A,\ A(x,y) \geq B(F(x), F(y)).$$

Considering $[0, +\infty]$-natural transformations one obtains then a preorder on non-increasing maps by $F \Rightarrow G : A \to B$ if and only if for all $x \in A$, $B(F(x), G(x)) = 0$. This preorder makes the category of general metric spaces and non-increasing maps, a locally preordered 2-category. Actually the sets of non-increasing maps between two general metric spaces inherits the structure of a general metric space.

$[0, +\infty]$-modules have a simple form. A $[0, +\infty]$-module $M : A \longrightarrow\!\!\!\!\!\circ\ B$ is a map $B \times A \to [0, +\infty]$ such that:

- for all $x, x' \in A$ and $y \in B$, $A(x, x') + M(y, x) \geq M(y, x')$;
- for all $x \in A$ and $y, y' \in B$, $M(y, x) + B(y', y) \geq M(y', x)$.

They compose as follows. Given $A \xrightarrow{M}\!\!\!\!\!\circ\ B \xrightarrow{N}\!\!\!\!\!\circ\ C$, the composite $N * M$ is defined for any $a \in A$ and $c \in C$ by $N * M(c, a) = \bigwedge_{b \in B} M(b, a) + N(c, b)$.

One has a 2-category $[0, +\infty]$-Mod of $[0, +\infty]$-modules and thus a formal notion of adjoints in this 2-category. It was observed in [14] that minimal Cauchy filters on a g.m.s. A are in one-to-one correspondence with left adjoint modules of the form $1 \xrightarrow{M}\!\!\!\!\!\circ\ A$. This is crucial to establish that the full subcategory of $[A^{op}, [0, +\infty]]$ of left adjoint modules is, in term of general metric spaces, the Cauchy-completion of A.

Of course one would like to adopt this categorical framework to treat quasi-uniform spaces and their bicompletions. Nevertheless there are a number of technical problems that we shall enumerate now.

The first point is to describe quasi-uniform spaces as enrichments. According to the coding below for metric spaces, our intuition would tell us that a quasi-uniform space (X, U) defines an enrichment with objects the $x, y ... \in X$ and homs given by:

$$(**) \quad X(x, y) = \{u \in U \mid (x, y) \in u\}.$$

Actually this is almost true but using this straightforward approach, one fails to define a monoidal category!

The second point is that there could not be one fixed base category for all quasi-uniform spaces. As one may suppose, there is description of the space (X, U) as a $\mathcal{V}$-category in the spirit of $(**)$ above. But here the base $\mathcal{V}$ would clearly depend from the quasi-uniformity U. Therefore one needs to consider different bases to code quasi-uniform spaces as enrichments.

A third point is to get a categorical characterization of the quasi-uniform

maps. Until now one only knows such one for non-increasing maps. Let us consider quasi-uniform spaces (X, U) and (Y, T). Any map $f : X \to Y$ defines a relation $R \subseteq U \times T$ by $R(u, t)$ if and only if for all $x, y \in X$, $(x, y) \in u \Rightarrow (f(x), f(y)) \in t$. Then one has

$$(1)\ \ \exists_R(X(x, y)) \subseteq Y(fx, fy), \quad for\ all\ x, y \in X,$$

where $\exists_R$ denotes the direct image of R. Note that the f above is uniformly continuous exactly when for all $t \in T$, there exists $u \in U$ such that for all x, y, $(x, y) \in u \Rightarrow (f(x), f(y)) \in t$ that is just when

$$(2)\ \ \exists_R(U) = T.$$

The inequality (1) may be seen as a generalization of the condition $(*)$ defining non-increasing maps. The idea is then to characterize the uniformly continuous maps f between (X, U) and (Y, T) as the ones with an adequate change of base — i.e. some $R \subseteq U \times T$ satisfying (1) — and such that this change of base satisfies moreover (2).

So far we have only defined the changes of base in terms of uniformities. Actually we shall show that they may be defined from particular monoidal functors, the so-called *super* ones. Then finally, uniformly continuous maps $(X, U) \to (Y, T)$ are maps $f : X \to Y$ defining a $\mathcal{V}_T$-functor $F_@(X) \to Y$ where $F : \mathcal{V}_U \to \mathcal{V}_T$ is a suitable monoidal functor with induced change of base $F_@$.

An unpleasant feature of this coding is that the bases depend from the set of objects of the enrichments. To get rid of this, we generalize slightly the quasi-uniform spaces by considering pairs (X, U) where U is no more a filter of binary relations on X but more generally a semi group with a lattice structure. This with a bunch of axioms defines our *abstract* quasi-uniformities.

We introduce also the notion of *canonical* abstract quasi-uniformities. They are needed to make the eventual link between abstract quasi-uniformities and enrichments. Canonical quasi-uniformities are exactly enrichments: monoidal structures may be derived from them so that the above condition $(**)$ holds. It happens also that any abstract quasi-uniformity is equivalent to a canonical one (in a 2-categorical sense!).

The last difficulty was to find the good change of base functors, i.e. they should induce the morphisms of a 2-category of enrichments over different bases that admits a Cauchy-completion. Their definition was found by a careful inspection of the changes of base induced by quasi-uniformly continuous maps.

To sum up our results we have established the following chain of corre-

spondences.

$$Enr(H) \rightharpoonup_{(1)} CAQUnif \rightharpoonup_{(2)} AQUnif \rightharpoonup_{(3)} QUnif.$$

$Enr(H)$ is a 2-category of enrichments over different bases. Bases considered are just partial orders or "quantales". $CAQunif$, $AQUinf$, $QUnif$ denote respectively the 2-categories of canonical abstract quasi-uniformities, abstract quasi-uniformities and quasi-uniformities. (1) is an isomorphism of 2-categories, (2) is an equivalence of 2-categories and (3) is a 2-coreflection. The 2-categorical Cauchy-completion in $Enr(H)$ migrates through (1) and (2) to $AQUnif$. We have also explicited this completion for abstract quasi-uniformities. From the latter and (3) one may retrieve the bicompletion for quasi-uniformities.

The rest of paper is technical and presents the above results. It is is organized as follows. There are two sides, one being enriched category theory, the other one topology, and the purpose is to reunify them. We start from enriched categories and do half of the way. Then start again from the quasi-uniformities and do the other half. The (canonical) abstract quasi-uniformities stand at the meeting point.

Section 2 recalls basic elements of quasi-uniformity theory. We introduce quasi-uniformities and their morphisms - that we call for short *uniformly continuous maps*. We point out that the set of uniformly continuous maps between two given quasi-uniformities may be preordered and hence quasi-uniformities and their morphisms form a locally preordered 2-category $QUnif$. Cauchy filters and the notions of separation and completeness for quasi-uniformities are recalled as well as the existence of a bicompletion for quasi-uniformities.

Chapter 3 recalls well-known very basic facts of the theory of $\mathcal{V}$-categories in the particular case where $\mathcal{V}$ is a monoidal partial order. The Cauchy-completion for $\mathcal{V}$-enrichments is detailed. The purpose of this chapter is to make the theory of the Cauchy completion accessible to non-specialists. The drawback is that fundamental notions of enriched category theory that are tight to the Cauchy-completion (functor categories, Yoneda, accessibility, free-cocompletions...) are just omitted. For any of these notions, we refer to Kelly's book [9]. The case of $\mathcal{V}$-enrichments with $\mathcal{V}$ non symmetric is treated in [4].

In chapter 4, we define a 2-category of enrichments over different bases. Here again, we only consider the case where the base monoidal categories are partial orders. We introduce the *super* monoidal functors between monoidal biclosed complete partial orders. They serve to define the 2-category $Enr(H)$, where H is a parameter 2-functor from a locally filtered 2-category to the 2-category $SMon$ of monoidal biclosed complete partial orders, super monoidal functors and monoidal transformations between them. For this 2-category

the definition of morphisms between enrichments over different bases involves a change base monoidal functor that is super. We show that the Cauchy completion of enrichments is really a completion in $Enr(H)$. Precisely the Cauchy-completion yields a left 2-adjoint to the inclusion 2-functor $SC-Enr(H) \to Enr(H)$ where $SC-Enr(H)$ denotes the 2-category with skeletal and Cauchy-complete objects and with arrows and local preorders inherited respectively from $Enr(H)$. An important point is that $SC-Enr(H)$ is a full replete subcategory of $Enr(H)$.

Section 5 introduces the locally partially ordered 2-category QUT of quasi-uniform triples and their morphisms. Quasi-uniform triples correspond to an axiomatization of the lattice of quasi-uniformities and may be seen as quasi-uniformities "without points". We show how to build monoidal biclosed categories from quasi-uniform triples and super monoidal functors from morphisms of quasi-uniform triples. The notion of *canonical* morphisms of quasi-uniform triple is defined. It is needed to show the existence of a 2-functor $QUT \to SMon$. Indeed this 2-functor factorizes through $CQUT$ that is the locally partially ordered 2-category of quasi-uniform triples and canonical morphisms.

Section 6 introduces the locally preordered 2-category $AQUnif$ of *abstract quasi-uniformities* and their morphisms. As mentioned below abstract quasi-uniformities lie in between quasi-uniformities and enrichments - this statement will be made precise! - An abstract quasi-uniformity consists in a set of points with a "general distance" valued in a quasi-uniform triple. Any quasi-uniformity defines in a natural way an abstract one, (but the nature of this correspondence is postponed to section 8). We show that the 2-category $AQUnif$ is 2-equivalent to the 2-category $CAQUnif$ of *canonical* abstract quasi-uniformities, that is 2-isomorphic to $Enr(H)$ for the 2-functor $H : CQUT \to SMon$ defined in section 5. Let us just underline here that the notion of canonical abstract quasi-uniformities is linked with the crucial axiom

$$\forall u \in U, \exists v \in U, v^2 \subseteq u$$

occurring in the definition of a quasi-uniformity U.

Section 7 concerns the completion of abstract quasi-uniformities. We introduce the notions of Cauchy filters, neighborhood filters, separation and completeness for abstract quasi-uniformities. They correspond to the usual notions for quasi-uniformities. Via the 2-equivalence $AQUnif \cong Enr(H)$, sending the quasi-uniformity A to the corresponding enrichment $C(A)$, we see that:

- minimal Cauchy filters on A are in bijective correspondence with left adjoint modules of the form $1 \longrightarrow C(A)$,

- the separation of A is equivalent to the skeletality of $C(A)$,
- the completeness of A is equivalent to the Cauchy-completeness of $C(A)$.

Since $AQUnif \cong Enr(H)$, one obtains as a direct application of the results of the section 4, a completion for abstract quasi-uniformities. Precisely, if $SC - AQUnif$ denotes the 2-category with objects the separated and complete abstract quasi-uniformities and arrows and local preorders inherited from $AQUnif$ then the inclusion 2-functor $SC - AQunif \to AQUnif$ has a left 2-adjoint that sends any abstract quasi-uniformity to its completion. Eventually we give an internal description of the completion of a quasi-uniformity.

Section 8 relates eventually usual quasi-uniformities to abstract ones. There is an obvious inclusion 2-functor $P : QUnif \to AQUnif$ that admits a right 2-adjoint Q such that $P \circ Q = 1$. Defining *concrete* abstract uniformities we show that $QUnif$ is 2-equivalent to the 2-category with objects the concrete abstract quasi-uniformities and arrows and local preorders inherited from $AQUnif$. Then we show how to retrieve the bicompletion of quasi-uniformities from the completion of abstract quasi-uniformities.

2 Quasi-uniformities

In this section we will recall briefly some elements of the theory of quasi-uniformities.

A quasi-uniformity U on a set X, also denoted (X, U), is a set of binary relations $U \subseteq \wp(X \times X)$ that satisfies:

(i) $\forall u \in U, \Delta \subseteq u$ (Δ is the diagonal relation),

(ii) U is a filter on $X \times X$,

(iii) $\forall u \in U, \exists v \in U, v^2 \subseteq u$.

A quasi-uniformity basis on X, is a filter basis for a quasi-uniformity (X, U).

Given quasi-uniformities (X, U) and (Y, V), a map $f : X \to Y$ is uniformly continuous when

$$\forall v \in V, \exists u \in U, \forall x, y \in X, (x, y) \in u \Rightarrow (fx, fy) \in v.$$

The set of uniformly continuous maps from (X, U) to (Y, V) is preordered by $\Rightarrow$ given for all $f, g : (X, U) \to (Y, V)$, by $f \Rightarrow g$ if and only if $\forall v \in V, \forall x \in X, (f(x), g(x)) \in v$. Quasi-uniformities, uniformly continuous maps and their preorders form a locally preordered 2-category $QUnif$.

Let (X, U) be a quasi-uniformity. A filter F on X is Cauchy (w.r.t U) when it satisfies:

$$\forall u \in U, \exists f \in F, f \times f \subseteq u.$$

The neighborhood filter of some $x \in X$ is the filter with basis the set of subsets of the form $\{y/(x,y) \in u \cap u^{-1}\}$, u ranging in U. Such a filter as above is Cauchy, it is also a minimal Cauchy one (This fact may be retrieved from results in subsections 7.2 and 8.2). (X,U) is said *complete* when any minimal Cauchy filter occurs as the neighborhood filter of some element of X. (X,U) is *separated* when the map sending the elements of X to their neighborhood filters is injective. Again we shall see in a slightly more general framework (7.10) that (X,U) is separated if and only if

$$\forall x, y \in X, x \neq y \Rightarrow \exists u \in U, (x,y) \notin u \cap u^{-1}.$$

We let $SC - QUnif$ denote the locally preordered 2-category with objects the separated and complete quasi-uniformities and with arrows and local preorders inherited from $QUnif$.

In section 8, we are going to show that the theorem below [6], may be obtained as a consequence of general results regarding enriched category theory 4.7 and 4.11.

Theorem 2.1 *The inclusion 2-functor $SC - QUnif \rightarrow QUnif$ has a left 2-adjoint.*

3 Enrichments over monoidal partial orders

This chapter recalls a few very basic elements of enriched category theory. All the results presented are well-known or belong to folklore. They are reproduced for completeness and to fix notations. This chapter introduces the Cauchy-completion of categories enriched over monoidal closed partial orders. Quite surprisingly this notion is accessible without referring to deep results in enriched category theory (Yoneda, indexed colimits, free cocompletions). We refer the reader to [9] for a general presentation of enriched category theory.

We call a monoidal category that is also a partial order, a *monoidal partial order*. In this section we will consider a monoidal biclosed and complete partial order $\mathcal{V}$, its tensor product is denoted $\otimes$, its identity I. An important fact is that $\mathcal{V}$ being complete is also cocomplete.

Definition 3.1 [$\mathcal{V}$-enrichments] An *enrichment* A over $\mathcal{V}$ or a *$\mathcal{V}$-category* is a set $Obj(A)$ — the *objects* of A — with a mapping $A(-,-) : Obj(A) \times Obj(A) \longrightarrow Obj(\mathcal{V})$ satisfying:

- for any object a of A, $I \leq A(a,a)$;
- for any objects a, b, c of A, $A(b,c) \otimes A(a,b) \leq A(a,c)$.

Given $\mathcal{V}$-categories A and B, a *$\mathcal{V}$-functor* f from A to B is a map $f : Obj(A) \longrightarrow Obj(B)$ such that for any objects a, a' of A, $A(a, a') \leq B(fa, fa')$.

Given two $\mathcal{V}$-functors $f, g : A \to B$ there is a *$\mathcal{V}$-natural transformation* from f to g, — which is denoted $f \Rightarrow g$, when for any object a of A, $I \leq B(fa, ga)$.

The usual definition of enrichment does not require $Obj(A)$ above to be a set. In this paper, we shall consider only "small" enrichments i.e. such that their sets of objects is small. The base categories will be also always small. This makes sense since the examples of enrichments that we will look at, are very "concrete", and the fact that Cauchy-completion (3.10) preserves the smallness of objects is essential.

Proposition 3.2 *$\mathcal{V}$-categories, $\mathcal{V}$-functors and, $\mathcal{V}$-natural transformations constitute a locally preordered 2-category $\mathcal{V} - Cat$.*

In $\mathcal{V} - Cat$ the (horizontal) composition of $\mathcal{V}$-functors is given by the composition of underlying maps, and the identity functors correspond to identity maps. $\hat{1}$ denotes the enrichment on $\mathcal{V}$, with one object, say $*$, and $\hat{1}(*, *) = I$. If B is a $\mathcal{V}$-category, for any of its object b, $b : \hat{1} \to B$ denotes the $\mathcal{V}$-functor sending $*$ to b.

Definition 3.3 [$\mathcal{V}$-modules] Given two $\mathcal{V}$-categories A and B, a *$\mathcal{V}$-module* φ from A to B — denoted $\varphi : A \longrightarrow\!\!\!\!\!\circ\; B$ — is a map $\varphi : Obj(B) \times Obj(A) \to Obj(\mathcal{V})$ such that:

- for any objects a, a' of A, and b of B, $A(a, a') \otimes \varphi(b, a) \leq \varphi(b, a')$;
- for any objects a of A and, b, b' of B, $\varphi(b, a) \otimes B(b', b) \leq \varphi(b', a)$.

For any two $\mathcal{V}$-modules $\varphi : A \longrightarrow\!\!\!\!\!\circ\; B$ and $\psi : B \longrightarrow\!\!\!\!\!\circ\; C$, their composite $\psi \bullet \varphi : A \longrightarrow\!\!\!\!\!\circ\; C$ is defined by: $\forall a \in Obj(A), c \in Obj(C)$, $(\psi \bullet \varphi)(c, a) = \bigvee_{b \in Obj(B)} \varphi(b, a) \otimes \psi(c, b)$. The set $\mathcal{V} - Mod(A, B)$ of $\mathcal{V}$-modules from A to B is partially ordered by: $\varphi \leq \varphi' \Leftrightarrow \forall (b, a) \in Obj(B) \times Obj(A), \varphi(b, a) \leq \varphi'(b, a)$.

Proposition 3.4 *$\mathcal{V}$-categories and $\mathcal{V}$-modules with partial orders defined above, constitute a locally partially ordered 2-category denoted $\mathcal{V} - Mod$.*

In $\mathcal{V} - Mod$, the identity in A is the module with underlying map $A(-, -) : Obj(A) \times Obj(A) \to Obj(\mathcal{V})$ sending any (a, a') to $A(a, a')$. For any $\mathcal{V}$-categories A and B, a $\mathcal{V}$-module $\varphi : A \longrightarrow\!\!\!\!\!\circ\; B$ has right adjoint $\psi : B \longrightarrow\!\!\!\!\!\circ\; A$ if and only if $A(-, -) \leq \psi \bullet \varphi$ and $\varphi \bullet \psi \leq B(-, -)$. For any left adjoint module φ, $\tilde{\varphi}$ will denote its (unique!) right adjoint. Any $\mathcal{V}$-functor $f : A \to B$ corresponds to a pair of adjoint modules $f_\diamond \dashv f^\diamond$, $f_\diamond : A \longrightarrow\!\!\!\!\!\circ\; B$, $f^\diamond : B \longrightarrow\!\!\!\!\!\circ\; A$,

as follows: for any objects a of A, and b of B, $f_\diamond(b,a) = B(b,fa)$ and $f^\diamond(a,b) = B(fa,b)$.

Further on $\mathcal{V}-AMod$ will denote the locally partially ordered 2-category with objects $\mathcal{V}$-categories, arrows: *left adjoint* $\mathcal{V}$-modules and local partial orders inherited from $\mathcal{V}-Mod$.

Proposition 3.5 *There is a 2-functor $J_\mathcal{V} : \mathcal{V}-Cat \to \mathcal{V}-Mod$ as follows. It is the identity on objects, and is the map $(-)_\diamond$ on arrows sending any $\mathcal{V}$-functor f to the $\mathcal{V}$-module $f_\diamond$. For any $\mathcal{V}$-functors $f,g : A \to B$, $f \Rightarrow g$ if and only if $f_\diamond \leq g_\diamond$.*

Definition 3.6 [Skeletality, Cauchy-completeness] A $\mathcal{V}$-category B is *skeletal*, respectively *Cauchy-complete* when for any $\mathcal{V}$-category A, the map $(-)_\diamond : \mathcal{V}-Cat(A,B) \to \mathcal{V}-AMod(A,B)$ is injective, respectively surjective.

Remark 3.7 For any $\mathcal{V}$-category B, the following assertions are equivalent:

- B is skeletal;
- The map $(-)_\diamond : \mathcal{V}-Cat(\hat{1},B) \to \mathcal{V}-AMod(\hat{1},B)$ is injective;
- For any of its objects a and b, if $I \leq B(a,b)$ and $I \leq B(b,a)$ then $a = b$.

Proposition 3.8 *A $\mathcal{V}$-category B is* Cauchy-complete *when the map $(-)_\diamond : \mathcal{V}-Cat(\hat{1},B) \to \mathcal{V}-AMod(\hat{1},B)$ is surjective. (i.e. for any left adjoint $\mathcal{V}$-module $\varphi : \hat{1} \longrightarrow\!\!\!\!\!\circ\; B$, $\varphi = b_\diamond$ for some object b of B).*

We define the following 2-categories.

- $\mathcal{V}-SkCcCat$: with objects skeletal and Cauchy-complete $\mathcal{V}$-categories, and arrows and local preorders inherited from $\mathcal{V}-Cat$.
- $\mathcal{V}-SkCcAMod$: with objects: skeletal and Cauchy-complete $\mathcal{V}$-categories, and arrows and local partial orders inherited from $\mathcal{V}-AMod$.

Proposition 3.9 *The restriction of $J_\mathcal{V}$ on $\mathcal{V}-SkCcCat$ is a 2-isomorphism onto $\mathcal{V}-SkCcAMod$.*

Definition 3.10 [Cauchy-completion] Let A be a $\mathcal{V}$-category. The *Cauchy-completion* of A is the $\mathcal{V}$-category $\bar{A}$ defined as follows. Its objects are the left-adjoint $\mathcal{V}$-modules of the form $\varphi : \hat{1} \longrightarrow\!\!\!\!\!\circ\; A$. For convenience we will consider modules in $Obj(\bar{A})$ as well as their adjoints, as maps with domains A. For any $\varphi,\psi \in Obj(\bar{A})$, $\bar{A}(\varphi,\psi) = (\tilde{\varphi} \bullet \psi)(*,*) = \bigvee_{a\in Obj(A)} \psi(a) \otimes \tilde{\varphi}(a)$. The map $Obj(A) \to Obj(\bar{A})$ sending any a to the $\mathcal{V}$-module $a_\diamond$, defines a $\mathcal{V}$-functor $j_A : A \to \bar{A}$.

Proposition 3.11 *For any $\mathcal{V}$-category A, $\bar{A}$ is skeletal and Cauchy-complete.*

Lemma 3.12 *For any $\mathcal{V}$-category A, $(j_A)_\diamond : A \longrightarrow\!\!\!\!\!\circ\ \bar{A}$ and $(j_A)^\diamond : \bar{A} \longrightarrow\!\!\!\!\!\circ\ A$ are inverse modules.*

With 3.9, 3.12 and 3.11, one may establish straightforwardly 3.13 and 3.14 below.

Proposition 3.13 *There is an equivalence of 2-categories $S : \mathcal{V} - AMod \cong \mathcal{V} - SkCcCat$, defined on objects by $S(A) = \bar{A}$ and on arrows by $S(\varphi) = f_\varphi$ where for any left adjoint $\mathcal{V}$-module $\varphi : A \longrightarrow\!\!\!\!\!\circ\ B$, $f_\varphi : \bar{A} \to \bar{B}$ is the unique $\mathcal{V}$-functor f satisfying $f_\diamond \bullet j_{A\diamond} = j_{B\diamond} \bullet \varphi$.*

Proposition 3.14 *The inclusion 2-functor $\mathcal{V} - SkCcCat \to \mathcal{V} - Cat$ has a left 2-adjoint.*

Here, the left 2-adjoint of the inclusion sends a $\mathcal{V}$-category A to $\bar{A}$ and the unit takes value $j_A : A \to \bar{A}$ in A.

According to 3.11 and 3.12, the following is coherent

Definition 3.15 [Morita-equivalence] Two $\mathcal{V}$-categories A and B are *Morita-equivalent* when one the following equivalent assertions is satisfied:

(i) They are isomorphic in $\mathcal{V} - Mod$;

(ii) Their Cauchy-completions are isomorphic in $\mathcal{V} - Cat$.

4 More on enrichments

In this chapter (4.1 and 4.2) we shall develop a 2-category of enrichments over different bases that will allow us to code quasi uniform spaces and their morphisms. As in the previous section we only treat enrichments over monoidal closed complete partial orders. We shall omit to recall the peculiar nature of the base monoidal categories.

4.1 About the change of base

The change of base for enrichments (over bicategories) has recently known some new developments [10]. Nevertheless we shall only use an old formulation of a classical result [5]. We introduce in this subsection the so-called *super* functors that were defined in [13]. These results will serve in the next section 4.2.

Proposition 4.1 *There is a 2-functor $(-)_@ : Mon \to 2 - CAT$ between the following 2-categories.*

Mon is the locally partially ordered 2-category with:

- *objects: small monoidal biclosed and complete partial orders,*
- *arrows: monoidal functors,*
- *2-cells: natural transformations (here, monoidal transformations are just natural transformations and there is a 2-cell $F \leq G : \mathcal{V} \to \mathcal{W}$ in Mon when for any object $v \in \mathcal{V}$, $Fv \leq Gv$ in $\mathcal{W}$).*

$2-CAT$ is the 2-category of large 2-categories, 2-functors and 2-natural transformations.

$(-)_@$ sends any category $\mathcal{V}$ to the 2-category $\mathcal{V}-Cat$. For any functor $F : \mathcal{V} \to \mathcal{W}$, $F_@ : \mathcal{V}-Cat \to \mathcal{W}-Cat$, is as follows. For any $\mathcal{V}$-category A, $F_@A$ is the $\mathcal{W}$-category with the same objects as A and such that for any objects a, b of A, $F_@A(a,b) = FA(a,b)$. For any $\mathcal{V}$-functor $f : A \to B$, $F_@f : F_@A \to F_@B$ is the $\mathcal{W}$-functor with the same underlying map as f. For any monoidal transformation $F \leq G$, there is a 2-natural transformation with value in A the $\mathcal{W}$-functor $F_@A \to G_@A$, with underlying map the identity of $Obj(A)$.

Proposition 4.2 *If $F : \mathcal{V} \to \mathcal{W}$ lies in Mon then there is a lax normal* [3] *functor $F_\sharp : \mathcal{V}-Mod \to \mathcal{W}-Mod$ that extends $F_@$, i.e. such that the diagram below commutes:*

$$\begin{array}{ccc} \mathcal{V}-Cat & \xrightarrow{F_@} & \mathcal{W}-Cat \\ {\scriptstyle J_\mathcal{V}}\downarrow & & \downarrow{\scriptstyle J_\mathcal{W}} \\ \mathcal{V}-Mod & \xrightarrow[F_\sharp]{} & \mathcal{W}-Mod \end{array}$$

For any $\mathcal{V}$-module $\varphi : A \longrightarrow\!\!\!\!\!\circ\; B$, $F_\sharp(\varphi) : F_@A \longrightarrow\!\!\!\!\!\circ\; F_@B$ is the map

$$\begin{cases} Obj(B) \times Obj(A) \to Obj(\mathcal{V}), \\ \forall a \in Obj(A), b \in Obj(B), F_\sharp(\varphi)(b,a) = F\varphi(b,a). \end{cases}$$

Proof. Let $\varphi : A \longrightarrow\!\!\!\!\!\circ\; B$. For any objects a, a' of A and b of B,

$$\begin{aligned} F_@A(a,a') \otimes F_\sharp(\varphi)(b,a) &= FA(a,a') \otimes F\varphi(b,a) \\ &\leq F(A(a,a') \otimes \varphi(b,a)) \\ &\leq F(\varphi(b,a')) \\ &= F_\sharp(\varphi)(b,a'). \end{aligned}$$

Similarly, for any object a of A and b, b' of B, $F\varphi(b,a) \otimes FB(b',b) \leq F\varphi(b',a)$. Thus $F_\sharp(\varphi)$ is a well defined module $F_@A \longrightarrow\!\!\!\!\!\circ\; F_@B$ in $\mathcal{W}-Mod$.

Trivially if $\varphi \leq \psi : A \longrightarrow\!\!\!\!\!\circ\; B$ then $F_\sharp\varphi \leq F_\sharp\psi : F_@A \longrightarrow\!\!\!\!\!\circ\; F_@B$.

Now given $\varphi : A \longrightarrow\!\!\!\!\!\circ\; B$ and $\psi : B \longrightarrow\!\!\!\!\!\circ\; C$, for any objects a of A and c

[3] A lax functor is *normal* when it preserves identities.

of C,

$$\begin{aligned} F_\sharp(\psi \bullet \varphi)(c,a) &= F(\psi \bullet \varphi)(c,a), \\ &= F(\textstyle\bigvee_{b\in Obj(B)} \varphi(b,a) \otimes \psi(c,b)), \\ &\geq \textstyle\bigvee_{b\in Obj(B)} F(\varphi(b,a) \otimes \psi(c,b)) \\ &\geq \textstyle\bigvee_{b\in Obj(B)} F\varphi(b,a) \otimes F\psi(c,b) \\ &= (F_\sharp\psi \bullet F_\sharp\varphi)(c,a). \end{aligned}$$

By definition $F_\sharp(f_\diamond) = (F_@ f)_\diamond$ for any $\mathcal{V}$-functor f, thus for any $\mathcal{V}$-category A, $F_\sharp(1_{A\diamond}) = (F_@ 1_A)_\diamond = (1_{F_@ A})_\diamond$. □

Lemma 4.3 *Let $F : \mathcal{V} \to \mathcal{W}$ lie in Mon and $f : A \to B$ be a $\mathcal{V}-functor$. Then:*
(i) $F_\sharp(f_\diamond) \dashv F_\sharp(f^\diamond)$ in $\mathcal{W}-Mod$;
(ii) If $f_\diamond$ is an isomorphism in $\mathcal{V}-Mod$ then $F_\sharp(f^\diamond) \bullet F_\sharp(f_\diamond) = 1$.

Proof. (i): Results from the facts that for any $\mathcal{V}$-functor f, $F_\sharp(f_\diamond) = (F_@(f))_\diamond$ and also $F_\sharp(f^\diamond) = (F_@(f))^\diamond$.
(ii): Suppose $f_\diamond \bullet f^\diamond = 1$ and $f^\diamond \bullet f_\diamond = 1$. Composing the second inequality by $F_\sharp$ which is lax and normal, one gets $F_\sharp(f^\diamond) \bullet F_\sharp(f_\diamond) \leq F_\sharp(f^\diamond \bullet f_\diamond) = F_\sharp(1) = 1$. Thus $F_\sharp(f^\diamond) \bullet F_\sharp(f_\diamond) = 1$ since $F_\sharp(f_\diamond) \dashv F_\sharp(f^\diamond)$. □

Definition 4.4 A monoidal functor $F : \mathcal{V} \to \mathcal{W}$ is *super* when for any family of pairs of objects $(v_i, w_i) \in Obj(\mathcal{V}) \times Obj(\mathcal{V})$, i ranging in I, if $I_\mathcal{V} \leq \bigvee_{i\in I} v_i \otimes w_i$ then $I_\mathcal{W} \leq \bigvee_{i\in I} F(v_i) \otimes F(w_i)$.

Note that the composition of super functors and the identity functors are super so we may define $SMon$ as the 2-category with objects the monoidal biclosed complete partial orders, with arrows the super monoidal functors and, with 2-cells inherited from Mon.

Lemma 4.5 *If F is super then $F_\sharp : \mathcal{V}-Mod \to \mathcal{W}-Mod$ preserves adjoint pairs: if $\varphi \dashv \tilde{\varphi}$ in $\mathcal{V}-Mod$ then $F_\sharp(\varphi) \dashv F_\sharp(\tilde{\varphi})$ in $\mathcal{W}-Mod$.*

Proof. Suppose that the monoidal $F : \mathcal{V} \to \mathcal{W}$ is super. Consider a left adjoint $\varphi : A \longrightarrow\!\!\!\!\!\!\circ\;\; B$ in $\mathcal{V}-Mod$.

Let us see that $1 \leq F_\sharp\tilde{\varphi} \bullet F_\sharp\varphi$. Since $1 \leq \tilde{\varphi} \bullet \varphi$, for any object a of A, $I_\mathcal{V} \leq A(a,a) \leq \bigvee_{b\in Obj(B)}(\varphi(b,a) \otimes \tilde{\varphi}(a,b))$ thus $I_\mathcal{W} \leq \bigvee_{b\in Obj(B)} F\varphi(b,a) \otimes F\tilde{\varphi}(a,b)$. Thus for any objects a, a' of A,

$$\begin{aligned} FA(a,a') &\leq FA(a,a') \otimes (\textstyle\bigvee_{b\in Obj(B)} F\varphi(b,a) \otimes F\tilde{\varphi}(a,b)) \\ &= \textstyle\bigvee_{b\in Obj(B)}((FA(a,a') \otimes F\varphi(b,a)) \otimes F\tilde{\varphi}(a,b)) \\ &\leq \textstyle\bigvee_{b\in Obj(B)}(F\varphi(b,a') \otimes F\tilde{\varphi}(a,b)) \\ &= (F_\sharp\tilde{\varphi} \bullet F_\sharp\varphi)(a,a'). \end{aligned}$$

That $F_\sharp\varphi \bullet F_\sharp\tilde{\varphi} \leq 1$ results from the facts that $F_\sharp$ is lax and normal and $\varphi \bullet \tilde{\varphi} \leq 1$:
$F_\sharp(\varphi) \bullet F_\sharp(\tilde{\varphi}) \leq F_\sharp(\varphi \bullet \tilde{\varphi}) \leq F_\sharp(1) = 1.$ □

Note that if $F : \mathcal{V} - Mod \to \mathcal{W}$ is super then according to the previous lemma $F_\sharp : \mathcal{V} - Mod \to \mathcal{W} - Mod$ will preserve in particular isomorphic modules, thus the Morita equivalence and the Cauchy-completion.

4.2 Enrichments over different bases

We investigate a kind of 2-categories with objects enrichments over different bases. These 2-categories are the $Enr(H)$ where $H : \mathcal{F} \to SMon$ is a parameter 2-functor with domain $\mathcal{F}$ locally filtered. The definition of the morphisms of $Enr(H)$ involves the super monoidal functors defined previously. It happens that for a particular H, $Enr(H)$ is 2-equivalent to the 2-category of abstract quasi-uniformities (see 6 further). Eventually we show that the Cauchy-completion yields again a left 2-adjoint to the inclusion $SC - Enr(H) \to Enr(H)$, where $SC - Enr(H)$ denotes the 2-category with objects the skeletal and Cauchy-complete enrichments and arrows and 2-cells inherited from $Enr(H)$. We introduce first the 2-category Enr^+ with objects enrichments over different bases. We establish a few technical results about it that will serve us further to reason conveniently about the 2-categories $Enr(H)$.

For convenience we write the "enrichment $(A, \mathcal{V})$" for the "$\mathcal{V}$-category A". Given a monoidal functor $F : \mathcal{V} \to \mathcal{W}$, enrichments $(A, \mathcal{V})$ and $(B, \mathcal{W})$, and a map $f : Obj(A) \to Obj(B)$, we say that F is *compatible* with the triple (f, A, B) when f defines a $\mathcal{W}$-functor $F_@A \to B$. Note

Remark 4.6 If $F \leq G : \mathcal{V} \to \mathcal{W}$ and G is compatible with (f, A, B) then so is F.

Let $\mathcal{E}$ be the locally preordered 2-category defined by the following data:

- objects: enrichments;
- arrows from $(A, \mathcal{V})$ to $(B, \mathcal{W})$: ordered pairs (f, F) where f is a map $Obj(A) \to Obj(B)$ and $F : \mathcal{V} \to \mathcal{W}$ is a monoidal functor compatible with (f, A, B);
- horizontal composition: the composite

$$(\,(B, \mathcal{W}) \xrightarrow{(g,G)} (C, \mathcal{X}\,)) \circ (\,(A, \mathcal{V}) \xrightarrow{(f,F)} (B, \mathcal{W})\,)$$

  is the pair $(g \circ f, GF : \mathcal{V} \to \mathcal{X})$;
- identities: $(1 : Obj(A) \to Obj(A), 1 : \mathcal{V} \to \mathcal{V})$ in $(A, \mathcal{V})$;

- 2-cells: the 2-cell $(f, F) \Rightarrow (g, G) : A \to B$ exists when there is a (monoidal) natural transformation $F \leq G$ and for any object a of A, there exists the arrow $I \to B(fa, ga)$ in $\mathcal{W}$.

Enr^+ is the locally preordered 2-category with objects enrichments, arrows those (f, F) of $\mathcal{E}$ with F *super*, and local preorders inherited from $\mathcal{E}$. It is straightforward to check that both $\mathcal{E}$ and Enr^+ are well-defined 2-categories. $SC - Enr^+$ will stand for the locally preordered 2-category with objects the skeletal and Cauchy-complete enrichments and with arrows and local preorders inherited from Enr^+.

Proposition 4.7 *The inclusion 2-functor $SC - Enr^+ \to Enr^+$ has a 2-left-adjoint. Precisely, for any enrichment $(A, \mathcal{V})$, the arrow $(A, \mathcal{V}) \xrightarrow{(j_A, 1)} (\bar{A}, \mathcal{V})$ (that lies in Enr^+) is such that:*

- *For any arrow $(A, \mathcal{V}) \xrightarrow{(f,F)} (B, \mathcal{W})$ in Enr^+ with B skeletal and Cauchy-complete, there is a unique arrow $(\bar{f}, \bar{F})$ that renders commutative the diagram in $\mathcal{E}$ below.*

$$\begin{tikzcd} (A, \mathcal{V}) \arrow[r, "(j_A,1)"] \arrow[dr, "(f,F)"'] & (\bar{A}, \mathcal{V}) \arrow[d, "(\bar{f},\bar{F})"] \\ & (B, \mathcal{W}) \end{tikzcd}$$

  This arrow $(\bar{f}, \bar{F})$ is in Enr^+. It is given by $\bar{F} = F$ and $\bar{f}_\diamond = f_\diamond \bullet F_\sharp(j_A{}^\diamond)$.
- *There is a 2-cell $(f, F) \Rightarrow (g, G) : (A, \mathcal{V}) \to (B, \mathcal{W})$ in Enr^+ if and only if there is one $(\bar{f}, F) \Rightarrow (\bar{g}, G) : (\bar{A}, \mathcal{V}) \to (B, \mathcal{W})$ in Enr^+.*

Proof. Let $(f, F) : (A, \mathcal{V}) \to (B, \mathcal{W})$ be in Enr^+, B be skeletal and Cauchy-complete. We are looking for some arrow $(\bar{f}, \bar{F}) : \bar{A} \to B$ in Enr^+ such that $(\bar{f}, \bar{F}) \circ (j_A, 1_\mathcal{V}) = (f, F)$. Necessarily $\bar{F} = F$. Since F is super, $F_\sharp(j_{A_\diamond})$ is an isomorphism in $\mathcal{W} - Mod$ (4.5) and the composition on the right by $F_\sharp(j_{A_\diamond})$ (see below),

$$\begin{tikzcd} F_@(A) \arrow[r, "F_\sharp(j_{A_\diamond})", "\circ"] \arrow[dr, "f"'] & F_@(\bar{A}) \arrow[d, "\bar{f}"] \\ & B \end{tikzcd}$$

yields an isomorphism of categories $\mathcal{W} - Mod(F_@\bar{A}, B) \cong \mathcal{W} - Mod(F_@A, B)$. Thus by Cauchy-completeness and skeletality of B, the composition on the right by $F_@(j_A)$ is an isomorphism $\mathcal{W} - Cat(F_@\bar{A}, B) \cong \mathcal{W} - Cat(F_@A, B)$. Therefore the required $\bar{f}$ exists and is unique, it satisfies $\bar{f}_\diamond = f_\diamond \bullet F_\sharp(j_A{}^\diamond)$. Given another functor $(g, G) : (A, \mathcal{V}) \to (B, \mathcal{W})$, we have to show that there is

a natural transformation $(f, F) \Rightarrow (g, G)$ if and only if there is one $(\bar{f}, F) \Rightarrow (\bar{g}, G)$ where $\bar{g}$ is such that $\bar{g} \circ G_{@}(j_A) = g : G_{@}A \to B$ in $\mathcal{W} - Cat$. Let us write $\theta_A : F_{@}A \to G_{@}A$ and $\theta_{\bar{A}} : F_{@}\bar{A} \to G_{@}\bar{A}$ respectively for the components in A and $\bar{A}$ of the $\mathcal{V}$-natural transformation $\theta : F_{@} \to G_{@}$ induced by $F \leq G$. θ_A and $\theta_{\bar{A}}$ are $\mathcal{W}$-functors with underlying maps the identities of $Obj(A)$ and $Obj(\bar{A})$. The diagram in $\mathcal{W} - Cat$ below commutes.

$$\begin{tikzcd}
F_{@}A \arrow[r, "F_{@}(j_A)"] \arrow[d, "\theta_A"'] & F_{@}\bar{A} \arrow[d, "\theta_{\bar{A}}"] \\
G_{@}A \arrow[r, "G_{@}(j_A)"'] & G_{@}\bar{A}
\end{tikzcd}$$

Then the situation in $\mathcal{W} - Mod$ is depicted below.

$$\begin{tikzcd}
 & F_{@}A \arrow[r, shift left, "F_\sharp(j_{A\diamond})"] \arrow[r, phantom, "\cong"] \arrow[dl, "(\theta_A)_\diamond"'] \arrow[dr] & F_{@}\bar{A} \arrow[l, shift left, "F_\sharp(j_A{}^\diamond)"] \arrow[dl, "f_\diamond"'] \arrow[d, "\bar{f}_\diamond"] \\
G_{@}A \arrow[r, shift left, "G_\sharp(j_{A\diamond})"] \arrow[r, phantom, "\cong"] \arrow[dr, "g_\diamond"'] & G_{@}\bar{A} \arrow[l, shift left, "G_\sharp(j_A{}^\diamond)"] \arrow[d, "\bar{g}_\diamond"] \arrow[ul, phantom, "(1)"] \arrow[ur, phantom, "(2)"] & B \arrow[dl, "1"] \\
 & B &
\end{tikzcd}$$

By 2-functoriality of $(-)_\diamond : \mathcal{W}-Cat \to \mathcal{W}-Mod$ and since $F_\sharp$ and $G_\sharp$ preserve isomorphisms, it is now straightforward that the 2-cell (1) exists if and only the 2-cell (2) exists. This is equivalent to the desired result according to 3.5 . $\square$

Consider 2-functors $H : \mathcal{C} \to \mathcal{A}$ and $K : \mathcal{B} \to \mathcal{A}$. Then one may construct the pullback $\mathcal{C} \times_{H,K} \mathcal{B}$ of H and K in the category $2 - Cat$, with projections π_H and π_K as below.

$$\begin{tikzcd}
\mathcal{C} \times_{H,K} \mathcal{B} \arrow[r, "\pi_K"] \arrow[d, "\pi_H"'] & \mathcal{B} \arrow[d, "K"] \\
\mathcal{C} \arrow[r, "H"'] & \mathcal{A}
\end{tikzcd}$$

If a 2-functor $H : \mathcal{C} \to SMon$ is such that $\mathcal{C}$ is locally partially ordered and U denotes the forgetful 2-functor $Enr^+ \to SMon$. $\mathcal{C} \times_{H,U} Enr^+$ is (2-isomorphic to) the locally preordered 2-category with:

- objects: (A, s) where s is an object of $\mathcal{C}$ and $(A, H(s))$ is an enrichment;
- arrows $(A, s) \to (B, t)$: the triples $(f, R : s \to t)$ where $R : s \to t$ is an arrow in $\mathcal{C}$ and $(f, H(R))$ is a morphism $(A, H(s)) \to (B, H(t))$ in Enr^+;
- 2-cells as follows: there is one 2-cell $(f, R : s \to t) \Rightarrow (g, P : s \to t) : (A, s) \to (B, t)$ if and only if $R \leq P$ in $\mathcal{C}$ and $I \leq B(fa, ga)$ in $H(t)$ for any object a of A.
- identity in (A, s): $(1 : Obj(A) \to Obj(A), 1 : s \to s)$;
- (horizontal) composition of

$$(A, s) \xrightarrow{(f,R)} (B, t) \xrightarrow{(g,P)} (C, u)$$

given by

$$(A, s) \xrightarrow{(gof, P \circ R)} (C, u)\,.$$

$\mathcal{C} \times_{H,U} SC - Enr^+$ is just (2-isomorphic) to the 2-category with objects the (A, s) of $\mathcal{C} \times_{H,U} Enr^+$ with $(A, H(s))$ skeletal and Cauchy-complete and arrows and 2-cells inherited from $\mathcal{C} \times_{H,U} Enr^+$.

From 4.7, it is straightforward that

Proposition 4.8 *Given a 2-functor $H : \mathcal{C} \to SMon$ with $\mathcal{C}$ locally partially ordered, the inclusion 2-functor $\mathcal{C} \times_{H,U} SC - Enr^+ \to \mathcal{C} \times_{H,U} Enr^+$ has a left 2 adjoint.*

Now we turn on to the 2-categories $Enr(H)$. For the rest of the section, we consider a 2-functor $H : \mathcal{F} \to SMon$ such that $\mathcal{F}$ is a locally partially ordered 2-category that is also *locally filtered* – i.e. the homsets of $\mathcal{F}$ are filtered partial orders.

The locally preordered 2-categories $Enr(H)$ and $SC - Enr(H)$ are as follows. $Enr(H)$ is the locally preordered 2-category with:

- objects: triples (A, s) where s is an object of $\mathcal{F}$ and $(A, H(s))$ is an enrichment;
- arrows $(A, s) \to (B, t)$: the maps $f : Obj(A) \to Obj(B)$ for which there exists an $R : s \to t$ in $\mathcal{F}$ such that $H(R)$ is compatible with $(f, (A, H(s)), (B, H(t)))$;
- 2-cells as follows: there is one 2-cell $f \Rightarrow g : (A, s) \to (B, t)$ if and only if for any object a, $I \leq B(fa, ga)$ in $H(t)$;
- identity in (A, s): the identity map on $Obj(A)$;

- horizontal composition of $(A,s) \xrightarrow{f} (B,t) \xrightarrow{g} (C,u)$ given by

$$(A,s) \xrightarrow{gof} (C,u)\,.$$

$SC - Enr(H)$ is the 2-category with objects the (A,s) of $Enr(H)$ where $(A,H(s))$ is skeletal and Cauchy-complete, and arrows and 2-cells inherited from $Enr(H)$.

It is important to note here that

Remark 4.9 If the enrichments $(A,\mathcal{V})$ and $(B,\mathcal{W})$ of are isomorphic in $Enr(H)$, then $(A,\mathcal{V})$ is Cauchy-complete if and only if $(B,\mathcal{W})$ is.

This results immediately from the fact that a change of base 2-functor $F_{@}$: $\mathcal{V} - Cat \to \mathcal{W} - Cat$ is super and thus preserves the Cauchy-completeness.

According to 4.6, and since $\mathcal{F}$ is locally filtered,

Proposition 4.10 *There is a 2-cell $f \Rightarrow g : (A,s) \to (B,t)$ in $Enr(H)$ if and only if there exists an $R : s \to t$ in $\mathcal{F}$ such that there is a 2-cell $(f,R) \Rightarrow (g,R) : (A,s) \to (B,t)$ in $\mathcal{F} \times_{H,U} Enr^{+}$.*

Theorem 4.11 *The inclusion 2-functor $SC - Enr(H) \to Enr(H)$ has a left 2-adjoint.*

Proof. First, note that for any object (A,s) there exists the arrow $(j_A,1)$: $(A,s) \to (\bar{A},s)$ in $\mathcal{F} \times_{H,U} Enr^{+}$. Thus the map $j_A : Obj(A) \to Obj(\bar{A})$ defines a morphism of $Enr(H)$.

Now given a morphism $f : (A,s) \to (B,t)$ in $Enr(H)$ with B skeletal and Cauchy-complete, we know from 4.8 that the map $\bar{f} : Obj(\bar{A}) \to Obj(B)$ as in 4.7, defines a morphism of $Enr(H)$ such that the diagram

$$\begin{tikzcd} (A,s) \arrow[r, "j_A"] \arrow[dr, "f"'] & (\bar{A},s) \arrow[d, "\bar{f}"] \\ & (B,t) \end{tikzcd}$$

commutes in $Enr(H)$. We are going to show that $\bar{f}$ is the unique morphism in $Enr(H)$ that makes this diagram commute.

Let $(f, R : s \to t) : (A,s) \to (B,t)$ and $(g, P : s \to t) : (\bar{A},s) \to (B,t)$ in $\mathcal{F} \times_{H,U} Enr^{+}$ such that $g \circ j_A = f$.

Since $\mathcal{F}$ is locally filtered, there is a $Q : s \to t$ in $\mathcal{F}$ with $Q \leq R$, $Q \leq P$. Then $H(Q)$ $(\leq H(R), H(P))$, is compatible both with $(\bar{f}, \bar{A}, B)$ and $(g, \bar{A}, B)$. So the following diagram commutes in Enr^{+}:

$$\begin{tikzcd}
(A, H(s)) \arrow[r, "(j_A,1)"] \arrow[d, "(j_A,1)"'] & (\bar{A}, H(s)) \arrow[d, "(\bar{f},H(Q))"] \\
(\bar{A}, H(s)) \arrow[r, "(g,H(Q))"'] & (B, H(t))
\end{tikzcd}$$

According to 4.7, one deduces that the maps $\bar{f}$ and g are equal.

Now let us see that for (A, s) and (B, t) with B skeletal and Cauchy-complete, the assignments $f \mapsto \bar{f}$ defines a preorder isomorphism:

$$Enr(H)[(A, s), (B, t)] \cong SC - Enr(H)[(\bar{A}, s), (B, t)].$$

For any morphisms $f, g : (A, s) \to (B, t)$ in $Enr(H)$, one has:

$f \Rightarrow g$

if and only if for some arrow $R : s \to t$ in $\mathcal{F}$,

$(f, H(R)) \Rightarrow (g, H(R)) : (A, H(r)) \to (B, H(t))$ in Enr^+ (according to 4.10)

if and only if for some arrow $R : s \to t$ in $\mathcal{F}$,

$(\bar{f}, H(R)) \Rightarrow (\bar{g}, H(R)) : (\bar{A}, H(s)) \to (B, H(t))$ in Enr^+ (according to 4.7)

if and only if $\bar{f} \Rightarrow \bar{g} : (\bar{A}, s) \to (B, t)$ in $Enr(H)$ (according to 4.10).

□

5 Quasi-uniform triples

This section introduces quasi-uniform triples and their morphisms. Quasi-uniform triples are just the algebraic version of the lattice of quasi-uniformities. They may be seen as quasi-uniformities without points. The purpose of this section is to establish the existence of a 2-functor $F_5 : QUT \to SMon$ from a locally partially ordered 2-category QUT of quasi-uniform triples and their morphisms to the locally partially ordered 2-category $SMon$ of monoidal partial orders and super monoidal functors between them. To get this result it appears necessary to introduce *canonical* morphisms of quasi-uniform triples. We obtain a locally partially ordered 2-category $CQUT$, with objects quasi-uniform triples and with arrows the canonical quasi-uniform triple morphisms. F_5 is obtained as a composition $QUT \to CQUT \to SMon$.

Definition 5.1 [Quasi-uniform triples] A *quasi-uniform triple* $(S, \cdot, \leq)$ consists of a semi-group $(S, \cdot)$ with base S, composition operation "$\cdot$", with S partially ordered by $\leq$. It satisfies moreover the following axioms:

(i) $\forall s, t \in S$, $(s \cdot t \geq s) \ \wedge \ (s \cdot t \geq t)$;

(ii) $\forall s, t, u \in S, t \leq u \Rightarrow (s \cdot t \leq s \cdot u) \wedge (t \cdot s \leq u \cdot s)$;

(iii) $\forall s \in S$, $\exists t \in S, t^2 \leq s$;

(iv) $\leq$ is filtered, i.e. $\forall s, t \in S, \exists u \in S, u \leq s \wedge u \leq t$.

5.1 From Quasi-uniform Triples to Complete Biclosed Categories

For the quasi-uniform triple $(S, \cdot, \leq)$, we define the following. The operation "$\cdot$" is extended to $\wp(S)$ by $V \cdot W = \{s \cdot t \mid s \in V, t \in W\}$. Further on, for some $V \subseteq S$, $[V]$ will denote the upper-closure of V with respect to $\leq$. $\mathcal{I}_S$ will stand for the set of upper-closed subsets of S. A binary operation $\otimes$ in $\mathcal{I}_S$ is defined by $V \otimes W = [V \cdot W]$. We define the operation ρ on $\mathcal{I}_S$ by $\rho(V) = S \otimes V \otimes S$. Note that according to 5.1-(iii), $S \otimes S = S$. From this, one checks that ρ is indeed a lower closure operation (it is monotonous and decreasing with respect to the inclusion and idempotent). $\mathcal{O}_S$ will stand for the sets of fixpoints for ρ. It is stable for $\otimes$ since for any $V, W \in \mathcal{O}_S$,

$$\begin{aligned} V \otimes W &= S \otimes V \otimes S \otimes S \otimes W \otimes S \\ &= S \otimes S \otimes V \otimes S \otimes S \otimes W \otimes S \otimes S \\ &= S \otimes V \otimes W \otimes S. \end{aligned}$$

Therefore $\mathcal{O}_S$ provided with the restriction of $\otimes$ – still denoted $\otimes$, is a monoid with S as neutral element.

Remark 5.2 Given a quasi-uniform triple $(S, \cdot, \leq)$, W and a family $(V_i)_{i \in I}$ both in $\mathcal{I}_S$, $W \otimes (\bigcup_{i \in I} V_i) = \bigcup_{i \in I} (W \otimes V_i)$ and $(\bigcup_{i \in I} V_i) \otimes W = \bigcup_{i \in I} (V_i \otimes W)$.

Proposition 5.3 *Given a quasi-uniform triple $(S, \cdot, \leq)$, a monoidal partial order $\mathcal{C}_S$ is defined by the following data:*
- *objects: elements of $\mathcal{O}_S$;*
- *order: inclusion;*
- *identity: S;*
- *tensor product: $\otimes$.*

$\mathcal{C}_S$ is cocomplete (and thus complete) and biclosed. The least upper bound a greatest lower bounds in $\mathcal{C}_S$ are as follows: for any family $(V_i)_{i \in I}$,

$$\bigvee_{i \in I} V_i = \bigcup_{i \in I} V_i,$$

$$\bigwedge_{i \in I} V_i = \rho(\bigcap_{i \in I} V_i).$$

Proof. Checking that $\mathcal{C}_S$ is a monoidal partial order is straightforward. Let us see that it is cocomplete. For any family $(V_i)_{i \in I}$ of elements $\mathcal{O}_S$, $\bigcup_{i \in I} V_i$ lies in $\mathcal{O}_S$, since it is upper closed and, according to 5.2, $S \otimes (\bigcup_{i \in I} V_i) \otimes S = \bigcup_{i \in I} S \otimes V_i \otimes S = \bigcup_{i \in I} V_i$. Therefore $\bigcup_{i \in I} V_i$ is the least upper bound of the V_i's in $\mathcal{O}_S$. Since $\rho : (\mathcal{I}_S, \subseteq) \to (\mathcal{O}_S, \subseteq)$ is a right adjoint, it preserves greatest lower bounds. Thus $\mathcal{C}_S$ has greatest lower bounds as above. $\mathcal{C}_S$ with inclusion ordering has a least upper bound S and greatest lower bound $\emptyset$. $\mathcal{C}_S$ is biclosed: for any $V, V', W \in \mathcal{O}_S$,

$V' \otimes V \subseteq W$ if and only if
$V' \subseteq \{s \in S \mid \forall t \in V, s \cdot t \in W\}$ if and only if
$V' \subseteq \rho(\{s \in S \mid \forall t \in V, s \cdot t \in W\})$.

Analogously, $V \otimes V' \subseteq W$ if and only if $V' \subseteq \rho(\{s \in S \mid \forall t \in V, t \cdot s \in W\})$.□

5.2 Morphisms of Quasi-uniform Triples

Recall that given a binary relation $R \subseteq S \times T$, the maps

$$\begin{cases} \exists_R : \wp(S) \to \wp(T), \\ V \mapsto \{u' \mid \exists u \in V, R(u, u')\} \end{cases}$$

and

$$\begin{cases} \forall_R : \wp(T) \to \wp(S), \\ V' \mapsto \{u \mid \forall u' \in T, R(u, u') \Rightarrow u' \in V'\} \end{cases}$$

defines an adjoint pair $\exists_R \dashv \forall_R : \wp(S) \to \wp(T)$, the powersets $\wp(S)$ and $\wp(T)$ being partially ordered with inclusion (For any $V \subseteq S$ and $V' \subseteq T$, $\exists_R(V) \subseteq V' \Leftrightarrow V \subseteq \forall_R(V')$).

The assignments $S \mapsto \wp(S)$ and $R \mapsto \exists_R$ defines a 2-functor from *Rel*, the locally partially ordered 2-category of relations with sets as objects, arrows as relations, and with relation inclusion for local partial orders, to *Cat* the 2-category of small categories. Moreover the local components of this 2-functor are full, i.e. for any $P, Q \subseteq S \times T$, $P \subseteq Q$ if and only if for the pointwise ordering $\exists_P \leq \exists_Q$. So $\exists$ as a functor is faithful. These local components also preserve least upper bounds: for any family of relations $(R_i \subseteq S \times T)_{i \in I}$, $\exists_{\bigcup_{i \in I} R_i} = \bigcup_{i \in I} \exists_{R_i}$.

Definition 5.4 Given two quasi-uniform triples S and T, a morphism from S to T is a relation $R \subseteq S \times T$ such that:

(i) $\exists_R(S) = T$;
(ii) $\forall s_1, s_2 \in S, \forall t \in T,\ R(s_2, t) \wedge (s_1 \leq s_2) \Rightarrow R(s_1, t)$;
(iii) $\forall s \in S, \forall t_1, t_2 \in T,\ R(s, t_1) \wedge (t_1 \leq t_2) \Rightarrow R(s, t_2)$;
(iv) $\forall s_1, s_2 \in S, \forall t_1, t_2 \in T,\ R(s_1, t_1) \wedge R(s_2, t_2) \Rightarrow R(s_1 s_2, t_1 t_2)$.

For any quasi-uniform triple S, its partial order $\leq \subseteq S \times S$ is a morphism of quasi-uniform triples. Quasi-uniform triples and their morphisms form a locally partially ordered bicategory *QUT* where:

- the horizontal composition of $S \xrightarrow{R} T \xrightarrow{P} U$ is $S \xrightarrow{P \circ R} U$;
- the identity at S is $\leq_S$ (thus *QUT* is NOT a subcategory of *Rel*);
- the local partial orders are the inclusion relations.

Proposition 5.5 *QUT is locally finitely complete. Precisely, given morphisms of quasi-uniform triples $P, Q : S \to T$, the relation $P \cap Q$ defines a morphism $S \to T$.*

Proof. For any $t \in T$, since P and Q satisfies (i), there are some $s_1, s_2 \in S$ such that $P(s_1, t)$ and $Q(s_2, t)$. Since S is filtered there is some $s \in S$ such that $s \leq s_1$ and $s \leq s_2$. Thus since P and Q satisfy (ii), $P(s, t)$ and $Q(s, t)$. That $P \cap Q$ satisfies axioms (ii), (iii) (iv) is trivial. □

5.3 Canonical morphisms

Consider a quasi-uniform triple morphism $R : S \to T$. Since R satisfies 5.4-(iii) for any subsets $P \subseteq S$, $\exists_R(P)$ is upper closed.

Also,

Proposition 5.6 *For any subsets P, Q of S,*

$$\begin{aligned}\exists_R(P) \otimes \exists_R(Q) &= [\{v_1 \cdot v_2 \mid (\exists u_1 \in P, R(u_1, v_1)) \wedge (\exists u_2 \in Q, R(u_2, v_2))]\\ &\subseteq [\{v \mid \exists u_1 \in P, u_2 \in Q, R(u_1 \cdot u_2, v)\}], \text{ by } 5.4\text{-}(iv)\\ &= \exists_R(P \otimes Q).\end{aligned}$$

Definition 5.7 A quasi-uniform triple morphism $R : S \to T$ is said *canonical* when one of the following equivalent assertions hold:

(1) For any $V \in \mathcal{I}_S$, $\exists_R(V)$ belongs to $\mathcal{O}_T$;
(2) For all $s \in S$, the upper set $\{t \in T \mid R(s, t)\}$ belongs to $\mathcal{O}_T$;
(3) $\forall s \in S, t \in T, R(s, t) \Rightarrow \exists t_1, t_2, t_3 \in T, t \geq t_1 t_2 t_3 \wedge R(s, t_2)$.

Let us just check the implication (2) $\Rightarrow$ (1), the other implications are trivial. Supposing (2), one has for any $V \in \mathcal{I}_V$,

$$\begin{aligned}\exists_R(V) &= \exists_R(\textstyle\bigcup_{s \in V} \uparrow s)\\ &= \textstyle\bigcup_{s \in V} \exists_R(\uparrow s)\\ &= \textstyle\bigcup_{s \in V} (S \otimes \exists_R(\uparrow s) \otimes S) \in \mathcal{O}_T \text{ according to } 5.3.\end{aligned}$$

Proposition 5.8 *Given a quasi-uniform triple morphism $R : U \to T$, there is a larger canonical morphism $\bar{R}$ amongst subrelations of R. It is defined for all $s \in S$ and $t \in T$ by:*

$$\bar{R}(s, t) \quad \Leftrightarrow \quad \exists t_1, t_2, t_3 \in T, t \geq t_1 t_2 t_3 \wedge R(s, t_2).$$

Proof. The $\bar{R}$ defined above is trivially a subrelation of R. Let us check that it is a morphism of quasi-uniform triples (5.4).

$\bar{R}$ satisfies (i). For any $t \in T$, there is some $t' \in T$ such that $t'^4 \leq t$. Since R satisfies axiom (i), there is an $s \in S$ such that $R(s, t')$, and thus $\bar{R}(s, t)$.

$\bar{R}$ trivially satisfies axioms (ii) and (iii). Let us see that $\bar{R}$ satisfies (iv). Suppose $s_1, s_2 \in S$, $t_1, t_2 \in T$, $\bar{R}(s_1, t_1)$ and $\bar{R}(s_2, t_2)$. Then there are $u_1, u_2, u_3, v_1, v_2, v_3 \in T$ such that $t_1 \geq u_1u_2u_3$, $R(s_1, u_2)$, $t_2 \geq v_1v_2v_3$ and $R(s_2, v_2)$. Then $t_1t_2 \geq u_1u_2u_3v_1v_2v_3 \geq u_1u_2v_2v_3$ and $R(s_1s_2, u_2v_2)$.

$\bar{R}$ is canonical. Suppose that $\bar{R}(s, t)$ then there are $t_1, t_2, t_3 \in T$ with $R(s, t_2)$ and $t \geq t_1t_2t_3$. There are t'_1, t'_3 such that $(t'_1)^2 \leq t_1$ and $(t'_3)^2 \leq t_3$. Thus for $t'_2 = t'_1t_2t'_3$, $\bar{R}(s, t'_2)$ and $t \geq t'_1t'_2t'_3$.

$\bar{R}$ is maximal amongst the canonical $P \subseteq R$. Given a canonical $P \subseteq R$, then for all $s \in S$, $t \in T$,

$$\begin{aligned} P(s,t) &\Rightarrow \exists t_1, t_2, t_3 \in T, P(s, t_2) \wedge t \geq t_1t_2t_3 \\ &\Rightarrow \exists t_1, t_2, t_3 \in T, R(s, t_2) \wedge t \geq t_1t_2t_3 \\ &\Leftrightarrow \bar{R}(s, t). \end{aligned}$$

□

Corollary 5.9 *For any quasi-uniform triple morphism R, $\bar{\bar{R}} = \bar{R}$.*

Proof. $\bar{\bar{R}} \subseteq \bar{R}$, and $\bar{\bar{R}}$ is the largest canonical morphism included in $\bar{R}$. Thus since $\bar{R}$ is canonical, $\bar{R} \subseteq \bar{\bar{R}}$. □

Consider the partial order $\leq_S \subseteq S \times S$, we will write $\sqsubseteq_S$ for the relation $\overline{\leq_S}$. $\sqsubseteq_S$ is the relation given for all $s, t \in S$ by: $s \sqsubseteq_S t \Leftrightarrow \exists t_1, t_3 \in S, t \geq t_1st_3$. Then for any quasi-uniform triple morphism $R : S \to T$, $\bar{R}$ is just the composite $\sqsubseteq_T \circ R$.

One has also

Proposition 5.10 *If the quasi-uniform triple morphism $R : S \to T$ is canonical then $R \circ \sqsubseteq_S = R$.*

Proof. Since $\sqsubseteq_S$ is a subrelation of $\leq_S$ then $(R \circ \sqsubseteq_S)$ is a subrelation of $(R \circ \leq_S) = R$. Conversely, suppose $R(s, t)$. Since R is canonical, there are $t_1, t_2, t_3 \in T$ such that $t \geq t_1t_2t_3$ and $R(s, t_2)$. Now there are some $s_1, s_3 \in S$ such that $R(s_1, t_1)$ and $R(s_3, t_3)$. Thus $R(s_1ss_3, t)$ showing $(R \circ \sqsubseteq_S)(s, t)$.□

Proposition 5.11 *The composition of canonical morphisms is canonical. Precisely, for any $\bullet \xrightarrow{P} \bullet \xrightarrow{Q} \bullet$ in QUT, $\overline{Q \circ P} = \bar{Q} \circ \bar{P}$.*

Proof.

$$\begin{aligned} \overline{Q \circ P} &= \sqsubseteq \circ Q \circ P \\ &= \bar{Q} \circ P \\ &= \bar{Q} \circ \sqsubseteq \circ P \text{ - according to 5.10} \\ &= \bar{Q} \circ \bar{P}. \end{aligned}$$

□

We define $CQUT$ as the partially ordered 2-category with quasi-uniform

triples as objects, canonical morphisms as arrows and with composition of arrows and local partial orders inherited from QUT. The identity at S in $CQUT$ is $\sqsubseteq_S$, thus $CQUT$ is NOT a subcategory of QUT. According to the results of this section, one gets

Proposition 5.12 *There is a 2-functor $\overline{(-)} : QUT \to CQUT$ that is the identity on objects and sends a morphism R to $\bar{R}$. The components $\overline{(-)}_{S,T} : QUT(S,T) \to CQUT(S,T)$ are closure relations, i.e. they are right adjoints to the inclusions $CQUT(S,T) \to QUT(S,T)$.*

5.4 Canonical morphisms and super functors

Consider a quasi-uniform triple morphism $R : S \to T$. Then for any $V \in \mathcal{I}_S$, $\exists_R(V) \in \mathcal{I}_T$. Thus we define the two maps $\delta(R) : \mathcal{I}_S \to \mathcal{I}_T$ and $\bar{\delta}(R) : \mathcal{O}_S \to \mathcal{O}_T$ as follows. $\delta(R)$ is the restriction of $\exists_R$ on $\mathcal{I}_S$, and $\bar{\delta}(R) = \rho \circ \delta(R) \circ i$, where i denotes there the embedding $\mathcal{O}_S \to \mathcal{I}_S$ (see below)

$$\begin{tikzcd} \mathcal{I}_S \arrow[r, "\delta(R)"] & \mathcal{I}_T \arrow[d, "\rho"] \\ \mathcal{O}_S \arrow[u, "i"] \arrow[r, "\bar{\delta}(R)"'] & \mathcal{O}_T \end{tikzcd}$$

By definition, for any morphism of quasi-uniform triple, $R : S \to T$, the following diagram commutes

$$\begin{tikzcd} \wp(S) \arrow[r, "\exists_R"] & \wp(T) \\ \mathcal{I}_S \arrow[u, "i"] \arrow[r, "\delta(R)"'] & \mathcal{I}_T \arrow[u, "i"'] \end{tikzcd}$$

From this one gets that for any $\bullet \xrightarrow{R} \bullet \xrightarrow{P} \bullet \in QUT$, $\delta(P \circ R) = \delta(P) \circ \delta(R)$. Also for any quasi-uniform triple S, $\delta(\leq_S)$ is the identity on $\mathcal{I}_S$, thus

Proposition 5.13 *The assignments $S \mapsto (\mathcal{I}_S, \subseteq)$ and $R \mapsto \delta(R)$ define a 2-functor $F_1 : QUT \to Cat$.*

Moreover the local components $F_{1S,T} : QUT(S,T) \to Cat(\mathcal{I}_S, \mathcal{I}_T)$ are full, thus F_1, as a functor, is faithful.

Given a quasi-uniform triple S, for any $V \in \mathcal{I}_S$, $\rho_S(V) = \exists_{\sqsubseteq_S}(V)$ i.e.

$$\delta(\sqsubseteq_S) = \mathcal{I}_S \xrightarrow{\rho} \mathcal{O}_S \xrightarrow{i} \mathcal{I}_S \,.$$

Lemma 5.14 *Any quasi-uniform triple morphism $R : S \to T$ is canonical if and only the diagram below commutes $(*)$*

$$\begin{tikzcd}
\mathcal{I}_S \arrow[r, "\delta(R)"] \arrow[d, "\rho"'] & \mathcal{I}_T \\
\mathcal{O}_S \arrow[r, "\bar{\delta}(R)"'] & \mathcal{O}_T \arrow[u, "i"']
\end{tikzcd}$$

Proof. Suppose that R is canonical, then by 5.10, $R = \sqsubseteq \circ R \circ \sqsubseteq$ thus by functoriality of F_1, (I) below commutes.

$$\begin{tikzcd}
 & \mathcal{O}_S \arrow[rr, "\bar{\delta}(R)"] \arrow[dd, "i"] \arrow[dr, phantom, "\text{(III)}"] & & \mathcal{O}_T \arrow[dr, "i"] \arrow[dd, "\rho"] \arrow[dl, phantom, "\text{(IV)}"] & \\
\mathcal{I}_S \arrow[ur, "\rho"] \arrow[rr, "\delta(\sqsubseteq)"'] \arrow[rrrr, bend right, "\delta(R)"'] \arrow[rrrrrr, bend right=60, "\text{(I)}"] & \arrow[ur, phantom, "\text{(II)}"] & \mathcal{I}_S \arrow[rr, "\delta(R)"'] & & \mathcal{I}_T \arrow[rr, "\delta(\sqsubseteq)"'] & & \mathcal{I}_S
\end{tikzcd}$$

(II), (IV), and (III) (by definition) also commute. Hence the diagram $(*)$ commutes. Conversely if $(*)$ commutes then $\delta(R)$ takes values in $\mathcal{O}_T$, i.e. R is canonical. $\square$

Composing on the right by $i : \mathcal{O}_S \to \mathcal{I}_S$ the diagram 5.14-$(*)$, one gets that for any canonical $R : S \to T$ the diagram below commutes

$$\begin{tikzcd}
\mathcal{I}_S \arrow[r, "\delta(R)"] & \mathcal{I}_T \\
\mathcal{O}_S \arrow[r, "\bar{\delta}(R)"'] \arrow[u, "i"] & \mathcal{O}_T \arrow[u, "i"']
\end{tikzcd}$$

From this one deduces

Proposition 5.15 *The assignments $S \mapsto (\mathcal{O}_S, \subseteq)$ and $R \mapsto \bar{\delta}(R)$ define a 2-functor $F_2 : CQUT \to Cat$.*

Proposition 5.16 *Given $S \xrightarrow{R} T$ in QUT, $\bar{R}$ is the unique canonical morphism $S \xrightarrow{P} T$ such that $\bar{\delta}(P) = \bar{\delta}(R)$, i.e. F_2 as a functor is faithful. Precisely the local components of F_2 are full.*

Proof. If $R : S \to T$ is canonical then

$$\begin{aligned}
\bar{\delta}(\bar{R}) &= \bar{\delta}(\sqsubseteq \circ R) \\
&= \rho \circ \delta(\sqsubseteq \circ R) \circ i
\end{aligned}$$

$= \rho \circ \delta(\sqsubseteq) \circ \delta(R) \circ i$, by functoriality of F_1,
$= \rho \circ \delta(R) \circ i$
$= \bar{\delta}(R)$.

Now suppose that $R, P : S \to T$ are both canonical. Then according to 5.14, one has: $\bar{\delta}(R) \subseteq \bar{\delta}(P)$ if and only if $\delta(R) \subseteq \delta(P)$. The result follows then from the fact that the local components of F_1 are full. □

Eventually we define $F_3 = F_2 \circ \overline{(-)} : QUT \to Cat$. Note that according to 5.16, for any morphism R of quasi-uniform triple $F_3(R) = \bar{\delta}(R)$.

Proposition 5.17 *Given a morphism $R : S \to T$ of quasi-uniform triple,*
- $\delta(R) : \mathcal{I}_S \to \mathcal{I}_T$ preserves least upper bounds.
- $\bar{\delta}(R) : \mathcal{O}_S \to \mathcal{O}_T$ preserves least upper bounds.

Proof. Since $\exists_R : (\wp(S), \subseteq) \to (\wp(T), \subseteq)$ is a left adjoint and any union of upper-closed subsets is upper-closed, $\delta(R) : (\mathcal{I}(S), \subseteq) \to (\mathcal{I}(T), \subseteq)$ preserves unions, i.e. least upper bounds. Since $i : (\mathcal{O}(S), \subseteq) \to (\mathcal{I}(S), \subseteq)$ and $\rho : (\mathcal{I}(T), \subseteq) \to (\mathcal{O}(T), \subseteq)$ also preserves least upper bounds (see 5.2), $\bar{\delta}(R) = \rho \circ \delta(R) \circ i : (\mathcal{O}(S), \subseteq) \to (\mathcal{O}(T), \subseteq)$ preserves least upper bounds.□

Proposition 5.18 *For any morphism of quasi-uniform triples $R : S \to T$, the map $\bar{\delta}(R)$ defines a normal super monoidal functor $\mathcal{C}_S \to \mathcal{C}_T$.*

Proof. Since $\delta(R)(S) = T$, $\bar{\delta}(R)(S) = \rho \circ \delta(R)(S) = \rho(T) = T$.
For any $V, W \in \mathcal{O}_S$,

$$\begin{aligned}\bar{\delta}(R)(V \otimes W) &= \rho \circ \delta(R) \circ i(V \otimes W)\\ &\supseteq \rho(\delta(R) \circ i(V) \otimes \delta(R) \circ i(W)), \text{ (see 5.6)}\\ &\supseteq (\rho \circ \delta(R) \circ i)(V) \otimes (\rho \circ \delta(R) \circ i)(W)\\ &= \bar{\delta}(R)(V) \otimes \bar{\delta}(R)(W)\end{aligned}$$

Let us see that $\bar{\delta}(R)$ is super. Let $(V_i, W_i)_{i \in I}$ be a family of pairs of objects in $\mathcal{C}_S$ such that $S \subseteq \bigcup_{i \in I} V_i \otimes W_i$. Given $t \in T$ there is a $t' \in T$ such that $t'^2 \leq t$. Since $T = \bar{\delta}(R)(S) = \bar{\delta}(R)(\bigcup_{i \in I} V_i \otimes W_i) = \bigcup_{i \in I} \bar{\delta}(R)(V_i \otimes W_i)$, for this t' there is some $i \in I$, such that $t' \in \bar{\delta}(R)(V_i \otimes W_i)$. Thus $t' \in \bar{\delta}(R)(V_i) \cap \bar{\delta}(R)(W_i)$ and $t \in \bar{\delta}(R)(V_i) \otimes \bar{\delta}(R)(W_i)$. □

Eventually,

Proposition 5.19 *The assignments $S \mapsto \mathcal{O}_S$ and $R \mapsto \bar{\delta}(R)$ define a 2-functor $F_4 : QUT \to SMon$. Considering its restriction F_5 to $CQUT$, one has the factorization in $2 - CAT$*

$$\begin{tikzcd} QUT \arrow[rr, "\overline{(-)}"] \arrow[dr, "F_4"'] & & CQUT \arrow[dl, "F_5"] \\ & SMon & \end{tikzcd}$$

The local components of F_5 are full.

6 Abstract Quasi-uniformities

This section introduces abstract quasi-uniformities and their morphisms. They form together a locally preordered 2-category $AQUnif$ that "lies in between" the 2-category $QUnif$ of uniformities and uniformly continuous maps and the 2-category of enrichments $Enr(F_5)$. Any quasi-uniformity defines naturally an abstract one and the nature of the correspondence between $QUnif$ and $AQUnif$ will be detailed in section 8. In the other direction, we define *canonical* quasi-abstract uniformities that correspond to enrichments. They form together with abstract quasi-uniformity morphisms between them a locally preordered 2-category $CAQUnif$ that is 2-isomorphic to $Enr(F_5)$, ($F_5 : QUT \to SMon$ was defined in section 5) and 2-equivalent to $AQUnif$.

6.1 A few definitions

Definition 6.1 An *abstract quasi-uniformity A over a quasi-uniform triple S* denoted also (A, S), is a set $Obj(A)$ together with a map: $A(-,-) : Obj(A) \times Obj(A) \to \mathcal{I}_S$ that satisfies:

(i) $\forall x, y, z \in Obj(A)$, $A(y,z) \otimes A(x,y) \subseteq A(x,z)$,

(ii) $\forall x \in Obj(A)$, $S \subseteq A(x,x)$.

Let S, T, U be quasi-uniform triples, and A, B and, C be abstract quasi-uniformities respectively over S, T and U.

Definition 6.2 Let f be a map $Obj(A) \to Obj(B)$. A relation $R \subseteq S \times T$ is said *compatible* with the triple (f, A, B) — or simply "with f", when there is no ambiguity — when for all $s \in S$, $t \in T$, for all $x, y \in Obj(A)$, $s \in A(x,y) \wedge R(s,t) \Rightarrow t \in B(fx, fy)$.

Note that if $R : S \to T$ is a morphism then it is compatible with (f, A, B) if and only if for all objects x, y of A, $\delta(R)A(x,y) \subseteq B(fx, fy)$.

The set of compatible relations with some (f, A, B) is closed under arbitrary unions and subrelations. The largest relation compatible with (f, A, B) is denoted $R_{f,A,B}$ (or simply R_f), it is defined for all $s \in S$ and $t \in T$ by $R_f(s,t)$ if and only if $\forall x, y \in Obj(A), s \in A(x,y) \Rightarrow t \in B(fx, fy)$. R_f

satisfies the conditions (ii) and (iii) of 5.4. If the relation P is compatible with (f, A, B) and Q is compatible with (g, B, C) then $Q \circ P$ is compatible $(g \circ f, A, C)$.

We define the locally preordered 2-categories $AQUnif^+$ and $AQUnif$ as follows. Both have objects abstract quasi-uniformities. A morphism of $AQUnif^+ : (A, S) \to (B, T)$ consists in a pair (f, R) where f is a map $Obj(A) \to Obj(B)$ and R is a morphism $S \to T$ of quasi-uniform triples that is compatible with (f, A, B). The composite of arrows

$$(A, S) \xrightarrow{(f,P)} (B, T) \xrightarrow{(g,Q)} (C, U)$$

is $(A, S) \xrightarrow{(g \circ f, Q \circ P)} (C, U)$. The identity morphism in (A, S) is the pair $(1_A, \leq_S \subseteq S \times S)$. The local preorders $AQUnif^+((A, S), (B, T))$ are given by $(f, R) \Rightarrow (g, P)$ if and only if $R \subseteq P$ and for all object x of A, $T \subseteq B(fx, gx)$.

Morphisms $(A, S) \to (B, T)$ in $AQUnif$, that we shall call *morphisms of quasi-uniformities*, are the maps $f : Obj(A) \to Obj(B)$ for which there exists an $R : S \to T$ in QUT compatible with. The composition of arrows in $AQUnif$ is given by the underlying map composition and the identity morphisms correspond to identity maps. The local preorders $AQUnif((A, S), (B, T))$ are given by: $f \leq g$ if and only if for all object x of A, $T \subseteq B(fx, gx)$. There is an obvious forgetful 2-functor $AQUnif^+ \to AQUnif$.

Remark 6.3 Given a morphism of $AQUnif^+$, $(f, R) : (A, S) \to (B, T)$, since R satisfies 5.4-(i) and is compatible with f, then

$$\forall t \in T, \exists s \in S, \forall x, y \in Obj(A), s \in A(x, y) \Rightarrow t \in B(fx, fy)$$

6.2 Canonical abstract quasi-uniformities

Let S be some quasi-uniform triple. An enrichment of the form $(A, \mathcal{C}_S)$ corresponds exactly to an abstract quasi-uniformity (A, S) where the map $A(-, -)$ takes its values in $\mathcal{O}_S$. An abstract quasi-uniformity satisfying the latter property is called *canonical*.

First we define the locally preordered 2-category $CAQUnif^+$ that is 2-isomorphic to $CQUT \times_{F_5,U} Enr^+$. Its objects are the canonical abstract quasi-uniformities, its morphisms, the $(f, R) : (A, S) \to (B, T)$ of $AQUnif^+$ with R canonical. The unit at (A, S) is the pair $(1_A, \sqsubseteq_S)$ where 1_A is the identity map on $Obj(A)$ (Thus $CAQUnif^+$ is NOT a sub-2-category of $AQUnif^+$). The local preorders on $CAQUnif^+$ are inherited from $AQUnif^+$.

We define for any abstract quasi-uniformity A over S, the canonical abstract quasi-uniformity $C(A)$ over S, as follows. It has the same objects as

A and for any $x, y \in Obj(A)$, $C(A)(x,y) = \rho(A(x,y))$. For any morphism $(f,R) : A \to B$ in $AQUnif^+$, since for all $x, y \in Obj(A)$,

$$\begin{aligned}\delta(\bar{R}) \circ \rho(A(x,y)) &= \rho \circ \delta(R) \circ i \circ \rho(A(x,y)) \\ &\subseteq \rho \circ \delta(R)(A(x,y)) \\ &\subseteq \rho(B(fx, fy)),\end{aligned}$$

we let $C(f,R)$ denote the morphism $(f, \bar{R}) : C(A) \to C(B)$ in $CAQUnif^+$.

According to the 2-functoriality of $\overline{(-)} : QUT \to CQUT$ (5.12),

Proposition 6.4 *The assignments $A \mapsto C(A)$ and $(f,R) \mapsto C(f,R)$ define a 2-functor $C : AQUnif^+ \to CAQUnif^+$.*

Remark 6.5 Given a morphism $(A,S) \xrightarrow{(f,R)} (B,T)$ of abstract quasi-uniformities, since $\bar{R} \subseteq R$, the pair $(f, \bar{R})$ defines also a morphism $(A,S) \to (B,T)$.

According to this, the local components of $C : AQUnif^+ \to CAQUnif^+$ are coreflections.

Eventually we define the 2-category $CAQUnif$ of $AQUnif$. Its objects are the canonical abstract quasi-uniformities, its arrows and local preorders are inherited from $AQUnif$. According to 6.5, it is 2-isomorphic to $Enr(F_5)$ (from 5.5, QUT is locally filtered) and there is a forgetful 2-functor $CAQUnif^+ \to CAQUnif$ that is full as a functor.

Let (A,S) be an abstract quasi-uniformity. Since for any objects x, y of A, $\rho(A(x,y)) \subseteq A(x,y)$, there is a morphism $C(A) \to A$ of $AQUnif^+$ defined by the pair $(1_A, \leq_S)$. Now $\sqsubseteq_S$ is compatible with $(1_A, A, C(A))$. Thus the pair $(1_A, \sqsubseteq_S)$ defines a morphism $A \to C(A)$ of $AQUnif^+$. This shows that $A \cong C(A)$ in $AQUnif$, thus

Proposition 6.6 *$AQUnif$ and $CAQUnif$ are 2-equivalent. The following diagrams in $2-CAT$ involving that equivalence, the above C and the forgetful 2-functors commute.*

$$\begin{tikzcd} AQUnif^+ \arrow[r, "C"] \arrow[d] & CAQUnif^+ \arrow[d] \\ AQUnif \arrow[r, "\cong"'] & CAQUnif \end{tikzcd}$$

7 Completions

In this section we define notions of Cauchy filter, neighborhood filter, separation and completeness for abstract quasi-uniformities, that correspond to

the classical notions for quasi-uniformities. We identify the equivalent notions in terms of enrichments. For an abstract quasi-uniformity A and the corresponding enrichment $C(A)$ via the equivalence $AQUnif \cong Enr(F_5)$: beginitemize

- Minimal Cauchy filters on A and left adjoint modules $1 \longrightarrow C(A)$ are in bijective correspondence;
- The separation and completeness of A are equivalent respectively to the skeletality and the Cauchy-completeness of $C(A)$.

Via the above equivalence C, we infer from the Cauchy-completion in $Enr(F_5)$, a completion for abstract quasi-uniformities (the completion of the abstract quasi-uniformity A corresponds via C to the Cauchy-completion of $C(A)$). We will prove in chapter 8 how retrieve the classical bicompletion of quasi-uniformities from the completion of abstract ones.

7.1 Cauchy filters / adjoint modules

In this subsection, (A, S) is an abstract quasi-uniformity.

Definition 7.1 [Cauchy filters] A filter F on $Obj(A)$ is *Cauchy* when it satisfies

$$(*)\ \forall u \in U, \exists f \in F, \forall x, y \in f, u \in A(x, y)$$

A filter basis is Cauchy when the filter that it generates is Cauchy.

In the definition above we could have replaced the condition $(*)$ by:

$$(**) \quad \forall u \in U, \exists f \in F, \forall x, y \in f, u \in A(x, y) \otimes A(y, x).$$

Proposition 7.2 *Let $f : (A, S) \to (B, T)$ be a morphism of abstract quasi-uniformities. If F is a Cauchy filter on (A, S) then $f(F)$, the direct image of F by f is a Cauchy basis on (B, T).*

Proof. Since the direct image of any filter is a filter basis, it remains to check that $f(F)$ is Cauchy. Let $t \in T$. According to 6.3, we may find some $s \in S$ such that $\forall x, y \in Obj(A), s \in A(x, y) \Rightarrow t \in B(fx, fy)$. Since F is Cauchy, there is $p \in F$ such that $\forall x, y \in Obj(A), x, y \in p \Rightarrow s \in A(x, y)$ Then $\forall x, y \in Obj(A), x, y \in p \Rightarrow t \in B(fx, fy)$, i.e. $\forall x', y' \in Obj(B), x', y' \in f(p) \Rightarrow t \in B(x', y')$ □

According to this and since the identity map defines two morphisms $1_A : A \to C(A)$ and $C(A) \to A$, Cauchy filters on A and $C(A)$ are the same.

To simplify we shall suppose from now that A is canonical. We shall also denote by A the corresponding $\mathcal{C}_S$-category. We shall relate adjoint pairs

$\varphi \dashv \tilde{\varphi} : 1 \multimap A$ and Cauchy-filters on the abstract quasi-uniformity A (7.5). A Cauchy filter F is said *minimal* when it is minimal for the inclusion relation on $\wp(\wp(Obj(A)))$.

Proposition 7.3 *To any adjoint pair of $\mathcal{C}_S$-modules $\varphi \dashv \tilde{\varphi}$ with $\varphi : \hat{1} \multimap A$, corresponds a Cauchy filter $\bar{\Gamma}(\varphi)$ on $Obj(A)$ with basis the family $\Gamma(\varphi)$ of sets $\Gamma(\varphi)(u) = \{x \in Obj(A) \mid u \in \varphi(x) \otimes \tilde{\varphi}(x)\}$, u ranging in S. $\bar{\Gamma}(\varphi)$ is a minimal Cauchy filter. Precisely any Cauchy filter compatible with $\Gamma(\varphi)$ is finer than $\Gamma(\varphi)$.*

Proof. Consider an adjoint pair as above. By definition, the maps $\varphi : Obj(A) \to \mathcal{O}_S$, $\tilde{\varphi} : Obj(A) \to \mathcal{O}_S$ are such that

- $\forall x, y \in Obj(A), \varphi(x) \otimes A(y, x) \subseteq \varphi(y)$;
- $\forall x, y \in Obj(A), A(x, y) \otimes \tilde{\varphi}(x) \subseteq \tilde{\varphi}(y)$.

and

(1) $S \subseteq \bigcup_{x \in Obj(A)} \varphi(x) \otimes \tilde{\varphi}(x)$;
(2) $\forall x, y \in Obj(A), \tilde{\varphi}(y) \otimes \varphi(x) \subseteq A(x, y)$.

According to (1), for all $u \in S$, $\Gamma(\varphi)(u)$ is non empty. Since S is filtered, for all $u, v \in S$, there is some $w \in S$ with $w \leq u$, $w \leq v$ and $\Gamma(\varphi)(w) \subseteq \Gamma(\varphi)(u) \cap \Gamma(\varphi)(v)$. Therefore the family $\Gamma(\varphi)$ is a filter basis. Let us see now that this basis is Cauchy. Let $u \in S$. Consider $v \in S$ such that $v^2 \leq u$ and let $x, y \in \Gamma(\varphi)(v)$. Then $v^2 \in \varphi(y) \otimes \tilde{\varphi}(y) \otimes \varphi(x) \otimes \tilde{\varphi}(x) \subseteq \tilde{\varphi}(y) \otimes \varphi(x)$. From (2) we have $u \in A(x, y)$.

Suppose now that F is a Cauchy filter compatible with $\Gamma(\varphi)$. We show that F is finer than $\Gamma(\varphi)$. Consider $u \in S$, we have to find some $f \in F$ such that $f \subseteq \Gamma(\varphi)(u)$. Let $v \in S$, such that $v^4 \leq u$. Choose some $f \in F$ such that $\forall x, y \in f, v \in A(x, y)$. Since F is compatible with $\Gamma(\varphi)$, $\Gamma(\varphi)(v) \cap f \neq \emptyset$. Let $\bar{x} \in \Gamma(\varphi)(v) \cap f$. Then for all $y \in f$, $u \geq v^4 \in \varphi(\bar{x}) \otimes A(y, \bar{x}) \otimes A(\bar{x}, y) \otimes \tilde{\varphi}(\bar{x}) \subseteq \varphi(y) \otimes \tilde{\varphi}(y)$. □

Proposition 7.4 *If F is a filter basis on A then the maps*

$$\begin{cases} \varphi : X \to \mathcal{O}_S, \\ x \mapsto \bigcup_{f \in F} \bigwedge_{y \in f} A(x, y) \end{cases} \text{and,}$$

$$\begin{cases} \tilde{\varphi} : X \to \mathcal{O}_S, \\ x \mapsto \bigcup_{f \in F} \bigwedge_{y \in f} A(y, x) \end{cases}$$

define respectively modules $\varphi : \hat{1} \multimap A$ and, $\tilde{\varphi} : A \multimap \hat{1}$. If F is moreover Cauchy then $\varphi \dashv \tilde{\varphi}$ and the Cauchy filter $\bar{\Gamma}(\varphi)$ (see 7.3) is less fine than F.

Proof. φ is a module from $\hat{1}$ to A since for all objects x, y of A,

$$\begin{aligned}\varphi(x) \otimes A(y,x) &= (\textstyle\bigcup_{f\in F} \bigwedge_{z\in f} A(x,z)) \otimes A(y,x)\\ &= \textstyle\bigcup_{f\in F}(\ (\bigwedge_{z\in f} A(x,z)) \otimes A(y,x)\ )\\ &\subseteq \textstyle\bigcup_{f\in F} \bigwedge_{z\in f}(A(x,z) \otimes A(y,x))\\ &\subseteq \varphi(y).\end{aligned}$$

Analogously $\tilde{\varphi}$ is a module $A \longrightarrow 1$.

Now suppose that F is moreover a filter basis. Let us see that $\varphi \dashv \tilde{\varphi}$. Let us check first $S \subseteq \bigcup_{x\in Obj(A)} \varphi(x) \otimes \tilde{\varphi}(x)$.

$$\begin{aligned}\bigcup_{x\in Obj(A)} \varphi(x) \otimes \tilde{\varphi}(x) &= \textstyle\bigcup_{x\in Obj(A)}((\bigcup_{f\in F} \bigwedge_{y\in f} A(x,y)) \otimes (\bigcup_{f\in F} \bigwedge_{y\in f} A(y,x)))\\ &= \textstyle\bigcup_{x\in Obj(A)} \bigcup_{f_1,f_2\in F}(\bigwedge_{y\in f_1} A(x,y) \otimes \bigwedge_{y\in f_2} A(y,x))\\ &= \textstyle\bigcup_{x\in Obj(A)} \bigcup_{f\in F}(\bigwedge_{y\in f} A(x,y) \otimes \bigwedge_{y\in f} A(y,x)),\end{aligned}$$

where the last equality follows from the fact that F is filtered. Let $u \in S$. There is some $v \in S$ such that $u \geq v^8$. Since F is Cauchy, there is some non-empty $f(v) \in F$ such that $x, y \in f(v) \Rightarrow v \in A(x,y)$. Then for some $x \in f(v)$, $v \in A(x,y) \cap A(y,x)$ for any $y \in f(v)$, $v^3 \in \bigwedge_{y\in f(v)} A(x,y) \cap \bigwedge_{y\in f(v)} A(y,x)$ and $u \in \bigwedge_{y\in f(v)} A(x,y) \otimes \bigwedge_{y\in f(v)} A(y,x)$.

Let us check that for any objects x, y of A, $\tilde{\varphi}(x) \otimes \varphi(y) \subseteq A(y,x)$. Let $x, y \in Obj(A)$,

$$\begin{aligned}\tilde{\varphi}(x) \otimes \varphi(y) &= (\textstyle\bigcup_{f\in F} \bigwedge_{z\in f} A(z,x)) \otimes (\bigcup_{f\in F} \bigwedge_{z\in f} A(y,z))\\ &= \textstyle\bigcup_{f_1,f_2\in F}(\bigwedge_{z\in f_1} A(z,x) \otimes \bigwedge_{z\in f_2} A(y,z))\\ &= \textstyle\bigcup_{f\in F}(\bigwedge_{z\in f} A(z,x) \otimes \bigwedge_{z\in f} A(y,z)).\end{aligned}$$

Let $f \in F$. Since f is non empty then for some $z_0 \in f$, $\bigwedge_{z\in f} A(z,x) \subseteq A(z_0,x)$, $\bigwedge_{z\in f} A(y,z) \subseteq A(y,z_0)$ and, $\bigwedge_{z\in f} A(z,x) \otimes \bigwedge_{z\in f} A(y,z) \subseteq A(z_0,x) \otimes A(y,z_0)$. Thus $\tilde{\varphi}(x) \otimes \varphi(y) \subseteq A(y,x)$.

Now we prove that F is finer than $\Gamma(\varphi)$. Let $u \in S$. There is some $v \in S$ such that $v^2 \leq u$. Since $\Gamma(\varphi)(v) \neq \emptyset$, we choose some $\bar{x} \in \Gamma(\varphi)(v)$ i.e. $v \in \varphi(\bar{x}) \otimes \tilde{\varphi}(\bar{x})$, i.e. $v \geq v_1 \cdot v_2$ for some $v_1 \in \varphi(\bar{x})$ and $v_2 \in \tilde{\varphi}(\bar{x})$. Thus there are some $f_1, f_2 \in F$ such that

$$\begin{aligned}&v_1 \in \textstyle\bigwedge_{y\in f_1} A(\bar{x},y) \subseteq \bigcap_{y\in f_1} A(\bar{x},y) \text{ and,}\\ &v_2 \in \textstyle\bigwedge_{y\in f_2} A(y,\bar{x}) \subseteq \bigcap_{y\in f_2} A(y,\bar{x}).\end{aligned}$$

Thus given $y \in f_1 \cap f_2$, $v_1 \in A(\bar{x},y)$ and $v_2 \in A(y,\bar{x})$ and $u \geq v^2 \geq v_1 v_2 v_1 v_2 \in \varphi(\bar{x}) \otimes A(y,\bar{x}) \otimes A(\bar{x},y) \otimes \tilde{\varphi}(\bar{x}) \subseteq \varphi(y) \otimes \tilde{\varphi}(y)$. Thus $f_1 \cap f_2 \subseteq \Gamma(\varphi)(u)$. $\square$

Proposition 7.5 *$\bar{\Gamma}$ defines a bijection between adjoint pairs of modules of the form $\varphi \dashv \tilde{\varphi} : 1 \longrightarrow A$ and minimal Cauchy-filters on A.*

This is a consequence of

Proposition 7.6 *Given an adjoint pair $\varphi \dashv \tilde{\varphi} : 1 \longrightarrow A$ then it is the pair corresponding to the Cauchy filter $\bar{\Gamma}(\varphi)$ as in 7.4, — i.e. for any object x of A, $\varphi(x) = \bigcup_{u \in S} \bigwedge_{y \in \Gamma(\varphi)(u)} A(x,y)$*

Proof. It is enough to show that for any object x of A,

$$\varphi(x) \subseteq \bigcup_{u \in S} \bigwedge_{y \in \Gamma(\varphi)(u)} A(x,y)$$

since one should get analogously

$$\tilde{\varphi}(x) \subseteq \bigcup_{u \in S} \bigwedge_{y \in \Gamma(\varphi)(u)} A(y,x)$$

and we are considering two adjoint pairs. Let $x \in Obj(A)$ and $u \in \varphi(x)$. $u \geq v_1 v_2 u' v_3$ for some $v_1, v_2, v_3 \in S$ and $u' \in \varphi(x)$. For any $y \in \Gamma(\varphi)(v_2)$, by definition $v_2 \in \varphi(y) \otimes \tilde{\varphi}(y)$ thus $v_2 \in \tilde{\varphi}(y)$ and $v_2 u' \in \tilde{\varphi}(y) \otimes \varphi(x) \subseteq A(x,y)$. Therefore $u \geq v_1 v_2 u' v_3 \in \bigwedge_{y \in \Gamma(\varphi)(v_2)} A(x,y)$. □

If F is a Cauchy-filter on A, we shall write φ_F for the corresponding left adjoint module $1 \longrightarrow A$ defined in 7.4. 7.3 and 7.4 show that

Proposition 7.7 *For a given Cauchy filter F, $\bar{\Gamma}(\varphi_F)$ is the only minimal Cauchy filter less fine than F.*

7.2 Completion and separation / Cauchy-completion and skeletality

In this subsection, (A,S) will denote an abstract quasi-uniformity. and $C(A)$ will denote the canonical abstract quasi-uniformity as well as the corresponding enrichment.

Given some object x of A, let $B_1(x)$, $B_2(x)$, $B_3(x)$, $B_4(x)$ denote respectively the set of subsets of $Obj(A)$,

- $\{y \in Obj(A) \mid u \in A(y,x) \cap A(x,y)\}$, u ranging in S;
- $\{y \in Obj(A) \mid u \in A(y,x) \otimes A(x,y)\}$, u ranging in S;
- $\{y \in Obj(A) \mid u \in \rho \circ A(y,x) \cap \rho \circ A(x,y)\}$ u ranging in S;
- $\{y \in Obj(A) \mid u \in \rho \circ A(y,x) \otimes \rho \circ A(x,y)\}$, u ranging in S.

Then $B_1(x)$, $B_2(x)$, $B_3(x)$, $B_4(x)$ are Cauchy bases for (A,S) If (A,S) is canonical then $B_1(x) = B_3(x)$ and $B_2(x) = B_4(x)$. These bases are equivalent in the sense that they generate the same filter on $Obj(A)$. This filter is denoted $V(x)$, an element of $V(x)$ is called a *neighborhood of* x (in (A,S)). For any object x of A, $V(x)$ is just $\bar{\Gamma}(x_\diamond)$ for $x_\diamond : 1 \longrightarrow C(A)$ Thus according to 7.3, $V(x)$ is a minimal Cauchy filter on (A,S).

This motivates the definition which is justified by 7.7

Definition 7.8 An abstract quasi-uniformity is said *complete* when one of the following equivalent assertion holds:

- any of its Cauchy filter is finer than a filter of neighborhoods for some element;
- any minimal Cauchy-filter occurs as a neighborhood filter.

Proposition 7.9 *The following assertions are equivalent:*

- *the abstract quasi-uniformity* (A, S) *is complete;*
- *the canonical abstract quasi-uniformity* $C(A, S)$ *is complete;*
- *the* $\mathcal{C}_S$*-enrichment* $C(A)$ *is Cauchy-complete.*

We define (dually)

Definition 7.10 (A, S) is said *separated* when one the following equivalent assertion holds:

- the map assigning any object x to the filter $V(x)$ is injective;
- For all objects x, y of A, if $x \neq y$ then there is $u \in S$ such that $u \notin A(x, y) \cap A(y, x)$.

The fact that the two assertions above are equivalent results from

Lemma 7.11 *For all objects* x*,* y *of* A*,* $V(x) \neq V(y)$ *if and and only if there is a* $u \in S$ *such that* $u \notin A(x, y) \cap A(y, x)$*.*

Proof. Clearly if there is a $u \in S$ such that $u \notin A(x, y) \cap A(y, x)$, then $y \notin \Gamma(x_\diamond)(u)$ or $x \notin \Gamma(y_\diamond)(u)$ and $V(x) \neq V(y)$.

To see the reverse implication, suppose $V(x) \neq V(y)$. Then by minimality of $V(y)$, $V(y)$ is not finer than $V(x)$. Thus there exists a $u \in S$ such that for any $v \in S$, $\Gamma(x_\diamond)(u) \not\supseteq \Gamma(y_\diamond)(v)$. For this u choose v such that $v^8 \leq u$. Then $v \notin A(x, y) \cap A(y, x)$ otherwise for any $z \in \Gamma(y_\diamond)(v)$, one would have $v^2 \in A(y, z) \otimes A(x, y) \subseteq A(x, z)$, $v^2 \in A(y, x) \otimes A(z, y) \subseteq A(z, x)$, thus $v^4 \in \rho(A(z, x)) \cap \rho(A(x, z))$ and $v^8 \in \rho(A(z, x)) \otimes \rho(A(x, z))$, and eventually $z \in \Gamma(x_\diamond)(u)$. □

Proposition 7.12 *The following assertions are equivalent:*

- *the abstract quasi-uniformity* (A, S) *is separated;*
- *the canonical abstract quasi-uniformity* $C(A, S)$ *is separated;*
- *the* $\mathcal{C}_S$*-enrichment,* $C(A)$ *is skeletal.*

7.3 A completion for abstract quasi-uniformities

Let us define the following 2-categories.

$SC - QAUnif$ with:
- objects: separated and complete abstract quasi-uniformities;
- arrows and local preorders inherited from $QAUnif$.

$SC - CQAUnif$ with:
- objects: separated and complete *canonical* abstract quasi-uniformities;
- arrows and local preorders inherited from $CQAUnif$.

Then

Theorem 7.13 *The inclusion functor $SC - QAUnif \to QAUnif$ has a left 2-adjoint.*

According to 6.6, 7.9 and 7.12, 7.13 is equivalent to

Theorem 7.14 *The inclusion functor $SC - CQAUnif \to CQAUnif$ has a left 2-adjoint.*

Which is just an instance of 4.11 for $H = F_5$, according again to 7.9 and 7.12.

Further on we shall write $\overline{\overline{(A,S)}}$ or simply $\overline{\overline{A}}$ for the "completion" of the abstract quasi-uniformity (A,S) that is its image by the left adjoint in 7.13. Let us describe $\overline{\overline{A}}$ up to isomorphism in $AQUnif$. Its elements are the Cauchy filters of A. We shall show that given Cauchy filters F_1 and F_2 and $u \in S$, $u \in \overline{\overline{A}}(F_1, F_2)$ if and only if there are some $f_1 \in F_1$ and $f_2 \in F_2$ such that $\forall x \in f_1, y \in f_2, u \in A(x,y)$.

Proof. First note that the map $\overline{\overline{A}}$ above defines an abstract quasi-uniformity. Then let us consider the enrichments $C(A)$ and $C(\overline{\overline{A}})$ corresponding respectively to the canonical abstract quasi-uniformities isomorphic to A and $\overline{\overline{A}}$. We show that $C(\overline{\overline{A}})$ is isomorphic to the Cauchy-completion $\overline{C(A)}$ of $C(A)$.

For two Cauchy filters F_1, F_2,

$$\overline{C(A)}(F_1,F_2) = \bigcup_{a\in A}\Big(\big(\bigcup_{f_2\in F_2} \bigwedge_{y\in f_2} A(a,y)\big) \otimes \big(\bigcup_{f_1\in F_1} \bigwedge_{y\in f_1} A(y,a)\big) \Big)$$

since for any family $V_i \in \mathcal{I}_S$, $\rho(\cap_{i\in I} V_i) = \rho(\cap_{i\in I}\rho(V_i))$. Which means that $u \in \overline{C(A)}(F_1,F_2)$ if and only there is $a \in A$, $f_1 \in F$, $f_2 \in F_2$, $u_1, v_1, t_1, u_2, v_2, t_2 \in S$ such that $u \geq u_2v_2t_2u_1v_1t_1$ and for all $y_1 \in f_1$, $y_2 \in f_2$, $u_1 \in A(y_1,a)$ and $u_2 \in A(a,y_2)$. Thus if $u \in \overline{C(A)}(F_1,F_2)$ considering the situation as above one has $u \geq u_2v_2v_1t_1$ with $v_2v_1 \in \overline{\overline{A}}(F_1,F_2)$, thus $u \in \rho(\overline{\overline{A}}(F_1,F_2))$.

Conversely, let $u \in \rho(\overline{\overline{A}}(F_1, F_2))$. Then there are $f_1 \in F_1$, $f_2 \in F_2$, $u_1, u_2, u_3 \in S$, such that $u \geq u_1 u_2 u_3$ and for all $y_1 \in f_1$, $y_2 \in f_2$, $u_2 \in A(y_1, y_2)$. Let t_1 such that ${t_1}^4 \leq u_3$. Choosing $g_1 \in F_1$ such that $\forall x, y \in g_1, t_1 \in A(x, y)$ and choosing $a \in f_1 \cap g_1$. Then $u \geq u_1 u_2 t_1^4$, for all $y_1 \in f_1 \cap g_1$, $t_1 \in A(y_1, a)$ and for all $y_2 \in f_2$, $u_2 \in A(a, y_2)$. This shows $u \in \overline{C(A)}(F_1, F_2)$. □

Eventually, the unit in A of 7.13 sends any object x to $V(x)$.

8 Quasi-uniformities

This section clarifies the connection between classical quasi-uniformities and abstract ones. The inclusion 2-functor $P : QUnif \to AQUnif$ has a right 2-adjoint Q. We define the *concrete* abstract uniformities. The 2-category with concrete abstract uniformities as objects and arrows and local preorders inherited from $AQUnif$ is 2-equivalent to $QUnif$. Both P, Q preserve the separation and completeness of objects. From this we can infer the classical bicompletion of quasi-uniformities from the completion of abstract ones.

8.1 Quasi-uniformities/abstract ones

Let (X, U) be a quasi-uniformity. One builds a quasi-uniform triple $U = (U, \circ, \subseteq)$, where $\circ$ denotes the composition of relations, and $\subseteq$ is the inclusion ordering. From (X, U), one can built $P(X, U)$, the abstract quasi-uniformity over U as follows. $P(X, U)$ has objects the elements of X, and for any $x, y \in X$, $P(X, U)(x, y) = (x, y)^{\in} = \{u \in U \mid (x, y) \in U\}$. Let (X, U), (Y, T) be quasi-uniformities, and let f be a map $X \to Y$. The relation $R_f = R_{f, P(X,U), P(Y,T)} \subseteq U \times T$ (see section 6.1) is for all $u \in U$ and $v \in T$

$$(u, v) \in R_f \Leftrightarrow (\ \forall x, y \in X, (x, y) \in u \Rightarrow (fx, fy) \in v\ ).$$

In this case, R_f also satisfies the condition 5.4-(iv). Now f is uniformly continuous exactly when $R_f(U) = T$, that is condition 5.4-(i). Thus f is uniformly continuous if and only if the map (f, R_f) is a morphism $P(X, U) \to P(Y, T)$ in $AQUnif^+$. So for an uniformly continuous map $f : (X, U) \to (Y, T)$, we let $P(f)$ denote the morphism $P(X, U) \to P(Y, T)$ in $AQUnif$ with underlying map $f : X \to Y$. Then

Proposition 8.1 *The assignments $(X, U) \mapsto P(X, U)$, $f : (X, U) \to (Y, T) \mapsto P(f)$, defines a 2-functor $P : QUnif \to AQUnif$. The local components of P are isomorphisms.*

Conversely, let (A, S) be an abstract quasi-uniformity, one may built a quasi-uniformity denoted $Q(A) = (A, \hat{S})$ with elements the objects of A, as

follows. $\hat{S}$ has basis the set, say B, of sets $\hat{s} = \{(x,y) \mid s \in A(x,y)\}$, s ranging in S. Let us check that B is a basis of a quasi-uniformity. For any $s \in S$, $\Delta \subseteq \hat{s}$ since for any $s \in S$, $s \in A(x,x)$. The assignment $s \mapsto \hat{s}$ defines a monotonous map $S \to \wp(Obj(A) \times Obj(A))$, thus B is filtered since S is. Note that for any $s,t \in S$, $\hat{s} \circ \hat{t} \subseteq \widehat{s \circ t}$. So, for any $p \in B$, since $p = \hat{s}$ for some $s \in S$ there is some $t \in S$ such that $t^2 \leq s$ and $(\hat{t})^2 \subseteq \widehat{t^2} \subseteq p$.

Let $(f,R) : (A,S) \to (B,T)$ be a morphism of $AQUnif^+$. According to 6.3, f is a uniformly continuous map $Q(A) \to Q(B)$. We let $Q(f)$ denote this arrow in $QUnif$.

Proposition 8.2 *The assignments $A \mapsto Q(A)$ and $f : (A,S) \to (B,T) \mapsto Q(f)$ define a 2-functor Q $AQUnif \to QUnif$. Trivially the $Q \circ P \cong 1$.*

Definition 8.3 An abstract uniformity (A,S) is said *concrete* when it satisfies: $(\forall s,t \in S)(\forall x,y \in Obj(A))$

$$t \circ s \in A(x,y) \quad \Rightarrow \quad (\exists z \in Obj(A))\ s \in A(x,z) \wedge t \in A(z,y).$$

Observe that (A,S) is concrete exactly when for all $s,t \in S$, $\widehat{s \circ t} \subseteq \hat{s} \circ \hat{t}$. For any quasi-uniformity (X,U), the abstract quasi-uniformity $P(X,U)$ is concrete.

Now given two abstract uniformities (A,S) and (B,T) with A concrete and a map $f : Obj(A) \to Obj(B)$, the relation $R_{f,A,B}$ (see section 6.1) satisfies conditions of 5.4-(iv), as shown below. For all $s,s' \in S$ and $t,t' \in T$, such that $R_f(s,t)$ and $R_f(s',t')$, one has:

$$\begin{aligned}\forall x,y \in Obj(A), s' \circ s \in A(x,y) &\Rightarrow \exists z \in Obj(A),\ s \in A(x,z) \wedge s' \in A(z,y)\\ &\Rightarrow t \in B(fx,fz) \wedge t' \in B(fz,fy)\\ &\Rightarrow t' \circ t \in B(fx,fy).\end{aligned}$$

From this one gets that for any quasi-uniformity (X,U) and any abstract quasi-uniformity (A,S), a map $f : X \to Obj(A)$ is uniformly continuous $(X,U) \to Q(A,S)$ if and only $R_{f,P(X,U),(A,S)}$ is a morphism of quasi-uniform triples $U \to S$. Thus one gets a 2-natural isomorphism

$$QUnif((X,U),Q(A,S)) \cong_{(X,U),(A,S)} AQUnif(P(X,U),(A,S))$$

i.e.,

Proposition 8.4 *There is a 2-adjunction $P \dashv Q : QUnif \rightharpoonup AQUnif$.*

Let (A,S) be an abstract quasi-uniformity, the component in (A,S), $P \circ Q(A,S) \to (A,S)$, of the co-unit of the previous equivalence, has underlying map, the identity $Obj(A) \to Obj(A)$. Suppose now that the abstract quasi-

uniformity (A, S) is moreover concrete. Then there is a morphism of quasi-uniform triples $R : S \to \hat{S}$ defined for all $s \in S$ and $p \in \hat{S}$ by $R(s,p) \Leftrightarrow \hat{s} \subseteq p$. It is straightforward that R satisfies conditions (i), (ii), (iii), of 5.4. It satisfies (iv) since for all $s, t \in S$, $p, q \in \hat{S}$ if $\hat{s} \subseteq p$ and $\hat{t} \subseteq q$ then $\widehat{s \circ t} = \hat{s} \circ \hat{t} \subseteq p \circ q$. This R is compatible with $(1_A, A, P \circ Q(A))$. This shows that the component in any concrete A of the co-unit of 8.4 is an isomorphism. Thus

Proposition 8.5 *$QUnif$ is 2-equivalent 2-category with objects concrete abstract uniformities, and morphisms and local preorders inherited from $AQUnif$.*

8.2 Bicompletion of quasi-uniformities

We show now how to retrieve 2.1 from 7.13 by proving that the inclusion $SC - QUnif \to QUnif$ has a left 2-adjoint with action on objects $(X, U) \mapsto Q\overline{\overline{P(X,U)}}$.

According to our definition of Cauchy filters and neighborhood filters on abstract quasi-uniformities (7.1 and section 7.2), P and Q preserve the separation and completeness of objects.

First we need to show

Lemma 8.6 *Given a quasi-uniformity (X, U), the component of the co-unit of the adjunction $P \dashv Q$ in $\overline{\overline{P(X,U)}}$ (8.4), $\epsilon_{\overline{\overline{P(X,U)}}} : PQ\overline{\overline{P(X,U)}} \to \overline{\overline{P(X,U)}}$ is an isomorphism.*

Proof. Since the components of unit of 8.4 are isomorphisms, the co-unit $\epsilon_{P(X,U)} : P \circ Q \circ P(X,U) \to P(X,U)$, in $P(X,U)$ is an isomorphism. Consider then the following commuting diagram in $QAUnif$.

$$\begin{tikzcd}
P \circ Q \circ P(X,U) \arrow[r, "\epsilon_{P(X,U)}", "\cong"'] \arrow[d, "P \circ Q(j_{P(X,U)})"'] & P(X,U) \arrow[d, "j_{P(X,U)}"] \\
P \circ Q \circ \overline{\overline{P(X,U)}} \arrow[r, "\epsilon_{\overline{\overline{P(X,U)}}}"] & \overline{\overline{P(X,U)}}
\end{tikzcd}$$

where j denotes the unit of the 2-adjunction 7.13. Since $P \circ Q \circ \overline{\overline{P(X,U)}}$ is separated and complete, there is a unique morphism of quasi-uniformities r such that $r \circ j_{P(X,U)} = P \circ Q(j_{P(X,U)}) \circ (\epsilon_{P(X,U)})^{-1}$. Thus $j_{P(X,U)} = \epsilon_{\overline{\overline{P(X,U)}}} \circ r \circ j_{P(X,U)}$ showing $\epsilon_{\overline{\overline{P(X,U)}}} \circ r = 1$. Since $\epsilon_{\overline{\overline{P(X,U)}}}$ is also a monomorphism, it is an isomorphism. $\square$

Now one has the following sequence of 2-natural isomorphisms between 2-functors $QUnif \times SC - QUnif \to Cat$

$$QUnif((X,U),(Y,V)) \cong_{(X,U),(Y,V)}$$

$$AQUnif(P(X,U),P(Y,V)) \cong_{(X,U),(Y,V)}$$
$$SC-AQUnif(\overline{\overline{P(X,U)}},P(Y,V)) \cong_{(X,U),(Y,V)}$$
$$SC-AQUnif(PQ\overline{\overline{P(X,U)}},P(Y,V)) \cong_{(X,U),(Y,V)}$$
$$SC-QUnif(Q\overline{\overline{P(X,U)}},(Y,V)).$$

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arXiv:math/0501383v1 [math.CT] 22 Jan 2005

Notes on enriched categories with colimits of some class

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September 25, 2018

Abstract

This work is on-going. Results presented here are issued mainly from unpublished notes of the first author and contain those in the notes http://arXiv.org/abs/math.CT/0309209 and http://arXiv.org/abs/math.CT/0403164 of the second author. The latter is also the only responsible for the typos, spelling mistakes and other errors that could occur in the present paper! We intend to add further sections to this work in the near future.

Given a class Φ of weights, we study the following classes: Φ^+ of Φ-*flat* weights which are the ψ for which ψ-colimits commute in the base $\mathcal{V}$ with limits with weight in Φ; and Φ^- - dually defined - of weights ψ for which ψ-limits commute in the base $\mathcal{V}$ with colimits with weight in Φ. We show that both these classes are *saturated* (i.e. *closed* with the terminology of [AK88]). We prove that for the class $\mathcal{P}$ of *all* weights $\mathcal{P}^+ = \mathcal{P}^-$. For any small $\mathcal{B}$, we defined an enriched adjunction à la Isbell $[\mathcal{B}, \mathcal{V}]^{op} \rightharpoonup [\mathcal{B}^{op}, \mathcal{V}]$ and show how it restricts to an equivalence $(\mathcal{P}^-(\mathcal{B}^{op}))^{op} \cong \mathcal{P}^-(\mathcal{B})$ between subcategories of *small projectives*.

1 Introduction

The present notes had their beginning in the analysis of the results obtained by Borceux, Quintero and Rosicky in their article [BQR98]. These authors were concerned with extending to the enriched case the notion of accessibility and its properties, described for ordinary categories in the book [MP89] of Makkai and Paré and [AR94] of Adamek and Rosicky. They were led to discuss categories - now meaning $\mathcal{V}$-categories - with finite limits (in a suitable sense), or more generally with α-small ones, or with filtered colimits (in a suitable sense), or more generally with α-filtered colimits and again to discuss the connexions between these classes of categories. When we looked in detail at their work, we observed that many of the properties they discussed hold in fact for categories having colimits of *any* given class Φ, while other holds when Φ is the class of colimits commuting in the base category $\mathcal{V}$ with the limits of some class Ψ - such properties as finiteness or filteredness being irrelevant to the *general* results. Approaching in this abstract way, not generalisations of accessibility as such, but the study of categories with colimits (or limits) of some class, brings considerable simplifications. A recent work by Adamek, Borceux, Lack and Rosicky [ABLR02] also treats a general notion of accessibility parameterised by pair of families of *ordinary* limits and colimits that commute in *Set*. Rather than families of ordinary -or conical limits- we consider *indexed or weighted* limits and colimits. As a result of this generalisation, we will avoid some technical complications even in the case when the base $\mathcal{V}$ is *Set*. Although our positive results are quite few in number, their value may be judged by their extra light they cast on the results in [BQR98] or [ABLR02].

We begin by reviewing and completing some known facts in the first sections 2, 3. Section 4 contains a slight extension of Kelly's cocompletion theorem. Section 5 treats generally the limit/colimit commutation in the base $\mathcal{V}$ and contains a study of families of the form Φ^+ of Φ-*flat* weights - i.e. weights whose colimits commute with Φ-weighted limits and families of the form Φ^-

of weights whose limits commute with Φ-weighted colimits. Section 6 focusses on the family $\mathcal{P}^-$ for $\mathcal{P}$ the class of all weights. $\mathcal{P}^-$ being the class of *small projective*, is shown to be also the class of $\mathcal{P}$-flat weights. More results regarding the Cauchy-completion and the Isbell adjunction in the enriched context are then presented.

2 Revision of the terminology

The necessary background knowledge about enriched categories is largely contained in [Kel82], augmented by Kelly's "Amiens" article [Kel82-2], and by the Albert-Kelly article [AK88].

We use "category", "functor", and "natural transformation" for "$\mathcal{V}$-category", "$\mathcal{V}$-functor", and "$\mathcal{V}$-natural transformation", except when more precision is needed. As usual the ordinary category $\mathcal{V}_o$ is supposed locally small, complete and cocomplete. $\mathcal{V}$-**CAT** is the 2-category of $\mathcal{V}$-categories whereas $\mathcal{V}$-**Cat** is that of small $\mathcal{V}$-categories. *Set* is the category of small sets, **Cat** = *Set*-**Cat** is the 2-category of small categories, **CAT** = *Set*-**CAT** the 2-category of locally small categories.

A *weight* is a functor $\phi : \mathcal{K} \to \mathcal{V}$ with domain $\mathcal{K}$ *small*; weights were called *indexing-types* in [Kel82] and in [Kel82-2]. Recall that the *ϕ-weighted limit* $\{\phi, T\}$ of a functor $T : \mathcal{K} \to \mathcal{A}$ is defined representably by

2.1 $\mathcal{A}(a, \{\phi, T\}) \cong [\mathcal{K}, \mathcal{V}](\phi, \mathcal{A}(a, T-))$,

while the *ϕ-weighted colimit* $\phi * S$ for $S : \mathcal{K}^{op} \to \mathcal{A}$ is defined dually by

2.2 $\mathcal{A}(\phi * S, a) \cong [\mathcal{K}, \mathcal{V}](\phi, \mathcal{A}(S-, a))$,

so that $\phi * S$ is equally the ϕ-weighted limit of $S^{op} : \mathcal{K} \to \mathcal{A}^{op}$. Of course the limit $\{\phi, T\}$ consists not just of the object $\{\phi, T\}$ but also of the representation 2.1, or equally of the corresponding *counit* $\mu : \phi \to \mathcal{A}(\{\phi, T\}, T-)$; it is by *abus de langage* that we usually mention only $\{\phi, T\}$. When $\mathcal{V} = Set$, we refind the classical (or "conical") limit of $T : \mathcal{K} \to \mathcal{A}$ and the classical colimit of $S : \mathcal{K} \to \mathcal{A}$ as

2.3 $lim\, T = \{\Delta 1, T\}$ *and* $colim\, S = \Delta 1 * S$

where $\Delta 1 : \mathcal{K} \to Set$ is the constant functor at the one point set 1. Recall that the weighted limits and colimits can be calculated using the classical ones when $\mathcal{V} = Set$: for then the presheaf $\phi : \mathcal{K} \to Set$ gives the discrete op-fibration $d : el(\phi) \to \mathcal{K}$ where $el(\phi)$ is the category of elements of ϕ, and now

2.4 $\{\phi, T\} = lim\{\ el(\phi) \xrightarrow{d} \mathcal{K} \xrightarrow{T} \mathcal{A}\ \}$,

2.5 $\phi * S = colim\{\ el(\phi)^{op} \xrightarrow{d^{op}} \mathcal{K}^{op} \xrightarrow{S} \mathcal{A}\ \}$

Also a functor $F : \mathcal{B} \to \mathcal{C}$ is said to *preserve the limit* $\{\phi, T\}$ as in 2.1 when $F(\{\phi, T\})$ is the limit of FT weighted by ϕ with counit

$$\phi \xrightarrow{\mu} \mathcal{A}(\{\phi, T\}, T-) \xrightarrow{F} \mathcal{B}(F\{\phi, T\}, FT-)$$

and F is said to preserve the colimit $\phi * S$ as in 2.2 when F^{op} preserves $\{\phi, S^{op}\}$.

We spoke above of a "class Φ of colimits" or a "class Ψ of limits"; but this is loose and rather dangerous language - the only thing that one can sensibly speak of is *a class Φ of weights*. Then a category $\mathcal{A}$ *admits Φ-limits* or is *Φ-complete*, if $\mathcal{A}$ admits the limit $\{\phi, T\}$ for each weight $\phi : \mathcal{K} \to \mathcal{V}$

of Φ and each $T : \mathcal{K} \to \mathcal{A}$, while $\mathcal{A}$ *admits* Φ-*colimits*, or is Φ-*cocomplete* when $\mathcal{A}$ admits the colimit $\phi * S$ for each $\phi : \mathcal{K} \to \mathcal{V}$ in Φ and for each $S : \mathcal{K}^{op} \to \mathcal{A}$ (and thus when $\mathcal{A}^{op}$ is Φ-complete). Of course a functor $\mathcal{A} \to \mathcal{B}$ between Φ-complete categories is said to be Φ-*continuous* when it preserves all Φ-limits and one defines Φ-*cocontinuous* dually. We write Φ-**Conts** for the 2-category of Φ-complete categories, Φ-continuous functors, and all natural transformations - which is a (non full) sub-2-category of $\mathcal{V}$-**CAT**; and similarly Φ-**Cocts** for the 2-category of Φ-cocomplete categories, Φ-cocontinuous functors, and all natural transformations.

To give a class Φ of weights is to give for each small $\mathcal{K}$, those $\phi \in \Phi$ with domain $\mathcal{K}^{op}$; let us use as in [AK88] the notation

2.6 $\Phi[\mathcal{K}] = \{\phi \in \Phi \mid dom(\phi) = \mathcal{K}^{op}\}$,

so that

2.7 $\Phi = \Sigma_{\mathcal{K}\ small}\Phi[\mathcal{K}]$.

In future, we look at $\Phi[\mathcal{K}]$ as a full subcategory of the functor category $[\mathcal{K}^{op}, \mathcal{V}]$. The smallest class of weights is the empty class 0, and 0-**Conts** is just $\mathcal{V}$-**CAT**. The largest class of weights consists of all weights with small domains, and we denote this class by $\mathcal{P}$; the 2-category $\mathcal{P}$-**Conts** is just the 2-category **Conts** of *complete* categories and *continuous* functors, and similarly $\mathcal{P}$-**Cocts** = **Cocts**.

There may well be different classes Φ and Ψ for which the sub-2-categories Φ-**Conts** and Ψ-**Conts** of $\mathcal{V}$-**CAT** coincide; which is equally to say that Φ-**Cocts** and Ψ-**Cocts** coincide. When $\mathcal{V} = Set$, for instance, **Conts** = $\mathcal{P}$-**Conts** coincides with Φ-**Conts** where Φ consists of the weights for products and for equalisers. We define the *saturation* Φ^* of a class Φ of weights as follows: the weight ψ belongs to Φ^* when every Φ-complete category is also ψ-complete and every Φ-continuous functor is also ψ-continuous. Actually we change here the name given in [AK88] where Φ^* was called the *closure* of Φ but we wish to avoid overusing this terminology. Clearly then, we have

2.8 Φ-**Conts** = Ψ-**Conts** $\Leftrightarrow$ Φ-**Cocts** = Ψ-**Cocts** $\Leftrightarrow$ $\Phi^* = \Psi^*$.

When $\mathcal{V} = Set$, we can of course consider Φ-**Conts** where Φ consists of the $\Delta 1 : \mathcal{K} \to Set$ for all $\mathcal{K}$ in some small class $\mathcal{D}$ of small categories; and we might write Δ-**Conts** for this 2-category Φ-**Conts** of $\mathcal{D}$-*complete* categories, $\mathcal{D}$-*continuous* functors, and all natural transformations. We underline the fact that however when $\mathcal{V} = Set$, the Φ-**Conts** generally do NOT occur as Δ-**Conts** as above. A simple example of this situation is in [AK88].

There is a distinction which is often important, but which is not covered by the notation above. We spoke of $\mathcal{V}$-**CAT** as a 2-category, the category $\mathcal{V}$-**CAT**$(\mathcal{A}, \mathcal{B})$ having as its objects the $\mathcal{V}$-functors $T : \mathcal{A} \to \mathcal{B}$ and as its arrows, the $\mathcal{V}$-natural transformations $\alpha : T \to S : \mathcal{A} \to \mathcal{B}$. When $\mathcal{A}$ is small, however, we also have the $\mathcal{V}$-category $[\mathcal{A}, \mathcal{B}]$, whose underlying category $[\mathcal{A}, \mathcal{B}]_0$ is $\mathcal{V}$-**CAT**$(\mathcal{A}, \mathcal{B})$. Thus *small* $\mathcal{V}$-categories form not only a 2-category $\mathcal{V}$-**Cat**, but also a $\mathcal{V}$-**Cat**-category (as $\mathcal{V}$-**Cat** is a monoidal closed 2-category and as any closed category, it is enriched over itself). Again, even when $\mathcal{A}$ is not small, we can (as in [Kel82] section 3.12) see $[\mathcal{A}, \mathcal{B}]$ as a $\mathcal{V}'$-category where $\mathcal{V}'$ is an extension of $\mathcal{V}$ to a Set'-category (with Set' a universe containing $Obj(Set)$ and $Obj(\mathcal{B})$ for all categories $\mathcal{B}$), such that $\mathcal{V} \to \mathcal{V}'$ preserves all limits that exist in $\mathcal{V}$ and all Set'-small colimits that exist in $\mathcal{V}$. So we take the view that $[\mathcal{A}, \mathcal{B}]$ always exists as a $\mathcal{V}'$-category, which is fact a $\mathcal{V}$-category when $\mathcal{A}$ is small. Thus, even for large $\mathcal{A}$, $\mathcal{A}$ is Set'-small and we may see $\mathcal{V}$-**CAT**$(\mathcal{A}, \mathcal{B})$ as the ordinary category underlying the $\mathcal{V}'$-category $[\mathcal{A}, \mathcal{B}]$, so that $\mathcal{V}$-categories form not only the 2-category $\mathcal{V}$-**CAT** cut also a $\mathcal{V}'$-**CAT**-category denoted $\underline{\mathcal{V}\text{-}\mathbf{CAT}}$. This has a full subcategory given by the Φ-complete categories; and if $\mathcal{A}$ and $\mathcal{B}$ are Φ-complete, the Φ-continuous functors $\mathcal{A} \to \mathcal{B}$ determine a full subcategory Φ-**Conts**$[\mathcal{A}, \mathcal{B}]$ of the $\mathcal{V}'$-category $[\mathcal{A}, \mathcal{B}]$. Note the use of *square* brackets in the name of the $\mathcal{V}'$-category Φ-**Conts**$[\mathcal{A}, \mathcal{B}]$ (which is a

$\mathcal{V}$-category for small $\mathcal{A}$); its underlying category is our earlier Φ-**Conts**$(\mathcal{A},\mathcal{B})$ with round brackets. Thus the 2-category Φ-**Conts** really underlies a $\mathcal{V}'$-**CAT**-category that we might note $\underline{\Phi\text{-}\mathbf{Conts}}$ with

2.9 $\underline{\Phi\text{-}\mathbf{Conts}}(\mathcal{A},\mathcal{B}) = \Phi\text{-}\mathbf{Conts}[\mathcal{A},\mathcal{B}]$

(Note that we suppress the mention of $\mathcal{V}$ in Φ-**Conts** or $\underline{\Phi\text{-}\mathbf{Conts}}$; one cannot know the class Φ without knowing $\mathcal{V}$. There may be a problem when $\Phi = \mathcal{P}$, so that Φ-**Conts** is just **Conts**; one can write $\mathcal{V}$-**Conts** if necessary.) Of Course the definition of $\underline{\Phi\text{-}\mathbf{Conts}}$ makes it clear that 2.8 can equally be written as

2.10 $\underline{\Phi\text{-}\mathbf{Conts}} = \underline{\Psi\text{-}\mathbf{Conts}} \;\Leftrightarrow\; \underline{\Phi\text{-}\mathbf{Cocts}} = \underline{\Psi\text{-}\mathbf{Cocts}} \;\Rightarrow\; \Phi^* = \Psi^*.$

3 Free cocompletions of categories and saturated classes of weights

Another piece of background knowledge that we need to recall concerns the "left bi-adjoint" to the forgetful 2-functor $U_\Phi : \Phi\text{-}\mathbf{Cocts} \to \mathcal{V}\text{-}\mathbf{CAT}$, or rather to $U_\Phi : \underline{\Phi\text{-}\mathbf{Cocts}} \to \underline{\mathcal{V}\text{-}\mathbf{CAT}}$. (Note that it is convenient to deal with colimits rather than limits.)

In the language of [Kel82], a presheaf $F : \mathcal{A}^{op} \to \mathcal{V}$ is said to be *accessible* if it is the left Kan extension of its restriction to some small full subcategory of $\mathcal{A}^{op}$. Even though $[\mathcal{A}^{op},\mathcal{V}]$ is when $\mathcal{A}$ is not small, only a $\mathcal{V}'$-category, its full subcategory $\mathcal{P}\mathcal{A}$ having as objects the accessible presheaves is a $\mathcal{V}$-category, as follows from [Kel82] 4.41; of course $\mathcal{P}\mathcal{A}$ coincides with $[\mathcal{A}^{op},\mathcal{V}]$ when $\mathcal{A}$ is small. Every representable $\mathcal{A}(-,a)$ is accessible, so that the Yoneda embedding $Y : \mathcal{A} \to [\mathcal{A}^{op},\mathcal{V}]$ takes its values in $\mathcal{P}\mathcal{A}$. Remember that the functor category $[\mathcal{A}^{op},\mathcal{V}]$ admits all small colimits, these being formed pointwise in $\mathcal{V}$. Each object of $\mathcal{P}\mathcal{A}$, being a left Kan extension of the given kind, is a small colimit of representables. Recall from [Kel82] p.154 that given a full subcategory $\mathcal{A}$ of $\mathcal{B}$ and a family Φ of weights, the *closure of $\mathcal{A}$ in $\mathcal{B}$ under Φ-colimits* is the smallest full replete subcategory of $\mathcal{B}$ containing $\mathcal{A}$ and closed under the formation Φ-colimits in $\mathcal{B}$. Since it is shown in [Kel82] Proposition 5.34 that $\mathcal{P}\mathcal{A}$ is closed in $[\mathcal{A}^{op},\mathcal{V}]$ under small colimits, we see that $\mathcal{P}\mathcal{A}$ is precisely the *closure of $\mathcal{A}$ in $[\mathcal{A}^{op},\mathcal{V}]$ under small colimits* (when $\mathcal{A}$ is embedded in $[\mathcal{A}^{op},\mathcal{V}]$ by the Yoneda embedding Y). For any class Φ of weights, and any category $\mathcal{A}$, write $\Phi(\mathcal{A})$ for the closure of $\mathcal{A}$ in $\mathcal{P}A$ (or equally in $[\mathcal{A}^{op},\mathcal{V}]$) under Φ-colimits, with $Z : \mathcal{A} \to \Phi(\mathcal{A})$ and $W : \Phi(\mathcal{A}) \to [\mathcal{A}^{op},\mathcal{V}]$ for the full inclusions, so that $Y : \mathcal{A} \to [\mathcal{A}^{op},\mathcal{V}]$ is the composite WZ. We now reproduce (the main point of) [Kel82] Theorem 5.35. The proof below is a little more direct than that given there, which referred back to earlier results. The result itself must be older still, at least for certain classes Φ.

Proposition 3.1 *For any Φ-cocomplete category $\mathcal{B}$ there is an equivalence of $\mathcal{V}'$-categories*

3.1 $[Z,1] : \Phi\text{-}\mathbf{Cocts}[\Phi(\mathcal{A}),\mathcal{B}] \to [\mathcal{A},\mathcal{B}]$

*with an equivalence inverse given by Lan_Z. It follows that Φ-**Cocts**$[\Phi(\mathcal{A}),\mathcal{B}]$ is like $[\mathcal{A},\mathcal{B}]$, a $\mathcal{V}$-category when $\mathcal{A}$ is small. Thus $\Phi(-)$ provides a left bi-adjoint, not only to the forgetful 2-functor Φ-**Cocts** $\to \mathcal{V}$-**CAT**, but even to the forgetful $\mathcal{V}'$-**Cat**-functor $\underline{\Phi\text{-}\mathbf{Cocts}} \to \underline{\mathcal{V}\text{-}\mathbf{CAT}}$.*

PROOF:The left Kan extension $Lan_Z G$, if it exists, is given by $Lan_Z G(F) = \tilde{Z}F * G$ where $\tilde{Z}(F) = \Phi(\mathcal{A})(Z-,F)$. However $\Phi(\mathcal{A})(Z-,F) = [\mathcal{A}^{op},\mathcal{V}](WZ-,WF) = [\mathcal{A}^{op},\mathcal{V}](Y-,WF)$, which by Yoneda is isomorphic to WF. Consider the full subcategory of $\Phi(\mathcal{A})$ given by those F for which $WF * G$ does exist; it contains the representables by Yoneda, it is also closed under Φ-colimits according to [Kel82] (3.22) since these colimits exist in $\mathcal{B}$; so it is all of $\Phi(A)$. What is more: $Lan_Z(G) = W - *G : \Phi(\mathcal{A}) \to \mathcal{B}$ preserves Φ-colimits as W preserves Φ-colimits and $- * G$ is cocontinuous (straightforward from [Kel82] (3.22) again). So one does have indeed a functor $Lan_Z : [\mathcal{A},\mathcal{B}] \to \Phi\text{-}\mathbf{Cocts}[\Phi(\mathcal{A}),\mathcal{B}]$, while one has trivially the restriction functor

$[Z,1] : \Phi\text{-}\mathbf{Cocts}[\Phi(\mathcal{A}),\mathcal{B}] \to [\mathcal{A},\mathcal{B}]$. The canonical $G \to Lan_Z(G)Z$ is invertible for all G since Z is fully faithful by [Kel82] (4.22). So that it remains to consider the canonical $\alpha : Lan_Z(SZ) \to S$ for a Φ-cocontinuous $S : \Phi(\mathcal{A}) \to \mathcal{B}$. The F-component of α for $F \in \Phi(\mathcal{A})$ is the canonical $\alpha_F : WF * SZ \to SF$; and clearly the collection of those F for which α_F is invertible contains the representables and is closed under Φ-colimits, therefore it is the totality of $\Phi(\mathcal{A})$. ∎

Remark 3.2 *It is often said that $\Phi(\mathcal{A})$ is the* free Φ-cocomplete category on $\mathcal{A}$*, but by this is usually meant an equivalence $\Phi\text{-}\mathbf{Cocts}(\Phi(\mathcal{A}),\mathcal{B}) \cong \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{A},\mathcal{B})$ of categories which is* weaker *than 3.1.*

Remark 3.3 *As a particular case $\mathcal{PA}$ itself is the free cocomplete category on $\mathcal{A}$; in other words $\Phi(\mathcal{A}) = \mathcal{PA}$ when Φ is the class of* all *weights - which is why (identifying $\mathcal{P}(\mathcal{A})$ with $\mathcal{PA}$) we use $\mathcal{P}$ as the name for this class of all weights.*

We can form $\Phi(\mathcal{A})$ by transfinite induction. Defining successively full replete subcategories $\mathcal{A}_\alpha$ of $\mathcal{PA}$ as α runs through the ordinals: $\mathcal{A}_0$ is the category $\mathcal{A} \subseteq \mathcal{PA}$ of representables, $\mathcal{A}_{\alpha+1}$ consists of $\mathcal{A}_\alpha$ together with all Φ-colimits in $\mathcal{PA}$ of diagrams in $\mathcal{A}_\alpha$, and for a limit ordinal we let $\mathcal{A}_\alpha = \bigcup_{\beta<\alpha} \mathcal{A}_\beta$, then $\Phi(\mathcal{A})$ is the smallest full replete subcategory of $[A^{op},\mathcal{V}]$ containing all the $\mathcal{A}_\alpha$ for all ordinals α. Note that this shows that $\Phi(\mathcal{A})$ *is small when $\mathcal{A}$ and Φ are small.* However $\Phi(\mathcal{A})$ may be small for all small $\mathcal{A}$ when Φ is not a small class: in this case we call the class Φ *locally small* (as usual in category theory, we call a category *small* when its set of *isomorphism classes* is a small set). In a number of important cases, one has $\Phi(\mathcal{A}) = \mathcal{A}_1$ in the notation above; but there is no special value in this condition, which (as we shall see) always holds for a small $\mathcal{A}$ when the class Φ is *saturated.*

An explicit description of the saturation Φ^* of a class Φ of weights was given by Albert and Kelly in [AK88], in the following terms:

Proposition 3.4 *The weight $\psi : \mathcal{K}^{op} \to \mathcal{V}$ lies in the saturation Φ^* of the class Φ if and only if the object ψ of $\mathcal{PK} = [\mathcal{K}^{op},\mathcal{V}]$ lies in the full subcategory $\Phi(\mathcal{K})$ of $[\mathcal{K}^{op},\mathcal{V}]$.*

There is another useful way of putting this. When $\mathcal{K}$ is small, both $\Phi[\mathcal{K}]$ and $\Phi(\mathcal{K})$ makes sense for any class Φ; and in fact we have

3.2 $\Phi[\mathcal{K}] \subseteq \Phi(\mathcal{K})$,

since for $\phi : \mathcal{K}^{op} \to \mathcal{V}$ the Yoneda isomorphism

3.3 $\phi \cong \phi * Y$

exhibits ϕ as an object of $\Phi(\mathcal{K})$ when $\phi \in \Phi$. We can write Proposition 3.4 as

3.4 $\Phi^*[\mathcal{K}] = \Phi(\mathcal{K})$

so that Φ is a saturated class precisely when

3.5 $\Phi[\mathcal{K}] = \Phi(\mathcal{K})$

for each small $\mathcal{K}$. In other words the class Φ is saturated precisely when, for each small $\mathcal{K}$, *the full subcategory $\Phi[\mathcal{K}]$ of $[\mathcal{K}^{op},\mathcal{V}]$ contains the representables $\mathcal{K}(-,k)$ and is closed in $[\mathcal{K}^{op},\mathcal{V}]$ under Φ-colimits.*

Note that the saturation 0^* of the empty class 0 consists precisely in the representables - now in the sense that $0^*[\mathcal{K}] = 0^*(\mathcal{K})$ consists exactly in the functors *isomorphic to* some $\mathcal{K}(-,k) : \mathcal{K}^{op} \to \mathcal{V}$.

It follows of course from the definitions of $\Phi(\mathcal{A})$ and of Φ^* that

3.6 $\Phi^*(\mathcal{A}) = \Phi(\mathcal{A})$

for any $\mathcal{A}$, small or not. We cannot write 3.5 when $\mathcal{K}$ is replaced by a non-small $\mathcal{A}$, since then $\Phi[\mathcal{A}]$ has no meanings; but a partial replacement for it is provided by the following which was Proposition 7.4 in [AK88]

Proposition 3.5 *If the presheaf $F : \mathcal{A}^{op} \to \mathcal{V}$ lies in $\Phi(\mathcal{A})$ for some closed class Φ then F is a Φ-colimit of representables; that is $F \cong \phi * ZS$ for some $\phi : \mathcal{K}^{op} \to \mathcal{V}$ in Φ and some $S : \mathcal{K} \to \mathcal{A}$.*

It may be useful to understand extreme special cases of one's notation: we mentioned above the empty class 0 of weights, with $0[\mathcal{K}] = 0$ and with $0(\mathcal{K}) = 0^*(\mathcal{K}) = 0^*[\mathcal{K}]$ consisting of the representables. Another extreme case involves the empty $\mathcal{V}$-category 0 with no objects. Of course $P0 = [0^{op}, \mathcal{V}]$ is the terminal category 1; its unique object is the unique functor $! : 0^{op} \to \mathcal{V}$ and $1(!, !)$ is the terminal object 1 of $\mathcal{V}$. (This differs in general from the *unit $\mathcal{V}$-category $\mathcal{I}$*, with one object $*$ but with $\mathcal{I}(*, *)=I$, the unit for $\otimes$.) So for any class Φ, we have $\Phi[0] = 0$ if $! : 0^{op} \to \mathcal{V}$ is not in Φ, and $\Phi[0] = 1$ otherwise. Now $\Phi(0)$ is the closure of 0 in $P0$ under Φ-colimits, and any diagram $T : \mathcal{K} \to 0$ has $\mathcal{K} = 0$, so that $\Phi(0) = 0$ if $! \notin \Phi$ and otherwise $\Phi(0)$ contains $! * Y =!$, giving $\Phi(0) = 1$. So in fact $\Phi(0) = \Phi[0]$, being 0 or 1. Both are possible for a closed Φ for $P0 = 1$, while the Albert-Kelly theorem (Proposition 3.4) gives $0^*[0] = 0(0) = 0[0] = 0$.

Before ending this section, we recall a result characterising Φ-cocomplete categories, along with a short proof. This was Proposition 4.5 in [AK88].

Proposition 3.6 *For any class Φ of weights, a category $\mathcal{A}$ admits Φ-colimits if and only if the fully faithful embedding $Z : \mathcal{A} \to \Phi(\mathcal{A})$ admits a left adjoint; that is, if and only if the full subcategory $\mathcal{A}$ given by the representables is reflective in $\Phi(\mathcal{A})$.*

PROOF:If $\mathcal{A}$ is reflective, it admits Φ-colimits because $\Phi(\mathcal{A})$ does so. Suppose conversely that $\mathcal{A}$ admits Φ-colimits, and write $\mathcal{B}$ for the full subcategory of $[\mathcal{A}^{op}, \mathcal{V}]$ given by those objects admitting a reflection into $\mathcal{A}$; then $\mathcal{B}$ contains $\mathcal{A}$ and $\mathcal{B}$ is closed in $[\mathcal{A}^{op}, \mathcal{V}]$ under Φ-colimits since $\mathcal{A}$ admits these; so that $\mathcal{B}$ contains $\Phi(\mathcal{A})$, as desired. ∎

4 The embedding $\mathcal{A} \to \Phi(\mathcal{A})$

We recall from Proposition 5.62 of [Kel82] a result characterising categories of the form $\Phi(\mathcal{A})$ - or more precisely functors of the form $Z : \mathcal{A} \to \Phi(\mathcal{A})$ (which we also write when necessary as $Z_{\mathcal{A}}$). At the same time, we give a direct proof; for the proof in [Kel82] refers back to earlier results in that book.

We begin with a piece of notation: for a category $\mathcal{A}$ and a class Φ of weights, we write $\mathcal{A}_\Phi$ for the full subcategory of $\mathcal{A}$ generated by those $a \in \mathcal{A}$ for which the representables $\mathcal{A}(a, -) : \mathcal{A}^{op} \to \mathcal{V}$ preserve all Φ-colimits (That is all Φ-colimits that *exist in* $\mathcal{A}$). There is no agreed name for $\mathcal{A}_\Phi$; the objects of $\mathcal{A}_\Phi$ are usually called *finitely presentable* when the Φ-colimits are the classical filtered colimits; while when Φ is the class P of all weights the objects of $\mathcal{A}_\Phi$ were called *small projective* in [Kel82], but have also been called *atoms* by some authors. Therefore we propose to name the objects of $\mathcal{A}_\Phi$ Φ-*atoms*. When $\mathcal{A}$ admits Φ-colimits and hence Φ^*-colimits, it follows from the definition of $\mathcal{A}_\Phi$ that

4.1 $\mathcal{A}_{\Phi^*} = \mathcal{A}_\Phi$.

The following is the characterisation result of Proposition 5.62 [Kel82] with a slightly expanded form of its statement.

Proposition 4.1 *In order that $G : \mathcal{A} \to \mathcal{B}$ be equivalent to the free Φ-cocompletion $Z : \mathcal{A} \to \Phi(\mathcal{A})$ of $\mathcal{A}$ for a class Φ of weights, the following conditions are necessary and sufficient:*

- *(i) G is fully faithful (allowing us to treat $\mathcal{A}$ henceforth as a full subcategory of $\mathcal{B}$);*

- *(ii) $\mathcal{B}$ is Φ-cocomplete;*
- *(iii) the closure of $\mathcal{A}$ in $\mathcal{B}$ under Φ-colimits is $\mathcal{B}$ itself;*
- *(iv) $\mathcal{A}$ is contained in the full subcategory $\mathcal{B}_\Phi$ of Φ-atoms of $\mathcal{B}$.*

When these conditions hold, the functor $\tilde{G} : \mathcal{B} \to [\mathcal{A}^{op}, \mathcal{V}]$ defined (in the usual notation of [Kel82]) by $\tilde{G}b = \mathcal{B}(G-, b)$ is fully faithful and takes its image in the full subcategory $\Phi(\mathcal{A})$ of $[\mathcal{A}^{op}, \mathcal{V}]$ thereby defining a fully faithful $K : \mathcal{B} \to \Phi(\mathcal{A})$; and this K is in fact an equivalence - inverse being given by $Lan_Z(G) : \Phi(\mathcal{A}) \to \mathcal{B}$, which by Proposition 3.1 is the unique Φ-cocontinuous extension of G to $\Phi(\mathcal{A})$.

PROOF:The necessity of the first three conditions is clear; for that of the forth, writing as before

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "Y"] \arrow[dr, "Z"'] & & {[\mathcal{A}^{op}, \mathcal{V}]} \\ & \Phi(\mathcal{A}) \arrow[ur, "W"'] & \end{tikzcd}$$

for the full inclusions, the point is that $\Phi(\mathcal{A})(Za, -) \cong [\mathcal{A}^{op}, \mathcal{V}](Ya, W-)$ preserves Φ-colimits since $W : \Phi(\mathcal{A}) \to [\mathcal{A}^{op}, \mathcal{V}]$ does so by definition of $\Phi(\mathcal{A})$, while $[\mathcal{A}^{op}, \mathcal{V}](Ya, -) : [\mathcal{A}^{op}, \mathcal{V}] \to \mathcal{V}$ preserves all small colimits, being isomorphic by Yoneda to the evaluation E_a.

Turning to the proof of sufficiency, to say that $\tilde{G}$ is fully faithful is just to say (see Theorem 5.1 [Kel82]) that $G : \mathcal{A} \to \mathcal{B}$ is dense; and in the present case that is so by Theorem 5.19 of [Kel82] since $\mathcal{B}$ is the closure of $\mathcal{A}$ in $\mathcal{B}$ under Φ-colimits and $\mathcal{A}$ lies in $\mathcal{B}_\Phi$. The proof is so short that we may recall it: we consider the full subcategory of $\mathcal{B}$ given by those b for which $\tilde{G}_{b,c} : \mathcal{B}(b, c) \to [\mathcal{A}^{op}, \mathcal{V}](\tilde{G}b, \tilde{G}c)$ is invertible for all c, observing that it is all of B since it contains $\mathcal{A}$ - since G is fully faithful - and is closed under Φ-colimits - as $\mathcal{A} \subseteq \mathcal{B}_\Phi$.

Next, to see that each $\tilde{G}b = \mathcal{B}(G-, b)$ lies in the full replete subcategory $\Phi(\mathcal{A})$ of $[\mathcal{A}^{op}, \mathcal{V}]$, consider the full subcategory of $\mathcal{B}$ given by those b for which this is so; this contains $\mathcal{A}$ since $\tilde{G} \cdot Ga \cong Ya$ because G is fully faithful, and it is closed in $\mathcal{B}$ under Φ-colimits since $\tilde{G}$ preserves these by (iv) and since $\Phi(\mathcal{A})$ is closed under these in $[\mathcal{A}^{op}, \mathcal{V}]$; so it is all of $\mathcal{B}$. Thus $\tilde{G}$ is indeed of the form $\tilde{G} = WK$ for some fully faithful $K : \mathcal{B} \to \Phi(\mathcal{A})$.

It remains to show that K and $S = Lan_Z G : \Phi(\mathcal{A}) \to \mathcal{B}$ are equivalence-inverses. Recall from Proposition 3.1 that S is equally the essentially unique Φ-cocontinuous functor with $SZ = G$. Since $\tilde{G} = WK$ and $G \cong SZ$, one has $WKSZ \cong \tilde{G}G \cong Y \cong WZ$, giving $KSZ \cong Z$ since W is fully faithful and then giving $KS \cong 1$ by Proposition 3.1 since KS and 1 are Φ-cocontinuous (K being so because $WK = \tilde{G}$ is so). Finally $KS \cong 1$ gives $KSK \cong K$ and $SK \cong 1$ since K (as we saw) is fully faithful. ∎

This is of course of particular interest in the case of a small $\mathcal{A}$ in Proposition 4.1. If we suppose the class Φ saturated for simplicity, we may cast the result for small $\mathcal{A}$'s in the form

Proposition 4.2 *For a saturated class Φ of weights, the following properties of a category $\mathcal{B}$ are equivalent:*

- *(i) For some small $\mathcal{K}$, there is an equivalence $\mathcal{B} \cong \Phi(\mathcal{K})$;*
- *(ii) $\mathcal{B}$ is Φ-cocomplete and has a small full subcategory $A \subseteq \mathcal{B}_\Phi$ such that every object of $\mathcal{B}$ is a Φ-colimit of a diagram in $\mathcal{A}$;*
- *(iii) $\mathcal{B}$ is Φ-cocomplete and has a small full subcategory $\mathcal{A} \subseteq \mathcal{B}_\Phi$ such that the closure of $\mathcal{A}$ in $\mathcal{B}$ under Φ-colimits is $\mathcal{B}$ itself.*

Under the hypothesis (iii) - and so a fortiori under (ii) - if $G : \mathcal{A} \to \mathcal{B}$ denotes the inclusion, the functor $\tilde{G} : \mathcal{B} \to [\mathcal{A}^{op}, \mathcal{V}]$ is fully faithful, with $\Phi(\mathcal{A})$ for its replete image.

Remarks 4.3

- *(a) When Φ is the class $\mathcal{P}$ of all weights, we get a characterisation here of the functor category $\mathcal{PK} = [\mathcal{K}^{op}, \mathcal{V}]$ for a small $\mathcal{K}$; note that it differs from the characterisation given in [Kel82] Theorem 5.26 which replaces the condition that $\mathcal{B}$ be the colimit closure of $\mathcal{A}$ by the condition that $\mathcal{A}$ be strongly generating in $\mathcal{B}$; but that these conditions are very similar in strength by [Kel82] Proposition 3.40.*
- *(b) When $\mathcal{V} = Set$, various special cases of this are well known, besides that where $\Phi = \mathcal{P}$; for instance the cases where Φ-colimits are finite coproducts, or finite colimits or filtered colimits, or absolute colimits. We will return to this last example in section 6.*
- *(c) Theorem 2.4 of [BQR98] is the special case where $\mathcal{V}$ is locally finitely presentable as a closed category in the sense of [Kel82-2] and Φ is the saturated class of flat presheaves.*

Of particular interest regarding Proposition 4.2 are the saturated families of the form Ψ^+ and Φ^-, for arbitrary classes of weights Ψ and Φ. In particular we shall see that when $\Phi = \mathcal{P}$ is the family of all weights, $\Phi^+(\mathcal{A})$ is the Cauchy-completion of $\mathcal{A}$. Also the fact of considering families of weighted limits/colimits instead of mere conical limits/colimits is crucial even in the case $\mathcal{V} = Set$. In [ABLR02], Adamek, Borceux, Lack and Rosicky investigated the notion of general commutation of families of limits/colimits - the limits and colimits considered are classical - i.e.defined in terms of diagrams - not weight. To obtain a result similar to 4.2 (that one may call "parameterised accessibility"), they needed to impose the technical condition of "soundness" on classes of limits. The next section should be a key argument in favour of weighted limits/colimits, that should also invalidate the wrong idea that "weighted limits/colimits in Sets are just the same as usual ones".

5 Limits and colimits commuting in $\mathcal{V}$

The new observations to which we now turn on begin with the general theory of the commutativity in $\mathcal{V}$ of limits and colimits. For a pair of weights $\psi : \mathcal{K}^{op} \to \mathcal{V}$ and $\phi : \mathcal{L}^{op} \to \mathcal{V}$, to say that

5.1 $\phi * - : [\mathcal{L}, \mathcal{V}] \to \mathcal{V}$ *preserves ψ-limits*

is equally to say that

5.2 $\{\psi, -\} : [\mathcal{K}^{op}, \mathcal{V}] \to \mathcal{V}$ *preserves ϕ-colimits*

because each in fact asserts the invertibility, for every functor $S : \mathcal{K}^{op} \otimes \mathcal{L} \to \mathcal{V}$, of the canonical comparison morphism

5.3 $\phi ? * \{\psi -, S(-, ?)\} \to \{\psi -, \phi * S(-, ?)\}$.

The reader who would not find the above statement immediate, would be happy to consider the following proposition.

Proposition 5.1 *Let $F : \mathcal{A}^{op} \to \mathcal{V}$, $P : \mathcal{K} \to \mathcal{V}$ and $G : \mathcal{A} \otimes \mathcal{K} \to \mathcal{V}$ equivalent to $G' : \mathcal{A} \to [\mathcal{K}, \mathcal{V}]$ and also to $G'' : \mathcal{K} \to [\mathcal{A}, \mathcal{V}]$. Then $F * - : [\mathcal{A}, \mathcal{V}] \to \mathcal{V}$ preserves the limit $\{P, G''\}$ if and only if $\{P, -\} : [\mathcal{K}, \mathcal{V}] \to \mathcal{V}$ preserves the colimit $F * G'$.*

PROOF:Let

$$P \xrightarrow{\eta} [\mathcal{A}, \mathcal{V}](\{P, G''\}, G''-)$$

be the unit of $\{P, G''\}$ and

$$F \xrightarrow{\lambda} [\mathcal{K}, \mathcal{V}](G'-, F * G')$$

be the unit of $F * G'$. We need to show that

$$(1) \quad P \xrightarrow{\eta} [\mathcal{A}, \mathcal{V}](\{P, G''\}, G''-) \xrightarrow{F*-} [F * \{P, G''\}, (F * G''-)]$$

exhibits $F * \{P, G''\}$ as $\{P, F * G'\}$ if and only if

$$(2) \quad F \xrightarrow{\lambda} [\mathcal{K}, \mathcal{V}](F * G', G'-) \xrightarrow{\{P,-\}} [\{P, F * G'\}, \{P, G'-\}]$$

exhibits $\{P, F * G'\}$ as $F * \{P, G''\}$.

First note that given $x \in \mathcal{V}$, any natural in k,

$$(1') \; Pk \to_k [x, (F * G')k]$$

corresponds via Yoneda to a natural in v, $[v, x] \to_v [\mathcal{K}, \mathcal{V}](P, [v, (F * G')-])$. Since

$$[\mathcal{K}, \mathcal{V}](P, [v, (F * G')-] \cong_v [v, \{P, F * G'\}],$$

it corresponds also to an arrow

$$(1'') \;\; x \to \{P, F * G'\}.$$

Also that $(1')$ exhibits x as the limit $\{P, F*G'\}$ is equivalent to the fact that $(1'')$ is iso. Analogously any natural in a,

$$(2') \;\; Fa \to [\{P, G''\}a, x]$$

corresponds to an arrow

$$(2'') \;\; F * \{P, G''\} \to x,$$

and $(2')$ exhibits x as the colimit $F * \{P, G''\}$ if and only if $(2'')$ is iso.

Now the result follows from the fact the arrow (1) above corresponds by the bijection $(1') - (1'')$ to the same arrow as (2) by $(2') - (2'')$. This is now left to the reader.

When these statements are true for every such S, we say that *ϕ-colimits commute with ψ-limits in $\mathcal{V}$*. For classes Φ and Ψ of weights, if 5.1 (equivalently 5.2) holds for all $\phi \in \Phi$ and all $\psi \in \Psi$, we say that *Φ-colimits commute with Ψ-limits in $\mathcal{V}$*. For any class Ψ of weights we may consider the class Ψ^+ of all weights ϕ for which ϕ-colimits commute with Ψ-limits in $\mathcal{V}$; and for any class Φ of weights we may consider the class Φ^- of all weights ψ for which Φ-colimits commute with ψ-limits in $\mathcal{V}$. This forms of course a Galois connection (with $\Phi \subseteq \Psi^+$ if and only if $\Psi \subseteq \Phi^-$). Note that $[\mathcal{L}, \mathcal{V}]$ and $\mathcal{V}$ in 5.1 admit all (small) limits; so that by the definition above of the saturation Ψ^* of a class Ψ of weights if $\phi * -$ preserves all Ψ-limits, it also preserves Ψ^*-limits. From this and a dual argument, one concludes that:

Proposition 5.2 *For any class Φ and Ψ of weights, the class Φ^- and Ψ^+ are saturated classes of weights; so that $\Psi^{+\,*} = \Psi^+$ and $\Phi^{-\,*} = \Phi^-$. Moreover $\Psi^+ = \Psi^{*\,+}$ and $\Phi^- = \Phi^{*\,-}$.*

When Ψ consists of the weights for finite limits (in the usual case for ordinary categories, or in the sense of [Kel82-2] when $\mathcal{V}$ is "locally finitely presentable as a closed category", its has been customary to call the elements of Ψ^+ the *flat* weights, as they are those ϕ having $\phi * - : [\mathcal{L}, \mathcal{V}] \to \mathcal{V}$ left exact. We might therefore for a general Ψ think of the elements of Ψ^+ as *Ψ-flat* weights.

Recall that the limit functor $\{\psi, -\}$ of 5.2 is just the representable functor $[\mathcal{K}^{op}, \mathcal{V}](\psi, -) : [\mathcal{K}^{op}, \mathcal{V}] \to V$. Accordingly $\psi : \mathcal{K}^{op} \to \mathcal{V}$ lies in Φ^- for a given class Φ if and only if it lies in the subcategory $[\mathcal{K}^{op}, \mathcal{V}]_\Phi$ of Φ-atoms:

5.4 $\Phi^-[\mathcal{K}] = \Phi^-(\mathcal{K}) = [\mathcal{K}^{op}, \mathcal{V}]_\Phi$

Part of the saturation of Φ^- - namely the closedness of $\Phi^-[\mathcal{K}]$ in $[\mathcal{K}^{op}, \mathcal{V}]$ under Φ^--colimits is the special case for $\mathcal{A} = [\mathcal{K}^{op}, \mathcal{V}]$ of the following more general result:

Proposition 5.3 *For any class Φ of weights and any category $\mathcal{A}$, the full subcategory $\mathcal{A}_\Phi$ of $\mathcal{A}$ is closed in $\mathcal{A}$ under Φ^--colimits that exists in $\mathcal{A}$.*

PROOF:Let the colimit $\psi * S$ exist, where $\psi : \mathcal{K}^{op} \to \mathcal{V}$ lies in Φ^- and $S : \mathcal{K} \to \mathcal{A}$ takes its values in $\mathcal{A}_\Phi$. Then by definition

$$\mathcal{A}(\psi * S, a) \cong [\mathcal{K}^{op}, \mathcal{V}](\psi, \mathcal{A}(S-, a)).$$

Since each $\mathcal{A}(SK, -)$ preserves Φ-colimits, and since $[\mathcal{K}^{op}, \mathcal{V}](\psi, -)$ preserves Φ-colimits by 5.4, it follows that $\mathcal{A}(\psi * S, -)$ preserves Φ-colimits, that is to say $\psi * S \in \mathcal{A}_\Phi$. ∎

Let us add ad-hoc proof of the closeness of $\Phi[\mathcal{K}]$ under Φ^+-colimits in $[\mathcal{K}^{op}, \mathcal{V}]$ (though this does not bring anything more than Proposition 5.2)

Lemma 5.4 *For any Ψ-flat $G : \mathcal{A}^{op} \to \mathcal{V}$ and any functor $H : \mathcal{A} \to [\mathcal{C}^{op}, \mathcal{V}]$ with values Ψ-flat functors, the colimit $G * H$ is again Ψ-flat.*

PROOF:Consider $F : \mathcal{K} \to \mathcal{V}$ in Ψ and $L : \mathcal{K} \to [\mathcal{C}, \mathcal{V}]$. One has the successive isomorphisms:

$$\begin{aligned}(G * H) * \{F, L\} &\cong G * (H - *\{F, L\}) \text{ ([Kel82], (3.23) “continuity of a colimit in its index”)}\\ &\cong G * (\{F, H - *L-\}) \text{ (since for all } a,\ Ha * - \text{ preserves } \Psi\text{-limits)}\\ &\cong \{F, G * (H - *L-)\} \text{ (since } G * - \text{ preserves } \Psi\text{-limits)}\\ &\cong \{F, (G * H) * L-\}) \text{ ([Kel82], (3.23)).}\end{aligned}$$

The resulting isomorphism $(G * H) * \{F, L\} \cong \{F, (G * H) * L-\})$ corresponds actually to the preservation of $\{F, L\}$ by $(G * H) * -$.

6 Cauchy completion and Isbell adjunction

This last section is devoted to the study of the particular class of weights $\mathcal{Q} = \mathcal{P}^-$ where $\mathcal{P}$ stands as before for the class of all weights. That is to say that $\mathcal{Q}$ is the class defined by:

6.1 *$\phi \in \mathcal{Q} \Leftrightarrow \phi$-limits commute with* all small *colimits in $\mathcal{V}$.*

and equivalently, as we shall see, that the weights in $\mathcal{Q}$ are the so-called *small projective* ones. According to Proposition 5.2, we already know that $\mathcal{Q}$ is a saturated class. We shall establish the following alternative characterisations for $\mathcal{Q}$. From Proposition 6.2:

6.2 *$\phi : \mathcal{A}^{op} \to \mathcal{V}$ lies in $\mathcal{Q}$ if and only the corresponding module $\mathcal{I} \longrightarrow\!\!\!\!\circ\ \mathcal{A}$ is left adjoint.*

This last result shows that the Cauchy-completion of a small category $\mathcal{A}$ is just $\mathcal{Q}(\mathcal{A}) = \mathcal{Q}[\mathcal{A}]$. Also from Proposition 6.11:

6.3 *$\mathcal{Q}$ is the class of $\mathcal{P}$-flat weights, i.e. $\mathcal{Q} = \mathcal{P}^+$.*

The end of the section treats an enriched adjunction à la Isbell that restricts to an equivalence between subcategories of small projectives.

We need to recall a little terminology and few facts regarding adjoints, Kan extensions and liftings in a bicategory. Let us consider a bicategory $\mathbf{B}$ with horizontal composition “$*$” and vertical one “.” to fix the notations. $\mathbf{B}^{op}$ will denote the bicategory obtained from $\mathbf{B}$ by reversing arrows, and $\mathbf{B}^{coop}$ will be the one obtained by reversing arrows and 2-cells. We borrow from [StWa78] the following terminology. Given a diagram as below in $\mathbf{B}$

$$\begin{tikzcd} & C \arrow[dl, "g"'] \arrow[dr, "h"] \arrow[d, phantom, "\overset{\epsilon}{\Leftarrow}"] & \\ B & & A \arrow[ll, "f"] \end{tikzcd}$$

the arrow h together with 2-cell ϵ constitutes the *right lifting of g through f* if and only if pasting with ϵ at h determines a bijection of 2-cells

$$\mathbf{B}(C,B)(f*k,g) \leftrightarrow \mathbf{B}(C,A)(k,h).$$

In this situation we shall loosely write $h = rlift(f,g)$. The right lifting is said *respected* by a $k : X \to C$ when the 2-cell $\epsilon * k$ exhibits $h * k$ as the right lifting of $g * k$ through f, it is *absolute* when it is respected by any such arrow. Note that a right lifting in $\mathbf{B}$ is a right Kan extension in $\mathbf{B}^{op}$ or a left Kan extension in $\mathbf{B}^{coop}$. One has also a dual notion of *left liftings* that correspond to left Kan extensions in $\mathbf{B}^{op}$.

The following result characterising adjoints in a bicategory goes back to early work of Lawvere and Bénabou on closed bicategories. In fact the full result is really present in the one object-closed-bicategory situation given by a monoidal closed category, and that was studied in the symmetric case by [KeLa80].

Proposition 6.1 *In ANY bicategory, the following statements are equivalent:*

- *(i) the arrow $s : A \to B$ has a right adjoint;*
- *(ii) for all $g : C \to B$, the right lifting $rlift(s,g)$ of g through s exists and is absolute;*
- *(iii) the right lifting $rlift(s,1)$ of $1 : B \to B$ through s exists and is respected by s.*

When s has a right adjoint, this one is $rlift(s,1)$ the right lifting of 1 through s.

PROOF:$(i) \Rightarrow (ii)$. Let $\mathbf{B}$ denote the ambient bicategory. Let also $\eta : 1 \Rightarrow t * s : A \to A$ and $\epsilon : s * t \Rightarrow 1 : B \to B$ denote respectively the unit and counit of $s \dashv t$. For any 1-cells $g : C \to B$ and $x : C \to A$ the "pasting with ϵ": $(\epsilon * g) \cdot (s * -) : \mathbf{B}(C,A)(x, t*g) \to \mathbf{B}(C,B)(s*x, g)$ admits for inverse the "pasting with η": $(t * -) \cdot (\eta * x)$. This shows that $\epsilon * g : s * t * g \Rightarrow g$ exhibits $t * g$ as the right lifting of g through s. This lifting is trivially absolute.
(ii) trivially implies (iii).
$(iii) \Rightarrow (i)$.
The 1-cell

$$\begin{tikzcd} & A \arrow[dl, "s"'] \arrow[dr, "1"] \arrow[d, phantom, "="] & \\ B & & A \arrow[ll, "s"] \end{tikzcd}$$

factorises as

$$\begin{tikzcd} & & A \arrow[dll, "s"'] \arrow[dd, equal] \\ B \arrow[dr, "t"] \arrow[dd, equal] \arrow[rr, phantom, "\Leftarrow\eta"] & & \\ & & A \arrow[dll, "s"] \arrow[rr, phantom, "\Leftarrow\epsilon"] & \\ B & & & \end{tikzcd}$$

since s respects the absolute right lifting

$$\begin{tikzcd}
 & B \arrow[dl, "1"'] \arrow[dr, "t"] \arrow[d, phantom, "\Leftarrow\epsilon"] & \\
B & & A. \arrow[ll, "s"]
\end{tikzcd}$$

Now

$$\begin{tikzcd}
B \arrow[dr, "t"] \arrow[dd, equal] & \\
 & A \arrow[dl, "s"] \arrow[dd, equal] \\
B \arrow[dr, "t"] \arrow[dd, equal] & \\
 & A \arrow[dl, "s"] \\
B &
\end{tikzcd}$$

equals

$$\begin{tikzcd}
B \arrow[dr, "t"] \arrow[dd, equal] & \\
 & A \arrow[dl, "s"] \\
B &
\end{tikzcd}$$

and because ϵ is a right lifting

$$\begin{tikzcd}
B \arrow[dr, "t"] \arrow[dd, equal] & \\
 & A \arrow[dl, "s"] \arrow[dd, equal] \\
B \arrow[dr, "t"'] & \\
 & A
\end{tikzcd}$$

equals

$$\begin{tikzcd}
B \arrow[r, "t"] \arrow[dd, equal] \arrow[dr, phantom, "="] & A \arrow[dd, equal] \\
B \arrow[r, "t"'] & A
\end{tikzcd}$$

As adjoints $F \dashv G$ in a bicategory $\mathbf{B}$ correspond to adjoints $G \dashv F$ in $\mathbf{B}^{op}$, there are dual statements obtained by using left or right Kan extensions or left liftings. In detail

Proposition 6.2 *In any bicategory, the following assertions are equivalent:*

- *the arrow $s : A \to B$ has a right adjoint;*
- *for all $g : A \to C$, the left Kan extension $lan_s(g)$ exists and is absolute;*
- *the left Kan extension $lan_s(1)$ of $1 : A \to A$ along s exists and is preserved by s.*

When s has a right adjoint, this one is $lan_s(1)$.

And

Proposition 6.3 *In any bicategory, the following assertions are equivalent:*

- *(i) the arrow $t : B \to A$ has a left adjoint;*
- *(ii) for all $f : C \to A$, the left lifting $llift(t, f)$ of f through t exists and is absolute;*
- *(iii) the lifting $llift(t, 1)$ of $1 : A \to A$ through t exists and is respected by t.*
- *(iv) for all $g : B \to C$, the right Kan extension $ran_t(g)$ of g along t exists and is absolute;*
- *(v) the right Kan extension $ran_t(1)$ of $1 : B \to B$ along t exists and is preserved by t.*

When t has a left adjoint, this one is $llift(t, 1)$ or also $ran_t(1)$.

There is one lemma that we shall use later

Proposition 6.4 *In a bicategory if an arrow $f : A \to B$ has a right adjoint $g : B \to A$ then for any $h : C \to B$, the right lifting of h through f exists and is given by $rlift(f, h) \cong g * h$.*

PROOF:The counit $\epsilon : f * g \to 1 : B \to B$ of the adjunction $f \dashv g$ exhibits g as the right lifting $rlift(f, 1)$ of $1 : B \to B$ through f, and this right lifting is absolute. Composing then with any $h : C \to B$ yields the right lifting $rlift(f, h)$. ∎

We shall now consider the more specific case when the considered bicategory is *closed*, i.e such a bicategory admits all right Kan extensions and all right liftings and is locally complete and cocomplete. The older terminology for this was *biclosed*, but the more modern terminology recognises that a bicategory may be *right-closed* or *left-closed*, or *both* and to be both is to be close.

In a closed bicategory $\mathbf{B}$, for any objects A and any arrow $h : C \to B$, one has the natural isomorphisms in f, g:

6.4

$$\mathbf{B}(A, B)(f, ran_g(h)) \cong \mathbf{B}(C, B)(f * g, h) \cong \mathbf{B}(C, A)(g, rlift(f, h))$$

and thus an adjunction

6.5

$$rlift(-, h) \dashv ran_{-}(h) : \mathbf{B}(A, B) \to \mathbf{B}(C, A)^{op}$$

Note also that according to 6.1 and 6.3

Remark 6.5 *the composite isomorphism 6.4 in the case $C = B$ and $h = 1_B : B \to B$, maps bijectively left adjoints $A \to B$ to right adjoints $B \to A$.*

We can now turn to the specific case of the closed bicategory $\mathcal{V}$-**Mod** of modules with domains and codomains *small* categories. For categories $\mathcal{A}$ and $\mathcal{B}$ not necessarily small, a module $\mathcal{A} \longrightarrow\!\!\!\!\circ\!\!\rightarrow \mathcal{B}$ is a functor $\mathcal{B}^{op} \otimes \mathcal{A} \to \mathcal{V}$ sending (b, a) to $f(b, a)$ and a morphism $\alpha : f \to g : \mathcal{A} \longrightarrow\!\!\!\!\circ\!\!\rightarrow \mathcal{B}$ of module is a natural transformation with components $\alpha_{b,a} : f(b, a) \to g(b, a)$. So the modules from $\mathcal{A}$ to $\mathcal{B}$ form an ordinary category $\mathcal{V}$-$\mathbf{MOD}(\mathcal{A}, \mathcal{B})$ that underlines a $\mathcal{V}'$-category $\mathcal{V}$-$\mathbf{MOD}[\mathcal{A}, \mathcal{B}]$ (for a suitable $\mathcal{V}'$ extending $\mathcal{V}$), which is a $\mathcal{V}$-category when $\mathcal{A}$ and $\mathcal{B}$ are small. Of course a module $\mathcal{A} \longrightarrow\!\!\!\!\circ\!\!\rightarrow \mathcal{B}$ can equally be seen as a functor $\mathcal{A} \to [\mathcal{B}^{op}, \mathcal{V}]$, and so (to within equivalence) as a cocontinuous functor $\mathcal{P}\mathcal{A} \to [\mathcal{B}^{op}, \mathcal{V}]$. Therefore when $\mathcal{A}$ and $\mathcal{B}$ are small, $\mathcal{V}$-$\mathbf{MOD}(\mathcal{A}, \mathcal{B})$ is equivalent to $\mathbf{Cocts}([\mathcal{A}^{op}, \mathcal{V}], [\mathcal{B}^{op}, \mathcal{V}])$ and the cocontinuous functors here are equally the left adjoint ones - see [Kel82], Theorem 4.82). For modules $f : \mathcal{A} \longrightarrow\!\!\!\!\circ\!\!\rightarrow \mathcal{B}$ and $g : \mathcal{B} \longrightarrow\!\!\!\!\circ\!\!\rightarrow \mathcal{C}$ we have a composite $gf : \mathcal{A} \longrightarrow\!\!\!\!\circ\!\!\rightarrow \mathcal{C}$ defined by

6.6

$$(gf)(c, a) = \int^b g(c, b) \otimes f(b, a)$$

whenever the integral exists - which surely does when $\mathcal{B}$ is small. This composition is associative up to coherent isomorphisms, with as identities-to-within-isomorphisms the $1_{\mathcal{A}} : \mathcal{A} \longrightarrow\!\!\!\!\circ\!\!\rightarrow \mathcal{A}$ given by the hom functor $\mathcal{A}(-, -) : \mathcal{A}^{op} \otimes \mathcal{A} \to \mathcal{V}$; so that the modules between *small* categories are the arrows of a bicategory $\mathcal{V}$-**Mod**. This $\mathcal{V}$-**Mod** is *closed* as it admits all right liftings and all right Kan extensions as follows. Given modules $f : \mathcal{A} \longrightarrow\!\!\!\!\circ\!\!\rightarrow \mathcal{B}$, $g : \mathcal{B} \longrightarrow\!\!\!\!\circ\!\!\rightarrow \mathcal{C}$ and $h : \mathcal{A} \longrightarrow\!\!\!\!\circ\!\!\rightarrow \mathcal{C}$, we consider new modules $[\![f, h]\!] : \mathcal{B} \longrightarrow\!\!\!\!\circ\!\!\rightarrow \mathcal{C}$ and $\{\!\{ g, h \}\!\} : \mathcal{A} \longrightarrow\!\!\!\!\circ\!\!\rightarrow \mathcal{B}$ by

6.7

$$[\![f, h]\!](c, b) = \int_a [f(b, a), h(c, a)]$$

and

6.8

$$\{\!\{ g, h \}\!\}(b, a) = \int_c [g(c, b), h(c, a)]$$

whenever these integrals exist - so that $[\![f, h]\!]$ certainly exists when $\mathcal{A}$ is small, while $\{\!\{ g, h \}\!\}$ certainly exists when $\mathcal{C}$ is small. When gf and $[\![f, h]\!]$ exist, to give a morphism $gf \to h$ is to give a family $g(c, b) \otimes f(b, a) \to h(c, a)$ natural (meaning $\mathcal{V}$-natural) in each variable; which is equally to give a family $g(c, b) \to [f(b, a), h(c, a)]$ natural in each variable, and so as morphism $g \to [\![f, h]\!]$ of modules. Accordingly so far as the modules in question exist we have natural bijections

6.9

$$\mathcal{V}\text{-}\mathbf{MOD}(\mathcal{A}, \mathcal{C})(gf, h) \cong \mathcal{V}\text{-}\mathbf{MOD}(\mathcal{B}, \mathcal{C})(g, [\![f, h]\!]) \cong \mathcal{V}\text{-}\mathbf{MOD}(\mathcal{A}, \mathcal{B})(f, \{\!\{ g, h \}\!\}).$$

exhibiting $[\![f, h]\!]$ as the *right Kan extension of* h *along* f and $\{\!\{ g, h \}\!\}$ as the *right lifting of* h *through* g.

Each functor $T : \mathcal{A} \to \mathcal{B}$ gives rise to modules $T_* : \mathcal{A} \longrightarrow\!\!\!\!\circ\!\!\rightarrow \mathcal{B}$ and $T^* : \mathcal{B} \longrightarrow\!\!\!\!\circ\!\!\rightarrow A$, where

6.10

$$T_*(b, a) = \mathcal{B}(b, Ta) \text{ and } T^*(a, b) = \mathcal{B}(Ta, b);$$

and a natural $\rho : T \to S : \mathcal{A} \to \mathcal{B}$ gives rise to evident morphisms of modules

$$\rho_* : T_* \to S_* \text{ and } \rho^* : S^* \to T^*.$$

Since $\rho \mapsto \rho_*$ and $\rho \mapsto \rho^*$ are fully faithful by Yoneda, any of ρ, ρ_* and ρ^* determines the others. For a functor $T : \mathcal{A} \to \mathcal{B}$, and modules $g : \mathcal{B} \longrightarrow\!\!\!\!\!\circ\!\!\!\!\rightarrow \mathcal{C}$ and $h : \mathcal{D} \longrightarrow\!\!\!\!\!\circ\!\!\!\!\rightarrow \mathcal{B}$ where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are not necessarily small, substituting the values 6.10 into 6.6, 6.7 and 6.8 and using the Yoneda isomorphisms gives natural isomorphisms

6.11

$$(gT_*)(c,a) \cong g(c,Ta) \cong [\![T^*, g]\!](c,a)$$

6.12

$$(T^*h)(a,d) \cong h(Ta,d) \cong \{\!\{ T_*, h \}\!\}(a,d).$$

In particular for $T : \mathcal{A} \to \mathcal{B}$ and $S : \mathcal{B} \to \mathcal{C}$, one has natural isomorphisms

$$(ST)_* \cong S_*T_* \text{ and } (ST)^* \cong T^*S^*;$$

so that there are locally fully faithful embeddings of bicategories

6.13

$$(-)_* : \mathcal{V}\text{-}\mathbf{Cat} \to \mathcal{V}\text{-}\mathbf{Mod} \quad \text{and} \quad (-)^* : \mathcal{V}\text{-}\mathbf{Cat} \to \mathcal{V}\text{-}\mathbf{Mod}^{coop}.$$

We shall now discuss *adjunctions* in $\mathcal{V}$-**Mod**. Consider a functor $T : \mathcal{A} \to \mathcal{B}$ in $\mathcal{V}$-**Cat** and a module $k : \mathcal{D} \longrightarrow\!\!\!\!\!\circ\!\!\!\!\rightarrow \mathcal{B}$ in $\mathcal{V}$-**Mod**. Taking the isomorphism 6.12, $\{\!\{ T_*, k \}\!\} \cong T^*k$ shows that the right lifting of k through T_* is absolute so by 6.1, T_* has right adjoint T^*. In any bicategory left adjoints are often called *maps* and it is common to denote by f^* the right adjoint of f. This is then consistent with the above notation $T_* \dashv T^*$ above provided that we identify functor with their associated module via the fully faithful $(-)_*$ of 6.13. However the functors are not the only maps in $\mathcal{V}$-**Mod** that we shall characterise now.

Consider the following situation in $\mathcal{V}$-**Mod**

$$\begin{tikzcd}
 & \mathcal{D} \arrow[d, "h"] & \\
 & \mathcal{C} \arrow[dl, "g"'] \arrow[dr, "rlift(f,g)=\{f,g\}"] \arrow[d, phantom, "\Leftarrow"] & \\
\mathcal{B} & & \mathcal{A} \arrow[ll, "f"]
\end{tikzcd}$$

where the bottom triangle is a right lifting. The module h gives by restriction for each object d of $\mathcal{D}$ a functor $H_d : \mathcal{C}^{op} \to \mathcal{V}$. Similarly f gives for each a in $\mathcal{A}$ a functor $F_a : \mathcal{B}^{op} \to \mathcal{V}$ and the module g corresponds to a functor $G : \mathcal{B}^{op} \to [\mathcal{C}, \mathcal{V}]$. One may check straightforwardly that

6.14 *the composition with h respects the right lifting $\{\!\{ f, g \}\!\}$ of g through f.*

if and only if

6.15 *for all a in $\mathcal{A}$ and all d in $\mathcal{D}$, $H_d * - : [\mathcal{C}, \mathcal{V}] \to \mathcal{V}$ preserves the limit $\{F_a, G\}$.*

Writing now G' for the functor $\mathcal{C} \to [\mathcal{B}^{op}, \mathcal{V}]$ corresponding to g, one gets according to 5.3 that the statement above is also equivalent to the fact that

6.16 *for all a in $\mathcal{A}$ and all d in $\mathcal{D}$, the colimit $H_d * G'$ is preserved by $\{F_a, -\} : [\mathcal{B}^{op}, \mathcal{V}] \to \mathcal{V}$.*

At the light of these observations, we may now reformulate Proposition 6.1 for the case when the ambient bicategory is $\mathcal{V}$-**Mod**. Note that the assertion (ii) below is the direct translation by 6.14-6.16 of the fact that for any module $g : \mathcal{C} \longrightarrow\!\!\!\!\circ\!\!\rightarrow \mathcal{B}$ the right lifting $\{\!\{ f, g \}\!\}$ is respected by composition with any module $\mathcal{I} \longrightarrow\!\!\!\!\circ\!\!\rightarrow \mathcal{C}$.

Proposition 6.6 *Given small categories $\mathcal{A}$ and $\mathcal{B}$ and a functor $F : \mathcal{A} \to [\mathcal{B}^{op}, \mathcal{V}]$ that corresponds to the module $f : \mathcal{A} \longrightarrow\!\!\!\!\circ\!\!\rightarrow \mathcal{B}$, the following conditions on F or f are equivalent:*

- *(i) f (as a module) has a right adjoint f^*;*
- *(ii) for each a in $\mathcal{A}$, the representable functor $\{Fa, -\} = [\mathcal{B}^{op}, \mathcal{V}](Fa, -) : [\mathcal{B}^{op}, \mathcal{V}] \to \mathcal{V}$ is cocontinuous;*
- *(iii) for each pair a,c in $\mathcal{A}$, the functor $Fc * - : [\mathcal{B}, \mathcal{V}] \to \mathcal{V}$ preserves the limit $\{Fa, Y'\}$ of $Y' : \mathcal{B}^{op} \to [\mathcal{B}, \mathcal{V}]$ weighted by $Fa : \mathcal{B}^{op} \to \mathcal{V}$.*
- *(iv) for each pair a,c in $\mathcal{A}$, the representable functor $[\mathcal{B}^{op}, \mathcal{V}](Fa, -) : [\mathcal{B}^{op}, \mathcal{V}] \to \mathcal{V}$ preserves the colimit $Fc * Y$ of $Y : \mathcal{B} \to [\mathcal{B}^{op}, \mathcal{V}]$ weighted by $Fc : \mathcal{B}^{op} \to \mathcal{V}$ giving (since $Fc * Y \cong Fc$ by Yoneda) an isomorphism $Fc * [\mathcal{B}^{op}, \mathcal{V}](Fa, Y-) \cong [\mathcal{B}^{op}, \mathcal{V}](Fa, Fc)$.*

When these are satisfied, we have

6.17

$$f^*(a, b) \cong [\mathcal{B}^{op}, \mathcal{V}](Fa, Yb).$$

Proposition 6.6 gives in the particular case $\mathcal{A} = \mathcal{I}$:

Proposition 6.7 *Given a small category $\mathcal{B}$ and a presheaf $F : \mathcal{B}^{op} \to \mathcal{V}$ that corresponds to the module $f : \mathcal{I} \longrightarrow\!\!\!\!\circ\!\!\rightarrow B$, the following conditions on F or f are equivalent:*

- *(i) f (as a module) has a right adjoint f^*;*
- *(ii) the representable functor $\{F, -\} = [\mathcal{B}^{op}, \mathcal{V}](F, -) : [\mathcal{B}^{op}, \mathcal{V}] \to \mathcal{V}$ is cocontinuous;*
- *(iii) the functor $F * - : [\mathcal{B}, \mathcal{V}] \to \mathcal{V}$ preserves the limit $\{F, Y'\}$ of $Y' : \mathcal{B}^{op} \to [\mathcal{B}, \mathcal{V}]$ weighted by $F : \mathcal{B}^{op} \to \mathcal{V}$.*
- *(iv) the representable functor $\{F, -\} = [\mathcal{B}^{op}, \mathcal{V}](F, -) : [\mathcal{B}^{op}, \mathcal{V}] \to \mathcal{V}$ preserves the colimit $F * Y$ of $Y : \mathcal{B} \to [\mathcal{B}^{op}, \mathcal{V}]$ weighted by $F : \mathcal{B}^{op} \to \mathcal{V}$ giving (since $F * Y \cong F$ by Yoneda) an isomorphism $F * \{F, Y-\} \cong \{F, F\}$;*

When these are satisfied, we have

6.18

$$f^*(b) \cong [\mathcal{B}^{op}, \mathcal{V}](F, Yb).$$

Once again there are dual statements to 6.6 and 6.7 obtained by translating part of 6.3. In detail:

Proposition 6.8 *For any functor $G : \mathcal{A}^{op} \to [\mathcal{B}, \mathcal{V}]$ that corresponds to the module $g : \mathcal{B} \longrightarrow\!\!\!\!\circ\!\!\rightarrow \mathcal{A}$, the following conditions on G or g are equivalent:*

- *(i) g (as a module) has a left adjoint g_*;*
- *(ii) for each a, the representable functor $\{Ga, -\} = [\mathcal{B}, \mathcal{V}](Ga, -) : [\mathcal{B}, \mathcal{V}] \to \mathcal{V}$ is cocontinuous;*

- *(iii) for each pair a,c in $\mathcal{A}$, the functor $Gc * - : [\mathcal{B}^{op}, \mathcal{V}] \to \mathcal{V}$ preserves the limit $\{Ga, Y\}$ of $Y : \mathcal{B} \to [\mathcal{B}^{op}, \mathcal{V}]$ weighted by $Ga : \mathcal{B} \to \mathcal{V}$;*
- *(iv) for each pair a,c in $\mathcal{A}$, the representable functor $[\mathcal{B}, \mathcal{V}](Ga, -) : [\mathcal{B}, \mathcal{V}] \to \mathcal{V}$ preserves the colimit $Gc * Y'$ where $Y' : \mathcal{B}^{op} \to [\mathcal{B}, \mathcal{V}]$.*

When these are satisfied, we have

6.19

$$g_*(a, b) \cong [\mathcal{B}, \mathcal{V}](Ga, Y'b).$$

Eventually for the case $\mathcal{A} = \mathcal{I}$ one has

Proposition 6.9 *For any weight $G : \mathcal{B} \to \mathcal{V}$ that corresponds to the module $g : \mathcal{B} \longrightarrow\!\!\!\!\circ\!\!\rightarrow \mathcal{I}$, the following conditions on G or g are equivalent:*

- *(i) g (as a module) has a left adjoint g_*;*
- *(ii) the representable functor $\{G, -\} = [\mathcal{B}, \mathcal{V}](G, -) : [\mathcal{B}, \mathcal{V}] \to \mathcal{V}$ is cocontinuous;*
- *(iii) the functor $G * - : [\mathcal{B}^{op}, \mathcal{V}] \to \mathcal{V}$ preserves the limit $\{G, Y\}$ of $Y : \mathcal{B} \to [\mathcal{B}^{op}, \mathcal{V}]$ weighted by $G : \mathcal{B} \to \mathcal{V}$.*
- *(iv) the representable functor $\{G, -\} : [\mathcal{B}, \mathcal{V}] \to \mathcal{V}$ preserves the colimit $G * Y'$.*

When these are satisfied, we have

6.20

$$g_*(b) \cong [\mathcal{B}, \mathcal{V}](G, Y'b).$$

For any presheaves $F, H : \mathcal{B}^{op} \to \mathcal{V}$ on a small $\mathcal{B}$, that correspond respectively to modules $f, h : \mathcal{I} \longrightarrow\!\!\!\!\circ\!\!\rightarrow \mathcal{B}$, $[\mathcal{B}^{op}, \mathcal{V}](F, H)$ is the right lifting of h through f. Therefore as a consequence of Proposition 6.4:

Proposition 6.10 *Given a module $f : \mathcal{I} \longrightarrow\!\!\!\!\circ\!\!\rightarrow \mathcal{B}$ corresponding to a presheaf $F : \mathcal{B}^{op} \to \mathcal{V}$, if f has a right adjoint $g : \mathcal{B} \longrightarrow\!\!\!\!\circ\!\!\rightarrow \mathcal{I}$ then for any left module $h : \mathcal{I} \longrightarrow\!\!\!\!\circ\!\!\rightarrow \mathcal{B}$ corresponding to a presheaf $H : \mathcal{B}^{op} \to \mathcal{V}$*

$$[\mathcal{B}^{op}, \mathcal{V}](F, H) \cong g * h.$$

Proposition 6.11 *For any $G : \mathcal{B} \to \mathcal{V}$ with $\mathcal{B}$ small, that corresponds to a module $g : \mathcal{B} \longrightarrow\!\!\!\!\circ\!\!\rightarrow \mathcal{I}$, the following assertions are equivalent:*

- *(i) g is a right adjoint;*
- *(ii) $- * G : [\mathcal{B}^{op}, \mathcal{V}] \to \mathcal{V}$ is representable;*
- *(iii) $- * G : [\mathcal{B}^{op}, \mathcal{V}] \to \mathcal{V}$ is continuous.*

PROOF:$(i) \Rightarrow (ii)$. If g has left adjoint $f : I \longrightarrow\!\!\!\!\circ\!\!\rightarrow \mathcal{V}$ that corresponds to a functor $F : \mathcal{B}^{op} \to \mathcal{V}$ then $\{F, -\} \cong - * G \cong G * -$ by 6.10. $G * -$ being representable is therefore continuous.
$(ii) \Rightarrow (iii)$ well-known.
$(iii) \Rightarrow (i)$. $G * - = - * G : [\mathcal{B}^{op}, \mathcal{V}] \to \mathcal{V}$ being continuous preserves in particular the limit $\{G, Y\}$ of $Y : \mathcal{B} \to [\mathcal{B}^{op}, \mathcal{V}]$ weighted by G and by Proposition 6.9 this equally asserts that g is a right

adjoint. ∎

In [Str83] Street characterised right adjoints modules $g : \mathcal{B} \longrightarrow\!\!\!\!\circ\!\!\rightarrow \mathcal{I}$ with $\mathcal{B}$ small as the ones for which *the corresponding presheaf* $G : \mathcal{B} \to \mathcal{V}$ *is a weight of absolute colimits.* A similar argument to the one used for the proof $(i) \Rightarrow (ii)$ above, yields half of the result. Actually if G is a weight of absolute colimits then the colimit $G * Y'$ is certainly preserved by $\{G, -\}$ and then by 6.9, g is right adjoint.

An object a of a category $\mathcal{A}$ (especially of a cocomplete $\mathcal{A}$) is said in [Kel82] to be *small projective* when the representable $\mathcal{A}(a, -)$ preserves all small colimits. Of course the representables in $[\mathcal{B}^{op}, \mathcal{V}]$ are among the small projectives: for $[\mathcal{B}^{op}, \mathcal{V}](Yb, -)$ is isomorphic to the evaluation E_b which preserves all colimits. Proposition 6.6 above asserts in particular that $f : \mathcal{A} \longrightarrow\!\!\!\!\circ\!\!\rightarrow \mathcal{B}$ has a right adjoint (or is a map) precisely when each $f(-, a) : \mathcal{B}^{op} \to \mathcal{V}$ is a small projective in $[\mathcal{B}^{op}, \mathcal{V}]$. Note the special case of a functor $T : \mathcal{A} \to \mathcal{B}$ where $T_*(-, a) = \mathcal{B}(-, Ta)$ and the representable $\mathcal{B}(-, Ta)$ is a small projective. There is another way of looking at the small projectives in the category $[\mathcal{B}^{op}, \mathcal{V}]$ when $\mathcal{B}$ is small: an object ϕ of $[\mathcal{B}^{op}, \mathcal{V}]$ is a weight, and $[\mathcal{B}^{op}, \mathcal{V}](\phi, \psi)$ is just the ϕ-weighted limit of $\psi : \mathcal{B}^{op} \to \mathcal{V}$. To say, therefore that ϕ is a small projective is equally to say that ϕ-limits commute in $\mathcal{V}$ with all small colimits. As mentioned earlier we shall write $\mathcal{Q}$ for the class of small projective weights. In the language of the previous section, $\mathcal{Q} = \mathcal{P}^-$, and according to 6.11 also $\mathcal{Q} = \mathcal{P}^+$. For any small $\mathcal{B}$, as $\mathcal{Q}$ is saturated, the closure $\mathcal{Q}(\mathcal{B})$ of $\mathcal{B}$ in $[\mathcal{B}^{op}, \mathcal{V}]$ under $\mathcal{Q}$-colimits is $\mathcal{Q}[\mathcal{B}]$, as $\mathcal{Q}$ is equally the class of weights corresponding in the sense of 6.6 to left modules, $\mathcal{Q}(\mathcal{B})$ is the *Cauchy-completion* of $\mathcal{B}$ in the sense of [Law73].

When the category $\mathcal{B}$ is not small, we can still consider the closure $\mathcal{Q}(\mathcal{B})$ of $\mathcal{B}$ in $[\mathcal{B}^{op}, \mathcal{V}]$ (or equally in $\mathcal{P}(\mathcal{B})$) under $\mathcal{Q}$-colimits. On the other hand, we can consider the full subcategory of $\mathcal{P}(\mathcal{B})$ given by its small projectives that is $\mathcal{P}(\mathcal{B})_{\mathcal{P}}$. By Proposition 5.3, $\mathcal{P}(\mathcal{B})_{\mathcal{P}}$ is closed in $\mathcal{P}(\mathcal{B})$ under $\mathcal{Q}$-colimits. Since the representables lie in $\mathcal{P}(\mathcal{B})_{\mathcal{P}}$, it follows that $\mathcal{Q}(\mathcal{B}) \subseteq \mathcal{P}(\mathcal{B})_{\mathcal{P}}$. When $\mathcal{V} = Set$ the converse is true, so that $\mathcal{Q}(\mathcal{B})$ consists even for large $\mathcal{B}$ of the small projectives in $\mathcal{P}(\mathcal{B})$. To check this last point one can adapt the proof of [Kel82] to show that any accessible $F : \mathcal{B}^{op} \to Set$ such that $[\mathcal{B}^{op}, Set](F, -)$ preserves small colimits is a retract of representables, thus a $\mathcal{Q} = \mathcal{P}^+$-weighted colimit of representables, and thus in $\mathcal{Q}(\mathcal{B})$.

For $\mathcal{B}$ small, the operations 6.18 and 6.20 above define actually two functors:

6.21

$$[\mathcal{B}^{op}, \mathcal{V}](-, Y) : [\mathcal{B}^{op}, \mathcal{V}] \to [\mathcal{B}, \mathcal{V}]^{op}$$

and

6.22

$$[\mathcal{B}, \mathcal{V}](-, Y') : [\mathcal{B}, \mathcal{V}] \to [\mathcal{B}^{op}, \mathcal{V}]^{op}.$$

Letting $\mathbf{B} = \mathcal{V}\text{-}\mathbf{Mod}$, $A = \mathcal{I}$ and $h : C \to B = 1_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ in 6.5 we get that their underlying Set-functors define an adjoint pair:

6.23

$$\mathcal{V}\text{-}\mathbf{Cat}(\mathcal{B}^{op}, \mathcal{V}) \rightharpoonup \mathcal{V}\text{-}\mathbf{Cat}(\mathcal{B}, \mathcal{V})^{op}.$$

The adjunction above goes back in the case $\mathcal{V} = Set$ to Isbell. Let us call it the *Isbell adjunction.* Also by Remark 6.5, Propositions 6.7 and 6.9,

Proposition 6.12 *The Isbell adjunction as in 6.23 restricts to an equivalence*

$$\mathcal{Q}(\mathcal{B}^{op})^{op} \cong \mathcal{Q}(\mathcal{B})$$

where $\mathcal{Q}(\mathcal{B})$ consists of the small projectives in $[\mathcal{B}^{op}, \mathcal{V}]$.

We shall further study more in detail the Isbell adjunction and shows that it lifts to a $\mathcal{V}$-adjunction.

Recall that for a small $\mathcal{B}$, the functor category $[\mathcal{B}^{op}, \mathcal{V}]$ is the free cocomplete category on $\mathcal{B}$, in the sense that, for each cocomplete $\mathcal{C}$, we have an adjoint equivalence of categories:

6.24

$$\mathbf{Cocts}[[\mathcal{B}^{op}, \mathcal{V}], \mathcal{C}] \underset{Lan_Y}{\overset{[Y,1]}{\rightleftarrows}} [\mathcal{B}, \mathcal{C}]$$

where $[Y, 1]$ sends the cocontinuous $S : [\mathcal{B}^{op}, \mathcal{V}] \to \mathcal{C}$ to its restriction $SY : \mathcal{B} \to \mathcal{C}$, while Lan_Y sends $G : \mathcal{B} \to \mathcal{C}$ to $Lan_Y(G) = - * G : [\mathcal{B}^{op}, \mathcal{V}] \to \mathcal{C}$ which in fact has the right adjoint $\tilde{G} : \mathcal{C} \to [\mathcal{B}^{op}, \mathcal{V}]$ where

6.25

$$\tilde{G}c = \mathcal{C}(G-, c)$$

as in the diagram

6.26

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, shift left, "\tilde{G}"] & & {[\mathcal{B}^{op}, \mathcal{V}]} \arrow[ll, shift left, "-*G"] \\
& \mathcal{B} \arrow[ul, "G"] \arrow[ur, "Y"'] &
\end{tikzcd}$$

Recall too that there is a dual version involving $\hat{G} : \mathcal{C}^{op} \to [\mathcal{B}, \mathcal{V}]$ where

6.27

$$\hat{G}c = \mathcal{C}(c, G-).$$

Now consider the case where G is the Yoneda embedding

6.28 $y = Y'^{op} : \mathcal{B} \to [\mathcal{B}, \mathcal{V}]^{op}$

whereupon 6.26 becomes

6.29

$$\begin{tikzcd}
{[\mathcal{B}, \mathcal{V}]^{op}} \arrow[rr, shift left, "\tilde{y}"] & & {[\mathcal{B}^{op}, \mathcal{V}]} \arrow[ll, shift left, "-*y"] \\
& \mathcal{B} \arrow[ul, "y"] \arrow[ur, "Y"'] &
\end{tikzcd}$$

Here

6.30

$$\tilde{y}(\varphi) = [\mathcal{B}, \mathcal{V}]^{op}(y-, \varphi) = [\mathcal{B}, \mathcal{V}](\varphi, Y'-),$$

so that

6.31

$$\tilde{y}(\varphi) = \psi \text{ where } \psi(b) = [\mathcal{B}, \mathcal{V}](\varphi, Y'b),$$

as in 6.22. Again, since $\psi * y$ is the colimit in $[\mathcal{B}, \mathcal{V}]^{op}$ of $y : \mathcal{B} \to [\mathcal{B}, \mathcal{V}]^{op}$ weighted by $\psi : \mathcal{B}^{op} \to \mathcal{V}$, it is equally the limit $\{\psi, Y'\}$ in $[\mathcal{B}, \mathcal{V}]$ of $y^{op} = Y' : \mathcal{B}^{op} \to [\mathcal{B}, \mathcal{V}]$ weighted by $\psi : \mathcal{B}^{op} \to \mathcal{V}$; and since limits in $[\mathcal{B}, \mathcal{V}]$ are formed pointwise from the limits in $\mathcal{V}$, its value $(\psi * y)b = \{\psi, Y'\}b$ at $b \in \mathcal{B}$ is given by

6.32

$$(\psi * y)(b) = \{\psi, Yb\} = [\mathcal{B}^{op}, \mathcal{V}](\psi, Yb).$$

An immediate consequence of this, using 6.27, is the observation that

6.33

$$- * y = \hat{Y}^{op} : [\mathcal{B}^{op}, \mathcal{V}] \to [\mathcal{B}, \mathcal{V}]^{op}$$

which may be combined with the trivially true

6.34

$$\tilde{y} = (Y'^{op})^{\sim} : [\mathcal{B}, \mathcal{V}]^{op} \to [\mathcal{B}^{op}, \mathcal{V}]$$

in order to write the adjunction in 6.29 as

6.35

$$\hat{Y}^{op} \dashv (Y'^{op})^{\sim}$$

Omitting the b in 6.32 gives us a form which we may compose with 6.30, namely

6.36

$$\psi * y = [\mathcal{B}^{op}, \mathcal{V}](\psi, Y-)$$

so that - compare 6.31 - we have

6.37

$$\psi * y = \varphi \text{ where } \varphi(b) = [\mathcal{B}^{op}, \mathcal{V}](\psi, Yb),$$

as in 6.21. Now we observe that there is an alternative analysis of this equivalence between the ψ's in $\mathcal{Q}(\mathcal{B})$ and the φ's in $\mathcal{Q}(\mathcal{B}^{op})$. Accordingly to the general picture in 6.26 referring to $[\mathcal{B}^{op}, \mathcal{V}]$ as the free cocompletion of $\mathcal{B}$, when $\psi \in \mathcal{Q}(\mathcal{B})$, the cocontinuous $[\mathcal{B}^{op}, \mathcal{V}](\psi, -) : [\mathcal{B}^{op}, V] \to \mathcal{V}$ is given by:

6.38

$$\begin{tikzcd}
\mathcal{V} \arrow[rr, shift left, "\tilde{\varphi}"] & & {[\mathcal{B}^{op}, \mathcal{V}]} \arrow[ll, shift left, "\{\psi,-\}=-*\varphi"] \\
& \mathcal{B} \arrow[ul, "\varphi=[\mathcal{B}^{op},\mathcal{V}](\psi,Y-)"] \arrow[ur, "Y"'] &
\end{tikzcd}$$

In more detail, ψ lies in $\mathcal{Q}(\mathcal{B})$ precisely when $[\mathcal{B}^{op}, \mathcal{V}](\psi, -)$ is cocontinuous (or equally left adjoint); and then $[\mathcal{B}^{op}, \mathcal{V}](\psi, -) = - * \varphi$ where $\varphi = [\mathcal{B}^{op}, \mathcal{V}](\psi, Y-) = \psi * y$. On the other hand, if we start with $\varphi : \mathcal{B} \to \mathcal{V}$ and ask whether it is of the form $[\mathcal{B}^{op}, \mathcal{V}](\psi, Y-)$ for some $\psi \in \mathcal{Q}(\mathcal{B})$, we observe that it so precisely when $- * \varphi : [\mathcal{B}^{op}, \mathcal{V}] \to \mathcal{V}$ is *representable*.

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METRIC SPACES, GENERALIZED LOGIC, AND CLOSED CATEGORIES

(Conferenza tenuta il 30 marzo 1973) *

SUNTO. — In questo articolo viene rigorosamente sviluppata l'analogia fra $\mathrm{dist}(a, b) + \mathrm{dist}(b, c) \geq \mathrm{dist}(a, c)$ e $\hom(A, B) \otimes \hom(B, C) \to \hom(A, C)$, giungendo a numerosi risultati generali sugli spazi metrici, come conseguenza di una « logica pura generalizzata » i cui « valori di verità » sono scelti in una arbitraria categoria chiusa.

INTRODUCTION.

It is a banality that all mathematical structures of a given kind constitute the objects of a category; the sequence: elements/structures/categories thus has led some people to attempt to characterize the philosophical significance of the theory of categories as that of a « third level of abstraction ». But the theory of categories actually penetrates much more deeply than that attempted characterizaion would suggest toward summing up the essence of mathematics. The kinds of structures which actually arise in the practice of geometry and analysis are far from being « arbitrary », and indeed in this paper we will investigate a particular case of the way in which logic should be specialized to take account of this experience of non-arbitrariness, as concentrated in the thesis that *fundamental* structures are themselves categories. Two cases of this thesis have been known for 30 years; an ordered set (often called poset) is a category in which for any ordered pair of objects there is at most one morphism from the first to the second, while a group is a category in which there is just one object and in which every morphism is a isomorphism. That the thesis has non-vacuous implications for these two cases follows from the facts that when the general idea

* Pervenuta in Redazione il 16 novembre 1973.

of functor between categories is restricted to the special categories (that is to the posets, respectively to the groups) it agrees with the correct idea of morphism between these fundamental structures (i.e. with order-preserving map, respectively group homomorphism) and that a functor from one of the special categories to some category (e.g. that of vector spaces) is itself an important structure (a direct system of vector spaces, respectively a linear group-representation; moreover the correct morphism between *these* structures are then just the natural transformations). Two further cases of the thesis have been developed in the past 10 years: functorial semantics, in which categories with special properties are identified as theories and special functors as interpretations or models, and the theory of topoi, in which certain categories correspond to (usefully generalized) topological spaces.

By taking account of a certain natural generalization of category theory within itself, namely the consideration of strong categories whose hom-functors take their values in a given « closed category » $\mathcal{V}$ (not necessarily in the category $\mathcal{S}$ of abstract sets), we will show below that it is possible to regard a metric space as a (strong) category and that moreover by specializing the constructions and theorems of general category theory we can deduce a large part of *general* metric space theory. The theory of closed categories (and of strong categories valued in them) was originally developed to deal with more complicated examples such as compactly generated topological spaces, Banach spaces, differential graded modules, etc., and moreover some of the publications on the subject seem forbiddingly technical to the beginner. I hope that this article can also be read as an introduction to closed categories on the basis of the guiding example of metric spaces considered as strong categories valued in the closed category of nonnegative real quantities.

Since closed categories are just what is suficient to have a reasonable theory of strong categories, we consider some examples of the latter first in order to bring out the elementary nature of the analogy on which the present work is based. In a *metric space* X, we will denote by $X(a, b)$ the non-negative real quantity (we will allow the value ∞) of X-distance from the point a to the point b. Then the laws satisfied by X are greater-than relations

$$\begin{aligned} X(a, b) + X(b, c) &\geq X(a, c) \\ 0 &\geq X(a, a) . \end{aligned}$$

(see below for remarks on the possible non-symmetry of the metric).

In a *category* X, we will denote by $X(a, b)$ the abstract set of X-morphisms from the object a to the object b. Then the composition law and specification of identity morphisms for X are mappings

$$\begin{array}{rcl} X(a, b) \times X(b, c) & \longrightarrow & X(a, c) \\ 1 & \longrightarrow & X(a, a) \end{array}$$

(which are subject to associativity and unity axioms which may be expressed as commutative diagrams of mappings betwen abstract sets, using elementary properties of the cartesian product $\times$ of abstract sets and of the one-element set 1). In a *poset* X, we will denote by $X(a, b)$ the truth-value of the X-dominance of the element a over the element b. Then the transitivity and reflexivity laws for X are entailments

$$\begin{array}{rcl} X(a, b) \wedge X(b, c) & \vdash & X(a, c) \\ \text{true} & \vdash & X(a, a)\,. \end{array}$$

If K is a commutative ring, then in a K-*additive category* X we will denote by $X(a, b)$ the K-module of X-morphisms from the object a to the object b. Then the composition and identity laws for X are K-linear mappings

$$\begin{array}{rcl} X(a, b) \otimes X(b, c) & \longrightarrow & X(a, c) \\ K & \longrightarrow & X(a, a) \end{array}$$

(again subject to associativity and unity axioms which may be expressed by commutative diagrams of K-linear mappings of K-modules, using elementary properties of the K-tensor product $\otimes$ of K-modules and of the K-module K). Thus we are led to consider that a greater-than-or-equal-to relation between nonnegative real quantities is analogous to a K-linear mapping between K-modules, since both are morphisms of possible hom-values for categories X, but in two different closed categories $\mathcal{V}$. Similarly, the sum of quantities is analogous to the tensor product of modules, both because they play the same role in the structure of a $\mathcal{V}$-valued category and also because they satisfy the same « elementary properties » (within $\mathcal{V}$ itself) of functoriality (i.e. monotonicity in the case of quantities), of associativity and commutativity up to $\mathcal{V}$-isomorphism (i.e. up to equality in the case of quantities), and of having a unit object K (the zero quantity). We have the table

X	hom-values for X	composition law and identity law for X	domain of composition law for X	domain of identity law for X
metric space	nonnegative real quantities	$\geqq$	sum	zero
category	abstract sets	mapping	cartesian product	one element set
poset	truth values	entailment	conjunction	true
$\mathcal{V}$-valued category	objects in $\mathcal{V}$	morphism in $\mathcal{V}$	« tensor » product in $\mathcal{V}$	unit object K for tensor product in $\mathcal{V}$

The associativity and unity axioms which the composition law and identity law of a $\mathcal{V}$-based category X must satisfy are automatic in the case of metric spaces or posets since *all* diagrams in $\mathcal{V}$ commute if $\mathcal{V} =$ reals or $\mathcal{V} =$ truth-values.

None of our results in this paper will depend on the additional Frechet axioms:

$$\begin{array}{llll} \text{if} & X(a, b) & = & 0 \text{ then } a = b \\ & X(a, b) & < & \infty \\ & X(a, b) & = & X(b, a)\,. \end{array}$$

The first of these is not very natural from the categorical viewpoint since it corresponds to requiring that isomorphic objects are equal; passage to the quotient can be avoided (as it *must* be for ordinary categories) by employing *equivalence* (broader than isomorphism) of strong categories. Allowing ∞ among the quantities is precisely analogous to including the empty set among abstract sets, and it is done for similar reasons of completeness; a metric space can be analyzed as a structured system of metric spaces with finite distances by considering the equivalence relation defined by « $X(a, b) < \infty$ and $X(b, a) < \infty$ ». The non symmetry is the more serious generalization, and moreover occurs in many naturally arising examples, such as $X(a, b) =$ work required to get from a to b in mountainous region X. Also within analysis itself a naturally arising metric is often

non-symmetric but traditionally symmetrized by one of the two procedures

$$X(a, b) + X(b, a)$$
$$\max(X(a, b) \quad , \quad X(b, a))$$

which in fact could be applied to any metric; we mention three common examples where this nonsymmetry exists. If M is any Boolean algebra equipped with an outer measure, then

$$M(a, b) \underset{\text{def}}{=} M(b - a)$$

(where $b - a = b \cap a'$ in the Boolean algebra and $M(c)$ denotes the measure of c) defines a metric space in which

$$0 \geq M(a, b) \quad \text{iff} \quad a \supseteq b$$

almost everywhere.

(For more about this example see the section below on closed functors).

If K is any convex set we may define a metric on it by

$$K(a, b) = \inf_{f: a \to b} \{ - \log(\alpha_f) \}$$

where $f: a \to b$ means that $f \in K$ with a on the open segment from f to b, with α_f then denoting the $0 < \alpha < 1$ with $a = (1 - \alpha) f + \alpha b$. (The proof of the triangle inequality follows from the fact that K is actually a « normed category », since if $g: b \to c$ we can define $fg: a \to c$ by

$$fg = \frac{1 - \alpha_f}{1 - \alpha_f \alpha_g} f + \frac{\alpha_f (1 - \alpha_g)}{1 - \alpha_f \alpha_g} g,$$

note that this is associative and that $\alpha_{fg} = \alpha_f \alpha_g$, and adjoin identities formally). The notion of « normed category » can also be related to the (nonsymmetric) Hausdorff metric

$$2^X(A, B) = \sup_{a \in A} \inf_{b \in B} X(a, b)$$

for subsets A, B of a metric space X: let $f: A \to B$ mean that f is any mapping from A to B and define

$$|f| = \sup_{a \in A} X(a, af);$$

then by the axiom of choice

$$2^X(A, B) = \inf_{f: A \to B} |f|$$

and the fundamental property of a normed category

$$|f| + |g| \geq |fg|$$

leads to a proof of the triangle inequality. We will leave as an exercise for the reader to define a closed category $\mathcal{S}(\mathbf{R})$ such that « normed categories » are just $\mathcal{S}(\mathbf{R})$-valued categories and a « closed functor » $\inf: \mathcal{S}(\mathbf{R}) \to \mathbf{R}$ which induces the passage from any « normed category » to a metric space with the same objects. Another approach to the Hausdorff metric will be evident from the discussion below of the comprehension scheme. The canonical symmetrization procedure actually applies to categories valued in any given closed category $\mathcal{V}$; in general, symmetry of a $\mathcal{V}$-valued category X has to mean (not a property but) a given structure consisting of $\mathcal{V}$-isomorphisms

$$\sigma_{ab}: X(a, b) \to X(b, a)$$

subject to suitable coherence axioms, and the first canonical procedure (specializing to the sum in the case of a metric space) is

$$\text{sym}\ (X)(a, b) = X(a, b) \otimes X(b, a).$$

We still have not, in this introduction, touched on the property of closed categories for which they are called « closed ». Basically, closed categories are closed with respect to the operation of forming the hom of two objects, so that $\mathcal{V}$ itself is a fundamental example of a $\mathcal{V}$-category. The internal Hom (a, c) in $\mathcal{V}$ is related to the internal tensor product in $\mathcal{V}$ by adjointness so that there is a natural one-to-one correspondence between the $\mathcal{V}$-morphisms

$$b \to \text{Hom}\ (a, c)$$

and the $\mathcal{V}$-morphisms

$$a \otimes b \to c.$$

Thus in the closed category $\mathbf{R} = [0; \infty]$ in which metric spaces are valued

$$\text{Hom}\ (a, c) = \begin{cases} c - a & \text{if} \quad c \geq a \\ 0 & \text{if} \quad a \geq c \end{cases}$$

so that the Hom-$\otimes$ adjointness reduces (if we denote by « minus » this truncated subtraction) to the if-and-only-if

$$\frac{b \geq c - a}{a + b \geq c},$$

whereas in the closed category $\mathcal{S}$ in which ordinary categories are valued, it reduces to the rule of lambda conversion

$$\frac{B \to C^A}{A \times B \to C}$$

and in the closed category **2** in which posets are valued, it reduces essentially to modus ponens and the « deduction theorem »

$$\frac{\beta \vdash \alpha \Rightarrow \gamma}{\alpha \wedge \beta \vdash \gamma}$$

i.e. the internal Hom for truth-values is implication. We will also assume that our closed categories $\mathcal{V}$ have cartesian products and coproducts over arbitrary index sets as well as equalizers and coequalizers, i.e. that they are complete and cocomplete. From general properties of adjoint functors it follows that $\otimes$ preserves direct limits in each variable separately, while

$$\mathrm{Hom}\,(\varinjlim a_i, c) \cong \varprojlim \mathrm{Hom}\,(a_i, c)$$

$$\mathrm{Hom}\,(a, \varprojlim c_j) \cong \varprojlim \mathrm{Hom}\,(a, c_j)$$

For example in $\mathbf{R}$, $\varprojlim$ means sup and $\varinjlim$ means inf; in particular 0 is the empty $\varprojlim$ and ∞ is the empty $\varinjlim$ so that

$$\begin{aligned} a + \infty &= \infty \\ c - \infty &= 0 \\ 0 - a &= 0\,. \end{aligned}$$

This completeness of $\mathcal{V}$ itself will be necessary for most of our general constructions. We will not consider (categorical) completeness of $\mathcal{V}$-valued categories; but on the other hand we will see that completeness in the *Cauchy* sense does have a meaning for categories valued in any closed $\mathcal{V}$.

The general constructions of functor categories, free categories, left and right Kan extensions, and « discrete » fibrations reduce in the very special case $\mathcal{V} = \mathbf{2}$ to higher types, transitive closure of a relation, existential and universal quantification, and the principle of set abstraction. Like the operation of implication mentioned above, the position of all these constructions in the general scheme as well as their fundamental properties of transformation are uniquely determined by adjointness. Since logic signifies formal relationships which are general in character, we may more precisely identify logic with this scheme of interlocking adjoints and then observe that all of logic applies *directly* to structures valued in an arbitrary closed category $\mathcal{V}$ (not only to structures valued in truth-values). For example, in quantitative logic (the case $\mathcal{V} = \mathbf{R}$ with which we will be mainly concerned in this paper) the isometric embedding of a metric space into a space of functions with sup metric is the application of the exactly same principle of logic which in the case $\mathcal{V} = \mathbf{2}$ gives Dedekind's representation of a poset by order ideals, and the upper and lower integrals of a real function are precisely cases of the generalized universal and existential quantification. This vast generalization is quite compatible with the specialization of logic called for in the first paragraph: although for any given $\mathcal{V}$ we could consider « arbitrary » $\mathcal{V}$ -valued structures, there is one type of such structure which is of first importance, namely for $\mathcal{V}$ respectively truth-values, quantities, abstract sets, abelian groups, the structure of respectively poset, metric space, category, additive category (a very natural generalization of ring) is the generally useful first approximation possible with $\mathcal{V}$-valued logic for analyzing various problems; it even seems that there is a natural second approximation, namely the structure of a « rigidly $\mathcal{V}$-closed $\mathcal{V}$-category » which in the four cases mentioned specializes *roughly* to partially ordered abelian group, normed abelian group, rigidly closed category, and (in the additive case) to a common generalization of the category of locally free modules on an algebraic space and the category of finite-dimensional representations of an algebraic group. Detailed discussion of this second approximation awaits further investigation, as does the extension of the results of this paper to an arbitrary base topos, i.e. the extension from the *constant* abstract sets which we consider here to continuously *variable* sets as «sets» of points for metric spaces, as index «sets» for products, etc.

Of course the really deep results in a subject depend very much on the particularity of that subject and the results we offer here in the field of metric spaces, taken individually, will justly appear

shallow to those with any experience. Indeed for me the surprising aspect was that methods originally devised to deal with quite different fields of algebra and geometry could yield any significant known theorems at all (for example the known theorem on extension of Lipschitz maps in section three). But there are many particularities, for example the special role of quadratic metrics, which I do not see how could be a result of « generalized logic ».

1. - Closed categories, strong categories, strong functors, closed functors.

In this paper we use the term « closed category » as short for « bicomplete symmetric monoidal closed category ». That is, a closed category $\mathcal{V}$ has equalizers, coequalizers, set-indexed products and coproducts plus a « monoidal » structure, which is symmetric and closed. A monoidal structure is a given functor

$$\mathcal{V} \times \mathcal{V} \xrightarrow{\otimes} \mathcal{V}$$

which is symmetric

$$u \otimes v \cong v \otimes u$$

and associative

$$u \otimes (v \otimes w) \cong (u \otimes v) \otimes w$$

and has a unit object k satisfying

$$k \otimes v \cong v \cong v \otimes k\,;$$

more precisely, the symmetry, associativity, and unitary isomorphisms are in general required to be given, natural, and «coherent», but since these conditions are automatic in the particular examples we consider, we do not emphasize them (MacLane). That the monoidal structure is closed means that we are further given a functor

$$\mathcal{V}^{op} \times \mathcal{V} \xrightarrow{\mathrm{Hom}} \mathcal{V}$$

and two natural transformations

$$u \xrightarrow{\lambda} \mathrm{Hom}\,(a, a \otimes u)$$

$$a \otimes \mathrm{Hom}\,(a, v) \xrightarrow{\epsilon} v$$

such that the two processses

$$a \otimes u \xrightarrow{f} v \rightsquigarrow u \xrightarrow{\lambda} \mathrm{Hom}\,(a, a \otimes u) \xrightarrow{\mathrm{Hom}\,(a,f)} \mathrm{Hom}\,(a, v)$$

$$u \xrightarrow{g} \mathrm{Hom}\,(a, v) \rightsquigarrow a \otimes u \xrightarrow{a \otimes g} a \otimes \mathrm{Hom}\,(a, v) \xrightarrow{\in} v$$

are inverse bijections,

$$\downarrow \uparrow \quad \frac{a \otimes u \to v}{u \to \mathrm{Hom}\,(a, v)}\,.$$

Thus in particular there is a natural bijection

$$\frac{a \to v}{k \to \mathrm{Hom}\,(a, v)}$$

between $\mathcal{V}$-morphism $a \to v$ and $\mathcal{V}$-morphisms from the unit object to the object $\mathrm{Hom}(a, v)$ for any two objects a, v of $\mathcal{V}$, in terms of which the natural transformation $\in$ behaves as an « evaluation » morphism, and two successive applications of appropriate instances of evaluation corresponds to a $\mathcal{V}$-morphism

$$\mathrm{Hom}\,(a, b) \otimes \mathrm{Hom}\,(b, c) \to \mathrm{Hom}\,(a, c)$$

which represents a « strong » form of composition. There is moreover a natural double-dualization morphism

$$u \to \mathrm{Hom}\,(\mathrm{Hom}\,(u, b), b)$$

for any fixed object b.

The simplest non-trivial example **2** of a closed category, which serves as the values for classical logic, has two objects false and $k =$ true and three morphisms

$$\begin{aligned} \text{false} &\vdash \text{false} \\ \text{false} &\vdash \text{true} \\ \text{true} &\vdash \text{true} \end{aligned}$$

with conjunction and implication as the tensor and Hom. Conjunction is of course symmetric and associative and since

$$u \vdash a \Longrightarrow (a \wedge u)$$

$$a \wedge (a \Longrightarrow v) \vdash v$$

always, we have the bijection

$$\frac{a \wedge u \vdash v}{u \vdash (a \Longrightarrow v)}$$

i.e. for any assignments of false, true to the three variables a, u, v either both or neither of these entailments exist. In particular

$$\frac{a \vdash v}{\text{true} \vdash a \Longrightarrow v}$$

and

$$(a \Longrightarrow b) \wedge (b \Longrightarrow c) \vdash a \Longrightarrow c$$

and

$$u \vdash (u \Longrightarrow b) \Longrightarrow b \,.$$

A somewhat more general and « dual » example is as follows: let $\mathcal{M}$ be any system of subsets of a given finite set which contains the empty subset $k = \emptyset$ and which is closed with respect to finite unions and with respect to set-theoretic difference; for $a, v \in \mathcal{M}$ let $a \to v$ mean that $a \supseteq v$, and let $a \otimes u$ mean $a \cup u$. Then $\mathcal{M}$ is a closed category since

$$\frac{a \cup u \supseteq v}{u \supseteq v - a}$$

(Even if $\mathcal{M}$ is not finite, our condition of bicompleteness can often be achieved by considering e.g. measurable sets *modulo* null sets).

Our central example $\mathbf{R}$ has as objects all non-negative real quantities (including ∞), as morphisms $a \to v$ the greater-than-or-equal-to relations $a \geq v$, and as « tensor » $a \otimes u = a + u$ the sum of quantities. Then Hom is forced to be truncated subtraction

$$\mathrm{Hom}(a, v) = \begin{cases} v - a & v \geq a \\ 0 & a \geq v \end{cases}$$

(denoted simply as subtraction) where in particular

$$\begin{aligned} \infty - \infty &= 0 \\ \infty - a &= \infty \quad \textit{if} \quad a \neq \infty \\ v - \infty &= 0 \end{aligned}$$

(bicompleteness holds since

$$\prod_{i \in I} v_i = \sup_{i \in I} v_i$$

$$\sum_{i \in I} a_i = \inf_{i \in I} a_i$$

and equalizers and coequalizers are trivial). We have in particular

$$\frac{a \geq v}{0 \geq v - a}$$

$$(b - a) + (c - b) \geq c - a$$

$$u \geq b - (b - u) .$$

The « original » example of a closed category is the category $\mathcal{S}$ of abstract sets and mappings with the (unique) closed structure in which tensor means cartesian product, $k = 1$ is a one-element set, and $\mathrm{Hom}(a, v)$ is the (abstract) set of (indices for) all the mappings $a \to v$.

Given a closed category $\mathcal{V}$, a strong category valued in $\mathcal{V}$, or simply a $\mathcal{V}$-category X is any structure consisting of a specified set of X-objects $a, b, c \ldots$ together with the assignment of an object $X(a, b)$ of $\mathcal{V}$ to every ordered pair of X-objects $\langle a, b \rangle$, the assignment of a $\mathcal{V}$-morphism

$$X(a, b) \otimes X(b, c) \xrightarrow{\mu_{abc}} X(a, c)$$

to every ordered triple $\langle a, b, c \rangle$ of X-objects, and the assignment of a $\mathcal{V}$-morphism

$$k \xrightarrow{\eta_a} X(a, a)$$

to every X-object a, subject to the conditions that the following diagrams in $\mathcal{V}$ always commute

$$\begin{tikzcd}
 & X(a,b) \otimes (X(b,c) \otimes X(c,d)) \arrow[r, "X(a,b) \otimes \mu_{bcd}"] \arrow[d, "\mathcal{V}\text{ — assoc } \cong"'] & X(a,b) \otimes X(b,d) \arrow[dd, "\mu_{abd}"] \\
 & (X(a,b) \otimes X(b,c)) \otimes X(c,d) \arrow[d, "\mu_{abc} \otimes X(c,d)"'] & \\
 & X(b,c) \otimes X(c,d) \arrow[r, "\mu_{bcd}"'] & X(a,d)
\end{tikzcd}$$

$$\begin{tikzcd}
X(a,b) \arrow[r, "\mathcal{V}\text{-unitary}\cong"] \arrow[d, "\mathcal{V}\text{-unitary}\cong"'] \arrow[ddr, "\mathrm{id}"] & k \otimes X(a,b) \arrow[r, "\eta_a \otimes X(a,b)"] & X(a,a) \otimes X(a,b) \arrow[dd, "\mu_{abd}"] \\
X(a,b) \otimes k \arrow[d, "X(a,b) \otimes \eta_b"'] & & \\
X(a,b) \otimes X(b,b) \arrow[rr, "\mu_{abb}"'] & & X(a,b)
\end{tikzcd}$$

Thus an $\mathcal{S}$-category is just an ordinary (small) category-whereas if $\mathcal{V} = Ab$ the category of abelian groups then an Ab-category with one object is just an ordinary ring (and more general Ab-categories arise in linear algebra just as naturally as do rings, e.g. *all* finite rectangular matrices over a given field form an Ab-category with the natural numbers as objects). The associativity and identity axioms (i.e. the above commutative diagrams) are (like coherence) automatic in case $\mathcal{V}$ itself is a poset, and hence, as claimed in the introduction, a $\mathbf{2}$-category is an arbitrary poset while an $\mathbf{R}$-category is an arbitrary (generalized) metric space.

Every $\mathcal{V}$-category X has an opposite X^{op} with the same objects and units but with

$$X^{op}(a, b) = X(b, a)$$

and

$$\begin{tikzcd}
X^{op}(a, b) \otimes X^{op}(b, c) \arrow[rr, "\mu^{op}_{abc}"] \arrow[d, equal] & & X^{op}(a, c) \arrow[d, equal] \\
X(b, a) \otimes X(c, b) \cong X(c, b) \otimes X(b, a) \arrow[rr, "\mu_{cba}"'] & & X(c, a)
\end{tikzcd}$$

If we define also

$$\mathrm{sym}\ (X)(a, b) = X(a, b) \otimes X(b, a)$$

with

$$\begin{tikzcd}
\mathrm{sym}\ (X)(a,b) \otimes \mathrm{sym}\ (X)(b,c) \arrow[rr, "\mu^{\mathrm{sym}}_{abc}"] \arrow[d, equal] & & \mathrm{sym}\ (X)(a,c) \arrow[dd, equal] \\
(X(a,b) \otimes X(b,a)) \otimes (X(b,c) \otimes X(c,b)) \arrow[d, "\cong"] & & \\
(X(a,b) \otimes X(b,c)) \otimes (X(c,b) \otimes X(b,a)) \arrow[rr, "\mu_{abc} \otimes \mu_{cba}"'] & & X(a,c) \otimes X(c,a)
\end{tikzcd}$$

and

$$\begin{tikzcd} k \arrow[r, "\eta_a^{\mathrm{sym}}"] \arrow[d, "\cong"'] & \mathrm{sym}\,(X)\,(a,a) \arrow[d, equal] \\ k \otimes k \arrow[r, "\eta_a \otimes \eta_a"] & X(a,a) \otimes X(a,a) \end{tikzcd}$$

then sym(X) is a $\mathcal{V}$-category isomorphic with its opposite in an object-preserving manner.

In case the objects of $\mathcal{V}$ can be indexed by a set (which by bicompleteness actually forces $\mathcal{V}$ to have a poset as its underlying category) then $\mathcal{V}$ itself is an example of a $\mathcal{V}$-category (more generally any small part of $\mathcal{V}$ becomes a $\mathcal{V}$-category) by setting

$$\mathcal{V}\,(a,b) \;=\; \mathrm{Hom}\,(a,b)$$

Thus **2** is an example of a poset, while **R** is a (highly non-symmetric) example of a metric space. But sym(**R**) is the usual metric on the reals.

If X and Y are two $\mathcal{V}$-categories, then by a $\mathcal{V}$-functor $X \xrightarrow{f} Y$ is meant any structure consisting of mapping of the objects of X into the objects of Y together with an assignment of a $\mathcal{V}$-morphism

$$X(a,b) \xrightarrow{f_{ab}} Y(fa, fb)$$

to every ordered pair $\langle a, b\rangle$ of X-objects, subject to the commutativity of the following diagrams in $\mathcal{V}$.

$$\begin{tikzcd} X(a,b) \otimes X(b,c) \arrow[r, "\mu^X_{abc}"] \arrow[d, "f_{ab} \otimes f_{bc}"'] & X(a,c) \arrow[d, "f_{ac}"] \\ Y(fa,fb) \otimes Y(fb,fc) \arrow[r, "\mu^Y_{fa\,fb\,fc}"'] & Y(fa,fc) \end{tikzcd}$$

$$\begin{tikzcd} & X(a,a) \arrow[dd] \\ k \arrow[ur, "\eta^X_a"] \arrow[dr, "\eta^Y_{fa}"'] & \\ & Y(fa,fa) \end{tikzcd}$$

Thus an $\mathcal{S}$-functor is just an ordinary functor between (small) categories, an Ab-functor is just an additive functor (e.g. a ring

homomorphism in case X has only one object) while a **2** functor (satisfying

$$X(a, b) \vdash Y(fa, fb))$$

is an arbitrary order-preserving mapping from the poset X to the poset Y, and an **R**-functor (satisfying

$$X(a, b) \geq Y(fa, fb))$$

is an arbitrary distance-decreasing map (Lipschitz map of Lipschitz constant ≤ 1) from the metric space X to the metric space Y.

If $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are $\mathcal{V}$-functors we can define $(gf)_{ab}$ as the composition

$$X(a, b) \xrightarrow{f_{ab}} Y(fa, fb) \xrightarrow{g_{fa, fb}} Z(gfa, gfb)$$

in $\mathcal{V}$ to obtain a $\mathcal{V}$-functor $X \xrightarrow{gf} Z$ and thus a category $\mathcal{V}$-Cat whose objects and morphisms are all the $\mathcal{V}$-categories and $\mathcal{V}$-functors. In the next section we will see that $\mathcal{V}$-Cat (e.g. the category of metric spaces and distance-decreasing maps) is itself a closed category.

A morphism between closed categories $\mathcal{V} \xrightarrow{\Phi} \mathcal{W}$ is basically the concentrated expression of a process which takes every $\mathcal{V}$-category X into a $\mathcal{W}$-category ΦX with the same objects, every $\mathcal{V}$-functor f into a $\mathcal{W}$-functor Φf and in general interprets $\mathcal{V}$-valued category-theory as $\mathcal{W}$-valued category-theory; in view of many examples it is too restrictive to require that Φ « strictly » preserves the tensor product and Hom, so we adopt the following definition. A closed functor is a triple consisting of a functor $\mathcal{V} \xrightarrow{\Phi} \mathcal{W}$, a $\mathcal{W}$-morphism

$$k_{\mathcal{W}} \xrightarrow{\varphi_0} \Phi(k_{\mathcal{V}})$$

and a natural transformation

$$\Phi(u) \underset{\mathcal{W}}{\otimes} \Phi(v) \xrightarrow{\varphi_{uv}} \Phi(u \underset{\mathcal{V}}{\otimes} v)$$

subject to compatibility between themselves and with the given symmetry, associativity and unitary isomorphisms in $\mathcal{V}$ and $\mathcal{W}$ (Eilenberg-Kelly).

For example, any locally small closed category $\mathcal{V}$ has a canonical closed functor

$$\mathcal{V} \xrightarrow{V} \mathcal{S}$$

to the category of abstract sets defined by

$$V(u) = \text{set of all } \mathcal{V}\text{-morphisms } k \to u$$
$$1 \xrightarrow{\varphi_0} V(k) = \text{the identity } \mathcal{V}\text{-morphism } k \to k$$

with

$$V(u_1) \times V(u_2) \xrightarrow{\varphi_{u_1 u_2}} V(u_1 \otimes u_2)$$

the mapping taking any ordered pair $k \xrightarrow{f_1} u_1$, $k \xrightarrow{f_2} u_2$ into

$$k \xrightarrow{\cong} k \otimes k \xrightarrow{f_1 \otimes f_2} u_1 \otimes u_2 .$$

If $\mathcal{M}$ is a Boolean ring of sets made into a closed category with union as tensor as above then a closed functor

$$\mathcal{M} \xrightarrow{M} \mathbf{R}$$

is any order-preserving real-valued function on $\mathcal{M}$ satisfying

$$0 = M(\emptyset)$$
$$M(a) + M(b) \geq M(a \cup b)$$

i.e. an « outer measure ». The closed functors $\mathbf{R} \xrightarrow{\lambda} \mathbf{R}$ are just the « subadditive » order-preserving functions, i.e.

$$u_1 \geq u_2 \implies \lambda u_1 \geq \lambda u_2$$
$$0 = \lambda 0$$
$$\lambda u_1 + \lambda u_2 \geq \lambda (u_1 + u_2) .$$

If $\mathcal{V} \xrightarrow{\Phi} \mathcal{W}$ is any closed functor and X any $\mathcal{V}$-category, then

$$(\Phi X)(a, b) = \Phi(X(a, b))$$

$$\begin{tikzcd}
(\Phi X)(a,b) \underset{\mathcal{W}}{\otimes} (\Phi X)(b,c) \arrow[r, "(\Phi\mu)_{abc}"] \arrow[d, "\varphi"] & (\Phi X)(a,c) \arrow[d, equal] \\
\Phi(X(a,b) \underset{\mathcal{V}}{\otimes} X(b,c)) \arrow[r, "\Phi(\mu_{abc})"'] & \Phi(X(a,c))
\end{tikzcd}$$

$$\begin{tikzcd}
k\mathcal{W} \arrow[rr, dashed, "(\Phi\eta)_a"] \arrow[dr, "\varphi_0"'] & & (\Phi X)(a,a) \\
& \Phi(k\mathcal{V}) \arrow[ur, "\Phi(\eta_a)"'] &
\end{tikzcd}$$

defines a $\mathcal{W}$-category ΦX with the same set of objects as X. Moreover if $X \xrightarrow{f} Y$ is any $\mathcal{V}$-functor by defining Φf to be the same mapping as f on objects and $(\Phi f)_{ab} = \Phi(f_{ab})$ we obtain a $\mathcal{W}$ — functor $\Phi X \to \Phi Y$, and moreover this process is compatible with composition of $\mathcal{V}$-functors (and more). For example, any $\mathcal{V}$-category X has an underlying $\mathcal{S}$-category VX because of the closed functor $\mathcal{V} \xrightarrow{V} \mathcal{S}$.

In particular if $\mathcal{V}$ is small and $\mathcal{V} \xrightarrow{\Phi} \mathcal{W}$ is a closed functor, then $\Phi\ \mathcal{V}$ is a $\mathcal{W}$-category. As an example, any Boolean ring of sets $\mathcal{M}$ equipped with an outer measure $\mathcal{M} \xrightarrow{M} \mathbf{R}$ becomes a metric space.

If $\mathbf{R} \xrightarrow{\lambda} \mathbf{R}$ is a closed functor X and Y are metric spaces, then an $\mathbf{R}$-functor

$$\lambda\ X \xrightarrow{f} Y$$

is just a function satisfying

$$\lambda(X(a,b)) \geq Y(fa, fb).$$

Since multiplication by a given constant is subadditive and thus a closed functor, we see that the study of (a very natural generalization of) arbitrary Lipschitz mappings $X \to Y$ is naturally incorporated into the functorial set-up.

A special class of closed categories are those in which the tensor product is actually the categorical product, which amounts to

$$\frac{u \to v_1 \otimes v_2}{u \to v_1\ ,\ u \to v_2}$$

further « rule of inference of propositional Logic »; such closed categories are usually called cartesian closed. $\mathcal{S}$ and **2** are cartesian closed, while Ab and **R** are not. If a category admits a cartesian closed structure, then that structure is essentially unique. Ab does not admit a cartesian closed structure, but the underlying category of **R** does admit the cartesian closed structure $\mathbf{R}_{cart}$ in which

$$u \otimes v = \max(u, v)$$

$$\mathrm{Hom}(a, v) = \begin{cases} v & v > a \\ 0 & a \geq v \end{cases}$$

An $\mathbf{R}_{cart}$-category is just an *ultrametric space*, so that all of our general results may be reinterpreted as applying to ultrametric spaces. Moreover, the identity mapping $\mathbf{R}_{cart} \to \mathbf{R}$ is a closed functor (i.e. $a + v \geq \max(a, v)$) which induces the inclusion

$$\mathbf{R}_{cart}\text{-Cat} \to \mathbf{R}\text{-Cat}$$

of ultrametric spaces into all metric spaces.

2. - Functor Categories, Yoneda embedding, adequacy, comprehension scheme.

If A and X are two $\mathcal{V}$-categories then there is a $\mathcal{V}$-category $A \times X$ whose objects are the ordered pairs $\langle a, x \rangle$ of objects and whose hom-values are

$$(A \times X)(\langle a, x \rangle, \langle a', x' \rangle) = A(a, a') \times X(x, x')$$

the last being the cartesian product (e.g. max. in the case of **R**) in $\mathcal{V}$. But if $\mathcal{V}$ is not cartesian closed, there is another more important $\mathcal{V}$-category structure on the same objects, defined by

$$(A \otimes X)(\langle a, x \rangle, \langle a', x' \rangle) = A(a, a') \otimes X(x, x').$$

This gives, for example, the l_1-style metric on the product of two metric spaces. The unit k for this tensor product has one object $*$ with $k(*, *) = k$.

This tensor product of $\mathcal{V}$-categories always has a « Hom » adjoint to it, making $\mathcal{V}$-cat itself into a closed category. This Hom always has a concrete interpretation in terms of « strong natural

transformations »; we are going to denote it by Y^A, and its objects are just all the $\mathcal{V}$-functors $A \to Y$. Recall that in the usual $\mathcal{S}$-valued case, a natural transformation $f_1 \to f_2$ (where $A \xrightarrow{f_1, f_2} Y$) is a family of Y-morphisms indexed by the objects of A, more exactly, an element of the set $\prod_{a \in A} Y(f_1 a, f_2 a)$ subject to equations indexed by the elements of the $A(a, b)$. In the $\mathcal{V}$-valued case, we define directly (not the set of but) the $\mathcal{V}$-object of natural transformations $Y^A(f_1, f_2)$ by the equalizer in $\mathcal{V}$ of the two morphisms

$$\prod_a \rightrightarrows \prod_{a,b}$$

in

$$Y^A(f_1, f_2) \to \prod_{a \in A} Y(f_1 a, f_2 a) \rightrightarrows \prod_{a,b \in A} \operatorname{Hom}(A(a, b), Y(f_1 a, f_2 b))$$

whose constructions we leave to the reader. A $\mathcal{V}$-morphism $k \to Y^A(f_1, f_2)$ may then be identified with a strong natural transformation $f_1 \to f_2$. In case $\mathcal{V}$ is itself a poset, as with $\mathcal{V} = \mathbf{R}$, the two morphisms are already equal so that $Y^A(f_1, f_2) = \prod_{a \in A} Y(f_1 a, f_2 a)$ in such a case. Since $\prod$ in $\mathbf{R}$ means sup, we deduce that

$$Y^A(f_1, f_2) = \sup_{a \in A} Y(f_1 a, f_2 a)$$

in the case of metric spaces, i.e. that the sup metric on the space of 1-Lipschitz maps is a special case of the general notion of $\mathcal{V}$-natural transformations. The reader should be able to verify that there is a bijection

$$\frac{A \otimes X \to Y}{X \to Y^A}$$

between the two indicated sets of $\mathcal{V}$-functors at least in the case $\mathcal{V} = \mathbf{R}$.

Of special interest is the case $\mathcal{V}A^{op}$, because of the existence of the basic Yoneda embedding

$$A \to \mathcal{V}^{A^{op}}$$

which is the $\mathcal{V}$-functor whose value at an object a of A is the $\mathcal{V}$-functor $A^{op} \to \mathcal{V}$ defined by

$$a' \rightsquigarrow A(a', a)$$

and which works on A-morphism objects precisely by use of A-composition. Saying that a $\mathcal{V}$-functor $A \xrightarrow{f} Y$ is $\mathcal{V}$-full-and-faithful iff each of the $\mathcal{V}$-morphisms

$$A(a, b) \xrightarrow{f_{ab}} Y(fa, fb)$$

is actually an isomorphism, we have the important *Yoneda Lemma* (Eilenberg-Kelley): The Yoneda embedding is, for any closed $\mathcal{V}$ and any $\mathcal{V}$-category A, a $\mathcal{V}$-full-and-faithful $\mathcal{V}$-functor.

As a simple example, note that for $\mathcal{V} = \mathbf{2}$, an order-preserving map $A^{op} \to \mathbf{2}$ is equivalent to an order-ideal in A, and the Yoneda embedding is simply Dedekind's representation of a poset by its principal ideals. In the case $\mathcal{V} = \mathcal{S}$ or $\mathcal{V} = k$-Modules, the Yoneda embedding is often (especially when A has one object) called the regular representation of A, and Yoneda's Lemma includes Cayley's theorem on representing an abstract group by transformations. For a metric space A and for each point a, the function assigning to any point a' its distance to a is a distance decreasing function, and Yoneda's Lemma states that assigning to each a the just-described function is an *isometric* embedding of A into the space $\mathbf{R}\, A^{op}$ where the last is equipped with the sup metric.

More generally, given any $\mathcal{V}$-functor $A \to X$, we can consider the Yoneda representation of X restricted to A, i.e. the composite $\mathcal{V}$-functor

$$X \to \mathcal{V}^{X^{op}} \xrightarrow{\mathcal{V}^{i^{op}}} \mathcal{V}^{A^{op}}$$

which assigns to each x the functor $A^{op} \to \mathcal{V}$ defined by $a' \rightsquigarrow X(ia', x)$. In case this restricted representation is still $\mathcal{V}$-full--and-faithful on all X, we say after Isbell that i is *$\mathcal{V}$-adequate*, or in case i is an inclusion that A is a $\mathcal{V}$-adequate subcategory of X. This concept is the basis of much representation theory of categories, especially for algebraic categories and topoi, since it often happens that a quite small category A is adequate in a quite large one X. (We hasten to point out that this paragraph is meaningful even when $\mathcal{V}$ is not small; for example $\mathcal{V}\, A^{op}$ can be interpreted in terms of « modules » as in the following section). For example, if $\mathcal{V}$ = abelian groups and A is any ring, then A (considered as a $\mathcal{V}$-category with only one object) is adequate in any category X of A-modules, no matter how large.

In particular, to say that a subspace A of a metric space X is adequate is to say that for any two points x_1, x_2 of X, the inequality

$$X(x_1, x_2) \geq \sup_{a \in A} [X(a, x_2) - X(a, x_1)]$$

is actually an equality, i.e. that for any $d > 0$ there exists $a \in A$ with

$$X(a, x_1) + X(x_1, x_2) \leq X(a, x_2) + d .$$

For example, the unit circle is adequate in the unit disc (this simple example was pointed out to me by Prof. Isbell). A more restricted notion is that of $\mathcal{V}$-density, by which we mean (here differing in terminology with some authors) that $i_* oi^* \cong 1_X$ in the sense of the next section on bimodules; this reduces in the case of metric spaces to the requirement that

$$X(x_1, x_2) = \inf_{a \in A} [X(x_1, ia) + X(ia, x_2)] .$$

Proposition. - If a distance-decreasing map $A \xrightarrow{i} X$ of metric spaces is $\mathbf{R}$-dense, then it is $\mathbf{R}$-adequate.

Proof. - We always have

$$X(a, x_1) + X(x_1, x_2) \leq X(a, x_1) + X(x_1, a) + X(a, x_2) .$$

But by taking $x_2 = x_1$ in the definition of density, we see that $X(a, x_1) + X(x_1, a) \leq d$ for suitable $a \in A$, as required.

Define a particular metric space A whose points are the natural numbers and for which

$$A(n, m) = \begin{cases} \infty & n \neq m \\ 0 & n = m \end{cases} .$$

Then $\mathbf{R}A^{op}$ is just the space of all sequences of nonnegative reals, with the (nonsymmetric) sup metric. Say that a metric space X is *separable* iff there exists an $\mathbf{R}$-dense map $A \to X$. Then

Corollary. - Any separable metric space X can be isometrically embedded in the space of all sequences of non-negative reals with the sup metric.

To discuss the « comprehension scheme » we will limit ourselves to those closed categories $\mathcal{V}$ in which $K = 1$, i.e. in which the unit object for the tensor product is also the terminal object of $\mathcal{V}$; thus for this form of the comprehension scheme, $\mathcal{V}$ may be car-

tesian closed, e.g. $\mathcal{S}$ or $\mathbf{2}$, but more generally e.g. $\mathbf{R}$ satisfies this condition, as does the category of so-called « affine modules » over a given commutative ring (but not the usual category of modules). This has the effect that for any $\mathcal{V}$-category E, there is a canonical « augmentation » $E(e, e') \to K$. Thus we may simply consider the category $\mathcal{V}$-Cat/B of all $\mathcal{V}$-categories equipped with with a $\mathcal{V}$--functor with codomain B, and compare it with the category $\mathcal{V}^B$ of all $\mathcal{V}$-functors with domain B and the fixed codomain $\mathcal{V}$. Namely, given any $E \xrightarrow{p} B$, we define $\varphi_p : B \to \mathcal{V}$ by the coequalizer diagram in $\mathcal{V}$

$$\sum_{e,e'} E(e, e') \otimes B(p(e'), b) \rightrightarrows \sum_{e \in E} B(p(e), b) \to \varphi_p(b)$$

where one of the two morphisms in induced by the functor p followed by composition in B, while the other is induced by the augmentation. In the case $\mathcal{V} = \mathbf{R}$, this simply means that for any distance-decreasing map p from a metric space E to the fixed metric space B, we define a real-valued function on B by

$$\varphi_p(b) = \inf_e B(p(e), b)$$

i.e. the distance from the image of p to the variable point b.

In the case $\mathcal{V} = \mathbf{2}$, φ_p is the order-preserving map $B \to \mathbf{2}$ defined by

$$\varphi_p(b) = \text{true} \quad \text{iff } \exists e [p(e) \text{ dominates } b \text{ in } B]$$

In the case $\mathcal{V} = \mathcal{S}$, $\varphi_p(b)$ may be interpreted as the set of components of the category p/b whose objects are pairs e, β with $p(e) \xrightarrow{\beta} b$ in B and whose morphisms are morphisms $e \xrightarrow{\xi} e'$ in E such that $\beta = p(\xi)\beta'$ in B.

The « comprehension scheme » then refers to the right adjoint of the functor

$$\mathcal{V}\text{—}Cat/B \to \mathcal{V}^B$$

defined by $p \rightsquigarrow \varphi_p$. Namely, given any $B\varphi \to \mathcal{V}$, define a category $\{B|\varphi\}$ whose objects are pairs $\langle b, x \rangle$ such that $k \xrightarrow{x} \varphi(b)$ in $\mathcal{V}$

and for which $\{B|\varphi\}$ $(\langle b, x\rangle, \langle b', x'\rangle)$ is defined by the following pullback diagram in $\mathcal{V}$

$$\begin{tikzcd}
\{B \mid \varphi\} \ (\langle b, x\rangle, \langle b', x'\rangle) \arrow[rr, "n"] \arrow[d, "\pi\, xx'"'] & & k \arrow[d, "x'"] \\
B(b, b') \cong k \otimes B(b, b') \arrow[r, "x \otimes id"'] & \varphi(b) \otimes B(b, b') \arrow[r] & \varphi(b')
\end{tikzcd}$$

where $\varphi(b) \otimes B(b, b') \to \varphi(b')$ is the « action » of B on φ adjoint to the functoriality of φ [note that $\mathcal{V}$-functors with codomain $\mathcal{V}$ always have an equivalent interpretation as « right modules » over the domain category]. The morphism n is unique since we have assumed $k = 1$, but without that assumption it would provide the $\mathcal{V}$-category $\{B|\varphi\}$ with an « augmentation » structure in addition to the structural functor π. In the case $\mathcal{V} = \mathcal{S}$, $\{B|\varphi\}$ is sometimes called the category of all elements of φ, and the functor $\{B|\varphi\} \xrightarrow{\pi} B$ is the discrete fibration corresponding to the set-valued functor φ. In case $\mathcal{V} = \mathbf{2}$

$$\{B \mid \varphi\} = \{\, b \in B \mid \varphi(b) = \text{true} \,\}$$

(accounting for the terminology « comprehension scheme ») and we have of course

$$\frac{Im(p) \subseteq \{B \mid \varphi\}}{\varphi_p \vdash \varphi}$$

But in the case $\mathcal{V} = \mathbf{R}$, we have for any « quantity-valued propositional function » φ on the metric space B that

$$\{B \mid \varphi\} = \{\, b \in B \mid 0 = \varphi(b) \}$$

and for any $E \xrightarrow{p} B$,

$$\frac{Im(p) \subseteq \{B \mid \varphi\}}{\inf B(p(e), b) \leq \varphi(b), \text{ all } b}\,.$$

To what extent are objects in one of the categories $\mathcal{V}$-Cat/B, $\mathcal{V}^B$ « equivalent » to objects in the other via this adjoint pair? For $\mathcal{V} = \mathcal{S}$, $\mathbf{2}$, $\mathbf{R}$ respectively, $E \xrightarrow{p} B$ must be respectively a discrete fibration, the inclusion of an order-ideal, the inclusion of a *closed* subset in order to have $p \cong \{|\varphi_p\}$. On the other side, for $\mathcal{V} = \mathbf{R}$ there are distance decreasing maps $B \to \mathbf{R}$ *not* of the form « di-

stance from a certain closed set », but for $\mathcal{V} = \mathcal{S}$ or $\mathcal{V} = \mathbf{2}$ *every* $\varphi \in \mathcal{V}^B$ is of the form φ_p.

3. - Bimodules, Kan quantification, Cauchy completeness.

For any $\mathcal{V}$-category Y, we have the canonical Yoneda embedding $Y \to \mathcal{V}^{Y^{op}}$ which means in particular that we can consider that the concept of a $\mathcal{V}$-functor $X \to \mathcal{V}^{Y^{op}}$ is a generalization of the concept of a $\mathcal{V}$-functor $X \to Y$. Such a generalized $\mathcal{V}$-functor $X \mapsto Y$ is equivalent to a $\mathcal{V}$-functor $Y^{op} \otimes X \xrightarrow{\varphi} \mathcal{V}$, and may be considered as a « $\mathcal{V}$-valued relation » from X to Y, $\varphi(y, x)$ being the « truth-value of the φ-relatedness of y to x ». In particular, every $\mathcal{V}$-functor $X \xrightarrow{f} Y$ thus yields a $X \overset{f_*}{\mapsto} Y$ defined by

$$f_*(y, x) = Y(y, f(x))$$

but also yields $Y \overset{f^*}{\mapsto} X$ defined by

$$f^*(x, y) = Y(f(x), y) .$$

We first give an alternate description, without recourse to the notion of $\mathcal{V}$-functor, of such $\mathcal{V}$-valued relations as *bimodules*. We are using, by the way, the notational convention that inside a $\mathcal{V}$-category composition is written from left to right, while composition of $\mathcal{V}$-functors and of bimodules (= « relations ») is written from right to left.

If X, Y are $\mathcal{V}$-categories, a *bimodule* $X \overset{\varphi}{\mapsto} Y$ (also called a right X, left Y-module) consists of a family $\varphi(y, x)$ of objects of $\mathcal{V}$ indexed by the objects of Y and X together with morphisms

$$Y(y', y) \otimes \varphi(y, x) \to \varphi(y', x)$$
$$\varphi(y, x) \otimes X(x, x') \to \varphi(y, x')$$

in $\mathcal{V}$ which behave as « actions » in the sense that the five axioms (commutative diagrams in $\mathcal{V}$) of X-unity, X-associativity, Y-unity, Y-associativity, and mixed associativity (= commutativity of X-action with Y-action) hold. Thus in case $\mathcal{V} = Ab$, this is just the usual notion of bimodule suitably extended to « rings » X and Y with more than one object. In case $\mathcal{V} = \mathcal{S}$, bimodules are someti-

mes called « profunctors » or « distributeurs ». In case $\mathcal{V} = \mathbf{R}$, a bimodule $X \overset{\varphi}{\mapsto} Y$ between two metric spaces is just a real-valued function on the product $Y \times X$ satisfying

$$\inf_y \, [Y(y', y) + \varphi(y, x)] \geq \varphi(y', x)$$

$$\inf_x \, [\varphi(y, x) + X(x, x')] \geq \varphi(y, x') \, .$$

The latter is clearly equivalent to putting a metric on the sum $X + Y$ extending the given metrics and having $d(x, y) = \infty$ for $x \in X$, $y \in Y$, and indeed this alternate mode of description works generally, recalling that the infinite quantity corresponds to the empty coproduct in $\mathcal{V}$.

If $X \overset{\varphi}{\mapsto} Y \overset{\psi}{\mapsto} Z$ are bimodules, then composition $X \overset{\psi \circ \varphi}{\mapsto} Z$ is defined by the following coequalizer diagram in $\mathcal{V}$

$$\sum_{y_1, y_2} \psi(z, y_1) \otimes Y(y_1, y_2) \otimes \varphi(y_2, x) \rightrightarrows \sum_{y \in Y} \psi(z, y) \otimes \varphi(y, x) \longrightarrow (\psi \circ \varphi)(z, x)$$

Thus in case $\mathcal{V}$ is itself a poset, $\psi \circ \varphi$ reduces to a « matrix product »

$$\sum_{y \in Y} \psi(z, y) \otimes \varphi(y, x)$$

where the sigma denotes coproduct in $\mathcal{V}$, but in general we take the quotient of the matrix product modulo the discrepancy between the two actions of Y in the middle, just as in the familiar case $\mathcal{V} = Ab$ where one often writes more explicitly

$$\psi \circ \varphi = \psi \underset{Y}{\otimes} \varphi \, .$$

In particular for $\mathcal{V} = \mathbf{2}$

$$(\psi \circ \varphi)(z, x) = \exists \, y \, [\psi(z, y) \wedge \varphi(y, x)]$$

is the usual relational product, while for $\mathcal{V} = \mathbf{R}$ we have

$$(\psi \circ \varphi)(z, x) = \inf \, [\psi(z, y) + \varphi(y, x)] \, .$$

It can be verified in general that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ are $\mathcal{V}$-functors, then

$$(gf)_* \cong g_* \circ f_*$$

where isomorphism of bimodules has an obvious sense; also 1_X defined by

$$1_X(x, x') = X(x, x')$$

plays the role of identity with respect to composition with any bimodules (not only those of the form f_*). Further, the composition of bimodules is associative up to (coherent) isomorphism, but in general not commutative. Note that the endobimodules of the one-object category k are in correspondence with the objects of $\mathcal{V}$, with the composition of these reducing to the tensor product in $\mathcal{V}$.
The concept Hom, right adjoint to $\otimes$ in $\mathcal{V}$ has two extensions to mixed Hom of bimodules, right adjoint on opposite sides to the composition of bimodules. That is, given any three $\mathcal{V}$-categories X, Y, Z and a bimodule $X \overset{\varphi}{\mapsto} Z$, there are two universal problems: given any $X \overset{\alpha}{\mapsto} Y$, there is a $Y \mapsto Z$ denoted by $\mathrm{Hom}^X(\alpha, \varphi)$ which is « best » in the sense that for any bimodule $Y \overset{\beta}{\mapsto} Z$ there is a natural bijection of morphisms of bimodules

$$\frac{\beta \longrightarrow \mathrm{Hom}^X(\alpha, \varphi)}{\beta \circ \alpha \longrightarrow \varphi}$$

likewise for given β there is a « best α », denoted by $\mathrm{Hom}_Z(\beta, \varphi)$ satisfying

$$\frac{\alpha \longrightarrow \mathrm{Hom}_Z(\beta, \varphi)}{\beta \circ \alpha \longrightarrow \varphi}$$

for any α.

Explicitly, $\mathrm{Hom}^X(\alpha, \varphi)$ is defined by an equalizer diagram in $\mathcal{V}$

$$\mathrm{Hom}^X(\alpha, \varphi)(z, y) \longrightarrow \prod_{x \in X} \mathrm{Hom}(\alpha(y, x)\varphi, (z, x) \rightrightarrows \prod_{x_1, x_2} \mathrm{Hom}(X(x_1, x_2), \mathrm{Hom}(\alpha(y, x_1), \varphi(z, x_2)))$$

and similarly $\mathrm{Hom}_Z(\beta, \varphi)(y, x)$ is defined by an equalizer condition as a subobject of

$$\prod_{z \in Z} \mathrm{Hom}(\beta(z, y), \varphi(z, x)) .$$

For *some* bimodules $X \overset{\alpha}{\mapsto} Y$, the operation of composing with α, $\beta \rightsquigarrow \beta \circ \alpha$, has not only the right adjoint $\varphi \rightsquigarrow \mathrm{Hom}^X(\alpha, \varphi)$ but has also a *left* adjoint. The typical such bimodule α is one of the form

$\alpha = f_*$, where $X \xrightarrow{f} Y$ is a $\mathcal{V}$-functor; presently we will also discuss the extent to which « all » such bimodules α are induced by functors f.

Lemma. - For any $\mathcal{V}$-functor $X \xrightarrow{f} Y$ and any $\mathcal{V}$-bimodule $Y \overset{\beta}{\mapsto} Z$, there is a natural isomorphism

$$\beta \circ f_* = \mathrm{Hom}^Y(f^*, \beta)$$

of bimodules $X \mapsto Z$.

Proof. - One verifies that both sides are naturally isomorphic to the bimodule whose typical component is

$$\beta(z, f(x)) .$$

Theorem. - For $\alpha = f_*$, composition $\beta \rightsquigarrow \beta \circ \alpha$ has as left adjoint the operation of composing with f^*

$$\varphi \rightsquigarrow \varphi \circ f^*$$

Proof. -

$$\frac{\dfrac{\varphi \circ f^* \longrightarrow \beta}{\varphi \longrightarrow \mathrm{Hom}^Y(f^*, \beta)}}{\varphi \longrightarrow \beta \circ f_*}$$

Corollary. - (Kan quantification). For any $\mathcal{V}$-functor $X \xrightarrow{f} Y$, the functor « composition with f »

$$\mathcal{V}^Y \xrightarrow{\mathcal{V}^f} \mathcal{V}^X$$

has both left and right adjoints. On $\varphi \in \mathcal{V}^X$, the left adjoint gives as a result the functor $\varphi \circ f^*$, whose value at y is the quotient of

$$\sum_X \varphi(x) \otimes Y(f(x), y)$$

« modulo the distinction between the two actions of X in the middle », while the right adjoint gives as a result the functor $\mathrm{Hom}^X(f_*, \varphi)$ whose value at y is that subobject of

$$\prod_X \mathrm{Hom}(Y(y, f(x)), \varphi(x))$$

« on which the two actions of X agree ». In the special case that f

is $\mathcal{V}$-full-and-faithful, $\mathcal{V}^f$ is also full and faithful, or equivalently the two Kan quantifications are actually *extensions*, in the sense that

$$(\varphi \circ f^*) \circ f_* \cong \varphi$$

$$\mathrm{Hom}^X(f_*, \varphi) \circ f_* \cong \varphi$$

for any φ.

Proof. - The corollary is for the most part just a summary of the preceding discussion for the special case $Z = k$ (Indeed the corollary remains valid if $\mathcal{V}$ is replaced by $\mathcal{V}^{Z^{op}}$ for any (small) $\mathcal{V}$-category Z).

The assertion that, under the assumption that f is $\mathcal{V}$-full-and-faithful, first « extending » φ along f (in either of the two adjoint ways) and then « restricting » along f gives again φ, is proved in very much the same way as the Lemma above.

Corollary. - Let $X \xrightarrow{f} Y$ be an isometric embedding of a metric space X into a metric space Y. Then any Lipschitz function $X \xrightarrow{\varphi} \mathbf{R}$ can be extended to Y, with the same Lipschitz constant λ. In fact, among all extensions there is a largest one and a smallest one, given respectively by

$$\overline{\varphi}(y) = \inf_X [\varphi(x) + \lambda\, Y(f(x), y)]$$

$$\underline{\varphi}(y) = \sup_X [\varphi(x) - \lambda\, Y(y, f(x)].$$

Proof. - Apply the preceding discussion to the diagram

$$\begin{tikzcd} \lambda X \arrow[rr, "\lambda f"] \arrow[dr, "\varphi"'] & & \lambda Y \arrow[dl, dashed, shift left] \arrow[dl, dashed, shift right] \\ & \mathcal{V} & \end{tikzcd}$$

noting that λf as a function is the same as the inclusion f.

As an example of the last corollary, we could take for X the space of nonnegative step functions on a probability space S and for Y the space of all nonnegative functions with the natural sup metric, (so that $Y(y_2, y_1) = 0$ iff $y_1 \geq y_2$), and consider as φ the

elementary integral; thus in general we might call y φ-integrable if $\overline{\varphi}(y) \cong \underline{\varphi}(y)$.

The (non standard) name Kan quantification was suggested by the case $\mathcal{V} = \mathbf{2}$, in which the adjointness rules reduce to the usual rules of inference for quantification, φ being thought of as property or relation, i.e.

$$(\varphi \circ f^*)(z, y) \equiv \exists\, x\, [\varphi(z, x) \wedge f(x) \geq y]$$

$$\mathrm{Hom}^X(f_*, \varphi)(z, y) \equiv \forall\, x\, [y \geq f(x) \Longrightarrow \varphi(z, x)]$$

But for other choices of $\mathcal{V}$, induced representations and relatively free universal algebras can be shown to arise as special cases of these two constructions.

The essential property of a bimodule of the form f_* is that there exists another bimodule f^* which is right adjoint to it, in the sense that there are morphisms

$$1_X \longrightarrow f^* \circ f_*$$

$$f_* \circ f^* \longrightarrow 1_Y$$

satisfying the usual two adjunction equations.

Proposition. - In order that a metric space Y be Cauchy-complete, it is necessary and sufficient that every adjoint pair of bimodules

$$X \underset{f^*}{\overset{f^*}{\rightleftarrows}} Y$$

be induced by a $\mathcal{V}$-functor $X \xrightarrow{f} Y$.

Proof. - It suffices to consider $X = 1$ and show that an adjoint pair of $\mathbf{R}$-valued bimodules $1 \leftrightarrows Y$ is essentially just a point in the completion of Y. But adjointness just means that

$$X(x, x') \geq \inf_y\, [f^*(x, y) + f^*(y, x')]$$

$$\inf_X\, [f_*(y, x) + f^*(x, y')] \geq Y(y, y')$$

in addition to the bimodule property for each of f_*, f^*. In case $X = 1$, we have then

$$0 = \inf_y\, [f^*(y) + f_*(y)]$$

$$f_*(y) + f^*(y') \geq Y(y, y')\,.$$

Thus for each n we can choose y_n satisfying, for example

$$f^*(y_n) + f_*(y_n) \leq \frac{1}{n}$$

and then

$$Y(y_n, y_m) \leq \frac{1}{n} + \frac{1}{m}$$

so that we have a Cauchy sequence, and any other choice y'_n satisfying the same condition would have

$$Y(y_n\, y'_n) \leq \frac{2}{n}$$

i.e. would be an equivalent Cauchy sequence. Conversely, any equivalence class of Cauchy sequences yields an adjoint pair of bimodules by the definition

$$f^*(y) = \lim_{n \to \infty} Y(y_n, y)$$

$$f_*(y) = \lim_{n \to \infty} Y(y, y_n).$$

It can be shown that the suggested definition of « $\mathcal{V}$-Cauchy-completeness » means in case $\mathcal{V} = Ab$ that a « point of the completion » of an additive category Y is simply any finitely generated projective module over Y, while in case $\mathcal{V} = \mathcal{S}$, it means that Y is Cauchy-complete iff all idempotents in Y split in Y.

4. - Free $\mathcal{V}$-categories.

Defining a $\mathcal{V}$-graph to be a pair consisting of any set X and any $X \times X$-indexed family of objects of $\mathcal{V}$, and a morphism $\langle X, \gamma \rangle \xrightarrow{f} \langle Y, \delta \rangle$ to be any pair consisting of a mapping $X \to Y$ and any family

$$\gamma(x, x') \xrightarrow{f_{x,x'}} \delta(f(x), f(x'))$$

of morphisms in $\mathcal{V}$, we have an obvious forgetful functor

$$\mathcal{V}\text{-Cat} \to \mathcal{V}\text{-Graph}.$$

This forgetful functor has a left-adjoint called taking the free $\mathcal{V}$-category generated by a $\mathcal{V}$-graph. Explicitly, the free category has as objects the vertices of the generating graph, and as hom from x to x' has the coproduct in $\mathcal{V}$

$$\sum_{x_1 \dots, x_n} \gamma(x, x_1) \otimes \gamma(x_1, x_2) \otimes \dots \otimes (x_n, x')$$

over all finite sequences $x_1 \dots x_n$ of vertices. For example with $\mathcal{V} = k$-modules, this formula contains the construction of the tensor algebra over a vector space, or with $\mathcal{V} = \mathcal{S}$, we have as a special case the word algebra (= free monoid) over a given set. For the case $\mathcal{V} = \mathbf{R}$, an $\mathbf{R}$-graph is just an arbitrary assignment γ of quantities to pairs of points, and in the « free metric space » over such, the distance from x to x' is

$$\inf_{x_1 \dots, x_n} [\gamma(x, x_1) + \gamma(x_1, x_2) + \dots\dots + \gamma(x_n, x')]$$

the well-known « least-cost » distance.

The adjointness of the free $\mathcal{V}$-category construction contains the essence of the notion of recursion, especially when one considers it in relation with bimodules, where it leads for example to the iteration of endobimodules.

5. - Further remarks.

We already remarked that the composition of bimodules is similar to matrix multiplication, and indeed the analogy with linear algebra goes further. For example, if bimodules rather than $\mathcal{V}$ functors are considered as the « morphisms » between $\mathcal{V}$-categories, then $A^{op} \otimes Y$ plays the role of « Hom ». Either further developing that remark in the case $A = Y$, or proceding directly, it is natural to define, for any endobimodule $A \xrightarrow{\alpha} A$, $Tr(\alpha)$ to be the object of $\mathcal{V}$ defined by coequalizer

$$\sum_{a,b \in A} A(b, a) \otimes \alpha(a, b) \rightrightarrows \sum_{a \in A} \alpha(a, a) \longrightarrow Tr(\alpha) .$$

The injections $\alpha(a, a) \to Tr(\alpha)$, denoted by tr_a, are a natural generalization of many examples of classical constructions such as trace of endomorphisms or Lefschitz numbers (when $\alpha = 1_A$) as has been

verified by A. Kock. In case A is a metric space and $\alpha = f_*$ where f is a distance decreasing endomap,

$$Tr(\alpha) = \inf_a A(a, fa)$$

showing that the vanishing of the trace is related to the existence of fixed points. It seems likely that there may be theorems holding for more general $\mathcal{V}$, relating the trace of an endobimodule to an infinite iteration of it, which would extend the Banach fixed-point theorem.

SUMMARY. — The analogy between $\mathrm{dist}(a, b) + \mathrm{dist}(b, c) \geq \mathrm{dist}(a, c)$ and $\hom(A, B) \otimes \hom(B, C) \to \hom(A, C)$ is rigorously developed to display many general results about metric spaces as consequences of a « generalized pure logic » whose « truth-values » are taken in an arbitrary closed category.

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