Question 1

$$C \subseteq D \subseteq \mathbb{R};$$
$$\left(f_n\right)_{n\in\mathbb{N}} \text{ is uniformly convergent on } C;$$
$$\forall i \in \mathbb{N},\ f_i : D \to \mathbb{R} \text{ is continous}$$

Show $\exists f$ such that $f_n \xrightarrow[\text{uniformly}]{\overline{C}\cap D} f$ and f is continous.

Proof. Fix an $\epsilon > 0$ so, by uniform continuity of f,

$$\exists M : p, m \geq M \implies \forall x \in C, |f_p(x) - f_m(x)| < \frac{\epsilon}{3}$$

Fix $p, m \geq M$. Now, fix a $k \in \overline{C} \cap D$ thus, $\exists (x_n) \subseteq C$ such that $x_n \to k$. As each f_i is continous ,

$$\exists \delta(i) \text{ s.t. } |x - y| \leq \delta \implies |f_i(x) - f_i(y)| < \frac{\epsilon}{3}$$

So, there's K such that

$$i \geq K \implies |f_p(x_i) - f_p(k)| < \frac{\epsilon}{3} \text{ and } |f_m(x_i) - f_m(k)| < \frac{\epsilon}{3}$$

Thus, by triangle inequality,

$$\begin{aligned}
2 \times \frac{\epsilon}{3} &> |f_p(x_i) - f_p(k)| + |f_m(x_i) - f_m(k)| \\
&\geq |f_p(x_i) - f_m(x_i) + f_p(k) - f_m(k)| \\
&\geq ||f_p(x_i) - f_m(x_i)| - |f_m(k) - f_p(k)||
\end{aligned}$$

Now, as $|f_p(x_i) - f_m(x_i)| < \frac{\epsilon}{3}$, $-|f_p(x_i) - f_m(x_i)| > \frac{-\epsilon}{3}$,

$$\begin{aligned}
2 \times \frac{\epsilon}{3} &\geq ||f_p(x_i) - f_m(x_i)| - |f_m(k) - f_p(k)|| \\
&\geq |f_p(x_i) - f_m(x_i)| - |f_m(k) - f_p(k)| \\
&\geq |f_p(x_i) - f_m(x_i)| - \frac{\epsilon}{3}
\end{aligned}$$

Therefore, $|f_p(x_i) - f_m(x_i)| \leq \epsilon$ $\square$

There's a much better way of doing this though,

Proof. Fix an $\epsilon > 0$ so, by uniform continuity of f,

$$\exists M : p, m \geq M \implies \forall x \in C, |f_p(x) - f_m(x)| < \frac{\epsilon}{4}$$

Fix $p, m \geq M$
Now, fix a $k \in \overline{C} \cap D$. Going to show that (f_n) uniformly converges at k.
As f_p and f_m are continous , in particular at k,

$$\exists \delta_1 \text{ s.t. } |x - k| \leq \delta_1 \implies |f_p(x) - f_p(k)| < \frac{\epsilon}{4}$$

$$\exists \delta_2 \text{ s.t. } |x - k| \leq \delta_2 \implies |f_m(x) - f_m(k)| < \frac{\epsilon}{4}$$

Take $\delta := min\{\delta_1, \delta_2\}$
Now, if $k \in C$, then by hypothesis,the given sequence uniformly converges at k. Else, k is a limit point of C.Thus,

$$\exists c \in C \text{ such that } |c - k| < \delta$$

Fix this c and consider the following inequality

$$|f_p(k) - f_m(k)| \leq |f_p(k) - f_p(c)| + |f_p(c) - f_m(c)| + |f_m(c) - f_m(k)|$$

Each of the right hand side term is $< \frac{\epsilon}{4}$ (1st,3rd term due to continuity, 2nd term due to uniform convergance on C), forcing the left side to be $< \epsilon$. $\square$

Question 2 Prove that $\Sigma x^n(1-x)$ converges pointwise on $[0,1]$ but not uniformly. While $\Sigma(-1)^n x^n(1-x)$ converges uniformly on $[0,1]$.

Proof. As $x^n(1-x) = x^n - x^{n+1}$, the first sum telescopes:

$$\sum_{i=1}^{k} x^n(1-x) = (x-x^2)+(x^2-x^3)+...+(x^k-x^{k+1}) = x-x^{k+1}$$

So, for $x=1$, every partial sum is 0, and for $0 \leq x < 1$,

$$\lim_{k\to\infty}\sum_{i=1}^{k} x^i(1-x) = \lim_{k\to\infty}(x-x^{k+1}) = x$$

Thus, the series converges pointwise on $[0,1]$. Suppose it also converges uniformly to f . Then, as the k^{th} partial sum is $x - x^{k+1}$, a polynomial, and hence continous on $[0,1]$, it's limit function, f must be continous on $[0,1]$. But, f is discontinous at 1 as

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} x = 1 \neq 0 = f(1)$$

The partial sums for the second series of functions,

$$\sum_{i=1}^{k}(-x)^n(1-x) = -x+2[(-x)^2+(-x)^3+...+(-x)^k]+(-x)^{k+1})$$

So, for $x = 1$, every partial sum is 0, and for $0 \leq x < 1$,

$$\begin{aligned}\lim_{k\to\infty}\sum_{i=1}^{k}(-x)^i(1-x) &= x + 2\lim_{k\to\infty}((-x)^{k+1}+\sum_{i=1}^{k}(-x)^i) \\ &= x + 2\lim_{k\to\infty}\frac{-x(1-(-x)^k)}{1+x} \\ &= x + \frac{-2x}{1+x}\end{aligned}$$

To show uniform convergence, going to use the Drichilet test:

I Take $b_n(x) := \frac{x^n}{2} = (\frac{x}{\sqrt[n]{2}})^n$ .Going to show that

(i) $\forall x \in$[0,1], $b_n(x) \geq b_{n+1}(x)$:
As $b_{n+1}(x) - b_n(x) = \frac{x^{n+1}}{\sqrt[n+1]{2}} - \frac{x^n}{\sqrt[n]{2}} = \frac{x^n}{\sqrt[n]{2}}(\frac{x}{\sqrt{2}} - 1) \leq 0$

(ii) $b_n \rightrightarrows 0(C)$

$$\begin{aligned}\text{Take } \delta > \frac{ln2}{ln(\epsilon+1)} &\implies ln(\epsilon+1) > \frac{1}{\delta}ln(2) \\ &\implies \epsilon > \sqrt[\delta]{2}\end{aligned}$$

$$\begin{aligned}|b_n(x) - b_m(x)| &= |\frac{x}{\sqrt[n]{2}} - \frac{x}{\sqrt[m]{2}}| = x(\frac{1}{\sqrt[n]{2}} - \frac{1}{\sqrt[m]{2}}) \\ \leq (\frac{1}{\sqrt[n]{2}} - \frac{1}{\sqrt[m]{2}}) &= \frac{\sqrt[m]{2} - \sqrt[n]{2}}{\sqrt[n]{2}\,\sqrt[m]{2}} \leq \sqrt[m]{2} - \sqrt[n]{2} \\ &\leq \sqrt[\delta]{2} - 1 \leq \epsilon\end{aligned}$$

II Take $a_n(x) := 2(-1)^n(1-x)$. So,
$|(A_n(x))| = |\sum_{i=1}^{n} a_i(x)| = 2(1-x)|\sum_{i=1}^{n}(-1)^n|$

$$= \begin{cases} 0 & n \text{ is even} \\ 2(1-x) & n \text{ is odd} \end{cases}$$

$\therefore$ for any n,x $|A_n(x)| \leq 2$, a_n is uniformly bounded.

III So, Drichilet's test is applicable and

$$\begin{aligned} \Sigma b_n a_n &= \Sigma \frac{x^n}{2} \times 2(-1)^n(1-x) \\ &= \Sigma(-1)^n x^n (1-x) \\ &\text{uniformly converges on [0,1]} \end{aligned}$$

$\square$

Question 3

$$A \text{ is closed and bounded ;}$$
$$(f_n) \text{ is a sequence of continous functions on } A;$$
$$(f_n) \xrightarrow{p.w.} f, \text{ with f continous on } A;$$
$$\forall x \in A, f_n(x) \geq f_{n+1}(x), \text{ with } n \in \mathbb{N};$$

Prove that $f_n \rightrightarrows f(A)$

Question 4 Construct a sequence of functions, (f_n) on [0,1] such that

(a) each f_i is discontinous at every point of [0,1]; and

(b) $\exists f$, a continous function on [0,1] such that $f_n \rightrightarrows f$

Proof. Define $f_n(x) = \begin{cases} \frac{1}{n} & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

(a) This sequence is discontinous on [0,1] as :

(i) For $q \in \mathbb{Q}$, take any sequence of irrational numbers, $(p_n) \to q$. So,

$$\lim_{k\to\infty} f_n(p_k) = 0 \neq \frac{1}{n} = f_n(q)$$

(ii) For $p \notin \mathbb{Q}$, take any sequence of rational numbers, $(q_n) \to p$. So,

$$\lim_{k\to\infty} f_n(q_k) = \frac{1}{n} \neq 0 = f_n(q)$$

(b) The defined sequence uniformly converges to 0 as: Fix $\epsilon = \frac{1}{n}$, and choose $\delta > n$,

$$m \geq \delta \implies |f_m(x)| = \frac{1}{m} \leq \frac{1}{\delta} < \frac{1}{n} = \epsilon$$

$\square$

Question 5 **Prove** If Σa_n is absolutely convergent then $\Sigma \frac{a_n x^n}{1+x^{2n}}$ converges uniformly on $\mathbb{R}$.