1 Yoneda Lemma

Theorem 1.1. Yoneda If $\mathcal{A}$ be a locally small category, then,

$$[\mathcal{A}^{op}, Set](H_A, X) \cong X(A) \text{ naturally in } A \in \mathcal{A} \text{ and } X \in [\mathcal{A}^{op}, Set]$$

Explaination:
First of all, we fix any category, $\mathcal{A}$. Now we choose two things (independent of each other):

i an object, A from the category $\mathcal{A} = \mathcal{A}^{op}$

ii an object, X of the category $[\mathcal{A}^{op}, Set]$ which is precisely a functor $X : A^{op} \to Set$

Here, $[\mathcal{A}^{op}, Set](H_A, X)$ denotes arrows $H_A \to X$ in $[\mathcal{A}^{op}, Set]$ i.e. natural transformations, $\alpha :$
$$\begin{tikzcd} A^{op} \arrow[r, bend left, "H_A"] \arrow[r, bend right, "X"'] \arrow[r, phantom, "\Downarrow\alpha"] & Set \end{tikzcd}$$

Each of these natural transformations is a collection of maps in Set, hence each of their components is exactly a function. i.e. $\forall \alpha \in [\mathcal{A}^{op}, Set](H_A, X), \forall K \in \mathcal{A}, \alpha_K$ is a function $: H_A(K) \to X(K)$

$X(A)$ is precisely a set, because $X(A)$ is the image of (our chosen object,) A, under (our chosen functor,) X.

So, the idea is that our choice of A and X completely determines all possible maps (i.e. natural transformations) from H_A to X. This answers out big question of "what are all the maps $H_A \to X$ " or "how does H_A see other presheaves on $\mathcal{A}$ ".

The theorem says not just that the two are isomorphic, but that they're ***naturally*** *isomorphic.*
This means that $[\mathcal{A}^{op}, Set](H_A, X)$ and $X(A)$ are *functorial* in *both* A and X

Also, So, the aforementioned collection of natural transformations also must be a set: As $\mathcal{A}$ is locally small, for each choice of $K \in \mathcal{A}$, $H_A(K) = Hom_{\mathcal{A}}(K, A)$ is a set. Thus, as α_K is a function,and hence a relation,
it's a subset of $Hom_{\mathcal{A}}(K, A) \times X(A)$, which is a set as it's a cartesian product of sets.
Thus, K-component of every natural transformation is a set. i.e. $\forall \alpha \in [\mathcal{A}^{op}, Set](H_A, X), \forall K \in \mathcal{A}, \alpha_K$ is a set

Now, the question is wether the set of all natural transformations is a set

Proof. Let a locally small category, $\mathcal{A}$ be given.

Denoting the category of all presheaves on $\mathcal{A}$ by $\mathcal{C}$, i.e. $\mathcal{C} := [\mathcal{A}^{op}, Set]$

Now, fix any object, $A \in \mathcal{A}$, and any object, $X \in \mathcal{C}$.
Need to show that $\mathcal{C}(H_A, X)$ is naturally isomorphic to $X(A)$.

Thus, need two mutually inverse natural transformations, such that : $\mathcal{C}(H_A, X) \underset{\phi}{\overset{\psi}{\leftrightarrows}} X(A)$
Need to show that the RHS is a set, and then, As both the RHS and LHS above are sets, the natural transformations between them are maps in Set i.e. functions.
Going to show the following in order,

1. Define ϕ and ψ
2. Show that ϕ and ψ are mutually inverse
3. Show naturality of ϕ in X
4. Show naturality of ϕ in A
5. Show naturality of ψ in X
6. Show naturality of ϕ in A

1. Defining ϕ and ψ Define ϕ (on natural transformations) as the A-component (of that natural transformation) at the identity of A. i.e. for $\alpha \in \mathcal{C}(H_A, X), \phi(\alpha) := \alpha_A(1_A)$

Define ψ on an object, $x \in X(A)$, by defining it's K-component for any $K \in \mathcal{A}$:

$$(\psi(x))_K : H_A(K) \to X(K) \text{ as, for each } p \in Hom_{\mathcal{A}}(K, A), p \mapsto \Big(X(p)\Big)(x)$$

That is to say that the K-component maps any arrow $p : K \to A$ to the image of x under the map X(p).

2. **Showing inverses** **Firstly**,to show $\psi \circ \phi(\alpha) = \alpha$, for any natural transformation $\alpha : H_A \to X$

$$\psi \circ \phi(\alpha) = \psi\Big(\alpha_A(1_A)\Big) =$$

   Secondly, for every $x \in X(A)$, need $\psi \circ \phi(x) = x$

3. **Naturality of ϕ in X** Need to show that the following square commutes
   i.e.

4. **Naturality of ϕ in A** Need to show that the following square commutes
   i.e.

5. **Naturality of ψ in X** Need to show that the following square commutes
   i.e.

6. **Naturality of ψ in A** Need to show that the following square commutes
   i.e.

$\square$

2 Embedding of a category in Presheaf category

Definiton 2.1 (Embedding of a category)**.** A category, $\mathcal{A}$ is said to be embedded in another $\mathcal{B}$

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