Question 1 For counter-example, consider

Question 3 **(a)** $f(x,y) := \frac{x^2y^2}{x^2y^2+(x-y)^2} = \frac{1}{1+(\frac{x-y}{xy})^2} = \frac{1}{1+(\frac{1}{y}-\frac{1}{x})^2}$

$$\lim_{x\to 0} f(x,y) = \lim_{x\to 0} \frac{1}{1+(\frac{1}{y}-\frac{1}{x})^2} = 0 \implies \lim_{y\to 0}\lim_{x\to 0} f(x,y) = 0$$

And as the expression is symmetric in x and y,

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{x\to 0}\lim_{y\to 0} f(x,y)$$

But the simultaneous limit at$(0,0)$ along $T(t) := (t,t)$ is

$$\lim_{t\to 0} f(T(t)) = \lim_{t\to 0} \frac{1}{1+(\frac{1}{t}-\frac{1}{t})^2} = 1$$

If the simultaneous limit existed, all the iterated limits would be equal to it. So, there is a curve, $S(t) := (\frac{1}{t}, \frac{1}{t+1})$ with

$$\lim_{t\to\infty} f(S(t)) = \lim_{t\to\infty} \frac{1}{1+(t+1-t)^2} = \frac{1}{2}$$

Thus, f is discontinous at $(0,0)$

(b) $f(x,y) := \frac{e^{\frac{-1}{x^2}}y}{e^{\frac{-1}{x^2}}+y^2} = \frac{ye^{\frac{1}{x^2}}}{1+(ye^{\frac{1}{x^2}})^2} = \frac{1}{\frac{1}{ye^{1/x^2}}+ye^{1/x^2}}$

$$\lim_{y\to 0}\lim_{x\to 0} \frac{1}{\frac{1}{ye^{1/x^2}}+ye^{1/x^2}} = \lim_{y\to 0} 0 = 0$$

$$\lim_{x\to 0}\lim_{y\to 0} \frac{ye^{\frac{1}{x^2}}}{1+(ye^{\frac{1}{x^2}})^2} = \lim_{x\to 0} 0 = 0$$

To show the non-existence of simulatenous limit at $(0,0)$, consider the curve $T(t) := (t, e^{-1/t^2})$

$$\lim_{t\to 0} f(T(t)) = \frac{e^{-1/t^2}\times e^{1/t^2}}{1+(e^{-1/t^2}\times e^{1/t^2})^2} = \frac{1}{2}$$

Question 4