

1 Definitions

Definiton 1.1. A prorelation is a partially ordered, down-directed, upper set of relations $X \rightarrow Y$, i.e a set, $P \subseteq \mathcal{P}(X \times Y)$ such that

- (i) (Partial Order) Containment of relations, $r \subseteq s$ only if for any $(x, y) \in X \times Y$, $(x, y) \in r \implies (x, y) \in s$
- (ii) (Down-directed), For any $r, s \in P$, $\exists t \in P$ such that $t \subseteq r$ and $t \subseteq s$
- (iii) (Up-set) for any relation $u : X \rightarrow Y$, if $\exists p \in P$ such that $p \leq u$ then $u \in P$

Definiton 1.2. Prorelations are composed by taking compositions of the relations belonging to them.

$$\text{For prorelations } P : X \rightarrow Y \text{ and } Q : Y \rightarrow Z, \quad Q.P := \{q \circ p : p \in P \text{ and } q \in Q\}$$

Lemma 1.2.1. Prorelations are closed under composition

Definiton 1.3. Prorelations with same domain and co-domain are said to be comparable when one of them contains all the relations of the other one.

$$\text{For prorelations } P, Q : X \rightarrow Y, \quad P \leq Q \text{ if } \forall q \in Q, \exists p \in P \text{ such that } p \subseteq q$$

Definiton 1.4. For a relation $r : X \rightarrow Y$, it's opposite relation $r^o : Y \rightarrow X$ is defined as

$$\forall (x, y) \in X \times Y, \quad (x, y) \in r \text{ if and only if } (y, x) \in r^o$$

Lemma 1.4.1. For any relation $r : X \rightarrow Y$, $r^o \circ r = \Delta_X$

Lemma 1.4.2. For any relation $r : X \rightarrow Y$, $r \circ r^o \subseteq \Delta_Y$

Lemma 1.4.3. For relations $r, s : X \rightarrow Y$ and $t : Y \rightarrow Z$, for any $x, x' \in X$, $r(x) \subseteq s(x') \implies (t \circ r)(x) \subseteq (t \circ s)(x')$

Lemma 1.4.4. For relations $r : X \rightarrow Y$ and $s, t : Y \rightarrow Z$, $s \subseteq t \implies (s \circ r) \subseteq (t \circ r)(x)$

2 Propositions

Definiton 2.1. A function, $f : (X, A) \rightarrow (Y, B)$ is said to be uniformly continuous if $f.A \leq B.f$ i.e.

$$\forall b \in B, \exists a \in A \text{ such that } f \circ a \subseteq b \circ f \quad \text{i.e.} \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ A \downarrow & \leq & \downarrow B \\ X & \xrightarrow{f} & Y \end{array}$$

Definiton 2.2. A prorelation on a set, $P : X \rightarrow X$ is a quasi-uniformity if it satisfies :

- (i) $\forall p \in P$, p is reflexive i.e. for any $x \in X$, $(x, x) \in p$
- (ii) $\forall p \in P, \exists p' \in P$ such that $p' \circ p' \subseteq p$

And in that case, (X, A) is called a *quasi-uniform space*.

Lemma 2.2.1. If A is a quasi-uniformity on a set X , then $A.A = A$

Proof. Fix any $a \in A$, as A is a quasi-uniformity, $\exists b \in A : bb \subseteq a$, we get that $A.A \leq A$. And as A is a prorelation, and is hence down-directed, $\exists c \in A : a.a \supseteq c$, giving that $A.A \geq A$ □

Proposition 2.1. QUnif is a category defined as having quasi-uniform spaces as objects, and uniformly continuous maps between them as morphisms, with composition defined as that of functions.

Proof. (i) (Associativity) The composition of functions is associative by definition.

- (ii) (Identity) For each object (X, A) , define it's identity to be the identity function $\Delta_X : (X, A) \rightarrow (X, A)$. This function is uniformly continuous as $\Delta_X.A = A \leq A = A.\Delta_X$. □

Definiton 2.3. A prorelation, $\phi : X \multimap Y$ is called a promodule $\phi : (X, A) \multimap (Y, B)$ if it satisfies:

$$\phi.A \leq \phi \text{ and } B.\phi \leq \phi$$

Definiton 2.4. Comparison of promodules is defined as that of prorelations.

Definiton 2.5. Promodules are composed as prorelations.

Proposition 2.2. ProMod is a 2-category defined as having quasi-uniform spaces as its 0-cells, promodules as 1-cells and containment of promodules as 2-cells.

Proof. In order to show that ProMod is a 2-category, need the following:

- (a) (1-Identities) For each quasi-uniform space (X, A) , define $A : (X, A) \multimap (X, A)$ to be the identity 1-cell for (X, A) . A is a promodule because $A.A = A$ (Lemma 2.2.1)
- (b) (1-Composition) Need promodules to be closed under composition.
Let $\phi : (X, A) \multimap (Y, B)$ and $\psi : (Y, B) \multimap (Z, C)$ be promodules. To show that $\psi.\phi : (X, A) \multimap (Z, C)$ is a promodule, need it to be a prorelation that satisfies the two conditions required to be a promodule:
 - (i) By Lemma 1.2.1, prorelations are closed under composition. Hence, $\psi.\phi$ is a prorelation
 - (ii) Need to show that $\psi.\phi.A \leq \psi.\phi$. So, Fix $p \in \psi$ and $q \in \phi$. As ϕ is a promodule, $\phi.A \leq \phi$ gives that there exists $q' \in \phi$ and $a \in A$ such that $q'a \subseteq q$. Thus, $pq'a \subseteq pq$.
 - (iii) Need to show that $C.\psi.\phi \leq \psi.\phi$. Fix $p \in \psi$ and $q \in \phi$. Because ψ is a promodule, $C.\psi \leq \psi$ gives that there exists $c \in C$ and $p' \in \psi$ such that $cp' \subseteq p$. Thus, $cp'q \subseteq pq$
- (c) (2-Identities) As every promodule is contained in itself, always have $\psi \leq \psi$. Define this comparison to be the identity 2-cell for ψ and denote it by \leq_ψ
- (d) (Vertical 2-composition) For promodules $\psi, \phi, \delta : (X, A) \multimap (Y, B)$, if there is a 2-cell from ψ to ϕ and another one from ϕ to δ i.e. $\psi \leq \phi \leq \delta$, then by transitivity of the partial order, $\psi \leq \delta$ i.e. there's a 2-cell from ψ to δ .
- (e) (Horizontal 2-composition) If there are promodules $\psi, \psi' : (X, A) \multimap (Y, B)$ and $\phi, \phi' : (Y, B) \multimap (Z, C)$ such that $\psi \leq \psi'$ and $\phi \leq \phi'$, need to show that $\psi.\phi \leq \psi'.\phi'$. Fix $p' \in \psi'$ and $q' \in \phi'$. As $\psi \leq \psi'$, $\exists p \in \psi : p \subseteq p'$ and as $\phi \leq \phi'$, $\exists q \in \phi : q \subseteq q'$. Thus, $pq \subseteq p'q'$
- (f) (1-Identity) Need to show that for any promodule $\phi : (X, A) \multimap (Y, B)$, $\phi.A = \phi = B.\phi$. By quasi-uniformity of A , every $a \in A$, is reflexive. Thus, for any $p \in \phi$ and $a \in A$, $p = p.\Delta_X \subseteq pa$ giving that $\phi \leq \phi.A$. And as ϕ is a promodule, $\phi \geq \phi.A$. Hence, $\phi = \phi.A$.
Similarly, By quasi-uniformity of B , every $b \in B$, is reflexive. Thus, for any $p \in \phi$ and $b \in B$, $p = \Delta_Y.p \subseteq bp$ giving that $\phi \leq B.\phi$. And as ϕ is a promodule, $\phi \geq B.\phi$. Hence, $\phi = B.\phi$.
- (g) (1-Associativity) As composition of relations is associative, so is the composition of prorelations. Hence composition of promodules i.e. 1-cells is associative.
- (h) (Vertical 2-Identity) Let $\leq : \psi \rightarrow \phi$ be a 2-cell i.e. $\psi \leq \phi$. By our definition of identity 2-cell, $\leq_\psi \leq_1$ means precisely that $\psi \leq \psi \leq \phi$, and this is equivalent to $\psi \leq \phi$. Similarly, $\leq_1 \leq_\phi$ means precisely that $\psi \leq \phi \leq \phi$, and this is equivalent to $\psi \leq \phi$.
- (i) (Vertical 2-Associativity) Associativity of the partial order on promodules directly gives the associativity of composition of 2-cells in ProMod.
- (j) (Horizontal 2-Identity) Let $\psi, \phi : (X, A) \multimap (Y, B)$ be promodules. For any 2-cell $\leq : \psi \rightarrow \phi$, need to show that the 2-cell given by the horizontal composition, $\leq * \leq_A$ is equal to \leq , as well as equal to $\leq_B * \leq$. So, it's required that $\psi.A \leq \phi.A \iff \psi \leq \phi \iff B.\psi \leq B.\phi$. And this holds as a direct consequence of (f).
- (k) (Horizontal 2-Associativity) As there's a unique 2-cell between any two promodules, horizontal composition of 2-cells is associative as a consequence of composition of promodules being associative.
- (l) (2-Identity) For promodules $\psi : (X, A) \multimap (Y, B)$ and $\phi : (Y, B) \multimap (Z, C)$ need $(\leq_\psi * \leq_\phi) = \leq_{\psi.\phi}$. Both sides of the equality to be shown are 2-cells $\leq : \psi.\phi \rightarrow \psi.\phi$. Thus, they are equal by the uniqueness of 2-cells between any two 1-cells.
- (m) (2-Interchange) Let $\psi, \phi, \delta : (X, A) \multimap (Y, B)$ and $\psi', \phi', \delta' : (Y, B) \multimap (Z, C)$ be promodules. For 2-cells $\leq_1 : \psi \rightarrow \phi, \leq_2 : \phi \rightarrow \delta, \leq_a : \psi' \rightarrow \phi'$ and $\leq_b : \phi' \rightarrow \delta'$, need to show $(\leq_b * \leq_a) * (\leq_2 * \leq_1) = (\leq_b * \leq_2) * (\leq_a * \leq_1)$. Both RHS and LHS are 2-cells from $\psi.\psi'$ to $\delta.\delta'$ and are hence equal.

□

Proposition 2.3. Functor, $(-)_* : \text{QUnif} \rightarrow \text{ProMod}$ is defined as fixing objects and taking morphisms to their image under $(-)_*$

- (a) for $(X, A) \in \mathbf{QUnif}$, $(X, A)_* := (X, A) \in \mathbf{ProMod}$
- (b) for $f : (X, A) \rightarrow (Y, B)$ in \mathbf{QUnif} , $f_* := f^o.B$

Proof. □

Proposition 2.4. Functor, $(-)^* : \mathbf{QUnif}^{op} \rightarrow \mathbf{ProMod}$ is defined as fixing objects and taking morphisms to their image under $(-)^*$

- (a) for $(X, A) \in \mathbf{QUnif}^{op}$, $(X, A)^* := (X, A) \in \mathbf{ProMod}$
- (b) for $f : (X, A) \rightarrow (Y, B)$ in \mathbf{QUnif} , $f^* := f^o.B$

Proof.

Showing that $f^o.B : (Y, B) \rightarrow (X, A)$ is a promodule

So, need to show $f^o.B$ a prorelation $Y \rightarrow X$ and that $(f^o.B).B \subseteq f^o.B$ and $A.(f^o.B) \subseteq f^o.B$

To show prorelation,

- (i) (Partial-order) Inclusion of relations i.e. for $k = f^o \circ b$ and $k' = f^o \circ b'$ in $f^o.B$, $k \subseteq k' \iff b \subseteq b'$

- (ii) (Down directed) for $k, k' \in f^o.B$, need that $\exists l \in f^o.B$ such that $l \subseteq k, k'$

Fix $k, k' \in f^o.B \implies \exists b, b' \in B : k = f^o \circ b$ and $k' = f^o \circ b'$

And as B is a quasi-uniformity, it's down directed so, $\exists c \in B : c \subseteq b, b' \implies l := f^o \circ c \subseteq k, k'$

- (iii) (Up-set) for a relation $l : Y \rightarrow X$ and $k \in f^o.B$ such that $l \supseteq k$, need $l \in f^o.B$

Let $b \in B$ be such that $k = f^o \circ b$ and define $b' := \{(y, y') : y \in \text{domain}(l) \text{ and } y' \in (f^o)^{-1}(l(y))\}$

As $l \supseteq k = f^o \circ b$, $\text{domain}(b') = \text{domain}(l) \supseteq \text{domain}(b)$

and $\text{range}(l) \supseteq \text{range}(f^o \circ b) \implies \forall y \in \text{domain}(b), \text{range}(b') = (f^o)^{-1}(l(y)) \supseteq (f^o)^{-1}(f^o \circ b) = \text{range}(b)$

Now, by definition of b' , $f^o \circ b' \supseteq l$. To show $f^o \circ b' \subseteq l$,

$(x, y) \in f^o \circ b' \implies \exists z \in Y : (x, z) \in b'$ and $(z, y) \in f^o \implies x \in \text{domain}(l)$ and $z \in l(x)$ i.e. $(x, z) \in l$

To show $(f^o.B).B \subseteq f^o.B$, need that $\forall b \in B, \exists b' \in B : f^o \circ b' \circ b' \subseteq f^o \circ b$,

Fix any $b \in B$, as B is a quasi-uniformity, $\exists b' \in B : b' \circ b' \subseteq b \implies f^o \circ b' \circ b' \subseteq f^o \circ b$

To show $A.(f^o.B) \subseteq f^o.B$, need that $\forall b \in B, \exists b' \in B, a \in A : a \circ f^o \circ b' \subseteq f^o \circ b$,

As f is uniformly continuous, $f.A \leq B.f$ i.e. $\forall b \in B, \exists a \in A : f \circ a \subseteq b \circ f \implies a = f^o \circ f \circ a \subseteq f^o \circ b \circ f$

Fix any $b \in B$, so, $\exists b' \in B : b' \circ b' \subseteq b$ (for brevity, omitting \circ to explicitly denote composition)

And, for this b' , $\exists a : a \subseteq f^o \circ b' \implies a \circ f^o \circ b' \subseteq f^o \circ b' \circ f^o \circ b' \subseteq f^o \circ b' \circ b' \subseteq f^o \circ b \implies a \circ f^o \circ b' \subseteq f^o \circ b$

Now, need to show that $(-)^*$ respects composition and identity.

- (i) (Composition) let f, g be uniformly continuous, $(X, A) \xrightarrow{f} (Y, B) \xrightarrow{g} (Z, C)$ need that $(g \circ f)^* = f^*.g^*$

LHS $= (g \circ f)^* = (g \circ f)^o.C = (f^o \circ g^o).C$ and RHS $= f^*.g^* = (f^o.B).(g^o.C)$

For equality, showing that LHS \geq RHS and LHS \leq RHS:

To show $(f^o \circ g^o).C \geq (f^o.B).(g^o.C)$, need that $\forall c \in C, \exists b \in B, c' \in C : f^o \circ g^o \circ c \supseteq f^o \circ b \circ g^o \circ c'$

Fix any $c \in C$, so, $\exists c' \in C : c' \circ c' \subseteq c \implies f^o \circ g^o \circ c \supseteq f^o \circ g^o \circ (c' \circ c') = f^o \circ g^o \circ (c' \Delta_Z c') \supseteq f^o \circ g^o \circ (g \circ g^o) \circ c'$

By uniform continuity of g , for $c' \in C, \exists b \in B : g \circ b \subseteq c' \circ g$

Thus, $f^o \circ g^o \circ c \supseteq f^o \circ g^o \circ (c' \circ g) \circ g^o \circ c' \supseteq f^o \circ (g^o \circ g) \circ b \circ g^o \circ c' = f^o \circ b \circ g^o \circ c'$

To show $(f^o \circ g^o).C \leq (f^o.B).(g^o.C)$, need that $\forall b \in B, c \in C, \exists c' \in C : f^o \circ g^o \circ c \subseteq f^o \circ b \circ g^o \circ c'$

Fix any $c \in C, b \in B$ will show that $c' := c$ works:

As B is a quasi-uniformity, $\Delta_Y \subseteq b \implies f^o \Delta_Y \circ g^o \circ c = f^o \circ g^o \circ c \subseteq f^o \circ b \circ g^o \circ c = f^o \circ b \circ g^o \circ c'$

- (ii) (Identity) let $(X, A) \in \mathbf{QUnif}^{op}$, and $1_{(X,A)} : (X, A) \rightarrow (X, A)$ as $x \mapsto x$ need that $(1_{(X,A)})^* = 1_{(X,A)}^*$

LHS $= (1_{(X,A)})^* = (1_{(X,A)})^o.A = 1_{(X,A)}.A = A$.

Now, it's required that A is the identity of (X, A) in \mathbf{ProMod} .

So, fix $\phi : (X, A) \rightarrow (Y, B)$, need to show $\phi.A = \phi$

As ϕ is a promodule, $\phi.A \leq \phi$ and as A is quasi-uniformity on X ,

$\forall a \in A, \Delta_X \subseteq a \implies \forall a \in A, \forall p \in \phi, p = p \Delta_X \subseteq p \circ a \implies \phi \leq \phi.A$

Also, fix $\psi : (Y, B) \rightarrow (X, A)$, need to show $A.\psi = \psi$

As ψ is a promodule, $A.\psi \leq \psi$ and as A is quasi-uniformity on X ,

$\forall a \in A, \Delta_X \subseteq a \implies \forall a \in A, \forall q \in \psi, q = \Delta_X \circ q \subseteq a \circ q \implies \psi \leq \psi.A$

□

Definiton 2.6. Let $f : (X, A) \rightarrow (Y, B)$ be a uniformly continuous function.

I f is said to be fully faithful if $f^*.f_* = A$

II f is said to be fully dense if $f_*.f^* = B$

Proposition 2.5. Fix a uniformly continuous map, $f : (X, A) \rightarrow (Y, B)$

- (a) f is fully faithful if and only if $A \geq f^o.B.f$
- (b) f is fully dense if and only if for any $b \in B$, $\exists b' \in B$ such that $b' \subseteq b f f^o b$
- (c) f is topologically dense if and only if for any $b \in B$, $b f f^o b$ is reflexive
- (d) f is fully dense if and only if f is topologically dense

Proof.

- (a) (i) (\implies) Let f be fully faithful i.e. $f^*.f_* = A \implies f^o.B.B.f = A$
 Need to show that $A = f^o.B.f$ i.e. $A \leq f^o.B.f$ and $A \geq f^o.B.f$
 By hypothesis and quasi-uniformity of B , $A \geq f^o.B.B.f \geq f^o.B.f$
 To show $A \leq f^o.B.f$, need that $\forall b \in B, \exists a \in A : a \subseteq f^o b f$
 Fix $b \in B$, hypothesis gives that $f^o.B.B.f \leq A$ so,
 $\exists a \in A : a \subseteq f^o b b f$ and also, by quasi-uniformity of B , for $b, \exists b' \in B : b' b' \subseteq b \implies f^o b' b' f \subseteq f^o b f$
 Combining the above two inequalities, $a \subseteq f^o b b f \subseteq f^o b f$
- (ii) (\impliedby) Let $A = f^o.B.f$ need to show $A = f^o.B.B.f$ i.e. $A \geq f^o.B.B.f$ and $A \leq f^o.B.B.f$
 To show $A \geq f^o.B.B.f$, need to show that $\forall a \in A, \exists b, b' \in B : a \supseteq f^o b b' f$
 Have that $A \geq f^o.B.f$ and $B.B \leq B$
 So, fix $a \in A$, now $\exists b \in B : a \subseteq f^o b f$ and for this b , $\exists b' \in B : b' b' \subseteq b$. Therefore, $a \supseteq f^o b f \supseteq f^o b' b' f$
 To show $A \leq f^o.B.B.f$, need $\forall b, b' \in B, \exists a \in A : a \subseteq f^o b b' f$
 Before that, uniform continuity of f along with Lemma 2.1.1 gives that
 $f.A \leq B.f \implies A = f^o f.A \leq f^o.B.f$
 So, fix $b, b' \in B$, now, as ,
 $A \leq f^o.B.f$ giving
 $\exists a \in A : a \subseteq f^o b f$ and $\exists a' \in A : a' \subseteq f^o b' f \implies \Delta_X \subseteq f^o b' f$.
 Therefore $a = a \Delta_X \subseteq (f^o b f)(f^o b' f) \subseteq f^o b b' f$
- (b) (i) (\implies) Let f be fully dense i.e. $B = f_*.f^* = B.f.f^o.B$. showing that $\forall b \in B, \exists b' \in B : b' \subseteq b f f^o b$:
 So, fix $b \in B$, as $B \leq B.f.f^o.B$, there exists $b' \in B$ such that $b' \subseteq b f f^o b$.
- (ii) (\impliedby) Suppose $\forall b \in B, \exists b' \in B : b' \subseteq b f f^o b$. This gives $B \leq B.f.f^o.B$, in order to show equality, also need $B \geq B.f.f^o.B$. By quasi-uniformity of B , for any $b \in B$, $\exists b' \in B : b' b' \subseteq b$. Now, by Lemma 1.9.2,

$$f f^o \subseteq \Delta_Y \implies b' f f^o b' \subseteq b' \Delta_Y b' = b' b' \subseteq b$$

- (c) (i) (\implies) Let f be topologically dense, going to show that $\forall b \in B, (y, y) \in b f f^o b$. So, fix any $b \in B$ and $y \in Y$. Now, by definition of $f(X) = Y$, we get

$$\exists x \in X \text{ such that } (f(x), y) \in b \text{ and } (y, f(x)) \in b$$

Re-writing the above statement in terms of relations, and considering f as a relation:

$$(f(x), y) \in b \text{ gives } x(b \circ f)y \text{ i.e. } y \in (b \circ f)(x) \quad (1)$$

$$(y, f(x)) \in b \text{ gives } f(x) \subseteq b(y) \quad (2)$$

Repeatedly applying Lemma 1.9.3 to (2),

$$f(x) \subseteq b(y) \implies (f \circ f^o)(f(x)) \subseteq (f \circ f^o)b(y) \implies (f \circ f^o \circ f)(x) \subseteq (f \circ f^o \circ b)(y)$$

Applying Lemma 1.9.1 to the above statement gives that

$$f(x) = (f \circ f^o \circ f)(x) \subseteq (f \circ f^o \circ b)(y)$$

Applying Lemma 1.9.3 and then using (1) to this inequality completes the result:

$$f(x) \subseteq (f f^o b)(y) \implies (b \circ f)(x) \subseteq (b f f^o b)(y) \implies y \in (b f f^o b)(y) \text{ i.e. } y(b f f^o b)y$$

- (ii) (\Leftarrow) Fix any $y \in Y$ and $b \in B$. Also, suppose that $\Delta_Y \leq bff^ob$. As f is a function with domain as X , $f^o : Y \rightarrow X$, $\phi \neq (f^o \circ b)(y) \subseteq X$. So, fix $x \in (f^o \circ b)(y)$, going to show that $(f(x), y) \in b$ and $(y, f(x)) \in b$. Again, while viewing f as a relation.

$$\Delta_Y \leq bff^ob \implies \Delta_Y(y) \subseteq bff^ob(y) = (bf)(f^ob(y))$$

Using Lemma 1.9.3 on the above statement, gives $y \in (bf)(x)$ i.e. $(f(x), y) \in b$.
Applying Lemma 1.9.3 to f , and then using Lemma 1.9.4,

$$ff^o \subseteq \Delta_Y \implies ff^ob \subseteq \Delta_Y b = b$$

Thus $ff^ob(y) \subseteq b(y)$ and hence $f(x) \subseteq b(y) \implies (y, f(x)) \in b$

- (d) (i) (\implies) Let f be topologically dense. As B is a quasi-uniformity, for any $b \in B$,

$$\exists b' \in B : b'b' \subseteq b \text{ and } \Delta_Y \subseteq b' \implies b' = b'\Delta_Y \subseteq b'b' \subseteq b \quad (3)$$

By the characterisation of topologically dense in (c), have that $\Delta_Y \subseteq b'ff^ob'$. Now, using the (3) and Lemma 1.9.2,

$$\Delta_Y \subseteq b'ff^ob' \implies b' = b'\Delta_Y \subseteq b'b'ff^ob' \subseteq bff^ob' \subseteq bff^ob$$

Hence, we have $b' \in B : b' \subseteq bff^ob$ giving us that f is fully dense (from (b)).

- (ii) (\Leftarrow) From (b), we have for $b \in B$, the existstence of $b' \in B$ such that $b' \subseteq bff^ob$. As B is a quasi-uniformity, $\Delta_Y \subseteq b'$. So, $\Delta_Y \subseteq bff^ob$, and from (c), this gives us that f is topologically dense. \square

Definiton 2.7. The set PX is defined to be the collection of all promodules from the quasi-uniform space (X, A) to the quasi-uniform space 1.

$$PX := \{\psi : \psi : (X, A) \multimap 1 \text{ is a promodule}\}$$

Proposition 2.6. For any $a \in A$, \tilde{a} is defined to be a relation $PX \rightarrow PX$ as

$$\text{for } \phi, \psi \in PX, \phi \tilde{a} \psi \text{ only if } \phi \leq \psi.a$$

The set, $\tilde{A} := \{\tilde{a} : a \in A\}$ defines a quasi-uniformity on PX .

Proof. First need to show that \tilde{A} is a prorelation,

- (i) (Partial order) For any two relations $\tilde{a}, \tilde{b} : PX \rightarrow PX$, define $\tilde{a} \leq \tilde{b}$ to be true only if $a \subseteq b$.
- (ii) (Down-Directed) Need for any $\tilde{a}, \tilde{b} \in \tilde{A}$, the existstence of some $\tilde{c} \in \tilde{A}$ such that $c \subseteq a, b$.
If $\tilde{a}, \tilde{b} \in \tilde{A}$ then there exist $a, b \in A$. By down-directedness of A , there exists a $c \in A$ such that $c \subseteq a, b$. Now the definition of \tilde{A} gives that $\tilde{c} \in \tilde{A}$. And the definition of the partial order on \tilde{A} ensures $\tilde{c} \leq \tilde{a}, \tilde{b}$.
- (iii) (Upset) For any relation $l : PX \rightarrow PX$, need that if \tilde{k} belongs to \tilde{A} such that $l \geq \tilde{k}$, then $l \in \tilde{A}$.
Fix any $k : PX \rightarrow PX$, and $\tilde{k} \in \tilde{A}$ such that $l \geq \tilde{k}$. As k is a relation between promodules $X \multimap 1$, it can be thought of as a relation a on X , defined as:

$$a := \{(x, y) : x \in \text{domain}(\psi) \text{ and } y \in \text{domain}(\phi) \text{ whenever } \exists \psi, \phi \in PX : \psi l \phi\}$$

So, $l \geq \tilde{k}$ gives that $\tilde{a} \geq \tilde{k}$ i.e. $a \supseteq k$. And as A is an upper-set, we get $a \in A$. Now, by definition of \tilde{A} , $l \in \tilde{A}$.

Secondly, need show that the other two conditions hold for \tilde{A} ,

- (i) For all $\tilde{a} \in \tilde{A}$, need \tilde{a} to be reflexive i.e if $\psi \in PX$ then $\psi \tilde{a} \psi$.
By definition of \tilde{a} , need to show that $\psi \leq \psi.a$. So, fix a $p \in \psi$, we will show that $p \subseteq p.a$. Quasi-uniformity of A gives that $\Delta_X \subseteq a$. Hence, by Lemma 1.9.3, $p = p\Delta_X \subseteq p.a$.
- (ii) For all $\tilde{a} \in \tilde{A}$, need to find $\tilde{b} \in \tilde{A}$ such that $\tilde{b}\tilde{b} \leq \tilde{a}$
Before showing the result, proving that for any $x, y \in A$, $\tilde{x}\tilde{y} \leq \widetilde{xy}$ i.e. $\forall \psi, \phi \in PX, \psi(\tilde{x}\tilde{y})\phi \implies \psi\widetilde{xy}\phi$. If $\psi_1(\tilde{a}.\tilde{b})\psi_3$, then, the definition of composition gives that $\exists \psi_2$ such that $\psi_1\tilde{b}\psi_2\tilde{a}\psi_3$. Now, the definition of \tilde{b} gives $\psi_1 \leq \psi_2.b$ and that of \tilde{a} gives $\psi_2 \leq \psi_3.a$. Combining these inequalities, $\psi_1 \leq \psi_2.b \leq \psi_3.ab$. Hence, by definition of ab , $\psi_1(ab)\psi_3$. Now, to show the result, fix any $\tilde{a} \in \tilde{A}$. Therefore, $a \in A$, and by quasi-uniformity of A , $\exists b \in A : b \circ b \subseteq a$. Thus, by the partial-order defined on \tilde{A} , $\tilde{b}\tilde{b} \leq \tilde{a}$. Now, transitivity of the partial order gives us the required result, $\tilde{b}\tilde{b} \leq \tilde{b}\tilde{b} \leq \tilde{a}$. \square

Proposition 2.7 (Yoneda Embedding).

For a quasi-uniform space (X, A) , function $y_X : X \rightarrow PX$ is defined by $x \mapsto x^*$ for $x \in X$.

- (a) $y_X : (X, A) \rightarrow (PX, \tilde{A})$ is a uniformly continuous map
- (b) $y_X : (X, A) \rightarrow (PX, \tilde{A})$ is fully faithful

Proof.

- (a) In order to show y_X is uniformly continuous, need to show that $y_X.A \leq \tilde{A}.y_X$. By definition of \leq , need $\forall a \in A, \exists b \in A : y_X \circ b \subseteq \tilde{a} \circ y_X$. Applying the relations to some element, x of the set X :

$$(y_X \circ b)(x) \subseteq (\tilde{a} \circ y_X)(x) \implies y_X(b(x)) \subseteq \tilde{a}(x^*) \quad (4)$$

So, for the condition given by (4) to hold, if $y \in b(x)$, then it's required that $y^* = y_X(y) \in \tilde{a}(x^*)$ i.e. $x^* \tilde{a} y^*$. Using the definition of x^*, y^* and \tilde{a} ,

$$x^* \tilde{a} y^* \iff x^o.A \leq y^o.A.a \iff \forall a' \in A, \exists a'' \in A : x^o a'' \subseteq y^o a' a \quad (5)$$

Now, fix any $a \in A, x \in X$. Thus, quasi-uniformity of A , gives $a'' \in A$ such that $a'' a'' \subseteq a$. Also, choose some $y \in a''(x)$. Hence, in order to show that the condition from (5) holds, need that $\forall b \in A, x^o a'' \subseteq y^o b a$, and by applying the relations to an element z gives the following condition

$$\forall b \in B, \forall x \in X, (x^o a'')(z) \subseteq (y^o b a)(z) \quad (6)$$

Examining the left side of (6),

$$(x^o a'')(z) = x^o(a''(z)) = \begin{cases} \phi & \text{if } x \notin a''(z) \\ \star & \text{if } x \in a''(z) \end{cases}$$

Thus, to show that (6) holds, need to show that (for any $b \in A$ and $z \in X$):

$$x \in a''(z) \implies z(y^o b a) \star \text{ i.e. } y \in (b a)(z) \quad (7)$$

To show that (7) holds, fix any $z \in X : x \in a''(z)$. Also, by our choice of y , have that $y \in a''(x)$. And as $b \in A$, it's reflexive, giving that $y \in b(y)$. So, by composition of relations, we get:

$$z a'' x, x a'' y \text{ and } y b y \implies z(a'' a'' b) y \implies z(a b) y \text{ i.e. } y \in (b a)(z)$$

- (b) By using Proposition 2.3(a), need to show that $A \geq y_X^o.\tilde{A}.y_X$ i.e. $\forall a \in A, \exists \tilde{b} \in \tilde{A} : a \supseteq y_X^o \tilde{b} y_X$. Applying to an element, $x \in X$ gives the condition

$$(y_X^o \tilde{b} y_X)(x) \subseteq a(x) \implies (y_X^o \tilde{b})(x^*) = y_X^o(\tilde{b}(x^*)) \subseteq a(x) \quad (8)$$

Thus, if $y^* \in PX$ such that $x^* \tilde{b} y^*$, then $y \in y_X^o(\tilde{b}(x^*))$. Now, for (8) to hold, $y \in a(x)$ i.e. $x a y$. Thus, need only to show that for any $a \in A, \exists b \in A$ such that $\forall x, y \in X, x^* \tilde{b} y^* \implies x a y$. So, fix $a \in A$, and take $b \in A : b b \subseteq a$. Now, let $x^* \tilde{b} y^*$ i.e. $x^o.A \leq y^o.A.b$. Hence, $\exists c \in A : x^o c \subseteq y^o b b$. And as c is reflexive,

$$x c x \implies x(c x^o) \star \implies x(b b y^o) \star \implies x(b b) y \implies x a y$$

□

Theorem 2.1 (Yoneda Lemma). *For every $\psi \in PX$, in the following digram,*

$$\begin{array}{ccc} X & \xrightarrow{(y_X)_*} & PX \\ & \searrow \psi & \downarrow \psi^* \\ & & 1 \end{array}$$

- (a) $\psi \geq \psi^*.(y_X)^*$
- (b) $\psi \in \overline{y_X(X)} \implies \psi \leq \psi^*.(y_X)_*$

Proof. (a) By definition, $(y_X)_* = \tilde{A}.y_X$, and $\psi^* = \psi^o.\tilde{A}$. Need that $\psi \geq (y_X)_*.\psi^* = \psi^o.\tilde{A}.\tilde{A}.y_X$. And applying Lemma 2.2.1 to \tilde{A} , the required condition becomes $\psi \geq \psi^o.\tilde{A}.y_X$. Fix $p \in \psi$, we will find $a \in A : p \supseteq \psi^o a y_X$. Examining the right side of the condition, (for any $a \in A, x \in X$)

$$(\psi^o.\tilde{a}.y_X)(x) = \psi^o.\tilde{a}(x^*) = \psi^o(\tilde{a}(x^*)) = \begin{cases} \phi & \text{if } \psi \notin \tilde{a}(x^*) \\ \star & \text{if } \psi \in \tilde{a}(x^*) \end{cases} \quad (9)$$

In case $\psi \notin \tilde{a}(x^*)$, the condition holds trivially. As ψ is a promodule, $\psi.A \leq \psi$ gives $\exists q \in \psi, a \in A : qa \subseteq p$. Thus, fix $x \in X$ and $\psi \in PX$ such that $x^* \tilde{a} \psi$. We will now show that $xp \star$. Using the definition of \tilde{a} ,

$$x^* \tilde{a} \psi \implies x^o.A \leq \psi.a \implies \exists b \in A : x^o b \subseteq qa \implies \forall z \in X, (x^o b)(z) \subseteq (qa)(z) \quad (10)$$

Thus, in particular for $z = x$, as b is reflexive, xbx , which gives:

$$(x^o b)(x) \subseteq (qa)(x) \implies x^o x \subseteq (qa)(x) \implies \star \in (qa)(x) \quad (11)$$

But, as $qa \subseteq p$, (11) gives that $xp \star$.

- (b) Suppose $\psi \in \overline{y_X(X)}$, need to show $\psi \leq \psi^*.(y_X)_* = \psi^o.\tilde{A}.y_X$ i.e. for $a \in A$, $\exists p \in \psi : p \subseteq \psi^o.\tilde{a}.y_X$. For any $x \in \text{domain}(p)$, the condition requires:

$$p(x) \subseteq \psi^o.\tilde{a}.y_X(x) = \psi^o(\tilde{a}(x^*)) \quad (12)$$

By definition of p , for (12) to hold, need that $xp \star \implies \psi \in \tilde{a}(x^*)$. Fix any $a \in A$, we will find $p \in \psi$ such that (12) holds. By quasi-uniformity of A , $\exists b \in A : bb \subseteq a$. From Proposition 2.5(a), y_X is uniformly continuous, $y_X.A \leq \tilde{A}.y_X$ giving that $\exists c \in A : y_X c \subseteq \tilde{b}y_X$. Thus, for any $z, w \in X$ such that $z c w$,

$$(y_X c)(z) \subseteq (\tilde{b}y_X)(z) \implies y_X(c(z)) \subseteq \tilde{b}(z^*) \implies w^* \in \tilde{b}(z^*) \text{ i.e. } z^* \tilde{b} w^* \quad (13)$$

As A is a quasi-uniformity, $\exists d \in A : dd \subseteq c$. Also, because A is a down-directed set, $\exists a' \in A : a' \subseteq b, d$. This along with (13) gives that for any $x, y \in X$

$$x(a'a')y \implies x(dd)y \implies xcy \implies x^* \tilde{b} y^* \quad (14)$$

Now, because $\psi \in \overline{y_X(X)}$, we get $\exists x^* \in y_X(X)$ such that $\psi \tilde{a}' x^*$ and $x^* \tilde{a}' \psi$. By definition of \tilde{a} , $\psi \tilde{a}' x^*$ gives

$$\psi \leq x^o.A.a' \implies \exists p \in \psi : p \subseteq x^o a' a' \quad (15)$$

Fix any $z \in X : zp \star$, using (15) and (14) gives:

$$zp \star \xrightarrow{\sim} (x^o a' a') \star \xrightarrow{(15)} z(a' a') x \xrightarrow{(14)} z^* \tilde{b} x^* \quad (16)$$

Finally, by definition of the partial order on $\tilde{A}, a' \subseteq b \implies \tilde{a}' \subseteq \tilde{b}$. Therefore, $x^* \tilde{a}' \psi \implies x^* \tilde{b} \psi$. Now, using (16), $z^* \tilde{b} x^*$ and $x^* \tilde{b} \psi$ gives the desired result $z^* \tilde{b} x^*$. \square

Corollary 2.1. For $\psi \in PX$, $\psi \in \overline{y_X(X)}$ if and only if ψ is a right-adjoint.

Proof. (i) (\implies)

- (ii) (\impliedby) Suppose ψ is a right adjoint. Need to show that for any $a \in A$, $\exists x^* \in y_X(X)$ such that $\psi \tilde{a} x^* \tilde{a} \psi$. Fix $a \in A$. Because ψ is a right-adjoint, there exists a promodule $\phi : 1 \dashv \Rightarrow X$ such that $\phi.\psi \leq A$ and $1 \leq \psi.\phi$. From $\phi.\psi \leq A$, we get that

$$\exists p \in \phi, q \in \psi \text{ such that } a \supseteq p.q \quad (1)$$

Because ϕ and ψ are promodules,

$$A.\phi \leq \phi \text{ gives the existence of } p' \in \phi \text{ such that } p \supseteq a' p' \quad (2)$$

$$A.\psi \leq \psi \text{ gives the existence of } q' \in \psi \text{ and } a'' \in A \text{ such that } q \supseteq a'' q' \quad (3)$$

Now, from $1 \leq \psi.\phi$, we get that $q' p'$ is reflexive i.e. $\star(q' p') \star$. By the definition of composition we get the existence of an $x \in X$ such that $\star p' x q' \star$. Now, considering x as a map, $x : 1 \rightarrow X$ defined as $\star \mapsto x$,

$$x q' \star \text{ i.e. } \star \in q'(x) \text{ gives that } q' \supseteq x^o \quad (4)$$

$$\star p' x \text{ i.e. } x \in p'(\star) \text{ gives that } p' \supseteq x \quad (5)$$

Thus, by using inequalities (1), (2) and (3), we get that

$$a \supseteq p q \supseteq a' p' q' a'' \quad (6)$$

By definition of \tilde{a} , to show $\psi \tilde{a} x^*$, we need that $\psi \leq x^* a = x^o.A.a$. Showing for any $b \in A$, $x^o b a \supseteq q$:

$$x^o b a \supseteq x^o b a' p' q' \supseteq x^o b a' x q' \supseteq x^o x q' = q'$$

Where the first inequality comes from (6) by using reflexiviness of a'' and then lef-multiplying by x^o . The second inequality comes from (5). Reflexiviness of b and a' gives the third inequality. And the equality is given by Lemma 1.9.2.

By definition of \tilde{a} , in order to show $x^* \tilde{a} \psi$, need that $x^o.A = x^* \leq \psi a$. We will show, for any $k \in \psi, k a \supseteq x^o a''$
 FOLLOWING IS WRONG APPROACH

$$a \supseteq a' p' q' a'' \supseteq p' q' a'' \supseteq x x^o a'' \supseteq a'' \quad (7)$$

Where the first inequality is given by (6) , second is due to reflexiviness of a' . The third one is given by (4) and (5) and the last one is by applying Lemma 1.9.2. Using Lemma 1.9.4 on (3) and (7) gives

$$q \supseteq q' a'' \implies q a \supseteq q'$$

□