1 Chapter 1

2 Functors(pg26)

Excercise (1.2.21)**.** As $A \cong$ A', there's a bijection between them in $\mathcal{A}$ i.e. $A \underset{f^{-1}}{\overset{f}{\longleftrightarrow}} A'$ with $f \in \mathcal{A}(A, A')$ and $f^{-1} \in \mathcal{A}(A', A)$. Thus, $F(f) \in \mathcal{B}(F(A), F(A'))$ and $F(f^{-1}) \in \mathcal{B}(F(A'), F(A))$. And as F is a functor,

$$F(f) \circ F(f^{-1}) = F(f \circ f^{-1}) = F(1_A) = 1_{F(A)} \text{ similarly, } F(f^{-1}) \circ F(f) = 1_{F(A')}$$

Hence, we have an isomorphism, $F(A) \underset{F(f^{-1})}{\overset{F(f)}{\longleftrightarrow}} A'$.

Excercise (1.2.22)**.** Let $a, a' \in A$ with $a \leq a'$, so that there's a morphism, $f : a \to a'$ in $\mathcal{A}$.
Now, $F : \mathcal{A} \to \mathcal{B}$ gives $F(a) \xrightarrow{F(f)} F(a')$ i.e. $F(a) \leq F(a')$.

Excercise (1.2.23)**.** (a) As $ob(G) = ob(G^{op})$,just need to ensure morphisms.

$$\forall f \in \mathcal{A}(A, B), \;\; f^{-1} \in \mathcal{A}(B, A) \implies f \in \mathcal{A}^{op}(B, A), \;\; f^{-1} \in \mathcal{A}^{op}(A, B)$$

Thus, for any two objects A, B , morphisms between them in $\mathcal{A}$ are also in $\mathcal{A}^{op}$. More precisely, define functors

$$G \xrightarrow{F} G^{op} \text{ and } G^{op} \xrightarrow{F^{-1}} G \text{ as } F(f) \mapsto f^{-1} \text{ and } G(f^{-1}) \mapsto f$$

Hence, $F^{-1} \circ F : G \to G$ and $F \circ F^{-1} : G^{op} \to G^{op}$. Giving an isomorphism, $G \cong G^{op}$ in **CAT**.

(b) Take the monoid, say M, consisting 2×2 matrices, $\left\{ I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ under matrix multiplication. So, M has two morphisms, the identity morphism, i and and a. Now, suppose $M \cong M^{op}$, then, a must have an inverse i.e. $k :=\in M : k.a = i$ i.e. a matrix $K : KA = I$ but this can't be as A is a singular matrix.

Excercise (1.2.25)**.** (a) Let $A \in \mathcal{A}$ and F^A be defined as mentioned. For unitality, as $F^A(B) = F(A, B)$

$$B \in \mathcal{B} \implies F^A(1_B) = F(1_A, 1_B) = 1_{F_{(A,B)}} = 1_{F^A(B)}$$

And for composability, let $f \in \mathcal{B}(X, Y)$ and $g \in \mathcal{B}(Y, Z)$,

$$F^A(f) \circ F^A(g) = (1_A, f) \circ (1_A, g) = (1_A, f \circ g) = F^A(f \circ g)$$

Hence, F^A is a functor. To prove F_B a functor,
For unitality, as $F_B(A) = F(A, B)$

$$A \in \mathcal{A} \implies F_B(1_A) = F(1_A, 1_B) = 1_{F_{(A,B)}} = 1_{F_B(A)}$$

And for composability, let $f \in \mathcal{A}(X, Y)$ and $g \in \mathcal{A}(Y, Z)$,

$$F_B(f) \circ F_B(g) = (f, 1_B) \circ (g, 1_B) = (f \circ g, 1_B) = F_B(f \circ g)$$

(b) Let $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then, $F^A(B) = F(A, B) = F_B(A)$. For $f \in \mathcal{A}(A, A')$ and $g \in \mathcal{B}(B, B')$,

$$F^{A'}(g) \circ F_B(f) = F(1_{A'}, g) \circ F(f, 1_B) = F(1_{A'} \circ f, g \circ 1_B) = F(f \circ 1_A, 1_{B'} \circ g) = F_{B'}(f) \circ F^A(g)$$

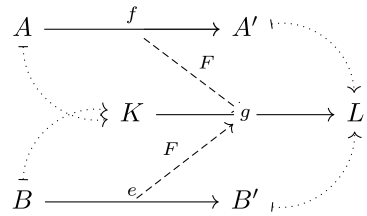
(c) Define functor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ as

$$\forall (A, B) \in \mathcal{A} \times \mathcal{B}, F((A, B)) := F^A(B) \text{ for objects}$$

And, for any morphism in the product category, $(A, B) \xrightarrow{(f,g)} (A', B')$, $F((f, g)) := F^{A'}(g) \circ F_B(f)$ Now, This functor exists, as $F((1_A, 1_B)) = F^{A'}(1_B) \circ F_B(1_A) =$

Excercise (1.2.26). need to read topology

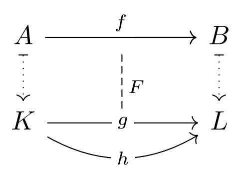
Excercise (1.2.27). Hom-set of each pair of objects must map injectively, but that condition need not hold for the objects themselves. So, consider a category, $\mathcal{A}$ with objects A, A', B, B', and morphisms $f_1 : A \to A'$ $f_2 : B \to B'$. And a category, $\mathcal{B}$ with objects K, L and morphism $g : K \to L$. And define F as taking f_1, f_2 to g.



This functor is faithful as every hom-set has at most one non-identity morphism. And the functor doesn't map any of those to identity.

Excercise (1.2.28). (a) To do 3

(b) Identity functor from any category to itself will be full and faithful, as it'll map each morphism only to itself(injectiveness) and it'll so map every morphism (surjectiveness). Also, F from excercise 1.2.27 is both full and faithful.
The following digram describes a functor F from $\mathcal{A}$ containing A, B and f to $\mathcal{B}$ containing K, L and g, h with $F(f) = g$. But, as h isn't in $Im(F)$, F isn't full. But it's faithful as it doesn't map f to identity.



For the same categories, take $G : \mathcal{B} \to \mathcal{A}$ as $K \mapsto A$; $L \mapsto B$ and $g, h \mapsto f$. Gives G as a non-faithful but full functor. For a functor that's neither full nor faithful, take $H : S_n \to S_n$ defined as taking all homomorphisms to the identity map.

Excercise (1.2.29).

To do 1

3 Natural Transformation (pg38)

Excercise (1.3.26).

Excercise (1.3.27).

Excercise (1.3.28).

Excercise (1.3.29).

Excercise (1.3.30).

Excercise (1.3.31).

Excercise (1.3.32).

Excercise (1.3.33).

Excercise (1.3.34).

TBD