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Theorem 1.1. **Yoneda** If $\mathcal{A}$ be a locally small category, then,

$$[\mathcal{A}^{op}, Set](H_A, X) \cong X(A) \text{ naturally in } A \in \mathcal{A} \text{ and } X \in [\mathcal{A}^{op}, Set]$$

Explaination:
First of all, we fix any category, $\mathcal{A}$. Now we choose two things (independent of each other):

i an object, A from the category $\mathcal{A} = \mathcal{A}^{op}$

ii an object, X of the category $[\mathcal{A}^{op}, Set]$ which is precisely a functor $X : A^{op} \to Set$

Here, $[\mathcal{A}^{op}, Set](H_A, X)$ denotes arrows $H_A \to X$ in $[\mathcal{A}^{op}, Set]$ i.e. natural transformations, $\alpha :$
$$\begin{tikzcd} A^{op} \arrow[rr, bend left, "H_A"] \arrow[rr, bend right, "X"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & Set \end{tikzcd}$$

Each of these natural transformations is a collection of maps in Set, hence each of their components is exactly a function. i.e. $\forall \alpha \in [\mathcal{A}^{op}, Set](H_A, X), \forall K \in \mathcal{A}, \alpha_K$ is a function :$H_A(K) \to X(K)$

$X(A)$ is precisely a set, because $X(A)$ is the image of (our chosen object,) A, under (our chosen functor,) X.

So, the idea is that our choice of A and X completely determines all possible maps (i.e. natural transformations) from H_A to X. This answers out big question of "what are all the maps $H_A \to X$ " or "how does H_A see other presheaves on $\mathcal{A}$ ".

The theorem says not just that the two are isomorphic, but that they're ***naturally*** *isomorphic.*
This means that $[\mathcal{A}^{op}, Set](H_A, X)$ and $X(A)$ are *functorial* in *both* A and X

Also, So, the aforementioned collection of natural transformations also must be a set: As $\mathcal{A}$ is locally small, for each choice of $K \in \mathcal{A}$, $H_A(K) = Hom_{\mathcal{A}}(K, A)$ is a set. Thus, as α_K is a function,and hence a relation,
it's a subset of $Hom_{\mathcal{A}}(K, A) \times X(A)$, which is a set as it's a cartesian product of sets.
Thus, K-component of every natural transformation is a set. i.e. $\forall \alpha \in [\mathcal{A}^{op}, Set](H_A, X), \forall K \in \mathcal{A}, \alpha_K$ is a set

Now, the question is wether the set of all natural transformations is a set

Proof. Let a locally small category, $\mathcal{A}$ be given.

Denoting the category of all presheaves on $\mathcal{A}$ by $\mathcal{C}$, i.e. $\mathcal{C} := [\mathcal{A}^{op}, Set]$

Now, fix any object, $A \in \mathcal{A}$, and any object, $X \in \mathcal{C}$.

Need to show that $\mathcal{C}(H_A, X)$ is naturally isomorphic to $X(A)$.
As X is a preasheaf, $X(A)$ is a set.
Thus, two mutually inverse natural transformations between the above functors are required:

$\square$