Question 1

$$C \subseteq D \subseteq \mathbb{R};$$
$$\big(f_n\big)_{n\in\mathbb{N}} \text{ is uniformly convergent on } C;$$
$$\forall i \in \mathbb{N},\ f_i : D \to \mathbb{R} \text{ is continous}$$

Show $\exists f$ such that $f_n \xrightarrow[\text{uniformly}]{\overline{C}\cap D} f$ and f is continous.

Proof. Fix an $\epsilon > 0$ so, by uniform continuity of f,

$$\exists M : p, m \geq M \implies \forall x \in C, |f_p(x) - f_m(x)| < \frac{\epsilon}{3}$$

Fix $p, m \geq M$
Now, fix a $k \in \overline{C} \cap D$ thus, $\exists (x_n) \subseteq C$ such that $x_n \to k$
As each f_i is continous ,

$$\exists \delta(i) \text{ s.t. } |x - y| \leq \delta \implies |f_i(x) - f_i(y)| < \frac{\epsilon}{3}$$

So, there's K such that

$$i \geq K \implies |f_p(x_i) - f_p(k)| < \frac{\epsilon}{3} \text{ and } |f_m(x_i) - f_m(k)| < \frac{\epsilon}{3}$$

Thus, by triangle inequality,

$$\begin{aligned} 2 \times \frac{\epsilon}{3} &> |f_p(x_i) - f_p(k)| + |f_m(x_i) - f_m(k)| \\ &\geq |f_p(x_i) - f_m(x_i) + f_p(k) - f_m(k)| \\ &\geq ||f_p(x_i) - f_m(x_i)| - |f_m(k) - f_p(k)|| \end{aligned}$$

Now, as $|f_p(x_i) - f_m(x_i)| < \frac{\epsilon}{3}$, $-|f_p(x_i) - f_m(x_i)| > \frac{-\epsilon}{3}$

Thus,

$$\begin{aligned}2 \times \frac{\epsilon}{3} &\geq ||f_p(x_i) - f_m(x_i)| - |f_m(k) - f_p(k)|| \\ &\geq |f_p(x_i) - f_m(x_i)| - |f_m(k) - f_p(k)| \\ &\geq |f_p(x_i) - f_m(x_i)| - \frac{\epsilon}{3}\end{aligned}$$

Therefore, $|f_p(x_i) - f_m(x_i)| \leq \epsilon$ $\square$

There's a much better way of doing this though,

Proof. Fix an $\epsilon > 0$ so, by uniform continuity of f,

$$\exists M : p, m \geq M \implies \forall x \in C, |f_p(x) - f_m(x)| < \frac{\epsilon}{4}$$

Fix $p, m \geq M$
Now, fix a $k \in \overline{C} \cap D$. Going to show that (f_n) uniformly converges at k.
As f_p and f_m are continous , in particular at k,

$$\exists \delta_1 \text{ s.t. } |x - k| \leq \delta_1 \implies |f_p(x) - f_p(k)| < \frac{\epsilon}{4}$$

$$\exists \delta_2 \text{ s.t. } |x - k| \leq \delta_2 \implies |f_m(x) - f_m(k)| < \frac{\epsilon}{4}$$

Take $\delta := min\{\delta_1, \delta_2\}$
Now, if $k \in C$, then by hypothesis,the given sequence uniformly converges at k. Else, k is a limit point of C.Thus,

$$\exists c \in C \text{ such that } |c - k| < \delta$$

Fix this c and consider the following inequality

$$|f_p(k) - f_m(k)| \leq |f_p(k) - f_p(c)| + |f_p(c) - f_m(c)| + |f_m(c) - f_m(k)|$$

Each of the right hand side term is $< \frac{\epsilon}{4}$ (1st,3rd term due to continuity, 2nd term due to uniform convergance on C), forcing the left side to be $< \epsilon$. $\square$

Question 3

$$A \text{ is closed and bounded ;}$$
$$(f_n) \text{ is a sequence of continous functions on } A;$$
$$(f_n) \xrightarrow{p.w.} f, \text{ with f continous on } A;$$
$$\forall x \in A, f_n(x) \geq f_{n+1}(x), \text{ with } n \in \mathbb{N};$$

Prove that $f_n \rightrightarrows f(A)$

Question 5 **Prove** If Σa_n is absolutely convergent then $\Sigma \frac{a_n x^n}{1+x^{2n}}$ converges uniformly on $\mathbb{R}$.