

## Quiz-4

$D \subseteq \mathbb{R}$  such that  $\exists a_D \in \mathbb{R}^+ : a_D > 0$  and  $\forall d \in D, d \geq a_D$

$$f_n(x) := \frac{nx}{1 + n^2x^2}$$

Need to show that this sequence converges uniformly on  $D$ , but not on  $[0, \infty)$

**Proof:**

$$f_n(x) := \frac{nx}{1 + n^2x^2} = \frac{\frac{1}{nx}}{\frac{1}{nx} + nx} < \frac{1}{nx}$$
$$[\because n, x > 0 \implies \frac{1}{nx} + nx > 1]$$

**Showing uniform convergence to 0 on  $D$**

Fix any set  $D$  as stipulated, with  $a := a_D$  ;

Also, let  $x \in D \implies x \geq a_D > 0 \implies \frac{1}{x} \leq \frac{1}{a_D}$

And let  $\epsilon = \frac{1}{n}$  where  $n \in \mathbb{N}$ . For this  $\epsilon$ , define  $\delta_\epsilon := \frac{2}{a\epsilon} = \frac{2n}{a}$ .

So that,

$$k > \delta_\epsilon \implies |f_k(x) - 0| < \frac{1}{kx} \leq \frac{1}{ka} < \frac{1}{\delta_\epsilon a}$$

$$\text{Thus, } |f_k(x)| < \frac{a\epsilon}{2} \cdot \frac{1}{a} = \epsilon/2 < \epsilon$$

**Showing non-uniform pointwise convergence to 0 on  $[0, \infty)$**

let  $x \in [0, \infty)$  , and  $\epsilon = \frac{1}{n}$

if  $x = 0$ , then  $f_n(x) = f_n(0) = 0 < \frac{1}{n} = \epsilon$

else, if  $x \in (0, \infty)$  ,then define  $\delta_x = \frac{2n}{x}$

$$k > \delta_x \implies |f_k(x) - 0| < \frac{1}{kx} < \frac{1}{\delta_x x} = \frac{x}{2nx} = \epsilon/2 < \epsilon$$

So, if the sequence of  $f_n(x)$  uniformly converges on  $[0, \infty)$ , then it does so to 0. Suppose, if at all possible, that  $f_n(x) \Rightarrow 0$ .

So, for  $\epsilon = \frac{1}{3}$ ,  $\exists \delta > 0$  such that

$$\forall x \in [0, \infty), n > \delta \implies |f_n(x)| < \frac{1}{3}$$

But, taking  $x = x_\delta := \frac{1}{(\delta+1)}$  gives

$$|f_{\delta+1}(x_\delta)| = \frac{(\delta+1)x}{1 + (\delta+1)^2 x^2} = \frac{1}{2}$$

Thus, for any  $\delta$ ,  $x_\delta$  is such that

$$|f_{\delta+1}(x_\delta)| = 0.5 > 0.\bar{3} = \epsilon$$

Hence, the sequence of functions can't uniformly converge on  $[0, \infty)$ .