Question 1

$$< a_n > \text{ is a real sequence; } \sigma_n := \frac{a_1 + a_2 + ... + a_n}{n}$$

Going to show that

I $\liminf a_n \leq \liminf \sigma_n$
If a_n is unbounded below, then $\liminf a_n = -\infty \leq \liminf \sigma_n$ So, let a_n be bounded below, thus LHS is a real number,

$$m := \liminf a_n$$

Now, if a_n is constant, and equal to a,

$$\forall n \in \mathbb{N} \text{ , } \inf\{a_i | i \geq n\} = a = \frac{na}{n} = \sigma_n$$

Otherwise, if a_n is not constant, then,

$$\exists i, j \in \mathbb{N} \text{ such that(wlog) } a_i < a_j$$

Suppose if possible, $\liminf a_n = m > \liminf \sigma_n$
But,

$$\forall n \in \mathbb{N} \text{ such that } n \geq i, j,$$

$$\begin{aligned}\sigma_n = \frac{a_1 + ...a_i + a_j + ... + a_n}{n} &\geq \frac{(n-2)m + a_i + a_j}{n} \\ &\geq \frac{(n-1)m + a_j}{n} \\ &> m \; [\because m \leq a_i < a_j \;]\end{aligned}$$

$$\text{Hence, } \inf \sigma_n \geq \frac{(n-1)m + a_j}{n}$$

$$\implies \liminf \sigma_n \geq \lim_{n\to\infty} \frac{(n-1)m + a_j}{n} = m$$

But this contradicts the initial assumption.

II $\limsup a_n \geq \limsup \sigma_n$ If a_n is unbounded above, then $\limsup a_n = \infty \geq \limsup \sigma_n$.

So, let a_n be bounded above, thus LHS is a real number,

$$M := \limsup a_n$$

Now, if a_n is constant, and equal to a,

$$\forall n \in \mathbb{N} \text{ , } \sup\{a_i | i \geq n\} = a = \frac{na}{n} = \sigma_n$$

Otherwise, if a_n is not constant, then,

$$\exists i, j \in \mathbb{N} \text{ such that(wlog) } a_i < a_j$$

Suppose if possible, $\limsup a_n = M <$ $\limsup \sigma_n$
But,

$$\forall n \in \mathbb{N} \text{ such that } n \geq i, j,$$

$$\begin{aligned}
\sigma_n = \frac{a_1 + ...a_i + a_j + ... + a_n}{n} &\leq \frac{(n-2)M + a_i + a_j}{n} \\
&\leq \frac{(n-1)M + a_j}{n} \\
&< M \ [\because M \geq a_j > a_i \ ]
\end{aligned}$$

$$\text{Hence, } \inf \sigma_n \leq \frac{(n-1)M + a_j}{n}$$

$$\implies \liminf \sigma_n \leq \lim_{n\to\infty} \frac{(n-1)M + a_j}{n} = M$$

But this contradicts the initial assumption.

Question 2

$$\liminf \frac{a_{n+1}}{a_n} \leq \liminf (a_n)^{\frac{1}{n}} \leq \limsup (a_n)^{\frac{1}{n}} \leq \limsup \frac{a_{n+1}}{a_n}$$

I Showing that $\liminf \frac{a_{n+1}}{a_n} \leq \liminf (a_n)^{\frac{1}{n}}$

Case 1: $\liminf \ \frac{a_{n+1}}{a_n} = 0$

$$\forall n \in \mathbb{N}, a_n > 0 \implies (a_n)^{\frac{1}{n}} > 0$$

$$\text{Thus , } \liminf (a_n)^{\frac{1}{n}} \geq 0 = \liminf \frac{a_{n+1}}{a_n}$$

Case 2: $\liminf \ \frac{a_{n+1}}{a_n} = \infty$

So, for any $a \in \mathbb{N}$,

$$\exists M_a \in \mathbb{N} : n \geq M_a \implies \frac{a_{n+1}}{a_n} > a$$

Fix any a, and choose $n > M_a$. For any such n,

$$\frac{a_{n+1}}{a_n} > a \implies a_{n+1} > aa_n$$

$$\implies a_n > aa_{n-1} > a^2a_{n-2} > ... > a^{n-M}a_M$$
$$\implies (a_n)^{\frac{1}{n}} > a(\frac{a_M}{a^M})^{\frac{1}{n}}$$

Now as for fixed a, $\frac{a_M}{a^M}$ is constant,

$$\lim_{n\to\infty}(\frac{a_M}{a^M})^{\frac{1}{n}} = 1 \implies \lim_{n\to\infty} a(\frac{a_M}{a^M})^{\frac{1}{n}} = a$$

Thus,

$$\forall\epsilon > 0, \exists K \in \mathbb{N} : n > K \implies a(\frac{a_M}{a^M})^{\frac{1}{n}} > a - \epsilon$$

And hence, in particular, $a(\frac{a_M}{a^M})^{\frac{1}{n}} > a - 1$ for any natural number a.

$$(a_n)^{\frac{1}{n}} > a(\frac{a_M}{a^M})^{\frac{1}{n}} > a - 1 \implies \lim_{n\to\infty}(a_n)^{\frac{1}{n}} = \infty$$

$$\therefore \text{lim inf } \frac{a_{n+1}}{a_n} = \infty = \text{lim inf } (a_n)^{\frac{1}{n}}$$

Case 3: lim inf $\frac{a_{n+1}}{a_n} = a \in \mathbb{R}$
So, $\forall\epsilon > 0, \exists M \in \mathbb{N} : n > M \implies \frac{a_{n+1}}{a_n} > a - \epsilon$

$$a_n > (a-\epsilon)a_{n-1} > (a-\epsilon)^2a_{n-2} > ... > (a-\epsilon)^{n-M}a_M$$
$$\implies (a_n)^{\frac{1}{n}} > (a - \epsilon)(\frac{a_M}{a^M})^{\frac{1}{n}}$$

But, as $\lim_{n\to\infty}(\frac{a_M}{a^M})^{\frac{1}{n}} = 1$,

$$(a_n)^{\frac{1}{n}} > (a-\epsilon)(\frac{a_M}{a^M})^{\frac{1}{n}} > (a-\epsilon)(1-\epsilon) = a-(1+a)\epsilon+\epsilon^2$$

$$\implies (a_n)^{\frac{1}{n}} > a - (1+a)\epsilon$$

But as this holds for every $\epsilon > 0$,

$$\liminf (a_n)^{\frac{1}{n}} \geq a - 0 = \liminf \frac{a_{n+1}}{a_n}$$

II $\limsup (a_n)^{\frac{1}{n}} \leq \limsup \frac{a_{n+1}}{a_n}$
Case 1: $\limsup \ \frac{a_{n+1}}{a_n} = \infty$
As $\limsup (a_n)^{\frac{1}{n}} \leq \infty = \limsup \frac{a_{n+1}}{a_n}$,done

Case 2: $\limsup \ \frac{a_{n+1}}{a_n} = -\infty$

$$\limsup \frac{a_{n+1}}{a_n} = -\infty \implies \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = -\infty$$

But, as all a_n are positive, so is their raito, and hence it cant be unbounded below.

Case 3: $\limsup \ \frac{a_{n+1}}{a_n} = a \in \mathbb{R}$
So, $\forall\epsilon > 0, \exists M \in \mathbb{N} : n > M \implies \frac{a_{n+1}}{a_n} > a - \epsilon$

$$a_n > (a-\epsilon)a_{n-1} > (a-\epsilon)^2 a_{n-2} > ... > (a-\epsilon)^{n-M} a_M$$

$$\implies (a_n)^{\frac{1}{n}} > (a-\epsilon)(\frac{a_M}{a^M})^{\frac{1}{n}} \implies \lim_{n\to\infty} (a_n)^{\frac{1}{n}} > (a-\epsilon)$$

Thus, $\liminf (a_n)^{\frac{1}{n}} \geq a - \epsilon$, but as this holds for every ϵ,

$$\liminf (a_n)^{\frac{1}{n}} \geq a = \liminf \frac{a_{n+1}}{a_n}$$

Question 3

I

Question 4

1.

2.

3.

4. Some convergant sequences are:

   (a) $a_n := 1$

5. Some divergent sequences are:

   (a) $a_n := n$

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n} = 1 + \frac{1}{n}$$

This tends to 1 [for $\epsilon = \frac{1}{n},$ take $\delta = \frac{1}{n+1}$]
But the sequence diverges [to ∞].

Appendix

1. also, make sure to show liminf $\leq$ limsup