

## Question 1

$$C \subseteq D \subseteq \mathbb{R};$$

$(f_n)_{n \in \mathbb{N}}$  is uniformly convergent on  $C$ ;

$\forall i \in \mathbb{N}, f_i : D \rightarrow \mathbb{R}$  is continuous

**Show**  $\exists f$  such that  $f_n \xrightarrow[\text{uniformly}]{\overline{C} \cap D} f$  and  $f$  is continuous.

*Proof.* Fix any  $\epsilon > 0$ . Need to show that

$$\exists K > 0 \text{ s.t. } k, l \geq K \implies \forall x \in \overline{C} \cap D, |f_k(x) - f_l(x)| < \epsilon$$

So, fix any  $c \in \overline{C} \cap D$ .

As each  $f_i$  is given continuous on  $D$ ,  $\exists \delta(i, c) > 0$  such that

$$\forall y \in D, |c - y| \leq \delta \implies |f_i(c) - f_i(y)| < \epsilon/2$$

So, in particular, for any sequence in  $C$ ,  $(c_n) \rightarrow c$ ,

$$\begin{aligned} \exists N((c_n), \delta) \text{ such that } n \geq N &\implies |c_n - c| < \delta \\ &\implies |f_i(c_n) - f_i(c)| < \epsilon/2 \end{aligned}$$

Now, as  $f_i \xrightarrow{C} f$ ,  $\exists \beta$  such that for any  $c \in C$ ,

$$k \geq \beta \implies |f_k(c) - f(c)| < \epsilon/2$$

By triangle inequality,

$$|f_i(c) - f_j(c)| \leq |f_i(c) - f_i(c_\alpha)| + |f_i(c_\alpha) - f_j(c_\alpha)| + |f_j(c_\alpha) - f_j(c)|$$

Thus, for  $L > \max\{\beta, N\}$ , both (1) and (2) will hold:

$$\begin{aligned} i \geq L &\implies \epsilon/2 + \epsilon/2 > |f_i(c_i) - f_i(c)| + |f_i(c_i) - f(c)| \\ &\geq |f_i(c) - f(c)| \end{aligned}$$

Hence, the sequence uniformly converges to  $f$  on  $\overline{C} \cap D$ .  
 And as  $(f_i)_{i \in \mathbb{N}}$  is a sequence of continuous functions on  $D$ , that uniformly converges to  $f$ ,  $f$  is continuous on  $\overline{C} \cap D$ .  $\square$

Note: In case  $D$  were bounded,  $\overline{C} \cap D$  would be compact. Then, Heine-Cantor Theorem would give the existence of  $f$ , and Uniform Limit Theorem would give its continuity.

**Question 2** Prove that  $\sum x^n(1-x)$  converges pointwise on  $[0, 1]$  but not uniformly. While  $\sum (-1)^n x^n(1-x)$  converges uniformly on  $[0, 1]$ .

*Proof.* As  $x^n(1-x) = x^n - x^{n+1}$ , the first sum telescopes:

$$\sum_{i=1}^k x^n(1-x) = (x-x^2) + (x^2-x^3) + \dots + (x^k-x^{k+1}) = x-x^{k+1}$$

So, for  $x = 1$ , every partial sum is 0, and for  $0 \leq x < 1$ ,

$$\lim_{k \rightarrow \infty} \sum_{i=1}^k x^i(1-x) = \lim_{k \rightarrow \infty} (x - x^{k+1}) = x$$

Thus, the series converges pointwise on  $[0, 1]$ . Suppose it also converges uniformly to  $f$ . Then, as the  $k^{th}$  partial sum is  $x - x^{k+1}$ , a polynomial, and hence continuous on  $[0, 1]$ , its limit function,  $f$  must be continuous on  $[0, 1]$ . But,  $f$  is discontinuous at 1 as

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} x = 1 \neq 0 = f(1)$$

The partial sums for the second series of functions,

$$\sum_{i=1}^k (-x)^i (1-x) = -x + 2[(-x)^2 + (-x)^3 + \dots + (-x)^k] + (-x)^{k+1}$$

So, for  $x = 1$ , every partial sum is 0, and for  $0 \leq x < 1$ ,

$$\begin{aligned} \lim_{k \rightarrow \infty} \sum_{i=1}^k (-x)^i (1-x) &= x + 2 \lim_{k \rightarrow \infty} ((-x)^{k+1} + \sum_{i=1}^k (-x)^i) \\ &= x + 2 \lim_{k \rightarrow \infty} \frac{-x(1 - (-x)^k)}{1+x} \\ &= x + \frac{-2x}{1+x} \end{aligned}$$

To show uniform convergence, going to use the Dirichlet test:

I Take  $b_n(x) := \frac{x^n}{2} = (\frac{x}{\sqrt[2]{2}})^n$ . Going to show that

(i)  $\forall x \in [0,1], b_n(x) \geq b_{n+1}(x)$  :

$$\text{As } b_{n+1}(x) - b_n(x) = \frac{x^{n+1}}{\sqrt[2]{2}} - \frac{x^n}{\sqrt[2]{2}} = \frac{x^n}{\sqrt[2]{2}} (\frac{x}{\sqrt[2]{2}} - 1) \leq 0$$

(ii)  $b_n \Rightarrow 0(C)$

$$\begin{aligned} \text{Take } \delta > \frac{\ln 2}{\ln(\epsilon + 1)} &\implies \ln(\epsilon + 1) > \frac{1}{\delta} \ln(2) \\ &\implies \epsilon > \sqrt[2]{2} \end{aligned}$$

$$\begin{aligned} |b_n(x) - b_m(x)| &= \left| \frac{x}{\sqrt[2]{2}} - \frac{x}{\sqrt[2]{2}} \right| = x \left( \frac{1}{\sqrt[2]{2}} - \frac{1}{\sqrt[2]{2}} \right) \\ &\leq \left( \frac{1}{\sqrt[2]{2}} - \frac{1}{\sqrt[2]{2}} \right) = \frac{\sqrt[2]{2} - \sqrt[2]{2}}{\sqrt[2]{2} \sqrt[2]{2}} \leq \sqrt[2]{2} - \sqrt[2]{2} \\ &\leq \sqrt[2]{2} - 1 \leq \epsilon \end{aligned}$$

II Take  $a_n(x) := 2(-1)^n(1 - x)$ . So,

$$\begin{aligned} |(A_n(x))| &= \left| \sum_{i=1}^n a_i(x) \right| = 2(1 - x) \left| \sum_{i=1}^n (-1)^i \right| \\ &= \begin{cases} 0 & n \text{ is even} \\ 2(1 - x) & n \text{ is odd} \end{cases} \end{aligned}$$

$\therefore$  for any  $n, x$   $|A_n(x)| \leq 2$ ,  $a_n$  is uniformly bounded.

III So, Dirichlet's test is applicable and

$$\begin{aligned} \sum b_n a_n &= \sum \frac{x^n}{2} \times 2(-1)^n(1 - x) \\ &= \sum (-1)^n x^n (1 - x) \\ &\text{uniformly converges on } [0, 1] \end{aligned}$$

□

### Question 3

- $A$  is closed and bounded ;  
 $(f_n)$  is a sequence of continuous functions on  $A$ ;  
 $(f_n) \xrightarrow{p.w.} f$ , with  $f$  continuous on  $A$ ;  
 $\forall x \in A, f_n(x) \geq f_{n+1}(x)$ , with  $n \in \mathbb{N}$ ;

**Prove** that  $f_n \rightrightarrows f(A)$

**Question 4** Construct a sequence of functions,  $(f_n)$  on  $[0,1]$  such that

- (a) each  $f_i$  is discontinuous at every point of  $[0,1]$ ;  
and  
(b)  $\exists f$ , a continuous function on  $[0,1]$  such that  $f_n \rightrightarrows f$

*Proof.* Define  $f_n(x) = \begin{cases} \frac{1}{n} & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

- (a) This sequence is discontinuous on  $[0,1]$  as :  
(i) For  $q \in \mathbb{Q}$ , take any sequence of irrational numbers,  $(p_n) \rightarrow q$ . So,

$$\lim_{k \rightarrow \infty} f_n(p_k) = 0 \neq \frac{1}{n} = f_n(q)$$

- (ii) For  $p \notin \mathbb{Q}$ , take any sequence of rational numbers,  $(q_n) \rightarrow p$ . So,

$$\lim_{k \rightarrow \infty} f_n(q_k) = \frac{1}{n} \neq 0 = f_n(q)$$

(b) The defined sequence uniformly converges to 0 as:  
Fix  $\epsilon = \frac{1}{n}$ , and choose  $\delta > n$ ,

$$m \geq \delta \implies |f_m(x)| = \frac{1}{m} \leq \frac{1}{\delta} < \frac{1}{n} = \epsilon$$

□

**Question 5 Prove** If  $\sum a_n$  is absolutely convergent then  $\sum \frac{a_n x^n}{1+x^{2n}}$  converges uniformly on  $\mathbb{R}$ .