

1 Definitions

Definiton 1.1 (Prorelation). A partial-ordered set of relations $X \rightarrow Y$, which is down-directed and an upper set. i.e A set, $P \subseteq \mathcal{P}(X \times Y)$ such that

- (i) A partial-order defined to be containment as relations, $r \subseteq s$ only if $\forall (x, y) \in X \times Y, (x, y) \in r \implies (x, y) \in s$
- (ii) (Down-directed), $\forall r, s \in P, \exists t \in P$ such that $t \subseteq r$ and $t \subseteq s$
- (iii) (Up-set) for any relation $u : X \rightarrow Y$, if $\exists p \in P$ such that $p \leq u$ then $u \in P$

Definiton 1.2 (Composition of prorelations). Prorelations can be composed by taking all compositions of their elements as relations: for prorelations $P : X \rightarrow Y$ and $Q : Y \rightarrow Z$,

$$Q.P := \{q \circ p : p \in P \text{ and } q \in Q\}$$

Definiton 1.3 (Comparison of Prorelations). Two prorelations with same domain, co-domain are comparable as

$$\text{for } P, Q : X \rightarrow Y, P \leq Q \text{ if } \forall q \in Q, \exists p \in P \text{ such that } p \subseteq q$$

Definiton 1.4 (Quasi-uniformity). A prorelation on a set X , $P : X \rightarrow X$ is a quasi-uniformity if it follows :

- i $\forall p \in P$, for any $x \in X$, $(x, x) \in p$ i.e. xpx
- ii $\forall p \in P, \exists p' \in P$ such that $p' \circ p' \subseteq p$

And in this case, (X, A) is called a *quasi-uniform space*.

Definiton 1.5 (Uniformly Continuous function). A function, $f : X \rightarrow Y$ is called a uniformly continuous function,

$$f : (X, A) \rightarrow (Y, B) \text{ if, } \forall b \in B, \exists a \in A \text{ such that } f \circ a \subseteq b \circ f. \text{ meaning that } f.A \leq B.f \text{ or } \begin{array}{ccc} X & \xrightarrow{f} & Y \\ A \downarrow & \leq & \downarrow B \\ X & \xrightarrow{f} & Y \end{array}$$

Definiton 1.6 (Promodule). A prorelation, $\phi : X \multimap Y$ is called a promodule $\phi : (X, A) \multimap (Y, B)$ if it obeys: $\phi.A \leq \phi$ and $B.\phi \leq \phi$ where $.$ denotes composition as prorelations.

Definiton 1.7 (Comparison of Promodules). Promodules with same domain and co-domain are compared as prorelations, for $\phi, \psi : (X, A) \multimap (Y, B)$, $\phi \sqsubseteq \psi$, only if $\phi \leq \psi$.

Definiton 1.8 (Composition of Promodules). Promodules are composed as prorelations.

For promodules $\phi : (X, A) \multimap (Y, B)$ and $\psi : (Y, B) \multimap (Z, C)$, $\psi\phi := \psi.\phi = \{q \circ p : p \in \phi \text{ and } q \in \psi\}$

Definiton 1.9 (Opposite relation). For relation $r : X \rightarrow Y$, r^o is defined to be a relation $r^o : Y \rightarrow X$ as

$$\forall (x, y) \in X \times Y, (x, y) \in r \iff (y, x) \in r^o$$

Lemma 1.9.1. For any relation $r : X \rightarrow Y$, $r^o \circ r = \Delta_X$

Lemma 1.9.2. For any relation $r : X \rightarrow Y$, $r \circ r^o \subseteq \Delta_Y$

Definiton 1.10 $((-)_*)$.

Definiton 1.11 $((-)^*)$.

Definiton 1.12 (Fully Faithful).

Definiton 1.13 (Fully Dense).

Definiton 1.14 (Topologically Dense).

2 Propositions

Definiton 2.1 (QUnif). QUnif is defined to be the category having quasi-uniform spaces as objects, and uniformly continous maps between them as morphisms.

Lemma 2.1.1. QUnif does define a category, as

- i Composition

ii Identity

Definiton 2.2 (ProMod).

Lemma 2.2.1. ProMod does define a category, as

i Composition

ii Identity

Proposition 2.1 $((-)^* : \text{QUnif} \rightarrow \text{ProMod}$ is a Functor).

Proposition 2.2 $((-)^* : \text{QUnif}^{\text{op}} \rightarrow \text{ProMod}$ is a Functor). Defined as fixing objects and taking morphisms to their image under $(-)^*$

(a) for $(X, A) \in \text{QUnif}^{\text{op}}$, $(X, A)^* := (X, A) \in \text{ProMod}$

(b) for $f : (X, A) \rightarrow (Y, B)$ in QUnif, $f^* := f \circ B$

Proof.

Showing that $f \circ B : (Y, B) \rightarrow (X, A)$ is a promodule

So, need to show $f \circ B$ a prorelation $Y \rightarrow X$ and that $(f \circ B).B \subseteq f \circ B$ and $A.(f \circ B) \subseteq f \circ B$

To show prorelation,

(i) (Partial-order) Inclusion of relations i.e. for $k = f \circ b$ and $k' = f \circ b'$ in $f \circ B$, $k \subseteq k' \iff b \subseteq b'$

(ii) (Down directed) for $k, k' \in f \circ B$, need that $\exists l \in f \circ B$ such that $l \subseteq k, k'$

Fix $k, k' \in f \circ B \implies \exists b, b' \in B : k = f \circ b$ and $k' = f \circ b'$

And as B is a quasi-uniformity, it's down directed so, $\exists c \in B : c \subseteq b, b' \implies l := f \circ c \subseteq k, k'$

(iii) (Up-set) for a relation $l : Y \rightarrow X$ and $k \in f \circ B$ such that $l \supseteq k$, need $l \in f \circ B$

Let $b \in B$ be such that $k = f \circ b$ and define $b' := \{(y, y') : y \in \text{domain}(l) \text{ and } y' \in (f \circ)^{-1}(l(y))\}$

As $l \supseteq k = f \circ b$, $\text{domain}(b') = \text{domain}(l) \supseteq \text{domain}(b)$

and $\text{range}(l) \supseteq \text{range}(f \circ b) \implies \forall y \in \text{domain}(b), \text{range}(b') = (f \circ)^{-1}(l(y)) \supseteq (f \circ)^{-1}(f \circ b) = \text{range}(b)$

Now, by definition of b' , $f \circ b' \supseteq l$. To show $f \circ b' \subseteq l$,

$(x, y) \in f \circ b' \implies \exists z \in Y : (x, z) \in b'$ and $(z, y) \in f \implies x \in \text{domain}(l)$ and $z \in l(x)$ i.e. $(x, z) \in l$

To show $(f \circ B).B \subseteq f \circ B$, need that $\forall b \in B, \exists b' \in B : f \circ b' \subseteq f \circ b$,

Fix any $b \in B$ as B is a quasi-uniformity, $\exists b' \in B : b' \subseteq b \implies f \circ b' \subseteq f \circ b$

To show $A.(f \circ B) \subseteq f \circ B$, need that $\forall b \in B, \exists b' \in B, a \in A : a \circ f \circ b' \subseteq f \circ b$,

As f is uniformly continuous, $f.A \subseteq B.f$ i.e. $\forall b \in B, \exists a \in A : f \circ a \subseteq b \circ f \implies a = f \circ f \circ a \subseteq f \circ b \circ f$

Fix any $b \in B$, so, $\exists b' \in B : b' \subseteq b$ (for brevity, omitting \circ to explicitly denote composition)

And, for this b' , $\exists a : a \subseteq f \circ b' \implies a \circ f \circ b' \subseteq f \circ b' \circ f \circ b' \subseteq f \circ b' \subseteq f \circ b \implies a \circ f \circ b' \subseteq f \circ b$

Now, need to show that $(-)^*$ respects composition and identity.

(i) (Composition) let f, g be uniformly continuous, $(X, A) \xrightarrow{f} (Y, B) \xrightarrow{g} (Z, C)$ need that $(g \circ f)^* = f^* \cdot g^*$

LHS $= (g \circ f)^* = (g \circ f) \circ C = (f \circ g \circ C)$ and RHS $= f^* \cdot g^* = (f \circ B) \cdot (g \circ C)$

For equality, showing that LHS \geq RHS and LHS \leq RHS:

To show $(f \circ g \circ C) \geq (f \circ B) \cdot (g \circ C)$, need that $\forall c \in C, \exists b \in B, c' \in C : f \circ g \circ c \supseteq f \circ b \circ g \circ c'$

Fix any $c \in C$, so, $\exists c' \in C : c' \subseteq c \implies f \circ g \circ c \supseteq f \circ g \circ (c' \circ c) = f \circ g \circ (c' \circ \Delta_Z c') \supseteq f \circ g \circ c' \circ (g \circ c')$

By uniform continuity of g , for $c' \in C, \exists b \in B : g \circ b \subseteq c' \circ g$

Thus, $f \circ g \circ c \supseteq f \circ g \circ (c' \circ g) \circ c' \supseteq f \circ (g \circ g) \circ b \circ g \circ c' = f \circ b \circ g \circ c'$

To show $(f \circ g \circ C) \leq (f \circ B) \cdot (g \circ C)$, need that $\forall b \in B, c \in C, \exists c' \in C : f \circ g \circ c \subseteq f \circ b \circ g \circ c'$

Fix any $c \in C, b \in B$ will show that $c' := c$ works:

As B is a quasi-uniformity, $\Delta_Y \subseteq b \implies f \circ \Delta_Y \circ g \circ c = f \circ g \circ c \subseteq f \circ b \circ g \circ c = f \circ b \circ g \circ c'$

(ii) (Identity) let $(X, A) \in \text{QUnif}^{\text{op}}$, and $1_{(X, A)} : (X, A) \rightarrow (X, A)$ as $x \mapsto x$ need that $(1_{(X, A)})^* = 1_{(X, A)^*}$

LHS $= (1_{(X, A)})^* = (1_{(X, A)}) \circ A = 1_{(X, A)} \cdot A = A$

Now, it's required that A is the identity of (X, A) in ProMod.

So, fix $\phi : (X, A) \rightarrow (Y, B)$, need to show $\phi.A = \phi$

As ϕ is a promodule, $\phi.A \leq \phi$ and as A is quasi-uniformity on X ,

$\forall a \in A, \Delta_X \subseteq a \implies \forall a \in A, \forall p \in \phi, p = p \Delta_X \subseteq p a \implies \phi \leq \phi.A$

Also, fix $\psi : (Y, B) \rightarrow (X, A)$, need to show $A.\psi = \psi$

As ψ is a promodule, $A.\psi \leq \psi$ and as A is quasi-uniformity on X ,

$\forall a \in A, \Delta_X \subseteq a \implies \forall a \in A, \forall q \in \psi, q = \Delta_X q \subseteq a q \implies \psi \leq \psi.A$

□

Proposition 2.3 (Proposition 1). Fix a uniformly continuous map, $f : (X, A) \rightarrow (Y, B)$

- (a) f is fully faithful $\iff A = f^o.B.f$
- (b) f is fully dense $\iff \forall b \in B, \exists b' \in B$ such that

Definiton 2.3 (PX). $PX := \{\psi : \psi : (X, A) \rightharpoonup 1 \text{ is a promodule}\}$

Definiton 2.4 (\tilde{a}). for any $a \in A$, \tilde{a} is defined to be a relation $PX \rightarrow PX$ as

$$\text{for } \phi, \psi \in PX, \phi \tilde{a} \psi \text{ only if } \phi \leq \psi.a$$

Proposition 2.4 (Prorelation \tilde{A}). The set, $\tilde{A} := \{\tilde{a} : a \in A\}$ defines a quasi-uniformity on PX .

Proof. First, need to show that \tilde{A} is a prorelation,

- (i) (Partial order) Define, for any two relations $\tilde{a}, \tilde{b} : PX \rightarrow PX$, that $\tilde{a} \leq \tilde{b}$ only if $a \subseteq b$
- (ii) (Down-Directed) Need that $\forall \tilde{a}, \tilde{b} \in \tilde{A}, \exists \tilde{c} \in \tilde{A} : \tilde{c} \subseteq \tilde{a}, \tilde{c} \subseteq \tilde{b}$
 $\tilde{a}, \tilde{b} \in \tilde{A} \implies a, b \in A \implies \exists c \in A : c \subseteq a, c \subseteq b \implies \tilde{c} \leq \tilde{a}, \tilde{b}$
- (iii) (Upset) Need that, for any relation $l : PX \rightarrow PX$, if $\exists \tilde{k} \in \tilde{A}$ such that $l \geq \tilde{k}$, then $l \in \tilde{A}$
Fix any $k : PX \rightarrow PX$, and $\tilde{k} \in \tilde{A}$ such that $l \geq \tilde{k}$
Now, k is a relation between promodules $X \rightharpoonup 1$. Thus, it can be thought of as a relation on X ,
 $a := \{(x, y) : x \in \text{domain}(\psi) \text{ and } y \in \text{domain}(\phi) \text{ whenever } \exists \psi, \phi \in PX : \psi l \phi\}$
So, $l = \tilde{a}$ and thus, $\tilde{a} \geq \tilde{k} \implies a \supseteq k \implies a \in A \implies l \in \tilde{A}$

Now to show that the other two conditions hold,

- (i) need that $\forall \tilde{a} \in \tilde{A}, \forall \psi \in PX, \psi \tilde{a} \psi$
So, need to show that $\psi \leq \psi.a$ i.e. $\forall p \in \psi, \exists q \in \psi : q \subseteq p.a$. Take $q := p$, and as A is a quasi-uniformity,
 $\Delta_X \subseteq a \implies p = p.\Delta_X \subseteq p.a$
- (ii) Need that $\forall \tilde{a} \in \tilde{A}, \exists \tilde{b} \in \tilde{A} : \tilde{b} \leq \tilde{a}$
Before that, showing, for any $x, y \in A, \tilde{x} \tilde{y} \leq \tilde{x} \tilde{y}$ i.e. $\forall \psi, \phi \in PX, \psi(\tilde{x}.\tilde{y})\phi \implies \psi \tilde{x} \tilde{y} \phi$
Let $\psi_1(\tilde{a}.\tilde{b})\psi_3 \implies \exists \psi_2 : \psi_1 \tilde{b} \psi_2 \tilde{a} \psi_3 \implies \psi_1 \leq \psi_2.b$ and $\psi_2 \leq \psi_3.a \implies \psi_1 \leq \psi_2.b \leq \psi_3.ab \implies \psi_1(\tilde{a}b)\psi_3$
Fix any $\tilde{a} \in \tilde{A} \implies a \in A \implies \exists b \in A : b \circ b \subseteq a \implies \tilde{b} \tilde{b} \leq \tilde{a} \implies \tilde{b} \tilde{b} \leq \tilde{b} \tilde{b} \leq \tilde{a}$

□

Proposition 2.5 (Yoneda Embedding).

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