1 Categories

Definiton 1.1 (Category)**.** A category, $\mathcal{A}$ is defined to have each of the following,

(i) A collection of objects, denoted by ob($\mathcal{A}$) and written A,B,C $\in \mathcal{A}$.
Such that each object has an 'identity', $1_A \in \mathcal{A}(A, A), 1_B \in \mathcal{A}(B, B), 1_C \in \mathcal{A}(C, C)$

(ii) For each pair of objects, a collection of 'links'/morphisms between them, denoted by $\mathcal{A}(A, B)$ and written as f $\in \mathcal{A}(A, B)$ $g \in \mathcal{A}(B, C)$. Such that,

  (a) morphisms with matching domain,co-domain can be 'chained'/composed $(g, f) = g \circ f$
  (b) with this composition being associative, $(h \circ g) \circ f = h \circ (g \circ f)$
  (c) and they are 'fixed' by the identity $f \circ 1_A = f = 1_B \circ f$

Example 1.1. Non-trivial Identity Consider the objects to be groups, and morphisms to be direct product between them:

i ob $(\mathcal{A}) = \{G|$ G is a group$\}$

ii $\mathcal{A}(A, B) := A \times B$

iii $\mathcal{A}(B, C)\ \circ \mathcal{A}(A, B) \mapsto \mathcal{A}(A, C)$

So, there's a unique morphism between any two objects i.e groups. And the identity morphism,

$\forall A, B \in \mathcal{A}$, if f $\in \mathcal{A}(A, B)$, then $f \circ 1_A \in \mathcal{A}(A, B) \times \mathcal{A}(A, A) \mapsto \mathcal{A}(A, B)$ and $1_B \circ f \in \mathcal{A}(B, B) \times \mathcal{A}(A, B) \mapsto \mathcal{A}(A, B)$

Thus, $ob(\mathcal{A})$ along with $\circ$ is actually a group. And hence has a unique inverse. But how exactly?

Example 1.2. Set The objects are defined to be sets, and morphisms are the functions between them, with the usual composition law:

i ob $(\mathcal{A}) = \{S|$ S is a set$\}$

ii $(f : A \mapsto B) \in \mathcal{A}(A, B)$

iii $(g \in \mathcal{A}(B, C)) \circ (f \in \mathcal{A}(A, B)) \mapsto g(f) \in \mathcal{A}(A, C)$

Example 1.3. Grp Objects are groups,with homomorphisms between them being the morphisms, and composition being as usual:

i $ob(\mathcal{A}) = \{G|G$ is a group $\}$

ii $\mathcal{A}(A, B) = Hom(A, B)$ i.e. all f such that $\forall x, y \in A f((x) \cdot_A (y)) = (f(x)) \cdot_B (f(y))$

iii composition is defined as that between two group homomorphisms

In this example, the set of all morphisms along with composition forms a group.

Example 1.4. Ring Objects are rings, and arrows are ring homomorphisms between them.

i $ob(\mathcal{A}) = \{G|G$ is a ring $\}$

ii $\mathcal{A}(A, B) = Hom(A, B)$

iii composition is defined as that between two ring homomorphisms

Example 1.5. Vect$_k$ Objects are vector spaces *over field k* , and the morphisms between them are linear transformations

i $ob(\mathcal{A}) = \{A|A$ is a vector space$\}$

ii $\mathcal{A}(A, B) = \mathcal{L}(A, B)$

iii composition is defined as that of linear transformations

Definiton 1.2 (Isomorphism)**.** An isomorphism, between objects, is a morphism between them such that it's 'inverse' is also a morphism. So,

$$f : A \mapsto B \text{ is an isomorphism } \iff \exists g \in \mathcal{A}(B, A) : gf = 1_A \text{ and } fg = 1_B$$

Definiton 1.3 (Product Category)**.** Somewhat like a cartesian product of categories. Given categories $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \times \mathcal{B}$ is defined as:

i $ob(\mathcal{A} \times \mathcal{B}) := ob(\mathcal{A}) \times ob(\mathcal{B})$

ii $(\mathcal{A} \times \mathcal{B})((A, B), (A', B')) := \mathcal{A}(A, A') \times \mathcal{B}(B, B')$

iii $(f, g) \in \mathcal{A} \times \mathcal{B}((A, B), (C, D))$, $(a, b) \in \mathcal{A} \times \mathcal{B}((C, D), (E, F)) \implies (a, b) \circ (f, g) := (a \circ f, b \circ g)$

iv $\forall (A, B) \in ob(\mathcal{A} \times \mathcal{B})$, $1_{(A,B)} := (1_A, 1_B)$

Example 1.6 (CAT)**.** The category of all categories with morphisms being functors.

i $ob(\mathcal{A}) = \{A | A \text{ is a category}\}$

ii $\mathcal{A}(A, B) = F(A, B)$

iii $F : \mathcal{A} \mapsto \mathcal{B}$, $G : \mathcal{B} \mapsto \mathcal{C} \implies G \circ F := H : \mathcal{A} \mapsto \mathcal{C}$

And thus, the identity of $\mathcal{A}$ is the functor, $1_{\mathcal{A}} : \mathcal{A} \mapsto \mathcal{A}$

2 Functors

Definiton 2.1 ((Covariant)Functor)**.** A functor is a map between categories, written $F : \mathcal{A} \mapsto \mathcal{B}$, consists :

(i) function taking objects of $\mathcal{A}$ to those of $\mathcal{B}$ i.e. $ob(\mathcal{A}) \mapsto ob(\mathcal{B})$. Written as $A \mapsto F(A)$.

(ii) associative, identity-preserving function taking links between objects of $\mathcal{A}$ to those for $\mathcal{B}$, $f \mapsto F(f)$, i.e.

$$\forall A, B \in \mathcal{A}, \mathcal{A}(A, B) \mapsto \mathcal{B}(F(A), F(B)) \text{ such that } (a)\ f \in \mathcal{A}(A, B)\, , g \in \mathcal{A}(B, C) \implies F(g \circ f) = F(g) \circ F(f) = F(g \circ f)$$
$$(b) A \in \mathcal{A} \implies F(1_A) = 1_{F(A)}$$

Example 2.1. Forgetful Functors They essentially ignore some of the structure of the 'domain'.

(a) $U : Grp \mapsto Set$ takes groups to their underlying set, and homomorphisms to maps between the sets.Similarly, $Ring \mapsto Set$ and $Vect_k \mapsto Set$

(b) Let Ab be the category of abelian groups, then, $U : Ring \mapsto Ab$ takes rings to their additive group, 'forgetting' the multiplicative group. And if Mon is the category of monoids, $U : Ring \mapsto Mon$ 'forgets' the additive group.

(c) $U : Ab \mapsto Grp$ just takes each abelian group to itself, and does the same for (homo)morphisms.

Example 2.2. Free Functors

(a) let $F(S)$ denote the free group on a set S. Then, $U : Set \mapsto Grp$ is a 'free' functor taking sets to their free group, and thus the maps between them becomes a homomorphism between their free groups.

(b) Same for Rings pp20

(c) Same for Vector Spaces

Example 2.3. Let $\mathcal{G}, \mathcal{H}$ be the one object categories of monoids G,H respectively. Then, due to composition being associative and identity preseving, possible functors are precisely the homomorphisms.

Definiton 2.2 (Contravariant Functor)**.** For categories $\mathcal{A} and \mathcal{B}$, $\mathcal{A}^{op} \mapsto \mathcal{B}$ is a contravariant functor from $\mathcal{A}$ to $\mathcal{B}$.

Example 2.4. Let k be a field and V, W be vector spaces over it. Then fixing W,

$$\forall f \in Hom(V, V'), \exists f^* : Hom(V', W) \mapsto Hom(V, W)$$

So, for each $V \in ob(Vect_k)$, $Hom(V, W)$ defines a contravariant functor from $Vect_k^{op}$ to $Vect_k$

3 Natural Isomorphisms

Example 3.1.