

# 1 Yoneda Lemma

In this section, we set forth some basic definitions, state and prove Yoneda lemma, and then show Cayley's Theorem and Yoneda Embedding as its consequences.

For any category  $\mathcal{A}$ , and its objects  $X, Y \in \mathcal{A}$ , we denote  $Hom_{\mathcal{A}}(X, Y)$  with  $\mathcal{A}(X, Y)$ .

**Definiton 1.1.** For any category  $\mathcal{A}$ , its opposite category,  $\mathcal{A}^{op}$  is the category having the objects of  $\mathcal{A}$ . And for objects  $A, B \in \mathcal{A}$ , a morphism  $f \in \mathcal{A}^{op}(A, B)$  if and only if there is a morphism  $g \in \mathcal{A}(B, A)$ .

**Proposition 1.2.** For a locally small category  $\mathcal{A}$ , fixing an object  $A \in \mathcal{A}$  gives a functor,  $H_A : \mathcal{A}^{op} \rightarrow Set$  defined as:

- (i) For any object  $B \in \mathcal{A}$ ,  $H_A(B) := \mathcal{A}(B, A)$ .
- (ii) For any morphism,  $g : X \rightarrow Y$  in  $\mathcal{A}$ ,

$$H_A(g) : \mathcal{A}(Y, A) \rightarrow \mathcal{A}(X, A) \text{ is given by } p \mapsto p \circ g.$$

*Proof.* Fix any objects  $K, L, M \in \mathcal{A}$ .

- I **(Composition)** As  $H_A$  is a contravariant functor, for any morphisms  $f \in \mathcal{A}(K, L)$  and  $g \in \mathcal{A}(L, M)$ , we need to show that  $H_A(g \circ f) = H_A(f) \circ H_A(g)$ . Note, the composition  $g \circ f$  on the left hand side is in  $\mathcal{A}^{op}$ . Hence, using the definition of  $H_A$  gives us that for any  $k \in H_A(M)$ , we must have

$$\begin{aligned} LHS &= (H_A(g \circ f))(k) = k \circ g \circ f \\ \text{and } RHS &= (H_A(f) \circ H_A(g))(k) = (H_A(f))(k \circ g) = (k \circ g) \circ f. \end{aligned}$$

- II **(Identity)** We will show that for any  $k \in \mathcal{A}(K, L)$ ,  $H_A$  respects the identities of  $K$  and  $L$  in  $\mathcal{A}$  (as they're equal to the identities of  $K$  and  $L$  in  $\mathcal{A}^{op}$ ). Using the definition of  $H_A$ , for any object  $L \in \mathcal{A}$  and morphism  $p \in H_A(L)$ , we get the following equations.

$$\begin{aligned} \text{Right Identity: } & ((H_A(1_K)) \circ (H_A(k)))(p) = (H_A(1_K))(p \circ k) = p \circ k \circ 1_K = p \circ k = (H_A(k))(p) \\ \text{Left Identity: } & ((H_A(k)) \circ (H_A(1_L)))(p) = (H_A(k))(p \circ 1_L) = (H_A(k))(p) \end{aligned}$$

Hence,  $H_A$  is indeed a functor. □

**Definiton 1.3.** For a locally small category  $\mathcal{A}$ , the category of presheaves on  $\mathcal{A}$ , denoted by  $[\mathcal{A}^{op}, Set]$  is defined to have functors from  $\mathcal{A}^{op}$  to  $Set$  as objects, and the natural transformations between them as morphisms.

**Lemma 1.4.** Lemma 1.3.11

**Lemma 1.5.** Exercrise 1.3.29

**Theorem 1.6. Yoneda Lemma** If  $\mathcal{A}$  is a locally small category then, for any object  $A \in \mathcal{A}$  and  $X \in [\mathcal{A}^{op}, Set]$ , there exists an isomorphism,

$$[\mathcal{A}^{op}, Set](H_A, X) \cong X(A) \text{ which is natural in } A \text{ and } X. \quad (1)$$

**Notation:**

- We denote the category of presheaves on  $\mathcal{A}$  by  $\mathcal{C}$ .
- For the map  $\hat{\phantom{a}}$ , instead of writing  $\hat{\phantom{a}}(a) = b$ , we use  $\hat{a} = b$  to denote  $a \mapsto b$ .
- For the map  $\tilde{\phantom{a}}$ , instead of writing  $\tilde{\phantom{a}}(a) = b$ , we use  $\tilde{a} = b$  to denote  $a \mapsto b$ .

•  $[\mathcal{A}^{op}, Set](H_A, X)$  denotes the collection of morphisms  $\alpha : \mathcal{A}^{op} \rightarrow Set$ .

$$\begin{array}{ccc} & \xrightarrow{H_A} & \\ & \Downarrow \alpha & \\ & \xrightarrow{X} & \end{array}$$

To prove the theorem, first, we show that  $[\mathcal{A}^{op}, Set](H_A, X)$  is isomorphic to  $X(A)$  as set, and then that this isomorphism is natural in  $X$  and  $A$ .

*Proof.* Let  $\mathcal{A}$  be a locally small category. Fix an object  $A \in \mathcal{A}$  and a presheaf  $X$  on  $\mathcal{A}$ .

## I Showing isomorphism between $[\mathcal{A}^{op}, Set](H_A, X)$ and $X(A)$

Define  $\hat{\cdot} : \mathcal{C}(H_A, X) \rightarrow X(A)$  for any  $\alpha : H_A \rightarrow X$ , as  $\hat{\alpha} := \alpha_A(1_A)$ . As  $1_A \in Set(A, A) = H_A(A)$ , definition of  $\alpha_A$  gives that  $\alpha_A(1_A) \in X(A)$ .

Define  $\tilde{\cdot} : X(A) \rightarrow [\mathcal{A}^{op}, Set](H_A, X)$  for any  $x \in X(A)$  as the natural transformation  $\tilde{x} : H_A \rightarrow X$  whose K-component is the function mapping each morphism  $p \in \mathcal{A}(K, A)$  to  $(X(p))(x)$ . That is,  $\tilde{x}_K(p) := (X(p))(x)$ .

We are going to show that  $\tilde{x}$  is a natural transformation. Fix objects  $K, L \in \mathcal{A}$  and morphism  $q \in \mathcal{A}^{op}(K, L)$ .

$$\begin{array}{ccc} H_A(K) & \xrightarrow{H_A(q)} & H_A(L) \\ \tilde{x}_K \downarrow & & \downarrow \tilde{x}_L \\ X(K) & \xrightarrow{X(q)} & X(L) \end{array}, \text{ that is } \begin{array}{ccc} \mathcal{A}(K, A) & \xrightarrow{- \circ q} & \mathcal{A}(L, A) \\ \tilde{x}_K \downarrow & & \downarrow \tilde{x}_L \\ X(K) & \xrightarrow{X(q)} & X(L) \end{array} \text{ commutes.}$$

So, for any  $f : K \rightarrow A$ , need that  $\tilde{x}_L(f \circ q) = X(q) \circ \tilde{x}_K(f)$ . Using the definition of  $\tilde{x}$  gives the following.

$$\begin{aligned} LHS &= \tilde{x}_L(f \circ q) = (X(f \circ q))(x) \\ RHS &= X(q) \circ \tilde{x}_K(f) = (X(q))(X(f)(x)) = (X(q) \circ X(f))(x) \end{aligned}$$

And as  $X$  is a contravariant functor,  $X(f \circ q) = X(q) \circ X(f)$ , giving that LHS=RHS. Now going to show that  $\hat{\cdot}$  and  $\tilde{\cdot}$  define an isomorphism. Need to show that  $\hat{\cdot}$  and  $\tilde{\cdot}$  are mutually inverse.

- (i) For any  $x \in X(A)$ ,  $\hat{\tilde{x}} = \tilde{x}_A(1_A) = (X(1_A))(x) = 1_{X(A)}(x) = x$ .
- (ii) For any  $\alpha \in \mathcal{C}(H_A, X)$ , need to show that  $\hat{\tilde{\alpha}} = \alpha$ . So, it's required that each of their component are equal. As both  $\hat{\tilde{\alpha}}$  and  $\alpha$  are natural transformations between functors that go to the category  $Set$ , each of the components is a function. So, need to show that for any  $f \in \mathcal{A}(K, A) = H_A(K)$ ,  $(\hat{\tilde{\alpha}})_K(f) = \alpha_K(f)$ . Using first the definition of  $\tilde{\cdot}$  and then that of  $\hat{\cdot}$  gives:

$$LHS = \hat{\tilde{\alpha}}_K(f) = (X(f))(\hat{\tilde{\alpha}}) = (X(f))(\alpha_A(1_A)) \quad (1)$$

And as  $f \in \mathcal{A}(K, A)$ , we also have the following.

$$RHS = \alpha_K(f) = \alpha_K(1_A \circ f) \quad (2)$$

Because  $\alpha$  is a natural transformation, the square following square commutes for  $1_A$ .

$$\begin{array}{ccc} \mathcal{A}(A, A) & \xrightarrow{- \circ f} & \mathcal{A}(K, A) \\ \alpha_A \downarrow & & \downarrow \alpha_K \\ X(A) & \xrightarrow{X(f)} & X(K) \end{array}$$

This gives that  $\alpha_K(1_A \circ f) = (X(f))(\alpha_A(1_A))$ . Hence, we have from (2) and (3), we get that  $RHS = LHS$ .

## II Showing naturality of this isomorphism

By Using Lemma 1.4 and 1.5, it's enough to show that  $\hat{\cdot}$  is natural in  $X$  and natural in  $A$ .

- (i) We are going to show the above isomorphism to be natural in  $X$ . Fix any  $A \in \mathcal{A}$ . Need that for presheaves  $X, Y \in \mathcal{C}$  and natural transformation  $\beta \in \mathcal{C}(X, Y)$ , the following square commutes.

$$\begin{array}{ccc} \mathcal{C}(H_A, X) & \xrightarrow{\beta \circ -} & \mathcal{C}(H_A, Y) \\ \hat{\cdot} \downarrow & & \downarrow \hat{\cdot} \\ X(A) & \xrightarrow{\beta_A} & Y(A) \end{array}$$

So, for any  $\alpha : H_A \rightarrow X$ , we need that  $(\hat{\cdot} \circ H_\beta)(\alpha) = (\beta_A \circ \hat{\cdot})(\alpha)$ . Using definition of  $H_\beta$  and  $\hat{\cdot}$  gives:

$$LHS = (\hat{\cdot} \circ H_\beta)(\alpha) = \widehat{(H_\beta(\alpha))} = \widehat{(\beta \circ \alpha)} = (\beta \circ \alpha)_A(1_A) \quad (3)$$

$$RHS = (\beta_A \circ \hat{\cdot})(\alpha) = \beta_A(\hat{\alpha}) = (\beta_A \circ \alpha_A)(1_A) \quad (4)$$

As  $\alpha \in \mathcal{C}(H_A, X)$  and  $\beta \in \mathcal{C}(X, Y)$  are morphisms in  $\mathcal{C}$ , composition in  $\mathcal{C}$  gives  $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$ . From (4) and (5), we directly get that  $RHS = LHS$ .

- (ii) We are going to show natural in  $A$ . Fix any  $X \in \mathcal{C}$ . Need that for objects  $A, B \in \mathcal{A}$  and morphism  $f \in \mathcal{A}^{op}(A, B)$ , the following square commutes.

$$\begin{array}{ccc} \mathcal{C}(H_A, X) & \xrightarrow{- \circ H_f} & \mathcal{C}(H_B, X) \\ \downarrow \hat{\cdot} & & \downarrow \hat{\cdot} \\ X(A) & \xrightarrow{X(f)} & X(B) \end{array}$$

So, for any  $\alpha : H_A \rightarrow X$ , we need that  $(\hat{\cdot} \circ H_f)(\alpha) = ((X(f)) \circ \hat{\cdot})(\alpha)$ . Using definition of  $H_f$  and  $\hat{\cdot}$ , we get:

$$LHS = (\hat{\cdot} \circ H_f)(\alpha) = \widehat{\alpha \circ H_f} = (\alpha \circ H_f)_B(1_B) = \alpha_B(f \circ 1_B) = \alpha_B(1_A \circ f) \quad (5)$$

$$RHS = ((X(f)) \circ \hat{\cdot})(\alpha) = (X(f))(\hat{\alpha}) = (X(f))(\alpha_A(1_A)) \quad (6)$$

The last equality in (6) is justified as  $f$  goes from  $B$  to  $A$  in  $\mathcal{A}$ . By using equality of (2) and (3) from I(i), for  $f \in \mathcal{A}(B, A)$ , we get that  $(X(f))(\alpha_A(1_A)) = \alpha_B(1_A \circ f)$ . Hence,  $RHS = LHS$ .  $\square$

## 1.1 Cayley's Theorem

Informally, given a locally small category  $\mathcal{A}$ , we can fix a presheaf  $X$  on  $\mathcal{A}$ , and for any object  $A \in \mathcal{A}$ , study the set  $X(A)$  and gain information about all possible natural transformations between  $H_A$  and  $X$ . Moreover, by part I(ii) of the proof of Yoneda Lemma, each of the natural transformations is determined by its action on the identity morphisms in  $\mathcal{A}$ . Thus, no matter how complicated  $\mathcal{A}$  is, if we choose  $X$  carefully, we can hope to understand the structure of  $\mathcal{A}$  by looking at how  $X(A)$  changes as we vary the chosen presheaf and object.

In group theory, Cayley's theorem says every group  $G$  is isomorphic to a subgroup of the symmetric group on  $G$ . Thus, instead of having to study a complicated group directly, we can study a subgroup of the symmetric group on it.

Cayley's theorem and Yoneda Lemma are similar in the sense that both allow us to change the environment that we study in by putting few restrictions on what we are allowed to study. Cayley allows us to change setting for groups, and Yoneda does that for locally small categories.

Also, as groups themselves can be considered as small categories, we can apply Yoneda Lemma to any group. In fact, we can get Cayley's theorem as a consequence of Yoneda Lemma by a suitable choice of  $X$  and  $A$ .

**Definition 1.7.** Symmetric group on a set  $X$  is the set of all bijections on  $X$ , with the binary operation defined as composition of bijections.

We will now use parts of proof of Theorem 1.6 in to prove Cayley's theorem. The notation  $(-)$  occurs as a placeholder for the element a map is applied to, that is,  $(- \circ f)(k)$  is defined to be  $k \circ f$ . And we use the notation  $g.f$  to mean the composition of  $g$  and  $f$  in the group.

**Theorem 1.8. Cayley's Theorem** Every group,  $(G, \cdot)$  is isomorphic to a subgroup of symmetric group on  $G$ .

*Proof.* Let a group  $G$  be given. Define category  $\mathcal{A}$  with a single object  $\star$ . And precisely one morphism in  $\mathcal{A}$  for each element of  $G$ , with the composition of said morphisms being as that of elements of  $G$ . That is, for morphisms  $f$  and  $g$  in  $\mathcal{A}$ ,  $f \circ g$  is defined to be the morphism  $(f.g)$ . Then,  $G$  and  $\mathcal{A}(\star, \star)$  have the same elements and rule of composition, so they're isomorphic as groups.

As  $\mathcal{A}^{op}$  is a category with single object, each natural transformation  $\alpha : \mathcal{A}^{op} \rightarrow \mathbf{Set}$  has only one component, that is  $\alpha_\star$ . Therefore, we can identify  $\alpha$  with  $\alpha_\star$ . Using naturality of  $\alpha$ , we get that

$$\begin{array}{ccc} H_\star(\star) & \xrightarrow{H_\star(f)} & H_\star(\star) & \mathcal{A}(\star, \star) & \xrightarrow{- \circ f} & \mathcal{A}(\star, \star) \\ \alpha_\star \downarrow & & \downarrow \alpha_\star & \alpha_\star \downarrow & & \downarrow \alpha_\star \\ H_\star(\star) & \xrightarrow{H_\star(f)} & H_\star(\star) & \mathcal{A}(\star, \star) & \xrightarrow{- \circ f} & \mathcal{A}(\star, \star) \end{array} \quad \text{the square commutes for any } f \in \mathcal{A}(\star, \star). \quad (1)$$

Using the identity of  $\star$  in  $\mathcal{A}$ , that is  $1_\star$  as the starting morphism in (1) gives us the following equation.

$$((- \circ f) \circ \alpha_\star)(1_\star) = (\alpha_\star \circ (- \circ f))(1_\star) \implies \alpha_\star(1_\star) \circ f = \alpha_\star(f) \quad (2)$$

Thus, we get that every natural transformation,  $\alpha$  is determined by its action on  $1_\star$ . As  $\mathcal{A}(\star, \star)$  is isomorphic to the group  $G$ , and  $\alpha_\star$  is a function from  $\mathcal{A}(\star, \star)$  to  $\mathbf{Set}$ , we can write the final equation from (2)

with the product being in  $G$ , that is,  $\alpha_*(1_*) \cdot f = \alpha_*(f)$ . Because multiplying every element of a group by a fixed element gives us a bijection on the group, each  $\alpha_*$ , and hence each  $\alpha$  can be thought of as a bijection on  $G$ .

Now we will show that the collection  $[\mathcal{A}^{op}, Set](H_*, H_*)$  forms a group, with composition being as morphisms. As  $[\mathcal{A}^{op}, Set]$  is a category, we get that this composition is associative. Also, because the aforementioned collection contains morphisms with the same source and destination, that is  $*$ , we get that it is closed under composition, and has an identity, that is  $1_*$ . In order to show inverses, we use the group isomorphism between  $\mathcal{A}(*, *)$  and the property of  $\alpha$  from (2), to bijectively map each element  $g \in G$  to the natural transformation  $\alpha$  such that  $\alpha_*(1_*)$  corresponds to  $g \in \mathcal{A}$ . Hence, for any  $\alpha$  corresponding to  $j \in G$ , there will exist a  $\beta \in [\mathcal{A}^{op}, Set](H_*, H_*)$  corresponding to  $j^{-1} \in G$ . Using (2) gives us that  $\alpha$  and  $\beta$  are inverses.

$$\text{For any } k \in \mathcal{A}(*, *), \quad (\alpha \circ \beta)_*(k) = \alpha_*(\beta_*(k)) = \alpha_*(\beta_*(1_*) \cdot k) = \alpha_*(j^{-1} \cdot k) = \alpha_*(1_*) \cdot j^{-1} \cdot k = j \cdot j^{-1} \cdot k = k$$

Thus, the collection  $[\mathcal{A}^{op}, Set](H_*, H_*)$  is a group.

As the collection of elements of  $G$  forms a set, we get that  $\mathcal{A}(*, *)$  is a set. Hence,  $\mathcal{A}$  is a locally small category. Because  $\mathcal{A}^{op}$  has the same number of morphism as  $\mathcal{A}$ , it is also a locally small category, and we may apply Yoneda Lemma to it. Taking  $A = * \in \mathcal{A}^{op}$  and  $X = H_*$  in Thm 1.6 (1), we get:

$$[\mathcal{A}^{op}, Set](H_*, H_*) \cong H_*(*) = \mathcal{A}(*, *) \cong G \quad (3)$$

where the first isomorphism above is between sets, and second one between groups. We will show that this isomorphism is also a homomorphism. The map  $\hat{\cdot}$ , defined in Theorem 1.6 will act here as  $\alpha \mapsto \alpha_*(1_*)$ . Hence, for any  $\alpha, \beta : H_* \rightarrow H_*$ ,

$$\widehat{\alpha \circ \beta} = (\alpha \circ \beta)_*(1_*) = (\alpha)_*((\beta)_*(1_*)) = ((\alpha)_*(1_*)) \cdot ((\beta)_*(1_*)). \quad (4)$$

Where the last equality is due to (2) being applicable as  $((\beta)_*(1_*))$  is an element of  $\mathcal{A}(*, *)$ , and hence corresponds to some element of  $G$ . We have shown that in statement (3), the isomorphism is between groups, with the left most expression being a group whose elements are bijections on  $G$ . This is precisely the statement of Cayley's theorem.

$$Sym(G) \geq [\mathcal{A}^{op}, Set](H_*, H_*) \cong G \quad \square$$

## 1.2 Yoneda Embedding

**Definition 1.9.** A category,  $\mathcal{A}$  is said to be embedded in a category,  $\mathcal{B}$  if and only if there exists a functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  such that  $F$  is full and faithful.

Note that a functor being fully faithful gives us that the functor is injective on objects upto isomorphism. Because fullness along with  $F(A) = F(B)$  gives us the existence of morphisms  $f \in \mathcal{A}(A, B)$  and  $g \in \mathcal{A}(B, A)$  such that  $F(f) = 1_{FA}$  and  $F(g) = 1_{FB}$ . Thus faithfulness gives us that  $F(f \circ g) = F(1_B)$  and  $F(g \circ f) = F(1_A)$  imply  $f \circ g = 1_B$  and  $g \circ f = 1_A$ , giving that  $A \cong B$ .

We are going to define a functor from locally small category  $\mathcal{A}$  to the presheaf category on  $\mathcal{A}$ , as taking any object  $A \in \mathcal{A}$  to the functor  $H_A$ . And for any  $X, Y, K \in \mathcal{A}$ , taking morphism  $f \in \mathcal{A}(X, Y)$  to the natural transformation  $f^*$ , whose  $K$ -component is defined as taking any  $k \in H_X(K)$  to  $f \circ k \in \mathcal{A}(K, Y)$ .

**Proposition 1.10.** The map defined above is a functor from  $\mathcal{A}$  to  $[\mathcal{A}^{op}, Set]$ . Henceforth, we denote it by  $H_\bullet$ .

*Proof.* Fix any objects  $K, L, M \in \mathcal{A}$ .

- I **(Composition)** Let  $f \in \mathcal{A}(K, L)$  and  $g \in \mathcal{A}(L, M)$ . As  $(g \circ f)^*$  and  $g^* \circ f^*$  are natural transformations from  $H_K$  to  $H_M$ , need to show that for any  $X \in \mathcal{A}^{op}$ ,  $X$ -components of  $(g \circ f)^*$  and  $g^* \circ f^*$  are equal. So, fix  $k \in H_K(X)$ , and using the definition of  $(-)^*$ , we get that

$$\begin{aligned} LHS &= ((g \circ f)^*)(k) = g \circ f \circ k \\ \text{and } RHS &= (f^* \circ g^*)(k) = g^*(f \circ k) = g \circ f \circ k. \end{aligned}$$

- II **(Identity)** We will show that for any  $g \in \mathcal{A}(K, L)$ ,  $H_\bullet$  respects the identities of  $K$  and  $L$  in  $\mathcal{A}$ . Thus, for any object  $X \in \mathcal{A}$ , we need to show that  $(H_\bullet(g) \circ H_\bullet(1_K))_X = (H_\bullet(g))_X = (H_\bullet(g) \circ H_\bullet(1_K))_X$ . So, fix any morphism  $p \in H_A(L)$ . Using the definition of  $(-)^*$ , we get the following equations.

$$\text{Right Identity: } ((H_\bullet(g)) \circ (H_\bullet(1_K)))(p) = (H_A(g))(1_K \circ p) = (H_A(g))(p)$$

$$\text{Left Identity: } ((H_\bullet(1_L)) \circ (H_\bullet(g)))(p) = (H_\bullet(1_L))(g \circ p) = g \circ p \circ 1_L = g \circ p = (H_\bullet(g))(p)$$

Hence,  $H_\bullet$  is indeed a functor.  $\square$

**Theorem 1.11.** Any locally small category  $\mathcal{A}$  can be embedded in the presheaf category on  $\mathcal{A}$ .

*Proof.* We will show that the functor from Proposition 1.10 is full and faithful. Fix any objects  $X, Y \in \mathcal{A}$ .

- I To show that  $H_\bullet$  is a full, we need to show that for every  $\alpha \in [\mathcal{A}^{op}, Set](H_X, H_Y)$ , there exists a morphism  $f \in \mathcal{A}(X, Y)$  such that  $H_\bullet(f) = \alpha$ . Thus, we need to show that their  $K$ -components are equal for every  $K \in \mathcal{A}$ . Using the definition of  $H_\bullet(f)$ , this amounts to showing that

$$\text{for any morphism } k \in H_X(K), \left(H_\bullet(f)\right)_K(k) = \alpha_K(k), \text{ that is } f \circ k = \alpha_K(k). \quad (1)$$

Because  $\alpha_X$  goes from  $H_X(X)$  to  $H_Y(X)$ ,  $\alpha_X(1_X)$  is a morphism in  $\mathcal{A}(X, Y)$ . We will show that choosing this morphism to be  $f$  will give us the required result, that is  $(\alpha_X(1_X)) \circ k = \alpha_K(k)$ . Using the naturality of  $\alpha$ ,

$$\begin{array}{ccc} H_X(X) & \xrightarrow{H_X(k)} & H_X(K) \\ \alpha_X \downarrow & & \downarrow \alpha_K \\ H_Y(X) & \xrightarrow{H_Y(k)} & H_Y(K) \end{array}, \text{ that is } \begin{array}{ccc} \mathcal{A}(X, X) & \xrightarrow{- \circ k} & \mathcal{A}(K, X) \\ \alpha_X \downarrow & & \downarrow \alpha_K \\ \mathcal{A}(X, Y) & \xrightarrow{- \circ k} & \mathcal{A}(K, Y) \end{array} \text{ commutes.}$$

Thus, for the identity morphism  $1_X \in \mathcal{A}(X, X)$ , we get the following

$$\left(H_Y(k) \circ \alpha_X\right)(1_X) = \left(\alpha_K \circ H_X(k)\right)(1_X) \implies \alpha_X(1_X) \circ k \implies \alpha_K(1_X) \circ k = \alpha_K(k)$$

Thus, we have that  $H_\bullet$  is a full functor.

- II Fix any morphisms  $f, g$  in  $\mathcal{A}(X, Y)$  and suppose  $H_\bullet(f) = H_\bullet(g)$ . In order to show  $H_\bullet$  is faithful, we need to show that  $f = g$ . As  $H_\bullet(f)$  and  $H_\bullet(g)$  are equal natural transformations, we have that the action of their  $X$ -components is equal. Thus, in particular, for the identity of  $X$ ,  $(H_\bullet(f))_X(1_X) = (H_\bullet(g))_X(1_X)$ . Using the definition of  $H_\bullet$ , we get that  $f \circ 1_X = g \circ 1_X$ . And as both  $g$  and  $f$  are morphisms from  $X$ , we get that  $f = g$ .  $\square$

## 2 Prorelations

**Definiton 2.1.** A prorelation is a partially ordered, down-directed, up-set of relations  $X \rightarrow Y$ .

That is,  $P \subseteq \mathcal{P}(X \times Y)$  is a prorelation if it satisfies the following conditions:

- (i) Partial Order: Containment of relations defines a partial order. That is,  $r \subseteq s$  meaning that for any  $(x, y) \in X \times Y$ , if  $(x, y) \in r$  then  $(x, y) \in s$ .
- (ii) Down-directed: For any  $r, s \in P$ , there exists  $t \in P$  such that  $t \subseteq r$  and  $t \subseteq s$ .
- (iii) Up-set: For any relation  $u : X \rightarrow Y$ , if there exists  $p \in P$  such that  $p \subseteq u$  then  $u \in P$ .

**Example 2.2.** (a) For any positive real number  $\epsilon$ , define a relation  $A_\epsilon$  on real numbers as  $A_\epsilon = \{(x, y) | x - y < \epsilon\}$ . The set  $K = \{A_\epsilon | \epsilon > 0\}$  forms a prorelation.  $K$  is down-directed, as  $\delta \geq \epsilon > 0$  implies  $A_\delta \supseteq A_\epsilon$ . It's an up-set because for any relation  $B$  such that  $B \supseteq A_\epsilon$ , for each  $(x, y) \in B$ , we take  $\delta > |x - y| \geq x - y$ . Thus, each element of  $K$  is in some  $A_{\delta_i}$ , and as the supremum of all such  $\delta_i$  is also a positive real number,  $B \in K$  by down-directedness of  $K$ .

(b)

**Definiton 2.3.** A prorelation  $P : X \rightarrow Y$  can be composed to a prorelation  $Q : Y \rightarrow Z$  by taking composition of the relations belonging to them. Then, the set  $Q.P$  is defined as  $Q.P = \{q \circ p : p \in P \text{ and } q \in Q\}$ .

**Lemma 2.4.** Composition of two prorelations is a prorelation.

*Proof.* For prorelations  $P : X \rightarrow Y$  and  $Q : Y \rightarrow Z$ , need to show that  $Q.P$  is a prorelation.

- (i) (Partial Order) Inclusion of relations gives a partial order.
- (ii) (Down-Directed) If  $k, k' \in Q.P$ , then  $k = qp$  and  $k' = q'p'$  for some  $q, q' \in Q$  and  $p, p' \in P$ . Because  $Q$  and  $P$  are prorelations, and hence down-directed sets there exists,  $a \in Q$  such that  $a \subseteq q, q'$  and  $b \in P$  such that  $b \subseteq p, p'$ . Thus, giving an element,  $a \circ b$  of  $Q.P$  such that  $a \circ b \subseteq k, k'$ .

(iii) (Up-Set) Let  $l : X \rightarrow Z$  be a relation, and  $k \in Q.P$  such that  $l \supseteq k$ . Define relations  $p : X \rightarrow Y$  and  $q : Y \rightarrow Z$  as,  $p = \{(x, y) : x \in \text{domain}(l) \text{ and } y \in Y\}$  and  $q = \{(y, z) : y \in Y \text{ and } z \in \text{range}(l)\}$ . Because  $k \in Q.P$ , there exist  $q' \in Q$  and  $p' \in P$  such that  $k = q' \circ p'$ . Thus by definition of  $p$  and  $q$ , we get that  $p \supseteq p'$  and  $q \supseteq q'$ . Hence  $p \in P$  and  $q \in Q$  because  $P$  and  $Q$  are up-sets, which gives us that  $q \circ p \in Q.P$ . For any  $(x, z) \in l$ , by definition of  $p$  and  $q$ , we get that for every  $y \in Y$ ,  $(x, y) \in p$  and  $(y, z) \in q$ . By definition of composition, this gives that  $(x, z) \in q \circ p$ , giving that  $l \subseteq q \circ p$ . And, by definition of  $q \circ p$  we get that  $l \supseteq q \circ p$ . Finally giving that  $l = q \circ p \in Q.P$ .  $\square$

**Definiton 2.5.** For prorelations  $P, Q : X \rightarrow Y$ , if for each  $q \in Q$ , there exists  $p \in P$  such that  $p \subseteq q$ , then we write  $Q \geq P$ .

**Definiton 2.6.** For a relation  $r : X \rightarrow Y$ , it's opposite relation  $r^o : Y \rightarrow X$  is defined as

$$(y, x) \in r^o \text{ if and only if } (x, y) \in r \text{ for } x \in X \text{ and } y \in Y.$$

**Lemma 2.7.** For any function  $f : X \rightarrow Y$ ,  $\Delta_X$  is contained in the composition  $f^o \circ f$ .

*Proof.* As  $f$  is a function, it must be defined on every element of it's domain. Thus, for every  $x \in X$ , there exists some  $(x, y)$  in  $f$ . By definition of  $f^o$ ,  $(y, x)$  is in  $f^o$ . Hence, by definition of composition,  $(x, x)$  is in  $f^o \circ f$ .  $\square$

**Lemma 2.8.** For any relation  $r : X \rightarrow Y$ , the composition  $r \circ r^o$  is contained in  $\Delta_Y$ .

*Proof.* Suppose there exist  $x \in X$  and  $y \in Y$  such that  $x r y$ . By definition of  $r^o$ , this gives us that  $y r^o x$ . Using definition of composition,  $y r^o x r y$  gives that  $y (r \circ r^o) y$ .  $\square$

**Lemma 2.9.** For relations  $r, s : X \rightarrow Y$  and  $t : Y \rightarrow Z$ , if  $r \subseteq s$  then  $(t \circ r) \subseteq (t \circ s)$ .

*Proof.* Suppose relations  $r, s$  and  $t$  are as given above, and let  $x (t r) z$ . By definition of composition, there exists,  $y \in Y$  such that  $x r y$  and  $y t z$ . Using the hypothesis, as  $r \subseteq s$ ,  $x r y$  gives  $x s y$ . And via composition of  $x s y$  with  $y t z$ , we get  $x (t s) z$ . We started with any element of  $(t \circ r)$  and showed that it must also be in  $t \circ s$  and thus have that  $(t \circ r) \subseteq (t \circ s)$ .  $\square$

**Lemma 2.10.** For relations  $r : X \rightarrow Y$  and  $s, t : Y \rightarrow Z$ , if  $s \subseteq t$  then  $(s \circ r) \subseteq (t \circ r)$ .

*Proof.* Suppose relations  $r, s$  and  $t$  are as given above, and let  $x (s r) z$ . By definition of composition of relations, we get that there exists some  $y \in Y$  such that  $x r y$  and  $y s z$ . Because  $s \subseteq t$ ,  $y s z$  implies that  $y t z$ . Taking the composition,  $x r y s z$  yields  $x (t r) z$ . We started with any element of  $(s \circ r)$  and showed that it must also be in  $t \circ r$  and thus have that  $(s \circ r) \subseteq (t \circ r)$ .  $\square$

### 3 Quasi-Uniform Spaces

**Definiton 3.1.** A prorelation  $P$  on a set  $X$  is said to be a quasi-uniformity if it satisfies the following conditions:

- (i) Every relation in  $P$  is reflexive. That is, for each  $p \in P$ , if  $x \in X$  then  $(x, x) \in p$ .
- (ii) For each  $p$  in  $P$ , there exists  $p'$  in  $P$  such that  $p' \circ p' \subseteq p$ .

**Definiton 3.2.** If  $X$  is a set, and  $A$  is a quasi-uniformity on  $X$ , then  $(X, A)$  is a quasi-uniform space.

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**Definiton 3.3.** A function,  $f : (X, A) \rightarrow (Y, B)$  is said to be uniformly continuous if  $f.A \leq B.f$ . That is, for each

$$b \in B, \text{ there exists } a \in A \text{ such that } f \circ a \subseteq b \circ f. \text{ Meaning that } \begin{array}{ccc} X & \xrightarrow{f} & Y \\ A \downarrow & \leq & \downarrow B \\ X & \xrightarrow{f} & Y \end{array}$$

**Lemma 3.4.** If  $A$  is a quasi-uniformity on a set  $X$ , then  $A.A = A$

*Proof.* Fix any  $a \in A$ , as  $A$  is a quasi-uniformity,  $\exists b \in A : b b \subseteq a$ , we get that  $A.A \leq A$ . And as  $A$  is a prorelation, and is hence down-directed,  $\exists c \in A : a.a \supseteq c$ , giving that  $A.A \geq A$ .  $\square$

**Proposition 3.5.** QUnif is a category defined as having quasi-uniform spaces as objects, and uniformly continuous maps between them as morphisms, with composition defined as that of functions.

*Proof.* (i) (Associativity) The composition of functions is associative by definition.

- (ii) (Identity) For each object  $(X, A)$ , define it's identity to be the identity function  $\Delta_X : (X, A) \rightarrow (X, A)$ . This function is uniformly continuous as  $\Delta_X.A = A \leq A = A.\Delta_X$ .  $\square$

**Definiton 3.6.** A prorelation,  $\phi : X \multimap Y$  is called a promodule  $\phi : (X, A) \multimap (Y, B)$  if it satisfies:

$$\phi.A \leq \phi \text{ and } B.\phi \leq \phi$$

**Definiton 3.7.** Containment of promodules is defined as that of prorelations.

**Definiton 3.8.** Promodules are composed as prorelations.

DEFINE PROMODULE CATEGORY HERE, BY COMBINING THE ABOVE TWO DEFINITIONS DONT FORGET TO WRITE ABOUT LEQ ALSO LOOK AT CITE

**Proposition 3.9.** ProMod is a 2-category defined as having quasi-uniform spaces as its 0-cells, promodules as 1-cells and containment of promodules as 2-cells.

*Proof.* In order to show that ProMod is a 2-category, need the following:

- (a) (1-Identities) For each quasi-uniform space  $(X, A)$ , define  $A : (X, A) \multimap (X, A)$  to be the identity 1-cell for  $(X, A)$ .  $A$  is a promodule because  $A.A = A$  (Lemma 2.2.1)
- (b) (1-Composition) Need promodules to be closed under composition.  
Let  $\phi : (X, A) \multimap (Y, B)$  and  $\psi : (Y, B) \multimap (Z, C)$  be promodules. To show that  $\psi.\phi : (X, A) \multimap (Z, C)$  is a promodule, need it to be a prorelation that satisfies the two conditions required to be a promodule:
  - (i) By Lemma 1.2.1, prorelations are closed under composition. Hence,  $\psi.\phi$  is a prorelation
  - (ii) Need to show that  $\psi.\phi.A \leq \psi.\phi$ . So, Fix  $p \in \psi$  and  $q \in \phi$ . As  $\phi$  is a promodule,  $\phi.A \leq \phi$  gives that there exists  $q' \in \phi$  and  $a \in A$  such that  $q'a \subseteq q$ . Thus,  $pq'a \subseteq pq$ .
  - (iii) Need to show that  $C.\psi.\phi \leq \psi.\phi$ . Fix  $p \in \psi$  and  $q \in \phi$ . Because  $\psi$  is a promodule,  $C.\psi \leq \psi$  gives that there exists  $c \in C$  and  $p' \in \psi$  such that  $cp' \subseteq p$ . Thus,  $cp'q \subseteq pq$
- (c) (2-Identities) As every promodule is contained in itself, always have  $\psi \leq \psi$ . Define this comparison to be the identity 2-cell for  $\psi$  and denote it by  $\leq_\psi$
- (d) (Vertical 2-composition) For promodules  $\psi, \phi, \delta : (X, A) \multimap (Y, B)$ , if there is a 2-cell from  $\psi$  to  $\phi$  and another one from  $\phi$  to  $\delta$  i.e.  $\psi \leq \phi \leq \delta$ , then by transitivity of the partial order,  $\psi \leq \delta$  i.e. there's a 2-cell from  $\psi$  to  $\delta$ .
- (e) (Horizontal 2-composition) If there are promodules  $\psi, \psi' : (X, A) \multimap (Y, B)$  and  $\phi, \phi' : (Y, B) \multimap (Z, C)$  such that  $\psi \leq \psi'$  and  $\phi \leq \phi'$ , need to show that  $\psi.\phi \leq \psi'.\phi'$ . Fix  $p' \in \psi'$  and  $q' \in \phi'$ . As  $\psi \leq \psi'$ ,  $\exists p \in \psi : p \subseteq p'$  and as  $\phi \leq \phi'$ ,  $\exists q \in \phi : q \subseteq q'$ . Thus,  $pq \subseteq p'q'$
- (f) (1-Identity) Need to show that for any promodule  $\phi : (X, A) \multimap (Y, B)$ ,  $\phi.A = \phi = B.\phi$ . By quasi-uniformity of  $A$ , every  $a \in A$ , is reflexive. Thus, for any  $p \in \phi$  and  $a \in A$ ,  $p = p.\Delta_X \subseteq pa$  giving that  $\phi \leq \phi.A$ . And as  $\phi$  is a promodule,  $\phi \geq \phi.A$ . Hence, by anti-symmetry of the partial order,  $\phi = \phi.A$ .  
Similarly, By quasi-uniformity of  $B$ , every  $b \in B$ , is reflexive. Thus, for any  $p \in \phi$  and  $b \in B$ ,  $p = \Delta_Y.p \subseteq bp$  giving that  $\phi \leq B.\phi$ . And as  $\phi$  is a promodule,  $\phi \geq B.\phi$ . Hence,  $\phi = B.\phi$ .
- (g) (1-Associativity) As composition of relations is associative, so too is the composition of prorelations directly giving that composition of promodules i.e. 1-cells is associative.
- (h) (Vertical 2-Identity) Let  $\leq : \psi \rightarrow \phi$  be a 2-cell i.e.  $\psi \leq \phi$ . By our definition of identity 2-cell,  $\leq_\psi . \leq_1$  means precisely that  $\psi \leq \psi \leq \phi$ , and by transitivity, this is equivalent to  $\psi \leq \phi$ . Similarly,  $\leq_1 . \leq_\phi$  means exactly that  $\psi \leq \phi \leq \phi$ , and this is equivalent to  $\psi \leq \phi$ .
- (i) (Vertical 2-Associativity) Associativity of the partial order on promodules directly gives the associativity of composition of 2-cells in ProMod.
- (j) (Horizontal 2-Identity) Let  $\psi, \phi : (X, A) \multimap (Y, B)$  be promodules. For any 2-cell  $\leq : \psi \rightarrow \phi$ , need to show that the 2-cell given by the horizontal composition,  $\leq * \leq_A$  is equal to  $\leq$ , as well as equal to  $\leq_B * \leq$ . So, it's required that  $\psi.A \leq \phi.A \iff \psi \leq \phi \iff B.\psi \leq B.\phi$ . And this holds as a direct consequence of (f).
- (k) (Horizontal 2-Associativity) As there's a unique 2-cell between any two promodules, and composition of promodules is associative, horizontal composition of 2-cells is associative.
- (l) (2-Identity) For promodules  $\psi : (X, A) \multimap (Y, B)$  and  $\phi : (Y, B) \multimap (Z, C)$  need  $(\leq_\psi * \leq_\phi) = \leq_{\psi.\phi}$ . Both sides of the required equality are 2-cells  $\leq : \psi.\phi \rightarrow \psi.\phi$ . Thus, they are equal by the uniqueness of 2-cells between any two 1-cells.

- (m) (2-Interchange) Let  $\psi, \phi, \delta : (X, A) \rightrightarrows (Y, B)$  and  $\psi', \phi', \delta' : (Y, B) \rightrightarrows (Z, C)$  be promodules. For 2-cells  $\leq_1 : \psi \rightarrow \phi, \leq_2 : \phi \rightarrow \delta, \leq_a : \psi' \rightarrow \phi'$  and  $\leq_b : \phi' \rightarrow \delta'$ , need to show  $(\leq_b \cdot \leq_a) * (\leq_2 \cdot \leq_1) = (\leq_b * \leq_2) \cdot (\leq_a * \leq_1)$ . Both RHS and LHS are 2-cells from  $\psi \cdot \psi'$  to  $\delta \cdot \delta'$  and are hence equal.  $\square$

We now define a functor from the category QUnif to ProMod, as fixing objects and taking uniformly continuous maps  $f : (X, A) \rightarrow (Y, B)$  to  $f.B$ .

**Proposition 3.10.** The mapping defined above,  $(-)_* : \text{QUnif}^{\text{op}} \rightarrow \text{ProMod}$  as

- (a) for  $(X, A) \in \text{QUnif}$ ,  $(X, A)_* := (X, A) \in \text{ProMod}$
- (b) for  $f : (X, A) \rightarrow (Y, B)$  in QUnif,  $f^* := B.f$

is indeed a functor.

*Proof.* We will first show that  $B.f = b \circ f : b \in B$  is a promodule, and then that  $(-)_*$  defines a functor.

- (i) (Partial-Order) Inclusion of relations acts as the partial order.
- (ii) (Down-Directed) Fix any  $k, k'$  belonging to  $B.f$ . Thus, there exist  $b, b'$  in  $B$  such that  $k = b.f$  and  $k' = b'.f$ . Using down-directedness of  $B$ , there exists a  $c \in B$  such that  $c \subseteq b, b'$ . Hence, by Lemma 2.4.4,  $c.f \subseteq k, k'$ .
- (iii) (Up-set) Let  $k$  belong to  $B.f$  and  $l : (X, A) \rightarrow (Y, B)$  be a uniformly continuous function such that  $l \supseteq k$ . Define a relation  $b' := \{(f(d), l(d)) : d \in \text{domain}(l)\}$ . By definition, for any  $x \in X$  and  $z \in Y$  such that  $(x, z) \in l$ , we get that  $(f(x), z) \in b'$ . And  $l \supseteq k = b.f$  implies  $\text{domain}(l) \supseteq \text{domain}(f)$  giving  $(x, f(x)) \in f$ . Thus, by definition of composition,  $(x, z) \in b'.f$ . Conversely, suppose  $(x, z) \in b'.f$ . By definition of composition, there exists  $f(x) \in Y$  such that  $(f(x), z) \in b'$ . Again using the definition of  $b'$ , we get that  $z = l(x)$  i.e.  $(x, z) \in l$ . Hence,  $l = b'.f$ . Now we will show that  $b' \supseteq b$ . Because  $b'.f = l \supseteq k = b.f$ , for any  $x \in X$  we have that  $b'(f(x)) \supseteq b(f(x))$ . Thus,  $b'|_{f(x)} \supseteq b|_{f(x)}$ . By down-directedness of  $B$ , the restriction  $b|_{f(x)} \subseteq b$  implies  $b(x)|_{f(x)} \in B$ . Finally,  $b' \supseteq b'|_{f(x)} \supseteq b|_{f(x)}$  gives  $b' \in B$ . Hence,  $b'.f \in B.f$ .
- (iv) Need to show that  $(B.f).A \leq B.f$ . So, fix any  $b \in B$ , we will find  $b' \in B$  and  $a \in A$  such that  $b'.f a \subseteq b.f$ . By quasi-uniformity of  $B$ , there exists  $b' \in B$  such that  $b' b' \subseteq b$ . Using Lemma 2.4.3, we get that  $b' b' f \subseteq b.f$ . As  $f$  is uniformly continuous,  $f.A \leq B.f$  gives that there is some  $a \in A$  such that  $f a \subseteq b'.f$ . Using this in the previous inequality, we get  $b'.f a \subseteq b' b' f \subseteq b.f$ .
- (v) Need to show that  $B.B.f \leq B.f$ . Fix any  $b \in B$ , we will find  $b' \in B$  such that  $b' b' f \subseteq b.f$ . By quasi-uniformity of  $B$ , there exists  $b \in B$  such that  $b' b' \subseteq b$ . Using Lemma 2.4.4, we get  $b' b' f \subseteq b.f$ .

Thus,  $B.f$  is a promodule. We now proceed to show that  $(-)_*$  defines a functor.

- (i) (Composition) Need to show that  $(g \circ f)_* = g_* f_*$  i.e.  $C.g.f = C.g.B.f$ .

In order to show  $C.g.f \leq C.g.B.f$ , fix any  $b \in B, c \in C$ . We will show that  $c g f \subseteq c g b.f$ . As  $f$  is uniformly continuous,  $f.A \leq B.f$  gives that there exists  $a \in A$  such that  $f a \subseteq b.f$ . Using Lemma 2.4.3, we get  $(c g) f a \subseteq (c g) b.f$ . Now, using reflexivity of  $a$ , we get  $c g f \subseteq c g b.f$ .

Now, to show that  $C.g.f \geq C.g.B.f$ . Fix any  $c \in X$ , we will find  $c' \in C$  and  $b \in B$  such that  $c g f \supseteq c g b.f$ . By quasi-uniformity of  $C$ , there exists  $c' \in C$  such that  $c \subseteq c' c'$ . Using Lemma 2.4.4 gives that  $c(g f) \supseteq c' c'(g f)$ . Because  $g$  is uniformly continuous,  $C.g \geq g.B$  gives us  $b \in B$  such that  $g c' \supseteq b g$ . Using this in the previous inequality gives that  $c g f \supseteq c' g b.f$ .

- (ii) (Identity) let  $(X, A)$  be in object of QUnif and  $1_{(X, A)} : (X, A) \rightarrow (X, A)$  be the identity function on  $(X, A)$ . That is,  $1_{(X, A)}$  is defined as  $x \mapsto x$ . Need to show that  $(1_{(X, A)})_* = 1_{(X, A)_*}$ . Using functor's definition,  $LHS = (1_{(X, A)})_* = A.(1_{(X, A)}) = A.1_{(X, A)} = A$  and  $RHS = 1_{(X, A)_*} = 1_{(X, A)}$  Using Proposition 3.2(f), we get that  $A = 1_{(X, A)} = RHS$ .  $\square$

Similar to the above functor, we define a contravariant functor from the category QUnif to ProMod, as fixing objects and taking uniformly continuous maps  $f : (X, A) \rightarrow (Y, B)$  to  $f^o.B$ .

**Proposition 3.11.** The mapping defined above,  $(-)^* : \text{QUnif}^{\text{op}} \rightarrow \text{ProMod}$  as

- (a) for  $(X, A) \in \text{QUnif}^{\text{op}}$ ,  $(X, A)^* := (X, A) \in \text{ProMod}$
- (b) for  $f : (X, A) \rightarrow (Y, B)$  in QUnif,  $f^* := f^o.B$

is indeed a functor.

*Proof.* Showing that  $f^o.B : (Y, B) \rightrightarrows (X, A)$  is a promodule.

So, need to show  $f^o.B$  a prorelation  $Y \rightarrow X$  and that  $(f^o.B).B \sqsubseteq f^o.B$  and  $A.(f^o.B) \sqsubseteq f^o.B$

To show prorelation,



(i) (Partial-order) Inclusion of relations i.e. for  $k = f^o \circ b$  and  $k' = f^o \circ b'$  in  $f^o.B$ ,  $k \subseteq k' \iff b \subseteq b'$

(ii) (Down directed) for  $k, k' \in f^o.B$ , need that  $\exists l \in f^o.B$  such that  $l \subseteq k, k'$

Fix  $k, k' \in f^o.B \implies \exists b, b' \in B : k = f^o \circ b$  and  $k' = f^o \circ b'$

By down-directedness of  $B$ , there exists  $c \in B$  such that  $c \subseteq b, b'$ , define  $l = f^o \circ c$ . Now, using Lemma 2.4.3 gives  $l = f^o \circ c \subseteq k, k'$ .

(iii) (Up-set) for a relation  $l : Y \rightarrow X$  and  $k \in f^o.B$  such that  $l \supseteq k$ , need  $l \in f^o.B$

Let  $b \in B$  be such that  $k = f^o \circ b$  and define  $b' := \{(y, y') : y \in \text{domain}(l) \text{ and } y' \in (f^o)^{-1}(l(y))\}$

As  $l \supseteq k = f^o \circ b$ ,  $\text{domain}(b') = \text{domain}(l) \supseteq \text{domain}(b)$

and  $\text{range}(l) \supseteq \text{range}(f^o \circ b) \implies \forall y \in \text{domain}(b), \text{range}(b') = (f^o)^{-1}(l(y)) \supseteq (f^o)^{-1}(f^o \circ b) = \text{range}(b)$

Now, by definition of  $b'$ ,  $f^o \circ b' \supseteq l$ . To show  $f^o \circ b \subseteq l$ ,

$(x, y) \in f^o \circ b \implies \exists z \in Y : (x, z) \in b'$  and  $(z, y) \in f^o \implies x \in \text{domain}(l)$  and  $z \in l(x)$  i.e.  $(x, z) \in l$

To show  $(f^o.B).B \leq f^o.B$ , need that  $\forall b \in B, \exists b' \in B : f^o \circ b' \circ b' \subseteq f^o \circ b$ ,

Fix any  $b \in B$ , as  $B$  is a quasi-uniformity,  $\exists b' \in B : b' \circ b' \subseteq b \implies f^o \circ b' \circ b' \subseteq f^o \circ b$

To show  $A.(f^o.B) \leq f^o.B$ , need that  $\forall b \in B, \exists b' \in B, a \in A : a \circ f^o \circ b' \subseteq f^o \circ b$ ,

As  $f$  is uniformly continuous,  $f.A \leq B.f$  i.e.  $\forall b \in B, \exists a \in A : f \circ a \subseteq b \circ f \implies a = f^o \circ f \circ a \subseteq f^o \circ b \circ f$

Fix any  $b \in B$ , so,  $\exists b' \in B : b'b' \subseteq b$  And, for this  $b'$ ,  $\exists a : a \subseteq f^o b' f \implies a f^o b' \subseteq f^o b' f f^o b' \subseteq f^o b' b' \subseteq f^o b \implies a f^o b' \subseteq f^o b$

Now, need to show that  $(-)^*$  respects composition and identity.

(i) (Composition) let  $f, g$  be uniformly continuous,  $(X, A) \xrightarrow{f} (Y, B) \xrightarrow{g} (Z, C)$  need that  $(g \circ f)^* = f^*.g^*$

LHS =  $(g \circ f)^* = (g \circ f)^o.C = (f^o \circ g^o).C$  and RHS =  $f^*.g^* = (f^o.B).(g^o.C)$

For equality, showing that LHS  $\geq$  RHS and LHS  $\leq$  RHS:

To show  $(f^o \circ g^o).C \geq (f^o.B).(g^o.C)$ , need that  $\forall c \in C, \exists b \in B, c' \in C : f^o g^o c \supseteq f^o b g^o c'$

Fix any  $c \in C$ , so,  $\exists c' \in C : c' \circ c' \subseteq c \implies f^o g^o c \supseteq f^o g^o (c' c') = f^o g^o (c' \Delta_Z c') \supseteq f^o g^o c' (g g^o) c'$

By uniform continuity of  $g$ , for  $c' \in C, \exists b \in B : g b \subseteq c' g$

Thus,  $f^o g^o c \supseteq f^o g^o (c' g) g^o c' \supseteq f^o (g^o g) b g^o c' = f^o b g^o c'$ .

To show  $(f^o \circ g^o).C \leq (f^o.B).(g^o.C)$ , need that  $\forall b \in B, c \in C, \exists c' \in C : f^o g^o c \subseteq f^o b g^o c'$

Fix any  $c \in C, b \in B$  will show that  $c' := c$  works:

As  $B$  is a quasi-uniformity,  $\Delta_Y \subseteq b \implies f^o \Delta_Y g^o c = f^o g^o c \subseteq f^o b g^o c = f^o b g^o c'$

(ii) (Identity) let  $(X, A) \in \text{QUnif}^{\text{op}}$ , and  $1_{(X, A)} : (X, A) \rightarrow (X, A)$  as  $x \mapsto x$  need that  $(1_{(X, A)})^* = 1_{(X, A)}^*$

LHS =  $(1_{(X, A)})^* = (1_{(X, A)})^o.A = 1_{(X, A)}.A = A$ .

And as  $RHS = 1_{(X, A)}^* = 1_{(X, A)}$  Using Proposition 3.2(f), we get that  $A = 1_{(X, A)} = RHS$ .  $\square$

**Definition 3.12.** Let  $f : (X, A) \rightarrow (Y, B)$  be a uniformly continuous function.

I  $f$  is said to be fully faithful if  $f^*.f_* = A$

II  $f$  is said to be fully dense if  $f_*.f^* = B$

NEED TO ADD TOPOLOGICALLY DENSE DEFINITION

Mention about topology so that below definition makes more sense Might as well put an example as well

**Proposition 3.13.** Fix a uniformly continuous map,  $f : (X, A) \rightarrow (Y, B)$

(a)  $f$  is fully faithful if and only if  $A = f^o.B.f$ , that is  $A \geq f^o.B.f$

(b)  $f$  is fully dense if and only if for any  $b \in B, \exists b' \in B$  such that  $b' \subseteq b f f^o b$

(c)  $f$  is topologically dense if and only if for any  $b \in B, b f f^o b$  is reflexive

(d)  $f$  is fully dense if and only if  $f$  is topologically dense

*Proof.*

(a) (i) ( $\implies$ ) Let  $f$  be fully faithful i.e.  $f^*.f_* = A \implies f^o.B.B.f = A$

Need to show that  $A = f^o.B.f$  i.e.  $A \leq f^o.B.f$  and  $A \geq f^o.B.f$

By hypothesis and quasi-uniformity of  $B$ ,  $A \geq f^o.B.B.f \geq f^o.B.f$

To show  $A \leq f^o.B.f$ , need that  $\forall b \in B, \exists a \in A : a \subseteq f^o b f$

Fix  $b \in B$ , hypothesis gives that  $f^o.B.B.f \leq A$  so,

$\exists a \in A : a \subseteq f^o b b f$  and also, by quasi-uniformity of  $B$ , for  $b, \exists b' \in B : b'b' \subseteq b \implies f^o b' b' f \subseteq f^o b f$

Combining the above two inequalities,  $a \subseteq f^o b b f \subseteq f^o b f$

- (ii) ( $\Leftarrow$ ) Let  $A = f^\circ.B.f$  need to show  $A = f^\circ.B.B.f$  i.e.  $A \geq f^\circ.B.B.f$  and  $A \leq f^\circ.B.B.f$

To show  $A \geq f^\circ.B.B.f$ , need to show that  $\forall a \in A, \exists b, b' \in B : a \supseteq f^\circ b b' f$

Have that  $A \geq f^\circ.B.f$  and  $B.B \leq B$

So, fix  $a \in A$ , now  $\exists b \in B : a \subseteq f^\circ b f$  and for this  $b$ ,  $\exists b' \in B : b' b' \subseteq b$ . Therefore,  $a \supseteq f^\circ b f \supseteq f^\circ b' b' f$

To show  $A \leq f^\circ.B.B.f$ , need  $\forall b, b' \in B, \exists a \in A : a \subseteq f^\circ b b' f$

Before that, uniform continuity of  $f$  along with Lemma 2.1.1 gives that

$$f.A \leq B.f \implies A = f^\circ f.A \leq f^\circ.B.f$$

So, fix  $b, b' \in B$ , now, as ,

$$A \leq f^\circ.B.f \text{ giving}$$

$$\exists a \in A : a \subseteq f^\circ b f \text{ and } \exists a' \in A : a' \subseteq f^\circ b' f \implies \Delta_X \subseteq f^\circ b' f.$$

$$\text{Therefore } a = a \Delta_X \subseteq (f^\circ b f)(f^\circ b' f) \subseteq f^\circ b b' f$$

- (b) (i) ( $\implies$ ) Let  $f$  be fully dense i.e.  $B = f_* f^* = B.f.f^\circ.B$ . showing that  $\forall b \in B, \exists b' \in B : b' \subseteq b f f^\circ b$ :

So, fix  $b \in B$ , as  $B \leq B.f.f^\circ.B$ , there exists  $b' \in B$  such that  $b' \subseteq b f f^\circ b$ .

- (ii) ( $\Leftarrow$ ) Suppose  $\forall b \in B, \exists b' \in B : b' \subseteq b f f^\circ b$ . This gives  $B \leq B.f.f^\circ.B$ , in order to show equality, also need  $B \geq B.f.f^\circ.B$ . By quasi-uniformity of  $B$ , for any  $b \in B, \exists b' \in B : b' b' \subseteq b$ . Now, by Lemma 2.4.2,

$$f f^\circ \subseteq \Delta_Y \implies b' f f^\circ b' \subseteq b' \Delta_Y b' = b' b' \subseteq b$$

- (c) (i) ( $\implies$ ) Let  $f$  be topologically dense. We will show that for any  $b \in B, y \in Y, (y, y) \in b f f^\circ b$ . Fix any  $b \in B$  and  $y \in Y$ . As  $f$  is topologically dense,  $\overline{f(X)} = Y$ , implying that  $y \in \overline{f(X)}$ , by definition giving that

$$\exists x \in X \text{ such that } (f(x), y) \in b \text{ and } (y, f(x)) \in b$$

Re-writing the above statement in terms of relations, and considering  $f$  as a relation:

$$(f(x), y) \in b \text{ gives } x(b \circ f)y \text{ i.e. } y \in (b \circ f)(x) \quad (1)$$

$$(y, f(x)) \in b \text{ gives } f(x) \subseteq b(y) \quad (2)$$

Repeatedly applying Lemma 2.4.3 to (2),

$$f(x) \subseteq b(y) \implies (f \circ f^\circ)(f(x) \subseteq (f \circ f^\circ)b(y) \implies (f \circ f^\circ \circ f)(x) \subseteq (f \circ f^\circ \circ b)(y)$$

Applying Lemma 2.4.1 to the final inequality in the above statement gives that

$$f(x) = (f \circ \Delta_X)(x) \subseteq (f \circ f^\circ \circ f)(x) \subseteq (f \circ f^\circ \circ b)(y)$$

Applying Lemma 2.4.3 and then using (1) on the above inequality completes the result:

$$f(x) \subseteq (f f^\circ b)(y) \implies (b \circ f)(x) \subseteq (b f f^\circ b)(y) \implies y \in (b f f^\circ b)(y) \text{ i.e. } y(b f f^\circ b)y$$

- (ii) ( $\Leftarrow$ ) Fix any  $y \in Y$  and  $b \in B$ . Also, suppose that  $\Delta_Y \leq b f f^\circ b$ . As  $f$  is a function with domain as  $X$ ,  $f^\circ : Y \rightarrow X$ ,  $\phi \neq (f^\circ \circ b)(y) \subseteq X$ . So, fix  $x \in (f^\circ \circ b)(y)$ , going to show that  $(f(x), y) \in b$  and  $(y, f(x)) \in b$ . Again, while viewing  $f$  as a relation.

$$\Delta_Y \leq b f f^\circ b \implies \Delta_Y(y) \subseteq b f f^\circ b(y) = (b f)(f^\circ b(y))$$

Last inequality of the above statement gives  $y \in (b f)(x)$  i.e.  $(f(x), y) \in b$ .

Applying Lemma 2.4.2 to  $f$ , and then using Lemma 2.4.4,

$$f f^\circ \subseteq \Delta_Y \implies f f^\circ b \subseteq \Delta_Y b = b$$

Thus  $f f^\circ b(y) \subseteq b(y)$  and hence  $f(x) \subseteq b(y) \implies (y, f(x)) \in b$

- (d) (i) ( $\implies$ ) Let  $f$  be topologically dense. As  $B$  is a quasi-uniformity, for any  $b \in B$ ,

$$\exists b' \in B : b' b' \subseteq b \text{ and } \Delta_Y \subseteq b' \implies b' = b' \Delta_Y \subseteq b' b' \subseteq b \quad (3)$$

By the characterisation of topologically dense in (c), have that  $\Delta_Y \subseteq b' f f^\circ b'$ . Now, using the (3) and Lemma 2.4.3,

$$\Delta_Y \subseteq b' f f^\circ b' \implies b' = b' \Delta_Y \subseteq b' b' f f^\circ b' \subseteq b f f^\circ b' \subseteq b f f^\circ b$$

Hence, we have  $b' \in B : b' \subseteq b f f^\circ b$  giving us that  $f$  is fully dense (from (b)).

- (ii) ( $\Leftarrow$ ) From (b), we have for  $b \in B$ , the existstence of  $b' \in B$  such that  $b' \subseteq b f f^\circ b$ . As  $B$  is a quasi-uniformity,  $\Delta_Y \subseteq b'$ . So,  $\Delta_Y \subseteq b f f^\circ b$ , and from (c), this gives us that  $f$  is topologically dense.  $\square$

## 4 Yoneda Lemma in Quasi-Uniform Spaces

In this section, we will look at Yoneda Lemma and Yoneda Embedding for Quasi-Uniform Spaces. We use 1 to denote the quasi-uniform space with one element, that is, the quasi-uniform space  $(\{*\}, \{(*, *)\})$ . Also, when unambiguous, we use 1 to denote the quasi-uniformity of the quasi-uniform space 1.

**Definiton 4.1.** The set  $PX$  is defined to be the collection of all promodules from the quasi-uniform space  $(X, A)$  to the quasi-uniform space 1.

$$PX := \{\psi : (X, A) \multimap 1 \mid \psi \text{ is a promodule}\}$$

**Proposition 4.2.** For any  $a \in A$ ,  $\tilde{a}$  is defined to be a relation  $PX \rightarrow PX$  as

$$\text{for } \phi, \psi \in PX, \phi \tilde{a} \psi \text{ only if } \phi \leq \psi.a$$

The set,  $\tilde{A} := \{\tilde{a} : a \in A\}$  defines a quasi-uniformity on  $PX$ .

*Proof.* First need to show that  $\tilde{A}$  is a prorelation,

- (i) (Partial order) For any two relations  $\tilde{a}, \tilde{b} : PX \rightarrow PX$ , define  $\tilde{a} \leq \tilde{b}$  to be true only if  $a \subseteq b$ .
- (ii) (Down-Directed) Need for any  $\tilde{a}, \tilde{b} \in \tilde{A}$ , the existstence of some  $\tilde{c} \in \tilde{A}$  such that  $\tilde{c} \subseteq \tilde{a}, \tilde{b}$ .  
If  $\tilde{a}, \tilde{b} \in \tilde{A}$  then there exist  $a, b \in A$ . By down-directedness of  $A$ , there exists a  $c \in A$  such that  $c \subseteq a, b$ . Now the definition of  $\tilde{A}$  gives that  $\tilde{c} \in \tilde{A}$ . And the definition of the partial order on  $\tilde{A}$  ensures  $\tilde{c} \leq \tilde{a}, \tilde{b}$ .
- (iii) (Upset) For any relation  $l : PX \rightarrow PX$ , need that if  $\tilde{k}$  belongs to  $\tilde{A}$  such that  $l \geq \tilde{k}$ , then  $l \in \tilde{A}$ .  
Fix any  $k : PX \rightarrow PX$ , and  $\tilde{k} \in \tilde{A}$  such that  $l \geq \tilde{k}$ . As  $k$  is a relation between promodules  $X \multimap 1$ , it can be thought of as a relation  $a$  on  $X$ , defined as:

$$a := \{(x, y) : x \in \text{domain}(\psi) \text{ and } y \in \text{domain}(\phi) \text{ whenever } \exists \psi, \phi \in PX : \psi l \phi\}$$

So,  $l \geq \tilde{k}$  gives that  $\tilde{a} \geq \tilde{k}$  i.e.  $a \supseteq k$ . And as  $A$  is an upper-set, we get  $a \in A$ . Now, by definition of  $\tilde{A}$ ,  $l \in \tilde{A}$ .

Secondly, need show that the other two conditions hold for  $\tilde{A}$ ,

- (i) For all  $\tilde{a} \in \tilde{A}$ , need  $\tilde{a}$  to be reflexive i.e if  $\psi \in PX$  then  $\psi \tilde{a} \psi$ .  
By definition of  $\tilde{a}$ , need to show that  $\psi \leq \psi.a$ . So, fix a  $p \in \psi$ , we will show that  $p \subseteq p.a$ . Quasi-uniformity of  $A$  gives that  $\Delta_X \subseteq a$ . Hence, by Lemma 2.4.3,  $p = p \Delta_X \subseteq p.a$ .
- (ii) For all  $\tilde{a} \in \tilde{A}$ , need to find  $\tilde{b} \in \tilde{A}$  such that  $\tilde{b} \tilde{b} \leq \tilde{a}$   
Before showing the result, proving that for any  $x, y \in A$ ,  $\tilde{x} \tilde{y} \leq \tilde{xy}$  i.e.  $\forall \psi, \phi \in PX, \psi(\tilde{x} \tilde{y})\phi \implies \psi \tilde{xy} \phi$ . If  $\psi_1(\tilde{a}.\tilde{b})\psi_3$ , then, the definition of composition gives that  $\exists \psi_2$  such that  $\psi_1 \tilde{b} \psi_2 \tilde{a} \psi_3$ . Now, the definition of  $\tilde{b}$  gives  $\psi_1 \leq \psi_2.b$  and that of  $\tilde{a}$  gives  $\psi_2 \leq \psi_3.a$ . Combining these inequalities,  $\psi_1 \leq \psi_2.b \leq \psi_3.ab$ . Hence, by definition of  $ab$ ,  $\psi_1(\tilde{ab})\psi_3$ . Now, to show the result, fix any  $\tilde{a} \in \tilde{A}$ . Therefore,  $a \in A$ , and by quasi-uniformity of  $A$ ,  $\exists b \in A : b \circ b \subseteq a$ . Thus, by the partial-order defined on  $\tilde{A}$ ,  $\tilde{b} \tilde{b} \leq \tilde{a}$ . Now, transitivity of the partial order gives us the required result,  $\tilde{b} \tilde{b} \leq \tilde{b} \tilde{b} \leq \tilde{a}$ .  $\square$

**Proposition 4.3** (Yoneda Embedding).

For a quasi-uniform space  $(X, A)$ , function  $y_X : X \rightarrow PX$  is defined by  $x \mapsto x^*$  for  $x \in X$ .

- (a)  $y_X : (X, A) \rightarrow (PX, \tilde{A})$  is a uniformly continuous map.
- (b)  $y_X : (X, A) \rightarrow (PX, \tilde{A})$  is fully faithful.

*Proof.*

- (a) In order to show  $y_X$  is uniformly continuous, need to show that  $y_X.A \leq \tilde{A}.y_X$ . By definition of  $\leq$ , need  $\forall a \in A, \exists \tilde{b} \in \tilde{A} : y_X \circ b \subseteq \tilde{b} \circ y_X$ . Applying the relations to some element,  $x$  of the set  $X$ :

$$(y_X \circ b)(x) \subseteq (\tilde{b} \circ y_X)(x) \implies y_X(b(x)) \subseteq \tilde{b}(x^*) \quad (1)$$

So, for the condition given by (4) to hold, if  $y \in b(x)$ , then it's required that  $y^* = y_X(y) \in \tilde{b}(x^*)$  i.e.  $x^* \tilde{a} y^*$ . Using the definition of  $x^*, y^*$  and  $\tilde{a}$ ,

$$x^* \tilde{a} y^* \iff x^o.A \leq y^o.A.a \iff \forall a' \in A, \exists a'' \in A : x^o a'' \subseteq y^o a' a \quad (2)$$

Now, fix any  $a \in A, x \in X$ . Thus, quasi-uniformity of  $A$ , gives  $a'' \in A$  such that  $a'' a'' \subseteq a$ . Also, choose some  $y \in a''(x)$ . Hence, in order to show that the condition from (5) holds, need that  $\forall b \in A, x^o a'' \subseteq y^o b a$ , and by applying the relations to an element  $z$  gives the following condition

$$\forall b \in B, \forall x \in X, (x^o a'')(z) \subseteq (y^o b a)(z) \quad (3)$$

Examining the left side of (6),

$$(x^o a'')(z) = x^o(a''(z)) = \begin{cases} \phi & \text{if } x \notin a''(z) \\ \star & \text{if } x \in a''(z) \end{cases}$$

Thus, to show that (6) holds, need to show that (for any  $b \in A$  and  $z \in X$ ):

$$x \in a''(z) \implies z(y^o b a) \star \text{ i.e. } y \in (b a)(z) \quad (4)$$

To show that (7) holds, fix any  $z \in X : x \in a''(z)$ . Also, by our choice of  $y$ , have that  $y \in a''(x)$ . And as  $b \in A$ , it's reflexive, giving that  $y \in b(y)$ . So, by composition of relations, we get:

$$z a'' x, x a'' y \text{ and } y b y \implies z(a'' a'' b) y \implies z(a b) y \text{ i.e. } y \in (b a)(z)$$

- (b) By using Proposition 2.3 (a), need to show that  $A \geq y_X^o \tilde{A} y_X$  i.e.  $\forall a \in A, \exists \tilde{b} \in \tilde{A} : a \supseteq y_X^o \tilde{b} y_X$ . Applying to an element,  $x \in X$  gives the condition

$$(y_X^o \tilde{b} y_X)(x) \subseteq a(x) \implies (y_X^o \tilde{b})(x^*) = y_x^o(\tilde{b}(x^*)) \subseteq a(x) \quad (5)$$

Thus, if  $y^* \in PX$  such that  $x^* \tilde{b} y^*$ , then  $y \in y_x^o(\tilde{b}(x^*))$ . Now, for (8) to hold,  $y \in a(x)$  i.e.  $x a y$ . Thus, need only to show that for any  $a \in A, \exists b \in A$  such that  $\forall x, y \in X, x^* \tilde{b} y^* \implies x a y$ . So, fix  $a \in A$ , and take  $b \in A : b b \subseteq a$ . Now, let  $x^* \tilde{b} y^*$  i.e.  $x^o . A \leq y^o . A . b$ . Hence,  $\exists c \in A : x^o c \subseteq y^o b b$ . And as  $c$  is reflexive,

$$x c x \implies x(c x^o) \star \implies x(b b y^o) \star \implies x(b b) y \implies x a y \quad \square$$

**Theorem 4.4** (Yoneda Lemma). *For every  $\psi \in PX$ , in the following digram,*

$$(a) \psi \geq \psi^* . (y_X)^*$$

$$(b) \psi \in \overline{y_X(X)} \implies \psi \leq \psi^* . (y_X)_*$$

*Proof.* (a) By definition,  $(y_X)_* = \tilde{A} . y_X$ , and  $\psi^* = \psi^o . \tilde{A}$ . Need that  $\psi \geq (y_X)_* . \psi^* = \psi^o . \tilde{A} . \tilde{A} . y_X$ . And applying Lemma 2.2.1 to  $\tilde{A}$ , the required condition becomes  $\psi \geq \psi^o . \tilde{A} . y_X$ . Fix  $p \in \psi$ , we will find  $a \in A : p \supseteq \psi^o a y_X$ . Examining the right side of the condition, (for any  $a \in A, x \in X$ )

$$(\psi^o . \tilde{a} . y_X)(x) = \psi^o . \tilde{a}(x^*) = \psi^o(\tilde{a}(x^*)) = \begin{cases} \phi & \text{if } \psi \notin \tilde{a}(x^*) \\ \star & \text{if } \psi \in \tilde{a}(x^*) \end{cases} \quad (1)$$

In case  $\psi \notin \tilde{a}(x^*)$ , the condition holds trivially. As  $\psi$  is a promodule,  $\psi . A \leq \psi$  gives  $\exists q \in \psi, a \in A : q a \subseteq p$ . Thus, fix  $x \in X$  and  $\psi \in PX$  such that  $x^* \tilde{a} \psi$ . We will now show that  $x p \star$ . Using the definition of  $\tilde{a}$ ,

$$x^* \tilde{a} \psi \implies x^o . A \leq \psi . a \implies \exists b \in A : x^o b \subseteq q a \implies \forall z \in X, (x^o b)(z) \subseteq (q a)(z) \quad (2)$$

Thus, in particular for  $z = x$ , as  $b$  is reflexive,  $x b x$ , which gives:

$$(x^o b)(x) \subseteq (q a)(x) \implies x^o x \subseteq (q a)(x) \implies \star \in (q a)(x) \quad (3)$$

But, as  $q a \subseteq p$ , (11) gives that  $x p \star$ .

- (b) Suppose  $\psi \in \overline{y_X(X)}$ , need to show  $\psi \leq \psi^* . (y_X)_* = \psi^o . \tilde{A} . y_X$  i.e. for  $a \in A, \exists p \in \psi : p \subseteq \psi^o . \tilde{a} . y_X$ . For any  $x \in \text{domain}(p)$ , the condition requires:

$$p(x) \subseteq \psi^o . \tilde{a} . y_X(x) = \psi^o(\tilde{a}(x^*)) \quad (4)$$

By definition of  $p$ , for (12) to hold, need that  $x p \star \implies \psi \in \tilde{a}(x^*)$ . Fix any  $a \in A$ , we will find  $p \in \psi$  such that (12) holds. By quasi-uniformity of  $A$ ,  $\exists b \in A : b b \subseteq a$ . From Proposition 2.5(a),  $y_X$  is uniformly continuous,  $y_X . A \leq \tilde{A} . y_X$  giving that  $\exists c \in A : y_x c \subseteq \tilde{b} y_X$ . Thus, for any  $z, w \in X$  such that  $z c w$ ,

$$(y_X c)(z) \subseteq (\tilde{b} y_X)(z) \implies y_X(c(z)) \subseteq \tilde{b}(z^*) \implies w^* \in \tilde{b}(z^*) \text{ i.e. } z^* \tilde{b} w^* \quad (5)$$

As  $A$  is a quasi-uniformity,  $\exists d \in A : d d \subseteq c$ . Also, because  $A$  is a down-directed set,  $\exists a' \in A : a' \subseteq b, d$ . This along with (13) gives that for any  $x, y \in X$

$$x(a' a') y \implies x(d d) y \implies x c y \implies x^* \tilde{b} y^* \quad (6)$$

Now, because  $\psi \in \overline{y_X(X)}$ , we get  $\exists x^* \in y_X(X)$  such that  $\psi \tilde{a}' x^*$  and  $x^* \tilde{a}' \psi$ . By definition of  $\tilde{a}$ ,  $\psi \tilde{a}' x^*$  gives

$$\psi \leq x^o.A.a' \implies \exists p \in \psi : p \subseteq x^o a' a' \quad (7)$$

Fix any  $z \in X : zp\star$ , using (15) and (14) gives:

$$zp\star \xrightarrow{\cong} (x^o a' a')\star \xrightarrow{(15)} z(a' a')x \xrightarrow{(14)} z^* \tilde{b} x^* \quad (8)$$

Finally, by definition of the partial order on  $\tilde{A}, a' \subseteq b \implies \tilde{a}' \subseteq \tilde{b}$ . Therefore,  $x^* \tilde{a}' \psi \implies x^* \tilde{b} \psi$ . Now, using (16),  $z^* \tilde{b} x^*$  and  $x^* \tilde{b} \psi$  gives the desired result  $z^* \tilde{b} x^*$ .  $\square$

**Lemma 4.5.** Composition of a right-adjoint and an equivalence is a right adjoint.

**Corollary 4.6.** For  $\psi \in PX$ ,  $\psi \in \overline{y_X(X)}$  if and only if  $\psi$  is a right-adjoint.

*Proof.* Fix any  $\psi \in PX$ .

- (i) ( $\implies$ ) Let  $\psi \in \overline{y_X(X)}$ , from Theorem 4.4, we get that  $\psi = \psi^*.(y_X)_*$ . In order to show  $\psi$  is a right-adjoint, by using Lemma 4.5, it is enough to show that  $\psi^*$  is a right adjoint and that  $(y_X)_*$  is an equivalence.

I In order to show that  $(y_X)_*$  is an equivalence, we need that  $A = (y_X)^*.(y_X)_*$  and  $\tilde{A} = (y_X)_*.(y_X)^*$ . From proposition 4.3 (b), we have that  $y_X$  is fully faithful, and by Proposition 3.13 (a), this gives us that  $A = (y_X)^*.(y_X)_*$ .

- We are now going to show that  $\tilde{A} \leq (y_X)_*.(y_X)^*$ . Fix any  $a, b \in A$ , we need to find  $c \in A$  such that  $\tilde{c} \subseteq \tilde{a} y_X y_X^o \tilde{b}$ .

$$(\tilde{a} y_X y_X^o \tilde{b})(\psi) = (\tilde{a} \tilde{b})(\psi) \supseteq \tilde{c} \tilde{c}(\psi) \supseteq \tilde{c}(\psi)$$

In the above equation, the equality holds because  $\psi \in \overline{y_X(X)}$ , gives the existence of  $x^* = \tilde{b}(\psi)$ . And the first inequality is given by down-directedness of  $\tilde{A}$ , whereas the second one holds because  $\tilde{c}$  is reflexive, as  $\tilde{A}$  is a quasi-uniformity.

- To show that  $\tilde{A} \geq (y_X)_*.(y_X)^*$ , fix any  $a \in A$ . By quasi-uniformity of  $\tilde{A}$ , there exists  $\tilde{b} \in \tilde{A}$  such that  $\tilde{b} \tilde{b} \subseteq a$ . We will show that  $\tilde{a} \supseteq \tilde{b} y_X y_X^o \tilde{b}$ .

$$\psi(\tilde{b} y_X y_X^o \tilde{b})\phi \implies \psi(\tilde{b} \tilde{b})\phi \implies \psi \tilde{a} \phi$$

II In order to show that  $\psi^*$  is a right adjoint to  $\psi_*$ , we need to show that  $\tilde{A} \geq \psi_*.\psi^*$  and  $\psi_*.\psi^* \geq 1$ .

- To show that  $\tilde{A} \geq \psi_*.\psi^* = \psi_*.\psi^o.\tilde{A}$ , fix any  $a \in A$ . We will show that  $\psi_*.\psi^o.\tilde{a} \subseteq \tilde{a}$ . Using definition of  $\psi_*$ , for any  $\phi \in y_X(X)$ , we get that

$$(\psi_*.\psi^o.\tilde{a})(\phi) = \psi_*.\psi^o(\tilde{a}(\phi)) = \begin{cases} \phi & \text{if } \tilde{a}(\phi) \neq \psi \\ \psi = \psi_*.\psi^o(\psi) & \text{if } \tilde{a}(\phi) = \psi \end{cases}$$

The above equation gives that  $\phi(\psi_*.\psi^o.\tilde{a})\psi$  implies  $\phi \tilde{a} \psi$ .

- We will show that  $\psi_*.\psi^* \geq 1$ , that is  $\star(\psi^o.\tilde{a}.\psi_*)\star$ . Using definition of  $\psi_*$ ,

$$(\psi^o.\tilde{a}.\psi_*)(\star) = (\psi^o.\tilde{a})(\psi_*(\star)) = (\psi^o.\tilde{a})(\psi) = \psi^o(\tilde{a}(\psi))$$

By the quasi-uniformity of  $\tilde{A}$ , we get that  $\tilde{a}$  is reflexive, and hence,  $\psi \tilde{a} \psi$ . So, from the above equation, we have that  $\star \in \psi^o(\psi) \subseteq (\psi^o.\tilde{a}.\psi_*)(\star)$ .

- (ii) ( $\impliedby$ ) Suppose  $\psi$  is a right adjoint. Need to show that for any  $a \in A$ ,  $\exists x^* \in y_X(X)$  such that  $\psi \tilde{a} x^* \tilde{a} \psi$ . Fix  $a \in A$ . Because  $\psi$  is a right-adjoint, there exists a promodule  $\phi : 1 \dashv\!\!\dashv\!\!\rightarrow X$  such that  $\phi.\psi \leq A$  and  $1 \leq \psi.\phi$ . From  $\phi.\psi \leq A$ , we get that

$$\exists p \in \phi, q \in \psi \text{ such that } a \supseteq p.q \quad (1)$$

Because  $\phi$  and  $\psi$  are promodules,

$$A.\phi \leq \phi \text{ gives the existence of } p' \in \phi \text{ such that } p \supseteq a' p' \quad (2)$$

$$A.\psi \leq \psi \text{ gives the existence of } q' \in \psi \text{ and } a'' \in A \text{ such that } q \supseteq a'' q' \quad (3)$$

Now, from  $1 \leq \psi.\phi$ , we get that  $q' p'$  is reflexive i.e.  $\star(q' p')\star$ . By the definition of composition we get the existence of an  $x \in X$  such that  $\star p' x q' \star$ . Now, considering  $x$  as a map,  $x : 1 \rightarrow X$  defined as  $\star \mapsto x$ ,

$$x q' \star \text{ i.e. } \star \in q'(x) \text{ gives that } q' \supseteq x^o \quad (4)$$

$$\star p' x \text{ i.e. } x \in p'(\star) \text{ gives that } p' \supseteq x \quad (5)$$

Thus, by using inequalities (1),(2) and (3), we get that

$$a \supseteq p q \supseteq a' p' q' a'' \quad (6)$$

By definition of  $\tilde{a}$ , to show  $\psi \tilde{a} x^*$ , we need that  $\psi \leq x^* a = x^o . A . a$ . Showing for any  $b \in A$ ,  $x^o b a \supseteq q'$ :

$$x^o b a \supseteq x^o b a' p' q' \supseteq x^o b a' x q' \supseteq x^o x q' \subseteq q'$$

Where the first inequality comes from (6) by using reflexivity of  $a''$  and then left-multiplying by  $x^o$ . The second inequality comes from (5), third one from reflexivity of  $b$  and  $a'$ , and the last one is given by Lemma 2.4.1.

In order to show  $x^* \tilde{a} \psi$ , by definition of  $\tilde{a}$ , need that  $x^o . A = x^* \leq \psi a$ . Fix  $k \in \psi$ . We will show  $k a \supseteq x^o a''$ .

$$a \supseteq a' p' q' a'' \supseteq p' q' a'' \supseteq p' x^o a'' \quad (7)$$

Where the first inequality is given by (6), second one is due to reflexivity of  $a'$  and the third inequality comes by using (4). Left-multiplying (7) with  $k$  gives the following.

$$k a \supseteq k p' x^o a'' \text{ that is, for any } z \in X, \quad z(k a) \star \implies z(k p' x^o a'') \star \quad (8)$$

As  $\psi$  is a right adjoint to  $\phi$ , we have  $1 \leq \psi . \phi$ , giving that  $\star(k p') \star$ . So, using the implication in(8), we get that  $z(k a) \star$  implies  $z(x^o a'') \star (k p') \star$ , which in turn gives that  $z(x^o a'') \star$ . Hence, we get that  $k a \supseteq x^o a''$   $\square$