A note on Free groups(borrowed from Hungerford Ch1)

Definiton 0.1 (Concrete category)**.** A category $\mathcal{A}$, along with a faithful functor to *Set*

Definiton 0.2 (Free object on a set)**.** Let A be an object of concrete category $\mathcal{A}$, $X \neq \phi$ a set, and a map $i : X \to A$. Then, A is said to be free on the set X iff

$$B \in \mathcal{B} \text{ and } f : X \to B \implies \begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & A \arrow[d, "g"] \\ & B \end{tikzcd} \text{ commutes i.e. } \exists! g \in \mathcal{A}(A, B) \text{ such that } g \circ i = f$$

Theorem 0.1 (Free objects depend only on the cardinality of the set they're free on)**.** If F and F' are objects of a concrete category $\mathcal{A}$ such that they're free on X and X' respectively. Then

$$|X| = |X'| \implies F \cong F'$$

Proof. Let A, B be free on X, Y; $i : X \to A$ and $j : Y \to B$. With $|X| = |Y|$, so, there's a bijection, $p : X \leftrightarrow Y$

$$\text{As } A \text{ is free on } X \text{ , } \begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "p \circ j"'] & A \arrow[d, "f"] \\ & B \end{tikzcd} \text{ i.e. } \begin{tikzcd} X \arrow[r, "i"] \arrow[d, "p"] & A \arrow[d, "f"] \\ Y \arrow[r, "j"] & B \end{tikzcd} \text{ must commute for some unique f.} \tag{1}$$

$$\text{Similarly, as } B \text{ is free on } Y \text{ , } \begin{tikzcd} Y \arrow[r, "j"] \arrow[dr, "p^{-1} \circ i"'] & B \arrow[d, "g"] \\ & A \end{tikzcd} \text{ i.e. } \begin{tikzcd} Y \arrow[r, "j"] \arrow[d, "p^{-1}"] & B \arrow[d, "g"] \\ X \arrow[r, "i"] & A \end{tikzcd} \text{ must commute for some unique g.} \tag{2}$$

$$\text{Combining (1): } \begin{tikzcd} A \arrow[r, "f"] & B \\ X \arrow[u, "i"] \arrow[r, "p"] & Y \arrow[u, "j"'] \end{tikzcd} \text{ and (2): } \begin{tikzcd} B \arrow[r, "g"] & A \\ Y \arrow[u, "j"] \arrow[r, "p^{-1}"] & X \arrow[u, "i"'] \end{tikzcd} \text{ gives } \begin{tikzcd} A \arrow[r, "f \circ g"] & A \\ X \arrow[u, "i"] \arrow[r, "p \circ p^{-1} = 1_X"] & X \arrow[u, "i"'] \end{tikzcd} \text{ So that } \begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "i \circ 1_X = i"'] & A \arrow[d, "f \circ g"] \\ & A \end{tikzcd}$$

But again, as A is free on X , there exists a unique ψ satisfying $\psi \circ i = i$. Thus, $\psi = f \circ g$, and as $1_A \circ i = i$, uniqueness of ψ gives $f \circ g = 1_A$. Now, via symmetry, $g \circ f = 1_B$. Hence, $A \cong B$ $\square$

Corollary 0.1.1. Two objects of a category are free on the same set only if they're isomorphic.