# 1 Categories

**Definition 1.1** (Category). A category, A is defined to have each of the following,

- (i) A collection of objects, denoted by ob(A) and written A,B,C  $\in A$ . Such that each object has an 'identity',  $1_A \in A(A, A), 1_B \in A(B, B), 1_C \in A(C, C)$
- (ii) For each pair of objects, a collection of 'links'/morphisms between them, denoted by  $\mathcal{A}(A, B)$  and written as  $f \in \mathcal{A}(A, B)$   $g \in \mathcal{A}(B, C)$ . Such that,
  - (a) morphisms with matching domain, co-domain can be 'chained'/composed  $(g, f) = g \circ f$
  - (b) with this composition being associative,  $(h \circ g) \circ f = h \circ (g \circ f)$
  - (c) and they are 'fixed' by the identity  $f \circ 1_A = f = 1_B \circ f$

**Example 1.1. Non-trivial Identity** Consider the objects to be groups, and morphisms to be direct product between them:

```
i ob (A) = \{G | G \text{ is a group}\}
ii A(A,B) := A \times B
iii A(B,C) \circ A(A,B) \to A(A,C)
```

So, there's a unique morphism between any two objects i.e groups. And the identity morphism,

$$\forall A, B \in \mathcal{A}$$
, if  $f \in \mathcal{A}(A, B)$ , then  $f \circ 1_A \in \mathcal{A}(A, B) \times \mathcal{A}(A, A) \to \mathcal{A}(A, B)$  and  $1_B \circ f \in \mathcal{A}(B, B) \times \mathcal{A}(A, B) \to \mathcal{A}(A, B)$ 

Thus, ob(A) along with  $\circ$  is actually a group. And hence has a unique inverse. But how exactly?

**Example 1.2. Set** The objects are defined to be sets, and morphisms are the functions between them, with the usual composition law:

```
i ob (A) = \{S | S \text{ is a set} \}
ii (f: A \to B) \in \mathcal{A}(A, B)
iii (g \in \mathcal{A}(B, C)) \circ (f \in \mathcal{A}(A, B)) \to g(f) \in \mathcal{A}(A, C)
```

**Example 1.3. Pre-ordered Set** A pre-ordered, can be made into a category via the binary operation, so that the morphism  $a \to b$  is defined iff  $a \le b$  where  $\le$  is the preorder. The interesting part about this category is that there's at most one morphism between any two objects.

**Example 1.4. Grp** Objects are groups, with homomorphisms between them being the morphisms, and composition being as usual:

```
i ob(A) = \{G | G \text{ is a group } \}

ii A(A, B) = Hom(A, B) i.e. all f such that \forall x, y \in Af((x) \cdot_A (y)) = (f(x)) \cdot_B (f(y))

iii composition is defined as that between two group homomorphisms
```

In this example, the set of all morphisms along with composition forms a group.

**Example 1.5. Ring** Objects are rings, and arrows are ring homomorphisms between them.

```
i ob(A) = \{G|G \text{ is a ring }\}
ii A(A,B) = Hom(A,B)
```

iii composition is defined as that between two ring homomorphisms

**Definition 1.2** (Dual Category). Given a category  $\mathcal{A}$ , it's opposite/dual,  $\mathcal{A}^{op}$  is a category with the same objects, but reversed arrows, while keeping the composition:

$$ob(\mathcal{A}^{op}) = ob(\mathcal{A})$$
 and  $\forall A, B \in ob(\mathcal{A})$ ,  $\mathcal{A}^{op}(A, B) = \mathcal{A}(B, A)$ 

**Example 1.6.** Vect<sub>k</sub> Objects are vector spaces over field k, and the morphisms between them are linear transformations

- i  $ob(A) = \{A | A \text{ is a vector space}\}\$
- ii  $\mathcal{A}(A,B) = \mathcal{L}(A,B)$
- iii composition is defined as that of linear transformations

**Definition 1.3** (Isomorphism). An isomorphism, between objects, is a morphism between them such that it's 'inverse' is also a morphism. So,

$$f: A \to B$$
 is an isomorphism  $\iff \exists g \in \mathcal{A}(B,A): gf = 1_A \text{ and } fg = 1_B$ 

**Definiton 1.4** (Product Category). Somewhat like a cartesian product of categories. Given categories  $\mathcal{A}$  and  $\mathcal{B}$ ,  $\mathcal{A} \times \mathcal{B}$  is defined as:

- i  $ob(\mathcal{A} \times \mathcal{B}) := ob(\mathcal{A}) \times ob(\mathcal{B})$
- ii  $(A \times B)((A, B), (A', B')) := A(A, A') \times B(B, B')$
- iii  $(f,g) \in \mathcal{A} \times \mathcal{B}((A,B),(C,D))$ ,  $(a,b) \in \mathcal{A} \times \mathcal{B}((C,D),(E,F)) \implies (a,b) \circ (f,g) := (a \circ f, b \circ g)$
- iv  $\forall (A, B) \in ob(A \times B)$ ,  $1_{(A,B)} := (1_A, 1_B)$

Example 1.7 (CAT). The category of all categories with morphisms being functors.

- $i\ ob(\mathcal{A}) = \{A|A \text{ is a category}\}\$
- ii  $\mathcal{A}(A,B) = F(A,B)$
- iii  $F: \mathcal{A} \to \mathcal{B}$ ,  $G: \mathcal{B} \to \mathcal{C} \implies G \circ F := H: \mathcal{A} \to \mathcal{C}$

And thus, the identity of  $\mathcal{A}$  is the functor,  $1_{\mathcal{A}}: \mathcal{A} \to \mathcal{A}$ 

**Example 1.8. Functor Category** Fix categories  $\mathcal{A}$  and  $\mathcal{B}$ . Take objects to be the functors  $F:A\to B$  and morphisms as the natural transformations between the objects. This **Functor category** is written as  $[\mathcal{A},\mathcal{B}]$  and  $\mathcal{B}^{\mathcal{A}}$ 

#### Example 1.9. Top

- i objects are topological spaces
- ii morphisms are continous functions

### 2 Functors

**Definition 2.1** ((Covariant)Functor). A functor is a map between categories, written  $F: A \to B$ , consists:

- (i) function taking objects of  $\mathcal{A}$  to those of  $\mathcal{B}$  i.e.  $ob(\mathcal{A}) \to ob(\mathcal{B})$ . Written as  $A \to F(A)$ .
- (ii) associative, identity-preserving function taking links between objects of  $\mathcal{A}$  to those for  $\mathcal{B}$ ,  $f \mapsto F(f)$ , i.e.

$$\forall A, B \in \mathcal{A}, \ \mathcal{A}(A, B) \mapsto \mathcal{B}(F(A), F(B)) \text{ such that } (a) \ f : A \to B \ , g : B \to C \implies F(g \circ f) = F(g) \circ F(f) = F(g \circ f)$$

$$(b) \ F(1_A) = 1_{F_A}$$

**Example 2.1. Forgetful Functors** They essentially ignore some of the structure of the 'domain'.

- (a)  $U: Grp \to Set$  takes groups to their underlying set, and homomorphisms to maps between the sets. Similarly,  $Ring \to Set$  and  $Vect_k \to Set$
- (b) Let Ab be the category of abelian groups, then,  $U: Ring \to Ab$  takes rings to their additive group, 'forgetting' the multiplicative group. And if Mon is the category of monoids,  $U: Ring \to Mon$  'forgets' the additive group.
- (c)  $U: Ab \to Grp$  just takes each abelian group to itself, and does the same for (homo)morphisms.

### Example 2.2. Free Functors

(a) let F(S) denote the free group on a set S. Then,  $F: Set \to Grp$  is a 'free' functor taking sets to their free group, and thus the maps between them become homomorphisms between their free groups. As,

$$f \in Set(S, S') \mapsto F(f) \in Grp(F(S), F(S'))$$
 i.e.  $f : s \to s'$  goes to  $F(f)$  defined as  $g := F(s) \mapsto f(g)$ 

- (b) Similarly, there's a 'free' functor  $F: Set \to CRing$  to the category of commutative rings. Defined as taking sets to polynomial rings having each element as a commuting variable, and coefficients from  $\mathbb{Z}$ .
- (c) Fix any field  $\mathbb{F}$ , and define F(S) to be a vector space over it with (Shrauder)basis S. As basis completely determines a vector space,

$$F(S) := \{L: S \to \mathbb{F} \mid \text{ L takes only finitely many s } \in \text{ S to a non-zero k } \in \mathbb{F} \} \text{ i.e. } F(S) \mapsto \sum_{s \in S} k_s s \in \mathbb{F} \}$$

and 
$$f \in Set(S, S')$$
 goes to  $F(f) : L(F(S), F(S'))$ 

**Example 2.3.** Let  $\mathcal{G}, \mathcal{H}$  be the one object categories of monoids G,H respectively. Then, due to composition being associative and identity preserving, possible functors are precisely the homomorphisms.

**Example 2.4.** Let monoid G be regarded as a one-object category,  $\mathcal{G}$ . Then, functor  $F: \mathcal{G} \to Set$  has one object, a set S. And,  $\forall g \in G$ ,  $F(g): S \to S$  is defined as (F(g))(s) = g \* s where \* is an associative identity-preserving function. Thus,  $(g, s) \mapsto g.s$  i.e. S is a left G-set.

**Definition 2.2** (Contravariant Functor). For categories  $\mathcal{A}$  and  $\mathcal{B}$ ,  $\mathcal{A}^{op} \mapsto \mathcal{B}$  is a contravariant functor from  $\mathcal{A}$  to  $\mathcal{B}$ .

**Example 2.5.** Let k be a field and V, V', W be vector spaces over it. Then fixing W,

$$\forall f \in Hom(V, V'), \exists f^* : Hom(V', W) \rightarrow Hom(V, W) \text{ as } g \in Hom(V', W) \implies V \xrightarrow{f} V' \xrightarrow{g} W$$

recheck the following argument So, for each  $V \in ob(Vect_k)$ , Hom(V, W) defines a (contravariant) functor on  $Vect_k$ , as, fixing W=V, the above argument can be restated as

$$f \in Vect_k^{op}(V', V) = Vect_k(V, V') \mapsto g \in Vect_k(V', V)$$

**Definition 2.3** (Faithful Functor). A functor  $F: \mathcal{A} \to \mathcal{B}$  is faithful iff the map  $\mathcal{A}(A, A') \mapsto \mathcal{B}(F(A), F(A'))$  is injective for any  $A, A' \in \mathcal{A}$  i.e. each arrow between A, A' goes to at most one arrow between F(A), F(A')

**Definition 2.4** (Full Functor). A functor  $F: \mathcal{A} \to \mathcal{B}$  is full iff the map  $\mathcal{A}(A, A') \mapsto \mathcal{B}(F(A), F(A'))$  is surjective for any  $A, A' \in \mathcal{A}$  i.e. each arrow between A, A' goes to at least one arrow between F(A), F(A')

**Definition 2.5** (Subcategory). A subcategory of  $\mathcal{A}$  is a category with objects from  $\mathcal{A}$ , but not necessarily all of them. Similarly for the morphisms.

**Definition 2.6** (Full Subcategory). A full subcategory of  $\mathcal{A}$  that retains as many morphisms of  $\mathcal{A}$  as possible.

## 3 Natural Transformation

**Definition 3.1** (Natural Transformation). Let  $\mathcal{A}$  and  $\mathcal{B}$  be categories and functors,  $F, G : \mathcal{A} \to \mathcal{B}$ . Then, a natural transformation,  $\alpha : F \to G$  is a family of arrows in  $\mathcal{B}$ ,  $\left(F(A) \xrightarrow{\alpha_A} G(A)\right)_{A \in \mathcal{A}}$  such that

$$F(A) \xrightarrow{F(f)} F(A')$$
 (Naturality Axiom) 
$$\forall f \in \mathcal{A}(A,A'), \text{ the square } \underset{\alpha_A}{\alpha_A} \downarrow \underset{G(A)}{\downarrow} \underset{G(A')}{\downarrow} G(A')$$

This is written as  $\mathcal{A} = \begin{pmatrix} \mathcal{B} \\ \alpha \end{pmatrix} = \mathcal{B}$ . And  $\alpha_A$ , are called the components of  $\alpha$ .

**Lemma 3.1** (Unique factorization through components). For any  $A, B \in \mathcal{A}$ 

$$\forall f \in \mathcal{A}(A,B) , \exists ! f' \in \mathcal{B}(F(A),G(B))$$

*Proof.* Because of the naturality axiom, there's at least one such map,  $f' = G(f) \circ \alpha_A$ . And if there exist two such maps, say a, b then

**Example 3.1. From a discrete category** The natural transformation has one component for every object,  $A \in \mathcal{A}$ , that takes  $1_{F(A)} \mapsto 1_{G(A)}$ .

**Example 3.2. Determinant (of an n**×n matrix) Let R be a commutative ring with unity. So, the matrices on it form a monoid under matrix multiplication. Also, a ring homomorphism,  $f: R \to S$  would induce a monoid homomorphism,  $g: M_n(R) \to M_n(S)$  as

$$f(rr') = f(r)f(r') \implies g(MM') = g(M)g(M')$$

Now, this defines a functor,  $M_n: CRing \to Mon$  which takes each ring to monoid of matrices with entries from it(And each ring homomorphism, h to a map that applies h pointwise to the matrices). Also, there's a forgetful functor,  $F: CRing \to Mon$  that retains only multiplication. Every  $n \times n$  matrix over X over R has a determinant in R which, due to linearity, is a monoid homomorphism,  $det_R: M_n(R) \to F(R)$ . In order to show that  $det_R$  is a natural transformation,

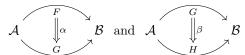
$$M_n(R) \xrightarrow{M_n(h)} M_n(S)$$

$$\forall h \in Cring(R, S), \text{ the square } \underset{det_R}{\underset{det_R}{\downarrow}} \qquad \underset{f(R)}{\underbrace{\downarrow_{det_S}}} \text{ must commute}$$

$$F(R) \xrightarrow{F(h)} F(S)$$

So, given any matrix M over R, and  $A := M_n(h)$ ; B := F(h), need to show that  $B(|M|_R) = |A(M)|_S$ . I.e. that taking the determinant, and then applying only the multiplicative part of h to it is equivalent to first applying, pointwise to the entries of M, the homomorphism h, and then taking the determinant.

Construction 3.1 (Composition of Natural Transforms). Given  $\mathcal{A}$ 



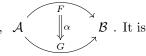
define their composition,  $\beta \circ \alpha$  as

$$\forall A \in \mathcal{A} , \ (\beta \circ \alpha)_A = \beta_A \circ \alpha_A \quad \text{i.e.} \quad \left( F(A) \xrightarrow{\alpha_A} G(A) \xrightarrow{\beta_A} H(A) \right)_{A \in \mathcal{A}}$$

**Example 3.3.** [  $2, \mathcal{B}$  ] Let 2 be the discrete category with two objects. So, a functor,  $F: 2 \to \mathcal{B}$  is a pair of objects in  $\mathcal{B}$  and a natural transformation is a pair of maps in  $\mathcal{B}$ . Thus, the functor category [2,  $\mathcal{B}$ ] a.k.a.  $\mathcal{B}^2$  is isomorphic to the product category  $\mathcal{B} \times \mathcal{B}$ .

**Definition 3.2** (Natural Isomorphism). Let  $\mathcal{A}$  and  $\mathcal{B}$  be categories, a natural isomorphism between functors from  $\mathcal{A}$  to  $\mathcal{B}$  is an isomorphism in  $[\mathcal{A}, \mathcal{B}]$ . I.e. a natural transformation such that it's 'inverse' is also a natural transformation between some functors in  $[\mathcal{A}, \mathcal{B}]$ .

**Lemma 3.2** (Alternate Defintion of Natural Isomorphism). Given a natural transformation,  $\mathcal{A}$ 



a natural isomorphism iff  $\forall A \in \mathcal{A}, \alpha_A : F(A) \to G(A)$  is an isomorphism.

Proof. soon

**Definition 3.3** (Isomorphy of functors). For functors  $\mathcal{A} \overset{F}{\underset{G}{\Longrightarrow}} \mathcal{B}$ , it's said that  $F(A) \cong G(A)$  naturally in A iff F and G are naturally isomorphic.

It gives not only that  $\forall A \in \mathcal{A}$ ,  $F(A) \cong G(A)$  but that there's a family of isomorphisms,  $\left(F(A) \stackrel{\alpha_A}{\longleftrightarrow} G(A)\right)_{A \in \mathcal{A}}$  in  $\mathcal{B}$  that satisfies the naturality axiom.

**Definition 3.4** (Equivalent categories). Categories  $\mathcal{A}$  and  $\mathcal{B}$  are said to be equivalent iff there's an *equivalence* between them. An equivalence is a pair of functors, F, G along with a natural isomorphisms  $\alpha, \beta$  such that:

$$\alpha: 1_{\mathcal{A}} \rightleftharpoons G \circ F \text{ and } \beta: F \circ G \rightleftharpoons 1_{\mathcal{B}}$$

And it's written  $\mathcal{A} \simeq \mathcal{B}$ 

**Definition 3.5** (Essentially Surjective on objects). A functor  $F : \mathcal{A} \to \mathcal{B}$  is said to be **essentially surjective** on objects iff  $\forall B \in \mathcal{B}$ ,  $\exists A \in \mathcal{A}$  such that  $F(A) \cong B$ .

Lemma 3.3. A functor ?? is an equivalence iff it is full, faithful and essentially surjective on objects.

Proof.

## 4 Representables

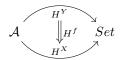
Hereon, only regarding locally small categories.

**Definition 4.1** (Functor  $H^A$  aka  $\mathcal{A}(f, \cdot)$ ). For a fixed  $A \in \mathcal{A}$ , functor  $H^A : \mathcal{A} \to \mathbf{Set}$  is defined:

- i on objects  $B \in \mathcal{A}, H^A(B) := \mathcal{A}(A, B)$
- ii for morphisms  $f \in \mathcal{A}(X,Y)$ ,  $H^A(g) : \mathcal{A}(A,X) \to \mathcal{A}(A,Y)$  mapping each arrow,  $p : A \to X$  as  $p \mapsto f \circ p$

**Definition 4.2** (Representable functor). Functor  $F : \mathcal{A} \to \mathbf{Set}$  is said to be representable iff it's isomorphic to  $H^A$  for some  $A \in \mathcal{A}$ . And in that case, the object A along with the isomorphism are called a arepresentation of F.

**Definition 4.3** ( $H^f$  aka  $\mathcal{A}(f, \_)$ ). Any morphism in  $\mathcal{A}, f: X \to Y$  induces a natural transformation  $H^Y \implies H^X$ :



At  $B \in \mathcal{A}$  for  $p \in Hom(Y, B)$  i.e.  $p: Y \to B$  as  $p \mapsto p \circ f$ 

**Definition 4.4**  $(H^{\bullet})$ . A functor,  $H^{\bullet}: \mathcal{A}^{op} \to [\mathcal{A}, Set]$  defined on

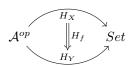
- i objects  $A \in \mathcal{A}$  as  $H^{\bullet}(A) = H^{A}$
- ii morphisms  $f: X \to Y$  as  $H^{\bullet}(f) = H^f$

**Definition 4.5** ( $H_A$  or  $\mathcal{A}(-,A)$  i.e. dual of  $H^A$ ). A functor,  $H_A: \mathcal{A}^{op} \to Set$  defined on:

- i objects  $B \in \mathcal{A}$  as Hom(B, A)
- ii on a morphism,  $f: X \to Y$  in  $\mathcal{A}$ ,  $H_A(g): \mathcal{A}(Y,A) \to \mathcal{A}(X,A)$  as, for each  $p \in \mathcal{A}(Y,A)$  as  $p \mapsto p \circ g$

**Definition 4.6** (contravariant representables). Functor  $X : \mathcal{A}^{op} \to Set$  is representable iff there is some object  $A \in \mathcal{A}$  such that  $X \cong H_A$ . And that choice of object and isomorphism is called a representation.

**Definition 4.7** ( $H_f$ ). Any map,  $f: X \to Y$  in  $\mathcal{A}$  induces a natural transformation  $H_f: \mathcal{A}^{op}$ 



with component for  $B \in \mathcal{A}$  being,  $p \in \mathcal{A}(B, X) \mapsto f \circ p \in \mathcal{A}(B, Y)$ 

**Definition 4.8**  $(H_{\bullet})$ . A functor,  $H_{\bullet}: \mathcal{A} \to [\mathcal{A}_{op}, Set]$  defined on

- i objects  $A \in \mathcal{A}$  as  $H_{\bullet}(A) := H_A$
- ii morphisms( of A) $f: X \to Y$  as  $H_{\bullet}(f) = H_f$

**Definition 4.9** (Generalized Element). A generalized element, of an object A (of some category) is a map with codomain A. The domain of that map is called the shape. So, a generalized element of A, of shape A is a map A is a map A.

**Definition 4.10** (Presheaf). A presheaf on a category  $\mathcal{A}$  is a functor  $\mathcal{A}^{op} \to Set$ .

To be continued.