1 Categories

Definiton 1.1 (Category)**.** A category, $\mathcal{A}$ is defined to have each of the following,

(i) A collection of objects, denoted by ob($\mathcal{A}$) and written A,B,C $\in \mathcal{A}$.
Such that each object has an 'identity', $1_A \in \mathcal{A}(A, A), 1_B \in \mathcal{A}(B, B), 1_C \in \mathcal{A}(C, C)$

(ii) For each pair of objects, a collection of 'links'/morphisms between them, denoted by $\mathcal{A}(A, B)$ and written as f $\in \mathcal{A}(A, B)$ $g \in \mathcal{A}(B, C)$. Such that,

   (a) morphisms with matching domain,co-domain can be 'chained'/composed $(g, f) = g \circ f$
   (b) with this composition being associative, $(h \circ g) \circ f = h \circ (g \circ f)$
   (c) and they are 'fixed' by the identity $f \circ 1_A = f = 1_B \circ f$

Example 1.1. Non-trivial Identity Consider the objects to be groups, and morphisms to be direct product between them:

i ob $(\mathcal{A}) = \{G|$ G is a group$\}$

ii $\mathcal{A}(A, B) := A \times B$

iii $\mathcal{A}(B, C)\ \circ \mathcal{A}(A, B) \to \mathcal{A}(A, C)$

So, there's a unique morphism between any two objects i.e groups. And the identity morphism,

$\forall A, B \in \mathcal{A}$, if f $\in \mathcal{A}(A, B)$, then $f \circ 1_A \in \mathcal{A}(A, B) \times \mathcal{A}(A, A) \to \mathcal{A}(A, B)$ and $1_B \circ f \in \mathcal{A}(B, B) \times \mathcal{A}(A, B) \to \mathcal{A}(A, B)$

Thus, $ob(\mathcal{A})$ along with $\circ$ is actually a group. And hence has a unique inverse. But how exactly?

Example 1.2. Set The objects are defined to be sets, and morphisms are the functions between them, with the usual composition law:

i ob $(\mathcal{A}) = \{S|$ S is a set$\}$

ii $(f : A \to B) \in \mathcal{A}(A, B)$

iii $(g \in \mathcal{A}(B, C)) \circ (f \in \mathcal{A}(A, B)) \to g(f) \in \mathcal{A}(A, C)$

Example 1.3. Pre-ordered Set A pre-ordered , can be made into a category via the binary operation, so that the morphism $a \to b$ is defined iff $a \leq b$ where $\leq$ is the preorder. The interesting part about this category is that there's at most one morphism between any two objects.

Example 1.4. Grp Objects are groups,with homomorphisms between them being the morphisms, and composition being as usual:

i $ob(\mathcal{A}) = \{G|G$ is a group $\}$

ii $\mathcal{A}(A, B) = Hom(A, B)$ i.e. all f such that $\forall x, y \in A f((x) \cdot_A (y)) = (f(x)) \cdot_B (f(y))$

iii composition is defined as that between two group homomorphisms

In this example, the set of all morphisms along with composition forms a group.

Example 1.5. Ring Objects are rings, and arrows are ring homomorphisms between them.

i $ob(\mathcal{A}) = \{G|G$ is a ring $\}$

ii $\mathcal{A}(A, B) = Hom(A, B)$

iii composition is defined as that between two ring homomorphisms

Definiton 1.2 (Dual Category)**.** Given a category $\mathcal{A}$, it's opposite/dual, $\mathcal{A}^{op}$ is a category with the same objects, but reversed arrows, while keeping the composition :

$$ob(\mathcal{A}^{op}) = ob(\mathcal{A}) \text{ and } \forall A, B \in ob(\mathcal{A}) \ , \ \mathcal{A}^{op}(A, B) = \mathcal{A}(B, A)$$

Example 1.6. $\mathbf{Vect}_k$ Objects are vector spaces *over field* k , and the morphisms between them are linear transformations

i $ob(\mathcal{A}) = \{A | A \text{ is a vector space}\}$

ii $\mathcal{A}(A, B) = \mathcal{L}(A, B)$

iii composition is defined as that of linear transformations

Definiton 1.3 (Isomorphism)**.** An isomorphism, between objects, is a morphism between them such that it's 'inverse' is also a morphism. So,

$$f : A \to B \text{ is an isomorphism} \iff \exists g \in \mathcal{A}(B, A) : gf = 1_A \text{ and } fg = 1_B$$

Definiton 1.4 (Product Category)**.** Somewhat like a cartesian product of categories. Given categories $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \times \mathcal{B}$ is defined as:

i $ob(\mathcal{A} \times \mathcal{B}) := ob(\mathcal{A}) \times ob(\mathcal{B})$

ii $(\mathcal{A} \times \mathcal{B})((A, B), (A', B')) := \mathcal{A}(A, A') \times \mathcal{B}(B, B')$

iii $(f, g) \in \mathcal{A} \times \mathcal{B}((A, B), (C, D))$, $(a, b) \in \mathcal{A} \times \mathcal{B}((C, D), (E, F)) \implies (a, b) \circ (f, g) := (a \circ f, b \circ g)$

iv $\forall (A, B) \in ob(\mathcal{A} \times \mathcal{B})$, $1_{(A,B)} := (1_A, 1_B)$

Example 1.7 (CAT)**.** The category of all categories with morphisms being functors.

i $ob(\mathcal{A}) = \{A | A \text{ is a category}\}$

ii $\mathcal{A}(A, B) = F(A, B)$

iii $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{C} \implies G \circ F := H : \mathcal{A} \to \mathcal{C}$

And thus, the identity of $\mathcal{A}$ is the functor, $1_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$

Example 1.8. Functor Category Fix categories $\mathcal{A}$ and $\mathcal{B}$. Take objects to be the functors $F : A \to B$ and morphisms as the natural transformations between the objects.
This **Functor category** is written as $[\mathcal{A}, \mathcal{B}]$ and $\mathcal{B}^{\mathcal{A}}$

Example 1.9. Top

i objects are topological spaces

ii morphisms are continous functions

2 Functors

Definiton 2.1 ((Covariant)Functor)**.** A functor is a map between categories, written $F : \mathcal{A} \to \mathcal{B}$, consists :

(i) function taking objects of $\mathcal{A}$ to those of $\mathcal{B}$ i.e. $ob(\mathcal{A}) \to ob(\mathcal{B})$. Written as $A \to F(A)$.

(ii) associative, identity-preserving function taking links between objects of $\mathcal{A}$ to those for $\mathcal{B}$, $f \mapsto F(f)$, i.e.

$$\forall A, B \in \mathcal{A},\ \mathcal{A}(A, B) \mapsto \mathcal{B}(F(A), F(B)) \text{ such that } (a)\ f : A \to B\,, g : B \to C \implies F(g \circ f) = F(g) \circ F(f) = F(g \circ f)$$
$$(b)\ F(1_A) = 1_{F_A}$$

Example 2.1. Forgetful Functors They essentially ignore some of the structure of the 'domain'.

(a) $U : Grp \to Set$ takes groups to their underlying set, and homomorphisms to maps between the sets.Similarly, $Ring \to Set$ and $Vect_k \to Set$

(b) Let Ab be the category of abelian groups, then, $U : Ring \to Ab$ takes rings to their additive group, 'forgetting' the multiplicative group. And if Mon is the category of monoids, $U : Ring \to Mon$ 'forgets' the additive group.

(c) $U : Ab \to Grp$ just takes each abelian group to itself, and does the same for (homo)morphisms.

Example 2.2. Free Functors

(a) let $F(S)$ denote the free group on a set S. Then, $F : Set \to Grp$ is a 'free' functor taking sets to their free group, and thus the maps between them become homomorphisms between their free groups. As,

$$f \in Set(S, S') \mapsto F(f) \in Grp(F(S), F(S')) \text{ i.e. } f : s \to s' \text{ goes to } F(f) \text{ defined as } g := F(s) \mapsto f(g)$$

(b) Similarly, there's a 'free' functor $F : Set \to CRing$ to the category of commutative rings. Defined as taking sets to polynomial rings having each element as a commuting variable, and coefficients from $\mathbb{Z}$.

(c) Fix any field $\mathbb{F}$, and define F(S) to be a vector space over it with (Shrauder)basis S. As basis completely determines a vector space,

$$F(S) := \{L : S \to \mathbb{F} \mid \text{ L takes only finitely many s } \in \text{ S to a non-zero k } \in \mathbb{F}\} \text{ i.e. } F(S) \mapsto \sum_{s \in S} k_s s$$

$$\text{and } f \in Set(S, S') \text{ goes to } F(f) : L(F(S), F(S'))$$

Example 2.3. Let $\mathcal{G}, \mathcal{H}$ be the one object categories of monoids G,H respectively. Then, due to composition being associative and identity preserving, possible functors are precisely the homomorphisms.

Example 2.4. Let monoid G be regarded as a one-object category, $\mathcal{G}$. Then, functor $F : \mathcal{G} \to Set$ has one object, a set S. And, $\forall g \in G,\ F(g) : S \to S$ is defined as $(F(g))(s) = g * s$ where * is an associative identity-preserving function.Thus, $(g, s) \mapsto g.s$ i.e. S is a left G-set.

Definiton 2.2 (Contravariant Functor)**.** For categories $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}^{op} \mapsto \mathcal{B}$ is a contravariant functor from $\mathcal{A}$ to $\mathcal{B}$.

Example 2.5. Let k be a field and V, V', W be vector spaces over it. Then fixing W,

$$\forall f \in Hom(V, V'), \exists f^* : Hom(V', W) \to Hom(V, W) \text{ as } g \in Hom(V', W) \implies V \xrightarrow{f} V' \xrightarrow{g} W$$

recheck the following argument So, for each $V \in ob(Vect_k)$, $Hom(V, W)$ defines a (contravariant) functor on $Vect_k$, as, fixing W=V, the above argument can be restated as

$$f \in Vect_k^{op}(V', V) = Vect_k(V, V') \mapsto g \in Vect_k(V', V)$$

Definiton 2.3 (Faithful Functor)**.** A functor $F : \mathcal{A} \to \mathcal{B}$ is faithful iff the map$\mathcal{A}(A, A') \mapsto \mathcal{B}(F(A), F(A'))$ is injective for any $A, A' \in \mathcal{A}$ i.e. each arrow between A, A' goes to at most one arrow between $F(A), F(A')$

Definiton 2.4 (Full Functor)**.** A functor $F : \mathcal{A} \to \mathcal{B}$ is full iff the map$\mathcal{A}(A, A') \mapsto \mathcal{B}(F(A), F(A'))$ is surjective for any $A, A' \in \mathcal{A}$ i.e. each arrow between A, A'goes to at least one arrow between $F(A), F(A')$

Definiton 2.5 (Subcategory)**.** A subcategory of $\mathcal{A}$ is a category with objects from $\mathcal{A}$, but not necessarily all of them. Similarly for the morphisms.

Definiton 2.6 (Full Subcategory)**.** A full subcategory of $\mathcal{A}$ that retains as many morphisms of $\mathcal{A}$ as possible.

3 Natural Transformation

Definiton 3.1 (Natural Transformation)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be categories and functors, $F, G : \mathcal{A} \to \mathcal{B}$. Then, a natural transformation, $\alpha : F \to G$ is a family of arrows in $\mathcal{B}$, $\left(F(A) \xrightarrow{\alpha_A} G(A)\right)_{A \in \mathcal{A}}$ such that

(Naturality Axiom) $\forall f \in \mathcal{A}(A, A')$, the square

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(A') \arrow[d, "\alpha_{A'}"] \\ G(A) \arrow[r, "G(f)"'] & G(A') \end{tikzcd}$$

commutes

This is written as $\begin{tikzcd} \mathcal{A} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] & \mathcal{B} \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$. And α_A,are called the components of α.

Lemma 3.1 (Unique factorization through components)**.** For any $A, B \in \mathcal{A}$

$$\forall f \in \mathcal{A}(A, B)\ ,\ \exists! f' \in \mathcal{B}\big(F(A), G(B)\big)$$

Proof. Because of the naturality axiom, there's at least one such map, $f' = G(f) \circ \alpha_A$. And if there exist two such maps, say a, b then □

Example 3.1. From a discrete category The natural transformation has one component for every object, $A \in \mathcal{A}$, that takes $1_{F(A)} \mapsto 1_{G(A)}$.

Example 3.2. Determinant (of an n×n matrix) Let R be a commutative ring with unity. So, the matrices on it form a monoid under matrix multiplication. Also, a ring homomorphism, $f : R \to S$ would induce a monoid homomorphism, $g : M_n(R) \to M_n(S)$ as

$$f(rr') = f(r)f(r') \implies g(MM') = g(M)g(M')$$

Now, this defines a functor, $M_n : \mathit{CRing} \to \mathit{Mon}$ which takes each ring to monoid of matrices with entries from it(And each ring homomorphism, h to a map that applies h pointwise to the matrices). Also, there's a forgetful functor, $F : \mathit{CRing} \to \mathit{Mon}$ that retains only multiplication. Every $n \times n$ matrix over X over R has a determinant in R which, due to linearity, is a monoid homomorphism, $det_R : M_n(R) \to F(R)$. In order to show that det_R is a natural transformation,

$$\forall h \in \mathit{Cring}(R, S),\ \text{the square}\ \begin{tikzcd} M_n(R) \arrow[r, "M_n(h)"] \arrow[d, "det_R"'] & M_n(S) \arrow[d, "det_S"] \\ F(R) \arrow[r, "F(h)"'] & F(S) \end{tikzcd}\ \text{must commute}$$

So, given any matrix M over R, and $A := M_n(h)\ ;\ B := F(h)$, need to show that $B(|M|_R) = |A(M)|_S$. I.e. that taking the determinant, and then applying only the multiplicative part of h to it is equivalent to first applying, pointwise to the entries of M, the homomorphism h , and then taking the determinant.

Construction 3.1 (Composition of Natural Transforms)**.** Given $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B} \end{tikzcd}$ and $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "G"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{B} \end{tikzcd}$,

define their composition, $\beta \circ \alpha$ as

$$\forall A \in \mathcal{A}\ ,\ (\beta \circ \alpha)_A = \beta_A \circ \alpha_A \quad \text{i.e.} \quad \left(\begin{tikzcd} F(A) \arrow[r, "\alpha_A"] \arrow[rr, bend left, "(\beta\circ\alpha)_A"] & G(A) \arrow[r, "\beta_A"] & H(A) \end{tikzcd} \right)_{A \in \mathcal{A}}$$

Example 3.3. $[\ 2, \mathcal{B}\ ]$ Let 2 be the discrete category with two objects. So, a functor, $F : 2 \to \mathcal{B}$ is a pair of objects in $\mathcal{B}$ and a natural transformation is a pair of maps in $\mathcal{B}$. Thus, the functor category $[2, \mathcal{B}]$ a.k.a. $\mathcal{B}^2$ is isomorphic to the product category $\mathcal{B} \times \mathcal{B}$.

Definiton 3.2 (Natural Isomorphism)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be categories, a natural isomorphism between functors from $\mathcal{A}$ to $\mathcal{B}$ is an isomorphism in $[\mathcal{A}, \mathcal{B}]$. I.e. a natural transformation such that it's 'inverse' is also a natural transformation between some functors in $[\mathcal{A}, \mathcal{B}]$.

Lemma 3.2 (Alternate Defintion of Natural Isomorphism)**.** Given a natural transformation, $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B} \end{tikzcd}$. It is a natural isomorphism iff $\forall A \in \mathcal{A}$, $\alpha_A : F(A) \to G(A)$ is an isomorphism.

Proof. soon □

Definiton 3.3 (Isomorphy of functors)**.** For functors $\mathcal{A} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{B}$, it's said that $F(A) \cong G(A)$ naturally in A iff F and G are naturally isomorphic.
It gives not only that $\forall A \in \mathcal{A}\ ,\ F(A) \cong G(A)$ but that there's a family of *isomorphisms*, $\left(F(A) \xleftrightarrow{\alpha_A} G(A) \right)_{A \in \mathcal{A}}$ in $\mathcal{B}$ that satisfies the naturality axiom.

Definiton 3.4 (Equivalent categories)**.** Categories $\mathcal{A}$ and $\mathcal{B}$ are said to be equivalent iff there's an *equivalence* between them. An equivalence is a pair of functors , F, G along with a natural isomorphisms α, β such that:

$$\alpha : 1_{\mathcal{A}} \rightleftarrows G \circ F \text{ and } \beta : F \circ G \rightleftarrows 1_{\mathcal{B}}$$

And it's writtten $\mathcal{A} \simeq \mathcal{B}$

Definiton 3.5 (Essentially Surjective on objects)**.** A functor $F : \mathcal{A} \to \mathcal{B}$ is said to be **essentially surjective** on objects iff $\forall B \in \mathcal{B}$, $\exists A \in \mathcal{A}$ such that $F(A) \cong B$.

Lemma 3.3. A functor ?? is an equivalence iff it is full, faithful and essentially surjective on objects.

Proof. $\square$

4 Representables

Hereon, only regarding locally small categories.

Definiton 4.1 (Functor H^A aka $\mathcal{A}(f, \_)$)**.** For a fixed $A \in \mathcal{A}$, functor $H^A : \mathcal{A} \to \mathbf{Set}$ is defined:

i on objects $B \in \mathcal{A}, H^A(B) := \mathcal{A}(A, B)$

ii for morphisms $f \in \mathcal{A}(X, Y)$, $H^A(g) : \mathcal{A}(A, X) \to \mathcal{A}(A, Y)$ mapping each arrow, $p : A \to X$ as $p \mapsto f \circ p$

Definiton 4.2 (Representable functor)**.** Functor $F : \mathcal{A} \to \mathbf{Set}$ is said to be representable iff it's isomorphic to H^A for some $A \in \mathcal{A}$. And in that case, the object A along with the isomorphism are called a arepresentation of F .

Definiton 4.3 (H^f aka $\mathcal{A}(f, \_)$)**.** Any morphism in $\mathcal{A}, f : X \to Y$ induces a natural transformation $H^Y \implies H^X$:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left=50, "H^Y"{name=U}] \arrow[r, bend right=50, "H^X"'{name=D}] & Set
\arrow[from=U, to=D, Rightarrow, "H^f"]
\end{tikzcd}$$

At $B \in \mathcal{A}$ for $p \in Hom(Y, B)$ i.e. $p : Y \to B$ as $p \mapsto p \circ f$

Definiton 4.4 ($H^\bullet$)**.** A functor, $H^\bullet : \mathcal{A}^{op} \to [\mathcal{A}, Set]$ defined on

i objects $A \in \mathcal{A}$ as $H^\bullet(A) = H^A$

ii morphisms $f : X \to Y$ as $H^\bullet(f) = H^f$

Definiton 4.5 (H_A or $\mathcal{A}(\_, A)$ i.e. dual of H^A)**.** A functor, $H_A : \mathcal{A}^{op} \to Set$ defined on:

i objects $B \in \mathcal{A}$ as $Hom(B, A)$

ii on a morphism, $f : X \to Y$ in $\mathcal{A}$, $H_A(g) : \mathcal{A}(Y, A) \to \mathcal{A}(X, A)$ as, for each $p \in \mathcal{A}(Y, A)$ as $p \mapsto p \circ f$

Definiton 4.6 (contravariant representables)**.** Functor $X : \mathcal{A}^{op} \to Set$ is representable iff there is some object ,$A \in \mathcal{A}$ such that $X \cong H_A$. And that choice of object and isomorphism is called a representation.

Definiton 4.7 (H_f)**.** Any map, $f : X \to Y$ in $\mathcal{A}$ induces a natural transformation H_f : $\begin{tikzcd}
\mathcal{A}^{op} \arrow[r, bend left=50, "H_X"{name=U}] \arrow[r, bend right=50, "H_Y"'{name=D}] & Set
\arrow[from=U, to=D, Rightarrow, "H_f"]
\end{tikzcd}$

with component for $B \in \mathcal{A}$ being, $p \in \mathcal{A}(B, X) \mapsto f \circ p \in \mathcal{A}(B, Y)$

Definiton 4.8 (Yoneda embedding)**.** A functor, $H_\bullet : \mathcal{A} \to [\mathcal{A}^{op}, Set]$ defined on

i objects $A \in \mathcal{A}$ as $H_\bullet(A) := H^A$

ii morphisms $f : X \to Y$ as $H_\bullet(f) = H_f$

To be continued.