Question 4 (b) $f(x,y) := \frac{2xy}{\sqrt{x^2+y^2}}$

For partial derivative along x at $(\delta, 0)$,

$$\frac{\partial f}{\partial x}(\delta,0) = \lim_{h\to 0}\frac{f(\delta+h,0)-f(\delta,0)}{h}$$
$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{2(\delta+h)\times 0}{\sqrt{(\delta+h)^2+0}} - \frac{2\delta\times 0}{\sqrt{\delta^2+0}}\right) = 0$$

In particular, if $\delta = 0$ then the second term is zero as $f(0,0) := 0$. And, due to symmetry, $f_y(0,\delta) = 0$. As :

$$\frac{\partial f}{\partial y}(0,\delta) = \lim_{h\to 0}\frac{f(0,\delta+h)-f(0,\delta)}{h}$$
$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{2\times 0\times(\delta+h)}{\sqrt{0+(\delta+h)^2}} - \frac{2\times 0\times\delta}{\sqrt{0+\delta^2}}\right) = 0$$

So, not only are the partial derivatives f_x and f_y 0 at (0,0), but also along x and y axes respectively.

To show the existence of second derivatives ,

$$f_{xx}(0,0) = \lim_{h\to 0}\frac{f_x(h,0)-f_x(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$$

Similarly, $f_{yy}(0,0) = 0$

Now, if the derivative of f existed at (0,0), then, the following limit must be 0

$$\lim_{h^2+k^2\to 0}\frac{f(0+h,0+k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}}$$
$$= \lim_{h^2+k^2\to 0}\frac{2hk}{h^2+k^2}$$

But this limit is equal to 1 along the line of slope 1, and thus f isn't differentiable.

$$\text{(a) } f(x,y) := \frac{(x^2y + xy^2)sin(x-y)}{x^2+y^2}$$

For partial derivative along x at $(\delta, 0)$,

$$\frac{\partial f}{\partial x}(\delta, 0) = \lim_{h\to 0} \frac{f(\delta+h,0) - f(\delta,0)}{h}$$
$$= \lim_{h\to 0} \frac{1}{h}\Big(\frac{\big((\delta+h)^2 \times 0 + (\delta+h)\times 0\big)sin((\delta+h))}{(\delta+h)^2+0} - 0\Big) = 0$$

In particular, if $\delta = 0$ then the second term is still zero as $f(0,0) := 0$. And, due to symmetry, $f_y(0,\delta) = 0$.

So, not only are the partial derivatives f_x and f_y 0 at (0,0), but also along x and y axes respectively.

To show the existence of second derivatives ,

$$f_{xx}(0,0) = \lim_{h\to 0} \frac{f_x(h,0) - f_x(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

Similarly, $f_{yy}(0,0) = 0$

Now, to show the existence of derivative of f at (0,0), then, the following limit must be 0

$$\lim_{h^2+k^2\to 0} \frac{f(0+h,0+k) - f(0,0) - hf_x(0,0) - kf_y(0,0)}{\sqrt{h^2+k^2}}$$
$$= \lim_{h^2+k^2\to 0} \frac{(h^2k + hk^2)(sin(h-k))}{(h^2+k^2)\sqrt{h^2+k^2})}$$

Now, as we want to evaluate differentiablity at (0,0), it's enough to consider all the lines of the form $y = mx$ i.e. (i, mi) with $i \in \mathbb{R}$. Also, for points along the x-axis, the limit is zero as,

$$\lim_{h^2+k^2\to 0} \frac{(0\times k + 0\times k^2)(sin(0-k))}{(0+k^2)\sqrt{0+k^2)}} = 0$$

By symmetry, the limit is zero along y-axis as well, so, let m be some finite, non-zero number, and consider the line (i, mi),

$$\begin{aligned}
&\lim_{i\to 0} \frac{(i^2\times mi + i\times m^2i^2)(sin(i-mi))}{(i^2+m^2i^2)\sqrt{i^2+m^2i^2)}} \\
&= \lim_{i\to 0} \frac{i^3(m+m^2)(sin(i(1-m))}{i^2(1+m^2)|i|\sqrt{1+m^2}} \\
&= \lim_{i\to 0} \frac{m+m^2}{(1+m^2)\sqrt{1+m^2}} \times sgn(i) \times sin(i(1-m))
\end{aligned}$$

For any particular line, the first term is a constant, second one is ± 1, while the third one goes to 0 as $i \to 0$. Thus, the limit is 0 for every line going through the origin.

Thus, the differential exists at the origin.

Question 5 (Marked)

$$f(x,y) = xy(1-x^2-y^2)$$

For each partial derivative to be zero,

$$f_x = y - 3x^2y - y^3 = 0 \text{ and } f_y = x - x^3 - 3xy^2 = 0$$
$$\implies f_x = y(1-3x^2-y^2) = 0 \text{ and } f_y = x(1-x^2-3y^2) = 0$$

So, a critical point is $(0,0)$.

And,
If $x = 0, y \neq 0$ then $f_x = y(1-y^2) = 0 \implies y = \pm 1$
gives two critical points: $(0, \pm 1)$

If $y = 0, x \neq 0$ then $0 = f_y = y(1-x^2) \implies x = \pm 1$
gives two critical points: $(\pm 1, 0)$

If $x \neq 0 \neq y$, then

$$\begin{aligned} f_x = y - 3x^2y - y^3 = 0 &\implies y(1-3x^2) = y^3 \\ &\implies 1 - 3x^2 = y^2 \qquad (equation A) \end{aligned}$$

And, also,

$$\begin{aligned} f_y = x - 3y^2x - x^3 = 0 &\implies x(1-3y^2) = x^3 \\ &\implies 1 - 3y^2 = x^2 \qquad (equation B) \end{aligned}$$

Thus,

Substituting B into A, $1 - 3 + 9y^2 = y^2 \implies y = \pm 0.5$
And, substituting A into B, $1 - 3 + 9x^2 = x^2 \implies x = \pm 0.5$

Thus, the four such possible points are also critical:

$$(0.5, 0.5), (0.5, -0.5), (-0.5, 0.5), (-0.5, -0.5)$$

Now, to classify these critical points, looking at $rt - s^2$

$$r = f_{xx} = -6xy = f_{yy} = t \text{ and } s = f_{xy} = 1 - 3x^2 - 3y^2$$

$$\text{So, } rt = 36x^2y^2 \text{ and } s^2 = (1 - 3(x^2 + y^2))^2$$

At $(0, 0), rt = 0$ and $s = 1$. Thus, $rt - s^2 = 0 - 1 < 0$
Thus, $(0, 0)$ is a saddle point.

At $(\pm 1, 0)$ and $(0, \pm 1)$,
$rt = 36 \times 1 \times 0 = 36 \times 0 \times 1 = 0$ and $s^2 = (1 - 3)^2 = 4$.
Thus, $rt - s^2 = 0 - 4 < 0$
Thus, $(\pm 1, 0), (0, \pm 1)$ are saddle points.

At $(0.5, 0.5), (0.5, -0.5), (-0.5, 0.5), (-0.5, -0.5)$,
$rt = 36x^2y^2 = 36 \times 0.25 \times 0.25 = \frac{9}{4} = 2.25$
and $s^2 = (1 - 3(0.25 + 0.25))^2 = (-0.5)^2 = 0.25$
Hence, $rt - s^2 = 2.25 - 0.25 = 2 > 0$
Thus, $(0.5, -0.5), (-0.5, 0.5)$ are minima(as $r = 1.5$),
while $(0.5, 0.5), (-0.5, -0.5)$ are maxima(as $r = 1.5$).

Question 3 (Marked) **(a)** $f(x,y) := \frac{x^2y^2}{x^2y^2+(x-y)^2} = \frac{1}{1+(\frac{x-y}{xy})^2} = \frac{1}{1+(\frac{1}{y}-\frac{1}{x})^2}$

$$\lim_{x\to 0} f(x,y) = \lim_{x\to 0} \frac{1}{1+(\frac{1}{y}-\frac{1}{x})^2} = 0 \implies \lim_{y\to 0}\lim_{x\to 0} f(x,y) = 0$$

And as the expression is symmetric in x and y,

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{x\to 0}\lim_{y\to 0} f(x,y)$$

But the simultaneous limit at$(0,0)$ along $T(t) := (t,t)$ is

$$\lim_{t\to 0} f(T(t)) = \lim_{t\to 0} \frac{1}{1+(\frac{1}{t}-\frac{1}{t})^2} = 1$$

If the simultaneous limit existed, all the iterated limits would be equal to it. So, there is a curve, $S(t) := (\frac{1}{t}, \frac{1}{t+1})$ with

$$\lim_{t\to\infty} f(S(t)) = \lim_{t\to\infty} \frac{1}{1+(t+1-t)^2} = \frac{1}{2}$$

Thus, f is discontinous at $(0,0)$

(b) $f(x,y) := \frac{e^{\frac{-1}{x^2}}y}{e^{\frac{-1}{x^2}}+y^2} = \frac{ye^{\frac{1}{x^2}}}{1+(ye^{\frac{1}{x^2}})^2} = \frac{1}{\frac{1}{ye^{1/x^2}}+ye^{1/x^2}}$

$$\lim_{y\to 0}\lim_{x\to 0} \frac{1}{\frac{1}{ye^{1/x^2}}+ye^{1/x^2}} = \lim_{y\to 0} 0 = 0$$

$$\lim_{x\to 0}\lim_{y\to 0} \frac{ye^{\frac{1}{x^2}}}{1+(ye^{\frac{1}{x^2}})^2} = \lim_{x\to 0} 0 = 0$$

To show the non-existence of simulatenous limit at $(0,0)$, consider the curve $T(t) := (t, e^{-1/t^2})$

$$\lim_{t\to 0} f(T(t)) = \frac{e^{-1/t^2}\times e^{1/t^2}}{1+(e^{-1/t^2}\times e^{1/t^2})^2} = \frac{1}{2}$$