## Question 1

$$C \subseteq D \subseteq \mathbb{R};$$
$$\big(f_n\big)_{n\in\mathbb{N}} \text{ is uniformly convergent on } C;$$
$$\forall i \in \mathbb{N},\ f_i : D \to \mathbb{R} \text{ is continous}$$

**Show** $\exists f$ such that $f_n \xrightarrow[\text{uniformly}]{\overline{C}\cap D} f$ and $f$ is continous.

*Proof.* Fix an $\epsilon > 0$ so, by uniform continuity of $f$,

$$\exists M(\epsilon) : p, m \geq M \implies \forall x \in C, |f_n(x) - f_m(x)| < \frac{\epsilon}{3}$$

Now, let $k \in \overline{C}$ so, $\exists (x_n) \subseteq C$ such that $x_n \to x$.
By continuity of each $f_i$,

$$\exists \delta(i, \epsilon) \text{ s.t. } |x - y| \leq \delta \implies |f_i(x) - f_i(y)| < \frac{\epsilon}{3}$$

So, there's $K(p, m, \epsilon, (x_n))$ such that

$$i \geq K \implies |f_p(x_i) - f_p(k)| < \frac{\epsilon}{3} \text{ and } |f_m(x_i) - f_m(k)| < \frac{\epsilon}{3}$$

Thus, by triangle inequality,

$$\begin{aligned} 2 \times \frac{\epsilon}{3} = \epsilon &> |f_p(x_i) - f_p(k)| + |f_m(x_i) - f_m(k)| \\ &\geq |f_p(x_i) - f_m(x_i) + f_p(k) - f_m(k)| \\ &\geq ||f_p(x_i) - f_m(x_i)| - |f_m(k) - f_p(k)|| \end{aligned}$$

Now, as $|f_p(x_i) - f_m(x_i)| < \frac{\epsilon}{3}$,
If $|f_m(k) - f_p(k)| < \frac{\epsilon}{3}$ then it's shown that $M$ works for any $k$. Else,

$$\begin{aligned} |f_m(k) - f_p(k)| \geq \frac{\epsilon}{2} &\implies \frac{2\epsilon}{3} > |f_m(k) - f_p(k)| - \frac{\epsilon}{3} \\ &\implies \epsilon > |f_m(k) - f_p(k)| \end{aligned}$$

□

**Question 3**

$$\begin{aligned} A \text{ is closed and bounded };\\ (f_n) \text{ is a sequence of continous functions on } A;\\ (f_n) \xrightarrow{p.w.} f, \text{ with f continous on } A;\\ \forall x \in A, f_n(x) \geq f_{n+1}(x), \text{ with } n \in \mathbb{N}; \end{aligned}$$

**Prove** that $f_n \rightrightarrows f(A)$

**Question 5** **Prove** If $\Sigma a_n$ is absolutely convergent then $\Sigma \frac{a_n x^n}{1+x^{2n}}$ converges uniformly on $\mathbb{R}$.