Final Exam

Question 1

Given that $c \in (a, b)$,with
(a) $f \in R[a, c]$ and
(b) $f \in R[b, c]$
Need that $f \in R[a, b]$

Proof. Need to show that,

$$\forall \epsilon > 0, \exists P \in \mathbb{P}([a, b]) : U(P, f) - L(P, f) < \epsilon$$

Fix any $\epsilon > 0$.
By (a), have that

$$\exists P \in \mathbb{P}([a, c]) : U(P, f) - L(P, f) < \frac{\epsilon}{2}$$

By (b), have that

$$\exists Q \in \mathbb{P}([c, b]) : U(Q, f) - L(Q, f) < \frac{\epsilon}{2}$$

Consider

$$U(P, f) + U(Q, f) = \sum_{i=1}^{p} M_i \delta t_i + \sum_{i=1}^{q} M_i' \delta t_i'$$

where M_i and t_i correspond to P, while where M_i' and t_i' correspond to Q.

Taking $S := P \cup Q$ so that $S \in \mathbb{P}([a,b])$,
giving S=$\{a < t_1 < t_2 < ... < c < t'_1 < t'_2 < ... < b\}$
where $t_i \in P$ and $t'_i \in Q$, so that,

$$U(S,f) = \sum_{i=1}^{p+q} M''_i \delta t_i = \sum_{i=1}^{p} M_i \delta t_i + \sum_{i=p+1}^{q} M'_{i-p} \delta t'_{i-p}$$
$$= U(P,f) + U(Q,f)$$

Repeating the same argument for lower sums,

$$L(P,f) + L(Q,f) = \sum_{i=1}^{p} m_i \delta t_i + \sum_{i=1}^{q} m'_i \delta t'_i$$

where m_i and t_i correspond to P, while where m'_i and t'_i correspond to Q. Taking $S := P \cup Q$ so that $S \in \mathbb{P}([a,b])$,
giving S=$\{a < t_1 < t_2 < ... < c < t'_1 < t'_2 < ... < b\}$
where $t_i \in P$ and $t'_i \in Q$, so that,

$$L(S,f) = \sum_{i=1}^{p+q} m''_i \delta t_i = \sum_{i=1}^{p} m_i \delta t_i + \sum_{i=p+1}^{q} m'_{i-p} \delta t'_{i-p}$$
$$= L(P,f) + L(Q,f)$$

Hence,

$$\begin{aligned} U(S,f) - L(S,f) &= (U(P,f) + U(Q,f)) - (L(P,f) + L(Q,f)) \\ &= U(P,f) - L(P,f) + U(Q,f) - L(Q,f) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

□

Question 2

Suppose $P \in \mathbb{P}([a, b])$ such that $U(P, f) = L(P, f)$.
Prove that f is a constant function.

Proof. As $U(P, f) = L(P, f)$,

$$\sum_{i=1}^{p} M_i \delta t_i = \sum_{i=1}^{p} m_i \delta t_i \implies \sum_{i=1}^{p} (M_i - m_i) \delta t_i = 0$$

But, as each $\delta_i > 0$, it must be that each $M_i - m_i = 0$.
Hence, f is constant in every $\Delta t_i = [t_{i-1}, t_i]$.
So, suppose(if possible) f is not constant over [a,b] ,
Then, there must be some $\alpha \in \mathbb{N}$ and $\Delta t_k, \Delta t_{k+\alpha}$ such that f takes distinct values in them.
Then, as f is constant on them, $f(t_k) \neq f(t_{k+\alpha})$
But, f is constant on $\Delta t_k = [t_{k-1}, t_k]$ and $\Delta t_{k+1} = [t_k, t_{k+1}]$
and as, t_k, is in both, f attains same value on both.

Now, Δt_{k+1} and Δt_{k+2} share t_{k+1} thus fixing the value of f to be the same over both of them. And, this is the same value as that on Δt_k, i.e. $f(t_k)$.

Repeating the above argument $\alpha - 2$ more times, the value on $\Delta t_{k+\alpha}$ also becomes $f(t_k)$, i.e. $f(t_k) = f(t_{k+\alpha})$
Hence, our supposition was incorrect, and f is constant over [a,b]. $\square$

Question 3

$a_n > 0$ need to show that $\limsup \frac{a_{n+1}}{a_n} \geq \limsup (a_n)^{1/n}$

Proof. Case 1: $\limsup \frac{a_{n+1}}{a_n} = \infty$
As $\limsup (a_n)^{\frac{1}{n}} \leq \infty = \limsup \frac{a_{n+1}}{a_n}$,done

Case 2: $\limsup \frac{a_{n+1}}{a_n} = -\infty$

$$\limsup \frac{a_{n+1}}{a_n} = -\infty \implies \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = -\infty$$

But, as all a_n are positive, so is their raito, and hence it cant be unbounded below.

Case 3: $\limsup \frac{a_{n+1}}{a_n} = a \in \mathbb{R}$
So, $\forall \epsilon > 0, \exists M \in \mathbb{N} : n > M \implies \frac{a_{n+1}}{a_n} < a + \epsilon.$
Fix an M. So, as $a \geq 0 \implies (a + \epsilon) > 0$, for any $n > M$,

$$a_n < (a + \epsilon)a_{n-1} < (a + \epsilon)^2 a_{n-2} < ... < (a + \epsilon)^{n-M} a_M$$

Now, as $\frac{a_M}{a^M}$ is a constant, it's n-th root goes to 1 i.e.

$$\exists K \in \mathbb{N} : n \geq K \implies (\frac{a_M}{a^M})^{\frac{1}{n}} < 1 + \epsilon$$

Thus, by the previous inequality,

$$(a_n)^{\frac{1}{n}} < (a + \epsilon)^{1-\frac{M}{n}}(a_M)^{\frac{1}{n}} < (a + \epsilon)(\frac{a_M}{(a + \epsilon)^M})^{\frac{1}{n}}$$

$$\implies (a_n)^{\frac{1}{n}} < (a+\epsilon)(\frac{a_M}{a^M})^{\frac{1}{n}} < (a+\epsilon)(1+\epsilon) = a+(a+1)\epsilon+\epsilon^2$$

Thus,

$$\limsup (a_n)^{\frac{1}{n}} \leq a + (a + 1)\epsilon + \epsilon^2 < a + (a + 1)(1 + \epsilon)\epsilon,$$

but as this holds for every $\epsilon > 0$,

$$\limsup (a_n)^{\frac{1}{n}} \leq a = \limsup \frac{a_{n+1}}{a_n}$$

$\square$

Question 4

Determine whether $x(1+\frac{1}{n})$ converges uniformly on $\mathbb{R}$.

Proof. To show pointwise convergance,

$$\lim_{n\to\infty} f_n = \lim_{n\to\infty} \left(x + \frac{x}{n}\right) = x$$

To show that it converges non uniformly,
need an ϵ for which every δ fails for some $x \in \mathbb{R}$

$$\exists\epsilon : \forall\delta, \exists x \text{ such that } \Big(\exists n, m > \delta \text{ with } |f_n(x) - f_m(x)| \geq \epsilon\Big)$$

Take $\epsilon = 0.5$, and fix any $\delta > 0$.

Now, take $n = \delta + 1$ and $m = \delta + 2$. Take $x = \frac{1}{(\frac{1}{n} - \frac{1}{m})}$
So,

$$\begin{aligned}
|f_n(x) - f_m(x)| &= |\Big(x(1+\frac{1}{n}) - x(1+\frac{1}{m})\Big)| \\
&= |x\Big((1+\frac{1}{n}) - (1+\frac{1}{m})\Big)| \\
&= |\frac{1}{(\frac{1}{n} - \frac{1}{m})}\Big(\frac{1}{n} - \frac{1}{m}\Big)| = 1 > 0.5 = \epsilon
\end{aligned}$$

□