1 Yoneda Lemma

Lemma 1.1 (H_A or $\mathcal{A}(\_, A)$)**.** *For any category $\mathcal{A}$, fixing an object, $A \in \mathcal{A}$, there's a functor, $H_A : \mathcal{A}^{op} \to Set$ defined as:*

i *For object $B \in \mathcal{A}$, $F(B) := Hom(B, A)$*

ii *For any morphism in $\mathcal{A}$, $g : X \to Y$,*

$$H_A(g) : \mathcal{A}(Y, A) \to \mathcal{A}(X, A) \ , \ as, \ \forall p \in \mathcal{A}(Y, A) \ , \ p \mapsto p \circ g \ \ i.e. \ \Big(H_A(g)\Big)(p) := p \circ g$$

Theorem 1.1. ***Yoneda*** *If $\mathcal{A}$ is a locally small category, for any object $A \in \mathcal{A}$ and $X \in [\mathcal{A}^{op}, Set]$, there's exists a natural isomorphism:*

$$[\mathcal{A}^{op}, Set](H_A, X) \cong X(A) \ \textit{naturally in} \ A \in \mathcal{A}$$

Explaination:
First , fix any category, $\mathcal{A}$. Now ,choose two things (independent of each other):

i an object, A from the category $\mathcal{A} = \mathcal{A}^{op}$

ii an object, $X \in [\mathcal{A}^{op}, Set]$, the presheaf category
i.e. a functor $X : A^{op} \to Set$

Here, $[\mathcal{A}^{op}, Set](H_A, X)$ denotes morphisms $H_A \to X$ in $[\mathcal{A}^{op}, Set]$, i.e. natural transformations, $\alpha :$

$$\begin{tikzcd} A^{op} \arrow[r, bend left, "H_A"] \arrow[r, bend right, "X"'] \arrow[r, phantom, "\Downarrow\alpha"] & Set \end{tikzcd}$$

Each of these natural transformations is a collection of, morphisms in Set, hence each of their components is exactly a function. i.e. $\forall \alpha \in [\mathcal{A}^{op}, Set](H_A, X), \forall K \in \mathcal{A}, \alpha_K$ is a function :$H_A(K) \to X(K)$

$X(A)$ is precisely a set, because $X(A)$ is the image of (our chosen object,) A, under (our chosen functor,) X.

The key idea is that the choice of A and X completely determines all possible maps (i.e. natural transformations) from H_A to X.
Moreover, this isomorphism is *natural* in A and X.
Meaning that $[\mathcal{A}^{op}, Set](H_A, X)$ and $X(A)$ are *functorial* in *both* A and X

Notation:

- Denoting the category of all presheaves on $\mathcal{A}$ by $\mathcal{C}$, i.e. $\mathcal{C} := [\mathcal{A}^{op}, Set]$
- using $\hat{}$ as a map i.e. $\hat{a} = b$ stands for $a \xrightarrow{\hat{}} b$
- using $\tilde{}$ as a map i.e. $\hat{a} = b$ stands for $a \xrightarrow{\tilde{}} b$

To prove the theorem, first, going to show that $[\mathcal{A}^{op}, Set](H_A, X)$ is isomorphic to $X(A)$. And then that this isomorphism is natural.

Proof. Let a locally small category, $\mathcal{A}$ be given.
Let $A \in \mathcal{A}$ and $X \in [\mathcal{A}^{op}, Set]$

1. Defining ϕ and ψ Define ϕ (on natural transformations) as the A-component (of that natural transformation) at the identity of A. i.e. for $\alpha \in \mathcal{C}(H_A, X), \phi(\alpha) := \alpha_A(1_A)$

Define ψ on an object, $x \in X(A)$, by defining it's K-component for any $K \in \mathcal{A}$:

$$(\psi(x))_K : H_A(K) \to X(K) \text{ as, for each } p \in Hom_{\mathcal{A}}(K, A), p \mapsto \Big(X(p)\Big)(x)$$

That is to say that the K-component maps any arrow $p : K \to A$ to the image of x under the map X(p).

□

2 Cayley's Theorem

3 Embedding of a category in Presheaf category

Definiton 3.1 (Embedding of a category)**.** A category, $\mathcal{A}$ is said to be embedded in a category, $\mathcal{B}$ if there exists a functor $F : \mathcal{A} \to \mathcal{B}$ such that F is full, faithful and injective (on objects).

Proof. Prove that H_A is indeed a functor

□

4 Quasi-Paper