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Theorem 1.1. Yoneda If $\mathcal{A}$ be a locally small category, then,

$$[\mathcal{A}^{op}, Set](H_A, X) \cong X(A) \text{ naturally in } A \in \mathcal{A} \text{ and } X \in [\mathcal{A}^{op}, Set]$$

Explaination:
First of all, we fix any category, $\mathcal{A}$. Now we choose two things (independent of each other):

i an object, A from the category $\mathcal{A} = \mathcal{A}^{op}$

ii an object, X of the category $[\mathcal{A}^{op}, Set]$ which is precisely a functor $X : A^{op} \to Set$

Here, $[\mathcal{A}^{op}, Set](H_A, X)$ denotes arrows $H_A \to X$ in $[\mathcal{A}^{op}, Set]$ i.e. natural transformations, $\alpha :$
$$\begin{tikzcd} A^{op} \arrow[r, bend left, "H_A"] \arrow[r, bend right, "X"'] \arrow[r, phantom, "\Downarrow\alpha"] & Set \end{tikzcd}$$

Each of these natural transformations is a collection of maps in Set, hence each of their components is exactly a function. i.e. $\forall \alpha \in [\mathcal{A}^{op}, Set](H_A, X), \forall K \in \mathcal{A}, \alpha_K$ is a function :$H_A(K) \to X(K)$

$X(A)$ is precisely a set, because $X(A)$ is the image of (our chosen object,) A, under (our chosen functor,) X.

So, the idea is that our choice of A and X completely determines all possible maps (i.e. natural transformations) from H_A to X. This answers out big question of "what are all the maps $H_A \to X$ " or "how does H_A see other presheaves on $\mathcal{A}$ ".

*The theorem says not just that the two are isomorphic, but that they're **naturally** isomorphic.*
This means that $[\mathcal{A}^{op}, Set](H_A, X)$ and $X(A)$ are *functorial* in *both* A and X

Also, So, the aforementioned collection of natural transformations also must be a set: As $\mathcal{A}$ is locally small, for each choice of $K \in \mathcal{A}$, $H_A(K) = Hom_{\mathcal{A}}(K, A)$ is a set. Thus, as α_K is a function,and hence a relation,
it's a subset of $Hom_{\mathcal{A}}(K, A) \times X(A)$, which is a set as it's a cartesian product of sets.
Thus, K-component of every natural transformation is a set. i.e. $\forall \alpha \in [\mathcal{A}^{op}, Set](H_A, X), \forall K \in \mathcal{A}, \alpha_K$ is a set

Now, the question is wether the set of all natural transformations is a set

Proof. Let a locally small category, $\mathcal{A}$ be given.

Denoting the category of all presheaves on $\mathcal{A}$ by $\mathcal{C}$, i.e. $\mathcal{C} := [\mathcal{A}^{op}, Set]$

Now, fix any object, $A \in \mathcal{A}$, and any object, $X \in \mathcal{C}$.
Need to show that $\mathcal{C}(H_A, X)$ is naturally isomorphic to $X(A)$.
Thus, need two mutually inverse natural transformations, such that : $\mathcal{C}(H_A, X) \underset{\phi}{\overset{\psi}{\rightleftarrows}} X(A)$
Need to show that the RHS is a set, and then, As both the RHS and LHS above are sets, the natural transformations between them are maps in Set i.e. functions.
Going to show the following in order,

i Define ϕ and ψ

ii Show that ϕ and ψ are mutually inverse

iii Show naturality of ϕ in X

iv Show naturality of ϕ in A

v Show naturality of ψ in X

vi Show naturality of ϕ in A

Defining ϕ and ψ Define ϕ (on natural transformations) as : for $\alpha \in \mathcal{C}(H_A, X), \phi(\alpha) := \alpha_A(1_A)$

Define ϕ on an object, $x \in X(A)$, by defining it's K-component for any $K \in \mathcal{A}$: $(\psi(x))_K$

□