1 Yoneda Lemma

Lemma 1.1. For a locally small category $\mathcal{A}$, fixing an object $A \in \mathcal{A}$ gives a functor, $H_A : \mathcal{A}^{op} \to Set$ defined as:

(i) For any object $B \in \mathcal{A}$, $H_A(B) := Hom_{Set}(B, A)$

(ii) For any morphism, $g : X \to Y$ in $\mathcal{A}$,

$$H_A(g) : \mathcal{A}(Y, A) \to \mathcal{A}(X, A) \text{ given by } p \mapsto p \circ g$$

Lemma 1.2. For a natural transformation α, it's a natural isomorphism iff each of its components is an isomorphism.

Lemma 1.3. naturality in two variables simultaneously is equivalent to naturalityin each variable independently(1.3.29 pg 39)

Definiton 1.1. For any category $\mathcal{A}$, it's opposite category is a category having the same objects as $\mathcal{A}$, with a morphism from object A to B, that is $f \in \mathcal{A}^{op}(A, B)$ only if there is a morphism $g \in \mathcal{A}(B, A)$.

Definiton 1.2. The category of presheaves on $\mathcal{A}$, denoted by $[\mathcal{A}^{op}, Set]$ is defined to have functors from $\mathcal{A}^{op}$ to Set as objects, and natural transformations between them as morphisms.

Theorem 1.1. ***Yoneda Lemma*** *If $\mathcal{A}$ is a locally small category then, for any object $A \in \mathcal{A}$ and $X \in [\mathcal{A}^{op}, Set]$, there's exists an isomorphism,*

$$[\mathcal{A}^{op}, Set](H_A, X) \cong X(A) \textit{such that it is natural in } A \textit{ and } X.$$

Explaination:
First , fix any category, $\mathcal{A}$. Now ,choose two things (independent of each other):

i an object, A from the category $\mathcal{A} = \mathcal{A}^{op}$

ii an object, $X \in [\mathcal{A}^{op}, Set]$, the presheaf category
i.e. a functor $X : A^{op} \to Set$

Here, $[\mathcal{A}^{op}, Set](H_A, X)$ denotes morphisms $H_A \to X$ in $[\mathcal{A}^{op}, Set]$, i.e. natural transformations, $\alpha :$
$$\begin{tikzcd} A^{op} \arrow[r, bend left, "H_A"] \arrow[r, bend right, "X"'] \arrow[r, phantom, "\Downarrow\alpha"] & Set \end{tikzcd}$$

The key idea is that the choice of A and X completely determines all possible maps (i.e. natural transformations) from functor H_A to functor$X$. Moreover, that this isomorphism is *natural* in A and X.
Meaning that $[\mathcal{A}^{op}, Set](H_A, X)$ and $X(A)$ are *functorial* in *both* A and X

Notation:

- We denote the category of presheaves on $\mathcal{A}$ by $\mathcal{C}$.
- For the map $\hat{}$, that is $\hat{a} = b$, we use $a \xrightarrow{\hat{}} b$
- For the map $\tilde{}$, that is $\tilde{a} = b$ we use $a \xrightarrow{\tilde{}} b$

To prove the theorem, first, we show that $[\mathcal{A}^{op}, Set](H_A, X)$ is isomorphic to $X(A)$. And then that this isomorphism is natural in X and A.

Proof. Let a locally small category $\mathcal{A}$ be given. Fix an object $A \in \mathcal{A}$ and a presheaf X.

Showing isomorphism between $[\mathcal{A}^{op}, Set](H_A, X)$ and $X(A)$

Define $\hat{}$: $\mathcal{C}(H_A, X) \to X(A)$ for any $\alpha : H_A \to X$, as $\hat{\alpha} := \alpha_A(1_A)$.

Define $\tilde{}$: $X(A) \to [\mathcal{A}^{op}, Set](H_A, X)$ for any $x \in X(A)$ as the natural transformation $\tilde{x} : H_A \to X$ whose K-component is the function mapping each morphsim $p \in \mathcal{A}(K, A)$ to $\Big(X(p)\Big)(x)$. That is, $\tilde{x}_K(p) := \Big(X(p)\Big)(x)$.

We are going to show that $\tilde{x}$ is a natural transformation. Fix objects $K, L \in \mathcal{A}$ and morphism $q \in \mathcal{A}^{op}(K, L)$.

Need to show that the square $\begin{tikzcd} H_A(K) \arrow[r, "H_A(q)"] \arrow[d, "\tilde{x}_K"'] & H_A(L) \arrow[d, "\tilde{x}_L"] \\ X(K) \arrow[r, "X(q)"'] & X(L) \end{tikzcd}$, that is $\begin{tikzcd} \mathcal{A}(K, A) \arrow[r, "- \circ q"] \arrow[d, "\tilde{x}_K"'] & \mathcal{A}(L, A) \arrow[d, "\tilde{x}_L"] \\ X(K) \arrow[r, "X(q)"'] & X(L) \end{tikzcd}$ commutes .

So, for any $f : K \to A$, need that $\tilde{x}_L(f \circ q) = X(q) \circ \tilde{x}_K(f)$. Using the definition of $\tilde{x}$ gives the following.

$$LHS = \tilde{x}_L(f \circ q) = \Big(X(f \circ q)\Big)(x)$$
$$RHS = X(q) \circ \tilde{x}_K(f) = \Big(X(q)\Big)\big(X(f)(x)\big) = \Big(X(q) \circ X(f)\Big)(x)$$

And as X is a contravariant functor, $X(f \circ q) = X(q) \circ X(f)$, giving that LHS=RHS.

Now going to show that $\hat{}$ and $\tilde{}$ define an isomorphism. Need to show that $\hat{}$ and $\tilde{}$ are mutually inverse.

(i) For any $x \in X(A)$, $\hat{\tilde{x}} = \tilde{x}_A(1_A) = \Big(X(1_A)\Big)(x) = 1_{X(A)}(x) = x$.

(ii) For any $\alpha \in \mathcal{C}(H_A, X)$, need to show that $\tilde{\hat{\alpha}} = \alpha$. So, it's required that each of their component are equal. As both $\tilde{\hat{\alpha}}$ and α are natural transformations between functors that go to the category *Set*, each of the components is a function. So, need to show that for any $f \in \mathcal{A}(K, A) = H_A(K)$, $\Big(\tilde{\hat{\alpha}}\Big)_K(f)$=$\alpha_K(f)$. Using first the definition of $\tilde{}$ and then that of $\hat{\alpha}$ gives:

$$LHS = \tilde{\hat{\alpha}}_B(f) = \Big(X(f)\Big)(\hat{\alpha}) = \Big(X(f)\Big)(\alpha_A(1_A)) \tag{1}$$

And as $f : k \to A$, we also have the following.

$$RHS = \alpha_K(f) = \alpha_K(1_A \circ f) \tag{2}$$

Because α is a natural transformation, the square following square commutes for 1_A.

$$\begin{array}{ccc} \mathcal{A}(A, A) & \xrightarrow{-\circ f} & \mathcal{A}(K, A) \\ {\scriptstyle \alpha_A}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_K} \\ X(A) & \xrightarrow[X(f)]{} & X(K) \end{array}$$

This gives that $\alpha_K(1_A \circ f) = \Big(X(f)\Big)\big(\alpha_A(1_A)\big)$. Hence, we have from (1) and (2), we get that $RHS = LHS$.

Showing naturality of this isomorphism

□

2 Cayley's Theorem

Informally, Yoneda Lemma gives us a stable proxy to study any presheaf on a locally small category. In group theory, Cayley's theorem is a result that similarly allows us to study any group by instead studying a subgroup of some symmetric group.

Definiton 2.1 (Symmetric group on a set)**.**

Theorem 2.1. ***Cayley's Theorem*** *Every group, (G,.) is isomorphic to a subgroup of symmetric group on G.*

3 Embedding of a category in Presheaf category

Definiton 3.1 (Embedding of a category)**.** A category, $\mathcal{A}$ is said to be embedded in a category, $\mathcal{B}$ if there exists a functor $F : \mathcal{A} \to \mathcal{B}$ such that F is full, faithful and injective (on objects).