

Question 1

$\langle a_n \rangle$ is a real sequence; $\sigma_n := \frac{a_1 + a_2 + \dots + a_n}{n}$

I $\liminf a_n \leq \liminf \sigma_n$

If a_n is unbounded below, then $\liminf a_n = -\infty \leq \liminf \sigma_n$. So, let a_n be bounded below, thus LHS is a real number,

$$m := \liminf a_n$$

Now, if a_n is constant, and equal to a ,

$$\forall n \in \mathbb{N}, \inf\{a_i | i \geq n\} = a = \frac{na}{n} = \sigma_n$$

Otherwise, if a_n is not constant, then,

$$\exists i, j \in \mathbb{N} \text{ such that (wlog) } a_i < a_j$$

Suppose if possible, $\liminf a_n = m > \liminf \sigma_n$. But, $\forall n \in \mathbb{N}$ such that $n \geq i, j$,

$$\sigma_n = \frac{a_1 + \dots + a_i + \dots + a_j + \dots + a_n}{n}$$

$$\implies \sigma_n \geq \frac{(n-2)m + a_i + a_j}{n} \geq \frac{(n-1)m + a_j}{n} > m$$

$$[\because m \leq a_i < a_j]$$

$$\text{Hence, } \inf \sigma_n \geq \frac{(n-1)m + a_j}{n}$$

$$\implies \liminf \sigma_n \geq \lim_{n \rightarrow \infty} \frac{(n-1)m + a_j}{n} = m$$

But this contradicts the initial assumption.

II $\limsup a_n \geq \limsup \sigma_n$ If a_n is unbounded above, then $\limsup a_n = \infty \geq \limsup \sigma_n$.

So, let a_n be bounded above, thus LHS is a real number,

$$M := \limsup a_n$$

Now, if a_n is constant, and equal to a ,

$$\forall n \in \mathbb{N}, \sup\{a_i | i \geq n\} = a = \frac{na}{n} = \sigma_n$$

Otherwise, if a_n is not constant, then,

$$\exists i, j \in \mathbb{N} \text{ such that (wlog) } a_i < a_j$$

Suppose if possible, $\limsup a_n = M < \limsup \sigma_n$ But,

$$\forall n \in \mathbb{N} \text{ such that } n \geq i, j,$$

$$\begin{aligned} \sigma_n &= \frac{a_1 + \dots + a_i + \dots + a_j + \dots + a_n}{n} \leq \frac{(n-2)M + a_i + a_j}{n} \\ &\leq \frac{(n-1)M + a_j}{n} \\ &< M \quad [\because M \geq a_j > a_i] \end{aligned}$$

$$\text{Hence, } \inf \sigma_n \leq \frac{(n-1)M + a_j}{n}$$

$$\implies \liminf \sigma_n \leq \lim_{n \rightarrow \infty} \frac{(n-1)M + a_j}{n} = M$$

But this contradicts the initial assumption.

Question 2

$$\liminf \frac{a_{n+1}}{a_n} \leq \liminf (a_n)^{\frac{1}{n}} \leq \limsup (a_n)^{\frac{1}{n}} \leq \limsup \frac{a_{n+1}}{a_n}$$

I Showing that $\liminf \frac{a_{n+1}}{a_n} \leq \liminf (a_n)^{\frac{1}{n}}$

Case 1: $\liminf \frac{a_{n+1}}{a_n} = 0$

$$\forall n \in \mathbb{N}, a_n > 0 \implies (a_n)^{\frac{1}{n}} > 0$$

$$\text{Thus, } \liminf (a_n)^{\frac{1}{n}} \geq 0 = \liminf \frac{a_{n+1}}{a_n}$$

Case 2: $\liminf \frac{a_{n+1}}{a_n} = \infty$

So, for any $a \in \mathbb{N}$,

$$\exists M_a \in \mathbb{N} : n \geq M_a \implies \frac{a_{n+1}}{a_n} > a$$

Fix any a , and choose $n > M_a$. For any such n ,

$$\frac{a_{n+1}}{a_n} > a \implies a_{n+1} > aa_n$$

$$\implies a_n > aa_{n-1} > a^2 a_{n-2} > \dots > a^{n-M} a_M$$

$$\implies (a_n)^{\frac{1}{n}} > a \left(\frac{a_M}{a^M} \right)^{\frac{1}{n}}$$

Now as for fixed a , $\frac{a_M}{a^M}$ is constant,

$$\lim_{n \rightarrow \infty} \left(\frac{a_M}{a^M} \right)^{\frac{1}{n}} = 1 \implies \lim_{n \rightarrow \infty} a \left(\frac{a_M}{a^M} \right)^{\frac{1}{n}} = a$$

Thus,

$$\forall \epsilon > 0, \exists K \in \mathbb{N} : n > K \implies a \left(\frac{a_M}{a^M} \right)^{\frac{1}{n}} > a - \epsilon$$

And hence, in particular, $a(\frac{a_M}{a^M})^{\frac{1}{n}} > a - 1$ for any natural number a .

$$(a_n)^{\frac{1}{n}} > a(\frac{a_M}{a^M})^{\frac{1}{n}} > a - 1 \implies \lim_{n \rightarrow \infty} (a_n)^{\frac{1}{n}} = \infty$$

$$\therefore \liminf \frac{a_{n+1}}{a_n} = \infty = \liminf (a_n)^{\frac{1}{n}}$$

Case 3: $\liminf \frac{a_{n+1}}{a_n} = a \in \mathbb{R}$

So, $\forall \epsilon > 0, \exists M \in \mathbb{N} : n > M \implies \frac{a_{n+1}}{a_n} > a - \epsilon$

$$a_n > (a - \epsilon)a_{n-1} > (a - \epsilon)^2 a_{n-2} > \dots > (a - \epsilon)^{n-M} a_M$$

$$\implies (a_n)^{\frac{1}{n}} > (a - \epsilon)(\frac{a_M}{a^M})^{\frac{1}{n}}$$

But, as $\lim_{n \rightarrow \infty} (\frac{a_M}{a^M})^{\frac{1}{n}} = 1$,

$$(a_n)^{\frac{1}{n}} > (a - \epsilon)(\frac{a_M}{a^M})^{\frac{1}{n}} > (a - \epsilon)(1 - \epsilon) = a - (1 + a)\epsilon + \epsilon^2$$

$$\implies (a_n)^{\frac{1}{n}} > a - (1 + a)\epsilon$$

But as this holds for every $\epsilon > 0$,

$$\liminf (a_n)^{\frac{1}{n}} \geq a - 0 = \liminf \frac{a_{n+1}}{a_n}$$

II $\limsup (a_n)^{\frac{1}{n}} \leq \limsup \frac{a_{n+1}}{a_n}$

Case 1: $\limsup \frac{a_{n+1}}{a_n} = \infty$

As $\limsup (a_n)^{\frac{1}{n}} \leq \infty = \limsup \frac{a_{n+1}}{a_n}$, done

Case 2: $\limsup \frac{a_{n+1}}{a_n} = -\infty$

$$\limsup \frac{a_{n+1}}{a_n} = -\infty \implies \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = -\infty$$

But, as all a_n are positive, so is their ratio, and hence it can't be unbounded below.

Case 3: $\limsup \frac{a_{n+1}}{a_n} = a \in \mathbb{R}$

So, $\forall \epsilon > 0, \exists M \in \mathbb{N} : n > M \implies \frac{a_{n+1}}{a_n} < a + \epsilon$

$$a_n < (a + \epsilon)a_{n-1} < (a + \epsilon)^2 a_{n-2} < \dots < (a + \epsilon)^{n-M} a_M$$

Now, as $(\frac{a_M}{a^M})^{\frac{1}{n}}$ is constant,

$$\exists K \in \mathbb{N} : n \geq K \implies (\frac{a_M}{a^M})^{\frac{1}{n}} < 1 + \epsilon$$

$$\implies (a_n)^{\frac{1}{n}} < (a + \epsilon)(\frac{a_M}{a^M})^{\frac{1}{n}} < (a + \epsilon)(1 + \epsilon) = a + (a + 1)\epsilon + \epsilon^2$$

Thus,

$$\limsup (a_n)^{\frac{1}{n}} \leq a + (a + 1)\epsilon + \epsilon^2 < a + (a + 1)(1 + \epsilon)\epsilon,$$

but as this holds for every $\epsilon > 0$,

$$\liminf (a_n)^{\frac{1}{n}} \leq a = \liminf \frac{a_{n+1}}{a_n}$$

Question 3

a_n & b_n are bounded, non-negative sequences; $a_n \rightarrow a > 0$

$$\limsup a_n b_n = a \limsup b_n$$

Question 4

I $\liminf \frac{a_{n+1}}{a_n} > 1 \implies a_n \not\rightarrow 0$ and $\limsup |a_n| > 0$

II $\limsup \frac{a_{n+1}}{a_n} < 1 \implies a_n \rightarrow 0$ where $a_n > 0$

III Some convergent sequences are:

(a) $a_n := 1$

IV Some divergent sequences are:

(a) $a_n := n$

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n} = 1 + \frac{1}{n}$$

This tends to 1 [for $\epsilon = \frac{1}{n}$, take $\delta = \frac{1}{n+1}$]

But the sequence diverges [to ∞].