1 Yoneda Lemma

In this section, we set forth some basic definitions, state and prove Yoneda lemma, and then show Cayley's Theorem and Yoneda Embedding as it's consequences.

For any category $\mathcal{A}$, and it's objects $X, Y \in \mathcal{A}$, we denote $Hom_{\mathcal{A}}(X,Y)$ with $\mathcal{A}(X,Y)$.

Definiton 1.1. For any category $\mathcal{A}$, it's opposite category, $\mathcal{A}^{op}$ is the category having the objects of $\mathcal{A}$. And for objects $A, B \in \mathcal{A}$, a morphism $f \in \mathcal{A}^{op}(A,B)$ if and only if there is a morphism $g \in \mathcal{A}(B,A)$.

Proposition 1.2. For a locally small category $\mathcal{A}$, fixing an object $A \in \mathcal{A}$ gives a functor, $H_A : \mathcal{A}^{op} \to Set$ defined as:

(i) For any object $B \in \mathcal{A}$, $H_A(B) := \mathcal{A}(B,A)$.

(ii) For any morphism, $g : X \to Y$ in $\mathcal{A}$,

$$H_A(g) : \mathcal{A}(Y,A) \to \mathcal{A}(X,A) \text{ is given by } p \mapsto p \circ g.$$

Proof. Fix any objects $K, L, M \in \mathcal{A}$.

I **(Composition)** As H_A is a contravariant functor, for any morphisms $f \in \mathcal{A}(K,L)$ and $g \in \mathcal{A}(L,M)$, we need to show that $H_A(g \circ f) = H_A(f) \circ H_A(g)$. Note, the composition $g \circ f$ on the left hand side is in $\mathcal{A}^{op}$. Hence, using the definition of H_A gives us that for any $k \in H_A(M)$, we must have

$$\begin{aligned} LHS &= \Big(H_A(g \circ f)\Big)(k) = k \circ g \circ f \\ \text{and } RHS &= \Big(H_A(f) \circ H_A(g)\Big)(k) = \Big(H_A(f)\Big)(k \circ g) = (k \circ g) \circ f. \end{aligned}$$

II **(Identity)** We will show that for any $k \in \mathcal{A}(K,L)$, H_A respects the identities of K and L in $\mathcal{A}$ (as they're equal to the identities of K and L in $\mathcal{A}^{op}$. Using the definition of H_A, for any object $L \in \mathcal{A}$ and morphism $p \in H_A(L)$, we get the following equations.

$$\begin{aligned} \text{Right Identity: } &\Big(\big(H_A(1_K)\big) \circ \big(H_A(k)\big)\Big)(p) = \big(H_A(1_K)\big)(p \circ k) = p \circ k \circ 1_K = p \circ k = \big(H_A(k)\big)(p) \\ \text{Left Identity: } &\Big(\big(H_A(k)\big) \circ \big(H_A(1_L)\big)\Big)(p) = \big(H_A(k)\big)(p \circ 1_L) = \big(H_A(k)\big)(p) \end{aligned}$$

Hence, H_A is indeed a functor. □

Definiton 1.3. For a locally small category $\mathcal{A}$, the category of presheaves on $\mathcal{A}$, denoted by $[\mathcal{A}^{op}, Set]$ is defined to have functors from $\mathcal{A}^{op}$ to Set as objects, and the natural transformations between them as morphisms.

Lemma 1.4. Let $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B} \end{tikzcd}$ be a natural transformation. If for every $A \in \mathcal{A}$, $\alpha_A : F(A) \to G(A)$ is an isomorphism then α is a natural isomorphism.

Proof. We are going to show that there exists a natural transformation $\beta : G \to F$. Fix any objects $A, B \in \mathcal{A}$ and morphism $k \in \mathcal{A}(A,B)$. As α is a natural transformation,

$$\alpha_B \circ F(k) = G(k) \circ \alpha_A. \tag{1}$$

Because α_A is an isomorphism, we get that there exists $\beta_A : G(A) \to F(A)$ such that

$$\alpha_A \circ \beta_A = 1_{G(A)} \text{ and } \beta_A \circ \alpha_A = 1_{F(A)}. \tag{2}$$

Similarly, α_B gives us the existence of $\beta_B : G(B) \to F(B)$ such that $\beta_B \circ \alpha_B = 1_{F(B)}$. Multiplying (1) with β_B and β_A,

$$\beta_B \circ \alpha_B \circ F(k) \circ \beta_A = \beta_B \circ G(k) \circ \alpha_A \circ \beta_A \implies F(k) \circ \beta_A = \beta_B \circ G(k). \tag{3}$$

Thus, β is a natural transformation from F to G. Using (2) gives us that $\big(\alpha \circ \beta\big)_A = 1_{G(A)}$ and $\big(\beta \circ \alpha\big)_A = 1_{F(A)}$ for any object $A \in \mathcal{A}$. Therefore,$\alpha \circ \beta = 1_G$ and $\beta \circ \alpha = 1_F$. Hence, α and β together give an isomorphism between F and G in the functor category $[\mathcal{A}, \mathcal{B}]$. □

Lemma 1.5. Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be categories. And suppose there are functors $F, G : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$. Thus, there are functors, $F^A, G^A : \mathcal{B} \to \mathcal{C}$ defined as taking $B \in \mathcal{B}$ to $F(A,B)$, $G(A,B)$ and morphism f to $F((1_A, f))$, $G((1_A, f))$. And functors $F_B, G_B : \mathcal{A} \to \mathcal{C}$ defined as taking $A \in \mathcal{A}$ to $F(A,B)$, $G(A,B)$ and morphism g to $F((g, 1_B))$, $F((g, 1_B))$. A family of maps, $\big(\alpha_{A,B} : F(A,B) \to G(A,B)\big)_{A \in \mathcal{A}, B \in \mathcal{B}}$, is a natural transformation $F \to G$ if the following conditions are satisfied:

(i) For each $A \in \mathcal{A}$, the family $\big(\alpha_{A,B} : F^A(B) \to G^A(B)\big)_{B\in\mathcal{B}}$ is a natural transformation $F^A \to G^A$;

(ii) For each $B \in \mathcal{B}$, the family $\big(\alpha_{A,B} : F_B(A) \to G_B(A)\big)_{A\in\mathcal{A}}$ is a natural transformation $F_B \to G_B$.

Proof. In order to show naturality of $\alpha_{(A,B)}$,

we need to show that for any $(f,g) \in \mathcal{A}\times\mathcal{B}\,(X,Y)$, the square $\begin{tikzcd} F(X) \arrow[r, "F((f,g))"] \arrow[d, "\alpha_X"'] & F(Y) \arrow[d, "\alpha_Y"] \\ G(X) \arrow[r, "G((f,g))"'] & G(Y) \end{tikzcd}$ commutes.

Fix any objects $A, A' \in \mathcal{A}$ and $B, B' \in \mathcal{B}$. Fix any morphism $(g,f) \in \mathcal{A}\times\mathcal{B}\,((A,B),(A',B'))$, where $g \in \mathcal{A}(A,A')$ and $f \in \mathcal{B}(B,B')$.

(I) The condition (ii) gives us that for any $g \in \mathcal{A}(A,A')$, the square $\begin{tikzcd} F_A(B) \arrow[r, "F_A(g)"] \arrow[d, "\alpha_{A,B}"'] & F_A(B') \arrow[d, "\alpha_{A,B'}"] \\ G_A(B) \arrow[r, "G_A(g)"'] & G_A(B') \end{tikzcd}$ commutes. By the definition of F_A and G_A, this square can be written as $\begin{tikzcd} F(A,B) \arrow[r, "F((g,1_B))"] \arrow[d, "\alpha_{A,B}"'] & F(A,B') \arrow[d, "\alpha_{A,B'}"] \\ G(A,B) \arrow[r, "G((g,1_B))"'] & G(A,B') \end{tikzcd}$.

(II) The condition (i) gives us that for any $f \in \mathcal{B}(B,B')$, the square $\begin{tikzcd} F^{A'}(B) \arrow[r, "F^{A'}(f)"] \arrow[d, "\alpha_{A',B}"'] & F^{A'}(B') \arrow[d, "\alpha_{A',B'}"] \\ G^{A'}(B) \arrow[r, "G_{A'}(f)"'] & G^{A'}(B') \end{tikzcd}$ commutes. By the definition of F^A and G^A, this square can be written as $\begin{tikzcd} F(A',B) \arrow[r, "F((1_{A'},f))"] \arrow[d, "\alpha_{A',B}"'] & F(A',B') \arrow[d, "\alpha_{A',B'}"] \\ G(A',B) \arrow[r, "G((1_{A'},f))"'] & G(A',B') \end{tikzcd}$.

(III) Composing the squares from (I) and (II), we get that the following rectangle commutes:

$$\begin{tikzcd} F(A,B) \arrow[r, "F((g,1_B))"] \arrow[d, "\alpha_{A,B}"'] & F(A,B') \arrow[r, "F((1_{A'},f))"] \arrow[d, "\alpha_{A',B}"'] & F(A',B') \arrow[d, "\alpha_{A',B'}"] \\ G(A,B) \arrow[r, "G((g,1_B))"'] & G(A,B') \arrow[r, "G((1_{A'},f))"'] & G(A',B') \end{tikzcd}. \tag{1}$$

Using (1), we get that:

$$\alpha_{A',B'} \circ \big(F((1_{A'},f)) \circ F((g,1_B))\big) = \big(G((1_{A'},f)) \circ G((g,1_B)) \circ \alpha_{A',B'}. \tag{2}$$

As F and G are functors, from (2), we have that

$$\alpha_{A',B'} \circ F\big((g,f)\big) = G\big((g,f)\big) \circ \alpha_{A',B'}.$$ □

Theorem 1.6. ***Yoneda Lemma*** *If $\mathcal{A}$ is a locally small category then, for any object $A \in \mathcal{A}$ and $X \in [\mathcal{A}^{op}, Set]$, there exists an isomorphism,*

$$[\mathcal{A}^{op}, Set](H_A, X) \cong X(A) \textit{ which is natural in } A \textit{ and } X. \tag{3}$$

Notation:

- We denote the category of presheaves on $\mathcal{A}$ by $\mathcal{C}$.
- For the map $\hat{}$, instead of writing $\hat{}(a) = b$, we use $\hat{a} = b$ to denote $a \mapsto b$.
- For the map $\tilde{}$, instead of writing $\tilde{}(a) = b$, we use $\tilde{a} = b$ to denote $a \mapsto b$.

- $[\mathcal{A}^{op}, Set](H_A, X)$ denotes the collection of morphisms $\alpha :$
$$\begin{tikzcd} \mathcal{A}^{op} \arrow[r, bend left, "H_A"] \arrow[r, bend right, "X"'] \arrow[r, phantom, "\Downarrow\alpha"] & Set \end{tikzcd}.$$

To prove the theorem, first, we show that $[\mathcal{A}^{op}, Set](H_A, X)$ is isomorphic to $X(A)$ as set, and then that this isomorphism is natural in X and A.

Proof. Let $\mathcal{A}$ be a locally small category. Fix an object $A \in \mathcal{A}$ and a presheaf X on $\mathcal{A}$.

I **Showing isomorphism between $[\mathcal{A}^{op}, Set](H_A, X)$ and $X(A)$**
Define $\hat{}: \mathcal{C}(H_A, X) \to X(A)$ for any $\alpha : H_A \to X$, as $\hat{\alpha} := \alpha_A(1_A)$. As $1_A \in Set(A, A) = H_A(A)$, definition of α_A gives that $\alpha_A(1_A) \in X(A)$.
Define $\tilde{}: X(A) \to [\mathcal{A}^{op}, Set](H_A, X)$ for any $x \in X(A)$ as the natural transformation $\tilde{x} : H_A \to X$ whose K-component is the function mapping each morphsim $p \in \mathcal{A}(K, A)$ to $\Big(X(p)\Big)(x)$. That is, $\tilde{x}_K(p) := \Big(X(p)\Big)(x)$.

We are going to show that $\tilde{x}$ is a natural transformation. Fix objects $K, L \in \mathcal{A}$ and morphism $q \in \mathcal{A}^{op}(K, L)$.

Need to show that the square
$$\begin{tikzcd} H_A(K) \arrow[r, "H_A(q)"] \arrow[d, "\tilde{x}_K"'] & H_A(L) \arrow[d, "\tilde{x}_L"] \\ X(K) \arrow[r, "X(q)"'] & X(L) \end{tikzcd}$$
, that is
$$\begin{tikzcd} \mathcal{A}(K, A) \arrow[r, "-\circ q"] \arrow[d, "\tilde{x}_K"'] & \mathcal{A}(L, A) \arrow[d, "\tilde{x}_L"] \\ X(K) \arrow[r, "X(q)"'] & X(L) \end{tikzcd}$$
commutes .

So, for any $f : K \to A$, need that $\tilde{x}_L(f \circ q) = X(q) \circ \tilde{x}_K(f)$. Using the definition of $\tilde{x}$ gives the following.

$$LHS = \tilde{x}_L(f \circ q) = \Big(X(f \circ q)\Big)(x)$$
$$RHS = X(q) \circ \tilde{x}_K(f) = \Big(X(q)\Big)\big(X(f)(x)\big) = \Big(X(q) \circ X(f)\Big)(x)$$

And as X is a contravariant functor, $X(f \circ q) = X(q) \circ X(f)$, giving that LHS=RHS. Now going to show that $\hat{}$ and $\tilde{}$ define an isomorphism. Need to show that $\hat{}$ and $\tilde{}$ are mutually inverse.

(i) For any $x \in X(A)$, $\hat{\tilde{x}} = \tilde{x}_A(1_A) = \Big(X(1_A)\Big)(x) = 1_{X(A)}(x) = x$.

(ii) For any $\alpha \in \mathcal{C}(H_A, X)$, need to show that $\tilde{\hat{\alpha}} = \alpha$. So, it's required that each of their component are equal. As both $\tilde{\hat{\alpha}}$ and α are natural transformations between functors that go to the category *Set*, each of the components is a function. So, need to show that for any $f \in \mathcal{A}(K, A) = H_A(K)$, $\Big(\tilde{\hat{\alpha}}\Big)_K(f)=\alpha_K(f)$. Using first the definition of $\tilde{}$ and then that of $\hat{\alpha}$ gives:

$$LHS = \tilde{\hat{\alpha}}_B(f) = \Big(X(f)\Big)(\hat{\alpha}) = \Big(X(f)\Big)(\alpha_A(1_A)) \tag{1}$$

And as $f \in \mathcal{A}(K, A)$, we also have the following.

$$RHS = \alpha_K(f) = \alpha_K(1_A \circ f) \tag{2}$$

Because α is a natural transformation, the square following square commutes for 1_A.

$$\begin{tikzcd} \mathcal{A}(A, A) \arrow[r, "-\circ f"] \arrow[d, "\alpha_A"'] & \mathcal{A}(K, A) \arrow[d, "\alpha_K"] \\ X(A) \arrow[r, "X(f)"'] & X(K) \end{tikzcd}$$

This gives that $\alpha_K(1_A \circ f) = \Big(X(f)\Big)(\alpha_A(1_A))$. Hence, we have from (2) and (3), we get that $RHS = LHS$.

II **Showing naturality of this isomorphism**
By Using Lemma 1.4 and 1.5, it's enough to show that $\hat{}$ is natural in X and natural in A.

(i) We are going to show the above isomorphism to be natural in X. Fix any $A \in \mathcal{A}$. Need that for presheaves $X, Y \in \mathcal{C}$ and natural transformation $\beta \in \mathcal{C}(X, Y)$, the following square commutes.

$$\begin{tikzcd} \mathcal{C}(H_A, X) \arrow[r, "\beta\circ -"] \arrow[d, "\hat{}"'] & \mathcal{C}(H_A, Y) \arrow[d, "\hat{}"] \\ X(A) \arrow[r, "\beta_A"'] & Y(A) \end{tikzcd}$$

So, for any $\alpha : H_A \to X$, we need that $\big(\hat{} \circ H_\beta\big)(\alpha) = \big(\beta_A \circ \hat{}\big)(\alpha)$. Using definition of H_β and $\hat{}$ gives:

$$LHS = \big(\hat{} \circ H_\beta\big)(\alpha) = \big(\widehat{H_\beta(\alpha)}\big) = \big(\widehat{\beta \circ \alpha}\big) = \big(\beta \circ \alpha\big)_A(1_A) \tag{3}$$

$$RHS = \big(\beta_A \circ \hat{}\big)(\alpha) = \beta_A(\widehat{\alpha}) = \big(\beta_A \circ \alpha_A\big)(1_A) \tag{4}$$

As $\alpha \in \mathcal{C}(H_A, X)$ and $\beta \in \mathcal{C}(X, Y)$ are morphisms in $\mathcal{C}$, composition in $\mathcal{C}$ gives $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$. From (4) and (5), we directly get that $RHS = LHS$.

(ii) We are going to show natural in A. Fix any $X \in \mathcal{C}$ Need that for objects $A, B \in \mathcal{A}$ and morphism $f \in \mathcal{A}^{op}(A, B)$, the following square commutes.

$$\begin{tikzcd} \mathcal{C}(H_A, X) \arrow[r, "- \circ H_f"] \arrow[d, "\hat{}"'] & \mathcal{C}(H_B, Y) \arrow[d, "\hat{}"] \\ X(A) \arrow[r, "X(f)"'] & X(B) \end{tikzcd}$$

So, for any $\alpha : H_A \to X$, we need that $\big(\hat{} \circ H_f\big)(\alpha) = \Big((X(f)) \circ \hat{}\Big)(\alpha)$. Using definition of H_f and $\hat{}$, we get:

$$LHS = \big(\hat{} \circ H_f\big)(\alpha) = \widehat{\alpha \circ H_f} = (\alpha \circ H_f)_B(1_B) = \alpha_B(f \circ 1_B) = \alpha_B(1_A \circ f) \tag{5}$$

$$RHS = \Big((X(f)) \circ \hat{}\Big)(\alpha) = (X(f))(\hat{\alpha}) = \Big(X(f)\Big)\big(\alpha_A(1_A)\big) \tag{6}$$

The last equality in (6) is justified as f goes from B to A in $\mathcal{A}$. By using equality of (2) and (3) from I(i), for $f \in \mathcal{A}(B, A)$, we get that $\Big(X(f)\Big)\big(\alpha_A(1_A)\big) = \alpha_B(1_A \circ f)$. Hence, $RHS = LHS$. □

1.1 Cayley's Theorem

Informally, given a locally small category$\mathcal{A}$, we can fix a presheaf X on $\mathcal{A}$, and for any object $A \in \mathcal{A}$, study the set $X(A)$ and gain information about all possible natural transformations between H_A and X. Moreover, by part I(ii) of the proof of Yoneda Lemma, each of the natural transformations is determined by its action on the identity morphisms in $\mathcal{A}$. Thus, no matter how complicated $\mathcal{A}$ is, if we choose X carefully, we can hope to understand the structure of $\mathcal{A}$ by looking at how $X(A)$ changes as we vary the chosen presheaf and object.

In group theory, Cayley's theorem says every group G is isomorphic to a subgroup of the symmetric group on G. Thus, instead of having to study a complicated group directly, we can study a subgroup of the symmetric group on it.

Cayley's theorem and Yoneda Lemma are similar in the sense that both allow us to change the environment that we study in by putting few restrictions on what we are allowed to study. Cayley allows us to change setting for groups, and Yoneda does that for locally small categories.

Also, as groups themselves can be considered as small categories, we can apply Yoneda Lemma to any group. In fact, we can get Cayley's theorem as a consequence of Yoneda Lemma by a suitable choice of X and A.

Definiton 1.7. Symmetric group on a set X is the set of all bijections on X, with the binary operation defined as composition of bijections.

We will now use parts of proof of Theorem 1.6 in to prove Cayley's theorem. The notation (_) occurs as a placeholder for the element a map is applied to, that is, $(\_ \circ f)(k)$ is defined to be $k \circ f$. And we use the notation $g.f$ to mean the composition of g and f in the group.

Theorem 1.8. ***Cayley's Theorem*** *Every group, (G,.) is isomorphic to a subgroup of symmetric group on G.*

Proof. Let a group G be given. Define category $\mathcal{A}$ with a single object $\star$. And precisely one morphism in $\mathcal{A}$ for each element of G, with the composition of said morphisms being as that of elements of G. That is, for morphisms f and g in $\mathcal{A}$, $f \circ g$ is defined to be the morphism $f.g$. Then, G and $\mathcal{A}(\star, \star)$ have the same elements and rule of composition, so they're isomorphic as groups.

As $\mathcal{A}^{op}$ is a category with single object, each natural transformation α : $\begin{tikzcd} \mathcal{A}^{op} \arrow[r, bend left, "H_\star"] \arrow[r, bend right, "H_\star"'] \arrow[r, phantom, "\Downarrow\alpha"] & Set \end{tikzcd}$ has only one component, that is $\alpha_\star$. Therefore, we can identify α with $\alpha_\star$. Using naturality of α, we get that

the square $\begin{tikzcd} H_\star(\star) \arrow[r, "H_\star(f)"] \arrow[d, "\alpha_\star"'] & H_\star(\star) \arrow[d, "\alpha_\star"] \\ H_\star(\star) \arrow[r, "H_\star(f)"'] & H_\star(\star) \end{tikzcd}$, that is $\begin{tikzcd} \mathcal{A}(\star, \star) \arrow[r, "\_ \circ f"] \arrow[d, "\alpha_\star"'] & \mathcal{A}(\star, \star) \arrow[d, "\alpha_\star"] \\ \mathcal{A}(\star, \star) \arrow[r, "\_ \circ f"'] & \mathcal{A}(\star, \star) \end{tikzcd}$ commutes for any $f \in \mathcal{A}(\star, \star)$. (1)

Applying the identity of $\star$ in $\mathcal{A}$ in (1) gives us the following equation:

$$\big((\_ \circ f) \circ \alpha_\star\big)(1_\star) = \big(\alpha_\star \circ (\_ \circ f)\big)(1_\star) \implies \alpha_\star(f) = \alpha_\star(1_\star) \circ f. \tag{2}$$

Thus, we get that every natural transformation α is defined in terms of it's value at $1_\star$. As $\mathcal{A}(\star,\star)$ is isomorphic to the group G, and $\alpha_\star$ is a function from $\mathcal{A}(\star,\star)$ to $\mathcal{A}(\star,\star)$, we can write the final equation from (2) with the product being in G, that is, $\alpha_\star(f) = \alpha_\star(1_\star).f$. This can be considered as left multiplication by $\alpha_\star(1_\star)$ in G, which we know is an automorphism of F. Thus, $\alpha_\star$, and hence α can be thought of as a bijection on G.

So far we have shown we have that shown that the collection of all such α, which is $[\mathcal{A}^{op}, Set](H_\star, H_\star)$ is indeed a collection of bijections on G. Moreover, the collection $[\mathcal{A}^{op}, Set](H_\star, H_\star)$ forms a group with the usual composition of the category. As $[\mathcal{A}^{op}, Set]$ is a category, we get that this composition is associative. Also, because this collection contains morphisms with the same source and destination, which is $H_\star$, we get that it is closed under composition, and has an identity. In order to show inverses, we use the group isomorphism between $\mathcal{A}(\star,\star)$ and the property of α from (2), to bijectively map each element $g \in G$ to the natural transformation α such that $\alpha_\star(1_\star)$ corresponds to $g \in \mathcal{A}$. Hence, for any α corresponding to $j \in G$, there will exist a $\beta \in [\mathcal{A}^{op}, Set](H_\star, H_\star)$ corresponding to $j^{-1} \in G$. Using (2) gives us that α and β are inverses.

$$\text{For any } k \in \mathcal{A}(\star,\star), \quad \big(\alpha \circ \beta\big)_\star(k) = \alpha_\star\big(\beta_\star(k)\big) = \alpha_\star\big(\beta_\star(1_\star).k\big) = \alpha_\star\big(j^{-1}.k\big) = \alpha_\star(1_\star).j^{-1}.k = j.j^{-1}.k = k$$

Thus, the collection $[\mathcal{A}^{op}, Set](H_\star, H_\star)$ is a group.

As the collection of elements of G form a set, we get that $\mathcal{A}(\star,\star)$ is a set. Hence, $\mathcal{A}$ is a locally small category. Because $\mathcal{A}^{op}$ has the same number of morphism as $\mathcal{A}$, it is also a locally small category, and we may apply Yoneda Lemma to it. Taking $A = \star \in \mathcal{A}^{op}$ and $X = H_\star$ in Thm 1.6 (1), we get:

$$[\mathcal{A}^{op}, Set](H_\star, H_\star) \hat{\cong} H_\star(\star) = \mathcal{A}(\star,\star) \cong G, \tag{3}$$

where the first isomorphism above is between sets, and second one between groups. We will show that this isomorphism is also a homomorphism. From the proof of Theorem 1.6, we know that the map $\hat{}$, acts as $\alpha \mapsto \alpha_\star(1_\star)$. Hence, for any $\alpha\beta : H_\star \to H_\star$,

$$\widehat{\alpha \circ \beta} = (\alpha \circ \beta)_\star(1_\star) = (\alpha)_\star\Big((\beta)_\star(1_\star)\Big) = \Big((\alpha)_\star(1_\star)\Big).\Big((\beta)_\star(1_\star)\Big). \tag{4}$$

Where the last equality is due to (2) being applicable as $\Big((\beta)_\star(1_\star)\Big)$ is an element of $\mathcal{A}(\star,\star)$. We have shown that in statement (3), the isomorphism is between groups, with the left most term being a group whose elements are bijections on G. This is precisely the statement of Cayley's theorem.

$$Sym(G) \geq [\mathcal{A}^{op}, Set](H_\star, H_\star) \cong G$$

□

1.2 Yoneda Embedding

Definiton 1.9. A category $\mathcal{A}$ is said to be embedded in a category $\mathcal{B}$ if and only if there exists a functor $F : \mathcal{A} \to \mathcal{B}$ such that F is full and faithful.

Lemma 1.10. If a functor is fully faithful, then it is injective on objects upto isomorphism.

Proof. Let functor $F : \mathcal{A} \to \mathcal{B}$ be fully faithful. Suppose, for objects $A, B \in \mathcal{A}$ that $F(A) = F(B)$. We are going to show that $A \cong B$. As F is full, there exists $f \in \mathcal{A}(A, B)$ such that $F(f) = 1_{F(A)} \in \mathcal{B}\big(F(A), F(B)\big)$. Similarly, we also have that there exists $g \in \mathcal{A}(B, A)$ such that $F(g) = 1_{F(B)} \in \mathcal{B}\big(F(B), F(A)\big)$. Because F is a functor,

$$F\big(g \circ f\big) = F(g) \circ F(f) = 1_{F(B)} \circ 1_{F(A)} = 1_{F(A)} \circ 1_{F(A)} = 1_{F(A)} = F(1_A); \tag{5}$$

$$F\big(f \circ g\big) = F(f) \circ F(g) = 1_{F(A)} \circ 1_{F(B)} = 1_{F(B)} \circ 1_{F(B)} = 1_{F(B)} = F(1_B). \tag{6}$$

As F is faithful, (5) gives us that $g \circ f = 1_A$ and (6) gives us $f \circ g = 1_B$. Hence, $A \cong B$.

□

We are going to define a functor $H_\bullet$, from locally small category $\mathcal{A}$ to the presheaf category on $\mathcal{A}$, as taking any object $A \in \mathcal{A}$ to the functor H_A. And for any $X, Y, K \in \mathcal{A}$, taking morphism $f \in \mathcal{A}(X, Y)$ to the natural transformation whose K^{th}-component is defined as taking any $k \in H_X(K)$ to $f \circ k \in \mathcal{A}(K, Y)$.

Proposition 1.11. $H_\bullet$ is a functor from $\mathcal{A}$ to $[\mathcal{A}^{op}, Set]$.

Proof. Fix any objects $K, L, M \in \mathcal{A}$.

I **(Composition)** Let $f \in \mathcal{A}(K, L)$ and $g \in \mathcal{A}(L, M)$. As $H_\bullet(g \circ f)$ and $H_\bullet(g) \circ H_\bullet(f)$ are natural transformations from H_K to H_M, need to show that their X-components are equal for any $X \in \mathcal{A}^{op}$. Fix $X \in \mathcal{A}^{op}$ and $k \in H_K(X)$, and using the definition of $H_\bullet$, we get that

$$LHS = \big(H_\bullet(g \circ f)\big)(k) = g \circ f \circ k$$
$$\text{and } RHS = \Big(H_\bullet(f) \circ H_\bullet(g)\Big)(k) = \Big(H_\bullet(g)\Big)\big(f \circ k\big) = g \circ f \circ k.$$

II **(Identity)** We will show that for any $g \in \mathcal{A}(K, L)$, $H_\bullet$ respects the identities of K and L in $\mathcal{A}$. Thus, for any object $X \in \mathcal{A}$, we need to show that $\Big(H_\bullet(g) \circ H_\bullet(1_K)\Big)_X = \Big(H_\bullet(g)\Big)_X = \Big(H_\bullet(g) \circ H_\bullet(1_K)\Big)_X$. Fix any morphism $p \in H_A(L)$. Using the definition of $H_\bullet$, we get the following equations.

$$\text{Right Identity: } \Big(\big(H_\bullet(g)\big) \circ \big(H_\bullet(1_K)\big)\Big)(p) = \big(H_A(g)\big)(1_K \circ p) = \big(H_A(g)\big)(p)$$
$$\text{Left Identity: } \Big(\big(H_\bullet(1_L)\big) \circ \big(H_\bullet(g)\big)\Big)(p) = \big(H_\bullet(1_L)\big)(g \circ p) = g \circ p \circ 1_L = g \circ p = \big(H_\bullet(g)\big)(p)$$

Hence, $H_\bullet$ is indeed a functor. □

Theorem 1.12. *Any locally small category $\mathcal{A}$ can be embedded in the presheaf category on $\mathcal{A}$.*

Proof. We will show that the functor from Proposition 1.10 is full and faithful. Fix any objects $X, Y \in \mathcal{A}$.

I To show that $H_\bullet$ is a full, we need to show that for every $\alpha \in [\mathcal{A}^{op}, Set](H_X, H_Y)$, there exists a morphism $f \in \mathcal{A}(X, Y)$ such that $H_\bullet(f) = \alpha$. Thus, we need to show that their K-components are equal for every $K \in \mathcal{A}$. Using the definition of $H_\bullet(f)$, this amounts to showing that

$$\text{for any morphism } k \in H_X(K), \Big(H_\bullet(f)\Big)_K(k) = \alpha_K(k), \text{ that is } f \circ k = \alpha_K(k). \tag{1}$$

Because α_X goes from $H_X(X)$ to $H_Y(X)$, $\alpha_X(1_X)$ is a morphism in $\mathcal{A}(X, Y)$. We will show that choosing this morphism to be f will give us the required result, that is $\big(\alpha_X(1_X)\big) \circ k = \alpha_K(k)$. Using the naturality of α,

$$\text{we get that } \begin{array}{ccc} H_X(X) & \xrightarrow{H_X(k)} & H_X(K) \\ {\scriptstyle \alpha_X}\downarrow & & \downarrow{\scriptstyle \alpha_K} \\ H_Y(X) & \xrightarrow[H_Y(k)]{} & H_Y(K) \end{array}, \text{ that is } \begin{array}{ccc} \mathcal{A}(X, X) & \xrightarrow{\_\circ k} & \mathcal{A}(K, X) \\ {\scriptstyle \alpha_X}\downarrow & & \downarrow{\scriptstyle \alpha_K} \\ \mathcal{A}(X, Y) & \xrightarrow[\_\circ k]{} & \mathcal{A}(K, Y) \end{array} \text{ commutes.}$$

Thus, for the identity morphism $1_X \in \mathcal{A}(X, X)$, we get the following

$$\Big(H_Y(k) \circ \alpha_X\Big)(1_X) = \Big(\alpha_K \circ H_X(k)\Big)(1_X) \implies \alpha_X(1_X) \circ k \implies \alpha_K(1_X) \circ k = \alpha_K(k)$$

Thus, we have that $H_\bullet$ is a full functor.

II Fix any morphisms f, g in $\mathcal{A}(X, Y)$ and suppose $H_\bullet(f) = H_\bullet(g)$. In order to show $H_\bullet$ is faithfull, we need to show that $f = g$. As $H_\bullet(f)$ and $H_\bullet(g)$ are equal natural transformations, we have that the action of their X-components is equal. Thus, in particular, for the identity of X, $\big(H_\bullet(f)\big)_X(1_X) = \big(H_\bullet(g)\big)_X(1_X)$. Using the difinition of $H_\bullet$, we get that $f \circ 1_X = g \circ 1_X$. And as both g and f are morphisms from X, we get that $f = g$.

□

2 Prorelations

Definiton 2.1. A prorelation is a partially ordered, down-directed, up-set of relations $X \to Y$.
That is, $P \subseteq \mathcal{P}(X \times Y)$ is a prorelation if it satisfies the following conditions:

(i) Partial Order: Containment of relations defines a partial order. That is, $r \subseteq s$ meaning that for any $(x, y) \in X \times Y$, if $(x, y) \in r$ then $(x, y) \in s$.

(ii) Down-directed: For any $r, s \in P$, there exists $t \in P$ such that $t \subseteq r$ and $t \subseteq s$.

(iii) Up-set: For any relation $u : X \to Y$, if there exists $p \in P$ such that $p \subseteq u$ then $u \in P$.

Example 2.2. We will define a prorelation on real numbers. For any positive real number ϵ, define a relation on $\mathbb{R}$ as $A_\epsilon = \{(x,y)|\ |x-y| < \epsilon\}$. The collection of all relations on $\mathbb{R}$ such that each of them contains some A_ϵ, is a prorelation, K on $\mathbb{R}$. That is, $K = \{a : \mathbb{R} \to \mathbb{R} \mid a \supseteq A_\epsilon \text{ for some } \epsilon > 0\ \}$ forms a prorelation. If $k, l \in K$, then there exist δ, $\epsilon > 0$ such that $k \supseteq A_\delta$ and $l \supseteq A_\epsilon$. Thus, the relation $A_{\frac{\delta+\epsilon}{2}}$ is in both k and l.

Definiton 2.3. A prorelation $P : X \to Y$ can be composed to a prorelation $Q : Y \to Z$ by taking composition of the relations belonging to them. Then, the set $Q.P$ is defined as $Q.P = \{q \circ p : p \in P \text{ and } q \in Q\}$.

Lemma 2.4. Composition of two prorelations is a prorelation.

Proof. For prorelations $P : X \to Y$ and $Q : Y \to Z$, need to show that Q.P is a prorelation.

(i) (Partial Order) Inclusion of relations gives a partial order.

(ii) (Down-Directed) If $k, k' \in Q.P$, then $k = q\,p$ and $k' = q'\,p'$ for some $q, q' \in Q$ and $p, p' \in P$. Because Q and P are prorelations, and hence down-directed sets there exists, $a \in Q$ such that $a \subseteq q, q'$ and $b \in P$ such that $b \subseteq p, p'$. Thus, giving an element, $a \circ b$ of $Q.P$ such that $a \circ b \subseteq k, k'$.

(iii) (Up-Set) Let $l : X \to Z$ be a relation, and $k \in Q.P$ such that $l \supseteq k$. Define relations $p : X \to Y$ and $q : Y \to Z$ as, $p = \{(x,y) : x \in domain(l) \text{ and } y \in Y\}$ and $q = \{(y,z) : y \in Y \text{ and } z \in range(l)\}$. Because $k \in Q.P$, there exist $q' \in Q$ and $p' \in P$ such that $k = q' \circ p'$. Thus by definition of p and q, we get that $p \supseteq p'$ and $q \supseteq q'$. Hence $p \in P$ and $q \in Q$ because P and Q are up-sets, which gives us that $q \circ p \in Q.P$. For any $(x,z) \in l$, by definition of p and q, we get that for every $y \in Y$, $(x,y) \in p$ and $(y,z) \in q$. By definition of composition, this gives that $(x,z) \in q \circ p$, giving that $l \subseteq q \circ p$. And, by definition of $q \circ p$ we get that $l \supseteq q \circ p$. Finally giving that $l = q \circ p \in Q.P$. □

Definiton 2.5. For prorelations $P, Q : X \to Y$, if for each $q \in Q$, there exists $p \in P$ such that $p \subseteq q$, then we write $P \leq Q$.

Definiton 2.6. For a relation $r : X \to Y$, it's opposite relation $r^o : Y \to X$ is defined as

$$(y,x) \in r^o \text{ if and only if } (x,y) \in r \text{ for } x \in X \text{ and } y \in Y.$$

Lemma 2.7. For any function $f : X \to Y$, Δ_X is contained in the composition $f^o \circ f$.

Proof. As f is a function, it must be defined on every element of it's domain. Thus, for every $x \in X$, there exists some (x,y) in f. By definition of f^o, (y,x) is in f^o. Hence, by definition of composition, (x,x) is in $f^o\,f$. □

Lemma 2.8. For any relation $r : X \to Y$, the composition $r \circ r^o$ is contained in Δ_Y.

Proof. Suppose there exist $x \in X$ and $y \in Y$ such that $x\,r\,y$. By definition of r^o, this gives us that $y\,r^o\,x$. Using definition of composition, $y\,r^o\,x\,r\,y$ gives that $y\,(r \circ r^o)\,y$. □

Lemma 2.9. For relations $r, s : X \to Y$ and $t : Y \to Z$, if $r \subseteq s$ then $(t \circ r) \subseteq (t \circ s)$.

Proof. Suppose relations r, s and t are as given above, and let $x\,(t\,r)\,z$. By definition of composition, there exists, $y \in Y$ such that $x\,r\,y$ and $y\,t\,z$. Using the hypothesis, as $r \subseteq s$, $x\,r\,y$ gives $x\,s\,y$. And via composition of $x\,s\,y$ with $y\,t\,z$, we get $x\,(t\,s)\,z$. We started with any element of $(t \circ r)$ and showed that it must also be in $t \circ s$ and thus have that $(t \circ r) \subseteq (t \circ s)$. □

Lemma 2.10. For relations $r : X \to Y$ and $s, t : Y \to Z$, if $s \subseteq t$ then $(s \circ r) \subseteq (t \circ r)$.

Proof. Suppose relations r, s and t are as given above, and let $x\,(s\,r)\,z$. By definition of composition of relations, we get that there exists some $y \in Y$ such that $x\,r\,y$ and $y\,s\,z$. Because $s \subseteq t$, $y\,s\,z$ implies that $y\,t\,z$. Taking the composition, $x\,r\,y\,s\,z$ yields $x(t\,r)z$. We started with any element of $(s \circ r)$ and showed that it must also be in $t \circ r$ and thus have that $(s \circ r) \subseteq (t \circ r)$. □

3 Quasi-Uniform Spaces

Definiton 3.1. A prorelation P on a set X is said to be a quasi-uniformity if it satisfies the following conditions:

(i) Every relation in P is reflexive. That is, for each $p \in P$, if $x \in X$ then $(x,x) \in p$.

(ii) For each p in P, there exists p' in P such that $p' \circ p' \subseteq p$.

Example 3.2. We will show that the prorelation defined in Example 2.2 is a quasi-uniformity. The definition $A_\epsilon = \{(x,y)|\ |x-y| < \epsilon\}$ implies that each A_ϵ is reflexive. And as every relation in K contains some A_ϵ, it must be reflexive as well, hence definition 3.1 (i) holds for K. Now we are going to show that definition 3.1 (ii) holds for K. Fix any relation $a \in K$, so, by definition of K, there exists ϵ such that $b \supseteq A_\epsilon$. Using $|x-y| = |y-x|$ we get that A_ϵ is symmetric. Thus, for any ϵ, $A_\epsilon \circ A_\epsilon \subseteq A_\epsilon \subseteq b$.

Definiton 3.3. If X is a set, and A is a quasi-uniformity on X, then (X,A) is a quasi-uniform space.

Definiton 3.4. A function, $f : (X, A) \to (Y, B)$ is said to be uniformly continuous if $f.A \leq B.f$. That is, for each $b \in B$, there exists $a \in A$ such that $f \circ a \subseteq b \circ f$. Meaning that
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "A"'] \arrow[dr, phantom, "\leq"] & Y \arrow[d, "B"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}.$$

Lemma 3.5. If A is a quasi-uniformity on a set X, then $A.A = A$

Proof. Fix any $a \in A$, as A is a quasi-uniformity, $\exists b \in A : bb \subseteq a$, we get that $A.A \leq A$. And as A is a prorelation, and is hence down-directed, $\exists c \in A : a.a \supseteq c$, giving that $A.A \geq A$ $\square$

We define a category called QUnif as having quasi-uniform spaces as objects and uniformly continuous maps between them as morphisms. With the composition of morphisms defined as that of functions, and identity of object (X, A) defined to be the identity function on set X.

Proposition 3.6. QUnif is a category.

Proof. (i) (Associativity) The composition of functions is associative by definition.

(ii) (Identity) For each object (X, A), the identity function $\Delta_X : (X, A) \to (X, A)$ is uniformly continuous as $\Delta_X.A = A \leq A = A.\Delta_X$. $\square$

Definiton 3.7. A prorelation, $\phi : X \twoheadrightarrow Y$ is called a promodule $\phi : (X, A) \twoheadrightarrow (Y, B)$ if it satisfies:

$$\phi.A \leq \phi \text{ and } B.\phi \leq \phi$$

Containment of promodules is defined as that of prorelations. Promodules are composed as prorelations.

Now, we define a 2-category called ProMod as having quasi-uniform spaces as 0-cells and the promodules between them being 1-cells. The identity 1-cell of (X, A) is defined to be the promodule A. For promodules $P, Q : (X, A) \twoheadrightarrow (Y, B)$, we define there to be a 2-cell from P to Q if and only if as prorelations, $P \leq Q$. Thus, the identity 2-cell of P is the 2-cell corresponding to $P \leq P$. We use the definition of a 2-category given in $CITE$.

Proposition 3.8. ProMod, as described above is a 2-category.

Proof. In order to show that ProMod is a 2-category, need the following:

(a) (1-Identities) For each quasi-uniform space (X, A), $A : (X, A) \twoheadrightarrow (X, A)$ a promodule because $A.A = A$ by Lemma 2.2.1.

(b) (1-Composition) Need composition of promodules to be a promodule.
Let $\phi : (X, A) \twoheadrightarrow (Y, B)$ and $\psi : (Y, B) \twoheadrightarrow (Z, C)$ be promodules. To show that $\psi.\phi : (X, A) \twoheadrightarrow (Z, C)$ is a promodule, need it to be a prorelation that satisfies the two conditions required to be a promodule:

  (i) By Lemma 1.2.1, prorelations are closed under composition. Hence, $\psi.\phi$ is a prorelation

  (ii) Need to show that $\psi.\phi.A \leq \psi.\phi$. So, Fix $p \in \psi$ and $q \in \phi$. As ϕ is a promodule, $\phi.A \leq \phi$ gives that there exists $q' \in \phi$ and $a \in A$ such that $q'\,a \subseteq q$. Thus, $p\,q'\,a \subseteq p\,q$.

  (iii) Need to show that $C.\psi.\phi \leq \psi.\phi$.Fix $p \in \psi$ and $q \in \phi$. Because ψ is a promodule, $C.\psi \leq \psi$ gives that there exists $c \in C$ and $p' \in \psi$ such that $c\,p' \subseteq p$. Thus, $c\,p'\,q \subseteq p\,q$

(c) (2-Identities) As every promodule is contained in itself, always have $\psi \leq \psi$. Define this comparison to be the identity 2-cell for ψ and denote it by $\leq_\psi$

(d) (Vertical 2-composition) For promodules $\psi, \phi, \delta : (X, A) \twoheadrightarrow (Y, B)$, if there is is a 2-cell from ψ to ϕ and another one from ϕ to δ i.e. $\psi \leq \phi \leq \delta$, then by transitivity of the partial order, $\psi \leq \delta$ i.e. there's a 2-cell from ψ to δ.

(e) (Horizontal 2-composition) If there are promodules $\psi, \psi' : (X, A) \twoheadrightarrow (Y, B)$ and $\phi, \phi' : (Y, B) \twoheadrightarrow (Z, C)$ such that $\psi \leq \psi'$ and $\phi \leq \phi'$, need to show that $\psi.\phi \leq \psi'.\phi'$. Fix $p' \in \psi'$ and $q' \in \phi'$. As $\psi \leq \psi'$, $\exists p \in \psi : p \subseteq p'$ and as $\psi \leq \psi'$, $\exists q \in \phi : q \subseteq q'$. Thus, $p\,q \subseteq p'\,q'$

(f) (1-Identity) Need to show that for any promodule $\phi : (X, A) \twoheadrightarrow (Y, B)$, $\phi.A = \phi = B.\phi$. By quasi-uniformity of A, every $a \in A$, is reflexive. Thus, for any $p \in \phi$ and $a \in A$, $p = p.\Delta_X \subseteq p\,a$ giving that $\phi \leq \phi.A$. And as ϕ is a promodule, $\phi \geq \phi.A$. Hence, by anti-symmetry of the partial order, $\phi = \phi.A$.

Similarly, By quasi-uniformity of B, every $b \in B$, is reflexive. Thus, for any $p \in \phi$ and $b \in B$, $p = \Delta_Y.p \subseteq b\,p$ giving that $\phi \leq B.\phi$. And as ϕ is a promodule, $\phi \geq B.\phi$. Hence, $\phi = B.\phi$.

(g) (1-Associativity) As composition of relations is associative, so too is the composition of prorelations directly giving that composition of promodules i.e. 1-cells is associative.

(h) (Vertical 2-Identity) Let $\leq\colon \psi \to \phi$ be a 2-cell i.e. $\psi \leq \phi$. By our definition of identity 2-cell, $\leq_\psi . \leq_1$ means precisely that $\psi \leq \psi \leq \phi$, and by transitivity, this is equivalent to $\psi \leq \phi$. Similarly, $\leq_1 . \leq_\phi$ means exactly that $\psi \leq \phi \leq \phi$, and this is equivalent to $\psi \leq \phi$.

(i) (Vertical 2-Associativity) Associativity of the partial order on promodules directly gives the associativity of composition of 2-cells in ProMod.

(j) (Horizontal 2-Identity) Let $\psi, \phi : (X, A) \multimap\!\!\rightarrow (Y, B)$ be promodules. For any 2-cell $\leq\colon \psi \to \phi$, need to show that the 2-cell given by the horizontal composition, $\leq * \leq_A$ is equal to $\leq$, as well as equal to $\leq_B * \leq$. So, it's required that $\psi.A \leq \phi.A \iff \psi \leq \phi \iff B.\psi \leq B.\phi$. And this holds as a direct consequence of (f).

(k) (Horizontal 2-Associativity) As there's a unique 2-cell between any two promodules, and composition of promodules is associative, horizontal composition of 2-cells is associative.

(l) (2-Identity) For promodules $\psi : (X, A) \multimap\!\!\rightarrow (Y, B)$ and $: \phi(Y, B) \multimap\!\!\rightarrow (Z, C)$ need $(\leq_\psi * \leq_\phi) =\leq_{\psi.\phi}$. Both sides of the required equality are 2-cells $\leq\colon \psi.\phi \to \psi.\phi$. Thus, they are equal by the uniqueness of 2-cells between any two 1-cells.

(m) (2-Interchange) Let $\psi, \phi, \delta : (X, A) \multimap\!\!\rightarrow (Y, B)$ and $\psi', \phi', \delta' : (Y, B) \multimap\!\!\rightarrow (Z, C)$ be promodules. For 2-cells $\leq_1\colon \psi \to \phi$,$\leq_2\colon \phi \to \delta$, $\leq_a\colon \psi' \to \phi'$ and $\leq_b\colon \phi' \to \delta'$,need to show $(\leq_b . \leq_a) * (\leq_2 . \leq_1) = (\leq_b * \leq_2).(\leq_a * \leq_1)$. Both RHS and LHS are 2-cells from $\psi.\psi'$ to $\delta.\delta'$ and are hence equal. □

We now define a functor from the category QUnif to ProMod, as fixing objects and taking uniformly continuous maps $f : (X, A) \to (Y, B)$ to $f.B$.

Proposition 3.9. The mapping defined above, $(\_)_* : \text{QUnif}^{op} \to \text{ProMod}$ as

(a) for $(X, A) \in$ QUnif, $(X, A)_* := (X, A) \in$ ProMod

(b) for $f : (X, A) \to (Y, B)$ in QUnif, $f^* := B.f$

is indeed a functor.

Proof. We will first show that $B.f = b \circ f : b \in B$ is a promodule, and then that $(\_)_*$ defines a functor.

(i) (Partial-Order) Inclusion of relations acts as the partial order.

(ii) (Down-Directed) Fix any k, k' belonging to $B.f$. Thus, there exist b, b' in B such that $k = b\, f$ and $k' = b\, f$. Using down-directedness of B, there exists a $c \in B$ such that $c \subseteq b, b'$. Hence, by Lemma 2.4.4, $c\, f \subseteq k, k'$.

(iii) (Up-set) Let k belong to $B.f$ and $l : (X, A) \to (Y, B)$ be a uniformly continous function such that $l \supseteq k$. Define a relation $b' := \{(f(d), l(d)) : d \in \mathit{domain}(l)\}$. By definition, for any $x \in X$ and $z \in Y$ such that $(x, z) \in l$, we get that $(f(x), z) \in b'$. And $l \supseteq k = b\, f$ implies $\mathit{domain}(l) \supseteq \mathit{domain}(f)$ giving $(x, f(x)) \in f$. Thus, by defniition of composition, $(x, z) \in b'.f$. Conversely, suppose $(x, z) \in b'.f$. By definition of composition, there exists $f(x) \in Y$ such that $(f(x), z) \in b'$. Again using the definition of b', we get that $z = l(x)$ i.e. $(x, z) \in l$. Hence, $l = b'\, f$. Now we will show that $b' \supseteq b$. Because $b'\, f = l \supseteq k = b\, f$, for any $x \in X$ we have that $b'\big(f(x)\big) \supseteq b\big(f(x)\big)$. Thus, $b'|_{f(x)} \supseteq b|_{f(x)}$. By down-directedness of B, the restriction $b|_{f(x)} \subset b$ implies $b(x)|_{f(x)} \in B$. Finally, $b' \supseteq b'|_{f(x)} \supseteq b|_{f(x)}$ gives $b' \in B$. Hence, $b'.f \in B.f$.

(iv) Need to show that $(B.f).A \leq B.f$. So, fix any $b \in B$, we will find $b' \in B$ and $a \in A$ such that $b'\, f\, a \subseteq bf$. By quasi-uniformity of B, there exists $b' \in B$ such that $b'\, b' \subseteq b$. Using Lemma 2.4.3, we get that $b'\, b'\, f \subseteq b\, f$. As f is uniformly continous, $f.A \leq B.f$ gives that there is some $a \in A$ such that $f\, a \subseteq b'\, f$. Using this in the previous inequality, we get $b'\, f\, a \subseteq b'\, b'\, f \subseteq b\, f$.

(v) Need to show that $B.B.f \leq B.f$. Fix any $b \in B$, we will find $b' \in B$ such that $b'\, b'\, f \subseteq b\, f$. By quasi-uniformity of B, there exists $b \in B$ such that $b'\, b' \subseteq b$. Using Lemma 2.4.4, we get $b'\, b'\, f \subseteq bf$.

Thus, $B.f$ is a promodule. We now proceed to show that $(\_)_*$ defines a functor.

(i) (Composition) Need to show that $(g \circ f)_* = g_* f_*$ i.e. $C.g.f = C.g.B.f$.

In order to show $C.g.f \leq C.g.B.f$, fix any $b \in B, c \in C$. We will show that $c\, g\, f \subseteq c\, g\, b\, f$. As f is uniformly continuous, $f.A \leq B.f$ gives that there exists $a \in A$ such that $f\, a \subseteq b\, f$. Using Lemma 2.4.3, we get $(c\, g) f\, a \subseteq (c\, g) b\, f$. Now, using reflexiviness of a, we get $c\, g\, f \subseteq c\, g\, b\, f$.

Now, to show that $C.g.f \geq C.g.B.f$. Fix any $c \in X$, we will find $c' \in C$ and $b \in B$ such that $c\, g\, f \supseteq c\, g\, b\, f$. By quasi-uniformity of C, there exists $c' \in C$ such that $c \subseteq c'\, c'$. Using Lemma 2.4.4 gives that $c\, (g\, f) \supseteq c'\, c'\, (g\, f)$. Because g is uniformly continuous, $C.g \geq g.B$ gives us $b \in B$ such that $g\, c' \supseteq b\, g$. Using this in the previous inequality gives that $c\, g\, f \supseteq c'\, g\, b\, f$.

(ii) (Identity) let (X, A) be in object of QUnif and $1_{(X,A)} : (X, A) \to (X, A)$ be the identity funtion on (X, A). That is, $1_{(X,A)}$ is defined as $x \mapsto x$. Need to show that $(1_{(X,A)})_* = 1_{(X,A)_*}$. Using functor's definition, $LHS = (1_{(X,A)})_* = A.(1_{(X,A)}) = A.1_{(X,A)} = A$ and $RHS = 1_{(X,A)_*} = 1_{(X,A)}$ Using Proposition 3.2(f), we get that $A = 1_{(X,A)} = RHS$. □

Similar to the above functor, we define a contravariant functor from the category QUnif to ProMod, as fixing objects and taking uniformly continuous maps $f : (X, A) \to (Y, B)$ to $B.f^o$.

Proposition 3.10. The mapping defined above, $(\_)^* : \mathrm{QUnif}^{op} \to \mathrm{ProMod}$ as

(a) for $(X, A) \in \mathrm{QUnif}^{op}$, $(X, A)^* := (X, A) \in \mathrm{ProMod}$

(b) for $f : (X, A) \to (Y, B)$ in QUnif, $f^* := f^o.B$

is indeed a functor.

Proof. Showing that $f^o.B : (Y, B) \nrightarrow (X, A)$ is a promodule.
So, need to show $f^o.B$ a prorelation $Y \to X$ and that $(f^o.B).B \sqsubseteq f^o.B$ and $A.(f^o.B) \sqsubseteq f^o.B$
To show prorelation,

(i) (Partial-order) Inclusion of relations i.e. for $k = f^o \circ b$ and $k' = f^o \circ b'$ in $f^o.B$, $k \subseteq k' \iff b \subseteq b'$

(ii) (Down directed) for $k, k' \in f^o.B$, need that $\exists l \in f^o.B$ such that $l \subseteq k, k'$

Fix $k, k' \in f^o.B \implies \exists b, b' \in B : k = f^o \circ b$ and $k' = f^o \circ b'$

By down-directedness of B, there exists $c \in B$ such that $c \subseteq b, b'$, define $l = f^o \circ c$. Now, using Lemma 2.4.3 gives $l = f^o \circ c \subseteq k, k'$.

(iii) (Up-set) for a relation $l : Y \to X$ and $k \in f^o.B$ such that $l \supseteq k$, need $l \in f^o.B$

Let $b \in B$ be such that $k = f^o \circ b$ and define $b' := \{(y, y') : y \in domain(l) \text{ and } y' \in (f^o)^{-1}(l(y))\}$
As $l \supseteq k = f^o \circ b$, $domain(b') = domain(l) \supseteq domain(b)$
and $range(l) \supseteq range(f^o \circ b) \implies \forall y \in domain(b), range(b') = (f^o)^{-1}(l(y)) \supseteq (f^o)^{-1}(f^o \circ b) = range(b)$
Now, by definition of b', $f^o \circ b' \supseteq l$. To show $f^o \circ b \subseteq l$,
$(x, y) \in f^o \circ b' \implies \exists z \in Y : (x, z) \in b'$ and $(z, y) \in f^o \implies x \in domain(l)$ and $z \in l(x)$ i.e. $(x, z) \in l$

To show $(f^o.B).B \leq f^o.B$, need that $\forall b \in B, \exists b' \in B : f^o \circ b' \circ b' \subseteq f^o \circ b$,
Fix any $b \in B$, as B is a quasi-uniformity, $\exists b' \in B : b' \circ b' \subseteq b \implies f^o \circ b' \circ b' \subseteq f^o \circ b$

To show $A.(f^o.B) \leq f^o.B$, need that $\forall b \in B$, $\exists b' \in B, a \in A : a \circ f^o \circ b' \subseteq f^o \circ b$,
As f is uniformly continuous, $f.A \leq B.f$ i.e. $\forall b \in B, \exists a \in A : f \circ a \subseteq b \circ f \implies a = f^o \circ f \circ a \subseteq f^o \circ b \circ f$
Fix any $b \in B$, so, $\exists b' \in B : b'b' \subseteq b$ And, for this $b', \exists a : a \subseteq f^o b' f \implies a f^o b' \subseteq f^o b' f f^o b' \subseteq f^o b' b' \subseteq f^o b \implies a f^o b' \subseteq f^o b$

Now, need to show that $(\_)^*$ respects composition and identity.

(i) (Composition) let f, g be uniformly continuous, $(X, A) \xrightarrow{f} (Y, B) \xrightarrow{g} (Z, C)$ need that $(g \circ f)^* = f^*.g^*$

LHS$=(g \circ f)^* = (g \circ f)^o.C = (f^o \circ g^o).C$ and RHS$=f^*.g^* = (f^o.B).(g^o.C)$
For equality, showing that LHS$\geq$RHS and LHS$\leq$RHS:

To show $(f^o \circ g^o).C \geq (f^o.B).(g^o.C)$, need that $\forall c \in C, \exists b \in B, c' \in C : f^o g^o c \supseteq f^o b g c'$
Fix any $c \in C$, so, $\exists c' \in C : c' \circ c' \subseteq c \implies f^o g^o c \supseteq f^o g^o (c'c') = f^o g^o (c' \Delta_Z c') \supseteq f^o g^o c' (g g^o) c'$
By uniform conntinuity of g, for $c' \in C, \exists b \in B : gb \subseteq c' g$
Thus, $f^o g^o c \supseteq f^o g^o (c' g) g^o c' \supseteq f^o (g^o g) b g^o c' = f^o b g^o c'$.

To show $(f^o \circ g^o).C \leq (f^o.B).(g^o.C)$, need that $\forall b \in B, c \in C, \exists c' \in C : f^o g^o c \subseteq f^o b g^o c$
Fix any $c \in C, b \in B$ will show that $c' := c$ works:
As B is a quasi-uniformity, $\Delta_Y \subseteq b \implies f^o \Delta_Y g^o c = f^o g^o c \subseteq f^o b g^o c = f^o b g^o c'$

(ii) (Identity) let $(X, A) \in$ QUnif ^{op}, and $1_{(X,A)} : (X, A) \to (X, A)$ as $x \mapsto x$ need that $(1_{(X,A)})^* = 1_{(X,A)^*}$
LHS$=(1_{(X,A)})^* = (1_{(X,A)})^o.A = 1_{(X,A)}.A = A$.
And as $RHS = 1_{(X,A)^*} = 1_{(X,A)}$ Using Proposition 3.2(f), we get that $A = 1_{(X,A)} = RHS$. □

Definiton 3.11. For any quasi-uniform space (X, A), an element $x \in X$ is said to belong in the topological closure of set $M \subseteq X$ if and only if for each $a \in A$, there exists $y \in M$ such that $x\,a\,y$ and $y\,a\,x$.

Definiton 3.12. Let $f : (X, A) \to (Y, B)$ be a uniformly continuous function.

I f is said to be fully faithful if and only if $f^*.f_* = A$.

II f is said to be fully dense if and only if $f_*.f^* = B$.

III f is said to be topologically dense of and only if $\overline{f(X)} = Y$.

Proposition 3.13. Fix a uniformly continuous map, $f : (X, A) \to (Y, B)$

(a) f is fully faithful if and only if $A = f^o.B.f$, that is $A \geq f^o.B.f$

(b) f is fully dense if and only if for any $b \in B$, $\exists b' \in B$ such that $b' \subseteq b\, f\, f^o\, b$

(c) f is topologically dense if and only if for any $b \in B$, $\;b\, f\, f^o\, b$ is reflexive

(d) f is fully dense if and only if f is topologically dense

Proof.

(a) (i) ($\Longrightarrow$) Let f be fully faithful i.e. $f^*.f_* = A \implies f^o.B.B.f = A$
Need to show that $A = f^o.B.f$ i.e. $A \leq f^o.B.f$ and $A \geq f^o.B.f$
By hypothesis and quasi-uniformity of B, $A \geq f^o.B.B.f \geq f^oB.f$
To show $A \leq f^o.B.f$, need that $\forall b \in B, \exists a \in A : a \subseteq f^obf$
Fix $b \in B$, hypothesis gives that $f^o.B.B.f \leq A$ so,
$\exists a \in A : a \subseteq f^obbf$ and also, by quasi-uniformity of B, for $b, \exists b' \in B : b'b' \subseteq b \implies f^ob'b'f \subseteq f^obf$
Combining the above two inequalities, $a \subseteq f^obbf \subseteq f^obf$

(ii) ($\Longleftarrow$) Let $A = f^o.B.f$ need to show $A = f^o.B.B.f$ i.e. $A \geq f^oB.B.f$ and $A \leq f^oB.B.f$
To show $A \geq f^o.B.B.f$, need to show that $\forall a \in A, \exists b, b' \in B : a \supseteq f^obb'f$
Have that $A \geq f^o.B.f$ and $B.B \leq B$
So, fix $a \in A$, now $\exists b \in B : a \subseteq f^obf$ and for this b, $\exists b' \in B : b'b' \subseteq b$. Therefore, $a \supseteq f^obf \supseteq f^ob'b'f$
To show $A \leq f^o.B.B.f$, need $\forall b, b' \in B, \exists a \in A : a \subseteq f^obb'f$
Before that, uniform continuity of f along with Lemma 2.1.1 gives that
$f.A \leq B.f \implies A = f^of.A \leq f^o.B.f$
So, fix $b, b' \in B$, now, as ,
$A \leq f^o.B.f$ giving
$\exists a \in A : a \subseteq f^obf$ and $\exists a' \in A : a' \subseteq f^ob'f \implies \Delta_X \subseteq f^ob'f$.
Therefore $a = a\Delta_X \subseteq (f^obf)(f^ob'f) \subseteq f^obb'f$

(b) (i) ($\Longrightarrow$) Let f be fully dense i.e. $B = f_*f^* = B.f.f^o.B$. showing that $\forall b \in B, \exists b' \in B : b' \subseteq bff^ob$:
So, fix $b \in B$, as $B \leq B.f.f^o.B$, there exists $b' \in B$ such that $b' \subseteq bff^ob$.

(ii) ($\Longleftarrow$) Suppose $\forall b \in B, \exists b' \in B : b' \subseteq bff^ob$. This gives $B \leq B.f.f^o.B$, in order to show equality, also need $B \geq B.f.f^o.B$. By quasi-uniformity of B, for any $b \in B$, $\exists b' \in B : b'b' \subseteq b$. Now, by Lemma 2.4.2,

$$ff^o \subseteq \Delta_Y \implies b'ff^ob' \subseteq b'\Delta_Yb' = b'b' \subseteq b$$

(c) (i) ($\Longrightarrow$) Let f be topologically dense. We will show that for any $b \in B, y \in Y,\ (y, y) \in bff^ob$. Fix any $b \in B$ and $y \in Y$. As f is topologically dense, $\overline{f(X)} = Y$, implying that $y \in \overline{f(X)}$, by definition giving that

$$\exists x \in X \text{ such that } (f(x), y) \in b \text{ and } (y, f(x)) \in b$$

Re-writing the above statement in terms of relations, and considering f as a relation:

$$(f(x), y) \in b \text{ gives } x(b \circ f)y \text{ i.e. } y \in (b \circ f)(x) \tag{1}$$
$$(y, f(x)) \in b \text{ gives } f(x) \subseteq b(y) \tag{2}$$

Repeatedly applying Lemma 2.4.3 to (2),

$$f(x) \subseteq b(y) \implies \big(f \circ f^o\big)(f(x) \subseteq \big(f \circ f^o\big)b(y) \implies \big(f \circ f^o \circ f\big)(x) \subseteq \big(f \circ f^o \circ b\big)(y)$$

Applying Lemma 2.4.1 to the final inequality in the above statement gives that

$$f(x) = (f \circ \Delta_X)(x) \subseteq \big(f \circ f^o \circ f\big)(x) \subseteq \big(f \circ f^o \circ b\big)(y)$$

Applying Lemma 2.4.3 and then using (1) on the above inequality completes the result:

$$f(x) \subseteq \big(ff^ob\big)(y) \implies \big(b \circ f\big)(x) \subseteq \big(bff^ob\big)(y) \implies y \in \big(bff^ob\big)(y) \text{ i.e. } y\big(bff^ob\big)y$$

(ii) ($\Longleftarrow$) Fix any $y \in Y$ and $b \in B$. Also, suppose that $\Delta_Y \leq bff^ob$. As f is a function with domain as X, $f^o : Y \to X$, $\phi \neq (f^o \circ b)(y) \subseteq X$. So, fix $x \in (f^o \circ b)(y)$, going to show that $(f(x), y) \in b$ and $(y, f(x)) \in b$. Again, while viewing f as a relation.

$$\Delta_Y \leq bff^ob \implies \Delta_Y(y) \subseteq bff^ob(y) = \big(bf\big)(f^ob(y))$$

Last inequality of the above statement gives $y \in (bf)(x)$ i.e. $(f(x), y) \in b$.
Applying Lemma 2.4.2 to f, and then using Lemma 2.4.4,

$$ff^o \subseteq \Delta_Y \implies ff^ob \subseteq \Delta_Y b = b$$

Thus $ff^ob(y) \subseteq b(y)$ and hence $f(x) \subseteq b(y) \implies (y, f(x)) \in b$

(d) (i) ($\Longrightarrow$) Let f be topologically dense. As B is a quasi-uniformity, for any $b \in B$,

$$\exists b' \in B : b'b' \subseteq b \text{ and } \Delta_Y \subseteq b' \implies b' = b'\Delta_Y \subseteq b'b' \subseteq b \tag{3}$$

By the characterisation of topologically dense in (c), have that $\Delta_Y \subseteq b'ff^ob'$. Now, using the (3) and Lemma 2.4.3,

$$\Delta_Y \subseteq b'ff^ob' \implies b' = b'\Delta_Y \subseteq b'b'ff^ob' \subseteq bff^ob' \subseteq bff^ob$$

Hence, we have $b' \in B : b' \subseteq bff^ob$ giving us that f is fully dense (from (b)).

(ii) ($\Longleftarrow$) From (b), we have for $b \in B$, the existstence of $b' \in B$ such that $b' \subseteq bff^ob$. As B is a quasi-uniformity, $\Delta_Y \subseteq b'$. So, $\Delta_Y \subseteq bff^ob$, and from (c), this gives us that f is topologically dense. ☐

4 Yoneda Lemma in Quasi-Uniform Spaces

In this section, we will look at Yoneda Lemma and Yoneda Embedding for Quasi-Uniform Spaces. We use 1 to denote the quasi-uniform space with one element, that is, the quasi-uniform space $(\{\star\}, \{(\star, \star)\})$. Also, when unambiguous, we use 1 to denote the quasi-uniformity of the quasi-uniform space 1.

Definiton 4.1. The set PX is defined to be the collection of all promodules from the quasi-uniform space(X,A) to the quasi-uniform space 1.

$$PX := \{\psi : (X, A) \multimapdotboth 1 | \psi \text{ is a promodule}\}$$

Proposition 4.2. For any $a \in A$, $\tilde{a}$ is defined to be a relation $PX \to PX$ as

$$\text{for } \phi, \psi \in PX,\ \phi\, \tilde{a}\, \psi \ \text{ only if } \ \phi \leq \psi.a$$

The set, $\tilde{A} := \{\tilde{a} : a \in A\}$ defines a quasi-uniformity on PX.

Proof. First need to show that $\tilde{A}$ is a prorelation,

(i) (Partial order) For any two relations $\tilde{a}, \tilde{b} : PX \to PX$, define $\tilde{a} \leq \tilde{b}$ to be true only if $a \subseteq b$.

(ii) (Down-Directed) Need for any $\tilde{a}, \tilde{b} \in \tilde{A}$, the existstence of some $\tilde{c} \in A$ such that $c \subseteq a, b$
If $\tilde{a}, \tilde{b} \in A$ then there exist $a, b \in A$. By down-directedness of A, there exists a $c \in A$ such that $c \subseteq a, b$. Now the definition of $\tilde{A}$ gives that $\tilde{c} \in \tilde{A}$. And the definition of the partial order on $\tilde{A}$ ensures $\tilde{c} \leq \tilde{a}, \tilde{b}$.

(iii) (Upset) For any relation $l : PX \to PX$, need that if $\tilde{k}$ belongs to $\tilde{A}$ such that $l \geq \tilde{k}$, then $l \in \tilde{A}$.
Fix any $k : PX \to PX$, and $\tilde{k} \in \tilde{A}$ such that $l \geq \tilde{k}$. As k is a relation between promodules $X \multimapdotboth 1$, it can be thought of as a relation a on X, defined as:

$$a := \{(x, y) : x \in domain(\psi) \text{ and } y \in domain(\phi) \text{ whenever } \exists \psi, \phi \in PX : \psi l \phi\}$$

So, $l \geq \tilde{k}$ gives that $\tilde{a} \geq \tilde{k}$ i.e. $a \supseteq k$. And as A is an upper-set, we get $a \in A$. Now, by definition of $\tilde{A}$, $l \in \tilde{A}$.

Secondly, need show that the other two conditions hold for $\tilde{A}$,

(i) For all $\tilde{a} \in \tilde{A}$, need $\tilde{a}$ to be reflexive i.e if $\psi \in PX$ then $\psi\, \tilde{a}\, \psi$.
By definition of $\tilde{a}$, need to show that $\psi \leq \psi.a$. So, fix a $p \in \psi$, we will show that $p \subseteq p.a$. Quasi-uniformity of A gives that $\Delta_X \subseteq a$. Hence, by Lemma 2.4.3, $p = p\,\Delta_X \subseteq p\,a$.

(ii) For all $\tilde{a} \in \tilde{A}$, need to find $\tilde{b} \in \tilde{A}$ such that $\tilde{b}\tilde{b} \leq \tilde{a}$
Before showing the result, proving that for any $x, y \in A$, $\tilde{x}\,\tilde{y} \leq \widetilde{xy}$ i.e. $\forall \psi, \phi \in PX$, $\psi(\tilde{x}\,\tilde{y})\phi \implies \psi\,\widetilde{xy}\,\phi$. If $\psi_1(\tilde{a}.\tilde{b})\psi_3$, then, the definition of composition gives that $\exists \psi_2$ such that $\psi_1\,\tilde{b}\,\psi_2\,\tilde{a}\,\psi_3$. Now, the definition of $\tilde{b}$ gives $\psi_1 \leq \psi_2\, b$ and that of $\tilde{a}$ gives $\psi_2 \leq \psi_3\, a$. Combining these inequalities, $\psi_1 \leq \psi_2.b \leq \psi_3.ab$. Hence, by definition of $\widetilde{ab}$, $\psi_1\,(\widetilde{ab})\,\psi_3$. Now, to show the result, fix any $\tilde{a} \in \tilde{A}$. Therefore,$a \in A$, and by quasi-uniformity of A, $\exists b \in A : b \circ b \subseteq a$. Thus, by the partial-order defined on $\tilde{A}$, $\widetilde{bb} \leq \tilde{a}$. Now, transitivity of the partial order gives us the required result, $\tilde{b}\,\tilde{b} \leq \widetilde{bb} \leq \tilde{a}$. ☐

Proposition 4.3 (Yoneda Embedding)**.**
For a quasi-uniform space (X, A), function $y_X : X \to PX$ is defined by $x \mapsto x^*$ for $x \in X$.

(a) $y_X : (X, A) \to (PX, \tilde{A})$ is a uniformly continuous map.

(b) $y_X : (X, A) \to (PX, \tilde{A})$ is fully faithful.

Proof.

(a) In order to show y_X is uniformly continuous, need to show that $y_X.A \leq \tilde{A}.y_X$. By definition of $\leq$, need $\forall a \in A, \exists b \in A : y_X \circ b \subseteq \tilde{a} \circ y_X$. Applying the relations to some element, x of the set X:

$$\big(y_X \circ b\big)(x) \subseteq \big(\tilde{a} \circ y_X\big)(x) \implies y_X(b(x)) \subseteq \tilde{a}(x^*) \tag{1}$$

So, for the condition given by (4) to hold, if $y \in b(x)$, then it's required that $y^* = y_X(y) \in \tilde{a}(x^*)$ i.e. $x^*\tilde{a}y^*$. Using the definition of x^*, y^* and $\tilde{a}$,

$$x^*\tilde{a}y^* \iff x^o.A \leq y^o.A.a \iff \forall a' \in A, \exists a'' \in A : x^o a'' \subseteq y^o a' a \tag{2}$$

Now, fix any $a \in A$, $x \in X$. Thus, quasi-uniformity of A, gives $a'' \in A$ such that $a''a'' \subseteq a$.Also, choose some $y \in a''(x)$. Hence, in order to show that the condition from (5) holds, need that $\forall b \in A, x^o a'' \subseteq y^o ba$, and by applying the relations to an element z gives the following condition

$$\forall b \in B, \forall x \in X \ , \ \big(x^o a''\big)(z) \subseteq \big(y^o ba\big)(z) \tag{3}$$

Examining the left side of (6),

$$\big(x^o a''\big)(z) = x^o(a''(z)) = \begin{cases} \phi & \text{if } x \notin a''(z) \\ \star & \text{if } \in a''(z) \end{cases}$$

Thus, to show that (6) holds, need to show that (for any $b \in A$ and $z \in X$):

$$x \in a''(z) \implies z(y^o ba)\star \ \text{ i.e. } y \in (ba)(z) \tag{4}$$

To show that (7) holds, fix any $z \in X : x \in a''(z)$. Also, by our choice of y, have that $y \in a''(x)$. And as $b \in A$, it's reflexive, giving that $y \in b(y)$. So, by composition of relations, we get:

$$za''x \ , \ xa''y \text{ and } yby \implies z(a''a''b)y \implies z(ab)y \text{ i.e. } y \in (ba)(z)$$

(b) By using Proposition2.3 (a), need to show that $A \geq y_X^o.\tilde{A}.y_X$ i.e. $\forall a \in A, \exists \tilde{b} \in \tilde{A} : a \supseteq y_X^o \ \tilde{b} \ y_X$. Applying to an element, $x \in X$ gives the condition

$$\Big(y_X^o \ \tilde{b} \ y_X\Big)(x) \subseteq a(x) \implies \Big(y_X^o \ \tilde{b}\Big)(x^*) = y_x^o\Big(\tilde{b}(x^*)\Big) \subseteq a(x) \tag{5}$$

Thus, if $y^* \in PX$ such that $x^*\tilde{b}y^*$, then $y \in y_x^o\Big(\tilde{b}(x^*)\Big)$. Now, for (8) to hold, $y \in a(x)$ i.e. xay. Thus, need only to show that for any $a \in A, \exists b \in A$ such that $\forall x, y \in X, x^*\tilde{b}y^* \implies xay$. So, fix $a \in A$, and take $b \in A : bb \subseteq a$. Now, let $x^*\tilde{b}y^*$ i.e. $x^o.A \leq y^o.A.b$. Hence, $\exists c \in A : x^o c \subseteq y^o bb$. And as c is reflexive,

$$xcx \implies x(cx^o)\star \implies x(bby^o)\star \implies x(bb)y \implies xay$$

□

Theorem 4.4 (Yoneda Lemma)**.** *For every $\psi \in PX$, in the following digram,*

(a) $\psi \geq \psi^*.(y_X)^*$

(b) $\psi \in \overline{y_X(X)} \implies \psi \leq \psi^*.(y_X)_*$

Proof. (a) By definition, $(y_X)_* = \tilde{A}.y_X$, and $\psi^* = \psi^o.\tilde{A}$. Need that $\psi \geq (y_X)_*.\psi^* = \psi^o.\tilde{A}.\tilde{A}.y_X$. And applying Lemma 2.2.1 to $\tilde{A}$, the required condition becomes $\psi \geq \psi^o.\tilde{A}.y_X$ Fix $p \in \psi$, we will find $a \in A : p \supseteq \psi^o a y_X$. Examining the right side of the condition, (for any $a \in A$, $x \in X$)

$$\Big(\psi^o.\tilde{a}.y_X\Big)(x) = \psi^o.\tilde{a}(x^*) = \psi^o\big(\tilde{a}(x^*)\big) = \begin{cases} \phi & \text{if } \psi \notin \tilde{a}(x^*) \\ \star & \text{if } \psi \in \tilde{a}(x^*) \end{cases} \tag{1}$$

In case $\psi \notin \tilde{a}(x^*)$, the condition holds trivially. As ψ is a promodule, $\psi.A \leq \psi$ gives $\exists q \in \psi, a \in A : qa \subseteq p$. Thus, fix $x \in X$ and $\psi \in PX$ such that $x^*\tilde{a}\psi$. We will now show that $xp\star$. Using the definition of $\tilde{a}$,

$$x^*\tilde{a}\psi \implies x^o.A \leq \psi.a \implies \exists b \in A : x^o b \subseteq qa \implies \forall z \in X, \big(x^o b\big)(z) \subseteq (qa)(z) \tag{2}$$

Thus, in particular for $z = x$, as b is reflexive, xbx, which gives:

$$\big(x^o b\big)(x) \subseteq (qa)(x) \implies x^o x \subseteq (qa)(x) \implies \star \in (qa)(x) \tag{3}$$

But, as $qa \subseteq p$, (11) gives that $xp\star$.

(b) Suppose $\psi \in \overline{y_X(X)}$, need to show $\psi \leq \psi^*.(y_X)_* = \psi^o.\tilde{A}.y_X$ i.e. for $a \in A$, $\exists p \in \psi : p \subseteq \psi^o.\tilde{a}.y_X$. For any $x \in domain(p)$, the condition requires:

$$p(x) \subseteq \psi^o.\tilde{a}.y_X(x) = \psi^o\big(\tilde{a}(x^*)\big) \tag{4}$$

By definition of p, for (12) to hold, need that $xp\star \implies \psi \in \tilde{a}(x^*)$. Fix any $a \in A$, we will find $p \in \psi$ such that (12) holds. By quasi-uniformity of A, $\exists b \in A : bb \subseteq a$. From Proposition 2.5(a), y_X is uniformly continuous, $y_X.A \leq \tilde{A}.y_X$ giving that $\exists c \in A : y_x c \subseteq \tilde{b} y_X$. Thus, for any $z, w \in X$ such that zcw,

$$\big(y_X c\big)(z) \subseteq \big(\tilde{b} y_X\big)(z) \implies y_X(c(z)) \subseteq \tilde{b}(z^*) \implies w^* \in \tilde{b}(z^*) \text{ i.e. } z^*\tilde{b}w^* \tag{5}$$

As A is a quasi-uniformity, $\exists d \in A : dd \subseteq c$. Also, because A is a down- directed set, $\exists a' \in A : a' \subseteq b, d$. This along with (13) gives that for any $x, y \in X$

$$x(a'a')y \implies x(dd)y \implies xcy \implies x^*\tilde{b}y^* \tag{6}$$

Now, because $\psi \in \overline{y_X(X)}$, we get $\exists x^* \in y_X(X)$ such that $\psi\tilde{a'}x^*$ and $x^*\tilde{a'}\psi$. By definition of $\tilde{a}$, $\psi\tilde{a'}x^*$ gives

$$\psi \leq x^o.A.a' \implies \exists p \in \psi : p \subseteq x^o a' a' \tag{7}$$

Fix any $z \in X : zp\star$, using (15) and (14) gives:

$$zp\star \overset{z}{\Rightarrow} (x^o a' a')\star \overset{(15)}{\Longrightarrow} z(a'a')x \overset{(14)}{\Longrightarrow} z^*\tilde{b}x^* \tag{8}$$

Finally, by definition of the partial order on $\tilde{A}$,$a' \subseteq b \implies \tilde{a'} \subseteq \tilde{b}$. Therefore, $x^*\tilde{a'}\psi \implies x^*\tilde{b}\psi$. Now, using (16), $z^*\tilde{b}x^*$ and $x^*\tilde{b}\psi$ gives the desired result $z^*\tilde{b}x^*$. ☐

Lemma 4.5. Composition of a right-adjoint and an equivalence is a right adjoint.

Corollary 4.6. *For $\psi \in PX$, $\psi \in \overline{y_X(X)}$ if and only if ψ is a right-adjoint.*

Proof. Fix any $\psi \in PX$.

(i) ($\implies$) Let $\psi \in \overline{y_X(X)}$, from Theorem 4.4, we get that $\psi = \psi^*.(y_X)_*$. In order to show ψ is a right-adjoint, by using Lemma 4.5, it is enough to show that ψ^* is a right adjoint and that $(y_X)_*$ is an equivalence.

I In order to show that $(y_X)_*$ is an equivalence, we need that $A = (y_X)^*.(y_X)_*$ and $\tilde{A} = (y_X)_*.(y_X)^*$. From proposition 4.3 (b), we have that y_X is fully faithful, and by Proposition 3.13 (a), this gives us that $A = (y_X)^*.(y_X)_*$.

- We are now going to show that $\tilde{A} \leq (y_X)_*.(y_X)^*$. Fix any $a, b \in A$, we need to find $c \in A$ such that $\tilde{c} \subseteq \tilde{a}\, y_X\, y_X^o\, \tilde{b}$.

$$\big(\tilde{a}\, y_X.y_X^o\, \tilde{b}\big)(\psi) = \big(\tilde{a}\, \tilde{b}\big)(\psi) \supseteq \tilde{c}\tilde{c}(\psi) \supseteq \tilde{c}(\psi)$$

  In the above equation, the equality holds because $\psi \in \overline{y_X(X)}$, gives the existence of $x^* = \tilde{b}(\psi)$. And the first inequality is given by down-directedness of $\tilde{A}$, whereas the second one holds because $\tilde{c}$ is reflexive, as $\tilde{A}$ is a quasi-uniformity.

- To show that $\tilde{A} \geq (y_X)_*.(y_X)^*$, fix any $a \in A$. By quasi-uniformity of $\tilde{A}$, there exists $\tilde{b} \in \tilde{A}$ such that $\tilde{b}\,\tilde{b} \subseteq a$. We will show that $\tilde{a} \supseteq \tilde{b}\, y_X\, y_X^o\, \tilde{b}$.

$$\psi\big(\tilde{b}\, y_X\, y_X^o\, \tilde{b}\big)\phi \implies \psi\big(\tilde{b}\tilde{b}\big)\phi \implies \psi\tilde{a}\phi$$

II In order to show that ψ^* is a right adjoint to ψ_*, we need to show that $\tilde{A} \geq \psi_\star.\psi^\star$ and $\psi_\star.\psi^\star \geq 1$.

- To show that $\tilde{A} \geq \underline{\psi_*.\psi^*} = \psi_*.\psi^o.\tilde{A}$, fix any $a \in A$. We will show that $\psi_*.\psi^o.\tilde{a} \subseteq \tilde{a}$. Using definition of ψ_*, for any $\phi \in y_X(X)$, we get that

$$\big(\psi_*.\psi^o.\tilde{a}\big)(\phi) = \psi_*.\psi^o(\tilde{a}(\phi)) = \begin{cases} \phi & \text{if } \tilde{a}(\phi) \neq \psi \\ \psi = \psi_*.\psi^o(\psi) & \text{if } \tilde{a}(\phi) = \psi \end{cases}$$

  The above equation gives that $\phi\big(\psi_*.\psi^o.\tilde{a}\big)\psi$ implies $\phi\tilde{a}\psi$.

- We will show that $\psi_\star.\psi^\star \geq 1$, that is $\star(\psi^o.\tilde{a}.\psi_*)\star$. Using definition of ψ_*,

$$\big(\psi^o.\tilde{a}.\psi_*\big)(\star) = \big(\psi^o.\tilde{a}\big)(\psi_*(\star)) = \big(\psi^o.\tilde{a}\big)(\psi) = \psi^o\big(\tilde{a}(\psi)\big)$$

  By the quasi-uniformity of $\tilde{A}$, we get that $\tilde{a}$ is reflexive, and hence, $\psi\tilde{a}\psi$. So, from the above equation, we have that $\star \in \psi^o(\psi) \subseteq \big(\psi^o.\tilde{a}.\psi_*\big)(\star)$.

(ii) ($\Longleftarrow$) Suppose ψ is a right adjoint. Need to show that for any $a \in A$, $\exists x^* \in y_X(X)$ such that $\psi\, \tilde{a}\, x^* \tilde{a}\, \psi$. Fix $a \in A$. Because ψ is a right-adjoint, there exists a promodule $\phi : 1 \nrightarrow X$ such that $\phi.\psi \leq A$ and $1 \leq \psi.\phi$. From $\phi.\psi \leq A$, we get that

$$\exists p \in \phi, q \in \psi \text{ such that } a \supseteq p.q \tag{1}$$

Because ϕ and ψ are promodules,

$$A.\phi \leq \phi \text{ gives the existence of } p' \in \phi \text{ such that } p \supseteq a'p' \tag{2}$$
$$A.\psi \leq \psi \text{ gives the existence of } q' \in \psi \text{ and } a'' \in A \text{ such that } q \supseteq a''q' \tag{3}$$

Now, from $1 \leq \psi.\phi$, we get that $q'\, p'$ is reflexive i.e. $\star(q'\, p')\star$. By the definition of composition we get the existence of an $x \in X$ such that $\star\, p'\, x\, q'\, \star$. Now, considering x as a map, $x : 1 \to X$ defined as $\star \mapsto x$,

$$x\, q'\, \star \;\text{ i.e. } \star \in q'(x) \text{ gives that } q' \supseteq x^o \tag{4}$$
$$\star\, p'\, x \text{ i.e. } x \in p'(\star) \text{ gives that } p' \supseteq x \tag{5}$$

Thus, by using inequalities (1),(2) and (3), we get that

$$a \supseteq p\, q \supseteq a'\, p'\, q'\, a'' \tag{6}$$

By definition of $\tilde{a}$, to show $\psi\, \tilde{a}\, x^*$, we need that $\psi \leq x^*\, a = x^o.\, A.\, a$. Showing for any $b \in A$, $x^o\, b\, a \supseteq q'$:

$$x^o\, b\, a \supseteq x^o\, b\, a'\, p'\, q' \supseteq x^o\, b\, a'\, x\, q' \supseteq x^o\, x\, q' \subseteq q'$$

Where the first inequality comes from (6) by using reflexiviness of a'' and then left-multiplying by x^o. The second inequality comes from (5), third one from reflexiviness of b and a', and the last one is given by Lemma 2.4.1.

In order to show $x^*\, \tilde{a}\, \psi$, by definition of $\tilde{a}$, need that $x^o.A = x^* \leq \psi\, a$. Fix $k \in \psi$. We will show $k\, a \supseteq x^o\, a''$.

$$a \supseteq a'\, p'\, q'\, a'' \supseteq p'\, q'\, a'' \supseteq p'\, x^o\, a'' \tag{7}$$

Where the first inequality is given by (6), second one is due to reflexiviness of a' and the third inequality comes by using (4). Left-multiplying (7) with k gives the following.

$$ka \supseteq k\, p'\, x^o\, a'' \text{ that is, for any } z \in X,\;\; z(k\, a)\star \implies z(k\, p'\, x^o\, a'')\star \tag{8}$$

As ψ is a right adjoint to ϕ, we have $1 \leq \psi.\phi$, giving that $\star(k\, p')\star$. So, using the implication in(8), we get that $z(k\, a)\star$ implies $z(x^o\, a'')\star(k\, p')\star$, which in turn gives that $z(x^o\, a'')\star$. Hence, we get that $ka \supseteq x^o\, a''$ □