1 Definitions

Definiton 1.1 (Prorelation)**.** A pre-ordered set of relations $X \to Y$, which is down-directed and an upper set. i.e A set, $P \subseteq \mathcal{P}(X \times Y)$ such that

(i) A pre-order defined to be containment as relations, $r \subseteq s$ only if $\forall (x, y) \in X \times Y$, $(x, y) \in r \implies (x, y) \in s$

(ii) (Down-directed), $\forall r, s \in P, \exists t \in P$ such that $t \subseteq r$ and $t \subseteq s$

(iii) (Up-set) for any relation $u : X \to Y$, if $\exists p \in P$ such that $p \leq u$ then $u \in P$

Definiton 1.2 (Composition of prorelations)**.** Prorelations can be composed by taking all compositions of their elements as relations: for prorelations $P : X \to Y$ and $Q : Y \to Z$,

$$Q.P := \{q \circ p : p \in P \text{ and } q \in Q\}$$

Definiton 1.3 (Comparison of Prorelations)**.** Two prorelations with same domain, co-domain are comparable as

$$\text{for } P, Q : X \to Y \text{ , } P \leq Q \text{ if } \forall q \in Q, \exists p \in P \text{ such that } p \subseteq q$$

Definiton 1.4 (Quasi-uniformity)**.** A prorelation on a set X, $P : X \to X$ is a quasi-uniformity if it follows :

i $\forall p \in P$, for any $x \in X$, $(x, x) \in p$ i.e. xpx

ii $\forall p \in P, \exists p' \in P$ such that $p' \circ p' \subseteq p$

And in this case, (X, A) is called a *quasi-uniform space*.

Definiton 1.5 (Uniformly Continuous function)**.** A function, $f : X \to Y$ is called a uniformly continuous function, $f : (X, A) \to (Y, B)$ if, $\forall b \in B, \exists a \in A$ such that $f \circ a \subseteq b \circ f$. meaning that $f.A \leq B.f$ or
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "A"'] \arrow[dr, phantom, "\leq"] & Y \arrow[d, "B"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}.$$

Definiton 1.6 (Promodule)**.** A prorelation, $\phi : X \mathrel{-\!\!\circ\!\!\!\rightarrow} Y$ is called a promodule $\phi : (X, A) \mathrel{-\!\!\circ\!\!\!\rightarrow} (Y, B)$ if it obeys: $\phi.A \leq \phi$ and $B.\phi \leq \phi$ where . denotes composition as prorelations.

Definiton 1.7 (Comparison of Promodules)**.** Promodules with same domain and co-domain are compared as prorelations, for $\phi, \psi : (X, A) \mathrel{-\!\!\circ\!\!\!\rightarrow} (Y, B)$, $\phi \sqsubseteq \psi$, only if $\phi \leq \psi$.

Definiton 1.8 (Composition of Promodules)**.** Promodules are composed as prorelations.
For promodules $\phi : (X, A) \mathrel{-\!\!\circ\!\!\!\rightarrow} (Y, B)$ and $\psi : (Y, B) \mathrel{-\!\!\circ\!\!\!\rightarrow} (Z, C)$, $\psi\phi := \psi.\phi = \{q \circ p : p \in \phi \text{ and } q \in \psi\}$

Definiton 1.9 (Opposite relation)**.** For relation $r : X \to Y$, r^o is defined to be a relation $r^o : Y \to X$ as

$$\forall (x, y) \in X \times Y, (x, y) \in r \iff (y, x) \in r^o$$

Definiton 1.10 ($(-)_*$)**.**

Definiton 1.11 ($(-)^*$)**.**

Definiton 1.12 (Fully Faithful)**.**

Definiton 1.13 (Fully Dense)**.**

Definiton 1.14 (Topologically Dense)**.**

2 Propositions

Definiton 2.1 (QUnif)**.** QUnif is defined to be the category having quasi-uniform spaces as objects, and uniformly continous maps between them as morphisms.

Lemma 2.1.1. QUnif does define a category, as

i Composition

ii Identity

Definiton 2.2 (ProMod)**.**

Lemma 2.2.1. QUnif does define a category, as

i Composition

ii Identity

Proposition 2.1 (Promodule Category)**.**

Proposition 2.2 ($(-)_*$ is a Functor)**.**

Proposition 2.3 ($(-)^*$ is a Functor)**.**

Proposition 2.4 (Proposition 1)**.** Fix a uniformly continuous map, $f : (X, A) \to (Y, B)$

(a) f is fully faithful $\iff A = f^o.B.f$

(b) f is fully dense $\iff \forall b \in B, \exists b' \in B$ such that asd