

1

Theorem 1.1. Yoneda If \mathcal{A} be a locally small category, then,

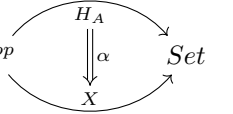
$$[\mathcal{A}^{op}, Set](H_A, X) \cong X(A) \text{ naturally in } A \in \mathcal{A} \text{ and } X \in [\mathcal{A}^{op}, Set]$$

Explanation:

First of all, we fix any category, \mathcal{A} . Now we choose two things (independent of each other):

- i an object, A from the category $\mathcal{A} = \mathcal{A}^{op}$
- ii an object, X of the category $[\mathcal{A}^{op}, Set]$ which is precisely a functor $X : \mathcal{A}^{op} \rightarrow Set$

Here, $[\mathcal{A}^{op}, Set](H_A, X)$ denotes arrows $H_A \rightarrow X$ in $[\mathcal{A}^{op}, Set]$ i.e. natural transformations, $\alpha : H_A \rightarrow X$



Each of these natural transformations is a collection of maps in Set , hence each of their components is exactly a function. i.e. $\forall \alpha \in [\mathcal{A}^{op}, Set](H_A, X), \forall K \in \mathcal{A}, \alpha_K$ is a function $: H_A(K) \rightarrow X(K)$

$X(A)$ is precisely a set, because $X(A)$ is the image of (our chosen object,) A , under (our chosen functor,) X .

So, the idea is that our choice of A and X completely determines all possible maps (i.e. natural transformations) from H_A to X . This answers out big question of "what are all the maps $H_A \rightarrow X$ " or "how does H_A see other presheaves on \mathcal{A} ".

*The theorem says not just that the two are isomorphic, but that they're **naturally** isomorphic.*

This means that $[\mathcal{A}^{op}, Set](H_A, X)$ and $X(A)$ are *functorial* in both A and X

Also, So, the aforementioned collection of natural transformations also must be a set: As \mathcal{A} is locally small, for each choice of $K \in \mathcal{A}$, $H_A(K) = Hom_{\mathcal{A}}(K, A)$ is a set. Thus, as α_K is a function, and hence a relation, it's a subset of $Hom_{\mathcal{A}}(K, A) \times X(K)$, which is a set as it's a cartesian product of sets. Thus, K -component of every natural transformation is a set. i.e. $\forall \alpha \in [\mathcal{A}^{op}, Set](H_A, X), \forall K \in \mathcal{A}, \alpha_K$ is a set

Now, the question is whether the set of all natural transformations is a set

Proof. Let a locally small category, \mathcal{A} be given.

Denoting the category of all presheaves on \mathcal{A} by \mathcal{C} , i.e. $\mathcal{C} := [\mathcal{A}^{op}, Set]$

Now, fix any object, $A \in \mathcal{A}$, and any object, $X \in \mathcal{C}$.

Need to show that $\mathcal{C}(H_A, X)$ is naturally isomorphic to $X(A)$.

Thus, need two mutually inverse natural transformations, such that : $\mathcal{C}(H_A, X) \xrightleftharpoons[\phi]{\psi} X(A)$

Need to show that the RHS is a set, and then, As both the RHS and LHS above are sets, the natural transformations between them are maps in Set i.e. functions.

Going to show the following in order,

- i Define ϕ and ψ
- ii Show that ϕ and ψ are mutually inverse
- iii Show naturality of ϕ in X
- iv Show naturality of ϕ in A
- v Show naturality of ψ in X
- vi Show naturality of ϕ in A

Defining ϕ and ψ Define ϕ (on natural transformations) as : for $\alpha \in \mathcal{C}(H_A, X), \phi(\alpha) := \alpha_A(1_A)$

Define ϕ on an object, $x \in X(A)$, by defining it's K -component for any $K \in \mathcal{A}$: $(\psi(x))_K$

□