Deflecting an asteroid

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1 Abstract

All the calculations and simulations described below are available in this Desmos spreadsheet. You can try out the entire strategy by clicking on the “play” button in the uppermost cell that handles time.

2 Introduction

A problem like this is a topic of many action movies of today’s entertainment industry. A huge asteroid approaching the Earth, potentially killing millions, is a threat to humanity and there’s a single group of scientists, probably somewhere in Florida, Washington or Moscow, that miraculously pull off an engineering feat that saves us all. Or at least that’s how I imagine action movies.

The problem I’ll be dealing with is of a very similar nature – an asteroid with certain properties is approaching the Earth and it’s up to me to come up with a way to deflect it. And provide the necessary calculations, too. More to that in the Problem Definition here.

I’d first define the variables I’ll be working with in order to better introduce the problem and be able to make a reference to them further in the paper.

3 Variables

R_E = the Radius of the Earth
R_{ef} = the Effective Radius of the Earth. Hitting the Earth itself is one thing (more to that later), but that can be as detrimental as having the asteroid traverse really close to Earth’s surface. Hence, I’ll define R_{ef} as $(1+R_{ratio})R_E$, where R_{ratio} is a cushion representing the atmosphere / gravitational field / etc. Let R_{ratio} be 0.3.
E = the Earth. For the purpose of this problem, it will be a curve defined by $(x - E_c x)^2 + (y - E_c y)^2 = R_E^2$ where E_{cx} and E_{cy} can be used to offset the Earth. With default values of E_{cx} and E_{cy} being 0 and 0, the center of the Earth will also be the center of our 2d plane.

E_{ef} = the Effective Earth. This is the object we will be making the asteroid avoid. Again, it will be a curve defined by $(x - E_{cx})^2 + (y - E_{cy})^2 = (R_{ef})^2$.
d = the initial distance between the asteroid and the Earth. It makes sense to define d as a multiple of R_E. In this case, I chose 100 as the default value, but it is easily changeable in the Desmos spreadsheet of the simulation.
A = the Asteroid. The position of it's center will be given by functions of time $(a_x(t), a_y(t))$. With a radius of R_a, the Asteroid will also be represented by a curve defined by $(x - a_x(t))^2 + (y - a_y(t))^2 = R_A^2$. The functions of positions are defined as $a_x(t) = x_0 + v_0 t + at^2 = -d + v_{0x} t$ and $y_x(t) = 0$ (before the collision)

4 "Hitting vs not hitting" or "How do I know I survived"?

The deflection was successful if the asteroid had not hit the Earth. As we are talking about two curves, one with a dynamic center (the Asteroid) and the other with a static (the Earth), if there's a collision, there must have been a time, in which both the curves shared at least one point.

So, in terms of mathematics, there must exist at least one t $\in (0, t_{max})$ such that

$$(x - a_x(t))^2 + (y - a_y(t))^2 = R_A^2$$
$$(x - E_{cx})^2 + (y - E_{cy})^2 = (R_{ef})^2.$$

there's a solution $s = (x_1, y_1)$ to this system of equations. If such a t and s exist, we can consider the two objects to have collided and therefore we failed our task.

Consequently, if s does not exist for any t $\in (0, t_{max})$, we can conclude that the deflection was successful and we saved the Earth.

5 The plan

I was thinking of multiple ways of deflecting the Asteroid. I'd generally say there are 2 distinct approaches: either hit the Asteroid head first (which sounds like it would require a lot of resources) or try to hit it from the side, steering it away from its original path. The first approach also requires us to completely stop / destroy / explode the Asteroid, and it being a pretty heavy object (with a mass of m_1!) with high velocity (v_{0x}!), we would need to supply the same amount of momentum as it has – which is a lot. So I resorted to the other one, as that seems far more realistic and will most likely not be as resource heavy.

The idea is to hit the Asteroid from the side as it travels towards the Earth. This way, not as much momentum will be needed in order to save the Earth. The rough visualization of this idea is provided below.

We can see that at any point of time t, we can draw a line between the current position of the Asteroid and the uppermost point of our Effective Earth. This is

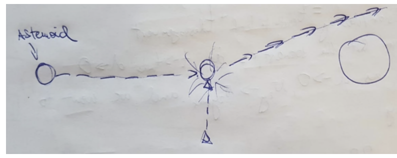

Figure 1: Rough sketch of the plan

the path the Asteroid will need to take if we want it to miss the Earth – let's call it l. l holds an angle α_{min} with the line $y = 0$, the initial path of our asteroid. High-level, our goal will be to deflect the asteroid by at least α_{min} degrees at the time of collision. The important thing to notice is that as $t \rightarrow t_{collision}$, $\alpha_{min} \rightarrow 90\,\text{deg}$. Similarly, as $\alpha_{min} \rightarrow 90\,\text{deg}$, the momentum we will need to provide to deflect the Asteroid by at least α_{min} deg, p increases. Hence, the best solution in terms of raw momentum is for us to perform the deflection as soon as possible / as far away from the Earth as possible, as that way we will need to deliver the least momentum.

The digital visualization of this concept:

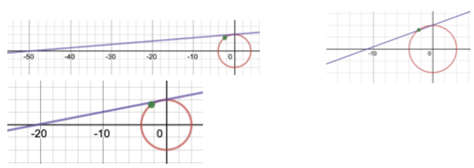

Figure 2: A digital visualization of the plan

Note: It would maybe make more sense to set the anchor point of deflection not at $(0, R_e)$ as is demonstrated here, but rather at $(cos(45)R_e, sin(45)R_e)$, because of the possible collisions that may happen if d is small. Having this limitation in mind, we will proceed further.

The line l is described by $y = tan(c)(x + a_x)$, where $c > \alpha_{min}$. α_{min} is essentially the ratio of the height (y) and width (x) of the triangle between the Asteroid, center of the Earth and the uppermost point of the Earth. Hence, $\alpha_{min} = atan(\frac{y}{x}) = atan(\frac{R_{ef}}{a_x})$

From here we can also see that as $a_x \rightarrow -\infty, c \rightarrow 0$. Therefore, as mentioned previously, the further away from Earth we perform the deflection, the less momentum will we need as the angle will be lower.

6 The execution

6.1 Getting our projectile launcher to our desired position.

As mentioned, my plan will be to vertically launch a projectile that will collide with the asteroid on its way to Earth and deflect it by such an angle, that the new path of the Asteroid will barely miss the Effective Earth. In order to do this, we first need to get our projectile launcher to such a position from which this will be possible.

We will be vertically launching the projectile from (z_x, z_y) as shown below. We will time our launch it in such a way that the projectile will hit the asteroid right as it travels on its y = 0 path and is at the point $(z_x, 0)$.

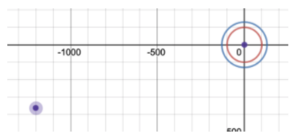

Figure 3: Z mid travel to Z_{launch}

I don't really want to get into the technicalities of this plan, but essentially the way i imagine it would be that a space device would bring the projectile (payload) to (z_x, z_y), where with the help of its engines it would launch the projectile on its desired path. Something like bringing a catapult to (z_x, z_y) and shooting the rock from there.

Let's call the space device Z. Getting Z to $Z_{launch} = (z_x, z_y)$ requires some small calculations, but essentially is a simple 2d motion problem. We will launch Z from Earth with the velocity l_v, therefore it's initial vertical and horizontal velocities will be defined by

$$l_{v0x} = \frac{z_x}{z_d} l_v$$

$$l_{v0y} = \frac{z_y}{z_d} l_v$$

Where z_d is the hypotenuse of our imaginary triangle $Z_t = ((0, 0), Z_{launch}, (0, z_y))$, $z_d = \sqrt{z_x^2 + z_y^2}$. We could also define the x and y components in terms of sines and cosines, but the above implementation works too.

The time it takes for Z to reach the point of launch will be easily determined as it is a linear, non accelerated motion.

$$t_z = \frac{z_d}{l_v}$$

.

Remember, we want to hit the asteroid as soon as possible. There's a certain optimization problem that I will later discuss, but so far we know that the earliest we can launch the deflection projectile will be after t_z "time".

6.2 Projectile travel

Now, naturally it's going to take some time for the projectile to travel after we launch it from Z_{launch} to the point of collision at $(z_x, 0)$. This time is dependent on the distance, which is just $-z_y$ and the launch velocity of our projectile v_{2y}, to which we will get soon. The travel time of the projectile itself will be then just $t_{dvert} = -\frac{z_y}{v_{2y}}$

It's important to note that v_{2y} can be viewed as a function of time or, similarly, the current asteroid's distance from z_x. More onto that later in the chapter about the collision itself.

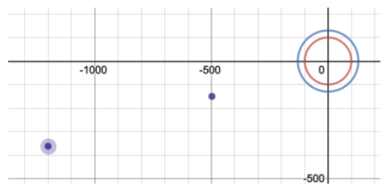

Figure 4: Z mid travel to Z_{launch}

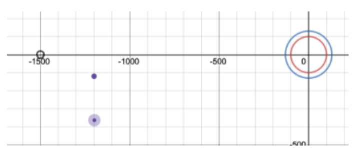

Figure 5: Z launched its projectile just in time for it to collide with the asteroid. Picture taken soon before the collision, circa 1978 (colorized).

6.3 The collision

This is by far the most confusing / explicit / mathematics-is part of the entire problem. I decided to approach it using momenta, as that seemed as the most straightforward option for me and for the purposes of this problem.

As we learnt in the book, momentum p is defined as $p = mv$. Moreover, we also know that it works very well in 1, but also multiple dimensions too. As momentum is a vector entity, we can decompose it into its respective x and y components. Working with the law of conservation of momentum, we know that

$$p_i = p_f =>$$
$$\sum p_{ix} = \sum p_{fx}$$
$$\sum p_{iy} = \sum p_{fy}$$

.

In order to simulate our collision, we then only need to know the velocities of the both of our projectiles and their masses. For the purposes of further calculations, let's label the asteroid as 1 and our projectile as 2. We know that

$$p_{1x} + p_{2x} = p'_{1x} + p'_{2x}$$
$$p_{1y} + p_{2y} = p'_{1y} + p'_{2y}$$

where p' is the final momentum after collision and subscripts 1 and 2 respectively denote the asteroid and the projectile.

$$m_1 v_{1x} + m_2 v_{2x} = m_1 v'_{1x} + m_2 v'_{2x} \tag{1}$$
$$m_1 v_{1y} + m_2 v_{2y} = m_1 v'_{1y} + m_2 v'_{2y} \tag{2}$$

2 – our projectile – will have no initial x velocity as we are launching it vertically, hence $v_{2x} = 0$

$$m_1 v_{1x} = m_1 v'_{1x} + m_2 v'_{2x}$$

Similarly, 1 – the asteroid – has no initial y velocity, as it only moves along the x-axis on $y = 0$ line, hence $v_{1y} = 0$

$$m_2 v_{2y} = m_1 v'_{1y} + m_2 v'_{2y}$$

Let's translate the x and y components of our velocities into a "common velocity" v'_1 and v'_2. We know that $v_{ny} = v_n sin(\alpha)$ and $v_{nx} = v_n cos(\alpha)$. Hence

$$m_1 v_{1x} = m_1 v'_1 cos(\alpha_1) + m_2 v'_2 cos(\alpha_2) \tag{1}$$
$$m_2 v_{2y} = m_1 v'_1 sin(\alpha_1) + m_2 v'_2 sin(\alpha_2) \tag{2}$$

Where α_1 and α_2 denote the angles with which our projectiles will "bounce off", as shown in the pic.

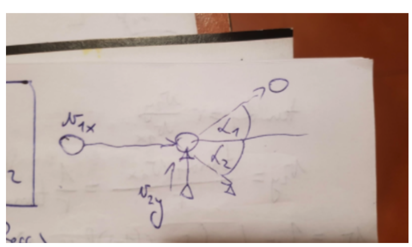

Figure 6: Visualization of α_1 and α_2.

Now this is where things start to get a little sketchy. I believe we can freely decide to have the projectile destroyed upon impact and therefore let $\alpha_2 = 90$ degrees. Followingly, $cos(\alpha_2) = 0$, $sin(\alpha_2) = 1$

$$m_1 v_{1x} = m_1 v_1' cos(\alpha_1) \tag{1}$$
$$m_2 v_{2y} = m_1 v_1' sin(\alpha_1) + m_2 v_2' \tag{2}$$

Similarly, let's assume our projectile will transfer all of its momentum and destroy itself after the collision, therefore let $v_2' = 0$.

$$m_1 v_{1x} = m_1 v_1' cos(\alpha_1) \tag{1}$$
$$m_2 v_{2y} = m_1 v_1' sin(\alpha_1) \tag{2}$$

From (1) and (2) we see that

$$cos(\alpha_1) = \frac{v_{1x}}{v_1'} \tag{3}$$
$$sin(\alpha_1) = \frac{m_2 v_{2y}}{m_1 v_1'} \tag{4}$$

Dividing (4) by (3) gives us

$$tan(\alpha_1) = \frac{m_2 v_{2y} v_1'}{m_1 v_1' v_{1x}} = \frac{m_2}{m_1} \frac{v_{2y}}{v_{1x}} \tag{5}$$

The ratio between the masses of the asteroid and our projectile seems like a good constant to define, so let's do $r = m_2/m_1$

$$tan(\alpha_1) = r \frac{v_{2y}}{v_{1x}} \tag{5}$$

Looking at the last picture, we see that α_1 is actually the desired angle $\alpha_{mi}n$ from "The plan" part. Let's recall that

$$\alpha_{min} = atan(\frac{R_{ef}}{a_x})$$

As we are looking to know the required launch velocity of our projectile, we can rearrange (5) to get an expression for v_{2y}.

$$v_{2y} = \frac{1}{r} tan(\alpha_1) v_{1x}$$

Plugging α_{min} for α_1 gives

$$v_{2y}(t) = \frac{v_{1x}}{r} \frac{R_{ef}}{a_x(t)} = \frac{v_{1x}}{r} \frac{R_{ef}}{-d + v_1 x t}$$

7 Values

In order to get some tangible values, let's plug in some values now.

m_1 = the mass of asteroid = $1500 * 10^{15}$ kg. From this NASA webpage about asteroids, it seems that most of them are in the range $k * 10^{15}$ kg, where $k \in (0.01, 900000)$. I'll select k to be 1500, similarly to asteroid Siwa. (Dr. David R. Williams, 2019)

m_2 = the mass of our projectile = $25 * 10^3 kg$ Judging from the masses of payloads of SpaceX Falcon 9 shuttle and NASA's Discovery Space Shuttle, respectively $22800kg$ and $26000kg$, our current realistic payload mass for bringing something to the Low Earth Orbit would be around $25000kg$. (Wikipedia contributors, 2020b) I'm sure that for the purposes of saving the Earth, we'd probably be able to do much more, but let's just stay with this number for now and we can reevaluate it further down the road if needed.

v_{1x} – "The mean collision velocity between an asteroid and the Earth is found to be $20.8km/s$", so I will use that as the velocity of the asteroid. (Harris & Hughes, 1994)

l_v - Launch velocity of the Z device. Apparently, the Apollo 10 capsule has managed to hit $39897km/h$ on their mission to the Moon. Voyager 1, the furthest man made object has clocked in a velocity of about 61197 kilometers per hour (38,026 mph) as of 2013. (Wikipedia contributors, 2020c) Let's use that for the velocity with which the Z can travel to s. These velocities, however, do not really fit into this model, so I will rather use an escape velocity on Earth, which is approximately $11.8km/s$ (Wikipedia contributors, 2020a).

7.1 Results

As described before, the velocity v_{2y} with which we need to launch the projectile to successfully deflect the asteroid is a function of time. Therefore, the closer the asteroid is to Earth, the more velocity we will need in order to perform the deflection. There are, however, still 2 points worth mentioning – the earliest and the latest we can fire our projectile ($t_{earliest}, t_{latest}$). Using our Desmos spreadsheet, we can easily figure the values for $v_{2y} = \frac{v_{1x}}{r}(\frac{R_{ef}}{-d+v_{1x}t})$.

$$v_{2y}(t_{earliest} \approx 12000) = 2.925114800410^{13} km/s$$

$$v_{2y}(t_{latest} \approx 22000) = 6.316552250210^{13} km/s$$

Both of these values are greater than the speed of light, so with the current configuration, it's not really plausible for us to deflect the asteroid. As we can notice, as all of the relationships within momenta are linear, with the masses being approximately 12 "degrees" apart (10^3 vs 10^{15}), so is the resulting velocity. Even though the current result is very unlikely physics wise, I think it makes sense empirically.

Moreover, we can notice that the difference between launching the projectile as soon as possible and as late as possible is only by a factor of 2 and something. Both values stay within the range of 10^{13} – this implies that we can make the problem a bit easier to solve by launching the projectile sooner, but not by that much.

Plugging in some “closer” values for masses of both objects, such as the one from the asteroid Castalia, $0.0005 * 10^{15}$ (Dr. David R. Williams, 2019) and increasing our payload capacity to, let's say to $25 * 10^5$ kg yields

$$v_{2y}(t_{earliest} \approx 12000) = 97503.82 km/s$$

$$v_{2y}(t_{latest} \approx 22000) = 210551.741 km/s$$

For a given Δv, the minimum energy needed to accelerate by a Δv is $E = 0.772 m_1 (\Delta v)^2$. Plugging in these latest values from the more ideal scenario, the required energy will be

$$E = (0.772)(25 * 10^5)(97503.82)^2 = 1.8348500210^{16} J = 18348500200 MJ$$

Taking the fuel value of 10.1 MJ/kg per the spacecraft propulsion article again yields a required $1.81668319 * 10^9$ kg of spacecraft fuel needed in order to accelerate our device to the needed speed.

Time spent: Calculations & Desmos & Write-up: Around 7 hours. Typing: Like 2 hours.

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