Convex Analysis Workshop

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1. Convex function

1.1. Henmi definition

Let $S \subseteq \mathbb{R}^n$. A function $f : S \to \mathbb{R} \cup \{\infty\}$ is *convex* if **dom** f is a convex set and

$$\begin{gathered} \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbf{dom}\ f, \forall t \in [0,1], \\ f(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \leq t f(\boldsymbol{x}) + (1-t) f(\boldsymbol{y}) \end{gathered} \tag{1}$$

where **dom** f is the *effective domain* of f:

$$\mathbf{dom} f \coloneqq \{\boldsymbol{x} \in S \mid f(\boldsymbol{x}) < \infty\} \tag{2}$$

- If the inequality condition holds with "$<$" everywhere, f is said to be *strictly convex*.
- If **dom** $f \neq \emptyset$, f is said to be *proper*.

1.2. Boyd definition

A function $f : \mathbb{R}^n \to \mathbb{R}$ is *convex* if **dom** f is a convex set and if $\forall x, y \in$ **dom** f, and θ with $0 \leq \theta \leq 1$, we have

$$f(\theta x + (1-\theta) y) \leq \theta f(x) + (1-\theta) f(y) \tag{3}$$

2. What is pointwise maximum?

2.1. Pointwise

When concepts (properties, operations, etc.) on a set Y are extended to functions $f : X \longrightarrow Y$ by treating each function value $f(x)$ in isolation, the extended concept is often qualified with the word *pointwise*. One example is pointwise convergence of functions – a sequence$\{f_n\}_{n=1}^{\infty}$ of functions $X \longrightarrow Y$ converges pointwise to a function f if $\lim_{n\to\infty} f_n(x) = f(x)$, $\forall x \in X$.

2.2. Pointwise maximum (boyd definition)

If f_1 and f_2 are convex functions then their *pointwise maximum* f, defined by

$$f(x) = \max\{f_1(x), f_2(x)\}, \tag{4}$$

with **dom** $f =$ **dom** $f_1 \cap$ **dom** f_2, is also convex. This property can be verified: If $0 \leq \theta \leq 1$ and $x, y \in$ **dom** f, then

$$\begin{aligned} f(\theta x + (1-\theta)y) &= \max\{f_1(\theta x + (1-\theta)y), f_2(\theta x + (1-\theta)y)\} \\ &\leq \max\{\theta f_1(x) + (1-\theta) f_1(y), \theta f_2(x) + (1-\theta) f_2(y)\} \\ &\quad (\because f_1, f_2 \text{ are convex functions}) \\ &\leq \theta \max\{f_1(x), f_2(x)\} + (1-\theta) \max\{f_1(y), f_2(y)\} \\ &= \theta f(x) + (1-\theta) f(y) \end{aligned} \tag{5}$$

which establishes convexity of f. If $a \leq c$ and $b \leq d$, then $\max\{a, b\} \leq \max\{c, d\}$ **note**: If you compare the pointwise maximum of two or more functions with the maximum of each individual function, the pointwise maximum will generally not be smaller than the maximum of either individual function.

- Let f and g be two functions. Define $h(x) = \max(f(x), g(x))$ for each x.
- The maximum of h over its domain will be at least as large as the maximum of f and the maximum of g.

2.2.1. We need to show...

$$\text{If } a \leq c \text{ and } b \leq d, \text{then} \max\{a, b\} \leq \max\{c, d\} \tag{6}$$

$$\text{Let } \theta \in [0, 1], \text{then} \max\{\theta a + (1 - \theta)b, \theta c(1 - \theta)d\} \leq \theta \max\{a, c\} + (1 - \theta) \max\{b + d\} \tag{7}$$

2.2.2. Proof

Proof: In case $a \leq b, c \leq d$

Let$\theta \in [0, 1]$,

$$\max\{\theta a + (1 - \theta)b\}, \max\{\theta c + (1 - \theta)d\} \leq \theta \max\{a, c\} + (1 - \theta) \max\{b, d\} \tag{8}$$

$$1) a \leq c, b \leq d \tag{9}$$

$$2) c \leq a, b \leq d \tag{10}$$

$$3) a \leq c, d \leq b \tag{11}$$

$$4) c \leq a, d \leq b \tag{12}$$

2.3. General definition

Pointwise maximum can be extended by defining recursively. Let $f_1, f_2, ..., f_n$ $(n \in \mathbb{N}) : X \longrightarrow \mathbb{R} \cup \{\infty\}$ be extended to real-valued functions. Then the *pointwise maximum of* $f_1, f_2, ..., f_n$, denoted $\max\{f_1, f_2, ..., f_n\}$, is defined by:

$$\begin{gathered} \max\{f_1, f_2, ..., f_n\} : X \longrightarrow \mathbb{R} \cup \{\infty\} : \\ \max\{f_1, f_2, ..., f_n\}(x) := \begin{cases} f_1(x) & (n = 1) \\ \max\{\max\{f_1(x), f_2(x), ..., f_{n-1}(x)\}, f_n(x)\} & (n \geq 2) \end{cases} \end{gathered} \tag{13}$$

3. Pointwise maximum convexity

3.1. Proposition

Let $f_1, ..., f_m$ $(m \in \mathbb{N})$ be convex functions and let

$$f(\boldsymbol{x}) := \max_{i=1,...,m} f_i(x) \tag{14}$$

with $\mathbf{dom}\ f = \bigcap_{i=1}^{m} \mathbf{dom}\ f_i$. Then f is convex.

3.2. Proof

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{dom}\ f$ and $t \in [0, 1]$, Then we have

$$
\begin{aligned}
f(t\boldsymbol{x} + (1-t)\boldsymbol{y}) &= \max_{i=1,\ldots,m} f_i(tx + (1-t)y) \\
&\leq \max_{i=1,\ldots,m} (tf_i(x) + (1-t)f_i(y)) \\
&\quad (\because f_i \text{ are convex functions}) \\
&\leq \max_{i=1,\ldots,m} tf_i(x) + \max_{k=1,\ldots,m} (1-t)f_k(y) \\
&\quad (\because \text{pointwise maximum is not smaller than maximum}) \\
&= t \max_{i=1,\ldots,m} f_i(x) + (1-t) \max_{k=1,\ldots,m} f_k(y) \\
&= tf(\boldsymbol{x}) + (1-t)f(\boldsymbol{y}) \quad (\because \text{from (7)})
\end{aligned} \tag{15}
$$

From the definiton, $f(\boldsymbol{x})$ is a convex function. Proposition 2.1 has been shown.■