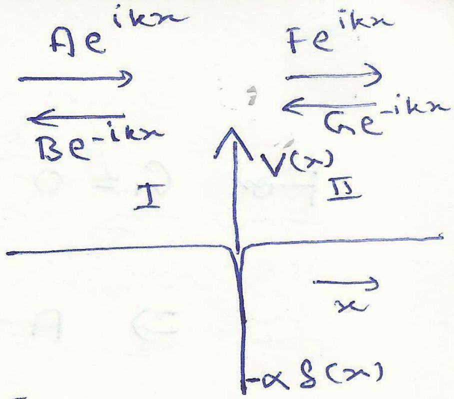
(1.) $V(x) = -\alpha\,\delta(x)$.

Consider $E>0$ solutions,



In region I & II,

$$\frac{d^2\Psi}{dx^2} = -\frac{2mE}{\hbar^2}\Psi = -k^2\Psi$$

$$\text{s.t.,}\quad k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$\therefore\quad \Psi_I(x) = Ae^{ikx} + Be^{-ikx}.$$

$$\&\quad \Psi_{II}(x) = Fe^{ikx} + Ge^{-ikx}.$$

By definition, $\begin{pmatrix} F \\ G \end{pmatrix} = \underset{\text{Transfer matrix}}{\overleftrightarrow{M}} \begin{pmatrix} A \\ B \end{pmatrix}$

& $\begin{pmatrix} B \\ F \end{pmatrix} = \underset{\text{Scattering matrix}}{\overleftrightarrow{S}} \begin{pmatrix} A \\ G \end{pmatrix}$

Continuity of Ψ : $F + G = A + B$

Discontinuity in $\frac{d\Psi}{dx}$: $ik(F - G - A + B) = -\frac{2m\alpha}{\hbar^2}(A+B)$, (derived in class).

Use $\frac{m\alpha}{\hbar^2 k} = \Gamma$

$$F + G = A + B \quad \text{— (A)}$$

$$\Rightarrow$$

$$F - G = (1+2i\Gamma)A - (1-2i\Gamma)B. \quad \text{— (B)}$$

For $G = 0$ (incoming particle from left)

$$\Rightarrow \quad A + B = (1+2i\Gamma)A - (1-2i\Gamma)B.$$

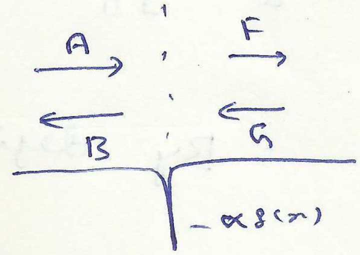
$$\therefore \quad \left.\begin{aligned} B &= \frac{i\Gamma}{1-i\Gamma}A. \\ \& \quad F &= \frac{1}{1-i\Gamma}A. \end{aligned}\right\} \text{— (a)} \quad (\text{using } F = A+B;\ G=0).$$

For $A = 0$ (incoming particle from right,

reflected ↓ incoming ↙

$$\left.\begin{aligned} F &= \frac{i\Gamma}{1-i\Gamma}G. \\ B &= \frac{1}{1-i\Gamma}G. \end{aligned}\right\} \text{— (b)}$$

↑ transmitted



When both $A \neq 0$ & $G \neq 0$, we can add r.h.s., contributions due to both (a) & (b), s.t.,

$$B = \frac{i\Gamma}{1-i\Gamma}A + \frac{1}{1-i\Gamma}G.$$

$$F = \frac{1}{1-i\Gamma}A + \frac{i\Gamma}{1-i\Gamma}G.$$

$$\therefore \begin{pmatrix} B \\ F \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{i\Gamma}{1-i\Gamma} & \frac{1}{1-i\Gamma} \\ \frac{1}{1-i\Gamma} & \frac{i\Gamma}{1-i\Gamma} \end{pmatrix}}_{\text{S-matrix}} \begin{pmatrix} A \\ G \end{pmatrix}$$

(2)

M - matrix

Continuity of $\Psi(x)$,

$$\Rightarrow \quad F + G = A + B. \quad \text{--- (I)}$$

Discontinuity of $\frac{d\Psi}{dx}$,

$$\Rightarrow ik\,(F - G - A + B) = -\frac{2m\alpha}{\hbar^2}(A+B).$$

$$\Rightarrow F - G = A(1 + 2i\Gamma) - B(1 - 2i\Gamma) \quad \text{--- (II)}$$

$$\left(\text{Let, } \Gamma = \frac{m\alpha}{\hbar^2 k}.\right)$$

Solving,

$$\Rightarrow \quad F = (1 + i\Gamma)A + i\Gamma B.$$

$$\& \quad G = -i\Gamma A + (1 - i\Gamma)B.$$

$$\therefore \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} 1+i\Gamma & i\Gamma \\ -i\Gamma & 1-i\Gamma \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}.$$

$$\Rightarrow M = \begin{pmatrix} 1+i\Gamma & i\Gamma \\ -i\Gamma & 1-i\Gamma \end{pmatrix}.$$

(3)

$$\begin{pmatrix} B \\ F \end{pmatrix} = \underbrace{\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}}_{\text{S-matrix}} \begin{pmatrix} A \\ G \end{pmatrix} \quad \text{— (i)}$$

$$\begin{pmatrix} F \\ G \end{pmatrix} = \underbrace{\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}}_{\text{M-matrix}} \begin{pmatrix} A \\ B \end{pmatrix} \quad \text{— (ii)}$$

by definition.

From (i), $F = S_{21} A + S_{22} G$

But $B = S_{11} A + S_{12} G$

i.e., $G = \dfrac{1}{S_{12}} B - \dfrac{S_{11}}{S_{12}} A$.

$$\therefore \quad F = S_{21} A + \frac{S_{22}}{S_{12}} B - \frac{S_{11} S_{22}}{S_{12}} A.$$

$$\therefore \quad F = -\frac{(S_{11} S_{22} - S_{21} S_{12})}{S_{12}} A + \frac{S_{22}}{S_{12}} B. \quad \text{— (iii)},$$

Similarly, $B = S_{11} A + S_{12} G$.

$$\therefore \quad G = -\frac{S_{11}}{S_{12}} A + \frac{1}{S_{12}} B. \quad \text{— (iv)}$$

From (iii) & (iv),

$$\Rightarrow \begin{pmatrix} F \\ G \end{pmatrix} = -\frac{1}{S_{12}} \begin{pmatrix} \det \overleftrightarrow{S} & -S_{22} \\ S_{11} & -1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}. \quad \text{— (v)}$$

Comparing (ii) & (v), $\overleftrightarrow{M} = -\dfrac{1}{S_{12}} \begin{pmatrix} \det \overleftrightarrow{S} & -S_{22} \\ S_{11} & -1 \end{pmatrix}$.