Two functions of Two Random Variables

In previous few lectures we are ~~dealt~~ dealing with the problem 'transformation of random variables in more than one dimension'. In particular we have evaluated prob. distn of $X+Y$, $X-Y$, $\frac{X}{Y}$, XY, $\max(X,Y)$, $\min(X,Y)$ and so on.

Another type of problem in this direction is that 'let two dimensional rv be given to you and suppose two transformations are considered, then how to compute the probability distribution of the new variables of interest'.

In such situation you can apply 'Multivariate Jacobian formula'. Note that this method is applicable for continuous case.

Multivariate Jacobian Formula:

Let (X,Y) be jointly distribution continuous rv with joint pdf $f_X(x,y)$; $(x,y) \subset R^2$. Consider two transformation $U_1 = g_1(X,Y)$, $U_2 = g_2(X,Y)$ such that this transformation is one-one.

Let the corresponding inverse functions be given by $x = h_1(u_1, u_2)$, $y = h_2(u_1, u_2)$

with jacobian of the transformation being

$$J = \begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial x}{\partial u_2} \\ \frac{\partial y}{\partial u_1} & \frac{\partial y}{\partial u_2} \end{vmatrix} .$$

Then joint PDF of (U_1, U_2) is given by

$$f_{U_1,U_2}(u_1, u_2) = f_{X,Y}(h_1(u_1,u_2), h_2(u_1,u_2))\,|J| ;$$
$$(u,v) \in R^2$$

So once you have joint pdf of (U_1, U_2). The other information like marginal pdf, Conditional pdfs etc. can easily be computed;

<u>Note</u>: The above Jacobian formula is general in nature it can be applied to n-dimensional rvs. also.

Ex: Suppose that (X,Y) is jointly distributed rv such that $X, Y \overset{iid}{\sim} exp(1)$. Consider the transformation $U = \frac{X}{X+Y}$, $V = X+Y$. Find joint pdf of (U,V). Also find marginal pdfs of U and V respectively.

Solution: we are given $X, Y \overset{iid}{\sim} exp(1)$. So we have

$$f_X(x) = e^{-x}, \quad 0 < x < \infty$$
$$f_Y(y) = e^{-y}, \quad 0 < y < \infty$$

Given transformations are

$$u = \frac{x}{x+y}, \quad v = (x+y)$$

we see that transformation is one-one. calculate inverse functions as

$$x = h_1(u,v) = uv$$
$$y = h_2(u,v) = v(1-u)$$

$\therefore$ The determinant of jacobian is

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = v - vu + vu = v.$$

Then joint pdf of (U,V) is given by

$$f_{U,V}(u,v) = f_{X,Y}(h_1(u,v), h_2(u,v))\,|J|$$

$$= f_{X,Y}(uv, v(1-u))\,v,$$

$$= f_X(uv)\,f_Y(v(1-u))\,v, \quad 0<v<\infty,\ 0<u<1$$

$$= e^{-uv}\,e^{-v(1-u)}\,v$$

$$= v e^{-v}, \quad 0<u<1,\ 0<v<\infty$$

$\therefore$ joint pdf of (u,v) is given by

$$\boxed{f_{U,V}(u,v) = v e^{-v}, \quad 0<u<1,\ 0<v<\infty}$$

We can easily compute marginal pdfs of U and V as

$$f_U(u) = 1, \quad 0<u<1$$

$$f_V(v) = v e^{-v}, \quad 0<v<\infty.$$

So if X, Y iid $\exp(1)$ variables then

$\frac{X}{X+Y}$ has uniform $U(0,1)$ distn, and

$X+Y$ has gamma $G(2,1)$ distribution.

Let us see one more example.

<u>Ex</u>: Suppose that $X \sim G(\alpha, 1)$, $Y \sim G(\beta, 1)$ and X & Y are independent rvs. Consider the transformation $U = \frac{X}{X+Y}$, $V = X+Y$. Find joint pdf of (U,V). Also compute marginal pdfs of U and V respectively.

<u>Sol^n</u>: Following previous example we find that the given transformations $u = \frac{x}{x+y}$, $v = x+y$ are one-one. So inverse transformations are

$$x = h_1(u,v) = uv, \quad y = h_2(u,v) = v(1-u)$$

Also $|J| = v$.

Then joint pdf of (u,v) is

$$f_{U,V}(u,v) = f_X(uv)\, f_Y(v(1-u))\, |J|, \quad 0<v<\infty, \; 0<u<1$$

$$= \frac{(uv)^{\alpha-1} e^{-uv}}{\Gamma\alpha} \cdot \frac{(v(1-u))^{\beta-1} e^{-v(1-u)}}{\Gamma\beta}$$

$$= \frac{1}{\Gamma\alpha\,\Gamma\beta}\, u^{\alpha-1} (1-u)^{\beta-1}\, v^{\alpha+\beta-1} e^{-v}$$

$\therefore$ joint PDF of (U,V) is

$$\boxed{f_{U,V}(u,v) = \frac{1}{\Gamma\alpha\,\Gamma\beta}\, u^{\alpha-1} (1-u)^{\beta-1}\, v^{\alpha+\beta-1} e^{-v} \quad \alpha>0, \beta>0, 0<v<\infty, 0<u<1}$$

Find marginal density of U & V without performing any computations.

Note that : $\boxed{\begin{array}{l} U \sim Beta(\alpha, \beta) \\ V \sim G(\alpha+\beta, 1) \end{array}}$

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Next we consider a case where the given transformations are not one-one.

Ex : Suppose that (X, Y) jointly distributed continuous random variables such that $X, Y \overset{iid}{\sim} N(0,1)$. Consider $U = \frac{X}{Y}$, $V\, |Y|$. Find joint pdf of (U,V). Also compute the Marginal pdf of U and V respectively.

Solution : Given that $X, Y \overset{iid}{\sim} N(0,1)$. So we have

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty.$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, \quad -\infty < y < \infty$$

Given transformations $U = \frac{X}{Y}$, $V = |Y|$ are not one-one. Since (x, y) and $(x, -y)$ are mapped to same (u,v)

Let us partition the range of (x, y) as

I

$\{(x,y);\ y>0\}$

$u = \frac{x}{y},\quad v = y$

Find inverse functions

$x = h_1(u,v) = uv$

$y = h_2(u,v) = v$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix}$$

$= v.$

Also note $\{(x,y);\ y>0\}$

$\Rightarrow -\infty < u < \infty$
$0 < v < \infty$

II

$\{(x,y),\ y<0\}$

$u = \frac{x}{y},\quad v = -y$

Find inverse functions

$x = h_2'(u,v) = -uv$

$y = h_2'(u,v) = -v$

Here also

$J = v$

Here also $\{(x,y),\ y<0\}$

$\Rightarrow -\infty < u < \infty$
$0 < v < \infty$

Thus joint pdf of (u,v) is given by

$$f_{U,V}(u,v) = f_{X,Y}(h_1(u,v), h_2(u,v))|J| + f_{X,Y}(h_1'(u,v), h_2'(u,v))|J|$$

$$= f_{X,Y}(uv, v)\cdot v + f_{X,Y}(-uv, -v)\, v;$$

$$= f_X(uv)\, f_Y(v)\cdot v + f_X(-uv)\, f_Y(-v)\, v$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2v^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2}\cdot v + \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2v^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} \cdot v$$

$$f_{U,V}(u,v) = \frac{v}{\pi} e^{-\frac{(u^2+1)v^2}{2}}, \quad \begin{array}{l} -\infty < u < \infty \\ 0 < v < \infty \end{array}$$

$\therefore$ joint pdf of (U,V) is

$$\boxed{f_{U,V}(u,v) = \frac{v}{\pi} e^{-\frac{(u^2+1)v^2}{2}}, \quad \begin{array}{l} -\infty < u < \infty \\ 0 < v < \infty \end{array}}$$

Let us compute marginal density of u.

$$f_U(u) = \frac{1}{\pi} \int_0^\infty v \cdot e^{-\frac{(u^2+1)v^2}{2}} dv$$

put $\frac{(u^2+1)v^2}{2} = t$

$v\,dv = \frac{dt}{u^2+1}$

$$= \frac{1}{\pi} \int_0^\infty e^{-t} \frac{dt}{u^2+1}$$

$$= \frac{1}{\pi} \cdot \frac{1}{u^2+1}$$

$\therefore$ Marginal density of u is

$$\boxed{f_U(u) = \frac{1}{\pi} \frac{1}{u^2+1}, \quad -\infty < u < \infty}$$

You can name it. It is standard Cauchy distn.

Similarly try to find $f_V(v)$.