(1.)

The wavefunction of the particle is given by

$$\Psi(x) = 2\alpha\sqrt{\alpha}\, x e^{-\alpha x} \quad \text{for } x > 0$$
$$= 0 \quad \text{otherwise.}$$

(a.) Probability density is given by

$$|\Psi(x)|^2 = 4\alpha^3 x^2 e^{-2\alpha x} \quad \text{for } x > 0$$
$$= 0 \quad \text{otherwise.}$$

As $x \to 0$, $|\Psi(x)|^2 \sim x^2$

As $x \to \infty$, $|\Psi(x)|^2 \to 0$.

At maximum probability location,

$$\frac{d}{dx}|\Psi(x)|^2 = 0 \quad \& \quad \frac{d^2}{dx^2}|\Psi(x)|^2 < 0.$$

$$\frac{d}{dx}|\Psi(x)|^2 = 0 \Rightarrow 2x(1-\alpha x)e^{-2\alpha x} = 0$$

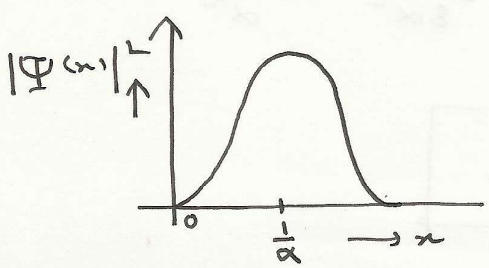
$$\therefore x = 0, \frac{1}{\alpha}.$$

$$\frac{d^2}{dx^2}|\Psi(x)|^2 = [2(1-\alpha x) - 2\alpha x - 4\alpha x(1-\alpha x)] \times e^{-2\alpha x}$$

$$= (2 - 4\alpha x - 4\alpha x + 4\alpha^2 x^2) e^{-2\alpha x}.$$

$$= (2 - 8\alpha x + 4\alpha^2 x^2) e^{-2\alpha x}.$$

$$< 0 \quad \text{for } x = \frac{1}{\alpha}. \quad \text{Maxima.}$$



(b) Expectation value of $x = \langle x \rangle$

$$= \int_{-\infty}^{\infty} \Psi^*(x)\, x\, \Psi(x)\, dx$$

$$= \int_0^{\infty} x\,(4\alpha^3 x^2 e^{-2\alpha x})\, dx$$

$$= 4\alpha^3 \int_0^{\infty} \frac{(2\alpha x)^3 e^{-2\alpha x}\, d(2\alpha x)}{(2\alpha)^4}$$

$$= \frac{1}{4\alpha} \int_0^{\infty} y^{4-1} e^{-y}\, dy \qquad (\because y = 2\alpha x.).$$

$$= \frac{1}{4\alpha}\Gamma(4) = \frac{3!}{4\alpha} = \frac{3}{2\alpha}.$$

$$\boxed{\therefore \langle x \rangle = \frac{3}{2\alpha}}$$

Similarly,

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \Psi^*(x)\, x^2\, \Psi(x)\, dx$$

$$= \int_0^{\infty} x^2\,(4\alpha^3 x^2 e^{-2\alpha x})\, dx$$

$$= 4\alpha^3 \int_0^{\infty} \frac{(2\alpha x)^4 e^{-2\alpha x}\, d(2\alpha x)}{(2\alpha)^5}$$

$$= \frac{1}{8\alpha^2} \int_0^{\infty} y^{5-1} e^{-y}\, dy \qquad (\because y = 2\alpha x.).$$

$$= \frac{1}{8\alpha^2}\Gamma(5) = \frac{4!}{8\alpha^2} = \frac{3}{\alpha^2}.$$

$$\boxed{\therefore \langle x^2 \rangle = \frac{3}{\alpha^2}}$$

(c) Probability of finding the particle between $x = 0$ & $x = \frac{1}{\alpha}$ is,

$$= \int_0^{\frac{1}{\alpha}} (4\alpha^3)\, x^2 e^{-2\alpha x}\, dx$$

$$= (4\alpha^3) \int_0^{\frac{1}{\alpha}} \frac{(2\alpha x)^2 e^{-2\alpha x}\, d(2\alpha x)}{(2\alpha)^3}$$

$$= \frac{1}{2} \int_0^2 y^2 e^{-y}\, dy$$

{ Note: can not use Γ-fn here!

$$= \frac{1}{2} (-y^2 - 2y - 2)\, e^{-y} \Big|_0^2$$

{ subsequent integration by parts.

$$= 0.3233 \text{ (approx.)}.$$

(d) Momentum space wave function can be written as

$$\phi(p_x) = \frac{1}{\sqrt{2\pi\hbar}} \int_0^\infty e^{-ip_x x/\hbar} (2\alpha\sqrt{\alpha})\, x\, e^{-\alpha x}\, dx.$$

(limits 0 to ∞ due to defn. of fn.)

$$= \sqrt{\frac{4\alpha^3}{2\pi\hbar}} \int_0^\infty x e^{-(\alpha + \frac{ip_x}{\hbar})x}\, dx$$

$$= -\sqrt{\frac{4\alpha^3}{2\pi\hbar}}\, \frac{\partial}{\partial\alpha} \left\{ \int_0^\infty e^{-(\alpha + \frac{ip_x}{\hbar})x}\, dx \right\}$$

$$= -\sqrt{\frac{4\alpha^3}{2\pi\hbar}}\, \frac{\partial}{\partial\alpha} \left\{ - \frac{e^{-(\alpha + \frac{ip_x}{\hbar})x}}{(\alpha + \frac{ip_x}{\hbar})} \Bigg|_0^\infty \right\}$$

$$= -\sqrt{\frac{4\alpha^3}{2\pi\hbar}}\, \frac{\partial}{\partial\alpha} \left\{ \frac{1}{(\alpha + \frac{ip_x}{\hbar})} \right\} = \sqrt{\frac{4\alpha^3}{2\pi\hbar}}\, \frac{1}{(\alpha + \frac{ip_x}{\hbar})^2}.$$

$$\therefore \langle p_x \rangle = \int_{-\infty}^{\infty} p_x |\phi(p_x)|^2 dp_x = \frac{4\alpha^3}{2\pi\hbar}\int_{-\infty}^{\infty} \frac{p_x \, dp_x}{(\alpha^2 + \frac{p_x^2}{\hbar^2})^2} = 0.$$

(Odd function)

$$\langle p_x^2 \rangle = \frac{4\alpha^3}{2\pi\hbar}\int_{-\infty}^{\infty} \frac{p_x^2 \, dp_x}{(\alpha^2 + \frac{p_x^2}{\hbar^2})^2} = \frac{8\alpha^3}{2\pi\hbar}\int_{0}^{\infty} \frac{p_x^2 \, dp_x}{(\alpha^2 + \frac{p_x^2}{\hbar^2})^2}$$

Let $p_x = \hbar\alpha \tan\theta$

$\therefore dp_x = \hbar\alpha \sec^2\theta \, d\theta.$

$p_x = -\infty \Rightarrow \theta = -\frac{\pi}{2}$

$p_x = \infty \Rightarrow \theta = \frac{\pi}{2}$

$$(\alpha^2 + \frac{p_x^2}{\hbar^2})^2 = \alpha^4 (1 + \tan^2\theta)^2$$
$$= \alpha^4 \sec^4\theta.$$

$$p_x^2 = \hbar^2\alpha^2 \tan^2\theta.$$

$$\therefore \langle p_x^2 \rangle = \frac{4\alpha^2\hbar^2}{\pi}\int_0^{\pi/2} \sin^2\theta \, d\theta = \alpha^2\hbar^2.$$

(e) $(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{3}{\alpha^2} - \frac{9}{4\alpha^2} = \frac{3}{4\alpha^2}.$

$(\Delta p_x)^2 = \langle p_x^2 \rangle - \langle p_x \rangle^2 = \alpha^2\hbar^2.$

$\Rightarrow (\Delta x \Delta p_x)^2 = \frac{3}{4}\hbar^2.$

$\therefore \Delta x \Delta p_x = \frac{\sqrt{3}}{2}\hbar > \frac{\hbar}{2}.$

(2) (a) $\rho(x) = A e^{-\lambda(x-a)^2}$

$$\therefore \int_{-\infty}^{\infty} \rho(x)\,dx = 1 \Rightarrow A\int_{-\infty}^{\infty} e^{-\lambda(x-a)^2}dx = 1.$$

Let $x - a = u \Rightarrow du = dx.$

$x \to \infty \Rightarrow u \to \infty$ & $x \to -\infty \Rightarrow u \to -\infty.$

$$\therefore A\int_{-\infty}^{\infty} e^{-\lambda u^2}du = A\sqrt{\left(\frac{\pi}{\lambda}\right)} = 1.$$

$$\therefore A = \sqrt{\left(\frac{\lambda}{\pi}\right)}.$$

(b) $$\langle x\rangle = \int_{-\infty}^{\infty} x\,\rho(x)\,dx = \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} x\,e^{-\lambda(x-a)^2}dx$$

using $x - a = u$ as in (a),

$$\Rightarrow \langle x\rangle = \sqrt{\frac{\lambda}{\pi}}\left[\underbrace{\int_{-\infty}^{\infty} u\,e^{-\lambda u^2}du}_{\int(\text{odd}^n\cdot f^n) = 0} + a\int_{-\infty}^{\infty} e^{-\lambda u^2}du\right].$$

$$= \sqrt{\frac{\lambda}{\pi}}\left[0 + a\sqrt{\frac{\pi}{\lambda}}\right]$$

$$= a.$$

Similarly, $$\langle x^2\rangle = \int_{-\infty}^{\infty} x^2\rho(x)\,dx$$

$$= \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty}(u^2 + 2ua + a^2)\,e^{-\lambda u^2}du.$$

$$= \sqrt{\frac{\lambda}{\pi}}\left[\frac{1}{2\lambda}\sqrt{\frac{\pi}{\lambda}} + 0 + a^2\sqrt{\frac{\pi}{\lambda}}\right]$$

$$= \frac{1}{2\lambda} + a^2.$$

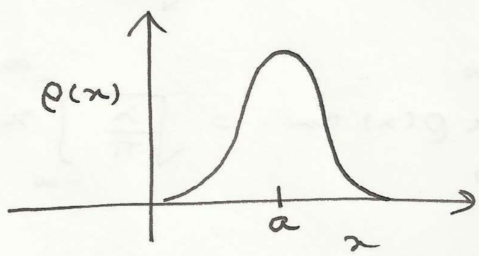
$$\Rightarrow (\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$= a^2 + \frac{1}{2\lambda} - a^2 = \frac{1}{2\lambda}.$$

$$\boxed{\therefore\ \Delta x = \frac{1}{\sqrt{2\lambda}}.}$$

(c.) Sketch : A Gaussian peaked at $x = a$.

Maximum probability for finding the particle at $x = a$.



③ ⓐ $\psi(x,t) = A e^{-a\left[\left(\frac{mx^2}{\hbar}\right) + it\right]}$

$\therefore$ Normalization requires,

$$\int_{-\infty}^{\infty} \psi^*(x,t)\,\psi(x,t)\,dx = 1$$

$$\Rightarrow \quad |A|^2 \int_{-\infty}^{\infty} e^{-\left(\frac{2am}{\hbar}\right)x^2}\,dx = 1.$$

$$\Rightarrow \quad |A|^2 \sqrt{\frac{\pi}{(2am/\hbar)}} = 1.$$

$$\therefore \quad A = \left(\frac{2am}{\pi\hbar}\right)^{1/4}.$$

ⓑ $\dfrac{\partial \psi}{\partial t} = -ia\psi.$

$$\frac{\partial \psi}{\partial x} = -\frac{2amx}{\hbar}\psi.$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{2am}{\hbar}\left(\psi + x\frac{\partial \psi}{\partial x}\right) = -\frac{2am}{\hbar}\left(1 - \frac{2amx^2}{\hbar}\right)\psi.$$

Using, $\quad -\dfrac{\hbar^2}{2m}\dfrac{\partial^2}{\partial x^2}\psi + V(x)\psi = i\hbar\dfrac{\partial \psi}{\partial t}.$

$$\Rightarrow V(x)\psi = \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + i\hbar\frac{\partial \psi}{\partial t}$$

$$= -\frac{\hbar^2}{2m} \times \frac{2am}{\hbar}\left(1 - \frac{2amx^2}{\hbar}\right)\psi - i^2\hbar a\psi.$$

$$= \left[-\hbar a\left(1 - \frac{2amx^2}{\hbar}\right) + \hbar a\right]\psi.$$

$$= 2a^2 m x^2 \qquad \boxed{\Rightarrow V(x) = (2a^2m)\,x^2}$$

"Harmonic oscillator" potential.