Tutorial - II

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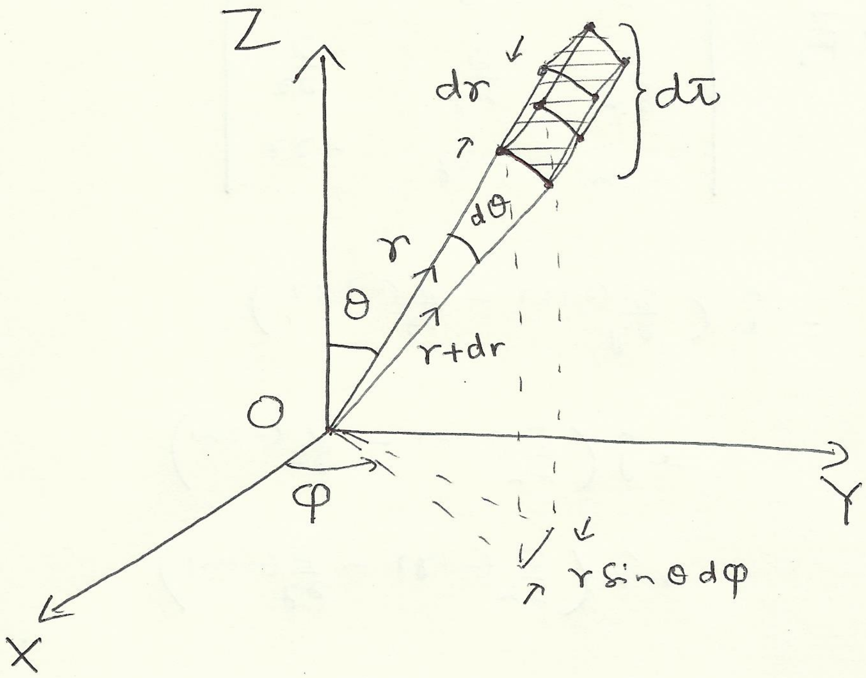
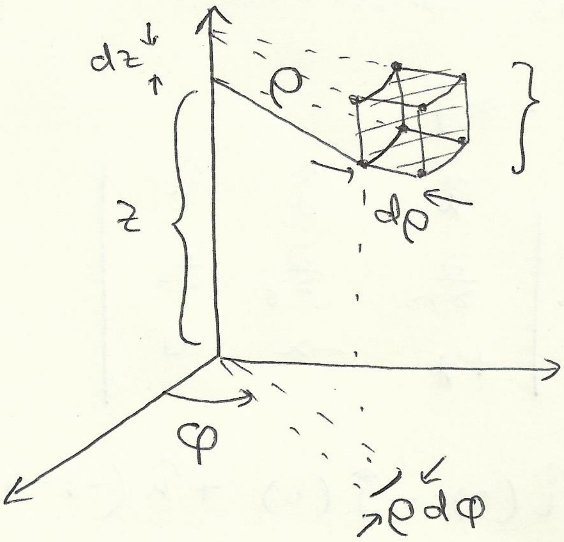
1. (a) (Done in class)



$d\tau = r^2 \sin\theta\, d\theta\, d\varphi\, dr$

(b)



$d\tau = \rho\, d\rho\, d\varphi\, dz$

Note: Algebra for estimating scale parameters was done in class.

Accordingly,

For (a) $h_r = 1$, $h_\theta = r$, $h_\varphi = r\sin\theta$

For (b) $h_\rho = 1$, $h_\varphi = \rho$, $h_z = 1$

2. (a) $\vec{F}_1 = -2x\hat{i} - 2y\hat{j} - 2z\hat{k}$

$$\therefore \quad \nabla \times \vec{F}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2x & -2y & -2z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(-2z) - \frac{\partial}{\partial z}(-2y)\right)$$

$$-\hat{j}\left(\frac{\partial}{\partial x}(-2z) - \frac{\partial}{\partial z}(-2x)\right)$$

$$+\hat{k}\left(\frac{\partial}{\partial x}(-2y) - \frac{\partial}{\partial y}(-2x)\right)$$

$$= 0.$$

$\therefore$ $\vec{F}_1$ is conservative.

(b) $\vec{F}_2 = +y\hat{i} - x\hat{j}$

$$\therefore \quad \nabla \times \vec{F}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ +y & -x & 0 \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(-1-1)$$

$$= -2\hat{k} \neq 0.$$

$\therefore$ $\vec{F}_2$ is non-conservative.

(3.) $\vec{F} = 3m\dot{r}\dot{\theta}\,\hat{\theta}$.

$\Rightarrow \left[(\ddot{r} - r\dot{\theta}^2)\,\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{\theta}\right]m$

$= 3m\dot{r}\dot{\theta}\,\hat{\theta}$

$\therefore\ \ddot{r} - r\dot{\theta}^2 = 0$

$\Rightarrow\ \ddot{r} = r\dot{\theta}^2$ —— (A.)

Also, $m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 3m\dot{r}\dot{\theta}$.

$\Rightarrow r\ddot{\theta} = \dot{r}\dot{\theta}$

$\therefore\ \dfrac{\ddot{\theta}}{\dot{\theta}} = \dfrac{\dot{r}}{r}$

A, B, C, D, E are constants

$\Rightarrow \displaystyle\int \frac{\ddot{\theta}}{\dot{\theta}}\,dt = \int \frac{\dot{r}}{r}\,dt$

$\therefore\ \ln\dot{\theta} = \ln r + C$

$\therefore\ \dfrac{\dot{\theta}}{r} = e^{C} = D.$ —— (B.)

Using (B.) in (A.),

$\Rightarrow\ \ddot{r} = rD^2r^2 = D^2r^3$.

Multiplying both sides by $\dot{r}$ & integrating,

$\Rightarrow \displaystyle\int \dot{r}\ddot{r}\,dt = \int D^2r^3\dot{r}\,dt$

$\therefore\ \dfrac{\dot{r}^2}{2} = \dfrac{D^2r^4}{4} + E$ $\qquad\therefore\ \dot{r} = \pm\sqrt{Ar^4 + B}$.

Relative to the fixed reference frame!

(4) Polar co-ordinates of the particle at time t are

$$\begin{cases} r = b\cosh \Omega t. \\ \theta = \Omega t. \end{cases}$$

The velocity is,

$$\vec{v} = \dot{r}\,\hat{r} + (r\dot{\theta})\,\hat{\theta}.$$

$$= (\Omega b \sinh \Omega t)\,\hat{r} + (\Omega b \cosh \Omega t)\,\hat{\theta}.$$

Speed is,

$$v = \sqrt{|\vec{v}|^2} = \sqrt{\Omega^2 b^2 \sinh^2 \Omega t + \Omega^2 b^2 \cosh^2 \Omega t}$$

$$= \Omega b \sqrt{\cosh 2\Omega t}.$$

The accleration is,

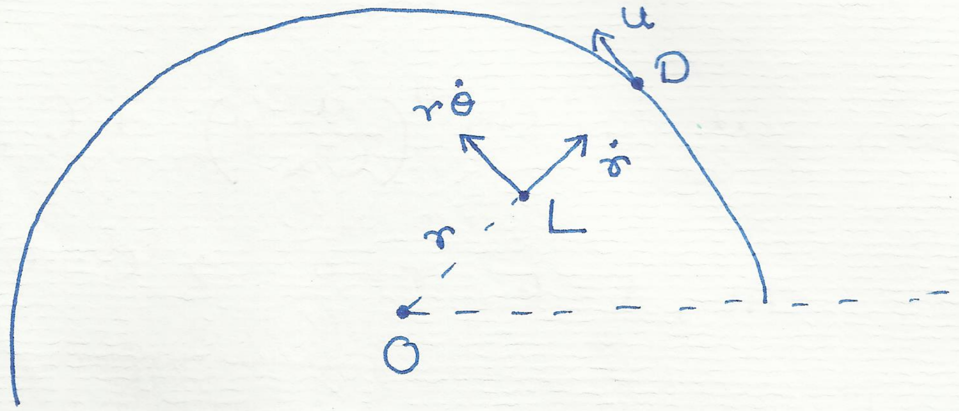
$$\bar{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\theta}.$$

$$= (\Omega^2 b \cosh \Omega t - \Omega^2 b \cosh \Omega t)\,\hat{r}$$

$$+ (0 + 2\Omega^2 b \sinh \Omega t)\,\hat{\theta}.$$

$$\therefore \vec{a} = (2\Omega^2 b \sinh \Omega t)\,\hat{\theta}.$$

5.



(i.) Let the lion (L) have polar coordinates (r, θ) as shown above.

The velocity vector of L is

$$\vec{v} = \dot{r}\,\hat{r} + (r\dot\theta)\,\hat\theta.$$

$$= \dot r\,\hat r + \left(\frac{ur}{a}\right)\hat\theta.$$

($\because$ the L stays on the radius $\vec{OD}$ which is rotating with angular velocity $\dot\theta = \frac{u}{a}$.)

Since, speed of L is U,

$$\Rightarrow \dot r^2 + \left(\frac{ur}{a}\right)^2 = U^2.$$

$$\boxed{\therefore\ \dot r^2 = \frac{u^2}{a^2}\left(\frac{U^2a^2}{u^2} - r^2\right).}$$

$\hookrightarrow$ Eqn satisfied by radial coordinate.

(ii.)

Thus, $\dot r = \left(\frac{u}{a}\right)\sqrt{\frac{U^2a^2}{u^2} - r^2}$. (Keeping +ve root).

Thus, $\frac{u}{a}\int dt = \int \frac{dr}{\sqrt{\frac{U^2a^2}{u^2} - r^2}}$.

$\Rightarrow \quad \frac{ut}{a} = \sin^{-1}\left(\frac{u}{Ua}r\right) + C.$

At, $t = 0$, $r = 0 \Rightarrow C = 0$.

$$\boxed{\therefore \quad r = \left(\frac{Ua}{u}\right)\sin\left(\frac{ut}{a}\right).}$$

(iii) Daniel will get caught when $r = a$.

i.e., when $\sin\left(\frac{ut}{a}\right) = \frac{u}{U}$.

If $U \geq u$, this eqn. has a real solution, $t = \left(\frac{a}{u}\right)\sin^{-1}\left(\frac{u}{U}\right)$.

$\therefore$ Daniel will get caught at

$t = \left(\frac{a}{u}\right)\sin^{-1}\left(\frac{u}{U}\right)$.

(iv) Since $\theta = \frac{ut}{a}$, the polar equation of the path of lion is,

$r = \frac{Ua}{u}\sin\theta$.

Multiply both sides by r,

$\Rightarrow \quad x^2 + y^2 = \left(\frac{Ua}{u}\right)y$.

i.e., $\boxed{x^2 + \left(y - \frac{Ua}{2u}\right)^2 = \left(\frac{Ua}{2u}\right)^2}$

Center: $(0, Ua/2u)$
Radius: $\left(\frac{Ua}{2u}\right)$

Eqn. of circle

Note: The Lion doesnot traverse full circle.

Daniel gets caught when the Lion has traversed an arc of length

$$\left(\frac{Ua}{u}\right) \sin^{-1}\left(\frac{u}{U}\right).$$

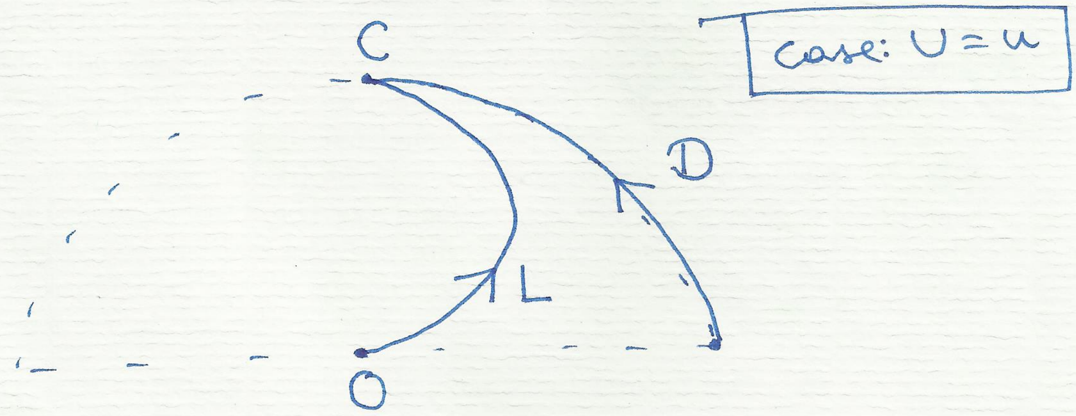
(v.) Special case: $U = u$.

Loci of Lion's path is

$$x^2 + \left(y - \frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2.$$

Daniel will get caught when the Lion has traversed half of this circle.

The point of capture is $(0, a)$.



⑥ The velocity of bee is

$$\vec{v} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta} = \frac{2b}{\tau^2}(\tau - t)\,\hat{r} + \frac{bt}{\tau^3}(2\tau - t)\,\hat{\theta}$$

$$\therefore\ |\vec{v}|^2 = \frac{b^2}{\tau^6}\left(t^4 - 4\tau t^3 + 8\tau^2 t^2 - 8\tau^3 t + 4\tau^4\right).$$

$$\frac{d}{dt}|\vec{v}|^2 = \frac{b^2}{\tau^6}\left(4t^3 - 12\tau t^2 + 16\tau^2 t - 8\tau^3\right)$$

$$= \frac{4b^2}{\tau^6}(t-\tau)\underbrace{(t^2 - 2\tau t + 2\tau^2)}_{\text{always positive}}.$$

$$\therefore\ \frac{d}{dt}|\vec{v}|^2 \begin{cases} < 0 & \text{for } t < \tau. \\ = 0 & \text{for } t = \tau. \\ > 0 & \text{for } t > \tau. \end{cases}$$

Also, $\frac{d^2}{dt^2}|\vec{v}|^2 > 0$. at $t = \tau$.

$\therefore$ $|\vec{v}|$ achieves its minimum value when $t = \tau$.

At this instant, $|\vec{v}| = \frac{b}{\tau}$.

(min. speed of bee).

The accl^n of the bee at time t is,

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\theta}.$$

$$= \left(-\frac{2b}{\tau^2} - \frac{bt}{\tau^4}(2\tau - t)\right)\hat{r} + \left(0 + \frac{4b}{\tau^3}(\tau - t)\right)\hat{\theta}.$$

$$= -\frac{3b}{t^2}\,\hat{r} \quad \text{at } t = \tau.$$

$\therefore$ when speed of the bee is minimum, its acceleration is $-\frac{3b}{\tau^2}\,\hat{r}$.