1. a:

$$\psi(x) = \frac{A}{x^2+a^2}, \quad E = 0.$$

From Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\,\psi(x) = E\,\psi(x).$$

$$\Rightarrow V(x)\,\psi(x) = \frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x), \quad \text{with } \psi(x) = \frac{A}{x^2+a^2}$$

$$\frac{d\psi(x)}{dx} = -\frac{2Ax}{(x^2+a^2)^2} \qquad \& \; E = 0.$$

$$\Rightarrow V(x)\frac{A}{x^2+a^2} = \frac{\hbar^2}{2m}\left[\frac{-2A}{(x^2+a^2)^2} + \frac{8Ax^2}{(x^2+a^2)^3}\right].$$

$$\therefore V(x) = \frac{\hbar^2}{2m}\left[-\frac{2}{(x^2+a^2)} + \frac{8x^2}{(x^2+a^2)^2}\right].$$

$$= \frac{\hbar^2}{2m}\left(\frac{-2x^2-2a^2+8x^2}{(x^2+a^2)^2}\right)$$

$$= \frac{\hbar^2}{m}\,\frac{3x^2-a^2}{(x^2+a^2)^2}.$$

b:

$$\psi(x) = \exp\left(-\sqrt{\frac{m\alpha^2}{2\hbar^2}}\,x^2\right), \quad V(x) = \alpha^2 x^2$$

α is a constant

$$\frac{d\psi(x)}{dx} = -2\sqrt{\frac{m\alpha^2}{2\hbar^2}}\,x\,\exp\left(-\sqrt{\frac{m\alpha^2}{2\hbar^2}}\,x^2\right).$$

$$\frac{d^2\psi(x)}{dx^2} = \frac{4m\alpha^2}{2\hbar^2}x^2\exp\left(-\sqrt{\frac{m\alpha^2}{2\hbar^2}}\,x^2\right) - 2\sqrt{\frac{m\alpha^2}{2\hbar^2}}\exp\left(-\sqrt{\frac{m\alpha^2}{2\hbar^2}}\,x^2\right).$$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = -\alpha^2x^2\exp\left(-\sqrt{\frac{m\alpha^2}{2\hbar^2}}\,x^2\right)$$
$$+\sqrt{\frac{\alpha^2\hbar^2}{2m}}\exp\left(-\sqrt{\frac{m\alpha^2}{2\hbar^2}}\,x^2\right).$$
$$+\alpha^2x^2\exp\left(-\sqrt{\frac{m\alpha^2}{2\hbar^2}}\,x^2\right)$$

$$= \sqrt{\frac{\alpha^2\hbar^2}{2m}}\exp\left(-\sqrt{\frac{m\alpha^2}{2\hbar^2}}\,x^2\right)$$

$$= E\,\psi(x).$$

$$\Rightarrow \boxed{E = \sqrt{\frac{\alpha^2\hbar^2}{2m}}}.$$

2:

For a particle constrained to move in $0 \le x \le L$,

$$\psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

$$\langle p_x \rangle = \left(\frac{2}{L}\right)\int_0^L \sin\left(\frac{n\pi x}{L}\right) (-i\hbar) \frac{\partial}{\partial x}\left\{ \sin\left(\frac{n\pi x}{L}\right)\right\} dx$$

$$= \frac{2}{L}(-i\hbar)\left(\frac{n\pi}{L}\right) \int_0^L \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx.$$

$$= -\frac{i\hbar n\pi}{L^2} \int_0^L \sin\left(\frac{2n\pi x}{L}\right) dx$$

$$= -\frac{i\hbar}{2L} \int_0^{2n\pi} \sin\xi \, d\xi \quad , \quad \xi = \frac{2n\pi}{L} x.$$

$$= -\frac{i\hbar}{2L} \left[-\cos\xi\right]\Big|_0^{2n\pi} = 0.$$

$$\langle p_x^2 \rangle = \left(\frac{2}{L}\right) \int_0^L \sin\left(\frac{n\pi x}{L}\right) (-i\hbar)^2 \frac{\partial^2}{\partial x^2}\left\{\sin\left(\frac{n\pi x}{L}\right)\right\} dx$$

$$= \frac{2\hbar^2}{L}\left(\frac{n\pi}{L}\right)^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx \qquad \text{Let, } \xi = \frac{n\pi x}{L}.$$

$$= 2n\pi \frac{\hbar^2}{L^2} \int_0^{n\pi} \sin^2\xi \, d\xi = -n\pi\frac{\hbar^2}{L^2}\int_0^{n\pi}\left[1 - \cos 2\xi\right] d\xi$$

$$= \left(\frac{n\pi\hbar}{L}\right)^2.$$

3.

From Schrödinger equation, $i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$.

$$\Rightarrow \frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar} V\Psi . \quad \text{— (i.)}$$

The complex conjugation leads to,

$$\frac{\partial \Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar} V\Psi^* \quad \text{— (ii.)}$$

(for a real V)

Now,

$$\frac{d\langle p_x \rangle}{dt} = \frac{d}{dt}\int \Psi^* \left(-i\hbar \frac{\partial}{\partial x}\right)\Psi \, dx$$

$$= -i\hbar \int \frac{\partial}{\partial t}\left\{\Psi^* \left(\frac{\partial \Psi}{\partial x}\right)\right\} dx . \quad \text{— (iii.)}$$

But $\frac{\partial}{\partial t}\left\{\Psi^*\left(\frac{\partial \Psi}{\partial x}\right)\right\} = \left(\frac{\partial \Psi^*}{\partial t}\right)\left(\frac{\partial \Psi}{\partial x}\right) + \Psi^* \frac{\partial^2 \Psi}{\partial t \partial x}$

$$= \left(\frac{\partial \Psi^*}{\partial t}\right)\left(\frac{\partial \Psi}{\partial x}\right) + \Psi^* \frac{\partial}{\partial x}\left(\frac{\partial \Psi}{\partial t}\right) \quad \text{— (iv.)}$$

(using $\frac{\partial^2 \Psi}{\partial t \partial x} = \frac{\partial^2 \Psi}{\partial x \partial t}$.)

Using (i.) & (ii.) in (iv.),

$$\Rightarrow \frac{\partial}{\partial t}\left\{\Psi^*\left(\frac{\partial \Psi}{\partial x}\right)\right\} = \left(-\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*\right)\frac{\partial \Psi}{\partial x} + \Psi^* \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi\right]$$

$$= \frac{i\hbar}{2m}\left[\Psi^*\left(\frac{\partial^3 \Psi}{\partial x^3}\right) - \left(\frac{\partial^2 \Psi^*}{\partial x^2}\right)\left(\frac{\partial \Psi}{\partial x}\right)\right] + \frac{i}{\hbar}\left[V\Psi^*\frac{\partial \Psi}{\partial x} - \Psi^*\frac{\partial}{\partial x}(V\Psi)\right]. \quad \text{— (v.)}$$

Using (iv.) & (v.) in (iii.),

$$\Rightarrow \frac{d}{dt}\langle p_x \rangle = -i\hbar \int \left[\text{R.H.S. of eqn. (v.)} \right] dx .$$

The first two term vanishes (boundary terms, $\Psi \to 0$ at $x \to \pm\infty$.)
same for Ψ^*.

The third & fourth terms lead to,

$$\frac{d}{dt}\langle p_x \rangle = -i\hbar\left(\frac{i}{\hbar}\right)\int \left[\underbrace{V\Psi^*\frac{\partial \Psi}{\partial x}}_{\text{Boundary terms} \to 0} - \Psi^*\frac{\partial}{\partial x}(V\Psi)\right] dx .$$

$$= \int \Psi^*\left(-\frac{\partial V}{\partial x}\right)\Psi \, dx - \int \underbrace{\Psi^* V \frac{\partial \Psi}{\partial x}}_{\text{Boundary terms} \to 0} dx$$

$$= \left\langle -\frac{\partial V}{\partial x}\right\rangle . \qquad \text{(Q.E.D.)}$$

4.

From Schrödinger equation, $i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi$.

$$\Rightarrow \frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m} \frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar} V\Psi. \quad \text{— (i.)}$$

The compl conjugation leads to,

$$\frac{\partial \Psi^*}{\partial t} = -\frac{i\hbar}{2m} \frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar} V\Psi^* \quad \text{— (ii.)}$$

(for a real V)

$$\Rightarrow \frac{\partial}{\partial t}|\Psi|^2 = \frac{\partial}{\partial t}(\Psi^*\Psi) = \left(\frac{\partial \Psi^*}{\partial t}\right)\Psi + \Psi^* \frac{\partial \Psi}{\partial t}.$$

$$= (\text{R.H.S. of (ii.)})\Psi + \Psi^*(\text{R.H.S. of (i.)})$$

$$= \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\Psi\right)\right]$$

$$= -\frac{\partial}{\partial x} J(x,t) \quad \text{(as defined in class)}$$

$$\therefore \frac{d}{dt} P_{ab}(t) = \frac{d}{dt}\left[\int_a^b |\Psi|^2 dx\right] = \int_a^b \frac{\partial}{\partial t}|\Psi|^2 dx$$

$$= -\int_a^b \frac{\partial J(x,t)}{\partial x} dx = J(a,t) - J(b,t).$$

(Q.E.D.)

5.

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1^*\Psi_2\,dx = \int_{-\infty}^{\infty}\frac{\partial}{\partial t}(\Psi_1^*\Psi_2)\,dx$$

$$= \int_{-\infty}^{\infty}\left[\left(\frac{\partial\Psi_1^*}{\partial t}\right)\Psi_2 + \Psi_1^*\left(\frac{\partial\Psi_2}{\partial t}\right)\right]dx$$

$$= \int_{-\infty}^{\infty}\left[\left(-\frac{i\hbar}{2m}\frac{\partial^2\Psi_1^*}{\partial x^2} + \frac{i}{\hbar}V\Psi_1^*\right)\Psi_2 + \Psi_1^*\left(\frac{i\hbar}{2m}\frac{\partial^2\Psi_2}{\partial x^2} - \frac{i}{\hbar}V\Psi_2\right)\right]dx$$

$$= -\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\left[\left(\frac{\partial^2\Psi_1^*}{\partial x^2}\right)\Psi_2 - \Psi_1^*\left(\frac{\partial^2\Psi_2}{\partial x^2}\right)\right]dx$$

$$= -\frac{i\hbar}{2m}\left[\left(\frac{\partial\Psi_1^*}{\partial x}\right)\Psi_2\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\left(\frac{\partial\Psi_1^*}{\partial x}\right)\left(\frac{\partial\Psi_2}{\partial x}\right)dx\right.$$

$$\left. - \Psi_1^*\left(\frac{\partial\Psi_2}{\partial x}\right)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\left(\frac{\partial\Psi_1^*}{\partial x}\right)\left(\frac{\partial\Psi_2}{\partial x}\right)dx\right]$$

$$= 0 \qquad (\text{Q.E.D.})$$