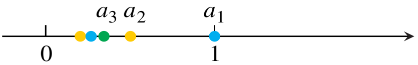

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Mathematics I

Infinite Sequences and Series

Amit K. Verma

Department of Mathematics
IIT Patna

1. Literal meaning of the word "Sequence"

   A following of one thing after another; succession.

   An order of succession; an arrangement.

2. A sequence is a list of numbers $\mathbf{a_1, a_2, a_3, \cdots, a_n, \cdots}$ in a given order. Each of $\mathbf{a_1, a_2, a_3,}$ and so on represents a number. These are the **terms** of the sequence.
3. For example $\mathbf{2, 4, 6, 8, 10, 12, \cdots, 2n, \cdots}$.
4. The sequence has first term $\mathbf{a_1 = 2}$, second term $\mathbf{a_2 = 4}$ and $\mathbf{n^{th}}$ term $\mathbf{a_n = 2n}$.
5. The integer $\mathbf{n}$ is called the **index** of $\mathbf{a_n}$ and indicates where $\mathbf{a_n}$ occurs in the list.
6. We can think of the sequence as a function that sends $\mathbf{1}$ to $\mathbf{a_1}$, $\mathbf{2}$ to $\mathbf{a_2}$, $\mathbf{3}$ to $\mathbf{a_3}$ and in general sends the positive integer $\mathbf{n}$ to the $\mathbf{n^{th}}$ term $\mathbf{a_n}$. This leads to the **formal definition** of a sequence.

Infinite Sequence

An infinite sequence of numbers is a function whose domain is the set of positive integers

$$\mathbf{f} : \mathbf{N} \to \mathbf{R}.$$

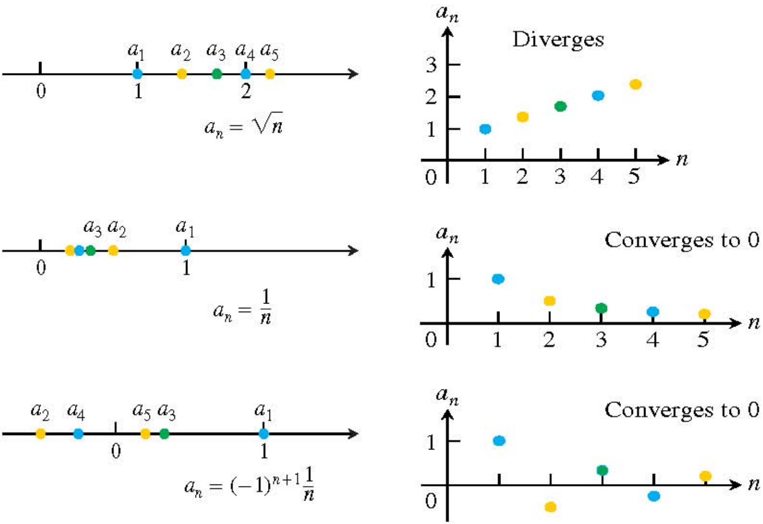
Where $\mathbf{N}$ is the set of natural numbers and $\mathbf{R}$ is the set of real numbers.

OR

Sequence of real numbers is a rule which assigns every natural number $\mathbf{n} \in \mathbf{N}$ a definite real number $\mathbf{a_n} \in \mathbf{R}$.

Notations

1. By writing rules that specify their terms, such as $\mathbf{a_n} = \sqrt{\mathbf{n}}$.
2. or by listing terms, $\{\mathbf{a_n}\} = \left\{\sqrt{\mathbf{1}}, \sqrt{\mathbf{2}}, \sqrt{\mathbf{3}}, \cdots, \sqrt{\mathbf{n}}, \cdots\right\}$.
3. We also sometimes write, $\{\mathbf{a_n}\} = \{\sqrt{\mathbf{n}}\}_{\mathbf{n=1}}^{\infty}$.

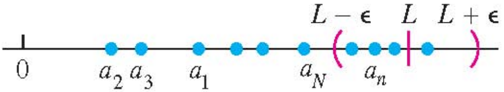


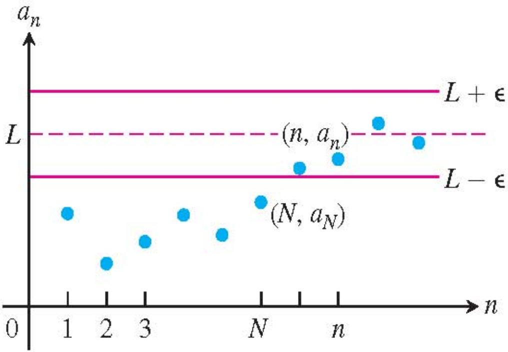
Sequences can be represented as points on the real line or as points in the plane where the horizontal axis n is the index number of the term and the vertical axis as is its value a_n.

Evolution of the definition of convergence

1. Sometimes the numbers in a sequence approach a single value as the index n increases. This happens in the sequence $\left\{\mathbf{1}, \frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{3}}, \cdots, \frac{\mathbf{1}}{\mathbf{n}}, \cdots\right\}$ whose terms approach $\mathbf{0}$ as $\mathbf{n}$ gets large.
2. On the other hand, sequences like $\left\{\sqrt{\mathbf{1}}, \sqrt{\mathbf{2}}, \sqrt{\mathbf{3}}, \cdots, \sqrt{\mathbf{n}}, \cdots\right\}$ have terms that get larger than any number as $\mathbf{n}$ increases.
3. Sequences like $\left\{\mathbf{1}, -\mathbf{1}, \mathbf{1}, -\mathbf{1}, \cdots, (-\mathbf{1})^{\mathbf{n}+\mathbf{1}}, \cdots\right\}$ bounce back and forth between $\mathbf{1}$ and $-\mathbf{1}$ never converging to a single value.
4. Meaning of having a sequence $\mathbf{a_n}$ converge to a limiting value $\mathbf{L}$ says that if we go far enough out in the sequence, by taking the index $\mathbf{n}$ to be larger then some value $\mathbf{N}$, the difference between $\mathbf{a_n}$ and the limit of the sequence $\mathbf{L}$ becomes less than any **preselected** number $\epsilon > \mathbf{0}$.

Convergence





Definition

The sequence **converges** $\{a_n\}$ to the number L if to every positive number ϵ there corresponds an integer N such that for all n,

$$n > N \Rightarrow |a_n - L| < \epsilon.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**. If a_n converges to L, we write

$$\lim_{n \to \infty} a_n = L,$$

or simply $a_n \to L$, and call L the limit of the sequence.

Theorem

Let $\lim_{n\to\infty} \mathbf{a_n} = \mathbf{A}$ and $\lim_{n\to\infty} \mathbf{b_n} = \mathbf{B}$. Then

1. **Sum Rule:** $\lim_{n\to\infty} (\mathbf{a_n} + \mathbf{b_n}) = \mathbf{A} + \mathbf{B}$.
2. **Difference Rule:** $\lim_{n\to\infty} (\mathbf{a_n} - \mathbf{b_n}) = \mathbf{A} - \mathbf{B}$.
3. **Product Rule:** $\lim_{n\to\infty} (\mathbf{a_n} \cdot \mathbf{b_n}) = \mathbf{A} \cdot \mathbf{B}$.
4. **Constant Multiple Rule:** $\lim_{n\to\infty} (k \cdot \mathbf{b_n}) = k \cdot \mathbf{B} \quad (\textit{any number } k)$.
5. **Quotient Rule:** $\lim_{n\to\infty} \dfrac{\mathbf{a_n}}{\mathbf{b_n}} = \dfrac{\mathbf{A}}{\mathbf{B}} \quad (\mathbf{B} \neq 0)$.

Theorem

Let $\{\mathbf{a_n}\}$, $\{\mathbf{b_n}\}$ and $\{\mathbf{c_n}\}$ be sequences of real numbers. If $\mathbf{a_n} \leq \mathbf{b_n} \leq \mathbf{c_n}$ holds for all $\mathbf{n}$ beyond some index $\mathbf{N}$, and if $\lim_{n\to\infty} \mathbf{a_n} = \lim_{n\to\infty} \mathbf{c_n} = \mathbf{L}$ then $\lim_{n\to\infty} \mathbf{b_n} = \mathbf{L}$ also.

Remark

An immediate consequence of Theorem 2 is that, if $|\mathbf{b_n}| \leq \mathbf{c_n}$ and $\mathbf{c_n} \to 0$ then $\mathbf{b_n} \to 0$ because $-\mathbf{c_n} \leq \mathbf{b_n} \leq \mathbf{c_n}$.

Applying the Sandwich Theorem

Since $\lim_{n\to\infty} \dfrac{1}{n} = 0$. We have

- $\frac{\cos n}{n} \to 0$ because $-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n} \to 0$;
- $\frac{1}{2^n} \to 0$ because $0 \leq \frac{1}{2^n} \leq \frac{1}{n} \to 0$;
- $\frac{(-1)^n}{n} \to 0$ because $-\frac{1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n} \to 0$.

Theorem : The Continuous Function Theorem for Sequences

Let $\{\mathbf{a_n}\}$ be a sequence of real numbers. If $\mathbf{a_n} \to \mathbf{L}$ and if $\mathbf{f}$ is a function that is continuous at $\mathbf{L}$ and defined at all $\mathbf{a_n}$, then $\mathbf{f(a_n)} \to \mathbf{f(L)}$.

Example: The Sequence $\{\mathbf{2^{1/n}}\}$

- The sequence $\{\mathbf{1/n}\}$ converges to $\mathbf{0}$.
- Consider $\mathbf{a_n = 1/n}$, $\mathbf{f(x) = 2^x}$ and $\mathbf{L = 0}$.
- By above theorem we have $\mathbf{2^{1/n} = f(1/n) \to f(L) = 2^0 = 1}$.
- The sequence $\{\mathbf{2^{1/n}}\}$ converges to $\mathbf{1}$.

Example:

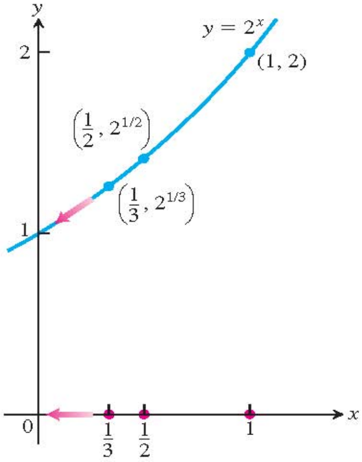


FIGURE 11.3 As $n \to \infty$, $1/n \to 0$ and $2^{1/n} \to 2^0$ (Example 6).

Using l'Hôpital's Rule

The next theorem enables us to use l'Hôpital's Rule to find the limits of some sequences. It formalizes the connection between $\lim_{n\to\infty} \mathbf{a_n}$ and $\lim_{x\to\infty} \mathbf{f(x)}$.

Theorem

Suppose that $\mathbf{f(x)}$ is a function defined for all $\mathbf{x} \geq \mathbf{n_0}$ and that $\{\mathbf{a_n}\}$ is a sequence of real numbers such that $\mathbf{a_n} = \mathbf{f(n)}$ for $\mathbf{n} \geq \mathbf{n_0}$. Then

$$\lim_{x\to\infty} \mathbf{f(x)} = L \Rightarrow \lim_{n\to\infty} \mathbf{a_n} = \mathbf{L}.$$

Using l'Hôpital's Rule

- Example: $\lim_{\mathbf{n}\to\infty} \frac{\ln \mathbf{n}}{\mathbf{n}} =?$ Answer 0.
- Example: $\lim_{\mathbf{n}\to\infty} \left(\frac{\mathbf{n}+\mathbf{1}}{\mathbf{n}-\mathbf{1}}\right)^{\mathbf{n}} =?$ Answer e^2.

Counter Example

Is $\mathbf{a_n} = \sin \mathbf{n}\pi$ convergent?

The following six sequences converge to the limits listed below:

Theorem :

- $\lim_{\mathbf{n}\to\infty} \frac{\ln \mathbf{n}}{\mathbf{n}} = \mathbf{0}.$
- $\lim_{\mathbf{n}\to\infty} \sqrt[n]{n} = \mathbf{1}.$
- $\lim_{\mathbf{n}\to\infty} \mathbf{x^{1/n}} = \mathbf{1} \quad (x > 0).$
- $\lim_{\mathbf{n}\to\infty} \mathbf{x^n} = \mathbf{0} \quad (|x| < 1).$
- $\lim_{\mathbf{n}\to\infty} \left(\mathbf{1} + \frac{\mathbf{x}}{\mathbf{n}}\right)^{\mathbf{n}} = \mathbf{e^x} \quad (\text{any } x).$
- $\lim_{\mathbf{n}\to\infty} \frac{\mathbf{x^n}}{\mathbf{n!}} = \mathbf{0} \quad (\text{any } x).$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

Other kinds of sequences

Recursive Definitions :

So far, we have calculated each a_n directly from the value of n. But sequences are often defined recursively by giving

1. **The value(s) of the initial term or terms, and**
2. **A rule, called a recursion formula, for calculating any later term from terms that precede it.**

Example

i. $a_{n+1} = a_n + 1$, $a_1 = 1$. Then $a_2 = 2$, $a_3 = 3$, and so on.

ii. **Fibonacci numbers :** $a_1 = 1,\ a_2 = 1$ and $a_{n+1} = a_n + a_{n-1}$. then $a_3 = 2$, $a_4 = 3$, $a_5 = 5$ and so on.

Bounded Nondecreasing Sequences

Definition : Nondecreasing Sequence

A sequence $\{a_n\}$ with the property that $a_n \leq a_{n+1}$ for all n is called a nondecreasing sequence.
$*$ There are two kinds of nondecreasing sequences - those whose terms increase beyond any finite bound and those whose terms do not.

Example

i. $1, 2, 3, \cdots, n, \cdots$ of natural numbers.
ii. The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots, \frac{n}{n+1}, \cdots$.
iii. The constant sequence $\{3\}$

Least Upper Bound

Definitions : Bounded, Upper Bound, Least Upper Bound

- **A sequence $\{a_n\}$ is bounded from above if there exists a number M such that $a_n \leq M$ for all n. The number M is an upper bound for $\{a_n\}$.**
- **If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$ then M is the least upper bound for $\{a_n\}$.**

Example

i. $1, 2, 3, \cdots, n, \cdots$ has no upper bound.

ii. The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots, \frac{n}{n+1}, \cdots$ is bounded above by $M = 1$.

The Nondecreasing Sequence Theorem

Theorem : The Nondecreasing Sequence Theorem*

A nondecreasing sequence of real numbers converges if and only if it is bounded from above. If a nondecreasing sequence converges, it converges to its least upper bound.

Convergence

Convergent Sequence a_n

A sequence $\{a_n\}$ of real numbers is said to converge to the real number L provided that "for each $\epsilon > 0$ there exists a number N such that $n > N(\epsilon)$ implies $|a_n - L| < \epsilon$."

Unique Limit

Convergent sequence has unique limit.

Boundedness

Convergent sequences are bounded.

Existence of monotone sequence

Every sequence has a monotonic subsequence.

Bolzano-Weierstrass Theorem

Every bounded sequence has a convergent subsequence.

Exercise

Exercise 1

Let $p \in \mathbb{N}$, $a > 0$ and $a_1 > 0$. Define sequence $\{a_n\}$ by setting

$$a_{n+1} = \frac{a_n(p-1) + \frac{a}{a_n^{p-1}}}{p}, \; n \in \mathbb{N}.$$

Determine

$$lim_{n\to\infty} a_n.$$

Exercise 2

Let $a_1 > b_1 > 0$. Define sequence $\{a_n\}$ and $\{b_n\}$ by setting

$$a_{n+1} = \frac{a_n^2 + b_n^2}{a_n + b_n}, \; b_{n+1} = \frac{a_n + b_n}{2}, \; n \in \mathbb{N}.$$

Show that both $\{a_n\}$ and $\{b_n\}$ are convergent and hence show that both has same limit.

Cauchy Sequence

Cauchy Sequence

A sequence $\{s_n\}$ of real numbers is called a Cauchy Sequence if for each $\epsilon > 0$ there exists a number N such that $m, n > N$ implies $|s_n - s_m| < \epsilon$.

Theorem

Convergent sequences are cauchy sequences.

Theorem

Cauchy sequences are bounded.

Theorem

A sequence is a convergent sequence if and only if it is a Cauchy Sequence.

- Read Example 3.5.6 Bartle & Sherbert Page 82-83, 3rd Edition

Contractive Sequence

If there exists a constant c, $0 < c < 1$,

$$|x_{n+2} - x_{n+1}| \leq c|x_{n+1} - x_n|, \ \forall n \in \mathbb{N}.$$

c is called constant of contractive sequence.

Theorem: Contractive Sequence

Every contractive sequence is Cauchy and hence convergent.

Proof. Discussed in class.

Application in finding root of higher degree polynomial

Use the concept of contractive sequences and find the root of $x^3 - 7x + 2 = 0$. Hint $x_{n+1} = \frac{x_n^3+2}{7}$.

Another problem

Show that $x_{n+1} = \frac{1}{x_n+2}$, $x_1 > 0$ is contractive and hence find out the limit of x_n as $n \to \infty$.