गणित विभाग, भारतीय प्रौद्योगिकी संस्थान पटना
DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY PATNA

B.Tech - I, MA-101
End Semester Examination
November 25, 2011

Time: 3 Hrs Max Marks: 50

Attempt all the questions. Write brief and precise solutions to each question.

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(1) Find the values of k for which the function $f(x, y) = x^2 + kxy + y^2$ will have a minima at origin? For what values of k is the test inconclusive? [2 + 1]

(2) If resistors of r_1, r_2, r_3 and r_4 ohms are connected in parallel to make an R ohm resistor, find the value of $\frac{\partial R}{\partial r_1}$ when $r_2 = 10, r_3 = 20$ and $r_4 = 15$ ohms. [3]

(3) Find the derivatives of the function $f(x, y, z) = x\,e^y + z^2$ in the direction in which it increases most rapidly at the point $(1, log_e 2, \frac{1}{2})$. [3 + 2]

(4) Find the linearization of the function $f(x, y, z) = x^2 - xy + 3\sin z$ at the point $(2, 1, 0)$. Find an upper bound for the error which may result on replacing f by its linearization on the domain defined by $|x-2| \leq 0.01$, $|y-1| \leq 0.02$ and $|z| \leq 0.01$. [3 + 2]

(5) Consider the function $f(x, y) = x^2 + y^2 + 2xy - x - y + 1$ over the region $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Show that f has an absolute minimum along the line $2x + 2y = 1$ in this region. Find the absolute maxima of f over this region. [3 + 2]

(6) Find the area enclosed by the cardioid $r = 2(1 + \cos 2\theta)$. [4]

(7) Evaluate the integral $\int\int_R (x-y)^4\, e^{2(x+y)}\, dx\, dy$ by applying the transformations $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$, where the region R is the square with vertices $(1, 0), (2, 1), (1, 2)$ and $(0, 1)$. [3 + 2]

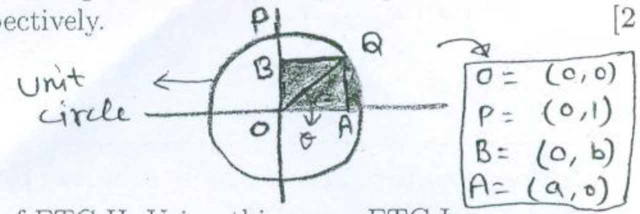
(8) If $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$ has a limit at $c \in \mathbb{R}$, then show that f is bounded on some neighborhood of c. Find $\lim_{x \to 2} \frac{Sin(x^2-4)}{x-2}$. [2 + 1]

(9) Let f and g be two functions defined as follows:

$$f(x) = \frac{x + |x|}{2}\ for\ all\ x,\ g(x) = \begin{cases} x, & \text{for } x < 0; \\ x^2, & \text{for } x \geq 0. \end{cases}$$

Find a formula for computing the composite function $h(x) = f(g(x))$. For what values of x is h continuous? [2]

(10) Using calculus technique, find the area of shaded portion in the following figure. Also find the arclength of arc PQ. Try to give final answers in terms of b and θ and a respectively. [2 + 1]



(11) Write the statement of FTC-II. Using this prove FTC-I. [2]

(12) Using first and second derivative tests, trace the curve of the function $F(x) = \int_0^x e^{-t^2}\,dt$. Write the power series of $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}F(x)$. [Given Data: $F(\infty) = \frac{\sqrt{\pi}}{2}$ and $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$] [3]

(13) Suppose you pick a point at random in the region $0 < y < \sqrt{1-x^2}$. What is the chance that $x > \frac{1}{2}$. [2]

(14) Discuss the convergence of the improper integral $\int_{100}^{\infty} \dfrac{dx}{\sqrt{x^2+1}}$. [Hint: Use comparison Test] [2]

(15) Consider the following function:

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

Prove or Disprove the followings: [1+2]

(a) For a fixed y, f is a continuous function of x.

(b) $f(x,y)$ is continuous at the point $(0,0)$.

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