Indian Institute of Technology Patna
MA101 (Mathematics-I)
B.Tech -I year
Autumn Semester: 2015-2016
(End Semester Examintaion)

Maximum Marks: 50 **Time: 3 Hours**

Note: This question paper has TWO pages and contain 17 questions. Please check all pages and report the discrepancy, if any.

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1. (a) Let I be an interval and $f: I \to \mathbb{R}$ be a strictly monotonic function such that $f(I)$ is an interval. Then show that f is one-one and continuous. [3]

   (b) Let $D = [a, \infty)$ and $f(x) = x^2$, for $x \in D$. Then f is continuous on D. Is it also uniformly continuous on D? Justify your answer. [2]

2. (a) Show that the series given below is convergent
   $(\frac{2^2}{1^2} - \frac{1}{1})^{-1} + (\frac{3^3}{2^3} - \frac{3}{2})^{-2} + (\frac{4^4}{3^4} - \frac{4}{3})^{-3} + ...$ [3]

   (b) Let g be a continuous function on $[0, 1]$ and differentiable on $(0, 1)$. Suppose that $g(0) = 0$ and $g(1) = 0$. Show that there exists a $d \in (0, 1)$ such that $g(d) + g'(d) = 0$. [2]

3. Prove that $f(x, y) = \sqrt{|xy|}$, is not differentiable at $(0, 0)$, but both the partial derivatives exist at $(0, 0)$ and have the values zero. [3]

4. If

   $$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, \ when \ (x,y) \neq (0,0) \\ 0, \ when \ (x,y) = (0,0) \end{cases}.$$

   Show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$. [3]

5. If $x^x y^y z^z = c$, then show that at $x = y = z$,

   $$\frac{\partial^2 z}{\partial x \partial y} = -(x(logex))^{-1}.$$

   [3]

6. Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s, if $w = x + 2y + z^2$; $x = \frac{r}{s}$; $y = r^2 \log s$; $z = 2r$. [3]

7. Find the derivative of $f(x, y) = xe^y + \cos(xy)$, at the point $P(2, 0)$ in the direction of the vector $\vec{u} = 3\hat{i} - 4\hat{j}$. [2]

8. Show that minimum value of $u = xy + \frac{a^3}{x} + \frac{a^3}{y}$, is $3a^2$. [2]

9. Let $\mathbb{R}^2 \to \mathbb{R}$ be defined by
   $f(x, y) = \frac{1}{\sqrt{x^2+y^2}}(||x| - |y|| - |x| - |y|)$, $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$.

   (i) Is f continuous at (0,0)? [2]

   (ii) Which directional derivative of f exist at (0,0)? [2]

   (iii) What can you say about the differentiability of the function f at (0,0)? [1]

10. If $f : [a,b] \to \mathbb{R}$ is continuous, then show that there exists $c \in (a,b)$ such that $\int\limits_a^b f(t)dt = f(c)(b-a)$. [2]

11. (a) If $\vec{v} = (yz)\hat{i} + (zx)\hat{j} + (xy)\hat{k}$, then show that $curl\,\vec{v}$=0. [1]

(b) Evaluate the line integral $\int_C F.ds$ for the vector field $F(x,y,z) = (cosz, e^x, e^y)$ over the curve $C(t) = (1, t, e^t)$ for $0 \le t \le 2$. [2]

12. Let $D = \{(x,y) \in \mathbb{R}^2 |\ 0 \le y \le 2, \frac{y}{2} \le x \le \frac{y+4}{2}\}$ and $f(x,y) = y^3(2x-y)e^{(2x-y)^2}$, for $(x,y) \in D$. Then evaluate $\iint_D f(x,y)dxdy$. [3]

13. Evaluate $\iint(x^2+y^2)dxdy$, over the region bounded between the circles $x^2+y^2 = 2x$ and $x^2+y^2 = 4x$. [3]

Attempt any two questions from the following:

14. (a) Find the equation of tangent plane for $z = x^2 + y^2 - 2xy + 3y - x + 4 - z$ at (2,-3,18). [2]

(b) Let $\int\limits_0^8 \int\limits_{\sqrt[3]{x}}^2 dydx = \int\limits_l^m \int\limits_r^s dxdy$. Find r, s, l and m. [2]

15. Discuss the maxima and minima of the function $u = \sin x \sin y \sin z$, where $x,\ y,\ z$ are the angles of a triangle by using Lagrange Multiplier Method. [4]

16. Find the extreme values of the function $f(x,y,z) = xy + z^2$, on the circle in which the plane $y - x = 0$ intersects the sphere $x^2 + y^2 + z^2 = 4$. [4]

17. Find the volume of tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,\ a, b, c \ge 0$. [4]