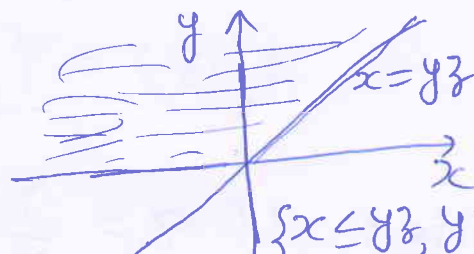
One Function of Two RVs (Part III) (1)

Till now we have evaluated pdf of $X+Y$, $X-Y$ using the CDF approach. Today we try to obtain pdf of XY and $\frac{X}{Y}$.

Result: Let (X,Y) be jointly distributed continuous rv with joint pdf being $f_{X,Y}(x,y)$. Consider the transformation $Z = \frac{X}{Y}$ and then find the pdf of Z when $-\infty < X < \infty$ and $-\infty < Y < \infty$.



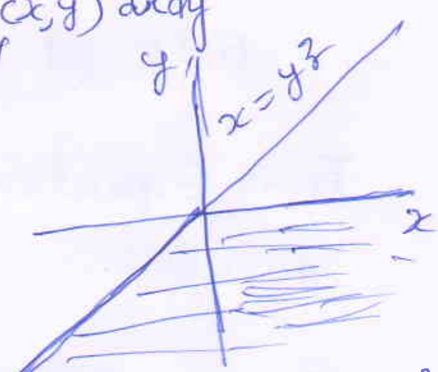
$\{x \le yz, y > 0\}$

CDF of Z is given by

$$F_Z(z) = P\left(\frac{X}{Y} \le z\right)$$

$$= P(X \le Yz, y > 0) + P(X \geq Yz, Y < 0)$$

$$= \int_0^{\infty}\int_{-\infty}^{yz} f_{X,Y}(x,y)\,dx\,dy + \int_{-\infty}^{0}\int_{yz}^{\infty} f_{X,Y}(x,y)\,dx\,dy$$



$\{x \geq yz, y < 0\}$

We differentiate this CDF with respect to z to get the pdf;

$$f_Z(z) = \frac{d}{dz}F_Z(z)$$

$$= \int_0^{\infty}\left\{\frac{\partial}{\partial z}\int_{-\infty}^{yz} f_{X,Y}(x,y)\,dx\right\}dy + \int_{-\infty}^{0}\left\{\frac{\partial}{\partial z}\int_{yz}^{\infty} f_{X,Y}(x,y)\,dx\right\}dy$$

$$f_Z(z) = \int_0^{\infty} y\, f_{X,Y}(yz, y)\,dy + \int_{-\infty}^{0} (-y) f_{X,Y}(yz, y)\,dy$$

$$= \int_{-\infty}^{\infty} |y|\, f_{X,Y}(yz, y)\,dy$$

$\therefore$ Thus pdf of $Z = \frac{X}{Y}$ is given by

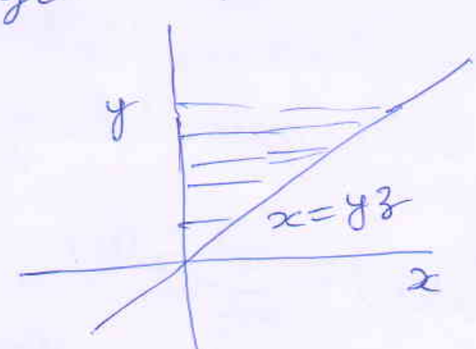
$$\boxed{f_Z(z) = \int_{-\infty}^{\infty} |y|\, f_{X,Y}(yz, y)\,dy} \quad -\text{(1)}$$

If X and Y independent then we have

$$f_Z(z) = \int_{-\infty}^{\infty} |y|\, f_X(yz)\, f_Y(y)\,dy$$

Note: If $Z = \frac{Y}{X}$ then $\boxed{f_Z(z) = \int_{-\infty}^{\infty} |x|\, f_{X,Y}(x, zx)\,dx}$

Remark: If $X > 0$, $Y > 0$ then what is the pdf of $Z = \frac{X}{Y}$ (note that pdf can be computed from Equation (1)). Here we get a special formula as well.



$$F_Z(z) = P(Z \le z) = P\left(\frac{X}{Y} \le z\right)$$

$$= P(X \le Yz)$$

$$= \int_{y=0}^{\infty} \int_{x=0}^{yz} f_{X,Y}(x, y)\,dx\,dy$$

the pdf of $Z = \frac{X}{Y}$ is given by

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz} \int_0^\infty \left\{ \int_0^{yz} f_{X,Y}(x,y)\,dx \right\} dy$$

$$= \int_0^\infty \frac{\partial}{\partial z} \left\{ \int_0^{yz} f_{X,Y}(x,y)\,dx \right\} dy$$

$$= \int_0^\infty y\, f_{X,Y}(yz, y)\,dy$$

So for $X>0$, $Y>0$, pdf of $Z = \frac{X}{Y}$ is given by

$$\boxed{f_Z(z) = \int_0^\infty y\, f_{X,Y}(yz,y)\,dy}$$

Also If $X>0, Y>0$, X and Y independent then

$$\cancel{f_Z(z) = \int_0^\infty y\, f_{X,Y}(yz, y)\,dz}$$

$$\boxed{f_Z(z) = \int_0^\infty y\, f_X(yz)\, f_Y(y)\,dy}$$

Ex: Let X, Y iid $U(0,1)$. Assume that $Z = \frac{X}{Y}$ then pdf to be evaluated.

Solution: Note that $f_X(x) = 1, \quad 0 < x < 1$
$f_Y(y) = 1, \quad 0 < y < 1$

Using Equation (1) we have (Assuming Independence)

$$f_Z(z) = \int_{-\infty}^{\infty} |y|\, f_X(yz)\, f_Y(y)\,dy$$

$$f_Z(z) = \int_{-\infty}^{\infty} |y| \cdot I(0 < yz < 1) \cdot I(0 < y < 1)\, dy$$

$$= \int_{-\infty}^{\infty} |y|\, I(0 < y < 1/z)\, I(0 < y < 1)\, dy$$

$$= \int_0^{\min(1, 1/z)} y\, dy = \left[\frac{y^2}{2}\right]_0^{\min(1, 1/z)}$$

$$= \frac{[\min(1, 1/z)]^2}{2} = \begin{cases} \frac{1}{2}, & 0 < z \leq 1 \\ \frac{1}{2z^2}, & z > 1 \end{cases}$$

Next

Consider the following problem.

<u>Result</u>: Let (X, Y) be jointly distributed rv with joint pdf $f_{X,Y}(x, y)$. Consider the transformation $Z = XY$ and find pdf of Z.

<u>Solution</u>: $F_Z(z) = P(Z \leq z) = P(XY \leq z)$

$$= P\left(X \leq \frac{z}{Y}, Y > 0\right) + P\left(X \geq \frac{z}{Y}, Y < 0\right)$$

$$= \int_{y=0}^{\infty} \int_{x=-\infty}^{z/y} f_{X,Y}(x, y)\, dx\, dy + \int_{-\infty}^{0} \int_{z/y}^{\infty} f_{X,Y}(x, y)\, dx\, dy$$

so pdf of Z is given by

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

$$= \int_0^{\infty} \frac{1}{y} f_{X,Y}\left(\frac{z}{y}, y\right) dy + \int_{-\infty}^{0} \left(-\frac{1}{y}\right) f_{X,Y}\left(\frac{z}{y}, y\right) dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{|y|} f_{X,Y}\left(\frac{z}{y}, y\right) dy$$

$\therefore$ we have pdf of Z as $(Z = XY)$

$$\boxed{f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{|y|} f_{X,Y}\left(\frac{z}{y}, y\right) dy}$$

If X and Y are independent then we have

$$\boxed{f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{|y|} f_X\left(\frac{z}{y}\right) f_Y(y)\, dy}$$

Also Note that $\underline{X > 0, Y > 0}$ & $Z = XY$ then we have $f_Z(z) = \int_0^{\infty} \frac{1}{y} f_{X,Y}\left(\frac{z}{y}, y\right) dy$

If X & Y independent then

$$f_Z(z) = \int_0^{\infty} \frac{1}{y} f_X\left(\frac{z}{y}\right) f_Y(y)\, dy.$$

Ex: Let $X, Y \overset{iid}{\sim} U(0,1)$. Find density function of $Z = XY$.

Solution: we are given $X, Y \overset{iid}{\sim} U(0,1)$ and so we have

$$f_X(x) = 1, \quad 0 < x < 1$$

$$f_Y(y) = 1, \quad 0 < y < 1$$

Recall the formula

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{|y|} f_X\left(\frac{z}{y}\right) \cdot f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{|y|} \cdot I\left(0 < \frac{z}{y} < 1\right) I(0 < y < 1)\, dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{|y|}\, I(z < y < \infty)\, I(0 < y < 1)\, dy$$

$$= \int_z^1 \frac{1}{y}\, dy = \ln y \Big|_z^1 = -\ln z, \quad 0 < z < 1$$

$$\boxed{f_Z(z) = -\ln z, \quad 0 < z < 1}$$

Students, till now we have discussed prob. distributions of $X+Y$, $X-Y$, $\frac{X}{Y}$ (also $\frac{Y}{X}$) XY.

Let us see one or two more important functions

Ex: Let (X,Y) be jointly distributed rv with joint pdf $f_{X,Y}(x,y)$. Consider the transformation $Z = \max(X,Y)$. find pdf of Z.

Ex: In the previous problem find pdf of $Z = \min(X,Y)$.

Problem: Let $X, Y \overset{iid}{\sim} Exp(\beta)$. Find pdf of $\max(X,Y)$ and $\min(X,Y)$.

Solution: we are given $X \sim Exp(\beta)$, $Y \sim exp(\beta)$ and

so, $f_X(x) = \frac{1}{\beta} e^{-x/\beta}$, $0 < x < \infty$, $\beta > 0$ $\mid F_X(\beta) = 1 - e^{-x/\beta}$

$f_Y(y) = \frac{1}{\beta} e^{-y/\beta}$, $0 < y < \infty$, $\beta > 0$ $\mid F_Y(\beta) = 1 - e^{-y/\beta}$

Let $Z = \max(X,Y)$.

Let us compute CDF of Z to obtain

$$F_Z(z) = P(Z \le z) = P(\max(X,Y) \le z)$$

$$= P(X \le z, Y \le z) = F_{X,Y}(z,z)$$

$$= F_X(z)\, F_Y(z) \quad [\because X \,\&\, Y \text{ independent}]$$

Thus pdf of Z is given by

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz}\{F_X(z)\, F_Y(z)\}$$

$$= f_X(z)\, F_Y(z) + F_X(z)\, f_Y(z)$$

$$\therefore f_Z(z) = \frac{1}{\beta} e^{-z/\beta} \cdot \left(1 - e^{-z/\beta}\right) + \left(1 - e^{-z/\beta}\right) \cdot \frac{1}{\beta} e^{-z/\beta}$$

$$= \frac{2}{\beta} e^{-z/\beta} \left(1 - e^{-z/\beta}\right)$$

$\therefore$ The pdf of $Z = \max(X, Y)$ when (X, Y) iid $Exp(\beta)$ is given by

$$\boxed{f_Z(z) = \frac{2}{\beta} e^{-z/\beta} \left(1 - e^{-z/\beta}\right), \quad \beta > 0;\ 0 < z < \infty}$$

You can verify above is a proper pdf.

Next consider $Z = \min(X, Y)$ then proceed as follows

$$F_Z(z) = P(Z \le z) = 1 - P(Z > z)$$

$$= 1 - P(\min(X, Y) > z)$$

$$= 1 - P(X > z, Y > z)$$

$$= 1 - P(X > z)\, P(Y > z) \quad [\because X \,\&\, Y \text{ independent}]$$

$$= 1 - [1 - P(X \le z)][1 - P(Y \le z)]$$

$$F_Z(z) = 1 - (1 - F_X(z))(1 - F_Y(z))$$

$$= 1 - e^{-z/\beta} \cdot e^{-z/\beta}$$

$$= 1 - e^{-2z/\beta}.$$

$\therefore$ pdf of $Z = \min(X, Y)$ is

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz}\left[1 - e^{-2z/\beta}\right]$$

$$= \frac{2}{\beta} e^{-2z/\beta}, \quad z > 0, \ \beta > 0$$

$\therefore$ pdf of $Z = \min(X, Y)$ when $(X, Y) \overset{iid}{\sim} \exp(\beta)$ is given by

$$\boxed{f_Z(z) = \frac{2}{\beta} e^{-2z/\beta}, \quad z > 0, \ \beta > 0}$$

Can you name it?? it is $\exp(\beta/2)$.

Then if X, Y iid $\exp(\beta)$ then $\min(X, Y)$ is again one parameter exponential $\mathcal{E}xp(\beta/2)$.

This is a characterizing property of the exponential distribution.