Midsem: MA 201, 2016 **IIT Patna**

Instructors: P. Das, A. Upadhyay & P. Srivastava Total Marks: 30, Time: 2 Hours

1. Show that the function $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$ for $z = x+iy \neq 0$ and $f(0) = 0$ is continuous at $z = 0$. [2]

2. Let $f(z) = (8x - x^3 - xy^2) + i(x^2y + y^3 - 8y)$ for $z = x + iy \in \mathbb{C}$. Determine the region S in $\mathbb{C}$ at which f is differentiable. What is the domain of analyticity of f in $\mathbb{C}$? [2]

3. Assume $Re(a)$, $Re(b) \leq 0$ where $a, b \in \mathbb{C}$. Show that $|e^a - e^b| \leq |a - b|$. [2]

4. Evaluate $\displaystyle\int_C \frac{e^{-iz}}{(z^2+1)^2} dz$, if C is the circle $|z - 3i| = 3$. [2]

5. Find the power series of the function $f(z) = \dfrac{1}{(z-1)(z-2)}$ in the domain $1 < |z| < 2$. [2]

6. Prove or disprove(by an example): If a function $f(z) = u(x,y) + iv(x,y)$ is analytic in $\mathbb{C}$ and satisfies $u(x.y) + v(x,y) = 1$ for all $z = x + iy$ in $\mathbb{C}$ then f is a constant function in $\mathbb{C}$. [1]

7. Find out the harmonic conjugate of the function $u(x,y) = y^3 - 3x^2y$. [2]

8. Write down the singular points with the of singularities of the functions

   $a.\quad f_1(z) = \displaystyle\sum_{k=-\infty}^{-1} z^k + \sum_{k=0}^{\infty} \frac{z^k}{2^{k+1}}$ and $b.\quad f_2(z) = \dfrac{e^{z^2} - 1}{z^4}$. [2]

9. Find the singularities of the function $f(z) = \cot(z)$. What are the residues at singular points? [2]

10. Show that $\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2)$, $\quad z_1, z_2 \in \mathbb{C}$. You may use the fact that it works when $z_1, z_2 \in \mathbb{R}$. [2]

11. Let $f(z) = \sin(z)$ and R be the rectangular region $0 \leq y \leq 1$, $0 \leq x \leq \pi$. Find the points in R where the function $|f(z)|$ reaches its maximum values. [3]

12. Evaluate $\displaystyle\int_C \frac{z^2+3z+2}{z^3 - z^2} dz$ where $C : |z| = 2$.(Hint: You may use Cauchy's residue theorem)[3]

13.a) Find the Fourier Series of the 2π periodic function $f(x) = x\sin(x)$, $-\pi \leq x < \pi$. Where does this obtained Fourier series converge to for $x = \pm\pi$? [2.5]

b) Given the Fourier Series of 2π periodic piecewise continuous function $f(x)$ for $0 < x < 2\pi$ as:

$$f(x) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos(nx) + B_n \sin(nx)),$$

where $A_0 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)dx$, $A_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\cos(nx)dx$, $B_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\sin(nx)dx$, $n = 1, 2, 3, \ldots$. Find the Fourier Series of $f(x)$ for $-\pi < x < \pi$ in terms of A_0, A_n's & B_n's. [1.5]

c) Is $\tan(x)$ a piecewise continuous function in $[-\pi, \pi]$? Justify. [1]