Indian Institute of Technology Patna
MA101, Mid Semester Exam: 2015

Maximum Marks: 30 **Time: 2 Hrs**

Note: This question paper has TWO pages and contain TWELVE questions. Please check all pages and report the discrepancy, if any. Attempt all questions. Use $\epsilon - \delta$ arguments wherever possible.

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1. (a) Let x denote an arbitrary real number. Show that there exists a unique integer n such that $n - 1 \leq x \leq n$.

   (b) If $x > 0$ is a real number and $p < q$ then show that there exists an irrational number r such that $p < ru < q$.

   [3]

2. Find the limit of the sequences (i) $\dfrac{n^2}{n!}$ and (ii) $((1 + \frac{1}{n})^{2n})$. [2]

3. If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$ then show that x_n is a Cauchy sequence. [2]

4. Consider the sequence defined by $a_1 = 1$ and $a_{n+1} = 1 + \frac{1}{a_n}$ for $n \in \mathbb{N}$. Show that the sequence is a Cauchy sequence and its limit is $\frac{1+\sqrt{5}}{2}$. [3]

5. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$ is convergent. Check the convergence of the series: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. [3]

6. For what values of θ and p the series $\displaystyle\sum_{k=1}^{\infty} \frac{(cos(k\theta))}{k^p}$ is (i) convergent and (ii) divergent. [2]

7. (a) Use the $\epsilon - \delta$ definition to establish that $f(x) = \frac{1}{x^2}$ $x > 0$, is not uniformly continuous at any point $c \in R$. Also show that the function $f(x) = \frac{1}{x^2}$ $x \geq a > 0$ is uniformly continuous. [$2\frac{1}{2}$]

(b) Let $f : \mathbb{R} \to (0, \infty)$, satisfy $f(x+y) = f(x)f(y)\ \forall\ x \in \mathbb{R}$. Suppose f is continuous at $x = 0$. Show that f is continuous at all $x \in \mathbb{R}$. [$2\frac{1}{2}$]

8. (a) Using Cauchy Mean Value theorem, show that $1 - \frac{x^2}{2!} < cosx$ for $x \neq 0$ [2]

(b) A right circular cone with a flat circular base is constructed of sheet material of uniform small thickness. Express the total area of the surface in terms of volume and semi-vertical angle θ. Show that for a given volume, the area of the surface is a minimum if $\theta = sin^{-1}(1/3)$ [3]

9. (a)Let $f : [0, 12] \longrightarrow \mathbb{R}$ be continuous and $f(0) = f(12)$. Show that there exists $x_1, x_2, x_3, x_4 \in [0, 12]$ such that $x_2 - x_1 = 6$ and $x_4 - x_3 = 3$, $f(x_1) = f(x_2)$ and $f(x_3) = f(x_4)$. (Use the intermediate value property (IVP)). [$2\frac{1}{2}$]

(b) Let $f : [1, 3] \to R$ be a continuous function that is differentiable on $(1, 3)$ with derivative $f'(x) = (f(x)^2) + 4$ for all $x \in (1, 3)$. Determine whether it is true or false that $f(3) - f(1) = 5$. Justify your answer. [$2\frac{1}{2}$]