# Stochastic Processes



Definition: A stochastic process is a collection of random variables $\{X(t), t \in T\}$ where $t$ is a parameter that runs over the index set $T$. Often $t$ is referred to as time.

Parameter Space: The set $T$ is called the parameter space where $t \in T$ may denote time length, distance or any other quantity.

State Space: The set of all possible values that $X(t), t \in T$ takes is known as state space of the stochastic process.

⊛ A Stochastic process is a collection of random variables that are used to describe the evolution of a physical phenomena over a time period.

Let us see some examples related to collection of random variables.

Ex: Toss a coin several times and let $S_n$ denote the total no of heads after $n$ tosses of coin then we have

$$S_n = X_1 + X_2 + \cdots + X_n$$

where ~~[illegible]~~ random variable $X_1$ denotes the result of first toss, $X_2$ result of the 2nd toss $X_3$ denotes the out come of the third toss and so on.

Also observe that $S_1$ is the total no of heads after the first toss, $S_2$ is the total no of heads after the 2nd toss and in general $S_n$, $n=1,2,\cdots$ is the cumulative no. of heads after $n^{th}$ toss. Thus $\{S_n, n=1,2,\cdots\}$ is a collection of random variable.

⊛ Let us modify this example as follows:

Suppose you have a total of 100 Rs with you. Try to play a coin tossing game: if a head turns up you win one rs., if a tail turns up then you loose one rs. Suppose that interest lies in the total winning after $n$ play of this game. Consider a random variable ~~[illegible]~~ $S_n$ such as

$$S_n = 100 + X_1 + X_2 + \cdots + X_n$$

where $X_1, X_2, \cdots, X_n$ are outcomes of respective tosses. ( $X_i, i=1,2,\cdots,n$ is like Ber(p) rvs).

$$\left\{ X_i \rightarrow 0, 1, \quad \left[ \begin{aligned} p_{X_i}(0) &= 1-p \\ p_{X_i}(1) &= p \end{aligned} \right] \right.$$

with this frame work

$S_1 \rightarrow$ total winning after first ~~game~~ game

$S_2 \rightarrow$ - - - - - - 2nd game.

$\vdots$

$S_n \rightarrow$ total winning after ~~winnin~~ n round of plays.

Then this is a modified collection of rvs $\{S_n, n=1,2,3,\cdots\}$ which represent total winning.

<u>Ex</u>: Consider a machine's working condition which may go out of order and is repairable. So the machine can be found either in working or in non working condition. Suppose status of the machine is checked every day. Define a random variable $X_n$ as

$X_n$: working condition of machine

Then $\{X_n, n=1,2,3,\cdots\}$ is a collection of random variables where $X_1$ denotes the status of machine on day one, $X_2$ is status on day 2. and so on.

Ex: Suppose that we are interested in the dynamics of the arrivals of telephone calls to a call center. To describe these arriving calls, consider a collection of random variables $\{X(t), t \geq 0\}$ where each $X(t)$, for fixed 't', represents the cumulative number of calls coming to the call center by the time point $t$. Note that telephone calls record a count, so the state space of this collection of rvs is given by $\{0, 1, 2, \cdots\}$. Thus arrival of telephone calls can be modeled using this collection of rvs.

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Thus in many field of practical studies we do not have the luxury of working with one, two, three or four dimensional random variables. We need to work with many rvs basically collection of random variables.

Many random phenomena evolve over a span of time. Their probabilistic behaviour changes as time varies. Stochastic process is a special type of prob. model that captures the evolution of a physical phenomena over time.

Based on state space and parameter space, we can classify a stochastic process as follows.

Note that a SP is some collection of rvs given as $\{X(t),\ t \in T\}$, $T$ is the parameter space.

Collection of all possible values of these rvs is called the state space of the process and is denoted as $S$ which can be discrete or continuous.

Similarly the parameter space $T$ can be discrete or continuous.

Based on these value we can classify a SP as follows:

| T \ S | Discrete | Continuous |
|---|---|---|
| Discrete | Discrete time discrete state Stochastic Process | Discrete time Continuous State Stochastic Process |
| Continuous | Continuous time Discrete state stochastic process | Continuous time continuous state Stochastic process |

Examples:

<1> Discrete time Discrete state Stochastic Process (DTDS - SP)

Consider a random variable $X_n$ defined as

$X_n$: no. of customers in a Shopping mall waiting for service after $n$th customer being served.

In this example both T and S are discrete.

Further we have $T = \{1, 2, 3, \ldots\}$

$S = \{0, 1, 2, 3, \ldots\}$

Then the collection of rvs $\{X_n;\ n = 1, 2, 3, \ldots\}$ is a SP. This process takes its value on discrete space. Also it changes its values at discrete time points. So this is an example of DTDS SP.

(2) Continuous time Discrete State Stochastic Process (CTDS − SP).

Define $X(t)$: no. of customers in the shop at any time $t$.

Then we have $T = \{t;\ t \geq 0\}$ and $S = \{0, 1, 2, 3, \cdots\}$. with this parameter space and state space the collection of random variables $\{X(t), t \geq 0\}$ is an example of CTDS stochastic process.

Another example: $X(t)$: number of customers taking food in a hotel at any time $t$.

then $T = \{t,\ t \geq 0\}$, $S = \{0, 1, 2, 3, \cdots\}$.

Here $\{X(t),\ t \in T\}$ is CTDS stochastic process.

(3) Discrete time continuous state stochastic process (DTCS − SP)

Let $X_n$: amount of water in a dam recorded at nth time unit.

$T = \{1, 2, 3, \cdots\}$ $\quad S = \{x \mid x \geq 0\}$.

Then $\{X_n;\ n = 1, 2, 3, \cdots\}$ is a collection of rvs which represents a DTCS stochastic process.

(4) Continuous time Continuous state Stochastic process (CTCS-SP).

Let $X(t)$: amount of water in a dam at any time $t$

then $T = \{t;\ t \geq 0\}$, $S = \{x;\ x \geq 0\}$.

The collection of rvs $\{X(t);\ t \in T\}$ is an example of CTCS stochastic process.

Another example:

$X(t)$: temperature of a city at any time $t$

Here $T = \{t;\ t \geq 0\}$, $S = \{x;\ -10 < x < 40\}$

then $\{X(t), t \in T\}$ is CTCS stochastic process.

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Once we have a collection of rvs our primary interest is derive required joint prob. distributions, marginal distributions among other things. Once we have prob distn then we can compute probabilities of many event of interest.

As a very simple example Consider the Bernoulli process $\{X_i;\ i = 1, 2, \cdots\}$

where $X_i$ iid random variables distributed as Bernoulli($p$). $\left[ \begin{array}{l} P(X_i = 1) = p \\ P(X_i = 0) = 1-p \end{array} \right]$

Define $S_n = \sum_{i=1}^{n} X_i = X_1 + X_2 + \cdots + X_n$

then $\{ S_n, n = 1, 2, \cdots \}$ is a s.p. this is an example of DT DS stochastic process. Among other quantities we know that

$$S_n \sim Bin(n, p).$$

Thus immediate summary of information for this process are

$$E(S_n) = np$$

$$V(S_n) = np(1-p).$$