Department of Mathematics
Indian Institute of Technology Patna
MA - 201:Autumn Semester: 2013-14

Assignment-3: Complex Analysis

1. Prove that the following functions are nowhere differentiable:
   (i) $f(z) = |z|$ (ii) $f(z) = Re(z)$ (iii) $f(z) = Im(z)$ (iv) $f(z) = \bar{z}$
   (v) $f(z) = z - \bar{z}$ (vi) $f(z) = 2x + ixy^2$ (vii) $f(z) = e^x e^{-iy}$
2. Prove that in each of the following cases U is a harmonic function. Further find a function V such that $f(z) = U + Vi$ is analytic.
   (i) $U = e^x(x \cos y - y \sin y)$, (ii) $U = x^3 - 3xy^2 - 3x^2 - 3y^2 + 1$,
   (iii) $U = \sin x \cosh y + 2 \cos x \sinh y + x^2 - y^2 + 4xy$, (iv) $U = 4xy - x^3 + 3xy^2$,
3. Prove that an analytic function with constant modulus is constant.
4. Show that if $u(x, y)$ and $v(x, y)$ are harmonic functions in a domain D then the function $f(z) = (u_y - v_x) + i(u_x + v_y)$ is analytic in D.
5. Verify the validity of the statement: "If the function $f(z) = u(x, y) + iv(x, y)$ is analytic at a point z, then necessarily the function $f(z) = v(x, y) - iu(x, y)$ is analytic at z."
6. Prove that if $f'(z) = 0$ everywhere in a domain D then $f(z)$ must be constant throughout D.
7. Use Cauchy-Riemann equations to check whether or not the function $f(z) = e^{\bar{z}}$ is analytic anywhere.
8. Verify the following inequalities.
   (i) $|e^{2z+i} + e^{iz^2}| \leq e^{2x} + e^{-2xy}$ (ii) $|e^{z^2}| \leq e^{|z|^2}$ (iii) $|e^{-2z}| < 1$ iff $Re(z) > 0$
9. Find all values of z such that:
   (i) $e^z = 2$ (ii) $e^z = 1 + \sqrt{3}\, i$ (iii) $e^{2z-1} = 1$ (iv) $e^z = -4$ (v) $e^z = \sqrt{3} - i$
10.
   (i) Show that $\overline{\exp(iz)} = exp(i\bar{z})$ if and only if $z = n\pi$, $(n = 0, \pm 1, \pm 2, \ldots)$
   (ii) e^z is real then what restriction is placed on z.
   (iii) e^z is imaginary then what restriction is placed on z.
11. Show that:
   (i) $\overline{\sin(z)} = \sin \bar{z}$ (ii) $\overline{\cos(z)} = \cos \bar{z}$ (iii) $\overline{\cos(iz)} = \cos i\bar{z}$
   (iv) $\overline{\sin(iz)} = \sin i\bar{z}$ iff $z = n\pi i$, $(n = 0, \pm 1, \pm 2, \ldots)$
   (v) $\sin \bar{z}$ and $\cos \bar{z}$ is nowhere analytic
12. Find the roots of the following equations:
   (i) $\sin z = \cosh 4$ (ii) $\cos z = 2$ (iii) $\sin z = i \sinh 1$ (iv) $\sinh z = -1$ (v) $\sinh z = e^z$
   (vi) $\cosh z = -2$ (vii) $\sinh z = i$
13. Show that:
   (i) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$ (ii) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$
   (iii) $|\sin z| \geq |\sin x|$ and $|\cos z| \geq |\cos x|$ (iv) $|\sinh y| \leq |\sin z| \leq \cosh y$
   (v) $|\sinh y| \leq |\cos z| \leq \cosh y$ (vi) $|\sinh x| \leq |\cosh z| \leq \cosh x$
   (vii) $\cosh^2 z - \sinh^2 z = 1$
14. Show that:
   (i) $Log(1 + i)^2 = 2Log(1 + i)$
   (ii) $Log(-1 + i)^2 \neq 2Log(-1 + i)$
   (iii) $log(i^2) = 2log(i)$ when $log(z) = ln(r) + i\theta(r > 0,\ \frac{\pi}{4} < \theta < \frac{9\pi}{4})$
   (iv) $log(i^2) \neq 2log(i)$ when $log(z) = ln(r) + i\theta(r > 0,\ \frac{3\pi}{4} < \theta < \frac{11\pi}{4})$
   (v) the set of values for $log(i^{1/2})$ and $(1/2)log(i)$ are same also find that common values
   (vi) if $Re(z_1) > 0$ and $Re(z_2) > 0$ then $Log(z_1 z_2) = Log(z_1) + Log(z_2)$
15. Find:
   (i) the values of $(1 + i)^i$ (ii) the values of $(-1)^{1/\pi}$
   (iii) principal value of i^i (iv) principal value of $[(e/2)(-1 - \sqrt{3}\, i)]^{3\pi i}$
   (v) all z for which $Log(z) = 1 - (\pi/4)i$ (vi) all z for which $e^z = -ie$
16. Derive formula for $\sin^{-1} z$, $\cos^{-1} z$, $\tan^{-1} z$, $\sinh^{-1} z$, $\cosh^{-1} z$, $\tanh^{-1} z$.
17. Find values of $\tan^{-1}(2i)$, $\cosh^{-1}(-1)$, $\tanh^{-1} 0$.
18. Solve the equations $\sin z = 2$ and $\cos z = \sqrt{2}$.