(1) If $\lim_{x\to -1} \frac{2x^2-\alpha x-14}{x^2-2x-3}$ exists and is equal to l (say), then we have $\lim_{x\to -1}(2x^2-\alpha x-14) = \lim_{x\to -1} \frac{2x^2-\alpha x-14}{x^2-2x-3}\cdot(x^2-2x-3)$

$$= \lim_{x\to -1} \frac{2x^2-\alpha x-14}{x^2-2x-3} \cdot \lim_{x\to -1}(x^2-2x-3) = l\cdot 0 = 0$$

$$\Rightarrow 2(-1)^2-\alpha\cdot(-1)-14 = 0 \Rightarrow \alpha = 12.$$

(2) $\lim_{x\to 0} \frac{f(x)}{x^2} = 5$ (given)

$$\lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0}\frac{f(x)}{x^2}\cdot x = \lim_{x\to 0}\frac{f(x)}{x^2}\cdot \lim_{x\to 0} x = 5\cdot 0 = 0$$

(3) $f(x) = \begin{cases} x & \text{if } x\in\mathbb{Q} \\ 0 & \text{if } x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}$

Choose $\varepsilon>0$ (arbitrary). Now let $\delta=\varepsilon$. Then $\forall x\in\mathbb{R}$ s/t $|x-0|<\delta \Rightarrow |f(x)-f(0)| = |f(x)| < \delta = \varepsilon$

[because, if $|x-0|<\delta$ s/t $x\in\mathbb{Q}$, then $|f(x)| = |x| < \delta = \varepsilon$
if $|x-0|<\delta$ s/t $x\in\mathbb{R}\setminus\mathbb{Q}$, then $|f(x)| = 0 < \delta = \varepsilon$]

$\therefore$ f is continuous at $x_0 = 0$.

Let, $a\in\mathbb{Q}$ s/t $a\neq 0$.

$\forall n\in\mathbb{N}$, $(a-\frac{1}{n}, a+\frac{1}{n})$ contains infinitely many rationals and irrationals.

$\therefore$ there exists $x_n\in\mathbb{R}\setminus\mathbb{Q}$ s/t $x_n\in(a-\frac{1}{n}, a+\frac{1}{n})$ $\forall n\in\mathbb{N}$. Then, we get a sequence $\{x_n\}$ in $\mathbb{R}\setminus\mathbb{Q}$ s/t $0\leq |x_n-a| < \frac{1}{n}$.

$\therefore \lim_{n\to\infty} x_n = a \Rightarrow \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0$.

Now, if f is continuous at a, then $\lim_{n\to\infty} f(x_n) = f(a) = a$

But $a\neq 0$, so f is not continuous at a.

Similarly, we can prove that f is not continuous for all $x\in\mathbb{R}\setminus\mathbb{Q}$.

An alternative way:

Alternative Procedure to Solve question NO. 3

(3) $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

Choose a sequence $x_n \to 0$

$|f(x_n)| \leq |x_n|$

$\Rightarrow f(x_n) \to 0 = f(0)$

$\therefore f$ is continuous at 0.

Let $x_0 \neq 0$, take a sequence of rationals $r_n \to x_0$

" " " of irrationals $i_n \to x_0$

$f(r_n) = \cancel{0 \forall} \; r_n \; \forall n$

$\therefore f(r_n) = r_n \to x_0$ (converges to x_0)

But $f(i_n) = 0 \to 0$

$\Rightarrow$ if $x_0 \neq 0$ then

$\boxed{\lim_n f(r_n) \neq \lim_n f(i_n)} \Rightarrow f$ is not continuo at x_0

(Because whatever sequence $\{x_n\}$ you take converges to x_0, all the sequence $\{f(x_n)\}$ converges to $f(x_0)$.

But here the case is different).

(4) $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

Let, $c \in \mathbb{Q}$. Choose ~~any~~ $\varepsilon = 1$. Then for any $\delta > 0$, $(c-\delta, c+\delta)$ contains infinitely many rationals and irrationals.

$\forall x \in \mathbb{R} \setminus \mathbb{Q}$ s/t $|x - c| < \delta \Rightarrow |f(x) - f(c)| = 2 > \varepsilon$.

$\therefore$ f is not continuous $\forall x \in \mathbb{Q}$.

Similarly, we can prove that f is not cont. $\forall x \in \mathbb{R} \setminus \mathbb{Q}$.

(5) Define, $f : [-1, 1] \to \mathbb{R}$ as follows —

$$f(x) = \begin{cases} x & \text{if } x \in (-1, 1) \\ 0 & \text{if } x \in \{-1, 1\}. \end{cases}$$

Then f is bounded on $[-1, 1]$ but does not have a maximum or a minimum.

(6) $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x+y) = f(x) + f(y)$ $\forall x, y \in \mathbb{R}$.

$f(0+0) = f(0) + f(0) \Rightarrow f(0) = f(0) + f(0) \Rightarrow f(0) = 0$ $\longrightarrow$ (i)

$\forall x \in \mathbb{R}$, $f(x - x) = f(0) = 0$ (by (i))

$$\Rightarrow f(x) + f(-x) = 0$$

$$\Rightarrow f(-x) = -f(x) \longrightarrow \text{(ii)}$$

Let, $c \in \mathbb{R}$ s/t $c \neq 0$. and $\{x_n\}$ be a sequence in $\mathbb{R}$ s/t $\lim_{n \to \infty} x_n = c$

$\therefore$ $\lim_{n \to \infty} (x_n - c) = 0$. Since f is continuous at 0,

so $\lim_{n \to \infty} f(x_n - c) = f(0) = 0$ (by (i))

[$\forall x, y \in \mathbb{R}$, $f(x - y) = f(x) + f(-y) = f(x) - f(y)$ by (ii)]

$\therefore$ $\lim_{n \to \infty} (f(x_n) - f(c)) = 0 \Rightarrow \lim_{n \to \infty} f(x_n) = f(c)$

$\therefore$ f is continuous at c.

$\therefore$ f is continuous at every point $c \in \mathbb{R}$.

(7) Let $c \in \mathbb{R}$.
Choose any $\varepsilon > 0$. Now choose $\delta = \varepsilon$
Then, $\forall x \in \mathbb{R}$ s/t $|x-c| < \delta \Rightarrow |f(x) - f(c)| \le |x-c| < \delta = \varepsilon$
(by given condn)
$\therefore$ f is continuous at c.
Since $c \in \mathbb{R}$ is arbitrary, f is continuous for all $x \in \mathbb{R}$.

(8) for $x \neq 0$, $\dfrac{\sin x}{\sqrt{1-\cos x}} = \dfrac{2 \sin x/2 \cos x/2}{\sqrt{2 \sin^2 x/2}} = \dfrac{2 \sin x/2 \cos x/2}{\sqrt{2} \sin x/2}$

Let, $D = (-\pi, \pi) \setminus \{0\}$, Then $D \subseteq \mathbb{R}$ and $0 \in D'$
(D' = limit points of D)

$\forall x \in D$, $\dfrac{\sin x}{\sqrt{1-\cos x}} = \sqrt{2} \cos x/2$

$\therefore \lim\limits_{x \to 0} \dfrac{\sin x}{\sqrt{1-\cos x}} = \lim\limits_{x \to 0} \sqrt{2} \cos x/2 = \sqrt{2} \lim\limits_{x \to 0} \cos x/2$

$= \sqrt{2} \cos \frac{0}{2} = \sqrt{2}$ (since $\cos x$ is continuous, $\lim\limits_{x \to 0} \cos x = \cos 0 = 1$)

(9) (i) $f(x) = x^2$ at $x = 3$, $x \in [0, 7]$
Choose any $\varepsilon > 0$, Now choose $\delta = \frac{\varepsilon}{10}$
Then $\forall x \in [0, 7]$ s/t $|x-3| < \delta \Rightarrow |f(x) - f(3)|$
$= |x^2 - 3^2|$
$= |x-3|\,|x+3|$
$< 10\delta = \varepsilon$

$\therefore$ f(x) is continuous at $x = 3$.

(ii) $f(x) = \frac{1}{x}$, $x \in [0, 1]$
Choose any $\varepsilon > 0$, Now let $\delta = \min\{1/4, \varepsilon/8\}$
Then $\forall x \in [0,1]$ s/t $|x - 1/2| < \delta \Rightarrow |f(x) - f(1/2)|$
$= |1/x - 2| = \dfrac{|1-2x|}{|x|}$
$< 2\delta \cdot 4$ ($\because |x - 1/2| < \delta \le 1/4 \Rightarrow |2x-1| < 2\delta$ and $1/4 < x < 3/4$)
$\le 8 \cdot \varepsilon/8$ ($\because \delta \le \varepsilon/8$)
$= \varepsilon$.

$\therefore$ f is continuous at $x = 1/2$.

(5)(iii) $f(x) = \sqrt{x}$, $x \geq 0$

Let $c \geq 0$ is arbitrary.

Case-I:- $c = 0$.

for any $\varepsilon > 0$, choose $\delta = \varepsilon^2$

then $\forall$ $x \geq 0$ such that $|x| < \delta$

$$\Rightarrow |f(x) - f(0)|$$

$$= |\sqrt{x}| < \varepsilon$$

Case II:-

when ~~for~~ $c \geq 1$.

Choose $\delta = \varepsilon$., then solve yourself.

Case-III

when $0 < c < 1$

Choose $\varepsilon > 0$

Let $\delta = \cancel{\min\{c/2, \varepsilon\}}$. $\min\left\{c/2, \dfrac{(\sqrt{2}+1)\cdot\sqrt{c}\,\varepsilon}{\sqrt{2}}\right\}$.

$\forall x \geq 0$ such that $|x-c| < \delta$

Now

$$\delta \leq c/2$$

$$-c/2 \leq -\delta.$$

$$\Rightarrow c/2 \leq c-\delta.$$

$$\therefore c/2 \leq c-\delta < x < c+\delta \leq 3c/2.$$

Now $|f(x) - f(c)|$

$$= |\sqrt{x} - \sqrt{c}|$$

$$= \frac{|x-c|}{|\sqrt{x}+\sqrt{c}|} \quad —(*)$$

[Now $x > c/2$

$$\sqrt{x} > \sqrt{\frac{c}{2}}.$$

$$\sqrt{x} + \sqrt{c} > \frac{(\sqrt{2}+1)\sqrt{c}}{\sqrt{2}}.$$

$$\Rightarrow \frac{1}{\sqrt{x}+\sqrt{c}} < \frac{\sqrt{2}}{(\sqrt{2}+1)\sqrt{c}}.\ ]$$

from (*) $|f(x) - f(c)| < \dfrac{\sqrt{2}\,\delta}{(\sqrt{2}+1)\sqrt{c}} \leq \varepsilon$

$$\left[\because \delta \leq \frac{(\sqrt{2}+1)\sqrt{c}\,\varepsilon}{\sqrt{2}}\right]$$

(10) $f(x) = \frac{x^2+x-6}{x-2}, \quad x \neq 2$

If f is continuous at c, then $\lim_{x\to c} f(x) = f(c)$

In this problem $\lim_{x\to 2} f(x) = f(2)$

Let, $D = \mathbb{R} \setminus \{2\}$, $D \subseteq \mathbb{R}$

$\forall x \in D, \quad f(x) = \frac{x^2+x-6}{x-2} = \frac{(x+3)(x-2)}{(x-2)} = x+3.$

$\therefore \lim_{x\to 2} f(x) = \lim_{x\to 2} (x+3) = 2+3 = 5.$

($\because g(x) = x$ is continuous, $\lim_{x\to 2} x = 2$)

Define, $f: \mathbb{R} \to \mathbb{R}$ as, $f(x) = \begin{cases} \frac{x^2+x-6}{x-2}, & x \neq 2 \\ 5, & x = 2. \end{cases}$

(11) (a) Let, $f(x) = x^2$. Let, $c \in \mathbb{R}$ is arbitrary s/t $c > 0$.

for any $\varepsilon > 0$, choose $\delta = \min\{c/2, \frac{2\varepsilon}{5c}\}$.

$\forall x > 0$ s/t $|x-c| < \delta$

$\Rightarrow |f(x) - f(c)| = |x^2 - c^2| = |x-c|\,|x+c| < \frac{5c}{2} \cdot \delta \leq \frac{5c}{2} \cdot \frac{2\varepsilon}{5c} = \varepsilon$

(because, $|x-c| < \delta \Rightarrow c - c/2 \leq c-\delta < x < c+\delta \leq c + c/2 = \frac{3c}{2}$

$\Rightarrow 0 < c/2 < x < 3c/2 \Rightarrow 0 < x+c < 5c/2$

Also, $\delta \leq \frac{2\varepsilon}{5c}$)

$\therefore f$ is continuous at $c \in \mathbb{R}$ s/t $c > 0$.

(11) (a) To show f is not uniformly continuous on $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ we have to prove for some $\varepsilon > 0$, for all $\delta > 0$ $\exists\, x_1, x_2 \in \mathbb{R}^+$ s/t $|x_1 - x_2| < \delta$ but $|f(x_1) - f(x_2)| \geq \varepsilon$.

Choose $\varepsilon = 1/2$,

When, $0 < \delta < 2$, Choose $x_1 = \dfrac{4-\delta^2}{4\delta} > 0$, $x_2 = x_1 + \delta/2 > 0$.

Then, $|x_1 - x_2| = \delta/2 < \delta$.

But $|f(x_1) - f(x_2)| = |x_1^2 - x_2^2| = |x_1 + x_2|\,|x_1 - x_2|$

$$= (x_1 + x_2)(x_2 - x_1)$$
$$= \left(\frac{4-\delta^2}{4\delta} + \frac{4-\delta^2}{4\delta} + \delta/2\right) \cdot \delta/2$$
$$= \left(\frac{4-\delta^2}{2\delta} + \delta/2\right) \cdot \delta/2$$
$$= \frac{4-\delta^2+\delta^2}{2\delta} \cdot \delta/2$$
$$= \frac{4}{2\delta} \cdot \delta/2 = 1 \geq 1/2 = \varepsilon$$

When, $\delta \geq 2$. Choose $x_1 = 2 > 0$, $x_2 = 1 > 0$.

Then, also $|x_1 - x_2| = 1 < \delta$

But $|f(x_1) - f(x_2)| = |4 - 1| = 3 \geq 1/2 = \varepsilon$.

$\therefore$ f is not uniformly continuous on $\mathbb{R}^+$.

Let, $g(x) = \frac{1}{x}$, $x > 0$.

Since, $g_1(x) = x \neq 0$ $\forall x > 0$. $g(x) = 1/g_1(x)$ is cont. on $x > 0$.

Let, $x_n = 1/n$ Then $\{x_n\}$ is cauchy sequence on $x > 0$.

But, $g(x_n) = n$, which is not cauchy sequence.

So, g is not uniformly continuous on $x > 0$.

Let, $h(x) = 1/x^2$, Let, $D = \{x \in \mathbb{R} : x > 0\}$.

Since, $h_1(x) = x^2 \neq 0$ on D & h_1 is cont. on D.

So, h is cont. on D.

Let, $x_n = 1/n$. Then $\{x_n\}$ is cauchy in D.

But $h(x_n) = n^2$ is not cauchy in D.

$\therefore$ h is not uniformly cont. on D.

11(b) Let $f(x) = x^2$ and $D = [-a, a]$, $a > 0$

for any $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{2a}$ Then, $\forall x, y \in D$

with $|x - y| < \delta \Rightarrow |f(x) - f(y)| = |x^2 - y^2| = |x+y||x-y| < 2a\delta = \varepsilon$

$[$ since, $|x+y| \leq |x| + |y| \leq a + a = 2a]$

$\therefore$ f is <u>uniformly continuous</u> on D.

Let, $f(x) = \frac{1}{x}$ and $D = \{ x \in \mathbb{R} : x \geq b > 0 \}$

for any $\varepsilon > 0$, Choose $\delta = \varepsilon \cdot b^2$ Then, $\forall x, y \in D$

with $|x - y| < \delta \Rightarrow |f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|y - x|}{|x||y|} < \frac{\delta}{b^2} = \varepsilon$

$\left[\text{because, } \frac{1}{|x||y|} \leq \frac{1}{b} \cdot \frac{1}{b} = \frac{1}{b^2}\right]$

$\therefore$ f is <u>uniformly continuous</u> on D.

Now, Let, $f(x) = \frac{1}{x^2}$ and $D = \{ x \in \mathbb{R} : x \geq b > 0 \}$

for any $\varepsilon > 0$, Choose $\delta = \min\left\{ 1, \frac{\varepsilon \cdot b^4}{2b+1} \right\}$ Then $\forall x, y \in D$

with $|x - y| < \delta \Rightarrow |f(x) - f(y)| = \left|\frac{1}{x^2} - \frac{1}{y^2}\right| < \frac{2b+1}{b^2} \cdot \frac{1}{b^2} \cdot \delta \leq \frac{2b+1}{b^4} \cdot \frac{\varepsilon \cdot b^4}{2b+1} = \varepsilon.$

Since, $\left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \frac{|y^2 - x^2|}{x^2 y^2} = \frac{|x+y||x-y|}{x^2 y^2}$

$|x - y| < \delta \leq 1 \Rightarrow y - 1 < x < y + 1 \Rightarrow x + y < 2y + 1.$

Now, $\frac{|x+y|}{y^2} = \frac{x+y}{y^2} < \frac{2y+1}{y^2} = \frac{2}{y} + \frac{1}{y^2} \leq \frac{2}{b} + \frac{1}{b^2} = \frac{2b+1}{b^2}$

$\forall x, y \in D.$

$\therefore$ f is <u>uniformly continuous</u> on D.

11(c) Let $f(x) = \sin x$ and $c \in \mathbb{R}$

Then for any $\varepsilon > 0$, choose $\delta = \varepsilon$.

$\forall x \in \mathbb{R}$ with $|x - c| < \delta \Rightarrow |f(x) - f(c)| = |\sin x - \sin c|$

$= 2\left|\sin\frac{x-c}{2}\right|\left|\cos\frac{x+c}{2}\right| \leq 2\left|\sin\frac{x-c}{2}\right| \leq 2 \cdot \frac{|x-c|}{2} < \delta = \varepsilon$

$[$ since, $|\sin x| \leq |x|$ and $|\cos x| \leq 1]$

$\therefore$ f is <u>continuous</u> at every point $c \in \mathbb{R}$.

(11) (a) Continued. To show, $\cos x$ is continuous at every point $c \in \mathbb{R}$.

$$|\cos x - \cos c| = \left|2\sin\frac{x+c}{2}\sin\frac{c-x}{2}\right| \le 2\left|\sin\frac{c-x}{2}\right| \le 2\cdot\frac{|c-x|}{2} = |x-c|.$$

(use this inequality)

To show, $|x|$ is continuous at every point $c \in \mathbb{R}$.

$||x|-|c|| \le |x-c|$ (Now use this inequality)

(12) In this problem, $f : [0,1] \to \mathbb{R}$ be a continuous function. which imply f attains maximum and minimum in $[0,1]$ i.e. there exists $a \,\&\, b \in [0,1]$ s/t $f(a) \le f(x) \le f(b)$ $\forall x \in [0,1]$.

$$\therefore \left.\begin{aligned} f(a) &\le f(1/4) \le f(b) \\ f(a) &\le f(1/2) \le f(b) \\ f(a) &\le f(3/4) \le f(b) \end{aligned}\right\} \Rightarrow \begin{aligned} &3f(a) \le f(1/4)+f(1/2)+f(3/4) \le 3f(b) \\ &\text{i.e. } f(a) \le \tfrac{1}{3}\left(f(1/4)+f(1/2)+f(3/4)\right) \le f(b) \end{aligned}$$

So, by Intermediate value theorem, $\exists\, x_0 \in [0,1]$ s/t

$f(x_0) = \frac{1}{3}\left(f(1/4)+f(1/2)+f(3/4)\right)$ (Actually $\exists x_0 \in [a,b]$ or $[b,a]$; $[a,b] \subseteq [0,1]$, $[b,a] \subseteq [0,1]$)

(13) $p(y) = a_n y^n + a_{n-1}y^{n-1} + \cdots + a_1 y + a_0$

In this problem, n is even $(n \ne 0)$, $a_n = 1$, $a_0 = -1$.

$\therefore p(0) = a_0 = -1 < 0$.

$p(y) = y^n + a_{n-1}y^{n-1} + \cdots + a_1 y - 1$.

$\therefore \lim\limits_{y\to+\infty} \frac{p(y)}{y^n} = 1$. Choose $\varepsilon = 1$, then $\exists M > 0$ s/t

$\forall y \ge M$, $\left|\frac{p(y)}{y^n} - 1\right| < 1 \Rightarrow 0 < \frac{p(y)}{y^n} < 2$

In particular, $0 < \frac{p(M)}{M^n} \Rightarrow p(M) > 0$ $(\because M > 0)$

Similarly, $\lim\limits_{y\to-\infty} \frac{p(y)}{y^n} = 1$, for $\varepsilon = 1$, $\exists M_1 < 0$ s/t

$\forall y \le M_1$, $\left|\frac{p(y)}{y^n} - 1\right| < 1 \Rightarrow 0 < \frac{p(y)}{y^n} < 2$

In particular, $0 < \frac{p(M_1)}{M_1^n} \Rightarrow p(M_1) > 0$ ($\because n$ is even)

Apply Intermideate value theorem on $[0, M]$ and $[M_1, 0]$, we can show that $p(y)$ has at least two real roots.

(14) $\lim_{x\to\infty} \left(x^2 - x^3 \sin\left(\frac{1}{x}\right)\right) = \lim_{x\to\infty} x^2\left(1 - x\sin\frac{1}{x}\right)$

$= \lim_{x\to\infty} x^2\left[1 - x\left\{\frac{1}{x} - \frac{1}{3!\,x^3} + \frac{1}{5!\,x^5} - \cdots\right\}\right]$

$= \lim_{x\to\infty} x^2\left[1 - 1 + \frac{1}{6x^2} - \frac{1}{5!\,x^4} + \cdots\right]$

$= \left[\frac{1}{6} - 0 + \cdots\right] = \frac{1}{6}$

(15) In this problem, $f, g : \mathbb{R} \to \mathbb{R}$ cont. function s/t $f(a) \neq g(a)$ for some $a \in \mathbb{R}$.

Let, $h(x) = f(x) - g(x)$ $\therefore h(a) \neq 0$

If $h(a) > 0$, Choose $\varepsilon > 0$ s/t $h(a) - \varepsilon > 0$ $\left(\varepsilon = \frac{h(a)}{2}\right)$

since h is cont. at a, $\exists\, \delta > 0$ s/t $\forall x \in \mathbb{R}$ with

$|x - a| < \delta \Rightarrow |h(x) - h(a)| < \varepsilon$

$\Rightarrow 0 < h(a) - \varepsilon < h(x) < h(a) + \varepsilon$

Similarly, if $h(a) < 0$, $\exists\, \delta > 0$ s/t $\forall x \in \mathbb{R}$ with

$|x - a| < \delta \Rightarrow h(x) < 0$.

(16) $f : \mathbb{R} \to \mathbb{R}$ continuous s/t $f(0) = -2$, $f(1) = 3$

$S = \{x \in [0,1] \mid f(x) = 0\}$

(a) Using IVP (Intermidiate value Property) on $[0,1]$,

$\exists\, c \in [0,1]$ s/t $f(c) = 0$.

$\therefore c \in S \Rightarrow S \neq \phi$.

(b) $S \subseteq [0,1]$ (it is obvious)

$\therefore \beta \in [0,1]$ (since $\beta = \text{Sup } S$)

since, $\exists\, c \in (0,1)$ s/t $f(c) = 0$.

$\Rightarrow \exists\, c \in S$ such that $c > 0$.

$\Rightarrow 0 \neq \text{Sup } S \Rightarrow \beta \in (0,1]$.

(c) since $\beta = \text{Sup } S$, $\forall n \in \mathbb{N}$, $\exists\, x_n$ in S such that

$\beta - \frac{1}{n} < x_n \leq \beta \Rightarrow \lim_{n\to\infty} x_n = \beta$

$\therefore \lim_{n\to\infty} f(x_n) = f(\beta) \Rightarrow f(\beta) = 0$ ($\because x_n \in S\ \forall n \in \mathbb{N}$)

$\Rightarrow \beta \in S$.

(17) $|f|(x) = |f(x)|$

$\big||f|(x) - |f|(c)\big| = \big||f(x)| - |f(c)|\big| \le |f(x) - f(c)|$.

Now using the defn. of continuity of f, $|f|$ is cont. at c.

But the reverse need not be true.

Define $f : \mathbb{R} \to \mathbb{R}$ as follows :—

$$f(x) = \begin{cases} -1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{Q} \end{cases}$$

In Prob. 4, we have shown that f is not cont. at every point. But $|f|$ is constant function hence continuous.

(18) f is cont. on $[0,2]$ and $f(0) = f(2)$.

If $f(0) = f(1)$ then $c = 0, 1$. So we have done.

If $f(0) \neq f(1)$. Let, $h(x) = f(x) - f(x+1)$.

Then h is cont. on $[0,2]$.

$h(0) = f(0) - f(1)$, $h(1) = f(1) - f(2)$.

$\therefore h(1) + h(0) = f(0) - f(2) = 0$.

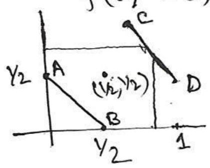
$\Rightarrow h(0) = -h(1)$. $\Rightarrow$ $h(0)$ and $h(1)$ is of opposite sign.

By Bolzano Theorem, $\exists\, c \in [0,1]$ s/t $h(c) = 0$

which implies $f(c) = f(c+1)$.

(19)(i) Define, $f : [0,1] \to \mathbb{R}$ as follows —

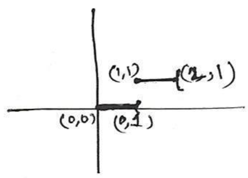
$f(0) = 0$, $f(1) = 1$, $f(1/2) = 1/2$ and $f(x) = \begin{cases} 1/2 - x, & 0 < x < 1/2 \\ 3/2 - x, & 1/2 < x < 1 \end{cases}$



$\therefore f$ is not cont. on $[0,1]$ (discontinuous at $0, 1/2, 1$)

But f satisfies IVP.

(ii) $f(x) = [x]$ $\forall x \in [0,2]$



$\therefore f$ is monotonic increasing on $[0,2]$

$f(0) = 0 < 1/2 < 1 = f(1)$

But, there does not exists $c \in [0,2]$ s/t $f(c) = 1/2$.

$\therefore f$ does not satisfy IVP on $[0,2]$.

(20) $f:[0,\pi] \to \mathbb{R}$ be defined by -

$$f(x) = \begin{cases} 0 & \text{is } x = 0 \\ x\sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x} & \text{if } x \neq 0. \end{cases}$$

Suppose if possible f is continuous at 0.

Then, $\lim_{x\to 0} f(x) = f(0) = 0$

$\Rightarrow \lim_{x\to 0}\left(x\sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}\right) = 0.$

$\Rightarrow \lim_{x\to 0} x\sin\frac{1}{x} - \lim_{x\to 0}\frac{1}{x}\cos\frac{1}{x} = 0$

$\Rightarrow 0 - \lim_{x\to 0}\frac{1}{x}\cos\frac{1}{x} = 0 \Rightarrow \lim_{x\to 0}\frac{1}{x}\cos\frac{1}{x} = 0 \longrightarrow (1)$

Choose $\varepsilon = 1$, then $\exists\ \delta > 0$ s/t $x \in [0,\pi]$ with $|x| < \delta$ i.e. $0 \leq x < \delta \Rightarrow |f(x)| < \varepsilon = 1$

$\Rightarrow \left|\frac{1}{x}\cos\frac{1}{x}\right| < 1$

But $\forall\ \delta > 0$, $\exists\ N \in \mathbb{N}$ s/t $1 < N(\pi\delta)$ (By Archimedean Property)

$\Rightarrow \frac{1}{N\pi} < \delta \Rightarrow \frac{1}{N\pi} \in [0,\pi]$ and $\frac{1}{N\pi} < \delta$.

But $\left|\frac{1}{\frac{1}{N\pi}}\cos\frac{1}{\frac{1}{N\pi}}\right| = |N\pi\cos N\pi| = N\pi > 1.$

This contradicts our initial assumption.

$\therefore$ f is not continuous at 0.

(21) $f:\mathbb{R} \to (0,\infty)$ satisfy $f(x+y) = f(x)f(y)\ \forall x \in \mathbb{R}$.

$\forall x \in \mathbb{R}$, $f(x) \neq 0$. ~~Also, $f(x+0) = f(x)f(0)$~~

Also, $f(x+0) = f(x)f(0) \Rightarrow f(x) = f(x)f(0) \Rightarrow f(0) = 1$.

Let, $c \in \mathbb{R}$ s/t $c \neq 0$.

~~Since, f is continuous at 0~~

Let, $\{x_n\}$ be a sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} x_n = c \Rightarrow \lim_{n\to\infty}(x_n - c) = 0$

$\because f$ is cont. at 0, $\lim_{n\to\infty} f(x_n - c) = f(0)$

$\Rightarrow \lim_{n\to\infty}\frac{f(x_n)}{f(c)} = 1$

$\Rightarrow \lim_{n\to\infty} f(x_n) = f(c)$ $[\because f(c) \neq 0]$

Hence, f is continuous $\forall c \in \mathbb{R}$.

$\forall x \in \mathbb{R}$, $f(x-x) = f(x)f(-x)$

$\Rightarrow f(0) = f(x)f(-x)$

$\Rightarrow 1 = f(x)f(-x)$

$\Rightarrow f(-x) = \frac{1}{f(x)}$

$\forall x, y \in \mathbb{R}$, $f(x-y) = f(x)f(-y) = \frac{f(x)}{f(y)}$.