Real Analysis (MA 101)
Tutorial Sheet- 7: Real Analysis, Partial Derivatives and Differentiability for function of several variables

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x_1, ..., x_n) = \sum_{i-1}^{n} a_i x_i$, where $a_1, ..., a_n \in \mathbb{R}$. Show that f is differentiable and find the total derivative of f.

2. Suppose $f(x, y) = 2x + 3y$, evaluate $Df(1, 2)$. What is the directional derivative of f at $(1, 2)$ in the direction $(1, 2)$?

3. Suppose $f(x, y) = (2x + 3y, xy)$. Show that f is differentiable and find the total derivative of f. Use this to find the directional derivative of f.

4. Suppose $S = \{(x, y) \in R^2 | (x - 1)^2 + (y - 1)^2 < \frac{1}{2}\}$ and $f : S \to R^2$ is the map $f(x, y) = (\frac{1}{x}, \frac{1}{y})$. Is f differentiable on S?

5. Suppose $f(x, y) = (x^2, y^2)$, then

   i) Find all the directional derivative of f.

   ii)Find all the partial derivative of f.

   iii) Find $Df(0, 0)$.

6. Suppose

$$f(x,y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, & \text{for } x \neq y \\ 0, & \text{for } x = y. \end{cases}$$

   Show that both the partial derivatives exist at $(0, 0)$, but the function is not continuous at $(0, 0)$.

7. Prove that $f(x, y) = \sqrt{|xy|}$ is not differentiable at $(0, 0)$, but both the partial derivatives exist at (0,0) and have the value 0. Hence deduce that these two partial derivatives are continuous except at the origin.

8. Suppose

$$f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & \text{for } x^2 + y^2 \neq (0,0) \\ 0, & \text{for } x^2 + y^2 = 0. \end{cases}$$

   Check the differentiability of the function at (0,0).

9. Suppose

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{for } x^2+y^2 \neq (0,0) \\ 0, & \text{for } x^2+y^2 = 0. \end{cases}$$

Check the differentiability of the function at (0,0).

10. Suppose

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & \text{for } x^2+y^2 \neq (0,0) \\ 0, & \text{for } x^2+y^2 = 0. \end{cases}$$

Check the differentiability of the function at (0,0).

11. If $f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}$ and $f(0,0) = 0$. Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

12. Show that if $w = f(u,v)$ satisfies the Laplace equation

$$f_{uu} + f_{vv} = 0$$

and if $u = \frac{x^2-y^2}{2}$ and $v = xy$ then w satisfies the Laplace equation

$$w_{xx} + w_{yy} = 0.$$

13. If $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$, $x^2+y^2+z^2 \neq 0$. Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

14. If $x^x y^y z^z = c$ (constant), Show that at (x,y,z) where $x = y = z$, $\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{x \log_e(ex)}$.

15. Use chain rule to find the derivative of $w = xy$, with respect to t along the path $x = \cos t,\ y = \sin t$. What is the derivative's value at $t = \frac{\pi}{2}$?

16. Evaluate $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ at the given point (x,y,z) for the function $u = \frac{p-q}{q-r}$, where $p = x+y+z,\ q = x-y+z,\ r = x+y-z$.

Note: Some of questions have been taken from the book of Calculus by Thomas and Finney. For more questions you can see the same book.