Mathematics I

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Syllabus

Properties of real numbers. Sequences of real numbers, montone sequences, Cauchy sequences, divergent sequences. Series of real numbers, Cauchy's criterion, tests for convergence. Limits of functions, continuous functions, uniform continuity, montone and inverse functions. Differentiable functions, Rolle's theorem, mean value theorems and Taylor's theorem, power series. Riemann integration, fundamental theorem of integral calculus, improper integrals. Application to length, area, volume, surface area of revolution. Vector functions of one variable and their derivatives. Functions of several variables, partial derivatives, chain rule, gradient and directional derivative. Tangent planes and normals. Maxima, minima, saddle points, Lagrange multipliers, exact differentials. Repeated and multiple integrals with application to volume, surface area, moments of inertia. Change of variables. Vector fields, line and surface integrals. Green's, Gauss' and Stokes' theorems and their applications.

Texts:
1) G. B. Thomas and R. L. Finney, Calculus and Analytic Geometry, 6th Ed/ 9th Ed, Narosa/ Addison Wesley/ Pearson, 1985/ 1996.
2) T. M. Apostol, Calculus, Volume I, 2nd Ed, Wiley, 1967.
3) T. M. Apostol, Calculus, Volume II, 2nd Ed, Wiley, 1969.
References:
4) R. G. Bartle and D. R. Sherbert, Introduction to Real Analysis, 5th Ed, Wiley, 1999.
5) J. Stewart, Calculus: Early Transcendentals, 5th Ed, Thomas Learning (Brooks/ Cole), Indian Reprint, 2003.

Advanced Calculus

- Convergence
- Continuity
- Differentiation
- Integration

Notations

- Set of Natural Numbers is written as $\mathbb{N}$.
- Set of Integers is written as $\mathbb{Z}$.
- Set of Rational Numbers is written as $\mathbb{Q}$.
- Set of Real numbers is written as $\mathbb{R}$.

The Set N of Natural Numbers

Peano Postulates

(N1) 1 belongs to N.

(N2) If n belongs to N, then its successor $n+1$ belongs to N.

(N3) 1 is not the successor of any element in N.

(N4) If m and n have same successor, then $n=m$.

(N5) A subset of N which contains 1 and which contains $n+1$ whenever it contains n, must equal N.

Mathematical Induction

(I_1) P_1 is true. (Basis for induction)

(I_2) P_{n+1} is true whenever P_n is true. (induction step)

Mathematical Induction : Further Generalization

A list $P_m, P_{m+1}, \cdots$ of Propositions is true provided

(i) P_m is true. (Basis for induction)

(ii) P_{n+1} is true whenever P_n is true and $n \geq m$. (induction step)

Example : (i) $n^2 > n + 1$, (ii) $2^n > n^2$.

- Rational numbers are inadequate for many purposes. For example,
- $p^2 = 2$ does not have solution in set of rational numbers.
- This leads to introduction of “Irrational Numbers”
- Consider the sequence $1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142, \cdots$. “What is it that this sequence tends to?”
- This leads to introduction of “Real number system”.

- Problem : Show that $p^2 = 2$ does not satisfy by any rational.
- The rational number system has certain gaps. The real number system fills this gap.

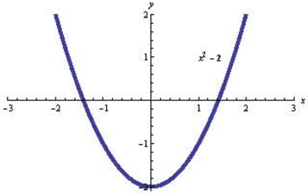

Figure: The graph of $x^2 - 2$ slips by all the rational numbers on $x - axis$.

True/False

- If x is rational and y is irrational, then $x + y$ is irrational.
- If x is rational and y is irrational, then xy is irrational.
- If x and y are irrational, then $x + y$ is irrational.
- If x and y are irrational, then xy is irrational.
- If x and y are irrational, then $x + y$ is rational.
- If x and y are irrational, then xy is rational.
- If $x \neq 0$ and x is rational and y is irrational, then xy is irrational.

Definition

Let S be a set. An order on S is a relation, denoted by $<$, with the following two properties:

- $x \in S$ and $y \in S$ then one and only one of the statements (**Tricotomy Law**)

$$x < y, \quad x = y, \quad x > y$$

  is true.

- If $x, y, z \in S$. If $x < y$ and $y < z$ then $x < z$. (**Transitive Law**)

  Note: The notation $x \leq y$ indicates $x < y$ or $x = y$ without specifying which of these two is to hold. $x \leq y$ is negation of $x > y$.

Ordered Set

An ordered set is a set in which an order is defined. Q the set of Rationals is an ordered set.

Upper and lower bounds

Definition

Suppose S is an ordered set, and $E \subseteq S$. If there exists a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that E is bounded above, and call β an upper bound of E.
Lower bounds are defined in the same way (with $\geq$ in place of $\leq$).

Supremum and Infimum

Definition

Suppose S is an ordered set, $E \subset S$ and E is bounded above. Suppose that there exists a $\beta \in S$ with the following properties

(i) β is an upper bound of E.

(ii) $\gamma < \beta$ then γ is not an upper bound of E.

Then β is called the least upper bound (**l.u.b.**) of E or the **Supremum** of E, and we write $\beta = \sup E$.
Similarly, we can define the greatest lower bound (**g.l.b**) or **Infimum** of the set E.

Least Upper Bound Property

Definition

An ordered set S is said to have least upper bound property if the following is true.

If $E \subset S$, E is not empty, and E is bounded above, then $\sup(E)$ exists in S. (Proof is not easy on the basis of Field and Ordered Set Properties)

Does Q has least upper bound property?

Examples

Ex.1 Consider A and B as ordered subset of Q, such that, $A = \{0 < p \in Q : p^2 < 2\}$ and $B = \{0 < p \in Q : p^2 > 2\}$. Upper bounds for A are elements of B but B has no smallest number so there is no least upper bound or supremum of A in Q. Similarly B does not have greatest lower bound in Q. [?????]

Does Q has least upper bound property? **No.**

Ex.2 Let $E = \{(n+1)/n : n = 1, 2, 3, \cdots\}$. Then $\sup(E) = 2$ which is in E and $\inf(E) = 1$ which is not in E.

Does set of real numbers has least upper bound property?

Completeness Axiom

Every nonempty subset S of R which is bounded above has least upper bound (l.u.b), i.e., $\sup S$ exists and is a real number.

Corollary

Every nonempty subset S of R which is bounded below has greatest lower bound (g.l.b), i.e., $\inf S$ exists and is a real number.

Note

For each real number there is point on the number line (or x-axis) and for each point on the number line there is real number.

Some Results

Archimedean Property

If $x \in R$, $y \in R$ and $x > 0$, then there exists a positive integer n such that

$$nx > y.$$

Corollary

If $y > 0$, $y \in R$ then $\exists\, m_y \in N$ such that

$$m_y - 1 \leq y < m_y.$$

Q is dense in R

If $x \in R$, $y \in R$ and $x < y$, then there exists $p \in Q$ such that $x < p < y$.

Problems

(i) Let S is some nonempty bounded subset of R, then $\sup(\mathbf{a}+\mathbf{S}) = \mathbf{a} + \sup(\mathbf{S})$ where a is some real number.

(ii) Let $a \leq b \; \forall \; a \in A$ and $\forall \; b \in B$ then $\sup \mathbf{A} \leq \inf \mathbf{B}$.

Problems

Let S be a non-empty bounded subset of R.

(iii) Let $a > 0$ and $aS = \{as : s \in S\}$. Prove then

$$\inf(\mathbf{aS}) = \mathbf{a}\inf(\mathbf{S}), \quad \sup(\mathbf{aS}) = \mathbf{a}\sup(\mathbf{S}).$$

(iv) Let $b < 0$ and $bS = \{bs : s \in S\}$. Prove then

$$\inf(\mathbf{bS}) = \mathbf{b}\sup(\mathbf{S}), \quad \sup(\mathbf{bS}) = \mathbf{b}\inf(\mathbf{S}).$$

Problem

Let A and B be bounded nonempty subsets of R and let $A+B=\{a+b : a\in A,\ b\in B\}$. Then show that

$$\sup(A+B)=\sup(A)+\sup(B),$$

$$\inf(A+B)=\inf(A)+\inf(B).$$

Problem

Let A be bounded nonempty subset of R and let $-A=\{-x : x\in A\}$. Then show that

$$\inf(A)=-\sup(-A).$$

sup and inf involving functions

$f(x) \leq g(x)$

Let $f(x)$ and $g(x)$ be two functions defined on $D \subseteq R$ with a range which is bounded and is subset of the real line, and $f(x) \leq g(x)\ \forall\, x \in D$. Then

$$\sup_{x \in D} f(x) \leq \sup_{x \in D} g(x),$$

and

$$\inf_{x \in D} f(x) \leq \inf_{x \in D} g(x).$$

Remark

What about

$$\sup_{x \in D} f(x) \leq \inf_{x \in D} g(x)?$$

Consider $f(x) = x^2$ and $g(x) = x$ on $D = [0, 1]$. (Horizontal line test ???)

sup and inf involving functions

$f(x) \leq g(y)\ \forall\, x,\ y \in D$

If $f(x) \leq g(y)\ \forall\, x, y \in D$. Then

$$\sup_{x\in D} f(x) \leq \inf_{y\in D} g(y).$$

$f(x) \leq g(x)\ \forall\, x \in X$

Let $f(x)$ and $g(x)$ be two functions defined on $X \subseteq R$ with bounded ranges. Then

$$\sup_{x\in D} (f(x) + g(x)) \leq \sup_{x\in D} f(x) + \sup_{x\in D} g(x),$$

and

$$\inf_{x\in D} f(x) + \inf_{x\in D} g(x) \leq \inf_{x\in D} (f(x) + g(x)).$$

Show by examples that there can be either equality or strict inequalities.

The Extended Real Number System

$\pm\infty$

- If A is a nonempty set of real numbers which has no upper bound and therefore no least upper bound in R, we express this by writing

$$\sup(A) = +\infty.$$

- If A is an empty subset of R we put

$$\sup(A) = -\infty.$$

- If A is a nonempty set of real numbers which has no lower bound and therefore no greatest lower bound in R, we express this by writing

$$\inf(A) = -\infty.$$

- If A is an empty subset of R we put $\inf(A) = +\infty$.

Finite, Countable and Uncountable Sets

Definitions

- The act of counting is undoubtedly one of the oldest human activity.
- **Cardinal Numbers** : Numbers used in counting.
- **Cardinality of a Set** : Number of elements in a set.
- 1-1 Correspondence : 1-1 Onto Mapping
- When two sets are said to have same Cardinal Number?
- Is the relation between two sets having same cardinal number an **Equivalence Relation**?
- Finite Set : A is finite if $A \sim \mathbb{N}_n$ for some $n \in \mathbb{N}$ ($\mathbb{N}_n = \{1, 2, \cdots, n\}$).
- Infinite Set : A is infinite if A is not finite.
- Countable Set : A is countable if $A \sim \mathbb{N}$.
- Uncountable Set : If A is neither finite nor countable.
- Countable sets are also referred as Enumerable or Denumerable.
- $A = \{0, 1, -1, 2, -2, 3, -3, \cdots\}$ is countable.

Definitions

- Finite set can not be equivalent to its subset. True
- Infinite sets can be equivalent to its subset. True
- Example : Z is equivalent to N.
- Another Definition : "A is infinite if A is equivalent to one of its proper subset."

Cardinality

- R is uncountable or uncountably infinite. (Proof was given by Cantor based on listing all real numbers. After listing is complete we discover a real number not present in the the list)
- All points on the real line represent higher type of of infinity than that of only naturals, integers or rational numbers.
- Set of Irrationals is uncountably infinite.
- Entire real line appears to be uncountable as it stretches from $-\infty$ to $+\infty$ but the truth is any open interval how so ever short it may be has precisely as many points as R itself.

Cardinality

- Open interval $[0, 1)$ is equivalent to $[0, 1) \times [0, 1)$.
- $1 < 2 < 3 \cdots < \aleph_0$, $\aleph_0 =$ number of elements in any countably infinite set.
- c is number of elements in $\boldsymbol{R}$.
- c is cardinal number of any subset of $\boldsymbol{R}$.
- Cantor's continuum Hypothesis: Every uncountable set of real numbers has c as its cardinal number $1 < 2 < 3 \cdots < \aleph_0 < c$.
- Are there any infinite cardinal number greater than c? Yes there are.
- Consider Power set of a set. [Think???]