Two-Dimensional (Bivariate) Normal Distribution

A jointly distributed two-dimensional continuous rv (X, Y) is said to have a bivariate normal dist^n if its joint PDF is given by

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \cdot e^{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_1}{\sigma_1}\right)^2+\left(\frac{y-\mu_2}{\sigma_2}\right)^2-2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)\right\}} \quad \text{——} (1)$$

$$-\infty < x < \infty, \quad -\infty < y < \infty$$
$$-\infty < \mu_1 < \infty, \quad -\infty < \mu_2 < \infty$$
$$0 < \sigma_1 < \infty, \quad 0 < \sigma_2 < \infty, \quad -1 < \rho < 1$$

<u>Notation:</u> Let us denote this probability distribution as $\boxed{(X,Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)}$

The joint PDF given in Equation (1) is a proper prob. distribution because $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1.$

(*) Next we compute marginal probability density functions of X and Y respectively.

<u>First we find marginal pdf of X.</u>

What is the formula for computing this pdf ??

It is $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$ ——— (2)

So integrate given joint PDF wrt y keeping x fixed.

So in this case we try to obtain a perfect square in variable y (similar to a normal distribution property) from the $f_{X,Y}(x,y)$ as given in Eq. (1).

So,

$$f_{X,Y}(x,y) = \frac{e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y-\mu_2}{\sigma_2}\right)^2 - 2\rho\frac{y-\mu_2}{\sigma_2}\cdot\frac{x-\mu_1}{\sigma_1} + \rho^2\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \rho^2\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x-\mu_1}{\sigma_1}\right)^2\right]}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left\{\frac{y-\mu_2}{\sigma_2} - \rho\left(\frac{x-\mu_1}{\sigma_1}\right)\right\}^2 + (1-\rho^2)\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right]}$$

$$= \frac{e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)\sigma_2^2}\left\{y-\mu_2-\rho\sigma_2\left(\frac{x-\mu_1}{\sigma_1}\right)\right\}^2} \qquad (3)$$

Let us use Eqn (3) in Eqn (2) to get

$$f_X(x) = \frac{e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}{\left(\sqrt{2\pi}\,\sigma_1\right)\left(\sqrt{2\pi}\,\sigma_2\sqrt{1-\rho^2}\right)} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)\sigma_2^2}\left[y-\mu_2-\rho\sigma_2\left(\frac{x-\mu_1}{\sigma_1}\right)\right]^2} dy \qquad (4)$$

Now apply the transformation

$$z = \frac{y-\mu_2-\rho\sigma_2\left(\frac{x-\mu_1}{\sigma_1}\right)}{\sqrt{1-\rho^2}\cdot\sigma_2}$$

$$\Rightarrow dz = \frac{dy}{\sqrt{1-\rho^2}\,\sigma_2}$$

Thus Eqn ④ is rewritten as (using the given transformation)

$$f_X(x) = \frac{e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}{\sqrt{2\pi}\,\sigma_1} \cdot \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz\right]$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}.$$

[$\because$ value of integral in square bracket is 1.]

So what is the result ??

Ans If (X, Y) follows bivariate normal $BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ then marginal probability distribution of X is such that

$$\boxed{X \sim N(\mu_1, \sigma_1^2)}$$

[So marginal pdf is one dimensional normal with mean μ_1 and variance σ_1^2.]

Similarly, we can verify that marginal pdf of Y is like

$$\boxed{Y \sim N(\mu_2, \sigma_2^2)}$$

So this is also univariate normal with mean μ_2 and variance σ_2^2.

Next we try to compute conditional Probability Density functions of X given Y = y and Y given X = x respectively.

Conditional PDF of X given Y = y is obtained as

$$f_{X|Y=y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \quad -\infty < x < \infty.$$

After simplifying the right hand side ratio we get that

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi}\,\sigma_1\sqrt{1-\rho^2}}\, e^{-\frac{1}{2\sigma_1^2(1-\rho^2)}\left[x-\left\{\mu_1+\rho\sigma_1\left(\frac{y-\mu_2}{\sigma_2}\right)\right\}\right]^2}$$

So $X|Y=y \sim N\left(\mu_1+\rho\sigma_1\left(\frac{y-\mu_2}{\sigma_2}\right),\ \sigma_1^2(1-\rho^2)\right)$

This is again a one-dimensional normal

Similarly;

$$f_{Y|X=x}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_2\sqrt{1-\rho^2}}\, e^{-\frac{1}{2\sigma_2^2(1-\rho^2)}\left[y-\left\{\mu_2+\rho\sigma_2\left(\frac{x-\mu_1}{\sigma_1}\right)\right\}\right]^2}$$

$Y|x \sim N\left(\mu_2+\rho\sigma_2\left(\frac{x-\mu_1}{\sigma_1}\right),\ \sigma_2^2(1-\rho^2)\right).$

<u>Result:</u> If $(X,Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ then marginal and conditional pdfs of X, Y, $X|y$, $Y|x$ are all univariate normal.

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Next we will discuss covariance, correlation coefficient and moment generating function under bivariate normal distribution.

Before we proceed further let us recall the following result: $E(X) = E[E(X|Y)]$.

A generalization of this result is presented below.

<u>Result:</u> Let (X,Y) be jointly distributed random variables and $g(X,Y)$ be any func^n of (X,Y) then

$$E[g(X,Y)] = E\{E[g(X,Y)\,|\,X]\}$$

$$\xrightarrow{\text{same as}} E\{E[g(X,Y)\,|\,Y]\}.$$

[That is you can condition either way.]

<u>Proof:</u> Note that $E[g(X,Y)|X] = \int_{-\infty}^{\infty} g(x,y) f_{Y|X}(y|x)\,dy$

So $E[E(g(X,Y)|X)] = \int_{-\infty}^{\infty} \left[\int g(x,y) f_{Y|X}(y|x)\,dy \right] f_X(x)\,dx$

that is,

$$E\left[E\left(g(X,Y)\mid X\right)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,\frac{f_{X,Y}(x,y)}{f_X(x)}\cdot dy\, f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\,dy\,dx$$

$$= E\left(g(X,Y)\right)$$ [by Definition of two dimensional expectation].

thus result is verified.

Next we compute covariance between X & Y.

Note that $\mathrm{Cov}(X,Y) = E(XY) - (EX)(EY)$

$$= E(XY) - \mu_1 \cdot \mu_2.$$

So we need to evaluate $E(XY)$. It can be complicated if we use the definition as in that case you have to simplify double integral. but using above expectation result $E(XY)$ is easily computed as follows:

$$E(XY) = E\left[X E(Y\mid X)\right] \quad \text{———} \; (*)$$

can you guess what is $E(Y/X)$. look at Conditional PDF of $Y|X$. So $E(Y|X)$ is the mean of that pdf.

so

$$E(Y|X) = \mu_2 + \rho\sigma_2\left(\frac{X-\mu_1}{\sigma_1}\right)$$

Now from $(*)$ we have

$$\cancel{E(XY) = E\left[E(Y|X)\right] = E\left[\mu_2 + \rho\sigma_2\left(\frac{X-\mu_1}{\sigma_1}\right)\right]}$$

$$= \cancel{\mu_2 +}$$

$$E(XY) = E\left[X\, E(Y|X)\right] = E\left[X\left(\mu_2 + \rho\sigma_2\left(\frac{X-\mu_1}{\sigma_1}\right)\right)\right]$$

$$= \mu_2 E(X) + \frac{\rho\sigma_2}{\sigma_1} E\{X(X-\mu_1)\}$$

$$= \mu_2\mu_1 + \rho\frac{\sigma_2}{\sigma_1}\left\{E(X^2) - \mu_1 E(X)\right\}$$

$$= \mu_1\mu_2 + \rho\frac{\sigma_2}{\sigma_1}\left\{\sigma_1^2 + \mu_1^2 - \mu_1^2\right\}$$

$$\left[\begin{array}{l} \because X \sim N(\mu_1, \sigma_1^2) \\ E(X) = \mu_1,\ V(X) = \sigma_1^2 \\ \therefore E(X^2) = \sigma_1^2 + \mu_1^2 \end{array}\right]$$

$$= \mu_1\mu_2 + \rho\sigma_1\sigma_2.$$

Put this in Eqn $(*)$ to get

$$\mathrm{Cov}(X,Y) = E(XY) - \mu_1\mu_2$$

$$= \mu_1\mu_2 + \sigma_1\sigma_2\rho - \mu_1\mu_2$$

$$= \rho\sigma_1\sigma_2$$

$$\therefore \boxed{\mathrm{Cov}(X,Y) = \rho\sigma_1\sigma_2}$$

Then Correlation coeff^n between X & Y is

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_1 \sigma_2} = \frac{\rho\sigma_1\sigma_2}{\sigma_1\sigma_2}$$

$$= \rho.$$

So the parameter ρ in Eqn 1 is nothing but the correlation coeff^n between X & Y.

Can you guess we are now able to specify all parameters of $BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$.

$\mu_1 \rightarrow$ mean of X. $\sigma_1^2 \rightarrow$ variance of X

$\mu_2 \rightarrow$ mean of Y. $\sigma_2^2 \rightarrow$ variance of Y

$\rho \rightarrow$ correlation coeff^n between X and Y.