INDIAN INSTITUTE OF TECHNOLOGY PATNA
MA-201 — FINAL EXAMINATION

TIME: 3 HOURS MARKS: 50

INSTRUCTIONS: *There are total twelve questions in this paper. Attempt all questions. Please do questions having more than one part in continuation. Calculators are not allowed.*

1. Let the function $f(z) = u(r,\theta)+iv(r,\theta)$ be analytic in a domain D does not include the origin. Using the Cauchy-Riemann equations in polar coordinates and assuming continuity of partial derivatives, show that, throughout D, the function $u(r,\theta)$ satisfies the partial differential equation
$$r^2u_{rr}(r,\theta) + ru_r(r,\theta) + u_{\theta\theta}(r,\theta) = 0.$$
[2]

2. Evaluate the integral:
$$\int_C \frac{\sin hz}{z^2(z-2)}dz$$
where $C = \{z : |z| = 1\}$, taken in the positive sense. [2]

3. Show that the singular point $z = 0$ of the function
$$f(z) = \frac{e^{az}}{1-e^{-z}}$$
is a simple pole. [1]

4. Find the Fourier series expansion of the function $f(t)$ where
$$f(t) = \begin{cases} \pi^2, & -\pi < t < 0 \\ (t-\pi)^2, & 0 \leq t < \pi \end{cases}$$
and it is known that $f(t)$ is periodic with period 2π. Evaluate the corresponding series when $t = \pi$. Also, determine the sine series for the given function $f(t)$. [3+1+2]

5. Consider two functions respectively defined as
$$(i)\ f(t) = \begin{cases} 1-t^2, & -1 < t < 1 \\ 0, & otherwise, \end{cases} \quad \text{and} \quad (ii)\ g(t) = \begin{cases} e^{-t}, & 0 < t < \infty \\ 0, & otherwise. \end{cases}$$
Compute Fourier transform of each of these functions. Further, evaluate the integral
$\int_0^\infty \frac{(\sin t - t\cos t)^2}{t^6}\,dt$ using Parseval's relation. [2+1+2]

6. Find the Fourier integral representation for the function
$$f(x) = \begin{cases} \cos x, & 0 < x < \pi \\ 0, & x < 0 \text{ and } x > \pi, \end{cases}$$
and then using it evaluate the integral $\int_0^\infty \frac{\alpha\sin(\alpha\pi)}{1-\alpha^2}\,d\alpha$. How can $f(0)$ and $f(\pi)$ be defined so that the corresponding Fourier integral converges to $f(x)$ for all real x. [2+1+1]

7. Consider a fluid, flowing with velocity, V, in a thin straight tube whose cross-sectional area is A. Suppose the fluid contains a contaminant whose concentration at position x at time t is given by $u(x,t)$. Find the concentration of the contaminant at any time t if it is given that initially the concentration of the contaminant at position x was $u(x,0) = e^x$. [5]

8. Use method of characteristics to find the general solution of the Cauchy problem of first order PDE.

$$xu(x,y)u_x(x,y) + yu(x,y)u_y(x,y) = xy, \quad 0 < x, y < \infty$$
$$u(x, 1-x) = x^2, \quad x \in (0, \infty)$$

[3]

9. (a) Solve the following quasilinear first order PDE by method of characteristics

$$xu_y - yu_x + u = 0, u(x,0) = 1, \text{ for } x > 0$$

[4]

(b) By drawing the characteristics, identify what problem may occur if the domain is extended to all real values of x. [2]

10. (a) Derive D'Alembert's formulae for the Initial Value Problem for Wave Equation given by

$$\begin{aligned} u_{tt}(t,x) &= c^2 u_{xx}(t,x), & 0 < t < \infty \\ u(0,x) &= f(x),\ u_t(0,x) = g(x), & -\infty < x < \infty \end{aligned}$$

[4]

(b) Suppose that an infinite string has an initial displacement

$$u(0,x) = f(x) = \begin{cases} x+1, & -1 \le x \le 0 \\ 1-2x, & 0 \le x \le 1/2 \\ 0, & \text{elsewhere.} \end{cases}$$

and zero velocity $u_t(0,x) = 0$. Write down the solution of the above wave equation with these initial conditions using D'Alembert's formulae. [2]

(c) Further for above solution sketch the solution profile $y = u(t,x)$ of the string displacement for $t = 0,$ and1. [1]

11. Using Duhamel's principle, solve

$$\begin{aligned} u_{tt}(t,x) - c^2 u_{xx}(t,x) &= \sin x, & 0 < t < \infty \\ u(0,x) &= 0,\ u_t(0,x) = 0, & -\infty < x < \infty \end{aligned}$$

[4]

12. Solve the Initial Boundary Value Problem for Heat Equation by separation of variables.

$$u_t(t,x) = ku_{xx}(t,x), \quad 0 < t < \infty, \quad 0 < x < 1$$
$$u(t,0) = 0, \text{ and } u(t,1) = 0, \quad 0 < t < \infty$$
$$u(0,x) = \begin{cases} 3x/2, & 0 < x \le 2/3 \\ 3 - 3x, & 2/3 \le x < 1 \end{cases}$$

[5]

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