Department of Mathematics, IIT Patna
MA - 101: Mathematics-I (MSE-Paper)

Time: 2 Hours Maximum Marks: 30 Roll No.:

There are **Twelve** questions in this paper. Attempt all questions. No mark will be awarded for answers without proper justification.

(1) Prove that the supremum of a non-empty bounded set which is not an element of the set is a cluster point of the set. Give an example of a non-empty set whose supremum is a member of the set but not the cluster point of the set. [3]

(2) Let $X = (x_n)$ be a sequence of reals that converges to x and suppose that $x_n \geq 0$. Then, prove that the sequence $(\sqrt{x_n})$ of positive square roots converges and $\lim(\sqrt{x_n}) = \sqrt{x}$. [3]

(3) Discuss the nature of the series

$$\frac{3}{7}x + \frac{3.6}{7.10}x^2 + \frac{3.6.9}{7.10.13}x^3 + \cdots ; \; x > 0$$

. [3]

(4) Let $I = [a, b]$ be closed bounded interval and let $f : I \to \mathbb{R}$ be a continuous function. Then prove that f is bounded on I. Further, give an example of a discontinuous function which is defined on a closed bounded interval but which is unbounded. [4]

(5) Let $f : [a, b] \to \mathbb{R}$ be continuous at a point c where $a < c < b$ and $f(c) \neq 0$ then there exists a $\delta > 0$ such that $f(x)$ has the same sign as $f(c)$ for all $x \in (c - \delta, c + \delta)$. [3]

(6) (a) Using the definition of a limit of a sequence show that $\lim(\frac{1}{\log(n+1)}) = 0$. [2]

(b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and that $f(r) = 0$ for every rational number r. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$. [2]

(7) Write the MVT. Using this prove that $f' = g' \Rightarrow f = g + c$, where c is a constant. [1.5]

(8) For a function f, it is given that $f(0) = 1$, $f'(x) = \dfrac{1}{1+x}$. Find constant bounds A and B such that $A \leq f(4) \leq B$. [Hint: Use MVT] [1.5]

(9) Using Cauchy's MVT prove the L'Hospital rule under the required conditions to evaluate $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ in case of $\frac{0}{0}$ and a is finite. [2]

(10) Using all the steps of the graphing procedure, trace the curve $f(x) = \dfrac{x}{\ln x}; x > 0$. [2].

(11) Assuming that the petrol burnt (per hour) in driving a motor boat varies as the cube of its velocity, show that the most economical speed when going against a current of c km/h is $\frac{3c}{2}$ km/h. [2].

(12) Using Newton method find positive root of equation $x^2 = 5$. (Apply 2 iterations only). [1]