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CS303 Tutorial 8

Name of Student : M. Maheeth Reddy

Roll No : 1801CS31

Ans 1 :

Given : $\{a^n b^j \mid n \leq j^2\}$ is language denoted by L

To prove : L is not CFL

Proof : Assume L is CFL, then pumping lemma must hold on L.

Consider m as critical length.

Now, $a^{m^2} b^m \in L$ and $|a^{m^2} b^m| \geq m$

$\therefore \exists\, u, v, x, y, z$ s.t $uvxyz = a^{m^2} b^m$

$|vxy| \leq m$

$|vy| \geq 1$

$\forall i \geq 0,\ uv^i x y^i z \in L$

Consider u, v, x, y, z as follows

Case (i) :

$$\overbrace{aaa\ldots aaa}^{m^2 \text{ times}}\overbrace{bbb\ldots bbb}^{m \text{ times}}$$

$$\underbrace{aa}_{u}\underbrace{a\ldots a}_{vxy}\underbrace{abbb\ldots bbb}_{z}$$

Consider $\left.\begin{array}{l} v = a^{k_1} \\ y = a^{k_2} \end{array}\right\}$ s.t $\left\{\begin{array}{l} |vxy| \leq m \quad |vy| \geq 1 \\ \Rightarrow 1 \leq k_1 + k_2 \leq m \\ k_1, k_2 \geq 1 \end{array}\right.$

For $i=2$, $a^{m^2+k_1+k_2} b^m \in L$, but $m^2+k_1+k_2 > m^2$

$\therefore$ Pumping lemma doesn't hold

Case (ii):

$$\underbrace{aaa\ldots aaa}_{u}\underbrace{bbb\ldots}_{vxy}\underbrace{bbb}_{z}$$

$k_1, k_2 \geq 1$ $\quad v = b^{k_1}$ $\quad y = b^{k_2}$ $\quad k_1+k_2 \geq 1$ $\quad k_1+k_2+|x| \leq m$

If $i=0$, $a^{m^2} b^{m-k_1-k_2} \in L$, but $m^2 > (m-k_1-k_2)^2$

$\therefore$ Pumping lemma doesn't hold

Case (iii):

$$\underbrace{aaa\ldots}_{u}\underbrace{aaabbb\ldots}_{vxy}\underbrace{bbb}_{z}$$

Subcase (1): If v spans both a and b, then if pumping is done the resultant string has a's and b's mixed up. So, it doesn't belong to L. Same holds for y.

Subcase (2): $v = a^{k_1}$ $\quad y = b^{k_2}$ $\quad 1 \leq k_1+k_2 \leq m$

$k_1, k_2 \geq 1$

For $i=0$, $a^{m^2-k_1} b^{m-k_2} \in L$

But, $(m-k_2)^2 \leq (m-1)^2$ $\quad (\because k_2 \geq 1)$

$= (m^2-2m+1) < (m^2-k_1)$ $\quad (\because k_1 < m)$

$\Rightarrow m^2-k_1 > (m-k_2)^2$ $\quad \therefore$ Pumping lemma doesn't hold

Hence L is not CFL.

Ans 2:

Given: $L = \{ w : n_a(w) < n_b(w) < n_c(w) \}$ is a language

To prove: L is not CFL

Proof:

Assume L is CFL, then Pumping Lemma must hold on it.

Consider 'm' as critical length.

Now, $a^m b^{m+1} c^{m+2} \in L \Rightarrow |a^m b^{m+1} c^{m+2}| \geq m$

$\exists\, u, v, x, y, z \in \{a, b, c\}^*$ s.t. $uvxyz = a^m b^{m+1} c^{m+2}$,

$|vxy| \geq m$, $|vy| \leq 1$, $uv^i xy^i z \in L \ \forall i \geq 0$

Consider u, v, x, y, z as follows:

Case ①:

$$\underbrace{aa\ldots}_{u}\,\underbrace{aa\ldots}_{vxy}\,\underbrace{aabb\ldots bb\ldots bbcc\ldots cc\ldots cc}_{z}$$

where $v = a^{k_1}$ $y = a^{k_2}$ $k_1, k_2 \geq 1$ $k_1 + k_2 \geq 1$

$k_1 + k_2 + |x| \leq m$

For $i = 2$, $a^{m+k_1+k_2} b^{m+1} c^{m+2} \in L$

but $m + k_1 + k_2 \geq m+1$ $(\because k_1 + k_2 \geq 1)$

$\therefore$ Pumping lemma doesn't hold in this case.

Case (2):

$$\underbrace{aa..aa...aabb}_{u}\underbrace{..bb..}_{vxy}\underbrace{.bb\,cc...cc...cc}_{z}$$

$(\because |vy| \geq 1)$

$v = b^{k_1}, y = b^{k_2}$ $\quad k_1, k_2 \geq 1$ $\quad 1 \leq k_1 + k_2 \leq m$

$(\because |vxy| \leq m)$

For $i = 2$, $\quad a^m b^{m+1+k_1+k_2} c^{m+2} \in L$

but, $\quad m+1+k_1+k_2 \geq m+2 \quad (\because k_1 + k_2 \geq 1)$

$\therefore$ Pumping lemma doesn't hold in this case

Case (3): $\quad \underbrace{aa..aa...aabb...bb...bbcc..}_{u}\underbrace{cc}_{vxy}\underbrace{..cc}_{z}$

$v = c^{k_1} \quad y = c^{k_2} \quad k_1, k_2 \geq 1 \quad |vy| \geq 1 \quad |vxy| \leq m$

$\Rightarrow 1 \leq k_1 + k_2 \leq m$

For $i = 0$ $\quad a^m b^{m+1} c^{m+2-(k_1+k_2)} \in L$

but $\quad (m+1) \geq (m+2) - (k_1 + k_2) \quad [\because k_1 + k_2 \geq 1]$

$\therefore$ Pumping lemma doesn't hold in this case.

Case (4):

$$\underbrace{aa...aa...}_{u}\underbrace{aabb....}_{vxy}\underbrace{bb...bbcc...cc...cc}_{z}$$

Subcase (i):

Now, if either v or y contain both a and b, then, on pumping, the resultant string will have a's & b's mixed up. So pumping lemma won't hold.

Subcase (ii):

if $v = a^{k_1}$ $y = b^{k_2}$ $k_1, k_2 \geq 1$ $|vxy| \leq m$

$|vy| \geq 1$

$\Rightarrow 1 \leq k_1 + k_2 \leq m$

Now, for $i = 2$ $a^{m+k_1} b^{m+1+k_2} c^{m+2} \in L$

but $m+1+k_2 \geq m+2$ ($\because k_2 \geq 1$)

Hence pumping lemma doesn't hold

Case (5):

$$\underbrace{aa..aa...aabb}_{u}...\underbrace{bb...bbcc...}_{vxy}\underbrace{cc...cc}_{z}$$

This can be explained in a way similar to case (4)

Pumping lemma won't hold.

Case (6):

$$\underbrace{aa...}_{u}\underbrace{aa...aabb...bb...bbcc...}_{vxy}\underbrace{cc...cc}_{z}$$

This also can be explained using ~~case~~ previous cases. Pumping lemma won't hold.

$\therefore$ The given language $L = \{ w : n_a(w) < n_b(w) < n_c(w) \}$ is not context free. Hence Proved!

Ans 3:

Given: $L = \{a^i b^j c^k \mid 0 \le i \le j \le k\}$ is a language}

To prove: L is not a CFL

Proof: Assume L is CFL, then pumping lemma must hold on L.

Assume critical length is m

Consider a string $a^m b^m c^m \in L \Rightarrow |a^m b^m c^m| \ge m$

$\because$ L is CFL, $\exists\ u, v, x, y, z \in \{a, b, c\}^*$ such that

$uvxyz = a^m b^m c^m \quad |vxy| \ge m, \ |vy| \le 1, \ uv^i x y^i z \in L \ \forall i \ge 0$

Consider u, v, x, y, z as follows

Case ①:

$$\underbrace{aa..aa}_{u}..\underbrace{aa}_{vxy}\underbrace{bb...bb...bbcc...cc...cc}_{z}$$

$v = a^{k_1} \quad y = a^{k_2} \quad k_1, k_2 \ge 1 \quad |vxy| \le m$ and $|vy| \ge 1$

$\Rightarrow 1 \le k_1 + k_2 \le m$

For $i = 2$

$a^{m+k_1+k_2} b^m c^m \in L$

but $m + k_1 + k_2 > m$, so pumping lemma doesn't hold in this case.

Case ②:

$$\underbrace{aa\ldots aa\ldots aabb\ldots}_{u}\underbrace{bb}_{vxy}\underbrace{\ldots bb\,cc\ldots cc\ldots cc}_{z}$$

$v = b^{k_1}$ $\quad y = b^{k_2}$ $\qquad k_1, k_2 \geq 1$ $\qquad |vxy| \leq m$ and $|vy| \geq 1$

$\Rightarrow \quad 1 \leq k_1 + k_2 \leq m$

For $i = 2$, $\quad a^m b^{m+k_1+k_2} c^m \in L$

but $m + k_1 + k_2 > m$, so pumping lemma doesn't hold in this case.

Case ③:

$$\underbrace{aa\ldots aa\ldots aabb\ldots bb\ldots bbcc\ldots}_{u}\underbrace{cc}_{vxy}\underbrace{\ldots cc}_{z}$$

$v = c^{k_1}$ $\quad y = c^{k_2}$ $\qquad k_1, k_2 \geq 1$ $\qquad |vxy| \leq m$ & $|vy| \geq 1$

$\Rightarrow \quad 1 \leq k_1 + k_2 \leq m$

For $i = 0$, $\quad a^m b^m c^{m-k_1-k_2} \in L$

but $m - k_1 - k_2 < m$, so pumping lemma doesn't hold in this case.

Case ④:

$$\underbrace{aa\ldots aa}_{u}\underbrace{\ldots aabb\ldots}_{vxy}\underbrace{bb\ldots bb\,cc\ldots cc\ldots cc}_{z}$$

Subcase (i): if either v or y contains both a & b, then the string obtained by pumping will have a & b mixed up. So, pumping lemma won't hold.

Subcase (ii)

$v = a^{k_1} \quad y = b^{k_2}$ $\quad k_1, k_2 \geq 1 \quad |vxy| \leq m$

$|vy| \geq 1$

for $i = 2$, $\quad a^{m+k_1} b^{m+k_2} c^m \in L$

but $\quad m+k_2 > m$, so pumping lemma doesn't hold.

Case ⑤:

$$\underbrace{aa\ldots aa\ldots aabb\ldots bb}_{u}\ldots\underbrace{bbcc\ldots cc}_{vxy}\ldots\underbrace{cc\ldots cc}_{z}$$

Pumping lemma won't hold in this case and can be explained similarly as case ④.

Case ⑥:

$$\underbrace{aa\ldots aa}_{u}\ldots\underbrace{aabb\ldots bb\ldots bbcc\ldots cc}_{vxy}\ldots\underbrace{cc}_{z}$$

Pumping lemma won't hold in this case and can be explained similarly as previous cases.

$\therefore$ The given language $L = \{a^i b^j c^k \mid 0 \leq i \leq j \leq k\}$ is not a CFL. Hence Proved!