Real Analysis (MA101), Tutorial Sheet-I

1. Let $x_0 \in \mathbb{R}$ and $x_0 \geq 0$. If $x_0 <\in$ for every positive real number $\in$, show that $x_0 = 0$

2. Let S be a non empty bounded above subset of $\mathbb{R}$. If α and β are supremum of S, show that $\alpha = \beta$

3. Prove that $\sqrt{2}$ is not a rational number and hence deduce that $\sqrt{2} + \sqrt{3}$ cannot be rational.

4. Show that if x and y are any two numbers of bounded set of real numbers S_1 and S_2, respectively, then prove that the set S whose elements are of the form $x+y$ is also bounded and $\sup S_1 + \sup S_2 = \sup S$, $\inf S_1 + \inf S_2 = \inf S$.

5. Prove that the set S whose elements are of the form $\frac{1}{p} + \frac{1}{q}$, where p and q are positive integers, is a bounded set for which $\inf S = 0$, $\sup S = 2$.

6. Find the supremum of the set $X = \{\pi - 1, \pi - \frac{1}{2}, \pi - \frac{1}{3} ...\}$.

7. Let E be a non-empty bounded above subset of $\mathbb{R}$. If $\alpha \in \mathbb{R}$ is an upper bound of E and $\alpha \in E$, show that α is the l.u.b of E.

8. Suppose that α and β are any two real numbers satisfying $\alpha < \beta$. Show that there exists $n \in \mathbb{N}$ such that $\alpha < \alpha + \frac{1}{n} < \beta$.

   Similarly, show that for any two real numbers s and t satisfying $s < t$, there exists $n \in \mathbb{N}$ such that $s < t - \frac{1}{n} < t$.

9. Let A be a non-empty bounded below subset of $\mathbb{R}$ and $\alpha \in \mathbb{R}$ is an lower bound of A and $\alpha \in A$. Suppose for every $n \in \mathbb{N}$, there exists $a_n \in A$ such that $a_n < \alpha + \frac{1}{n}$. show that α is the infimum of A.

10. If S_1 and S_2 are two bounded sets of real numbers. Prove that the bounds of the set $S_1 \cup S_2$ are $\max\{sup\ S_1, sup\ S_2\}$ and $\min\{inf\ S_1, inf\ S_2\}$.

11. Let $\{p_n\}$ be a sequence of rationals such that $p_1 < p_2 < p_3 < ....$ and $p_n \to 0$ as $n \to \infty$. Let $A = \cup_{i=1}^{\infty}(p_i, p_{i+1})$. What is $\sup A$ and $\inf A$?

12. Let $A = \{x \in \mathbb{R} | 3x^2 + 8x - 3 < 0\}$. Find sup and inf of A.

13. Let S and T are non-empty subsets of $\mathbb{R}$, such that $s \in S$, $t \in T \Rightarrow s \leq t$. Prove that $\sup S \leq \inf T$.

14. Find sup and inf of A.

    (a) $A = \{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N}$.

    (b) $A = \{\frac{n+(-1)^n}{n} : n \in \mathbb{N}\}$.

    (c) $A = \{x \in \mathbb{R} | \sin 1/x = 0\}$.

    (d) $\{n^{(-1)^n} : n \in \mathbb{N}\}$.

    (e) $\bigcap_{n=1}^{\infty}[\frac{-1}{n}, 1 + \frac{1}{n}]$.

    (f) $\{1 - \frac{1}{3^n} : n \in \mathbb{N}\}$.

(g) $\{\cos(\frac{n\pi}{3}) : n \in \mathbb{N}\}$.

(h) $\{\frac{1}{n} : n \in \mathbb{N}$ and n is prime $\}$.

(i) $\{1 - (-1)^n/n : n \in N\}$.

(j) $\{\frac{1}{n} - \frac{1}{m} : m, n \in N\}$

15. Let $A_n = \{x \in \mathbb{R} | x \leq -\frac{1}{n}$ or $x \geq \frac{1}{n}\}$, $n \in N$ $A = \cup_{i=1}^{\infty} A_n$, $B = \cap_{i=1}^{\infty} B_n$. Find sup and inf of A and B if exist.

16. Show that if m, n are rational numbers then $m + n$ and mn are rational numbers.

17. If $x > -1$, then show that $(1 + x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

18. Give an example of a set which is
(a) bounded above but not bounded below,
(b) bounded below but not bounded above,
(c) bounded (above as well as below),
(d) neither bounded above nor bounded below.

19. Show that supremum of a set (if it exists) is unique.

20. Let S and T be nonempty bounded subsets of $\mathbb{R}$.

    (i) Prove that if $S \subseteq T$, then $\inf T \leq \inf S \leq \sup S \leq \sup T$

    (ii) Prove that $\sup(S \bigcup T) = \max\{\sup S, \sup T\}$.

21. If $y > 0$, show that there exists $n \in N$ such that $\frac{1}{2^n} < y$.

22. Let $x_0 \in \mathbb{R}$ and $x_0 > 0$. If $x_0 < \in$ for every positive real number $\in$, show that $x_0 = 0$.

23. Let S be a non empty bounded above subset of $\mathbb{R}$. If α and $\beta$are supremums of S, show that $\alpha = \beta$.

24. If S_1 and S_2 are two bounded sets of real numbers. Prove that the bounds of the set $S_1 \cup S_2$ are $\max\{sup\ S_1, sup\ S_2\}$ and $\min\{inf\ S_1, inf\ S_2\}$.

25. If x and y are members of bounded sets A and B of real numbers, prove that bounds of the set C of numbers $\frac{y}{x}$ are the $\frac{supB}{supA}$ and $\frac{infB}{infA}$, provided $\inf A \neq 0$ and $\sup A \neq 0$ and the members of A and B are all positive.