End Semester Examination, MA102
Indian Institute of Technology Patna
25 April 2011

All twelve problems are compulsory. Maximum marks to each problem is given. Total marks are 50 and total time is 180 minuets.

1. Show that the set of all real valued continuous functions $y = f(x)$ satisfying the differential equation
$$y''' + 6y'' + 11y' + 6y = 0$$
is a subspace of the space of all real valued continuous functions over the field of real numbers. Find a basis of this subspace. [4]

2. Let $A = (a_{ij})_{n\times n}$ be an $n \times n$ matrix such that $\sum_j a_{ij} = 1$ for each i. Using the definition of a characteristic value for the linear transformations prove that the associated linear transformation of the matrix A has a characteristic value 1. [3]

3. Using the Cayley-Hamilton theorem find the inverse of the matrix
$$A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$
[3]

4. Let p be a fixed real number. Determine the general solution of the differential equation
$$(1 - x^2)\, y'' - 2xy' + p(p+1)y = 0$$
near origin, using series solution method. [5]

5. Find a Frobenius series solution of the following differential equation
$$x\, y'' + 2y' + xy = 0$$
Can you find two linearly independent series solutions for the given differential equation? What are they? [2+1+2]

6. Find the general solution of the following differential equations
   (i) $(x + y)y' = 1, \quad x + y \neq -1$
   (ii) $yy'' = (y')^2, \quad y' \neq 0$
   (iii) $(1 + y^2)dx = (\arctan y - x)dy$
   (iv) $x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \log x$

[2 + 2 + 2 + 2]

7. Find the orthogonal trajectories of the system of circles touching y-axis at origin. [3]

8. Using variation of parameter find the general solution of

$$y'' - 2y' + y = \frac{e^{-x}}{1+x^2}.$$

[4]

9. A circuit contains a 1 farad capacitor, 1 henry inductor, and an AC generator whose voltage at time t is $\sin(t)$. Find the oscillation of the charge in the circuit as a function of time t. Assume that charge $q(t)$ at time $t = 0$ is $q(0) = 0$ and current $i(t) = q'(t)$ at time $t = 0$ is $q'(0) = 1$.

   Hint: The voltage drop caused by a capacitor is proportional to the charge; the reciprocal of the proportionality constant is called the capacitance of the capacitor and is measured in units called farads. The voltage drop caused by an inductor is proportional to $\frac{d}{dt}i(t)$; the proportionality constant is called the inductance of the inductor and is measured in units called henrys. One fundamental fact about circuits is Kirchoff's Law which states that the sum of the voltage drops in any circuit loop must be 0. [5]

10. State and proof the second translation theorem for Laplace transform. Then find the Laplace transform of the function $f(t)$ defined as

$$f(t) = \begin{cases} 0, & \text{if } 0 \le t \le a; \\ t^5, & \text{if } t > a. \end{cases}$$

where $a(\geq 0)$ is a constant. [1+1+2]

11. Use Laplace transform technique to solve the initial value problem:

$$y''(t) - 5y'(t) + 6y(t) = te^{2t} + e^{3t},\ y(0) = 0,\ y'(0) = 1.$$

[3]

12. Find the inverse transform $f(t)$ of the Laplace transform:

(i)

$$\frac{e^6}{(s+6)^{5/6}}$$

[1]

(ii)

$$\frac{(e^{-as} - e^{-bs})^2}{s^2},\ b > a > 0.$$

Here, the function $f(t)$ should be properly illustrated for different values of t. [1+1]