Riemann Integral

Amit K. Verma

Department of Mathematics, IIT Patna

October 10, 2017

Outline

Definition: Darboux Integral

Let f be a bounded function on closed interval $[a, b]$. For $S \subseteq [a, b]$, we adopt the notation $M(f, S) = \sup\{f(x) : x \in S\}$ and $m(f, S) = \inf\{f(x) : x \in S\}$.
A partition of $[a, b]$ is any finite ordered subset p having the form

$$\mathbf{P = \{a = t_0 < t_1 < \cdots < t_n = b\}.}$$

The upper Darboux sum $U(f, P)$ of f with respect to P is the sum

$$\mathbf{U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k])(t_{k-1} - t_k),}$$

and the lower Darboux sum $L(f, P)$ is

$$\mathbf{L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k])(t_{k-1} - t_k).}$$

Remark

Note that

$$U(f,P) \leq \sum_{k=1}^{n} M(f,[a,b])(t_{k-1}-t_k) = M(f,[a,b])(b-a)$$

likewise $L(f,P) \geq m(f,[a,b])(b-a)$, so

$$\mathbf{m(f,[a,b])(b-a) \leq L(f,P) \leq U(f,P) \leq M(f,[a,b])(b-a)} \tag{1}$$

The upper Darboux integral $U(f)$ of f over $[a, b]$ is defined by

$$\mathbf{U(f)} = \inf\{\mathbf{U(f,P)} : \mathrm{P \text{ is a partition of }} [\mathbf{a}, \mathbf{b}]\}$$

and the lower Darboux integral is

$$\mathbf{L(f)} = \sup\{\mathbf{L(f,P)} : \mathrm{P \text{ is a partition of }} [\mathbf{a}, \mathbf{b}]\}.$$

In view of (1), $U(f)$ and $L(f)$ are real numbers.

Definition

If $L(f) = U(f)$ then $f(x)$ is Darboux Integrable and we write

$$L(f) = U(f) = \int_a^b f(x)dx.$$

Lemma

Let f be a bounded function on $[a, b]$. If P and Q are partitions of $[a, b]$ and $P \subseteq Q$, then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

Lemma

Let f be a bounded function on $[a, b]$. If P and Q are partitions of $[a, b]$, then

$$L(f, P) \leq U(f, Q).$$

Theorem

Let f be a bounded function on $[a, b]$. Then

$$L(f) \leq U(f).$$

Theorem

A bounded function f on $[a, b]$ is integrable if and only if for each $\epsilon > 0$ there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon.$$

Definition

The mesh of a partition P is the maximum length of the subintervals comprising P. Thus if

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\},$$

then

$$mesh(P) = \max\{t_k - t_{k-1} : k = 1, 2, \cdots, n\}.$$

Here is another "Cauchy criterion" for integrability

Theorem

A bounded function f on $[a, b]$ is integrable if and only if for each $\epsilon > 0$ there exists a $\delta > 0$ such that
$mesh(P) < \delta$ implies $U(f, P) - L(f, P) < \epsilon$
for all partitions P of $[a, b]$.

Riemann Integral

Let f be a bounded function on $[a, b]$, and let $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition of $[a, b]$. A Riemann sum of f associated with the partition P is a sum of the form

$$\sum_{k=1}^{n} f(x_k)(t_k - t_{k-1})$$

where $x_k \in [t_{k-1}, t_k]$ for $k = 1, 2, 3 \cdots n$ The choice of x_k's is quit arbitrary, so there are infinitely many Riemann sums associated with a single function and partition.

The function f is **Riemann integrable** on $[a, b]$ if there exists a number r with the following property. For each $\epsilon > 0$ there exists $\delta > 0$ such that

$$|S - r| < \epsilon$$

for every Riemann sum S of f associated with a partition P having $mesh(P) < \delta$. The number r is the Riemann integral off on [a, b] and will be provisionally written as $R \int_a^b f$

Properties of Riemann Integral

Theorem

A bounded function f on $[a, b]$ is Riemann integrable if and only if it is $[Darboux]$ integrable, in which case the values of the integrals agree.

Properties of Riemann Integral

Theorem

Every monotonic function f on $[a, b]$ is integrable.

Theorem

Every continuous function f on $[a, b]$ is integrable.

Theorem

Let f and g be integrable functions on $[a, b]$, and let c be a real number. Then

(i) cf is integrable and $\int_a^b cf = c\int_a^b f$;

(i) $f + g$ is integrable and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

Theorem

If f is integrable on $[a, b]$, then If $|f|$ is integrable on $[a, b]$ and

$$|\int_a^b f| \leq \int_a^b |f|$$

Theorem

If f and g are integrable on $[a, b]$ and if $f(x) \leq g(x)$ for $x \in [a, b]$, then

$$\int_a^b f \leq \int_a^b g.$$

Theorem

Let f be a function defined on $[a, b]$. If $a < c < b$ and f is integrable on $[a, c]$ and on $[c, b]$, then f is integrable on $[a, b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Definition

A function f on [a, b] is piecewise monotonic if there is a partition

$$P = \{a = t_0 < t_1 < \ldots < t_n = b\}$$

of $[a, b]$ such that f is monotonic on each interval (t_{k-1}, t_k). The function f is piecewise continuous if there is a partition P of $[a, b]$ such that f is uniformly continuous on each interval (t_{k-1}, t_k).

Theorem

If f is a piecewise continuous function or a bounded piecewise monotonic function on $[a, b]$, then f is integrable on $[a, b]$.

Intermediate Value Theorem for Integrals

If f is a continuous function on [a, b], then for at least one x in $[a, b]$ we have

$$f(x) = \frac{1}{b-a}\int_a^b f.$$

Fundamental Theorem of calculus I

If g is a continuous function on $[a, b]$ that is differentiable on (a, b), and if g' is integrable on $[a, b]$, then

$$\int_a^b g' = g(b) - g(a)$$

Theorem : Integration by Parts

If u and v are continuous functions on $[a, b]$ that are differentiable on (a, b), and if u' and v' are integrable on [a, b], then $\int_a^b u(x)v'(x)dx + \int_a^b u'(x)v(x)dx = u(b)v(b) - u(a)v(a)$

Fundamental Theorem of calculus II

Let f be an integrable function on $[a, b]$. For x in $[a, b]$, let

$$F(x) = \int_a^x f(t)dt.$$

Then F is continuous on $[a, b]$. If f is continuous at x_0 in (a, b), then F is differentiable at x_0 and

$$F'(x_0) = f(x_0).$$

Theorem : Change of variable

Let u be a differentiable function on an open interval J such that u' is continuous, and let I be an open interval such that $u(x) \in I$ for all $x \in J$. If f is continuous on I, then $f\ o\ u$ is continuous on J and

$$\int_a^b f\ o\ u(x)u'(x)dx = \int_{u(a)}^{u(b)} f(u)du$$