Mathematics III (MA201)
Mid Semester Examination (19-09-2011)

Time: 2 Hrs **Max. Marks: 30**

Q1. Prove that for any integer $n > 1$,

$$(z+1)^{2n} + (z-1)^{2n} = 2\left(\prod_{k=0}^{n-1}\left[z^2 + \cot^2\left(\frac{(2k+1)\pi}{4n}\right)\right]\right)$$

Hence deduce the following:

(i) $\cot^2\left(\frac{\pi}{32}\right)\cot^2\left(\frac{3\pi}{32}\right)...\cot^2\left(\frac{15\pi}{32}\right) = 1.$

(ii) $\csc^2\left(\frac{\pi}{32}\right)\csc^2\left(\frac{3\pi}{32}\right)...\csc^2\left(\frac{15\pi}{32}\right) = 2^{15}.$

(3)

Q2. If the points P_1 and P_2, represented by z_1 and z_2 respectively, are such that $|z_1 + z_2| = |z_1 - z_2|$, prove that (i) z_1 / z_2, $(z_2 \neq 0)$ is a pure imaginary number, (ii) $< P_1 0 P_2 = 90^0$.

(2)

Q3. For the function defined by $f(z) = \sqrt{|xy + 2y - 3x - 6|}$, show that the Cauchy Riemann equations are satisfied at (-2, 3), but the function is not differentiable at that point.

(3)

Q4. State two similarities and two dissimilarities between $f(x) = e^x, f : R \to R$ and $g(z) = e^z,\ g : C \to C.$

(2)

Q5. Solve the equation: $\tanh z - \coth z + 2i = 0.$ **(3)**

Q6. Let $f(z)$ be an entire function which satisfies any one of the conditions for all $z \in C$.

(i) $\operatorname{Re} f(z)$ has upper bound.

(ii) $|f(z)| \geq 1.$

Then prove that $f(z)$ is constant. **(3)**

Q7. Find a bilinear transformation which maps the points

$$z_1 = -i, z_2 = 0, z_3 = i \text{ onto } w_1 = -1, w_2 = i, w_3 = 1.$$

Into what point $z = 2 + 3i$ is transformed ? **(2)**

Q8. Find the residue of the function

$$f(z) = \frac{z^2 + \sin z}{\cos z - 1}$$

at its all singular points. **(3)**

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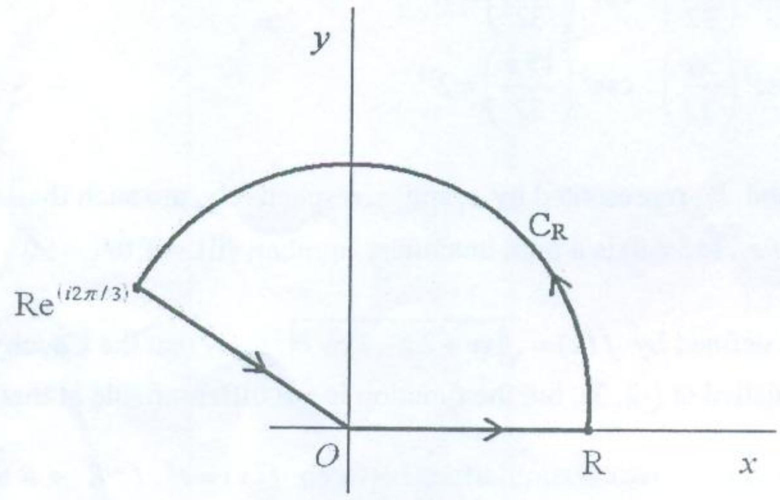
Q9. Use residue and the contour shown in Figure below, where $R>1$ to establish the integration formula

$$\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}.$$

(4)



(Here, $Re^{(i2\pi/3)}$ stands for $R\exp(i2\pi/3)$).

Q10. Obtain the Laurent series expansion of

$$f(z) = \frac{z}{z^2-4z+3}$$

in the region (a) $1<|z|<3$ (b) $|z-1|>2$.

(3)

Q11. With the help of Cauchy Integral formula, evaluate the integral:

$$\int_C \frac{\cos 2\pi z}{(2z-1)(z-1)}dz$$

where $C:|z|=2$ is a positively oriented circle.

(2)

ALL THE BEST