INDIAN INSTITUTE OF TECHNOLOGY PATNA

MATHEMATICS-I - MA 101
B.TECH.- I, JULY - DEC. 2009 - 2010
END SEMESTER EXAMINATION

TIME: 3 HRS | MAX MARKS : 50

Note: Attempt all the questions. Give precise and to the point solutions for each question with necessary reason and justifications where ever required. Marks against each question are indicated.

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1. Write the statement of any two of the following theorems -
   (*a*) Green's theorem, (*b*) Stoke's theorem, (*c*) Divergence theorem. [1 + 1]

2. Evaluate the integral $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$. [3]

3. (*a*) Find linearization of $f(x,y) = \sin(x)\cos(y)$ at the point $(\pi/4, \pi/4)$. (*b*) Find the extreme values of $f(x,y,z) = x(z+y)$ on the curve of intersection of the right circular cylinder $x^2+y^2 = 4$ and $xz = 4$. [2 + 3]

4. Let $D_u f$ denote the derivative of $f(x,y) = (x^2+y^2)/2$ in the direction of the unit vector $\mathbf{u} = u_1\hat{i} + u_2\hat{j}$. Find the average value of $D_u f$ over the triangular region cut from the second quadrant by the line $x + y = -1$. [5]

5. (*a*) Evaluate the double integral $\int_0^2 \int_{\sqrt{2x-x^2}}^{\sqrt{2x}} y\,dy\,dx$, by changing the order of integration. (*b*) Find the center of mass of the solid $x^2+y^2 = 4$ and the planes $z = 1$ and $z = 4$. [3 + 2]

6. A slender metal arch denser at the bottom than at the top, lies along the semicircle $y^2 + z^2 = 1$, $z \geq 0$ in the $y - z$ plane. Find the center of mass of arch if the density at the point (x,y,z) on the arch is $\delta(x,y,z) = 4 - z$. [5]

7. Find a parametrization of the cone $z = \sqrt{(x^2+y^2)}$, $0 \leq z \leq 1$. Hence evaluate the surface integral of $G(x,y,z) = x^2$ over this cone. [1 + 2]

8. Show that the polar equation

$$r = \frac{ke}{1 + e\cos\theta},\quad 0 < e < 1,\quad k \in \mathbb{R} - \{0\}$$

represent equation of an ellipse. [2]

9. Derive the expression for tangential (a_T) and normal (a_N) component of acceleration. Also, find them for the space curve

$$r(t) = (t\cos t)\hat{i} + (t\sin t)\hat{j} + t^2\hat{k},$$

at $t = 0$ without finding T and N. [4]

10. Let $\{a_n\}$ be a decreasing sequence, $a_n \geq 0$ and $\lim_{n\to\infty} a_n = 0$. For each $n \in N, b_n = \frac{a_1+a_2+\cdots+a_n}{n}$. Show that $\sum_{n\geq 1}(-1)^n b_n$ converges. [4]

11. Let $f : R \to R$ satisfy $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at 0, show that f is continuous at every point $c \in \mathbb{R}$. [2]

12. Let D be a region in space such that the volume of this region $V = \iiint_D r dr dz d\theta$

$= 2\int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-z^2}} r dr\, dz\, d\theta$. Sketch this region. A ray along +ve x-axis, with initial point at origin enters and exits this surface at points $x = x_1$ and $x = x_2$. Find the length of the two segments joining origin to these two points. Also evaluate this integral. [2 + 3 + 3 + 2]