Tutorial I

1. $V(\neq \phi)$, is a set $\oplus : V \times V \to V$; F is a field $\odot : F \times V \to V$
$u \oplus v \in V \ \forall \ u, v \in V$ $\quad a \cdot v \in V \ \forall \ a \in F$ & $u \in V$
$(V, \oplus, \odot)$ is called vector space.
$V = R^2 \{(a, b) / a \in R, b \in R\}$ R is set of real numbers.
$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, 0)$.
Now, let the additive identity be (e_1, e_2)
Now, $(a_1, a_2) + (e_1, e_2) = (a_1, a_2) = (e_1, e_2) + (a_1, a_2)$

$(a_1 + e_1, 0) = (a_1, a_2)$

$a_1 + e_1 = a_1$ & $0 = a_2$

$e_1 = 0 = a_2$ not possible.

So, additive identity does not exist.
$a_2 \neq 0$ but the relation gives us $a_2 = 0$ thats why it doesn't exist

ii) Now, let the additive identity be (e_1, e_2, e_3)
Now, $(a_1, a_2, a_3) + (e_1, e_2, e_3) = (a_1 e_1, a_2 e_2, a_3 e_3) = (a_1, a_2, a_3)$
$a_1 e_1 = a_1$; $a_2 e_2 = a_2$; $a_3 e_3 = a_3$.
$e_1 = 1$; $e_2 = 1$; $e_3 = 1$.
and there is no condition on other numbers & is applicable for all the element of R^3
so, additive identity exist & equals to $(1, 1, 1)$.
For inverse
let the inverse of additivity be (b_1', b_2', b_3') Now

if V is a vector space & S is a subset of V & is also a vector space then S is a subspace.

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$(a_1, a_2, a_3) + (b_1', b_2', b_3') = (e_1, e_2, e_3)$ (by defination)

$a_1 b_1' = 1$, $a_2 b_2' = 1$, $a_3 b_3' = 1$

so, $b_1' = 1/a_1$, $b_2' = 1/a_2$, $b_3' = 1/a_3$

it only exist when none of a_1, a_2, a_3 are not zero

2. $u \oplus v = uv \ \forall u, v \in R^+$ & $\alpha \otimes u = u^\alpha \ \forall u \in R^+$ and $\alpha \in R$.

i) $(R^+, \oplus)$ is an abelian group.

$(u \oplus v) \oplus w = uv \oplus w = uvw$

$u \oplus (v \oplus w) = u \oplus (vw) = uvw$.

Now, $u \oplus e = u = ue \Rightarrow e = 1$

Now, $u \oplus u' = e \Rightarrow uu' = 1 \Rightarrow u' = 1/u$. & also $u \oplus v = v \oplus v$

So, its an abelian group.

ii) $(\alpha \oplus \beta) \otimes u = (\alpha + \beta) \otimes u = u^{\alpha+\beta}$

Now, $(\alpha \otimes u) \oplus (\beta \otimes u) \Rightarrow u^\alpha \oplus u^\beta = u^{\alpha+\beta}$.

so, its hold.

iii) $\alpha \otimes (u \oplus v) = \alpha \otimes u \oplus \alpha \otimes v \Rightarrow u^\alpha \oplus v^\alpha = (uv)^\alpha$

$\alpha \otimes uv \Rightarrow (uv)^\alpha$ so, its hold.

iv) $\alpha(\beta u) = (\alpha\beta) u \Rightarrow$

$u^{\alpha\beta} = (\alpha\beta) u$ so, its hold.

v) Now, $e \otimes \alpha = \alpha \Rightarrow \alpha = e^\alpha$ $e = \sqrt[\alpha]{\alpha}$ so, identity exist so, its a vector space.

4. $\alpha \otimes 0 = 0$. $(V, \oplus, \otimes)$ is a vector space

i) $\alpha \otimes (0 \oplus 0) = \alpha \otimes 0 \oplus \alpha \otimes 0 \Rightarrow 2\alpha \otimes 0 = \alpha \otimes 0 \Rightarrow \alpha \otimes 0 = 0$

ii) $(0 \oplus 0) \otimes u = 0 \otimes u \oplus 0 \otimes u \Rightarrow 2\, 0 \otimes u = 0 \otimes u \Rightarrow 0 \otimes u = 0$.

3. $(V, \oplus)$ is an abelian group.

ii) Now. $\alpha \otimes [(a_1, a_2) \oplus (b_1, b_2)] = \alpha \otimes [a_1 + b_2, b_1 + a_2] =$

$\alpha(a_1 + b_2), \alpha(b_1 + a_2)$

Now, $\alpha \otimes (a_1, a_2) \oplus \alpha \otimes (b_1, b_2)$.

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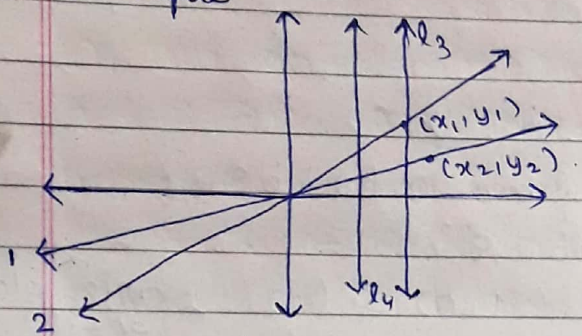
Tutorial I.

11\. V is a vectorspace. W_1, W_2 subspaces of V, $V = W_1 + W_2$, $W_1 \cap W_2 = \{0\}$
let us suppose that u can be expressed as $u = u_1' + u_2'$ (ii), $u_1' \in W_1$ & $u_2' \in W_2$. Now, $u = u_1 + u_2$ (i) subtracting (ii) − (i)
$(u_2' - u_2) + (u_1' - u_1) = 0$. Now, $u_1' - u_1 \in W_1$ if we multiply $(u_1' - u_1)$ by −1 it will also lies in W_1
Now, $-(u_1' - u_1) \in W_1$ Now adding it in $(u_1' - u_1) + (u_2' - u_2)$
then $u_2' - u_2 \in W_1$ & we assume that $u_2' - u_2 \in W_2$
but $W_1 \cap W_2 = \{0\}$. So, $u_2' - u_2 = 0$ similarly $u_1' - u_1 = 0$
So, there exist only one unique element in W_1 & W_2 such that $u = u_1 + u_2$ $\forall u \in W$ & $u_1 \in W_1$ & $u_2 \in W_2$.
example.



equation of line 1 $\Rightarrow y = \left(\tan^{-1} \frac{y_1}{x_1}\right) x$.
" " " 2 $\Rightarrow y = \left(\tan^{-1} \frac{y_2}{x_2}\right) x$
$l_3 = c_1 l_1 + d l_2$. & $l_4 = d_1 l_2 + d_2 l_1$
so, any vertical line can be written an a linear combination of l_1 & l_2 with unique points.

10\. let us suppose that $W_1 \not\subseteq W_2$ then there must exist an element in W_1 which is not an element of W_2 & vice-versa
Now, $x \in W_1$, $x \notin W_2$ & $y \in W_2$ & $y \notin W_1$
Now, $a, b \in F$. so, $ax \in W_1$ & $by \in W_2$.
Now, W_1 & W_2 are subspaces of vector space V
so, $ax + by \in W_1 \cup W_2$. or $ax + by \in W_1$, or $ax + by \in W_2$.
Now add $-ax$ then $by \in W_1$ but we assume that $by \in W_2$
So, our assumption is wrong
so. $W_1 \subseteq W_2$
Another method :-
let $a, b \in F$ & $x \in W_1$ & $y \in W_2$.
Now $ax + by \in W_1 \cup W_2$
Case I $ax + by \in W_1$ or $ax + by \notin W_2$
$y \in W_1$ & $y \in W_2$ & this emply $W_2 \subseteq W_1$

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Case II $ax+by \notin \omega_1$, $ax+by \in \omega_2$ so, $x \in \omega_2$ & $x \in \omega_1$ $\omega_1 \subseteq \omega_2$

Case III $ax+by \in \omega_1$, $ax+by \in \omega_2$ " $\omega_1 = \omega_2$.

8.(i) $\{P \in P : \deg P \leq 4\}$:- it is subspace of P

iv) $\{P \in P : P(1) = 0\}$:- Now let $P_1, P_2 \in P$ then $P_1(1) + P_2(1) = 0$.
so, $P_1(1) + P_2(1) \in P$ & $cP_1(1) = 0$ & $cP_2(1) = 0$.
so, $cP_1(1) P_2(1) \in P$. so, its a vectorspace so, its a subspace of P.

v) $\{P \in P : P(2) = 1\}$ Now let P_1, P_2 be $\in P$
then $P_1(2) + P_2(2) = 1+1 = 2 = P_3(2) \notin P$.
so, its not a vectorspace. so, its not a subspace of P.

vi) $\{P \in P : P'(1) = 0\}$. Now, $P_1; P_2 \in P$ Now, $(P_1+P_2)(x) = P_1(x) + P_2(x)$ $\Rightarrow (P_1+P_2)'(x) = P_1'(x) + P_2'(x) = 0$
$(P_1+P_2) \in x$ & $cP_1 \Rightarrow cP_1'(x) = cp_1'(1) = 0$ so, $cp_1(x) \in P$

6 (iv) $\{f \in V : f(x) = 0$ only at a finite number of points$\}$
let $c \in$ Field Now, if we take $c = 0$. then $cf(x) = 0$ $\forall x \in [0,1]$ so, there are infinetly many point where $cf(x) = 0$ thus the given set is not a vectorspace.

* Linear span of a subset of a vectorspace $V(F)$.
let S be a non-empty subset of a vector space $V(F)$. Then linear span S is defined by $L(S)$ n $[S]$ or $\langle S \rangle$ and defined as. $L(S) = \{\sum_{i=1}^{n} a_i x_i \mid a_i \in F$ & $x_i \in S; 1 \leq i \leq n$ (finite)

→ Justification for $L(S)$:. S is a non-empty set then $L(S)$ is also a non-empty set because whatever in S is in $L(S)$ so, now, $a, b \in F$ $a_i x_i \in L(S)$ & $b_j y_j \in L(S)$ $x = \sum_{i=1}^{n} a_i x_i$ $y = \sum_{j=1}^{n} b_j y_j$ Now $ax+by = a\sum_{i=1}^{n} a_i x_i + b\sum_{j=1}^{n} b_j y_j$ & scalar will affect only scalar so, $a\sum_{i=1}^{n} a_i$ & $b\sum_{j=1}^{n} b_j$ are another scalar & we are taking n as finite so, $\sum_{i=1}^{n} a_i x_i + \sum_{j=1}^{n} b_j y_j$ is also finite & S is a subspace so finite subspace i.e subset of S is a subspace.

* $L(S)$ is the smallest subspace of the vectorspace $V(F)$ containing S. Therefore S is also contain in all W_i
$L(S) = \bigcap_{i=1} W_i$

Tutorial II

1 i) a) $\{e^x, e^{2x}\}$ in $C^\infty(R)$.

Now. $\alpha_1 e^x + \alpha_2 e^{2x} = 0$ (i) Now differentiate. $\alpha_1 e^x + 2\alpha_2 e^{2x} = 0$ (ii)

e^x & e^{2x} can not be zero from $(-\infty, \infty)$. It is linearly Independent

b) Now. $\{\sin x, \sin 2x, \sin nx\}$ $[-\pi, \pi]$ Ex.

→ for $c_1 \sin x + c_2 \sin 2x + c_3 \sin nx = 0$.

Now, $c_1 \cos x + 2c_2 \cos 2x + nc_3 \sin nx = 0$

$\sin x, \sin 2x, \sin nx$ is a continuous function from $(-\pi, \pi)$

e) $= \{(x, x^3 - x, x^4 + x^2, x + x^2 + x^4 + \frac{1}{2})\}$ in P_4.

$c_1 x + c_2(x^3 - x) + c_3(x^4 + x^2) + c_4(x + x^2 + x^4 + \frac{1}{2}) = 0$.

$c_1 - c_2 + c_4 = 0$, $c_2 = 0$. , $c_4 = 0$.

$c_3 + c_4 = 0$ $c_3 + c_4 = 0$

2iv) Let S be LI subset of V $\notin L[S]$

S is linearly independent set & $\{v\} \cup S$ has one extra element v in S.

Now, $cv + c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0$

Case I. if $c = 0$, then $\{v\} \cup S$ is L.I.

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Case I If $c \neq 0$ then $v = -\frac{c_1}{c}\alpha_1 - \frac{c_2}{c}\alpha_2 - \cdots - \frac{c_n}{c}\alpha_n$

this imply $v \in L[S]$ so, $c = 0$ so, $\{v\} \cup S$ is L.I.

Q. If a vectorspace V(F) is finitely dimensional what do you say about dimension of the subspace W of V(F)?

→ $R^2(R)$ is a vectorspace. $W_1 = \{(a, 0)\ a \in R\}$ subspace of $R^2(R)$ dimension of subspace $\leq \dim V$

* $F[x] = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \mid a_i \in F, 1 \leq i \leq n\}$.

$\alpha F(x) = \alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 + \cdots + \alpha a_n x^n$.

Basis $= \{1, x, x^2, x^3, \cdots \}$.

Th:- let W_1 and W_2 be 2 subspaces of a finite dimensional vectorspace V(F). Then dimension of $W_1 + W_2 = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$.

→ Since V(F) is finite dimensional so dimension of W_1 and W_2 must be finite dimensional

We know $\cap$ of 2 subspace is also a subspace. $W_1 \cap W_2 \subseteq W_1 \subseteq W_1 + W_2$; $W_1 \cap W_2 \subseteq W_2 \subseteq W_1 + W_2$

$W_1 + W_2$ is also a subspace of V.

Now, dim subspace $\leq \dim V(F)$ & $\dim W_1 + W_2 \leq \dim V(F)$

$\dim W_1 \leq \dim V(F)$ & $\dim W_2 \leq \dim V(F)$.

Everysubspace is a vectorspace. ⇒ Every vectorspace has a basis so, $W_1 \cap W_2$ has a basis let it be $B = \{x_1, x_2, x_3, \cdots x_r\}$.

$B \subseteq W_1 \cap W_2 \subseteq W_1$ & $B \subseteq W_1 \cap W_2 \subseteq W_2$.

then B can be extended to the basis of W_1 & W_2. let the dimension of B_1 & B_2 be m and n.

then $B_1 = \{x_1, x_2, x_3, \cdots y_{m-r}\}$.

then $B_2 = \{x_1, x_2, x_3, \cdots z_{n-r}\}$.

Now $B_3 = \{x_1, x_1 + x_2 + x_3 + \cdots + y_1 + \cdots + y_{m-r} + z + \cdots + z_{n-r}\}$ be the basis of $W_1 + W_2$

$z \in W_1 + W_2 \Rightarrow z = x + y$ $x \in W_1$ & $y \in W_2$.

since B_1 is the basis of W_1 so, x can be written as a linear combination of elements of B_1 & similar for y

$z = \sum_1^n a_i x_i + \sum_1^n b_i y_i \Rightarrow z = \{a_1x_1 + \cdots a_r y_r + \cdots a_m y_{m-r} + b_r z \cdots + \cdots b_n z_{n-r}\}$.

Tutorial basis.

3 ii) In a vector space V if a set $\{v_1, v_2 \ldots v_n\}$ is L.I & $S = \{w_1, \ldots w_n\}$ generates the space,

since, $v_1 \in L(S) = V \Rightarrow v_1 = c_1 w_1 + c_2 w_2 + \ldots + c_m w_m$ where $c_i \in F$, say the field $(1 \leq i \leq m)$

since, $v_i \neq 0$, $\exists$ atleast one c_i which is not equal to zero

assume $c_j \neq 0$. Then $w_j = \left(\frac{-c_i}{c_j}\right) w_1 + \left(\frac{-c_2}{c_j}\right) w_2 + \ldots \left(\frac{-c_m}{c_j}\right) w_m$

$w_j \in L(\{w_1, w_2, \ldots w_{j-1}, \ldots w_m\}) = L(\{w_1, \ldots w_m\})$.

Similarly $v_i \in L(\{w_1, \ldots w_{j-1}, v_1, w_{j+1}, \ldots, w_m\})$

$v_i = b_1 w_1 + \ldots b_{j} w_{j-1} + b_j w_j + \ldots b_m w_m$.

Since $v_i \neq 0$, $\exists$ at least one non zero term among $b_1, b_2, \ldots b_m$ except b_j

Assume $b_j \neq 0$, $i \neq j$

$w_i \in L(\{w_1, \ldots w_{j-1}, v_1, w_{j+1}, \ldots w_{i-1}, v_2, w_{i+1}, \ldots w_m\})$

$= L(\{w_1, w_2, \ldots w_m\})$

Now, if possible let $n > m$ following the above described procedure we can show that $L\{v_1, v_2 \ldots v_t\} = L\{w_1, w_2 \ldots w_m\}$?

$t \leq m$. Since $v_{t+1} \in V$, $v_{t+1} \in L\{v_1, v_2 \ldots v_t\}$.

$\{v_1, v_2, \ldots v_m\}$ is LD, contradiction

$\therefore n \leq m$.

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3 iv) $\{a+bx+cx^3 \text{ in } P_3 \; a-2b+c=0\}$.

$a+bx+cx^3 = 2b-c+bx+cx^3 = b(2+x)+c(x^3-1)$

so, basis is $\{(2+x, x^3-1)\}$

6) $P = x-2y+3z=0$, $x-2y+3z=0$, $x=2y-3z$

$(x,y,z) = (2y-3z, y, z) = y(2,1,0)+z(-3,0,1)$

$S = C\{(2,1,0), (-3,0,1)\}$.

7 i) $S=\{(4,5,6), (a,2,4), (4,3,2)\}$.

$\Rightarrow c_1(4,5,6)+c_2(a,2,4)+c_3(4,3,2) = (0,0,0)$.

$4c_1+ac_2+4c_3=0$. $6c_1+4c_2+2c_3=0$

$5c_1+2c_2+3c_3=0$

$$\underbrace{\begin{bmatrix} 4 & 5 & 6 \\ a & 2 & 4 \\ 4 & 3 & 2 \end{bmatrix}}_{A}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

take $\det A = 0$

$4(4-12)-5(2a-16)+6(3a-8)=0$

$-32-10a+80+18a-48=0 \Rightarrow a=0$.

10 i) $U=\{(x_1,x_2,x_3) \mid x_1+x_2-x_3=0\}$.

$W=\{(x_1,x_2,x_3) \mid 2x_1+x_2=0\}$.

$U \cap V = \{(x_1,x_2,x_3) \mid x_1+x_2-x_3=0, \; 2x_1+x_2=0\}$.

$\rightarrow U=(x_1,x_2,x_3) = (x_1,x_2,x_1+x_2) = (1,0,1)x_1+(0,1,1)x_2$

$W=(x_1,x_2,x_3) = (x_1,-2x_1,0) = x_1(1,-2,0)+x_3(0,0,1)$

$U+W = \{u+w \mid u \in U \; w \in W\}$.

$L\{(1,0,1),(0,1,1),(1,-2,0),(0,0,1)\}$.