MA101

MULTIPLE INTEGRALS

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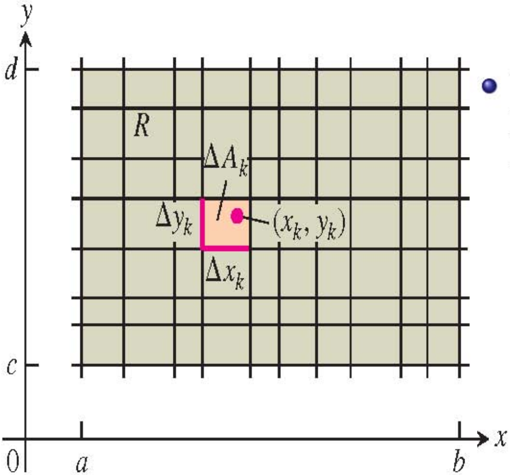
Figure: Rectangular grid partitioning the region R into small rectangles of area $A_k = \Delta x_k \Delta y_k$.

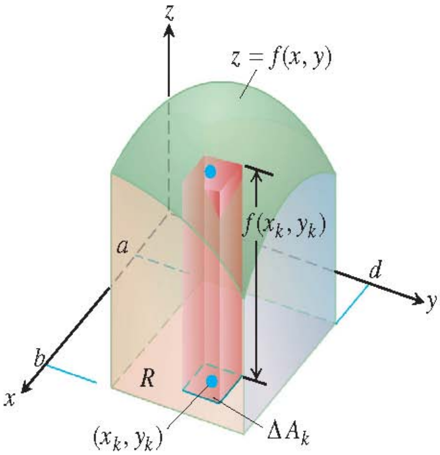


- There are many choices involved in a limit of this kind.

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

- When a limit of the sums s_n exists, giving the same limiting value no matter what choices are made, then the function f is said to be integrable and the limit is called the double integral of f over R, written as
$$\iint_R f(x,y)\,dA \quad \text{or} \quad \iint_R f(x,y)\,dx\,dy.$$
- It can be shown that if $f(x, y)$ is a **continuous function** throughout R, then f is integrable, as in the single-variable case. Many **discontinuous functions** are also integrable, including functions which are discontinuous only on a finite number of points or smooth curves.

When $f(x,y)$ is a positive function over a rectangular region R in the $xy-$plane, we may interpret the double integral of f over R as the **volume** of the 3-dimensional solid region over the $xy-$plane bounded below by R and above by the surface $f(x,y)$.



Calculate the volume under the plane $z = 4 - x - y$ over the rectangular region $R : 0 \leq x \leq 2,\ 0 \leq y \leq 1$ in the xy-plane.

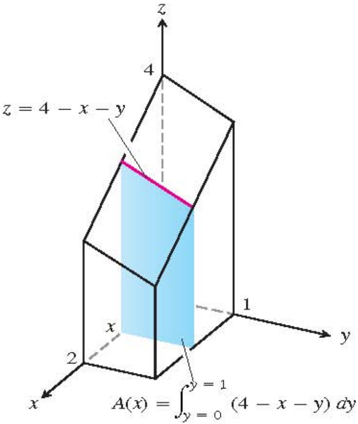


Figure:

$$V = \int_{x=0}^{x=2} \int_{y=0}^{y=1} (4 - x - y)\,dy\,dx$$

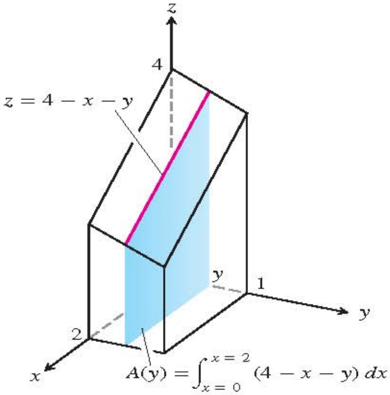


Figure:

$$V = \int_{y=0}^{y=1} \int_{x=0}^{x=2} (4 - x - y)\,dx\,dy$$

THEOREM 1 Fubini's Theorem

Theorem

If $f(x, y)$ is continuous throughout the rectangular region then

$$\iint\limits_R \mathbf{f(x,y)dA} = \int_c^d \int_a^b \mathbf{f(x,y)dxdy} = \int_a^b \int_c^d \mathbf{f(x,y)dydx}.$$

Fubini's Theorem says that double integrals over rectangles can be calculated as ***iterated integrals***. Thus, we can evaluate a double integral by integrating with respect to one variable at a time.

Note: Fubini proved his theorem in greater generality, but this is what it says in our setting.

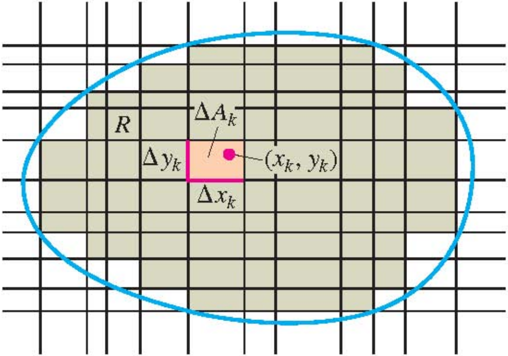
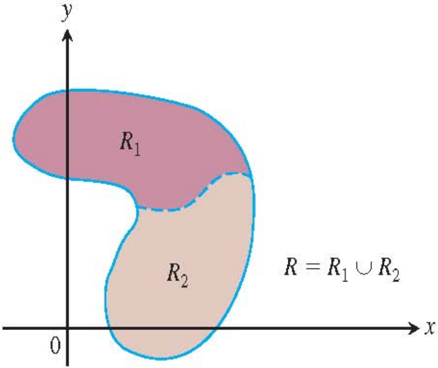


FIGURE 15.6 A rectangular grid partitioning a bounded nonrectangular region into rectangular cells.

Figure: The Additivity Property for rectangular regions holds for regions bounded by continuous curves.



$$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$$

Figure: The area of the vertical slice shown here is $\mathbf{A(x)} = \int_{\mathbf{g_1(x)}}^{\mathbf{g_2(x)}} \mathbf{f(x,y)dy}$. To calculate the volume of the solid, we integrate this area from $x = a$ to $x = b$ and get $\int_{\mathbf{a}}^{\mathbf{b}} \int_{\mathbf{g_1(x)}}^{\mathbf{g_2(x)}} \mathbf{f(x,y)dydx}$.

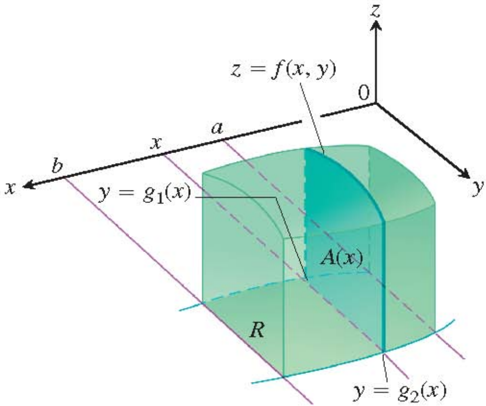
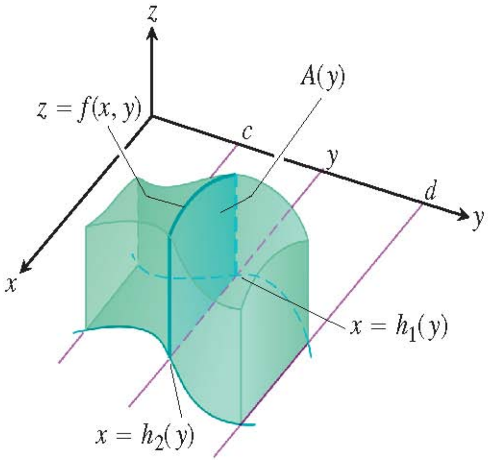


Figure: The volume of the solid shown here is $\int_c^d \mathbf{A(y)dy} = \int_c^d \int_{h_1(y)}^{h_2(y)} \mathbf{f(x,y)dxdy}$.



THEOREM 2 Fubini's Theorem Stronger Form

Theorem

Let $f(x,y)$ be continuous on a region R.

1. *If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$ with g_1 and g_2 continuous on $[a,b]$, then*

$$\iint\limits_{\mathbf{R}} \mathbf{f(x,y)dA} = \int_{\mathbf{a}}^{\mathbf{b}} \int_{\mathbf{g_1(x)}}^{\mathbf{g_2(x)}} \mathbf{f(x,y)dydx}.$$

2. *If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$ with h_1 and h_2 continuous on $[c,d]$, then*

$$\iint\limits_{\mathbf{R}} \mathbf{f(x,y)dA} = \int_{\mathbf{c}}^{\mathbf{d}} \int_{\mathbf{h_1(y)}}^{\mathbf{h_2(y)}} \mathbf{f(x,y)dxdy}.$$

Problem

Find the volume of the prism whose base is the triangle in the $xy-$plane bounded by the $x-$axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane $z = f(x, y) = 3 - x - y$.

Important Remark

Note:

Although Fubini's Theorem assures us that a double integral may be calculated as an iterated integral in either order of integration, the value of one integral may be easier to find than the value of the other.

Problem

Evaluate the following double integral in two ways (both iterated integrals)

$$\iint_R \frac{\sin x}{x} dA,$$

where R is the triangle in the xy-plane bounded by the x-axis, the line $y = x$ and the line $x = 1$.

Like single integrals, double integrals of continuous functions have algebraic properties that are useful in computations and applications. If $f(x,y)$ and $g(x,y)$ are continuous, then

1. Constant Multiple:

$$\iint\limits_{\mathbf{R}} \mathbf{cf(x,y)dA} = \mathbf{c} \iint\limits_{\mathbf{R}} \mathbf{f(x,y)dA}. \ (\textit{any number } c)$$

2. Sum and Difference:

$$\iint\limits_{\mathbf{R}} \mathbf{(f(x,y) \pm g(x,y))dA} = \iint\limits_{\mathbf{R}} \mathbf{f(x,y)dA} \pm \int\int_{\mathbf{R}} \mathbf{g(x,y)dA}.$$

3. Domination:

(a)

$$\iint\limits_R f(x,y)\,dA \geq 0 \quad \textit{if} \quad f(x,y) \geq 0$$

on R.

(b)

$$\iint\limits_R f(x,y)\,dA \geq \iint\limits_R g(x,y)\,dA$$

if

$$f(x,y) \geq g(x,y)$$

on R.

4. Additivity:

$$\iint\limits_R f(x,y)\,dA = \iint\limits_{R_1} f(x,y)\,dA + \iint\limits_{R_2} f(x,y)\,dA$$

if R is the union of non overlapping regions R_1 and R_2.

Ex. 15.1 Problem 61

What region R in the $xy-$plane maximizes the value of

$$\iint_R (4 - x^2 - 2y^2)\,dA.$$

Give reasons for your answer.

Ex. 15.1 Problem 48

Find the volume of the solid cut from the square column $|x| + |y| \leq 1$ by the planes $z = 0$ and $3x + z = 3$.

Remark

Can we use symmetry of the region $R: \ |x| + |y| \leq 1$?

Area, Moments and Center of Mass

Definition

The area of a closed, bounded plane region R is

$$Area = \iint_{\mathbf{R}} \mathbf{dA}.$$

Finding Area

Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

Must to do!

Changing order of Integration

Change the order of the integration of the following integration

$$\int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\,dy\,dx,$$

where $f(x,y)$ is defined over the shaded region. What if area of shaded region is asked?