Indian Institute of Technology Patna
MA201: Mathematics III
End Semester Exam (20-11-2015)

Time: 3hrs **Max. Marks: 50**

<u>**Note:**</u> There are total **SIX** questions. Answer all questions. Give precise and brief answer. Standard formulae may be used.

Que 1. Answer all parts of this question at one place. **[1x10]**

(a.) Write the statement of *Cauchy Integral Formula.*

(b.) State *Cauchy-Goursat Integral Theorem* for multiple connected domain.

(c.) Define a first order *semilinear pde* for $z = z(x, y)$.

(d.) Obtain a *linear first order pde* for the family of spheres: $x^2 + y^2 + (z - c)^2 = r^2$.

(e.) If $z = z(x, y)$ is integral surface of the pde $P(x, y, z)p + Q(x, y, z)q = R(x, y, z)$, then (geometrically) vector $(p, q, -1)$ represents ....

(f.) $\int_{-\pi}^{\pi} \sin mx \cos nx dx = \cdots\cdots$, m, n are integers.

(g.) Fourier Transform of $f(x) = \begin{cases} 1, & |x| < 1; \\ 0, & |x| > 1. \end{cases}$

(h.) Fourier Series of an odd 2π periodic function contains **sine** terms only. TRUE/ FALSE.

(i.) Show that $f * g(x) = g * f(x)$ where $*$ is ***convolution*** of two functions.

(j.) Let $z = e^{-4x}G(2y - 3x)$ be solution of $2p + 3q + 8z = 0$. How many solutions are possible passing through initial curve $\Gamma: \ z = e^{-4x}$ on the line $2y = 3x$.

Que 2. a) Obtain Fourier Series for the function :

$$f(x) = \begin{cases} x, & -\pi/2 < x < \pi/2; \\ \pi - x, & \pi/2 < x < 3\pi/2, \end{cases} \qquad f(x + 2\pi) = f(x), \forall x.$$

Discuss the convergence of the Fourier Series obtained. [3+1]

b) Find Fourier Integral of the function $f(x) = \begin{cases} 1, & \text{if } |x| < 1; \\ 0, & \text{if } |x| > 1. \end{cases}$ [2]

c) The Fourier series for $f(x) = x, \ -\pi < x < \pi, \ \ f(x + 2\pi) = f(x)$ is given by

$$x = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - ...\right]$$

which converges for all x. Can we differentiate the series term-by-term so as to get Fourier Series of $f'(x) = 1$? Justify. [2]

Que 3. a) Let $f(x)$ be piecewise smooth function in $\mathbb{R}$ and $\lim_{x\to\infty} f(x), \ f'(x) = 0$. Show that $\mathcal{F}_c(f''(x)) = -w^2\mathcal{F}_c(f(x)) - \sqrt{\frac{2}{\pi}}\ f'(0)$. Hence obtain, $\mathcal{F}_c(e^{-ax}) = \sqrt{\frac{2}{\pi}}\ \frac{a}{a^2+w^2}$. [4]

b) Using Fourier Transform solve the heat equation: [4]

$$\begin{array}{ll} u_t = ku_{xx}, & -\infty < x < \infty, \ t > 0, \\ u(x, 0) = f(x), & -\infty < x < \infty, \\ u \text{ and } u_x \to 0 & \text{as } |x| \to \infty. \end{array}$$

Que 4. **a)** Obtain general solution of the PDE: $x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$ with initial data $z=1$ on $x+y=0$. [4]

b) Classify the PDE $y^2 u_{xx} - x^2 u_{yy} = 0$. Obtain the corresponding canonical form (*formulae given on back*). [4]

Que 5. **a)** Use D'Alembert formula to solve the wave equation $u_{tt} = u_{xx}$, $x \in \mathbb{R}$, $t > 0$ with ICs $u(x,0) = x$ and $u_t(x,0) = \sin x$. Hence, giving justification, solve $u_{tt} - u_{xx} = x + t$, $x \in \mathbb{R}$, $t > 0$ with ICs $u(x,0) = x$ and $u_t(x,0) = \sin x$. [2+3]

b) Use method of separation of variables to solve heat equation $u_t = u_{xx}$, with IC, $u(x,0) = \sin 3\pi x - 2\sin 5\pi x$, $0 < x < 1$, and BCs, $u(0,t) = 0 = u(1,t)$. *Do not use formula.* [3]

Que 6. **a)** Expand $f(z) = \frac{5z-2}{z(z-1)}$ in a Laurent series valid in the region $0 < |z-1| < 1$. [3]

b) Determine value of the contour integral $\int_C \frac{z^3+2z}{(z-i)^3}\, dz$ where C is given as $|z| = \frac{3}{2}$. [3]

c) Determine the nature of isolated singular point and corresponding residue of the function $\frac{1-e^{2z}}{z^4}$. [2]

Important Formulae:

- The second order general PDE : $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$ can be transformed using $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$ into following canonical form $\overline{A}u_{\xi\xi} + \overline{B}u_{\xi\eta} + \overline{C}u_{\eta\eta} + \overline{D}u_\xi + \overline{E}u_\eta + \overline{F}u = \overline{G}$ where

$\overline{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$
$\overline{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$
$\overline{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$
$\overline{D} = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y$
$\overline{E} = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y$
$\overline{F} = F$, $\overline{G} = G$.

- Fourier Transform of $f(x)$, $\mathcal{F}(f(x)) = F(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx$
- $\mathcal{F}(e^{-ax^2}) = \frac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}}$.

Good Luck