Recall of a previous lecture

Suppose (X,Y) is a jointly distributed random variable. Then

1 → (X,Y) is said to discrete if both X & Y are discrete random variables.

2 → (X,Y) is said to continuous if both X & Y are continuous random variables.

In Case 1 we further define joint PMF as collection of probabilities $p_{XY}(x_i, y_j)$ such that

(i) $p_{X,Y}(x_i, y_j) \geq 0, \ \forall \ x_i \in R_X, \ y_j \in R_Y$

(ii) $\sum_{x_i} \sum_{y_j} p_{X,Y}(x_i, y_j) = 1$

In Case 2 joint pdf is $f_{X,Y}(x,y)$ provided it satisfies (i) $f_{X,Y}(x,y) \geq 0 \ \forall \ (x,y) \subset \mathbb{R}^2$

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = 1$

Marginal PMFs of X and Y (Discrete case)

$$p_X(x_i) = \sum_{y_j \in R_Y} p_{X,Y}(x_i, y_j), \quad x_i \in R_X$$

$$p_Y(y_j) = \sum_{x_i \in R_X} p_{X,Y}(x_i, y_j), \quad y_j \in R_Y$$

Marginal PDFs of X & Y (continuous case)

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

Conditional PMFs: (Discrete case)

$$p_{X|Y=y_j}(x_i|y_j) = \frac{p_{X,Y}(x_i,y_j)}{p_Y(y_j)}, \quad x_i \in R_X, \quad p_Y(y_j) > 0$$

$$p_{Y|X=x_i}(y_j|x_i) = \frac{p_{X,Y}(x_i,y_j)}{p_X(x_i)}, \quad y_j \in R_Y, \quad p_X(x_i) > 0.$$

Conditional PDFs (continuous case)

$$f_{X|Y=y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \quad x \in R_X$$

$$f_{Y|X=x}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}, \quad y \in R_Y.$$

joint MGF, Independence, Covariance

In this lecture, dear students, we try to cover the above mentioned topics.

joint moment generating function.

Let (X,Y) be jointly distributed random variables. Then joint MGF of (X,Y) is defined as

$$M_{X,Y}(t_1,t_2) = E[e^{t_1 X + t_2 Y}] \quad \text{(1)}$$

provided expectation exists for $|t_j| < h_j$, $j=1,2$ for some $h_j > 0$, $j=1,2$.

$\boxed{\text{(1) is equal to } \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1x+t_2y} f(x,y)\,dx\,dy \text{ when } (X,Y) \text{ is continuous}}$

(*) The joint MGF uniquely determines the joint probability distⁿ of (X,Y).

(*) The joint MGF also completely determines the marginal distributions X and Y respectively. indeed

$$M_{X,Y}(t_1,0) = E[e^{t_1X}] = M_X(t_1).$$

$$M_{X,Y}(0,t_2) = E(e^{t_2Y}) = M_Y(t_2).$$

(*) If $M_{X,Y}(t_1,t_2)$ exists then Expected value of (X,Y) of all order exist and can be computed as follows:

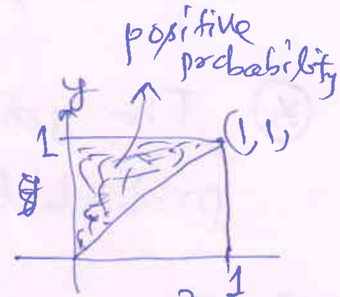
$$\left.\frac{\partial^{m+n} M_{X,Y}(t_1,t_2)}{\partial t_1^m \, \partial t_2^n}\right|_{t_1=0=t_2} = E[X^m Y^n]$$

Thus $\left.\dfrac{\partial M_{X,Y}(t_1, t_2)}{\partial t_1}\right|_{\substack{t_1=0\\t_2=0}} = E(X)$, $\left.\dfrac{\partial M(t_1,t_2)}{\partial t_2}\right|_{\substack{t_1=0\\t_2=0}} = E(Y)$.

$$\left.\frac{\partial^2 M(t_1,t_2)}{\partial t_1 \partial t_2}\right|_{\substack{t_1=0\\t_2=0}} = E(XY) \text{ and so on.}$$

Let us compute MGF for a continuous problem

Ex: $f_{X,Y}(x,y) = 2, \; 0 \le x \le y \le 1$
$\qquad\qquad = 0$, elsewhere



Solution: $M_{X,Y}(t_1, t_2) = E\left(e^{t_1 X + t_2 Y}\right)$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 x + t_2 y} f_{XY}(x,y)\, dx\, dy$$

$$= 2\int_{x=0}^{1} e^{t_1 x} \int_{y=x}^{1} e^{t_2 y}\, dy\, dx = 2\int_0^1 e^{t_1 x}\left[\frac{e^{t_2 y}}{t_2}\right]_x^1 dx$$

$$= \frac{2}{t_2}\int_0^1 e^{t_1 x}\left(e^{t_2} - e^{t_2 x}\right) dx$$

$$= \frac{2}{t_2}\left[\left\{\frac{e^{t_1+t_2} - e^{t_2}}{t_1}\right\} - \frac{e^{t_1+t_2}}{t_1+t_2} + \frac{1}{t_1+t_2}\right] \text{(Check!)}$$

Ex: Discrete case:

Find joint MGF of (X,Y) with joint PMF given in the table.

$Y \backslash X$	0	1	2	3
0	0.05	0.21	0	0
1	0.20	0.26	0.08	0
2	0	0.06	0.07	0.02
3	0	0	0.03	0.04

formula for joint MGF here is

$$M_{X,Y}(t_1,t_2) = \sum_{x_i}\sum_{y_j} e^{t_1 x_i + t_2 y_j} p_{XY}(x_i, y_j)$$

$$= \sum_{x=0}^{3}\sum_{y=0}^{3} e^{(t_1 x + t_2 y)} p_{X,Y}(x,y).$$

Now simplify it. Note that ultimately it will be a function of (t_1, t_2) only.

<u>Independence of Two Random Variables X & Y.</u>

Let (X,Y) be jointly distributed random variables

⊛ Suppose that (X,Y) a bivariate discrete rV then X and Y are independent provided

$$p_{X,Y}(x_i, y_j) = p_X(x_i)\, p_Y(y_j) \;\forall\; (x_i, y_j) \in R_{X \times Y}$$

⊛ Let (X,Y) be continuous random variable then X & Y are independent provided

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) \;\forall\; (x,y) \in R^2.$$

Ex: In previous discrete example verify that X and Y are not independent.

Ex: continuous case $f_{X,Y}(x,y) = 2, \; 0 \le x \le y \le 1$

X & Y are dependent.

Ex: $f_{X,Y} = 2e^{-x-2y}, \; \begin{matrix} x > 0 \\ y > 0 \end{matrix}$

$\Rightarrow$ Check $f_X(x) = e^{-x}$
$f_Y(y) = 2e^{-2y}$

$f_X(x) = \int_0^\infty 2e^{-x-2y}\,dy = e^{-x}$

$f_Y(y) = \int_0^\infty 2e^{-x-2y}\,dx = 2e^{-2y}$

Now

$$f_{X,Y}(x,y) = 2e^{-x-2y}$$
$$= e^{-x}\cdot 2e^{-2y} = f_X(x)\cdot f_Y(y)$$

Then X & Y are independent. $\forall \; \begin{matrix} x > 0 \\ y > 0. \end{matrix}$

Properties: <i> If X and Y are independent then

$$E[g(X)\,h(Y)] = E(g(X))\cdot E(h(Y)).$$

~~(ii) If X and~~ Converse of this result is not true in general. That is $E(g(X)\,h(Y)) = E(g(X))\,E(h(Y))$ may not imply that X and Y are independent.

Ex: Let $X \sim N(0,1)$ and $Y = X^2$. Then

$$E(XY) = E(X\cdot X^2) = E(X^3) = 0.$$

but X, Y are not independent.

<ii> If X and Y are independent then joint MGF factors into marginal MGFs. That is,

$$M_{X,Y}(t_1, t_2) = M_X(t_1) \cdot M_Y(t_2)$$

~~Also note: [illegible]~~

<u>Ex</u>: Find joint MGF of (X, Y) where

$$f_{XY}(x, y) = 2e^{-x-2y}, \quad x>0,\ y>0.$$

$\Rightarrow$ Note that X and Y are independent here.

Also $f_X(x) = e^{-x}$, $f_Y(y) = 2e^{-2y}$

$$M_X(t_1) = \frac{1}{t_1 - 1}. \qquad M_Y(t_2) = \frac{2}{t_2 - 2}.$$

Thus joint MGF of (X, Y) is

$$M_{X,Y}(t_1, t_2) = M_X(t_1)\, M_Y(t_2) = \frac{2}{(t_1-1)(t_2-2)}$$