Indian Institute of Technology Patna
MA-225: B.Tech. II year
Spring Semester: 2016-17 (End Semester Examination)

Maximum Marks: 50 Total Time: 3 Hours

Note: This question paper contains **Eight** questions. Answer all questions.

1. Let X denotes waiting time (in minutes) at a bus stop. The probability density function (pdf) of X is $f_X(x) = x/25,\ 0 \le x < 5,\ = (2/5) - (x/25),\ 5 \le x \le 10,\ = 0$, otherwise. Find the probability that the waiting time is between 3 and 8 minutes. What is the probability that waiting time is either less than 2 minutes or more than 6 minutes. **[1.5+1.5]**

2. Two people have agreed to meet for lunch between noon (12:00 PM) to 1:00 PM. Let X and Y respectively denote their arrival time and suppose that X and Y are independent with respective density functions $f_X(x) = 3x^2,\ 0 \le x \le 1,\ = 0$, otherwise; $f_Y(y) = 2y,\ 0 \le y \le 1,\ = 0$, otherwise. Determine corresponding joint distribution function of X and Y. Determine the probability $P(X - Y < 0.25)$. What is the expected amount of time that the one who arrives first must wait for the other person. **[3+2+3]**

3. Consider two light bulbs for a particular lamp with X and Y respectively denoting their lifetimes (in thousands of hours). Let X ad Y be independent and that each has an exponential distribution with mean lifetime 1. Find the probability that total lifetime (i.e., sum of lifetimes) of the two bulbs is at most 2. Determine the probability $P(X - Y < 0.5)$. **[2+2]**

4. (i) Define convergence in probability. **[1]**
   (ii) Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of iid random variables from a normal population with mean μ and variance σ^2. Obtain distribution of $(n-1)\sigma^{-2}S^2$ where S^2 is the sample variance. Then show that the sample variance converges to population variance in probability. **[3+2]**

5. A bank operates both a drive-up facility and a walk-up window. On a randomly selected day let X denotes the proportion of time the drive-up facility is in use and Y denotes the proportion of time the walk-up window is in use. Let the joint pdf of (X, Y) be given by $f(x, y) = 24xy, 0 \le x \le 1,\ 0 \le y \le 1,\ x + y \le 1$ and $f(x, y) = 0$, otherwise. Find the marginal densities of X and Y. Find the probabilities $P(X + Y \le 0.5)$ and $P(Y \le 0.25 \mid X = 0.5)$. Compute the conditional expected value $E(Y \mid X = 0.5)$ and the conditional variance $V(Y \mid X = 0.5)$. **[1+1+1+2+1+2]**

6. Consider a Poisson process with rate 0.2 per unit time. Then find the density function of the random variable 'waiting time for the tenth event'. Compute the corresponding moment generating function. Determine the probability the 10th event occurs 2 or more time units after the 9th event. **[2+2+1]**

7. (i) The amount X of a drink dispensed by certain machine is normally distributed with mean 64 ml and standard deviation 0.78 ml. What container size c will ensure that overflow occur only 5% of the time. What is the probability that a cup contains between 63 and 66 mls? **[2+2]**
   (ii) Suppose that X_1 and X_2 are independent standard normal random variables and consider the transformation $Y_1 = X_1, Y_2 = X_1 + X_2$. Find the joint pdf of (Y_1, Y_2) and properly name it. Now given this joint pdf use MGF technique to find the marginal density of Y_2. Further determine the covariance between Y_1 and Y_2. **[2+2+2+2]**

8. Show that the correlation coefficient $\rho_{X,Y}$ between two random variables X (mean μ_1 and variance σ_1^2) and Y (mean μ_2 and variance σ_2^2) always lies in the interval $[-1, 1]$. **[4]**