Indian Institute of Technology Patna
MA201: Mathematics III
End Semester Examination(27-11-2012)

Time: 3hrs | Max. Marks: 50

Note: Answer all questions. Give precise and brief answer. Standard formulae may be used.

Que 1. a. Evaluate $\oint_C \frac{e^z}{z^2-5z+6}dz$, where C is circle $|z|=1$ oriented in positive direction. [2]

b. Let the rectangular region R in $z-$plane be bounded by $x=0, y=0, x=2, y=1$. Determine the region R' in w plane into which R is mapped under transformation
(i) $w=f(z)=z+(1+2i)$, (ii) $w=f(z)=\sqrt{2}e^{i\pi/4}z$, and
(iii) $w=f(z)=\sqrt{2}e^{i\pi/4}z+(1+2i)$. [1+1+1]

c. Find the radius of convergence of the power series: $\sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2}(z-3i)^n$. [2]

Que 2. a. State Cauchy Residue Theorem and using it, evaluate the integral

$$\oint \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz$$

around the circle $C:|z|=3$ oriented in positive direction. [4]

b. Find the Laurent Series for $f(z)=\frac{1}{(z+1)(z+3)}$ in the following regions
(i) $1<|z|<3$ and (ii) $0<|z+1|<2$. [4]

Que 3. a. Using Fourier Integral show that [3]

$$\int_0^{\infty}\frac{w^3\sin xw}{w^4+4}dw=\frac{\pi}{2}e^{-x}\cos x,\quad x>0.$$

b. Let $f(x)$ be continuous and $f'(x)$ be integrable on the $x-$axis and $f(x)\to 0$ as $|x|\to\infty$. Then show that

$$\mathcal{F}\{f'(x)\}=iw\mathcal{F}\{f(x)\}$$

where $\mathcal{F}(f)$ represents Fourier Transform of $f(x)$.
Hence or otherwise find $\mathcal{F}(xe^{-x^2})$. [5]

Que 4. Find the Fourier series of the function:

$$f(x)=\begin{cases}0, & -2\le x<0;\\ 2-x, & 0<x\le 2,\end{cases}$$

and discuss its convergence. Where does the series converge when $x=0$ and hence obtain the sum $s=1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}\cdots$ [5]

Que 5. a. Determine the integral surface of the equation

$$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z,$$

with the data $x+y=0$, $z=1$. [3]

b. Use Duhamel principle to solve:
$u_{tt} = u_{xx} + t\sin\pi x,\ 0<x<1,\ t>0$
$u(x,0) = \sin\pi x,\ u_t(x,0) = 2\sin\pi x + 4\sin 3\pi x,\ 0<x<1,$
$u(0,t)=0,\ u(1,t)=0,\ t>0$ [5]

Que 6. a. Let u be harmonic in $\Omega = \{(x,y) : x^2+y^2<1\}$, and $u(x,y) = 1-x$ for $(x,y)\in\partial\Omega$. Without solving, show that $u(x,y)>0,\ \forall(x,y)\in\Omega$. [3]

b. Solve with details: [5]
$u_{xx}+u_{yy}=0,\ \ 0<x<1,\ \ 0<y<2,$
$u(x,0)=0,\ u(x,2)=x,\ 0<x<1;\ \ u(0,y)=0,\ \ u(1,y)=0,\ 0<y<2.$

Que 7. Solve the following equation of vibrating membrane: [6]

PDE: $u_{tt} = u_{xx}+u_{yy},\ 0<x<a,\ 0<y<b, t>0,$

ICs: $u(x,y,0)=f(x,y),\ u_t(x,y,0)=g(x,y), 0<x<a,\ 0<y<b,$

BCs: $u(0,y,t)=0, u(a,y,t)=0,\ t>0,\ 0<y<b,$
$u(x,0,t)=0, u(x,b,t)=0,\ \ t>0,\ 0<x<a.$

Good Luck