Tutorial:

(Two-dimensional rvs).

Ex: Consider the function $F_{X,Y}(x,y)$ as

$$F_{X,Y}(x,y) = \begin{cases} 0, & x<0, y<0 \\ \frac{1}{16}xy(x+y), & 0\le x\le 2, 0\le y\le 2 \\ 1, & x>2, y>2 \end{cases}$$

(i) Verify that $F_{X,Y}(x,y)$ is a joint CDF of X & Y.

(ii) Determine the corresponding PDF

(iii) Find the probability $P(0\le X\le 1, 1\le Y\le 2)$

Soln: Before we solve this problem let us recall joint CDF $F_{X,Y}(x,y)$ and its properties, in general.

$$F_{X,Y}(x,y) = P(X\le x, Y\le y)$$

Now note that

$$\lim_{y\to\infty} F_{X,Y}(x,y) = F_X(x) \text{ (marginal CDF of X)}$$

$$\lim_{x\to\infty} F_{X,Y}(x,y) = F_Y(y) \text{ (marginal CDF of Y)}$$

* $F_{X,Y}(-\infty, -\infty) = 0 = F_{X,Y}(-\infty, y) = F_{X,Y}(x, -\infty)$.

* $F_{X,Y}(x,y)$ is nondecreasing in each of x & y variable

$$\begin{cases} \text{If } x_1 \le x_2 \text{ then for given } y \\ \quad F_{X,Y}(x_1, y) \le F_{X,Y}(x_2, y) \\ \text{If } y_1 \le y_2 \text{ then for given } x \\ \quad F_{X,Y}(x, y_1) \le F_{X,Y}(x, y_2). \end{cases}$$

* $F_{X,Y}(x,y)$ is continuous from right in each of x and y variable.

$$\begin{cases} \lim_{h \to 0^+} F_{X,Y}(x+h, y) = F_{X,Y}(x,y); \text{ given } y \\ \lim_{k \to 0^+} F_{X,Y}(x, y+k) = F_{X,Y}(x,y); \text{ given } x \end{cases}$$

* If $x_1 \le x_2$, $y_1 \le y_2$ then

$$F_{X,Y}(x_2, y_2) + F_{X,Y}(x_1, y_1) - F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2) \geq 0.$$

* If (X,Y) jointly distributed continuous rv then

$$\frac{\partial^2 F_{X,Y}(x,y)}{\partial x \, \partial y} = f_{X,Y}(x,y).$$

we are given that

$$F_{X,Y}(x,y) = 0, \; x<0, y<0$$
$$= \frac{1}{16} xy(x+y), \; 0 \le x \le 2, \; 0 \le y \le 2$$
$$= 1, \; x>2, y>2.$$

we observe that $F_{X,Y}(-\infty, -\infty) = 0$

In fact $F_{X,Y}(x,y)$ is continuous and hence right continuous in each of variable x and y.

$$F_{X,Y}(\infty, \infty) = 1$$

It is easily verified that $F_{X,Y}(x,y)$ is nondecreasing in x and in y also.

~~Let us~~ Let us also verify the last property.

for any $x_1 \le x_2$, $y_1 \le y_2$ we have

$$\frac{1}{16}\left[F_{X,Y}(x_1,y_1) + F_{X,Y}(x_2,y_2) - F_{X,Y}(x_1,y_2) - F_{X,Y}(x_2,y_1)\right]$$

$$= \frac{1}{16}\left[x_1 y_1(x_1+y_1) + x_2 y_2(x_2+y_2) - x_1 y_2(x_1+y_2) - x_2 y_1(x_2+y_1)\right]$$

$$= \frac{1}{16}\left[(x_2^2 - x_1^2)(y_2 - y_1) + (x_2 - x_1)(y_2^2 - y_1^2)\right]$$

$$= \frac{1}{16}\left[(x_2-x_1)(x_2+x_1)(y_2-y_1) + (x_2-x_1)(y_2-y_1)(y_2+y_1)\right]$$

$$\geq 0 \quad (\because x_1 \le x_2, \; y_1 \le y_2)$$

$\therefore F_{X,Y}(x,y)$ is a joint CDF of (X,Y).

$F_{X,Y}(x,y)$ is continuous and thus joint pdf of (X,Y) is

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x\,\partial y} = \frac{\partial}{\partial x\,\partial y}\left\{\frac{1}{16}xy(x+y)\right\}$$

$$= \begin{cases} \frac{1}{8}(x+y), & 0 \le x \le 2,\ 0 \le y \le 2 \\ 0, & \text{elsewhere.} \end{cases}$$

Verify it is a joint PDF of (X,Y).

$$\frac{1}{8}\int_0^2\int_0^2 (x+y)\,dx\,dy = 1.$$

(iii) $P(0 \le X \le 1,\ 1 \le Y \le 2) = \int_1^2\int_0^1 f_{X,Y}(x,y)\,dx\,dy$

$$= \int_1^2\int_0^1 \left[\frac{1}{8}(x+y)\right] dx\,dy$$

$$= \frac{1}{4}. \text{ (After simplification)}$$

Ex: Let X and Y have joint PDF defined as

$$f_{X,Y}(x,y) = K\,\frac{1+x+y}{(1+x)^4(1+y)^4}, \quad \begin{matrix} x>0 \\ y>0 \end{matrix}$$

Find K.

Ans: $9/2$.

Ex: In a sociological project, families with 0, 1, and 2 children are studied. Suppose that no. of children occur with frequencies: 0 children 30%; 1 children 40%. 2 children 30%. A family is choosen from target population. Let X and Y be the Random variables denoting the number of children in the family and no. of boys among those children, respectively. Finally assume that $P(\text{boy}) = \frac{1}{2} = P(\text{Girl})$. Calculate the joint probability mass function (PMF)

$$p_{X,Y}(x,y) = P(X=x, Y=y), \quad 0 \le y \le x, \; x = 0,1,2.$$

Soln: X: no. of children in the family

Y: no of boys among those children

$P(0 \text{ children}) = 0.3$, $P(1 \text{ children}) = 0.4$

$P(2 \text{ children}) = 0.3$

We are interested to evaluate the joint PMF of (X,Y). Recall the following

$$p_{X,Y}(x,y) = P(Y=y \mid X=x)\, P(X=x)$$

$$P(X=x, Y=y) = P(Y=y \mid X=x)\, P(X=x)$$

$$= \begin{cases} P(Y=y \mid X=0)\, P(X=0), & y=0 \\ P(Y=y \mid X=1)\, P(X=1), & y=0,1 \\ P(Y=y \mid X=2)\, P(X=2), & y=0,1,2 \end{cases}$$

$$P(X=0, Y=0) = P(Y=0 \mid X=0)\, P(X=0) = 1 \times 0.3 = 0.3$$

$$P(X=1, Y=0) = P(Y=0 \mid X=1)\, P(X=1) = \tfrac{1}{2} \cdot (0.4) = 0.2$$

$$P(X=1, Y=1) = P(Y=1 \mid X=1)\, P(X=1) = \tfrac{1}{2} \cdot (0.4) = 0.2$$

$$P(X=2, Y=0) = P(Y=0 \mid X=2)\, P(X=2) = \tfrac{1}{4} \cdot (0.3) = 0.075$$

$$P(Y=1, X=2) = P(Y=1 \mid X=2)\, P(X=2) = \tfrac{1}{2}(0.3) = 0.15$$

$$P(Y=2, X=2) = P(Y=2 \mid X=2)\, P(X=2) = \tfrac{1}{4}(0.3) = 0.075$$

Y \ X	0	1	2
0	0.3	0.2	0.075
1	0	0.2	0.15
2	0	0	0.075

Ex: Let RV Y be distributed as $P(\lambda)$ (Poisson) distn and the conditional pdf of a RV X given $Y=y$ is Binomial $B(y,p)$. Then show that

(i) Marginal PMF of X is Poisson $P(\lambda p)$.

(ii) The conditional PMF $p_{Y|X=x}(y|x)$ is Poisson with parameter $\lambda(1-p)$ over the set $x, x+1, \ldots$

Soln: (i) Given that $Y \sim P(\lambda)$

$$p_{X|y}(x|y) \sim Bin(y,p)$$

Marginal PMF of X is

$$p_X(x) = \sum_{y \in R_Y} p_{X,Y}(x,y)$$

$$= \sum_{y \in R_Y} p_{X|y}(x|y)\, p_Y(y)$$

$$= \sum_{y=x}^{\infty} \binom{y}{x} p^x (1-p)^{y-x} \frac{e^{-\lambda}\lambda^y}{y!}$$

put $y-x=t$

$$= \sum_{t=0}^{\infty} \binom{t+x}{x} p^x (1-p)^t \frac{e^{-\lambda}\lambda^{t+x}}{(t+x)!}$$

$$= \frac{(\lambda p)^x e^{-\lambda}}{x!} \sum_{t=0}^{\infty} \frac{1}{t!} (1-p)^t \lambda^t$$

$$p_X(x) = \frac{(\lambda p)^x e^{-\lambda}}{x!} \sum_{t=0}^{\infty} \frac{(\lambda(1-p))^t}{t!}$$

$$= \frac{(\lambda p)^x e^{-\lambda}}{x!} e^{\lambda(1-p)}$$

$$= \frac{(\lambda p)^x e^{-\lambda p}}{x!}, \quad x = 0, 1, 2, \cdots$$

$\boxed{\therefore X \sim P(\lambda p)}$ proved.

try part (ii).

<u>Ex</u>: If the joint pdf $f(x,y)$ is defined by

$$f_{X,Y}(x,y) = cx^2y, \quad 0 < x^2 < y < 1$$
$$= 0, \quad \text{otherwise},$$

then find the constant c. Also Evaluate the prob. $P(0 < X \le \frac{3}{4}, \frac{1}{4} \le y \le 1)$.

<u>Ex</u>: Consider the joint pdf $f_{X,Y}(x,y) = e^{-(x+y)}$, $x>0, y>0$.

Calculate $P(X>1)$, $P(X<Y \mid X<2Y)$

$P(1<X+Y<2)$.

Ex: Consider $f_{X,Y}(x,y) = 8xy,\ 0 < x \le y < 1$.

Find $f_X(x)$, $f_Y(y)$, $f_{X|Y}(x|y)$, $f_{Y|X}(y|x)$

Also evaluate $E(X)$, $E(X^2)$, $V(X)$, $E(XY)$

$E(X|y)$, $E(Y|x)$, $V(X|y)$.

Ex: For the joint PMF

$y \backslash x$	0	1	2	3
0	0.05	0.21	0	0
1	0.2	0.26	0.08	0
2	0	0.06	0.07	0.02
3	0	0	0.03	0.02

Find $F_{X,Y}(2,1)$.

$P(2 \le X \le 3, 0 \le Y \le 2)$

$E(X|Y=0)$

$E(Y|X=2)$.