Poisson Process II

Result: Let $X_1(t)$ and $X_2(t)$ be two independent Poisson processes, then show that the conditional distⁿ of $X_1(t)$ given $X_1(t)+X_2(t)$ is Binomial random variable.

$\Rightarrow$ Let $X_1(t)$ and $X_2(t)$ be the Poisson processes with rates λ_1 and λ_2 respectively; That is,

$$P(X_1(t)=k) = e^{-\lambda_1 t}\frac{(\lambda_1 t)^k}{k!}, \quad k=0,1,2,\ldots \quad \lambda_1>0$$

$$P(X_2(t)=l) = e^{-\lambda_2 t}\frac{(\lambda_2 t)^l}{l!}, \quad l=0,1,2,3,\ldots \quad \lambda_2>0.$$

Using addition property of Poisson random variables we observe that

$$X_1(t)+X_2(t) \sim \text{Poisson}((\lambda_1+\lambda_2)t).$$

ie,
$$P(X_1(t)+X_2(t)=n) = e^{-(\lambda_1+\lambda_2)t}\frac{[(\lambda_1+\lambda_2)t]^n}{n!}$$
$$n=0,1,2,\ldots$$

The required conditional distⁿ is obtained as

$$P(X_1(t)=k \mid X_1(t)+X_2(t)=n)$$
$$= \frac{P(X_1(t)=k,\ X_1(t)+X_2(t)=n)}{P(X_1(t)+X_2(t)=n)}$$

$$= \frac{P(X_1(t) = k, X_2(t) = n-k)}{P(X_1(t) + X_2(t) = n)}$$

$$= \frac{P(X_1(t) = k)\, P(X_2(t) = n-k)}{P(X_1(t) + X_2(t) = n)} \quad \left\{ \because X_1(t) \,\&\, X_2(t) \text{ independent.} \right\}$$

$$= \frac{e^{-\lambda_1 t} \dfrac{(\lambda_1 t)^k}{k!} \, e^{-\lambda_2 t} \dfrac{(\lambda_2 t)^{n-k}}{(n-k)!}}{e^{-(\lambda_1+\lambda_2) t} \dfrac{((\lambda_1+\lambda_2) t)^n}{n!}}$$

$$= \frac{n!}{k!(n-k)!} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

$$= \binom{n}{k} p^k (1-p)^{n-k}, \quad p = \frac{\lambda_1}{\lambda_1+\lambda_2}.$$

this is a binomial dist^n.

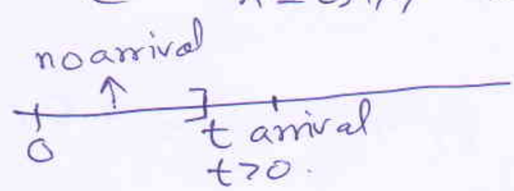
Result is proved

Result: Consider a Poisson process $X(t)$ with rate λ $(\lambda > 0)$. Suppose that random variable T denotes the time of the first occurrence. What is the probability distribution of T.

$\Rightarrow$ We use the CDF approach to obtain the prob. distribution. Note that

$$P(T > t) = P(X(t) = 0) = \begin{cases} e^{-\lambda t}, & t > 0 \\ 1, & t \leq 0 \end{cases}$$

$$\left[\because P(X(t) = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \cdots \right]$$



$\therefore$ CDF of T is

$$F_T(t) = 1 - P(T > t) = \begin{cases} 0, & t \leq 0 \\ 1 - e^{-\lambda t}, & t > 0 \end{cases}$$

The pdf of T is given by

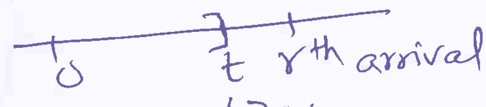
$$f_T(t) = \frac{d F_T(t)}{dt} = \begin{cases} \lambda e^{-\lambda t}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

So this is one parameter exponential dist^n.

Thus one parameter exponential distribution arises as the distribution of the waiting time in a Poisson process for the first arrival or first occurrence of the event of interest.

<u>Result</u>: Consider a Poisson process with rate λ. Let T_r denote the time of rth arrival/occurrence in the process. Find the probability distribution of T_r.

$\Rightarrow$ Let us compute the following probability



$$P(T_r > t) = \begin{cases} P(X(t) \le r-1), & t > 0 \\ 1, & t \le 0 \end{cases}$$

$$= \begin{cases} \sum_{j=0}^{r-1} e^{-\lambda t} \frac{(\lambda t)^j}{j!}, & t > 0 \\ 1, & t \le 0 \end{cases}$$

$$F_{T_r}(t) = 1 - P(T_r > t) = \begin{cases} 0, & t \le 0 \\ 1 - \sum_{j=0}^{r-1} e^{-\lambda t} \frac{(\lambda t)^j}{j!}, & t > 0. \end{cases}$$

The pdf of T_r is given by

$$f_{T_r}(t) = \frac{d F_{T_r}(t)}{dt}$$

$$= \begin{cases} 0, & t < 0 \\ -\frac{d}{dt}\left[e^{-\lambda t} + \lambda t\, e^{-\lambda t} + \frac{(\lambda t)^2}{2!} e^{-\lambda t} + \cdots \right. \\ \left. \cdots + e^{-\lambda t} \frac{(\lambda t)^{r-1}}{(r-1)!} \right], & t > 0 \end{cases}$$

$$f_{T_r}(t) = \lambda e^{-\lambda t} - \lambda e^{-\lambda t} + \lambda^2 t e^{-\lambda t} - \lambda^2 t e^{-\lambda t} + \cdots$$

$$\cdots + \frac{\lambda^r t^{r-1}}{(r-1)!} e^{-\lambda t}$$

$$= \frac{\lambda^r t^{r-1} e^{-\lambda t}}{(r-1)!}$$

$$\therefore f_{T_r}(t) = \frac{\lambda^r t^{r-1}}{(r-1)!} e^{-\lambda t}, \quad t > 0, \ \lambda > 0$$

r is a positive integer.

pdf of T_r is gamma $G(r, \lambda)$, when r is a positive integer, gamma is also known as Erlang distn.

Thus gamma distribution arises in practice as a waiting time for a given rth arrival in a Poisson process.

— x —

Suppose we are interested in whether or not event T_r has occurred for some given values of r and time t. That is we want to evaluate $P(T_r \le t)$.

To obtain this probability we have to integrate the pdf of T_r over the specified range of the event of interest. Sometimes it is easier to sum the corresponding discrete Probabilities. Note that event $\{T_r \le t\}$ is equivalent to the event $\{X(t) \geq r\}$, In other words, the rth arrival occurs before time t iff there are r or more arrivals occurred in the time interval $(0, t]$. With this background, the equivalence of these events we can switch from a prob. statement dealing with a continuous rv to a prob. statement dealing with discrete rv.

Thus
$$P(T_r \le t) = P(X(t) \geq r) = 1 - \sum_{j=0}^{r-1} e^{-\lambda t} \frac{(\lambda t)^j}{j!}$$

<u>EX</u>: The time between accidents, near a busy road intersection, follows exponential distribution with a mean time of five days between accidents. An accident has just occurred, what is the prob. that next accident will occur within next 48 hours.

$\Rightarrow$ $T \sim \exp(\frac{1}{5})$ $\quad f_T(t) = \frac{1}{5} e^{-t/5}, \quad t > 0.$

$\therefore$ $P(T \le 2) = 1 - e^{-2/5} \simeq 0.33.$

* In the previous example determine the probability that there will be at least four accidents next week.

$\Rightarrow$ If $X(t)$: no of accident between t days

Then $$P(X(t)=n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

Here $\lambda = 0.5$, $t = 7$ (days) $\therefore \lambda t = 1.4$

$$\therefore P(X(t)=n) = \frac{e^{-1.4}(1.4)^n}{n!}, \quad n = 0,1,2,\ldots$$

$$\therefore P(X(7)\geq 4) = 1 - P(X(7)\leq 3)$$

$$= 1 - \left(e^{-1.4} + 1.4\,e^{-1.4} + \frac{(1.4)^2 e^{-1.4}}{2} + \frac{(1.4)^3 e^{-1.4}}{3!}\right)$$

$$= 1 - (0.2466 + 0.3452 + 0.2417 + 0.1128)$$

$$= 0.0537 \longrightarrow *$$

Alternative Solution:

Note that $\{X(7)\geq 4\}$ is equivalent to the event $(T_4 \leq 7)$ where

$$f_{T_4}(t) = \frac{\lambda^r t^{r-1} e^{-\lambda t}}{\Gamma r} = \frac{(0.2)^4}{\Gamma 4} e^{-0.2t} t^3, \quad t>0.$$

$$\therefore P(T_4 \le 7) = 1 - P(T_4 > 7)$$

$$= 1 - \frac{(0.2)^4}{\Gamma 4} \int_7^{\infty} e^{-0.2t}\, t^3\, dt$$

$$= 1 - \frac{(0.2)^4}{4} \left\{ \left. \frac{e^{-0.2t}}{-0.2}\, t^3 \right|_7^{\infty} + \frac{3}{0.2} \int_7^{\infty} t^2 e^{-0.2t}\, dt \right\}$$

$$= 1 - \frac{(0.2)^4}{4} \left\{ \frac{7^3 e^{-1.4}}{0.2} + \frac{3}{0.2} \left[\left. \frac{e^{-0.2t}}{-0.2}\, t^2 \right|_7^{\infty} + \frac{2}{0.2} \int_7^{\infty} t e^{-0.2t}\, dt \right] \right\}$$

$$= 1 - \frac{(0.2)^4}{\Gamma 4} \frac{(7)^3}{0.2} e^{-1.4} - \frac{(0.2)^3}{\Gamma 4}\, 3 \cdot \left[\frac{e^{-1.4t}}{0.2}\, 7^2 + \frac{2}{0.2} \int_7^{\infty} t e^{-0.2t}\, dt \right]$$

Integrate the last term to obtain

$$P(T_4 \le 7) = 1 - \frac{(1.4)^3}{6} e^{-1.4} - \frac{(1.4)^2}{2} e^{-1.4} - 1.4 e^{-1.4} - e^{-1.4}$$

$$\simeq 0.0537 \longleftarrow (**)$$

Note: $(*)$ & $(**)$ have same values as expected.

Ex: Assume that the arrivals to the call center is modeled using a Poisson process with rate 25 per hour. Let 20 calls arrive during 8AM–9AM. Find the probability there will no calls arrive from 9:00 AM to 9:06 AM. Also compute the probability that more than 225 calls arrive during the eight-hour shift.

$\Rightarrow$ $\lambda = 25$ calls per hour

$X(t)$: no of calls during an interval of t hours

· Prob that no calls between 9:00AM to 9:06 AM is given by (here $t = 0.1$ hour)

$$P(X(0.1) = 0) = \frac{e^{-25\times 0.1}(25\times 0.1)^0}{0!}$$

$$= e^{-2.5} \simeq 0.08208.$$

for the 2nd Part

$$P(X(8) > 225)$$

$$= \sum_{j=226}^{\infty} e^{-200}\frac{(200)^j}{j!}$$

$$\left[\begin{aligned} &X(8) \sim P(25\times 8) \\ &\quad \simeq P(200). \\ &E(X(8)) = 200 \\ &V(X(8)) = 200 \end{aligned}\right.$$

It is difficult to evaluate this sum. In such situation we can use CLT to get the prob.

So using CLT we have

$$P(X(8) > 225) = P\left(\frac{X(8) - 200}{\sqrt{200}} > \frac{225 - 200}{\sqrt{200}} \right)$$

$$= P(Z > 1.80)$$

$$\simeq 0.0359.$$