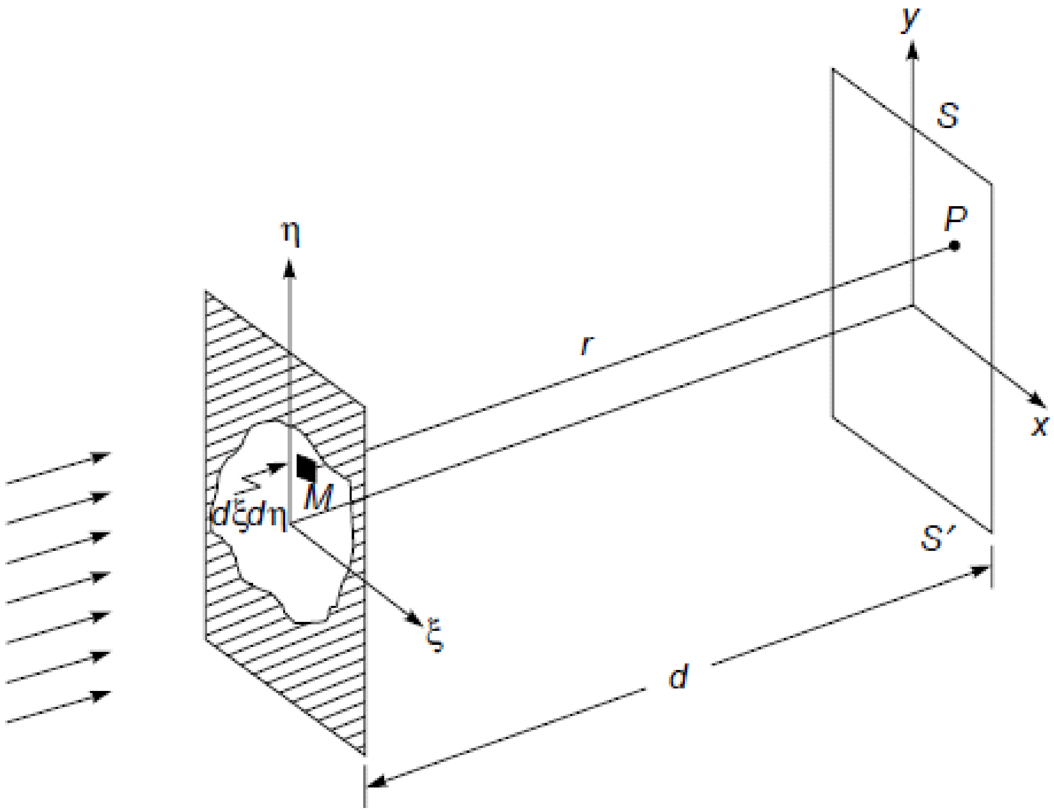
# PH 201

# OPTICS & LASERS

## Lec_Fresnel Diffraction_2

# Fresnel Diffraction

❖ Consider a plane wave of amplitude $A$ incident normally.




**A plane wave incident normally on an aperture.**

- Field produced at point P is given by

$$u(P) = \frac{A}{i\lambda} \iint \frac{e^{ikr}}{r} d\xi d\eta$$

where integration is over area of aperture.

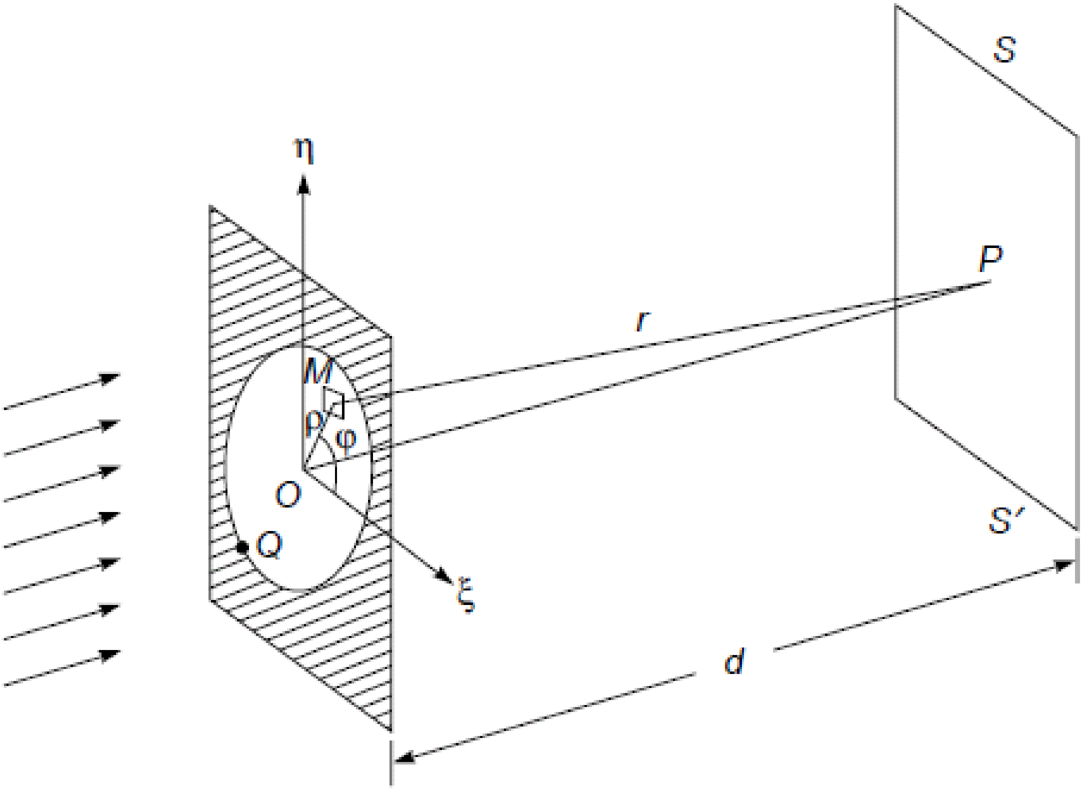
- Now, if amplitude & phase distribution on plane $z = 0$ is given by $A(\xi,\eta)$, then above integral is modified as

$$u(P) = \frac{1}{i\lambda} \iint A(\xi,\eta) \frac{e^{ikr}}{r} d\xi d\eta$$

- In Fresnel approximation, above integral takes the form

$$u(x,y,z) \approx \frac{1}{i\lambda z} e^{ikz} \iint A(\xi,\eta) \times \exp\left\{ \frac{ik}{2z} [(x-\xi)^2 + (y-\eta)^2] \right\} d\xi d\eta$$

# Diffraction of a Plane Wave Incident Normally on a Circular Aperture




**Diffraction of a plane wave incident normally on a circular aperture of radius *a*; point *Q* is an arbitrary point on periphery of aperture.**

- Consider a plane wave incident normally on a circular aperture of radius *a*.

- Z axis is normal to plane of aperture, & screen SS' is assumed to be normal to z axis.

- It is obvious from symmetry of problem that we will obtain circular fringes on screen SS'; however, it is very difficult to calculate actual intensity variation on screen.

- For the sake of mathematical simplicity, we will calculate variation of intensity only along z axis. It will be more convenient to use circular system of coordinates.

- Coordinates of an arbitrary point M on aperture will be ($\rho$,$\Phi$), where $\rho$ is distance of point M from centre O & $\Phi$ is angle that OM makes with $x$ axis, & a small element area $dS$ surrounding point M will be $\rho\, d\rho\, d\Phi$.

$$u(P) \approx -\frac{A}{i\lambda}\int_0^{2a}\int_0^{a}\frac{e^{ikr}}{r}\rho d\rho d\phi$$

❖ Now, $$\rho^2 + d^2 = r^2$$

Thus, $$\rho d\rho = r dr$$

$$u(P) \approx -\frac{iA}{\lambda}\int_0^{2\pi}\int_d^{\sqrt{a^2+d^2}} e^{ikr}\, dr\, d\phi \qquad k = \frac{2\pi}{\lambda}$$

$$\Rightarrow \qquad u(P) \approx Ae^{ikd}(1 - e^{ip\pi})$$

$$where \qquad k\left(\sqrt{a^2+d^2} - d\right) = p\pi$$

$$\Rightarrow \qquad QP - OP = \frac{p\lambda}{2}$$

where Q is a point on periphery of circular aperture.
Taking intensity,

$$I(P) = 4I_0 \sin^2 \frac{p\pi}{2}$$

$I_0$ is intensity associated with incident plane wave.

- Intensity is zero or maximum when *p* is an even or odd integer, i.e., when QP-OP is an even or odd multiple of $\lambda/2$.

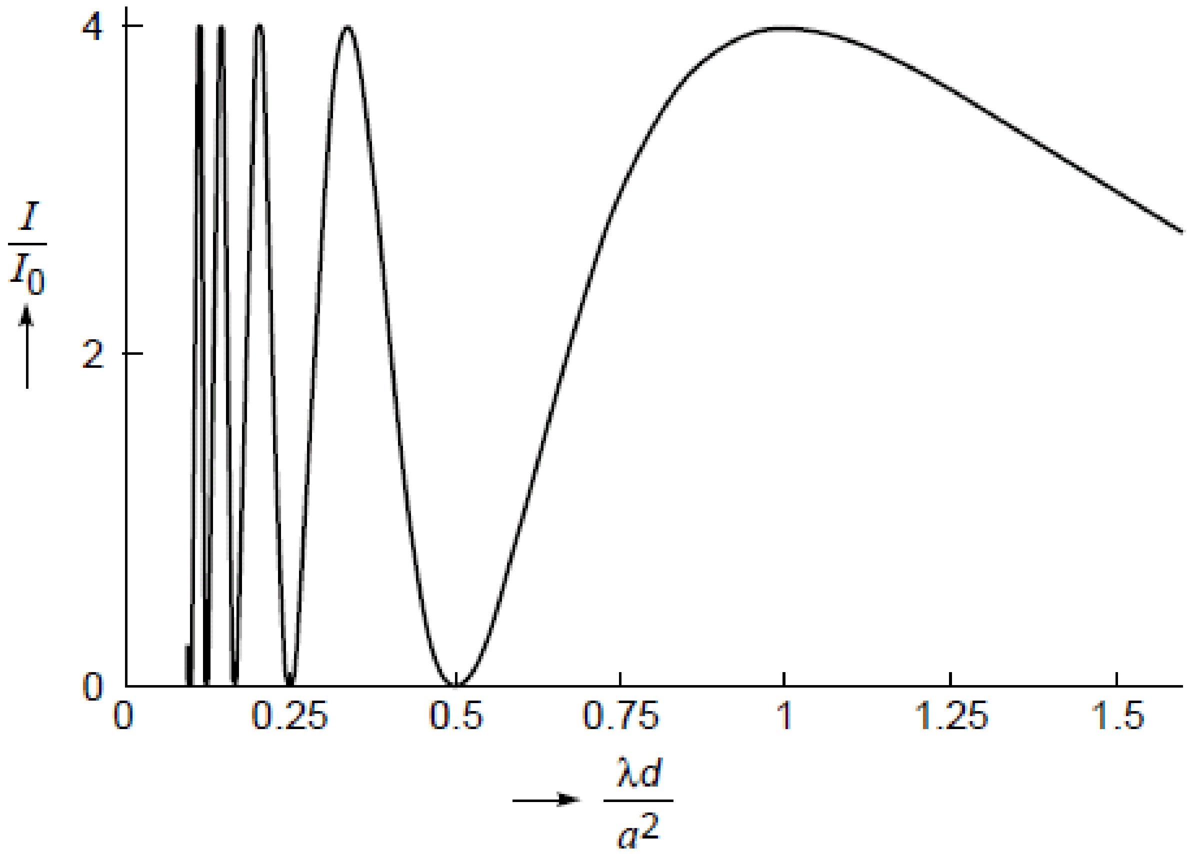
- If aperture contains an even number of half-period zones, intensity at P will be negligibly small; & conversely, if circular aperture contains an odd number of zones, intensity at P will be maximum.

- When $d \ll a$,

$$p \approx \frac{k}{\pi}\left[d\left(1+\frac{a^2}{2d^2}\right)-d\right]$$

$$or \qquad p \approx \frac{a^2}{\lambda d}$$

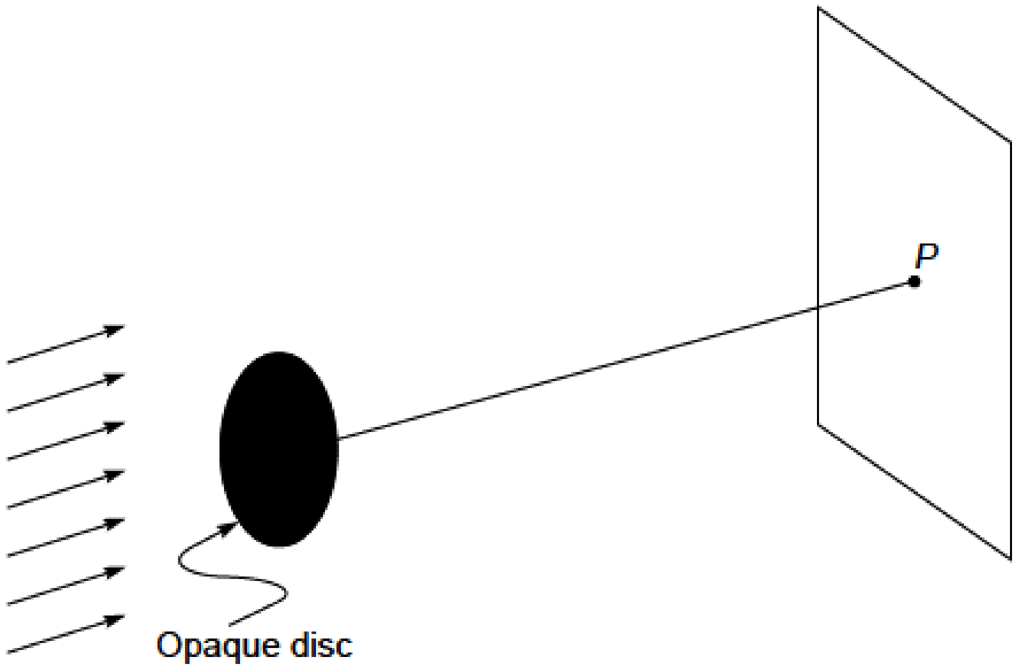
which is known as Fresnel number of aperture.




**Intensity variation on an axial point corresponding to a plane wave incident on a circular aperture of radius *a*.**

# Diffraction by a Circular Disc

- Assuming that observation point lies on the axis of disc.
- Carry out integration over open region of aperture.




**Diffraction pattern produced by an opaque disc of radius *a*.**

- If $u_1$(P) & $u_2$(P), respectively, represent fields at P due to a circular aperture & an opaque disc (of same radius), then

$$u_1(P)+u_2(P)=u_0(P)$$

where $u_0$(P) represents field in absence of any aperture. This Eq. is known as **Babinet's principle**. Thus,

$$\begin{aligned} u_2(P) &= u_0(P)-u_1(P) \\ &= u_0(P)-u_0(P)(1-e^{ip\pi}) \\ u_2(P) &= u_0(P)e^{ip\pi} \end{aligned}$$

- Intensity at P on the axis of a circular disc is

$$I_2(P)=\left|u_2(P)\right|^2=I_0(P)$$

- Intensity at a point on axis of an opaque disc is equal to intensity at point in absence of disc! This is the Poisson spot.

# Gaussian Beam Propagation

- When a laser oscillates in its fundamental transverse mode, transverse amplitude distribution is Gaussian.

- Also, output of a single mode fiber is very nearly Gaussian.

- Assuming a Gaussian beam propagating along $z$ direction whose amplitude distribution on plane $z = 0$ is given by

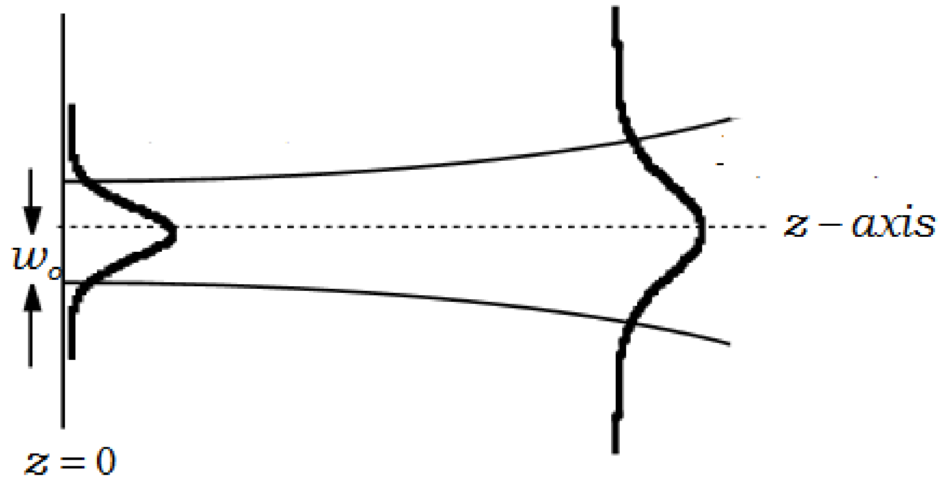
$$A(\xi,\eta) = a\exp\left(-\frac{\xi^2+\eta^2}{w_0^2}\right)$$

implying that phase front is plane at $z = 0$. From this Eq. it follows that at a distance $w_0$ from $z$ axis, amplitude falls by a factor $1/e$ (i.e., intensity reduces by a factor $1/e^2$).

This quantity $w_0$ is called *spot size* of beam.

$$u(x,y,z) \approx \frac{1}{i\lambda z} e^{ikz} \iint A(\xi,\eta) \times \exp\left\{\frac{ik}{2z}[(x-\xi)^2 + (y-\eta)^2]\right\} d\xi d\eta$$

$$A(\xi,\eta) = a \exp\left(-\frac{\xi^2+\eta^2}{w_0^2}\right)$$




**Diffraction of a Gaussian field profile**

After substitution & solving integral,

$$u(x,y,z) \approx \frac{a}{1+i\gamma}\exp\left[-\frac{x^2+y^2}{w^2(z)}\right]e^{i\phi}$$

$$\gamma = \frac{\lambda z}{\pi w_0^2}$$

$$w(z) = w_0(1+\gamma^2)^{1/2} = w_0\left(1+\frac{\lambda^2 z^2}{\pi^2 w_0^4}\right)^{1/2}$$

$$\phi = kz + \frac{k}{2R(z)}(x^2+y^2)$$

$$R(z) \equiv z\left(1+\frac{1}{\gamma^2}\right) = z\left(1+\frac{\pi^2 w_0^4}{\lambda^2 z^2}\right)$$

Thus, intensity distribution is given by,

$$I(x,y,z) = \frac{I_0}{1+\gamma^2}\exp\left[-\frac{2(x^2+y^2)}{w^2(z)}\right]$$

- It is proved that transverse intensity distribution remains Gaussian with beam width increasing with *z* essentially implies diffraction divergence.

- For small values of *z*, width increases quadratically with *z*, but for large values of *z*

$$z >> \frac{w_0^2}{\lambda}$$

$$w(z) \approx w_0 \frac{\lambda z}{\pi w_0^2} = \frac{\lambda z}{\pi w_0}$$

which shows that width increases linearly with z.

- Diffraction angle is defined as,

$$\tan\theta = \frac{w(z)}{z} \approx \frac{\lambda}{\pi w_0}$$

showing that rate of increases in width is proportional to wavelength & inversely proportional to initial width of beam.

❖ Assuming $\lambda = 0.5\ \mu m$, For $w_0 = 1\ mm$

$2\theta \approx 0.018^o$ $\quad$ $w \approx 0.018\ mm$ $\quad$ at $z = 10\ m$

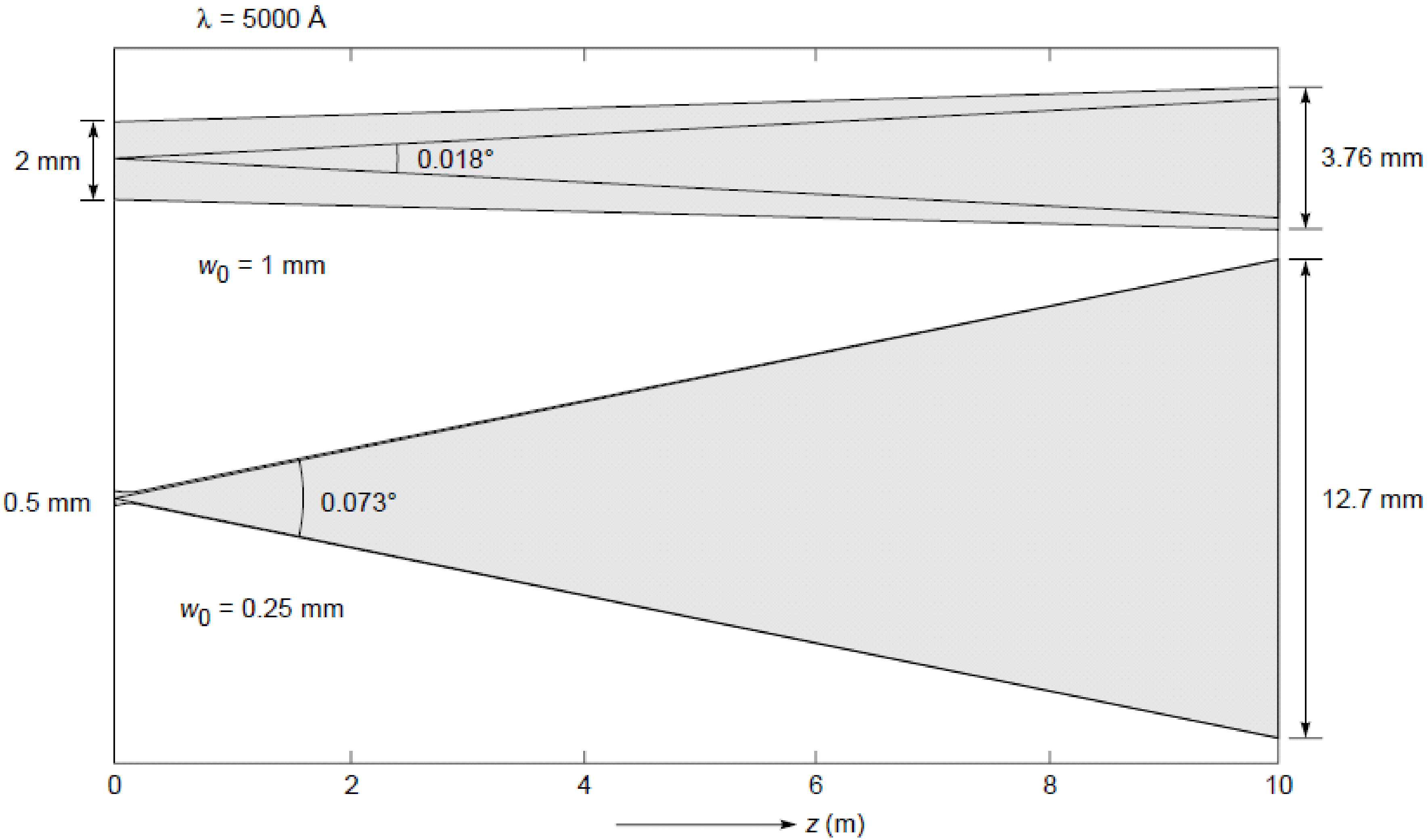
❖ Similarly, for $w_0 = 0.25\ mm$

$2\theta \approx 0.073^o$ $\quad$ $w \approx 6.35\ mm$ $\quad$ at $z = 10\ m$

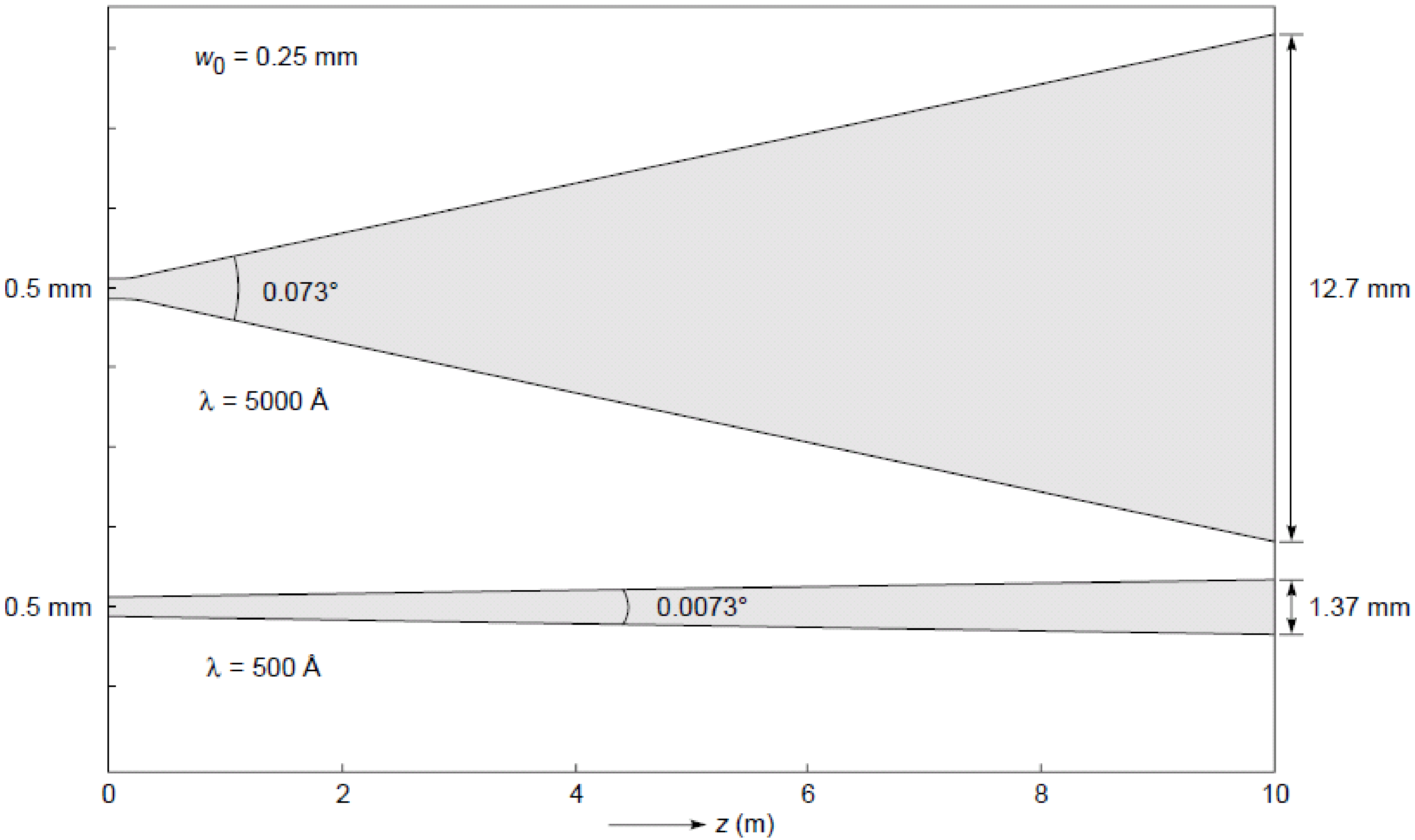
❖ $\theta$ increases with a decrease in $w_0$ (the smaller the size of aperture, the greater the diffraction).

❖ For a given value of $w_0$, diffraction effects decrease with $\lambda$.

❖ Fig.: decrease in diffraction divergence for $w_0$ = 0.25 mm as wavelength is decreased from 5000 to 500 Å; indeed as $\lambda \rightarrow 0$, $\theta \rightarrow 0$ & there is no diffraction which is geometric optics limit.




Diffraction divergence of a Gaussian beam whose phase front is plane at $z = 0$. Fig. shows increase in diffraction divergence as initial spot size is decreased from 1 to 0.25 *mm*; wavelength is assumed to be 5000 Å.




Diffraction divergence of a Gaussian beam whose phase front is plane at $z = 0$. Fig. shows decrease in divergence as wavelength is decreased from 5000 to 500 Å; initial spot size $w_0$ is assumed to be 0.25 *mm*.

- It can be shown that

$$\int_{-\infty}^{+\infty}\int I(x,y,z)dxdy = \frac{\pi w_0^2}{2} I_0$$

which is independent of *z*. This is to be expected, as the total energy crossing the entire *xy* plane will not change with *z*.

- For a spherical wave diverging from origin, the field distribution is given by

$$u \sim \frac{1}{r} e^{ikr}$$

- On the plane *z* = *R*

$$r = (x^2 + y^2 + R^2)^{1/2}$$

$$= R\left(1 + \frac{x^2 + y^2}{R^2}\right)^{1/2}$$

$$\approx R + \frac{x^2 + y^2}{2R}$$

$$Assu\min g$$

$$|x|, |y| << R$$

- Thus, on the plane $z = R$, phase distribution (corresponding to a spherical wave of radius $R$) is given by

$$e^{ikr} \approx e^{ikR} e^{\frac{ik}{2R}(x^2+y^2)}$$

- A phase variation of type

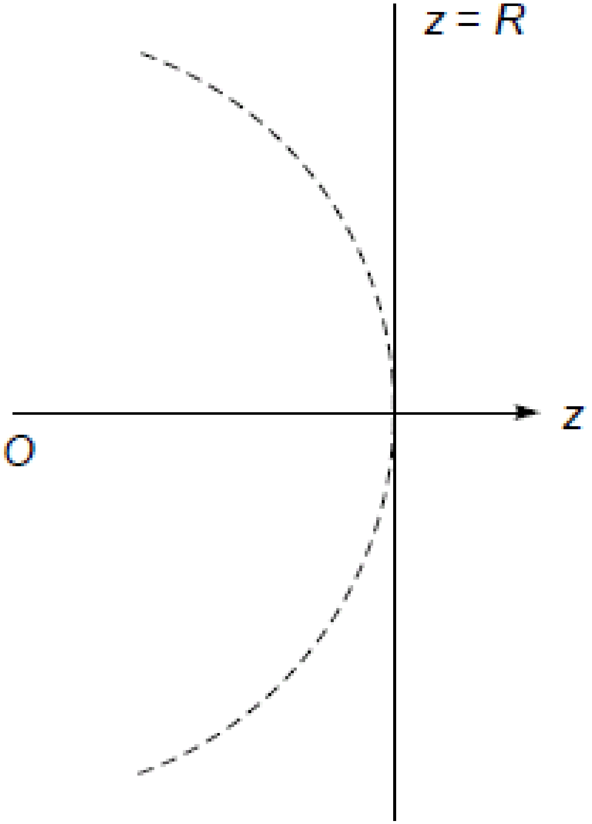
$$\exp\left[ i\frac{k}{2R}(x^2+y^2) \right]$$

on $xy$ plane represents a diverging spherical wave of radius $R$.

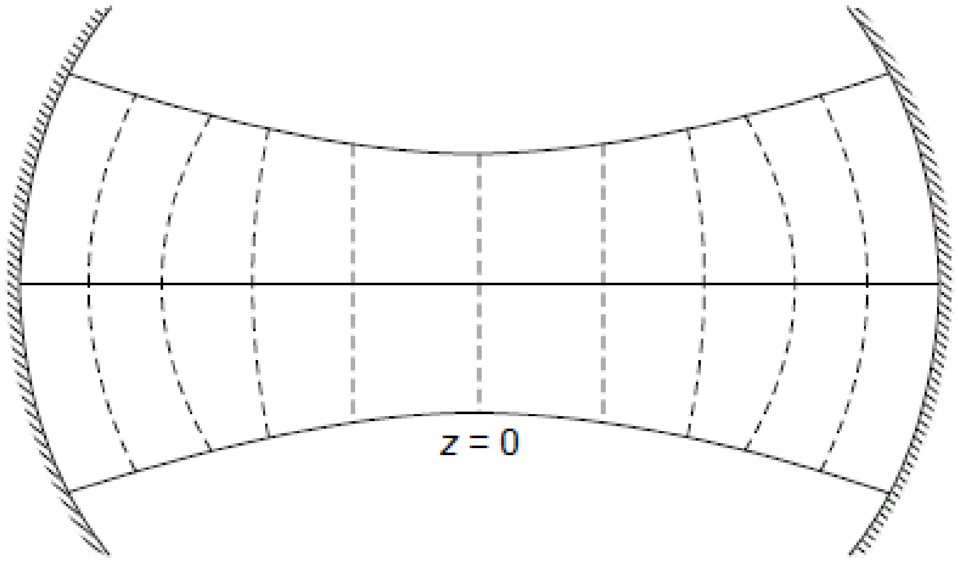
Comparing two expressions, radius of curvature of phase front

$$R(z) \approx z\left( 1+\frac{\pi w_0^4}{\lambda^2 z^2} \right)$$

- Thus, as the beam propagates, the phase front which was plane at $z = 0$ becomes curved.




A spherical wave diverging from point O. Dashed curve represents a section of spherical wave front at a distance *R* from source.




Diffraction divergence of a Gaussian beam whose phase front is plane at *z* = 0. Dashed curves represent phase fronts.

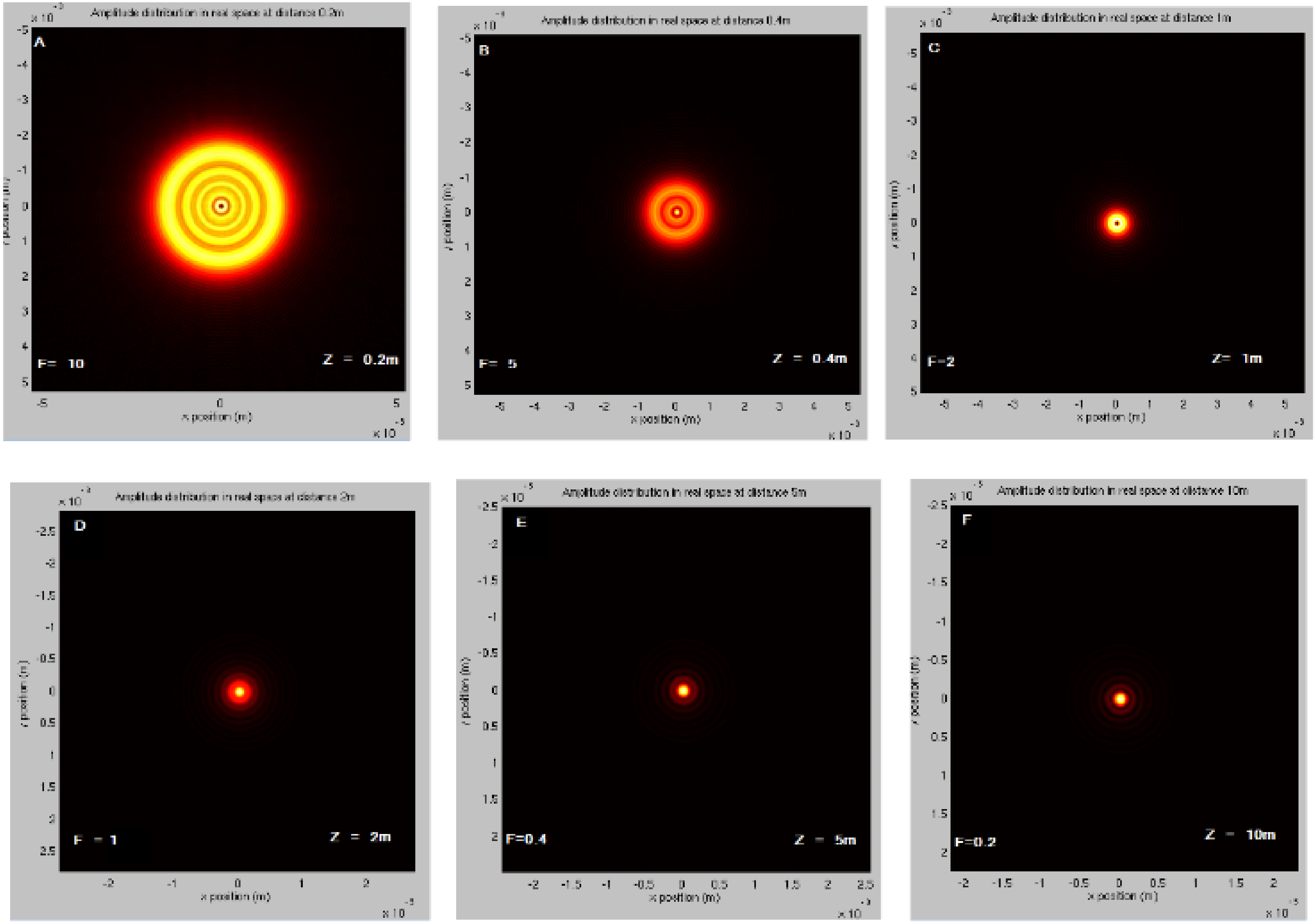
- Gaussian beam resonating between two identical spherical mirrors of radius *R*, plane $z$ = 0, where phase front is plane & beam has minimum spot size, referred to as waist of Gaussian beam.

- For the beam to resonate, the phase front must have a radius of curvature equal to *R* on the mirrors.

- For this to happen, we must have

$$R \approx \frac{d}{2}\left(1+\frac{4\pi w_0^4}{\lambda^2 d^2}\right)$$

where *d* is the distance between the two mirrors.




**Transition from Fresnel to Fraunhofer regimes with increasing distance**