Indian Institute of Technology, Patna
MA102, B.Tech -I year
Spring Semester: 2012-2013
(Mid Semester Examintaion)

Maximum Marks: 30 **Time:** 2 Hours

Note:

(i) This question paper has TWO pages and contain ELEVEN questions. Please check all pages and report the discrepancy, if any.

(ii) Attempt all questions.

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1. Let A be a real skew-symmetric matrix of order n.

(a) If n is odd, show that $det(A) = 0$

(b) If n is even, show that $det(A) \geq 0$

(c) For every n, show that $det(I + A) \geq 1$ [3]

2. Consider the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ in $\mathbb{R}^2$. Let A be the 2×2 matrix with (a_i, b_i) as the ith row, $i = 1, 2$. Let B be the 2×3 matrix with (a_i, b_i, c_i) as the ith row, $i = 1, 2$. Show the following:

(a) The lines are identical iff $rank(B) = 1$.

(b) The lines are parallel but not identical iff $rank(A) = 1$ and $rank(B) = 2$.

(c) The lines intersect but are not identical iff $rank(A) = 2$. [2]

3. Find the non-zero solutions of the following system of linear equations. $[2\frac{1}{2}]$
$2x - 2y + 5z + 3w = 0,\ 4x - y + z + w = 0,\ 3x - 2y + 3z + 4w = 0,\ x - 3y + 7z + 6w = 0$

4. Determine the value of λ for which the following equations have non-zero solutions. $[2\frac{1}{2}]$
$x + 2y + 3z = \lambda \cdot x,\ 3x + y + 2z = \lambda \cdot y,\ 2x + 3y + z = \lambda \cdot z$

5. (i) Show that the vectors $v = (1 + i, 2i)$ and $w = (1, 1 + i)$ in $\mathbb{C}^2$ are linearly dependent over the complex field $\mathbb{C}$ but linearly independent over the real number $\mathbb{R}$ $[1\frac{1}{2}]$

(ii) Find the dimension of the subspaces $S \bigcap T$ of $\mathbb{R}^4$ where $S = \{(x, y, z, w) \in \mathbb{R}^4 : x + y + z + w = 0\}$ $T = \{(x, y, z, w) \in \mathbb{R}^4 : 2x + y - z + w = 0\}$. $[1\frac{1}{2}]$

6. (i) Let T be a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}$ such that $T(1, 0, 0, 0) = 1, T(1, -1, 0, 0) = 0$, $T(1, -1, 1, 0) = 1$, $T(1, -1, 1, -1) = 0$. Determine $T(a, b, c, d)$. $[1\frac{1}{2}]$

(ii) Let T be linear transformation from a vectors space V of dimension n to itself that satisfies $T^2 = 0$. Prove that the image of T is contained in the Kernel of T and hence that the rank of T is at most $n/2$. $[1\frac{1}{2}]$

7. Prove of disprove the following statements. Justify your answers.

   (i) Let A be an $n \times n$ matrix with integer entries. Suppose n is an integer and each row of A has sum n. $\Rightarrow n$ is an eigen value of A.

   (ii) Let T be linear transformation from a vectors space V into itself. Suppose $x \in V$ is such that $T^m x = 0$, but $T^{m-1}x \neq 0$ for some positive integer m. $\Rightarrow x, Tx, ..., T^{m-1}x$ are linearly independent.

   (iii) The matrix $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 5 & 0 & 5 & 0 \\ 1 & 5 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ has two positive and two negative eigen values.

   (iv) The matrix $A = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix}$, where a, b, c are non-zero real numbers, has only one non-zero eigen value. $[4 \times 1\frac{1}{2} = 6]$

8. Let V be a vector space and T be a linear transformation from V into V. Prove that the following statements are equivalent:

   (i) The intersection of range of T and the null space of T is the zero subspace of V.

   (ii) $T(Tu) = \theta \Rightarrow Tu = \theta$. [2]

9. Determine whether the maps given below is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ over $\mathbb{R}$.
$$f(x_1, x_2) = \begin{cases} (x_1, x_2) & \text{if } x_2 = 0 \\ \left(\frac{{x_1}^2}{x_2}, x_2\right) & \text{if } x_2 \neq 0 \end{cases}$$
$[1\frac{1}{2}]$

10. A sequence of real numbers $(a_1, a_2, a_3, ...)$ is called a Fibonacci sequence if $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 3$. Show that the set of all Fibonacci sequences form a vector space under component-wise addition and scalar multiplication defined in a natural way. $[3\frac{1}{2}]$

11. Check whether the functions $f(t) = sint, g(t) = cost, h(t) = t$ from $\mathbb{R}$ to $\mathbb{R}$ are linearly independent or dependent. Justify your answer. [1]