Department of Mathematics
Indian Institute of Technology Patna
MA - 201: B.Tech. II year
Autumn Semester: 2013-14

Assignment-4: Complex Analysis

1. Evaluate $\int_C |z|\bar{z}\mathrm{d}z$ where C contain $z = Re^{it},\ 0 \leqslant t \leqslant \pi$ and straight line $-R \leq Re(z) \leq R,\ Im(z) = 0$.

2**. Let $f(z)$ be continuous in a simply connected domain D and if $\oint_C f(z)\mathrm{d}z = 0$ for every closed contour C in D then $f(z)$ is analytic in D. (Morera Theorem)

3**. If a function $f(z)$ is analytic at a given point, then its derivatives of all orders are analytic there too.

4**. If $P(z)$is a non constant polynomial then prove that the equation $P(z) = 0$ has at least one root.

5. Evaluate $\int_C f(z)dz$, when
(i) $f(z) = \frac{z+2}{z},\ \ C:\ z = 2e^{i\theta},\ \pi \leq \theta \leq 2\pi$
(ii) $f(z) = \frac{z+2}{z},\ \ C:\ z = 2e^{i\theta},\ 0 \leq \theta \leq 2\pi$
(iii) $f(z) = \pi e^{\pi\bar{z}},\ \ C:$ boundary of the square with vertices at the points $0, 1, 1+i, i,$ orientation of C is in positive direction
(iv)** $f(z) = \bar{z},\ \ C: z = \sqrt{4-y^2} + iy\ (-2 \leq y \leq 2)$
(v)$f(z) = x + y^2 - ixy,\ \ C: z(t) = (t-2i),\ 1 \leq t \leq 2,$ and $z(t) = 2-(4-t)i,\ 2 \leq t \leq 3$
(vi)$f(z) = z^{-1+i},\ (|z| > 0, 0 < \arg z < 2\pi),\ \ C: |z| = 1$ taken anticlockwise

6. Find an upper bound for the absolute value of the integral $\int_C f(z)dz$, when
(i) $f(z) = e^{1/z},\ \ C:$ quarter circle $|z| = 1,\ 0 \leq arg(z) \leq \pi/2$ from the point 1 to the point i
(ii) $f(z) = e^{z^2},\ \ C:$ broken lines from $z = 0$ to $z = 1$ and then from $z = 1$ to $z = 1+i$
(iii) $f(z) = \frac{2z^2-1}{z^4+5z^2+4},\ \ C:$ upper half of the circle $|z| = r\ (r > 2)$ taken in counterclockwise direction
(iv)** $f(z) = Log(z)/z^2,\ \ C:\ |z| = r\ (r > 1)$ taken in counterclockwise direction
(v) $f(z) = x^2 + iy^2\ \ C:$ is the line segment joining $-i$ to i

7. Let $z^{1/2}$ denote the function $z^{1/2} = \sqrt{r}e^{i\theta/2},\ (r > 0, -\pi/2 < \theta < 3\pi/2)$. Without actually finding the value of the integral, show that $\lim_{R\to\infty} \int_{C_R} \frac{z^{1/2}}{z^2+1}dz = 0$, where C_R denotes the semicircular path $z = Re^{i\theta}, (0 \leq \theta \leq \pi)$.

8. Let C denotes a positively oriented circle $|z - z_0| = r$, (z_0 is any complex number), then show that $\int_C (z-z_0)^{n-1}dz = \begin{cases} 0, & \text{if } n = \pm 1, \pm 2, \ldots \\ 2\pi i, & \text{if } n = 0 \end{cases}$

9. Evaluate $\int_B f(z)dz$, when $f(z)$ is: $(i)\ \frac{1}{3z^2+1}\quad (ii)\ \frac{z+2}{\sin\frac{z}{2}}\quad (iii)\ \frac{z}{1-e^z}$
where B forms the positively oriented boundary curve of the domain between $|z| = 4$ and the square with sides along $x = \pm 1,\ y = \pm 1$

10. Examine whether Cauchy-Goursat theorem can be applied to evaluate the integral $\int_C f(z)dz$ where $C: |z| = 1$ is in anticlockwise direction and $f(z)$ is:
$(i)\ \frac{z^2}{z-3}\ \ (ii)\ ze^{-z}\ \ (iii)\ sechz\ \ (iv)\ \tan z\ \ (v)\ Log(z+2)\ \ (vi)\ |z|^2e^z\ \ (vii)\ \frac{1}{|z|^3}\ \ (viii)\ \bar{z}$

11. Let C be positively oriented boundary of the square whose sides along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate the integral $\int_C f(z)dz$ when:
$(i)\ f(z) = \frac{e^{-z}}{(z-(i\pi/2))}\quad (ii)\ \frac{\cos z}{z(z^2+8)}\quad (iii)\ \frac{\cosh z}{z^4}\quad (iv)\ \frac{\tan(z/2)}{(z-x_0)^2},\ (-2 < x_0 < 2)$

12. Integrate $\frac{1}{z^4-1}$ over $(i)\ |z+1| = 1,\quad (ii)\ |z-i| = 1$, each curve being taken in anticlockwise direction.

13. Let C be the unit circle centered at zero traversed in positive direction. Integrate over C:
$(i)\ \frac{e^z-1}{z}$ $\quad (ii)\ \frac{z^3}{2z-i}$ $\quad (iii)\ \frac{\cos z}{z-\pi}$ $\quad (iv)\ \frac{\sin z}{z^4}$ $\quad (v)\ \frac{1}{z\cos z}$ $\quad (vi)\ \frac{e^z}{z^2(z^2-16)}$ $\quad (vii)\ \frac{\sinh z^2}{z^3}$.

14. Find the value of the integral of $f(z)$ around the circle $|z-i|=2$ taken in the anticlockwise direction when: $(i)\ f(z)=\frac{1}{z^2+4}$ $\quad (ii)\ f(z)=\frac{1}{(z^2+4)^2}$

15. Evaluate $\int_C (2z-1)(z^2-z)^{-1}dz$ when:
$(i)\ C:|z|=2$, positive direction $\quad (ii)\ C:|z|=\frac{1}{2}$, positive direction

16. Evaluate $\int_C (4z^2+4z-3)^{-1}dz$ when:
$(i)\ C:|z|=1$, positive direction $\quad (ii)\ C:|z+\frac{2}{3}|=1$, positive direction $\quad (iii)\ |z|=3$, positive direction

17**. Suppose that $|f(z)| \leq |f(z_0)|$ at each point z in some neighborhood $|z-z_0|<\epsilon$ in which $f(z)$ is analytic. Then $f(z)$ has the constant value $f(z_0)$ throughout that neighborhood.

18. Find the maximum modulus of following functions over the region prescribed.
$(i)\ 2z+5i,\ |z|\leq 2$ $\quad (ii)\ -iz+i,\ |z|\leq 5$
$(iii)\ z^2,\ \{z=x+iy: 2\leq x\leq 3 \text{ and } 1\leq y\leq 3\}$ $\quad (iii)\ Re(z^2),\ \{z=x+iy: 2\leq x\leq 3 \text{ and } 1\leq y\leq 3\}$

19. Find a power series representation of the following functions centered at a point z_0. Also find their radius of convergence.
(i) $\frac{1}{z^2-5z+6},\ z_0=0$ (ii) $\frac{1}{1-z},\ z_0=2i$ (iii) $\frac{1}{z},\ z_0=1$
(iv) $\cos z,\ z_0=\frac{\pi}{4}$ (v) $\frac{i}{(z-i)(z-2i)},\ z_0=0$ (vi) $\frac{1}{1+z},\ z_0=-i$ (vii) $\frac{1-z}{z-3},\ z_0=1$

20. Find the radius of convergence of Taylor series of given function centered at the indicated point z_0, without expanding the function.
(i) $\frac{3-i}{1-i+z},\ z_0=4-2i$ (ii) $\frac{4+5z}{1+z^2},\ z_0=2+5i$
(iii) $\cos z,\ z_0=\frac{\pi}{4}$ (iv) $\frac{i}{(z-i)(z-2i)},\ z_0=0$

21. Find Laurent series representation for the following functions in specified region:
(i) $z^2\sin(\frac{1}{z^2}),\ 0<|z|<\infty$ (ii) $\frac{e^z}{(z+1)^2},\ 0<|z+1|<\infty$ (iii) $\frac{1}{(z+1)},\ 1<|z|<\infty$

22. Give two Laurent series representation for the following functions and also specify the region of validity:
(i) $\frac{1}{z^2(1-z)}$ (ii) $\frac{1}{z^3-z^4}$ (iii) $\frac{1}{z(z^2+1)}$ (iv) $\frac{1}{z(4-z)^2}$

23. Expand the following functions in a Laurent series valid for specified region:
(i) $\frac{z}{(z-1)(z-3)},\ 0<|z-1|<2$ (ii) $\frac{\cosh z-\cos z}{z^5},\ 0<|z|$ (iii) $\frac{1}{z(z-3)},\ 0<|z|<3$
(iv) $\frac{1}{z(z-3)},\ 3<|z-3|$ (v) $\frac{1}{z(z-3)},\ 1<|z+1|<4$ (vi) $\frac{z}{(z+1)(z-2)},\ 1<|z|<2$
(v) $\frac{7z-3}{z(z-1)},\ 0<|z-1|<1$