One function of Two RVs (Part II). (1)

In the last lecutre we started our discussion on the Transformation of Random variables in 2-dim cases. In particular we are dealing with the transformation 'One function of two rVs'.

In the last class we considered one such transformation like $Z = X+Y$ and obtained its pdf by assuming various consideration on the range space of (X, Y). Two examples were discussed as well in support of the result.

Let us consider $Z = X-Y$ and try to obtain pdf of Z. Note that (X, Y) is a 2-dim continuous Random Variable.

Result(2) Let (X, Y) be jointly distributed continuous random variable. Consider the transformation $Z = X - Y$ and then find the ~~joint~~ pdf of Z when $-\infty < x < \infty$ and $-\infty < y < \infty$.

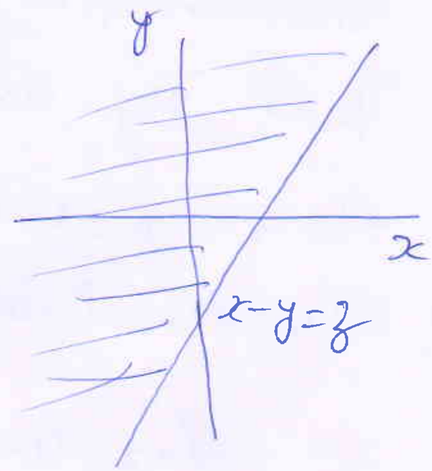
~~Proof~~ Solution: (Please recall the method that we apply to get the result. Basically it is the well known CDF approach, very useful for continuous case.

Cumulative Distribution of function of Z is

$$F_Z(z) = P(Z \le z) = P(X - Y \le z)$$

$$= P(X \le z + Y)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z+y} f_{X,Y}(x, y)\, dx\, dy \quad \text{——} (1)$$



Recall Leibnitz formula

If $F(t) = \int_{a(t)}^{b(t)} f(x, t)\, dx$ then

$$\frac{dF(t)}{dt} = b'(t) f(b(t), t) - a'(t) f(a(t), t) + \int_{a(t)}^{b(t)} \left[\frac{\partial}{\partial t} f(x, t)\right] \cdot dt$$

So from Equation (1) we have the pdf of Z as

$$f_Z(z) = \frac{d F_Z(z)}{dz} = \frac{d}{dz}\left[\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{z+y} f_{X,Y}(x, y)\, dx \right\} dy \right]$$

$$f_Z(z) = \int_{-\infty}^{\infty}\left[\frac{\partial}{\partial z}\int_{-\infty}^{z+y} f_{X,Y}(x,y)\,dx\right]dy$$

$$= \int_{-\infty}^{\infty}\left[\frac{\partial}{\partial z}(y+z)\cdot f_{X,Y}(z+y,y) - 0 + \int_{-\infty}^{z+y}\frac{\partial}{\partial z} f_{X,Y}(x,y)\,dx\right]dy$$

$$= \int_{-\infty}^{\infty}\left[1\cdot f_{X,Y}(z+y,y) - 0 + 0\right]dy$$

$$= \int_{-\infty}^{\infty} f_{X,Y}(z+y,y)\,dy.$$

So pdf of $Z = X+Y$ is given by

$$\boxed{f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(z+y,y)\,dy} \quad \text{--- } ②$$

Likewise we see that

$$\boxed{f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z+x)\,dx} \quad \text{--- } ③$$

Remark: In the previous result if X and Y are independent then ~~$f_Z(z) =$~~ Equation ② becomes

$$\boxed{f_Z(z) = \int_{-\infty}^{\infty} f_X(z+y) f_Y(y)\,dy}$$

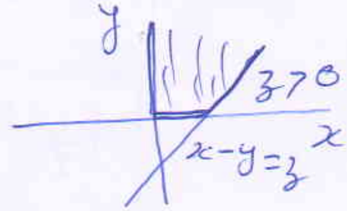
Simily Equation ③ can be modified under the assumption X and Y being independent RVs.

Previous results are derived for $-\infty < X < \infty, -\infty < Y < \infty$.
Next we discuss a special case of (X, Y) where $0 < X < \infty$ and $0 < Y < \infty$.

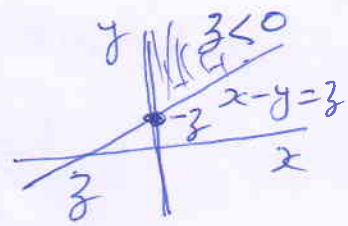
<u>Result(2)</u> Let (X, Y) be jointly distributed RV where $X > 0$ and $Y > 0$. Consider $Z = X - Y$ and then find the pdf of Z.

$\Rightarrow$ For $X > 0$, $Y > 0$, note that $Z = X - Y$ is a real number. ( )

Then CDF of Z is



$$F_Z(z) = \begin{cases} \int_{y=0}^{\infty} \int_{x=0}^{y+z} f_{X,Y}(x, y)\, dx\, dy, & z \geq 0 \\ \int_{y=-z}^{\infty} \int_{x=0}^{y+z} f_{X,Y}(x, y)\, dx\, dy, & z < 0 \end{cases}$$



$\therefore$ Now pdf of Z is given by

$$f_Z(z) = \begin{cases} \int_0^{\infty} f_{X,Y}(z+y, y)\, dy, & z \geq 0 \\ \int_{-z}^{\infty} f_{X,Y}(z+y, y)\, dy, & z < 0. \end{cases}$$

(*) If X & Y be independent then

$$f_Z(z) = \begin{cases} \int_0^{\infty} f_X(z+y)\, f_Y(y)\, dy, & z \geq 0 \\ \int_{-z}^{\infty} f_X(z+y)\, f_Y(y)\, dy, & z < 0. \end{cases}$$

Let us discuss some examples related to the previous results.

Ex: Let (X, Y) be jointly distributed rv such that X, Y iid $\exp(1)$. Consider $Z = X - Y$ find the pdf of Z.

$\Rightarrow$ Sol^n ① Given that X, Y iid $\exp(1)$ That is we have

$$f_X(x) = e^{-x}, \quad 0 < x < \infty$$
$$f_Y(y) = e^{-y}, \quad 0 < y < \infty.$$

Now using Equation ④ we have

$$f_Z(z) = \begin{cases} \int_0^\infty f_X(z+y) f_Y(y)\, dy, & z \geq 0 \\ \int_{-z}^\infty f_X(z+y) f_Y(y)\, dy, & z < 0 \end{cases}$$

$$= \begin{cases} \int_0^\infty e^{-(z+y)} \cdot e^{-y}\, dy, & z \geq 0 \\ \int_{-z}^\infty e^{-(z+y)} e^{-y}\, dy, & z \leq 0. \end{cases}$$

$$= \begin{cases} e^{-z} \int_0^\infty e^{-2y}\, dy, & z \geq 0 \\ e^{-z} \int_{-z}^\infty e^{-2y}\, dy, & z < 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2} e^{-z}, & z \geq 0 \\ \frac{1}{2} e^{z}, & z < 0 \end{cases} = \frac{1}{2} e^{-|z|}, \quad -\infty < z < \infty.$$

Solution ②: Let us solve this problem using Equation ② as well. From equation ② we have

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z+y)\, f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} e^{-(z+y)} I(0<z+y<\infty)\, e^{-y} I(0<y<\infty)\, dy$$

$$= e^{-z}\int_{-\infty}^{\infty} e^{-2y} I(-z<y<\infty)\, I(0<y<\infty)\, dy$$

$$= e^{-z}\int_{\max(-z,0)}^{\infty} e^{-2y}\, dy = \begin{cases} e^{-z}\int_0^{\infty} e^{-2y}\, dy, & z\geq 0 \\ e^{-z}\int_{-z}^{\infty} e^{-2y}\, dy, & z<0 \end{cases}$$

$$= \begin{cases} \frac{1}{2}e^{-z}, & z\geq 0 \\ \frac{1}{2}e^{z}, & z<0 \end{cases} = \frac{1}{2}e^{-|z|} \quad -\infty<z<\infty$$

this is standard double exponential (Laplace distn).

Ex: Let $X, Y \overset{iid}{\sim} U(0,1)$, $Z = X-Y$. Find pdf of Z.

Soln: Using Equation ④ we have

$$f_Z(z) = \begin{cases} \int_0^{\infty} f_X(z+y)\, f_Y(y)\, dy, & z\geq 0 \\ \int_{-z}^{\infty} f_X(z+y)\, f_Y(y)\, dy, & z<0. \end{cases}$$

Note that $X\sim U(0,1) \Rightarrow f_X(x) = 1,\ 0<x<1$

$Y\sim U(0,1) \Rightarrow f_Y(y) = 1,\ 0<y<1$

$$\therefore f_Z(z) = \begin{cases} \int_0^\infty I(0<z+y<1)\, I(0<y<1)\, dy, & z \geq 0 \\ \int_{-z}^\infty I(0<z+y<1)\, I(0<y<1)\, dy, & z<0 \end{cases}$$

$$= \begin{cases} \int_0^\infty I(-z< y<1-z)\, I(0<y<1)\, dy, & z \geq 0 \\ \int_{-z}^\infty I(-z<y<1-z)\, I(0<y<1)\, dy, & z<0 \end{cases}$$

$$= \begin{cases} \int_{\max(-z,0)}^{\min(1,1-z)} dy, & z \geq 0 \\ \int_{-z}^{\min(1,1-z)} dy, & z<0 \end{cases}$$

$$= \begin{cases} \int_0^{1-z} dy, & z \geq 0 \\ \int_{-z}^{1} dy, & z<0 \end{cases} = \begin{cases} 1-z, & 0<z<1 \\ 1+z, & -1<z<0 . \end{cases}$$