Poisson Process

This process arises in situations where we are interested in the total no. of arrivals of a particular type of events upto a specified time t, $t \geq 0$ such as no. of telephone calls to a call center, no of customers arriving to a shop, no. of defective products in a lot, printing errors and so on.

In many such fields of practical studies Poisson process has found wide applications. To analyze such process in a meaningful manner we need to put certain assumptions on ~~phen~~ phenomean under investigation.

Let $X(t)$ denote no. of customers in the system in a time interval of length t.

We are interested to evaluate

$$P(X(t) = n) = P_n(t), \quad n = 0, 1, 2, 3, \ldots \qquad (1)$$

under the following assumptions.

Assumption for Poisson Process:

(i) The number of arrivals of customers during disjoint time intervals are independent. That is no. of occurrence ~~of~~ during the time interval $(t, t+h]$ is independent of the no. of occurrences in $(0, t]$ for all $h > 0$.

(ii) Probability of exactly one occurrence during a small time interval is proportional to the length of the interval. That is probability of exactly one occurrence in $(t, t+h]$ is λh where λ is ~~proportionality~~ proportionality constant. In fact λ is the rate at which customer arrives. $\left[\therefore P(X(h)=1) = P_1(h) = \lambda h\right]$

(iii) Probability of more than one occurrence during a small time intervals is negligible. Thus we have

$$P(X(h) > 1) = P(X(h) \geq 2) = o(h)$$
$$\Rightarrow P_2(h) + P_3(h) + \cdots = o(h)$$

$f(x)$ is order $o(x)$ if $\frac{f(x)}{x} \to 0$ as $x \to 0$

Note that $P_0(h) + P_1(h) + P_2(h) + P_3(h) + \cdots = 1$

$$\therefore 1 - P_0(h) - P_1(h) = o(h)$$
$$\Rightarrow P_0(h) = 1 - P_1(h) - o(h) = 1 - \lambda h - o(h).$$

Under assumptions (i), (ii) and (iii) we find that we have $X(t)$ distributed as Poisson distn with parameter λt, that is,

$$P(X(t)=n) = P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n=0,1,2,3,\dots$$

Proof: This result is proved using the mathematical induction. Let us look at the probability $P_0(t+h)$,

$P_0(t+h) = P($no customer arrived during the time interval $(0, t+h])$

$= P($no customer arrive during $(0,t] \cap$ no customer arrive during $(t, t+h])$

$= P($no customer arrive in $(0,t]) \cdot P($no customer arrive during the time $(t, t+h])$

$$= P_0(t)\, P_0(h) = P_0(t)\left(1 - P_1(h) - o(h)\right)$$

$$= P_0(t)\left(1 - \lambda h - o(h)\right)$$

$$\frac{P_0(t+h) - P_0(t)}{h} = -\lambda P_0(t) - \frac{o(h)}{h} P_0(t)$$

as $h \to 0$ we have $\dfrac{dP_0(t)}{dt} = -\lambda P_0(t)$

Solving this we get

$$P_0(t) = c\, e^{-\lambda t}$$

Find the constant c: note that $P_0(0) = 1$

$$\Rightarrow c = 1$$

$$\therefore \boxed{P_0(t) = e^{-\lambda t}}$$

Equation (1) holds for $n = 0$.

Next consider

$$P_1(t+h) = P(\text{single customer arrives in time interval } (0, t+h])$$

$$= P(\{\text{one customer arrives in } (0,t]\} \cap \{\text{no customers arrive in } (t, t+h]\}) + P(\{\text{no customer arrives in } (0,t]\} \cap \{\text{one customers arrives in } (t, t+h]\})$$

$$= P(\text{one customer in } (0,t])\, P(\text{no customer in } (t, t+h]) + P(\text{no customer in } (0,t])\, P(\text{one customer in } (t, t+h])$$

$$= P_1(t)\, P_0(h) + P_0(t)\, P_1(h)$$

$$= P_1(t)(1 - \lambda h - o(h)) + \lambda h e^{-\lambda t}$$

$$\frac{P_1(t+h) - P_1(t)}{h} = -\lambda P_1(t) - P_1(t)\frac{o(h)}{h} + \lambda e^{-\lambda t}$$

As $h \to 0$ we get that

$$\frac{dP_1(t)}{dt} = -\lambda P_1(t) + \lambda e^{-\lambda t}$$

$$\Rightarrow \quad \frac{dP_1(t)}{dt} + \lambda P_1(t) = \lambda e^{-\lambda t}$$

Solving we get $P_1(t) e^{\lambda t} = \lambda t + c$

To find the constant c we use the condition $P_1(0) = 0$. This implies that $c = 0$.

$$\therefore \boxed{P_1(t) = \lambda t e^{-\lambda t}}$$

Thus result stated in Eqn (1) holds for $n = 1$ as well.

Let us assume that this result is true up to $n = k$.

Then we look ~~the~~ at following probability:

$P_{k+1}(t+h) = P((k+1)$ customers arrive in $(0, t+h])$

$= P[\{(k+1)$ customers arrive in $(0,t]\} \cap \{$no customers arrives in $(t, t+h]\}] + P[\{k$ Customer in $(0,t]\}$

$\cap \{$one customer in $(t, t+h]\}]$

$+ \sum_{i=1}^{k} P[\{(k-i)$ customer in $(0,t]\} \cap \{(i+1)$ customer in $(t, t+h]\}]$

$$P_{k+1}(t+h) = P_{k+1}(t)\, P_0(h) + P_k(t)\, P_1(h) + \sum_{i=1}^{k} P_{k-i}(t)\, P_{i+1}(h)$$

$$= P_{k+1}(t)\left(1-\lambda h - o(h)\right) + P_k(t)\,\lambda h + \sum_{i=1}^{k} P_{k-i}(t)\, o(h)$$

$$= P_{k+1}(t)\left(1-\lambda h - o(h)\right) + \frac{e^{-\lambda t}(\lambda t)^k}{k!}\lambda h + \left(\sum_{i=1}^{k} P_{k-i}(t)\right)\left(o(h)\right),$$

$$\frac{P_{k+1}(t+h) - P_{k+1}(t)}{h} = -\lambda P_{k+1}(t) - \frac{o(h)}{h} P_{k+1}(t) + \frac{e^{-\lambda t}(\lambda t)^k}{k!}\lambda + \left(\sum_{i=1}^{k} P_{k-i}(t)\right)\frac{o(h)}{h}.$$

as $h \to 0$ we have

$$\frac{dP_{k+1}(t)}{dt} = -\lambda P_{k+1}(t) + \frac{e^{-\lambda t}(\lambda t)^k}{k!}\lambda.$$

$$\Rightarrow \quad \frac{dP_{k+1}(t)}{dt} + \lambda P_{k+1}(t) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}\lambda.$$

we have to solve this ODE. note that IF is $e^{\lambda t}$.

Solving it we get

$$P_{K+1}(t)\, e^{\lambda t} = \int e^{-\lambda t} \frac{(\lambda t)^K}{K!} \lambda \cdot e^{\lambda t}\, dt$$

$$= \frac{(\lambda t)^{K+1}}{(K+1)!} + c$$

To find the constant c, we note that

$P_{K+1}(0) = 0$ which implies that $c = 0$.

$$\therefore\ P_{K+1}(t) = \frac{e^{-\lambda t} (\lambda t)^{K+1}}{(K+1)!}$$

So Equation (1) holds for $n = K+1$ also.

Thus it holds for all $n = 0, 1, 2, 3, \ldots$.

Ex: Let average no. of telephone calls arriving at a call center is 30 calls per hour.

(i) what is the probability that no call arrives in a 3 minute period

(ii) what is the probability that more than 5 calls arrive in a 5 minute period.

Solution:

$\lambda = 30$ call per hour
$= \frac{1}{2}$ call per minute.

~~(i) $P(X(3)=0)$~~

$X(t)$: no of calls during time interval of length t.

$$P(X(t)=n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n=0,1,2,3,\ldots$$
$$\lambda = 1/2$$

(i) $P(X(3)=0) = \frac{e^{-\frac{1}{2}\cdot 3}\left(\frac{1}{2}\cdot 3\right)^0}{0!} = e^{-3/2}$.

(ii) $P(X(5)>5) = 1-P(X(5)\le 5)$

$$= 1-\sum_{j=0}^{5} \frac{e^{-5/2}(5/2)^j}{j!}$$

$\simeq 0.42$.