## $$\operatorname{MA}$ 101 Real Analysis Limit and continuity for function of several variables

1. Find the following limits, if they exist:

$$(i) \lim_{(x,y)\to(-1,2)} \frac{xy^3}{x+y} \qquad (ii) \lim_{(x,y)\to(0,0)} \frac{x-y}{x^2+y^2} \qquad (iii) \lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}$$
 
$$(iv) \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} \qquad (v) \lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^4+z^4} \qquad (vi) \lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^3+y^3+z^3}$$
 
$$(vii) \lim_{t\to 0} F(t) \text{ where } F(t) = (2\cos t, \frac{\sin t}{t}, t^2)$$

- 2. Consider the function  $f(x,y) = \frac{x+y}{x-y}$  for  $(x,y) \in \mathbb{R}^2$  with  $x+y \neq 0$ . What can you say about the existence of  $\lim_{(x,y)\to(0,0)} f(x,y)$ .
- 3. Let  $f(x,y) = \frac{x^2y^2}{x^2y^2 + (x-y)^2}$  whenever  $x^2y^2 + (x-y)^2 \neq 0$ . Compute the limit  $\lim_{(x,y)\to(0,0)} f(x,y)$ , if it exists.
- 4. Show that  $\lim_{(x,y)\to(0,0)} \frac{xy(x^2-y^2)}{x^2+y^2} = 0$ .
- 5. Show that  $\lim_{(x,y)\to(0,0)} \frac{\sqrt{(x^2y^2+1)}-1}{x^2+y^2} = 0$ .
- 6. Show that  $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$  does not exist.
- 7. Show that  $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x-y}$  does not exist.
- 8. Suppose

$$f(x,y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2}, & \text{for } (x,y) \neq (0,0) \\ 0, & \text{for } (x,y) = (0,0) \end{cases}$$

Is f continuous at (0,0)?

9. Suppose

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}}, & \text{for } (x,y) \neq (0,0) \\ \\ 0, & \text{for } (x,y) = (0,0) \end{cases}$$

Is f continuous at (0,0)?

10. Suppose

$$f(x,y) = \begin{cases} \frac{xy}{x^2 + y^2}, & \text{for } (x,y) \neq (0,0) \\ 0, & \text{for } (x,y) = (0,0) \end{cases}$$

Show that  $\lim_{(x,y)\to(0,0)} f(x,y)$  does not exist.

## 11. Suppose

$$f(x,y) = \begin{cases} x\sin\frac{1}{y} + y\sin\frac{1}{x}, & \text{for } (x,y) \neq (0,0) \\ 0, & \text{for } (x,y) = (0,0) \end{cases}$$

Then show that  $\lim_{(x,y)\to(0,0)} f(x,y)$  exist.

- 12. Examine the following functions for continuity at (0,0). The expression below give the value at  $(x,y) \neq (0,0)$ . At (0,0), the function value should be taken as zero.
  - $(i) \frac{x^2}{x^2+y^2}$
  - (ii)  $x^p y^q \frac{x^2 y^2}{x^2 + y^2}$ , p > 0, q > 0
  - $(iii) \ xy \log (x^2 + y^2)$
  - (iv) |x| + |y|
- 13. Explain why the function  $F:[0,\infty]\to R^2$  defined by

$$F(t) = (t+1, 2t-3), \text{ if } 0 \le t < 1$$

$$= (t^2 + 1, t) \text{ if } 1 \le t < 2$$

$$= (7 - t, |t-4|) \text{ if } 2 \le t$$

is discontinuous. At which points, it is discontinuous?