MA101

Limits and Continuity in Higher Dimensions

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Level Curve: Problem

Example

Find an equation for the level curve of the function

$$\mathbf{f(x,y)} = \int_{\mathbf{x}}^{\mathbf{y}} \frac{\mathbf{t\,dt}}{\mathbf{1+t^2}}, \quad \text{at the point} \quad (\mathbf{0},\mathbf{0}).$$

Answer:

$$\mathbf{y} = \pm\, \mathbf{x}$$

Definition of Limit: Two variables

"It is similar to the definition of the limit of a function of a single variable but with a crucial difference."

If the values of $\mathbf{f(x, y)}$ lie arbitrarily close to a fixed real number $\mathbf{L}$ for all points $(\mathbf{x}, \mathbf{y})$ sufficiently close to a point $(\mathbf{x_0}, \mathbf{y_0})$ we say that $\mathbf{f(x, y)}$ approaches the limit $\mathbf{L}$ as $(\mathbf{x}, \mathbf{y})$ approaches $(\mathbf{x_0}, \mathbf{y_0})$.

Definition

Let $\mathbf{f(x, y)}$ be a function with domain D. Then $\mathbf{f(x, y)}$ approaches to the limit $\mathbf{L}$ as $(\mathbf{x}, \mathbf{y})$ approaches $(\mathbf{x_0}, \mathbf{y_0})$ and is written $\lim\limits_{(\mathbf{x},\mathbf{y})\to(\mathbf{x_0},\mathbf{y_0})} \mathbf{f(x, y)} = \mathbf{L}$ if, for every number $\epsilon > \mathbf{0}$ there exists a corresponding number $\delta > \mathbf{0}$ such that for all $(\mathbf{x}, \mathbf{y})$ in the domain of $\mathbf{f}$,

$$|\mathbf{f(x, y)} - \mathbf{L}| < \epsilon \quad \text{whenever} \quad \mathbf{0} < \sqrt{(\mathbf{x} - \mathbf{x_0})^2 + (\mathbf{y} - \mathbf{y_0})^2} < \delta.$$

An alternative definition

Definition

Let $\mathbf{f}(\mathbf{x}, \mathbf{y})$ be a function with domain D. Then $\mathbf{f}(\mathbf{x}, \mathbf{y})$ approaches to the limit $\mathbf{L}$ as $(\mathbf{x}, \mathbf{y})$ approaches $(\mathbf{x_0}, \mathbf{y_0})$ and write $\lim\limits_{(\mathbf{x},\mathbf{y}) \to (\mathbf{x_0},\mathbf{y_0})} \mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{L}$ if, for every number $\epsilon > \mathbf{0}$ there exists a corresponding number $\delta > \mathbf{0}$ such that for all $(\mathbf{x}, \mathbf{y})$ in the domain of $\mathbf{f}$,

$$|\mathbf{f}(\mathbf{x}, \mathbf{y}) - \mathbf{L}| < \epsilon \quad \text{whenever} \quad \mathbf{0} < \sqrt{(\mathbf{x} - \mathbf{x_0})^2 + (\mathbf{y} - \mathbf{y_0})^2} < \delta.$$

or

$$\mathbf{0} < |\mathbf{x} - \mathbf{x_0}| < \delta, \quad \mathbf{0} < |\mathbf{y} - \mathbf{y_0}| < \delta \implies |\mathbf{f}(\mathbf{x}, \mathbf{y}) - \mathbf{L}| < \epsilon.$$

Properties of Limits of Functions of Two Variables

The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L, \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$$

1. Sum Rule: $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$
2. Difference Rule: $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$
3. Multiplication Rule: $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) \cdot g(x,y)) = L \cdot M$
4. Constant Multiple Rule: $\lim_{(x,y)\to(x_0,y_0)} (k\, f(x,y)) = kL$ (Any number k)
5. Quotient Rule: $\lim_{(x,y)\to(x_0,y_0)} \left(\frac{f(x,y)}{g(x,y)}\right) = \frac{L}{M}, M \neq 0$
6. Power Rule: If r and s are integers with no common factors, and $s \neq 0$ then
$$\lim_{(x,y)\to(x_0,y_0)} (f(x,y))^{r/s} = L^{r/s}.$$
provided $L^{r/s}$ is a real number.

Calculating Limits

Find out the following limits:

(i) $\lim_{(x,y)\to(-3,4)} f(x,y) = \sqrt{x^2+y^2}$.

(ii) $\lim_{(x,y)\to(2,2)} f(x,y) = \dfrac{x^2-xy}{\sqrt{x}-\sqrt{y}}$.

(iii) $\lim_{(x,y)\to(0,0)} f(x,y) = \dfrac{4xy^2}{x^2+y^2}$.

Definition

A $f(x, y)$ is said to be **continuous** at (x_0, y_0)

(i) $f(x, y)$ exists at (x_0, y_0),

(ii) $\lim\limits_{(x,y)\to (x_0,y_0)} f(x, y)$ exists,

(iii) $\lim\limits_{(x,y)\to (x_0,y_0)} f(x, y) = f(x_0, y_0)$.

Remark

- ***Sums, differences, products, constant multiples, quotients, and powers of continuous functions are continuous where defined.***
- ***Polynomials are continuous at every point at which they are defined.***
- ***Rational functions of two variables are continuous at all points where denominator is non-zero.***

Problem

Problem

If $\mathbf{f}(\mathbf{x_0}, \mathbf{y_0}) = \mathbf{3}$ *what can you say about*

$$\lim_{(\mathbf{x},\mathbf{y}) \to (\mathbf{x_0},\mathbf{y_0})} \mathbf{f}(\mathbf{x}, \mathbf{y}) \ \ ?$$

- *If* $\mathbf{f}(\mathbf{x}, \mathbf{y})$ *is continuous at* $(\mathbf{x_0}, \mathbf{y_0})$.
- *If* $\mathbf{f}(\mathbf{x}, \mathbf{y})$ *is not continuous at* $(\mathbf{x_0}, \mathbf{y_0})$.

Remark

- If
$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L,$$
i.e., limit as $(x,y) \to (x_0,y_0)$ exists. Then along any path in the domain $f(x,y)$ limit of $f(x,y)$ as $(x,y) \to (x_0,y_0)$ must exist and equal to L. (Discuss)
- **Two Path Test:** If $f(x,y)$ has different limits along two different paths in the domain approaching (x_0,y_0), then
$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) \text{ DOES NOT EXIST.}$$
- Two path test can only be used to prove **NON EXISTENCE OF THE LIMIT**.

Problem

1. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 - y^2}{x^4 + y^2},$
2. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^y \sin x}{x},$
3. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{|xy|},$
4. $\displaystyle\lim_{(x,y)\to(2,2)} \frac{x + y - 4}{\sqrt{x+y} - 2},$

Theorem

Let

$$\mathbf{g}(\mathbf{x},\mathbf{y}) \leq \mathbf{f}(\mathbf{x},\mathbf{y}) \leq \mathbf{h}(\mathbf{x},\mathbf{y})$$

for all $(\mathbf{x},\mathbf{y}) \neq (\mathbf{x_0},\mathbf{y_0})$ *in an open disk centered at* $(\mathbf{x_0},\mathbf{y_0})$ *and which completely lie in domain of* $\mathbf{f}(\mathbf{x},\mathbf{y})$. *If*

$$\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{x_0},\mathbf{y_0})} \mathbf{g}(\mathbf{x},\mathbf{y}) = \lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{x_0},\mathbf{y_0})} \mathbf{h}(\mathbf{x},\mathbf{y}) = \mathbf{L},$$

where $\mathbf{L}$ *is a real number then*

$$\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{x_0},\mathbf{y_0})} \mathbf{f}(\mathbf{x},\mathbf{y}) = \mathbf{L}.$$

Problem

Find the limit if it exists

1. $$\lim_{(x,y)\to(0,0)} \left(\frac{3x^2y}{x^2+y^2}\right).$$
2. $$\lim_{(x,y)\to(0,0)} \left(\frac{x^2-y^2}{x^2+y^2}\right).$$
3. $$\lim_{(x,y)\to(0,0)} \left(\frac{xy}{x^2+y^2}\right).$$
4. $$\lim_{(x,y)\to(0,0)} \left(\frac{xy^2}{x^2+y^4}\right).$$

Problem

Discuss the region where where the following function is continuous

$$\mathbf{f(x,y)} = \frac{\mathbf{x^2 - y^2}}{\mathbf{x^2 + y^2}}?$$

Solution: **Hint: Use the fact that it is a Rational function.**
Domain where the function is continuous is

$$\{(\mathbf{x},\mathbf{y}) : (\mathbf{x},\mathbf{y}) \neq (\mathbf{0},\mathbf{0})\}$$

Problem

$$\mathbf{f(x,y)} = \begin{cases} \frac{\mathbf{x^2-y^2}}{\mathbf{x^2+y^2}}, & ;(\mathbf{x},\mathbf{y}) \neq (\mathbf{0},\mathbf{0}) \\ \mathbf{0}, & ;(\mathbf{x},\mathbf{y}) = (\mathbf{0},\mathbf{0}) \end{cases}$$

Is $\mathbf{f(x,y)}$ *continuous at* $(\mathbf{0},\mathbf{0})$*?*

Answer: NO Why?

Problem

$$\mathbf{f}(\mathbf{x},\mathbf{y}) = \begin{cases} \frac{3\mathbf{x}^2\mathbf{y}}{\mathbf{x}^2+\mathbf{y}^2}, & ;(\mathbf{x},\mathbf{y}) \neq (\mathbf{0},\mathbf{0}) \\ \mathbf{0}, & ;(\mathbf{x},\mathbf{y}) = (\mathbf{0},\mathbf{0}) \end{cases} \tag{1}$$

Is $\mathbf{f}(\mathbf{x},\mathbf{y})$ *continuous* $(\mathbf{0},\mathbf{0})$*?*

Solution:

- $\mathbf{f}(\mathbf{x},\mathbf{y})$ is continuous at $(\mathbf{0},\mathbf{0})$ in fact,
- $\mathbf{f}(\mathbf{x},\mathbf{y})$ is continuous throughout $\mathbb{R} \times \mathbb{R}$.

Problem

Find the limit if it exists

1. $$\lim_{(x,y)\to(0,0)} \left(x \sin\frac{1}{y} + y \sin\frac{1}{x} \right).$$

   Answer: Exists and is equal to zero.

2. $$\lim_{(x,y)\to(0,0)} \left(x \sin\frac{1}{y} \right).$$

   Answer: Exists and is equal to zero.

3. $$\lim_{(x,y)\to(0,0)} \left(y \sin\frac{1}{x} \right).$$

   Answer: Exists and is equal to zero.

Some specific paths for specific functions

Problem

1. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{y-x}$; *Try Path:* $y = x - mx^3, \quad m \neq 0.$
2. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4+y^4}{y-x}$; *Try Path:* $y = x - mx^4, \quad m \neq 0.$

- **If you cannot make any headway $\lim_{(x,y)\to(0,0)} f(x,y)$ with in rectangular coordinates, try changing to polar coordinates.**
- **Substitute $x = r\cos\theta$ and $y = r\sin\theta$ and investigate the limit of the resulting expression as $r \to 0$.**
- **To decide whether there exists a number L we use $\epsilon - \delta$ defintion:**
- Definition of Limit in Polar system using $\epsilon - \delta$ concept:

Definition

Given ϵ there exist a δ such that for all r and θ,

$$0 < |r| < \delta \quad \Rightarrow \quad |f(r,\theta) - L| < \epsilon.$$

If such an L exists then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} f(r,\theta) = L$$

- Let
$$\mathbf{f}(\mathbf{x},\mathbf{y}) : \mathbf{D} \to \mathbb{R} \quad \& \quad \mathbf{g}(\mathbf{z}) : \mathbb{R} \to \mathbb{R}$$
for subset $\mathbf{D}$ of plane. Let
$$\lim_{(\mathbf{x},\mathbf{y})\to (\mathbf{x}_0,\mathbf{y}_0)} \mathbf{f}(\mathbf{x},\mathbf{y}) = L$$
and $\mathbf{g}(\mathbf{z})$ is continuous at $\mathbf{z} = \mathbf{L}$. If $\mathbf{h} = \mathbf{g} \text{ o } \mathbf{f}$ defined by $\mathbf{h}(\mathbf{x},\mathbf{y}) = \mathbf{g}(\mathbf{f}(\mathbf{x},\mathbf{y}))$ is composite function from $D \to \mathbb{R}$ then
$$\lim_{(\mathbf{x},\mathbf{y})\to (\mathbf{x}_0,\mathbf{y}_0)} \mathbf{h}(\mathbf{x},\mathbf{y})$$
exists and is equal to
$$\lim_{(\mathbf{x},\mathbf{y})\to (\mathbf{x}_0,\mathbf{y}_0)} \mathbf{h}(\mathbf{x},\mathbf{y}) = \mathbf{g}(\mathbf{L}).$$

- If $\mathbf{f}(\mathbf{x},\mathbf{y})$ is continuous at $(\mathbf{x}_0,\mathbf{y}_0)$ and $\mathbf{g}$ is a single-variable function continuous at $\mathbf{f}(\mathbf{x}_0,\mathbf{y}_0)$ then the composite function $\mathbf{h} = \mathbf{g} \text{ o } \mathbf{f}$ defined by $\mathbf{h}(\mathbf{x},\mathbf{y}) = \mathbf{g}(\mathbf{f}(\mathbf{x},\mathbf{y}))$ is continuous at $(\mathbf{x}_0,\mathbf{y}_0)$.

Problem

Find

$$\lim_{(x,y)\to(0,0)} \cos\left(\frac{x^3 - y^3}{x^2 + y^2}\right).$$

Problem

Use Mathematica and plot the following Graphs near $(0, 0)$ and try to visualize the limit.

$$\lim_{(x,y)\to(0,0)} -\frac{x}{\sqrt{x^2+y^2}}.$$

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2}.$$

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}.$$

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}.$$