Real Analysis (MA101), Tutorial Sheet 3 (Limit and Continuity)

1. Find the value of α such that
$$\lim_{x\to -1} \frac{2x^2 - \alpha x - 14}{x^2 - 2x - 3}$$
exists. Find the limit.

2. Let $\lim_{x\to 0} \frac{f(x)}{x^2} = 5$. Show that $\lim_{x\to 0} \frac{f(x)}{x} = 0$.

3. Let $f(x) = x$ if $x \in Q$ and $f(x) = 0$ if $x \in R \setminus Q$. Show that f is continuous at $x_o = 0$. Also show that it is discontinuous at any other point.

4. Let $f(x) = 1$ if $x \in Q$ and $f(x) = -1$ if $x \in R \setminus Q$. Show that f is discontinuous at every point.

5. Give an example of a bounded function on $[-1, 1]$ which does not have a maximum or a minimum.

6. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x + y) = f(x) + f(y)$ for all $x, y \in R$. If f is continuous at 0 then show that f is continuous at every point $c \in R$.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be such that for every $x, y \in R$, $|f(x) - f(y)| \le |x - y|$. Show that f is continuous for all $x \in R$.

8. Use properties of limit to evaluate $\displaystyle\lim_{x\to 0} \left(\frac{\sin x}{\sqrt{1 - \cos x}}\right)$.

9. Use the definition to establish the continuity of the following functions: (i) $f(x) = x^2$ at $x = 3$, $x \in [0, 7]$ (ii) $f(x) = \frac{1}{x}$ at $x = 1/2$, $x \in [0, 1]$ (iii)$f(x) = \sqrt{x}, x \ge 0$

10. Let $f(x) = \frac{x^2+x-6}{x-2}, x \ne 2$. Define $f(x)$ in a way such that it becomes continuous at $x = 2$.

11. (a) Show that the functions $x^2, \frac{1}{x}, \frac{1}{x^2}, x > 0$ are continuous at any point $c \in R$ but not uniformly.
(b) Show that the function x^2, $x \in [-a, a], a > 0$ and functions $\frac{1}{x}, \frac{1}{x^2}, x \ge b > 0$ are uniformly continuous on respective domain.
(c) Show that $\sin x$, $\cos x$, $|x|$ are continuous at every point $c \in \mathbb{R}$

12. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Show that $\exists\ x_0 \in [0, 1]$ such that
$f(x_0) = \frac{1}{3}(f(\frac{1}{4}) + f(\frac{1}{2}) + f(\frac{3}{4}))$.

13. Let $p(y)$ be a polynomial
$p(y) = a_n y^n + a_{n-1} y^{n-1} + ... + a_1 y + a_0$.
Suppose n is even $(n \ne 0)$, $a_n = 1, a_0 = -1$. Show that $p(y)$ has at least two real roots.

14. Compute the limit $\displaystyle\lim_{x\to\infty} \left(x^2 - x^3 \sin\left(\frac{1}{x}\right)\right)$.

15. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that $f(a) \ne g(a)$ for some $a \in \mathbb{R}$. Show that $\exists$ a $\delta > 0$ such that $f(x) \ne g(x)$, $\forall\ x$ such that $|x - a| < \delta$.

16. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with $f(0) = -2, f(1) = 3$. Let $S = \{x \in [0, 1] | f(x) = 0\}$
(a) Show that S is non empty.
(b) Let β be the supremum of the set S. Show that $\beta \in (0, 1]$.
(c) Show that $f(\beta) = 0$

17. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at $c \in \mathbb{R}$. Then $|f|$ is continuous at c. Give an example to show that the reverse is not true.

18. A real function f is continuous on $[0, 2]$ and $f(0) = f(2)$. Prove that there exists at least a point c in $[0, 1]$ such that $f(c) = f(c + 1)$.

19. (i) Give an example of a function f which satisfies the initial value problem (IVP) on a closed and bounded interval $[a, b]$, but is not continuous on $[a, b]$.

    (ii) Give an example of a function f which is monotone increasing on a closed and bounded interval $[a, b]$ but does not satisfy the IVP on $[a, b]$.

20. Let $f : [0, \pi] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if} \quad x = 0, \\ x \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x} & \text{if} \quad x \neq 0. \end{cases}$$

    Is f continuous?

21. Let $f : \mathbb{R} \to (0, \infty)$, satisfy $f(x + y) = f(x)f(y)\ \forall\ x \in \mathbb{R}$. Suppose f is continuous at $x = 0$. Show that f is continuous at all $x\ \in \mathbb{R}$.