Indian Institute of Technology Patna

MA201 (Mathematics III)

B. Tech. II Year (Autumn Semester: 2018-19)

Mid Semester Examination- 2018

Maximum Marks: 30

Total Time: 2 Hours

Attempt all questions. Notations have their usual meaning.

1. Show that the function $u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$ satisfies Laplace's equation and find the corresponding analytic function $u + iv$ by using *Milne-Thomson* method. [3]

2. Show that the function $f(z) = e^{-z^{-4}}$, $(z \neq 0)$ and $f(0) = 0$ is not analytic at $z = 0$, but the *Cauchy – Riemann* equations are satisfied at that point. [3]

3. State and prove *Cauchy* Theorem and evaluate $\int_{(0,3)}^{(2,4)} (2y + x^2)dx + (3x - y)dy$ along the parabola $x = 2t$, $y = t^2 + 3$. [4]

4. State maximum modulus theorem and find the maximum modulus of the function z^2 over the region $\{z = x + iy : 2 \leq x \leq 3; 1 \leq y \leq 3\}$. [3]

5. State Laurent's theorem and expand the following functions in a Laurent series valid for specified region: [4]

   (i) $\frac{z}{(z-1)(z-3)}$, $0 < |z - 1| < 2$ (ii) $\frac{\cosh z - \cos z}{z^5}$, $0 < |z|$.

6. If the function $f(z)$ is analytic when $|z| < R$ and has the Taylor's expansion $\sum_{n=0}^{\infty} a_n z^n$, then show that for $r < R$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

   Hence, prove that if $|f(z)| \leq M$ when $|z| < R$, then

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \leq M^2.$$ [4]

7. Find the residues of the following functions at the poles:

   (a) $f(z) = \frac{z^3}{(z-1)^4(z-2)(z-3)}$.

   (b) $f(z) = \frac{\cot \pi z}{(z-a)^2}$. [2+2= 4]

8. By using contour integration, prove that

$$\int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta} d\theta = \frac{2\pi}{b^2}\{a - \sqrt{a^2 - b^2}\}, \text{ where } a > b > 0.$$ [5]