PH 201

OPTICS & LASERS

Lecture_Polarization_2

Superposition of two Disturbances

- Consider propagation of two linearly polarized *em* waves (both propagating along *z* axis) with their electric vectors oscillating along *x* axis.

$$E_1 = \hat{x} a_1 \cos(kz - \omega t + \theta_1)$$

$$E_2 = \hat{x} a_2 \cos(kz - \omega t + \theta_2)$$

where a_1 & a_2 represent amplitudes of waves, $\hat{x}$ represents unit vector along *x* axis, & θ_1 & θ_2 are phase constants.

Resultant of these two waves:

$$E = E_1 + E_2$$

$$\Rightarrow \quad E = \hat{x} a \cos(kz - \omega t + \theta)$$

$$where \quad a = [a_1^2 + a_2^2 + 2a_1 a_2 \cos(\theta_1 - \theta_2)]^{1/2}$$

- Resultant is a linearly polarized wave with its electric vector oscillating along the same axis.

- Consider superposition of two linearly polarized em waves (both propagating along z axis) but with their electric vectors along two mutually perpendicular directions.

$$E_1 = \hat{x} a_1 \cos(kz - \omega t)$$
$$E_2 = \hat{y} a_2 \cos(kz - \omega t + \theta)$$

For $\theta = n\pi$, resultant will also be a linearly polarized wave with its electric vector oscillating along a direction making a certain angle with x axis; this angle will depend on relative values of a_1 & a_2.

- To find state of polarization of resultant field, we consider time variation of resultant electric field at an arbitrary plane perpendicular to z axis, which can be assumed to be z = 0.

- If E_x & E_y represent x & y component of resultant field $\boldsymbol{E}$ (= $\boldsymbol{E}_1$ + $\boldsymbol{E}_2$), then

$$E_x = a_1 \cos \omega t$$
$$E_y = a_2 \cos(\omega t - \theta)$$

For $\theta = n\pi$,

$$E_x = a_1 \cos \omega t$$

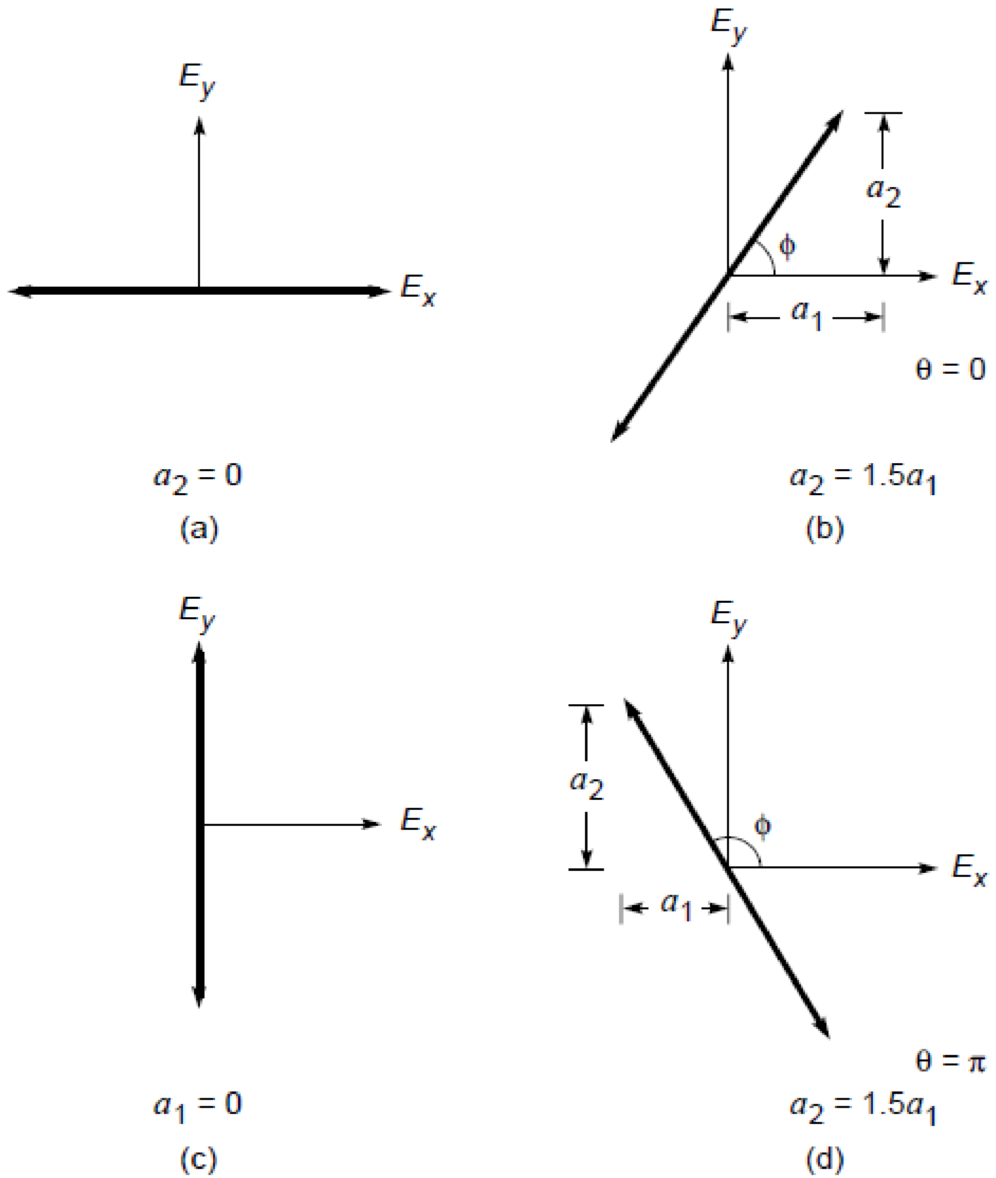
$$E_y = (-1)^n a_2 \cos \omega t$$

$$\Rightarrow \quad \frac{E_y}{E_x} = \pm \frac{a_2}{a_1} \qquad (\textit{independent of } t)$$

where upper & lower signs correspond to n even & n odd, respectively. In E_xE_y plane, this Eq. represents a **straight line**; angle Φ that this line makes with E_x axis depends on ratio a_2/a_1.

$$\phi = \tan^{-1}\left(\pm \frac{a_2}{a_1} \right)$$

Condition $\theta = n\pi$ implies that the two vibrations are either in phase (n = 0, 2, 4, ….) or out of phase (n = 1, 3, 5, ….).



Superposition of two linearly polarized waves with their electric fields oscillating in phase. Resultant is again a linearly polarized wave with its electric vector oscillating in a direction making an angle *Φ* with *x* axis.

- Superposition of two linearly polarized *em* waves with their electric fields at right angles to each other & oscillating in phase is again a linearly polarized wave with its electric vector, in general, oscillating in a direction which is different from the fields of either of the two waves.

For $\theta \neq n\pi$ (n = 0, 1, 2, ….), resultant electric vector does not oscillate along a straight line.

Ex. Consider $\theta = \pi/2$ with $a_1 = a_2$. Thus,

$$E_x = a_1 \cos \omega t$$

$$E_y = a_1 \sin \omega t$$

- If we plot time variation of resultant electric vector we find that **tip of electric vector rotates on the circumference of a circle** (of radius a_1) in **counterclockwise direction**, & propagation is in +z direction which is coming out of page. Such a wave is known as a **right circularly polarized wave**.

- Tip of resultant electric vector should lie on circumference of a circle.

$$E_x^2 + E_y^2 = a_1^2 \qquad (independent\ of\ t)$$

For $\theta = 3\pi/2$,

$$E_x = a_1 \cos \omega t$$

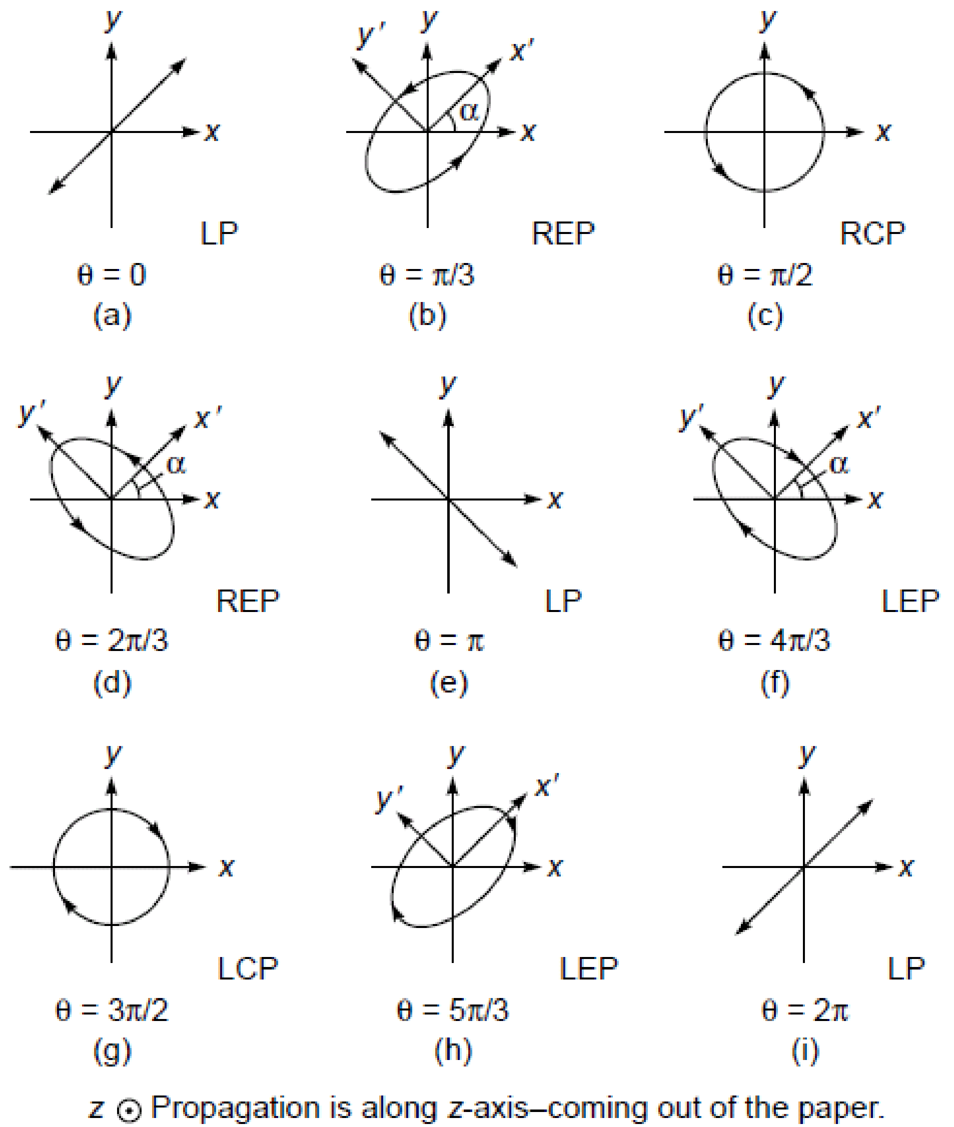
$$E_y = -a_1 \sin \omega t$$

which would also represent a circularly polarized wave; however, the electric vector will rotate in **clockwise direction**. Such a wave is known as a **left circularly polarized wave**.

- For $\theta \neq m\pi/2$ (m = 0, 1, 2, …), the tip of electric vector rotates on circumference of an ellipse.

- This ellipse will degenerate into a straight line or a circle when θ becomes an even or an odd multiple of $\pi/2$.

- When $a_1 \neq a_2$, one obtains an elliptically polarized wave which degenerates into a straight line for $\theta = 0, \pi, 2\pi$, … etc.



Propagation is out of the page.

States of polarization for various values of θ corresponding to $a_1 = a_2$. Ex. (c) & (g) correspond to right circularly & left circularly polarized light, respectively; similarly, (b) & (d) correspond to right elliptically polarized light, & (f) & (h) correspond to left elliptically polarized light.

- Different states of polarization are a characteristic of any transverse wave.

- Ex. If we move a stretched string up & down, we generate a linearly polarized wave with its displacement confined to vertical plane.

- Similarly, a linearly polarized wave with its displacement confined to horizontal plane can be generated.

- Further, we may rotate end of string on circumference of a circle (or an ellipse) to produce a circularly polarized (or an elliptically polarized) wave; similar to the case of an em wave, one may produce an elliptically polarized wave by allowing two linearly polarized waves to propagate through string.

- For such a wave, particles of string actually move on circumference of a circle (or an ellipse).

- For an elliptically polarized em wave, it is the electric field which changes its magnitude & direction at a particular point; the presence of these fields can be felt by their interaction with a charged particle.

- For a circularly polarized wave, magnitude of field remains same; direction changes with an angular frequency ω.

- For a linearly polarized wave, direction of field does not change; it is the magnitude which keeps on oscillating about the zero value with the angular frequency of the wave.

Mathematical Analysis

$$E_x = a_1 \cos \omega t$$

$$E_y = a_2 \cos(\omega t - \theta)$$

- Assume that major axis of ellipse is along x' or y' axes & that x' axis makes an angle *α* with *x* axis; i.e.,

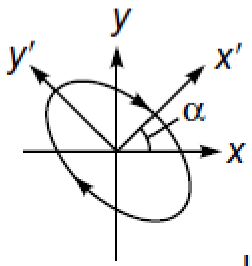
$$E'_x = E_1 \cos(\omega t - \phi)$$

$$\frac{E'_x}{E_1} = \cos(\omega t - \phi)$$

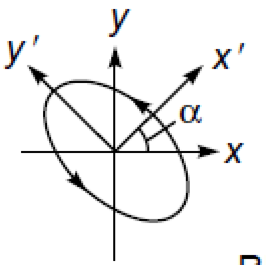
$$E'_y = E_2 \sin(\omega t - \phi)$$

$$\frac{E'_y}{E_2} = \sin(\omega t - \phi)$$

$$\Rightarrow \left(\frac{E'_x}{E_1}\right)^2 + \left(\frac{E'_y}{E_2}\right)^2 = 1$$



LEP



REP

which represents Eq. of an ellipse.

❖ For rotated coordinates,

$$E_x = E'_x \cos\alpha - E'_y \sin\alpha$$

$$E_y = E'_x \sin\alpha - E'_y \cos\alpha$$

If we multiply 1st Eq. by *cosα* & 2nd Eq. by *sinα* & add,

$$E'_x = E_x \cos\alpha + E_y \sin\alpha$$

Similarly,

$$E'_y = -E_x \sin\alpha + E_y \cos\alpha$$

Substituting above Eqs., we get

$$E_1 \cos(\omega t - \phi) = a_1 \cos\omega t \cos\alpha + a_2 \cos(\omega t - \theta) \sin\alpha$$

$$E_2 \sin(\omega t - \phi) = -a_1 \cos\omega t \sin\alpha + a_2 \cos(\omega t - \theta) \cos\alpha$$

These Eqs. have to be valid at all times; thus we equate coefficients of *cosωt* & *sinωt* on both sides of Eq.

$$E_1 \cos\phi = a_1 \cos\alpha + a_2 \cos\theta \sin\alpha$$
$$E_1 \sin\phi = a_2 \sin\theta \sin\alpha$$

$$\& \qquad -E_2 \sin\phi = -a_1 \sin\alpha + a_2 \cos\theta \cos\alpha$$
$$E_2 \cos\phi = a_2 \sin\theta \cos\alpha$$

If we square the four equations & add, we get

$$E_1^2 + E_2^2 = a_1^2 + a_2^2$$

Further,

$$\frac{E_2}{E_1} = \frac{a_2 \sin\theta \cos\alpha}{a_1 \cos\alpha + a_2 \cos\theta \sin\alpha} = \frac{a_1 \sin\alpha - a_2 \cos\theta \cos\alpha}{a_2 \sin\theta \sin\alpha}$$

$$\Rightarrow \qquad a_2^2 \sin^2\theta \sin\alpha \cos\alpha = a_1^2 \sin\alpha \cos\alpha - a_2^2 \cos^2\theta \sin\alpha \cos\alpha$$
$$- a_1 a_2 \cos\theta (\cos^2\alpha - \sin^2\alpha)$$

With simple manipulations,

$$\tan 2\alpha = \frac{2a_1 a_2 \cos\theta}{a_1^2 - a_2^2}$$

Examples. For

$$a_1 = a_2 \qquad 2\alpha = \frac{\pi}{2} \quad \Rightarrow \quad \alpha = \frac{\pi}{4}$$

implying that major (or minor) axis of ellipse makes 45° with *x* axis. Further,

$$\frac{E_2}{E_1} = \frac{\sin\theta}{1+\cos\theta} = \tan\frac{\theta}{2}$$

Thus, for $a_1 = a_2$ & for

$$\theta = \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}$$

$$\frac{E_2}{E_1} = +0.577, 1, 1.732, -1.732, -1, -0.577$$

which correspond to **REP, RCP, REP, LEP, LCP, & LEP**.

For $\theta = 4\pi/3$,

$$E'_x = E_1 \cos(\omega t - \phi)$$

$$E'_y = -1.732 E_1 \sin(\omega t - \phi)$$

Thus major axis of ellipse is along y' axis. To determine the state of polarization, we may choose $t = 0$ at the instant so that Φ may be assumed to be zero:

$$E'_x = E_1 \cos \omega t$$

$$E'_y = -1.732 E_1 \sin \omega t$$

Thus at

$$t = 0 \qquad E'_x = E_1 \qquad E'_y = 0$$

$$t = \frac{\pi}{2\omega} \qquad E'_x = 0 \qquad E'_y = -1.732 E_1$$

$$t = \frac{\pi}{\omega} \qquad E'_x = -E_1 \qquad E'_y = 0$$

etc., & the electric vector will rotate in clockwise direction.