# Indian Institute of Technology Patna

## MA-225: B.Tech. II year

## Spring Semester: 2011-12

## End Semester Examination

Maximum Marks: 50 Total Time: 3 Hours

**Note:** This question paper has **TWO** pages and contains **Twelve** questions. Answer all questions.

1. (*i*) For an $Exp(\mu, \sigma)$ distribution evaluate its $p^{th}, 0 < p < 1$ quantile. [1]
   (*ii*) For a $LN(\mu, \sigma^2)$ distribution evaluate its mean, variance and median. [1+1+1]

2. Suppose that time between breakdowns of a generator is the exponential distribution with mean 15 days.
   (*i*) Let generator has just broken down, find the probability that it will break down in next 25 days. (*ii*) Evaluate the median of the corresponding distribution. [1+1]

3. Consider three urns such that Urn I contains 4 white and 5 black balls, Urn II contains 3 white and 5 black balls, Urn III contains 5 white and 3 black balls. One ball is randomly selected from Urn I and is put in Urn II. Then one ball is randomly selected from Urn II and is put in Urn III. Now suppose that 2 balls are drawn (one by one) from Urn III. Find the probability that both balls selected from Urn III are black. [3]

4. (*i*) Write properly the statement of De Moivre-Laplace central limit theorem. Suppose that $X_1, X_2, \ldots, X_n$ are independently and identically distributed $\chi^2(1)$ random variables and define $S_n = \sum_{i=1}^{n} X_i$. Use moment generating function technique to show that $\frac{S_n - E(S_n)}{\sqrt{V(S_n)}}$ will follow a standard normal distribution as $n \to \infty$. Then from this result find the corresponding probability distribution of $S_n$ and write its density function properly. [2.5+1.5]

   (*ii*) Let the random variable $X \sim Bin(n, p)$ distribution. Use central limit theorem to derive a formula for finding the value of $n$ such that $P(X > (n/2)) = (1 - \alpha)$. Particularly when $\alpha = 0.1, p = .45$ find the value of corresponding $n$. [1+1]

5. Consider three independent standard normal random variables $X, Y, Z$ such that $U = X + Y + Z, V = X - Y + 2Z$. Find the joint distribution of $(U, V)$. What is the correlation coefficient between these two random variables. What is the expected value of $U$ when $V$ is equal to 1. [2+1+1]

6. A soft drink machine is regulated so that it discharges an average of 190 milliliters per cup. Let amount of drink is normally distributed with a standard deviation equal to 16 milliliters,
   (*i*) What fraction of the cup will contain more than 210 milliliters ? [1.5]
   (*ii*) Find the probability that a cup contains between 175 and 202 milliliters [1.5]
   (*iv*) Below what value do we get the smallest 60% of the drinks? [1]

7. Suppose that amounts $X$ and $Y$ (in milligrams) of two toxic chemicals in a liter of water selected at random from a river near a certain manufacturing plant can be modeled by the joint density function $f_{X,Y}(x,y) = 6\theta^{-3}(x-y),\ 0 < y < x < \theta$.
(*i*) Find the conditional density of $X$ when $Y = y$ and that of $Y$ given that $X = x$. Further evaluate the correlation coefficient between $X$ and $Y$. [1.5+1.5+3]
(*ii*) Assume that $\theta = 2$ in the defined joint density function. Then set up appropriate integrals (with proper limits) that are required to evaluate $P(X+Y < 2 \mid X+2Y > \frac{1}{2})$ (do not simplify integrals). [3]

8. Various research study show that the eldest child in a family with multiple children generally has higher IQ than his or her siblings. In a given population with two children let random variable $X$ denote the IQ of the older child and random variable $Y$ denotes the IQ of the younger child. Suppose $(X,Y) \sim BVN(110, 100, 210, 220, 0.75)$.
(*i*) Suppose five families are randomly chosen from the given population. What is the probability that the older child has an IQ at least 10 points more than the younger child for at least three of these five families? [2]
(*ii*) For a randomly chosen family from the population, if the older child is known to have an IQ of 115, what is the probability that the younger child has an IQ greater than 115? [2]

9. Consider a warehouse in which auto parts are stocked in order to satisfy consumer demands. Suppose orders for parts arrive at the warehouse according to a Poisson process at the rate $\lambda = 1.5$ per hour. The warehouse is open 10 hours per day. The warehouse begins the day with an initial inventory of 20 parts.
(*i*) What is the probability that the warehouse exhausts its inventory after 6 hours of operation (*ii*) What is the probability that at the end of 10 hours: the warehouse has exactly 6 parts in inventory? at least 6 parts in inventory? [1+1+1]

10. Suppose that $X$ and $Y$ are independent random variables respectively distributed as $N(0,1)$ and $\chi^2(n)$ distributions. Find the probability density function of $U$ defined as $U = \frac{X}{\sqrt{Y/n}}$. [3]

11. Let $X$ and $Y$ be independent random variables with respective probability mass functions as $NB(r_1, p)$ and $NB(r_2, p)$ (here $X$ and $Y$ are defined as number of failures that precede $r_1^{th}$ and $r_2^{th}$ success respectively). Find the conditional probability distribution of $X$ given that $X + Y = n$, $n$ is a nonnegative integer. Further illustrate the corresponding probability distribution for the case $r_1 = r_2 = 1$. [3+1]

12. A certain study describing the risk $Y$ of an adult developing leukemia as a function of lifetime exposure $X$ to radiation (in microsieverts) is given by the equation $Y = g(X) = 1 - \alpha e^{-\beta X^2},\ X > 0, 0 < \alpha < 1, 0 < \beta < \infty$ where $X$ has the density function as $f_X(x) = \sqrt{\frac{2}{\pi\theta}}e^{-\frac{x^2}{2\theta}}, x > 0, \theta > 0$. Find average risk $E(Y)$. Comment on how the average risk varies as a function of $\alpha$, also as a function $\beta$ and $E(X)$. [2+1+1]