Indian Institute of Technology Patna
MA201: Mathematics III
END SEMESTER EXAM (26-11-2013)

Time: 3hrs | Max. Marks: 50

Note: Answer all questions. Give precise and brief answer. Standard formulae may be used.

Q.1. Obtain the Laurent Series expansions of $f(z) = \frac{z}{(z+1)(z-2)}$ in the following regions:

(a) $1 < |z| < 2$, (b) $|z| > 2$, (c) $1 < |z-1| < 2$. [3]

Q.2. Let $f(z) = \frac{z^2+2z-5}{(z^2+4)(z^2+2z+2)}$ and C is the circle $|z| = R$, then show that

$$\lim_{R\to\infty} \int_C f(z)dz = 0.$$

Hence find $\int_{C_1} f(z)dz$ where C_1 is the circle $|z-2| = 5$. [3]

Q.3. If $f(z) = z^5 - 3iz^2 + 2z - 1 + i$, then evaluate $\displaystyle\int_C \frac{f'(z)}{f(z)}dz$ where C is closed contour which contains all zeros of $f(z)$. [2]

Q.4. Determine singularities and their type for the function $f(z) = \frac{1}{z^3(z+4)}$. Expand the function in Laurent Series in the region $0 < |z| < 4$ and hence evaluate the integral $\int_C f(z)dz$, where C is unit circle centered at origin. [3]

Q.5. Evaluate $\displaystyle\int_0^\infty \frac{x \sin x}{x^2+4}dx$ using Calculus of Residues. [3]

Q.6. Find Fourier Integral of the function: $f(x) = \begin{cases} 1, & \text{if } |x| \le a; \\ 0, & \text{if } |x| > a. \end{cases}$ What can be said about the convergence of the integral for $x = \pm a$. [3]

Q.7. Find Fourier Transform of $f(x) = e^{-x^2}$. [3]

Q.8. Find the Fourier Series expansion of the following function:
$f(x) = x + x^2, \quad -\pi < x < \pi, \quad \text{and} \quad f(x+2\pi) = f(x), \quad \forall x.$

Discuss its convergence and obtain the value where this series converges to at $x = \pi$, and hence obtain the sum $s = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} \cdots$ [6]

Q.9.a. Formulate the PDE corresponding to following surface: $z = axe^x + \frac{1}{2}a^2e^{2y} + b$ where a, b are arbitrary constants. [2]

b. Solve the PDE: $z(x+y)p + z(x-y)q = x^2 + y^2$, with Cauchy data $z = 0$ on $y = 2x$. [3]

Q.10. Find the general solution of $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0$ by transforming into canonical form. [3]

Q.11. Use Duhamel to solve: [6]

$$u_{tt} = u_{xx} + \pi^2 \sin \pi x, \quad 0 < x < 1,\ t > 0,$$
IC: $u(x,0) = 2\sin \pi x - 2\sin 3\pi x, \quad u_t(x,0) = 4\pi \sin 2\pi x, \quad 0 < x < 1,$
BCs: $u(0,t) = 0,\ u(1,t) = 0,\ t > 0.$

Q.12. Solve the heat equation [4]

$$u_t = u_{xx}, \quad 0 < x < l,\ t > 0,$$
IC: $u(x,0) = f(x), \quad 0 < x < l,$
BCs: $u(0,t) = 0,\ u(l,t) = 0,\ t > 0.$

Q.13. Solve $u_{xx} + u_{yy} = \pi^2 \sin \pi x, \quad 0 < x < 1, \quad 0 < y < 2,$ [4]
$u(x,0) = 2\sin 3\pi x,\ u(x,2) = -\sin \pi x,\ 0 < x < 1,$
$u(0,y) = 0, \quad u(1,y) = 0,\ 0 < y < 2.$

Q.14. Let u be harmonic in $\Omega = \{(x,y) : x^2 + y^2 < 1\}$, and $u(x,y) = 1 + 3x$ for $(x,y) \in \partial\Omega$. Without solving, determine $\max_{(x,y)\in\Omega} u$ and $\min_{(x,y)\in\Omega} u$. [2]

Important Formulae:

The second order general PDE
$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$
can be transformed using $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$ into following canonical form
$\overline{A}u_{\xi\xi} + \overline{B}u_{\xi\eta} + \overline{C}u_{\eta\eta} + \overline{D}u_\xi + \overline{E}u_\eta + \overline{F}u = \overline{G}$
where
$\overline{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$
$\overline{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$
$\overline{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$
$\overline{D} = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y$
$\overline{E} = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y$
$\overline{F} = F, \quad \overline{G} = G.$

Good Luck