# Real Analysis (MA101), Tutorial Sheet-II (Sequence)

1. Let $\{u_n\}$ be a convergent sequence of real numbers converging to $u$. Then the sequence $\{|u_n|\}$ converges to $|u|$.

2. Let $\{u_n\}$ be a convergent sequence of real numbers and there exists a natural number $m$ such that $u_n > 0$ for all $n \geq m$. Then $\lim u_n \geq 0$.

3. Let $\{u_n\}$ and $\{v_n\}$ be two convergent sequences that converge to $u$ and $v$, respectively. Then
   (i) $\lim(u_n + v_n) = u + v$;
   (ii) If $c \in \mathbb{R}$ then $\lim(cu_n) = cu$;
   (iii) $\lim(u_n v_n) = uv$;
   (iv) $\lim \frac{u_n}{v_n} = \frac{u}{v}$, provided $\{v_n\}$ is a sequence of nonzero real numbers and $v \neq 0$.

4. Prove that $0 < \alpha < 1$, $\lim\limits_{n\to\infty} \alpha^n = 0$

5. Use the definition of limit of a sequence to establish the following limits:
   (i) $\lim\, (\ln\, (\frac{n}{n+1})) = 0$
   (ii) $\lim(\frac{3n+1}{2n+5}) = \frac{3}{2}$
   (iii) $\lim(\frac{n^2-1}{3n^2+5}) = \frac{1}{3}$
   (iv) If $c > 0$, then $\lim c^{\frac{1}{n}} = 1$.

6. A sequence $\{x_n\}$ is defined by $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$ for $n \geq 1$ and $x_1 = 0$ and $x_2 = 1$. Prove that the sequence $\{x_n\}$ converges and prove that the limit is $\frac{2}{3}$.

7. Let $x_n = (a^n + b^n)^{\frac{1}{n}}$ for all $n \in \mathbb{N}$ and $0 < a < b$, show that $\lim x_n = b$.

8. Prove that a monotone decreasing sequence, if bounded below, is convergent and it converges to the greatest lower bound.

9. Use Sandwich theorem to prove that
   (i) $\lim(2^n + 3^n)^{\frac{1}{n}} = 3$;
   (ii) $\lim \frac{1.3.5....(2n-1)}{2.4.6....2n} = 0$;
   (iii) $\lim(\frac{1}{1+n^2} + \frac{2}{2+n^2} + ... + \frac{n}{n+n^2}) = \frac{1}{2}$;
   (iv) $\lim((\sqrt{2} - 2^{\frac{1}{3}})(\sqrt{2} - 2^{\frac{1}{5}})...(\sqrt{2} - 2^{\frac{1}{2n+1}})) = 0$.

10. Show that the following sequence $x_n$ is bounded and monotone for $n \in N$. Also, find the limit.
    (i) $x_1 = 2$ and $x_{n+1} = 2 - \frac{1}{x_n}$
    (ii) $x_1 = 1$ and $x_{n+1} = \sqrt{2 + x_n}$
    (iii) $x_1 \geq 2$ and $x_{n+1} = 1 + \sqrt{x_n - 1}$
    (iv) $x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ..... + \frac{1}{n!}$
    (iv) $x_n = \frac{3n-1}{n+2}$

11. Let $A$ be a nonempty bounded subset of $\mathbb{R}$ and $\alpha = inf A$. Show that there exists a sequence $(a_n)$ such that $a_n \in A$ for all $n \in \mathbb{N}$ and $a_n \to \alpha$.

12. Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Prove or disprove the following statements:
    (i) Let $x_n = (-1)^n\ \forall n \in \mathbb{N}$. The sequence $(x_n)$ does not converge.
    (ii) If $x_n \to l (l \neq 0)$ and $\{y_n\}$ is a bounded sequence, then $(x_n y_n)$ converges.
    (iii) If the sequence $(x_n^2 + \frac{1}{n}x_n)$ converges then $(x_n)$ converges.

13. Let $(x_n)$ be a sequence defined by
$$x_n = (1+\alpha)^{-n} n^{\beta} \cos n.$$
    for all $n \in \mathbb{N}$ where $\alpha$ and $\beta$ are fixed positive real numbers. Show that $(x_n)$ converges.

14. Show directly from definition that if $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences, then $\{x_n + y_n\}$ and $\{x_n y_n\}$ are Cauchy sequences.

15. If $x_n = \sqrt{n}$, show that $\{x_n\}$ satisfies $\lim |x_{n+1} - x_n| = 0$, but that is not a Cauchy Sequence.

16. If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that $\{x_n\}$ is a Cauchy Sequence.

17. If $x_1 = 2$ and $x_{n+1} = 2 + \frac{1}{x_n}$ for $n \geq 1$. Show that $\{x_n\}$ is a contractive sequence. Find the limit.

18. The polynomial equation $x^3 - 5x + 1 = 0$ has root $r$ with $0 < r < 1$. Use an appropriate contractive sequence to calculate $r$ with in $10^{-4}$.