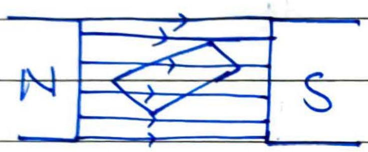
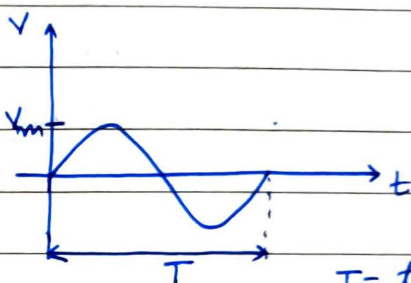
Chapter - AC Analysis

→ PRODUCTION OF AC →

- When a conductor coil is rotated in a magnetic field, a current is produced through it



$\Longrightarrow$



T = time period

- we rotate the coil with ω angular velocity

where $\omega = 2\pi f = 2\pi/T$

& we can write this wave as

$$V = V_m \sin(\omega t)$$

- now we find root mean square voltage

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T (V_m \sin\omega t)^2 \, dt}$$

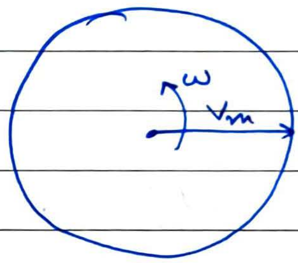
$$\Rightarrow \boxed{V_{rms} = V_m/\sqrt{2}}$$

NOTE: The voltage displayed in electical circiuts ie 250V is actually V_{rms}.

→ REPRESENTATION OF AC →

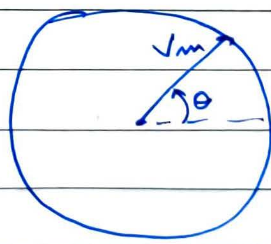
- We represent voltage ~~vectors~~ as a rotating phasor vector.

- We take anti-clockwise as +ve direction and clockwise as -ve direction by convention

- 

- ~~this phasor vector ...~~

- We assume this phasor vector to lie in complex plane for ease of calculations.

eg- 

We represent this as

$V = V_m \angle \theta$

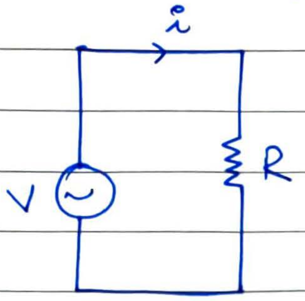
$$\boxed{V = V_m \angle \theta}$$

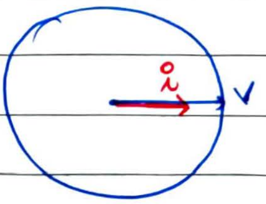
$$\Rightarrow \boxed{V = V_m e^{j\theta}} \qquad j = \sqrt{-1}$$

$$\Rightarrow \boxed{V = V_m \cos\theta + j V_m \sin\theta}$$

→ RESISTOR IN AC CIRCUITS →

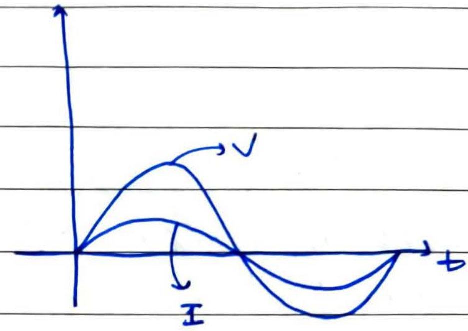
• now consider





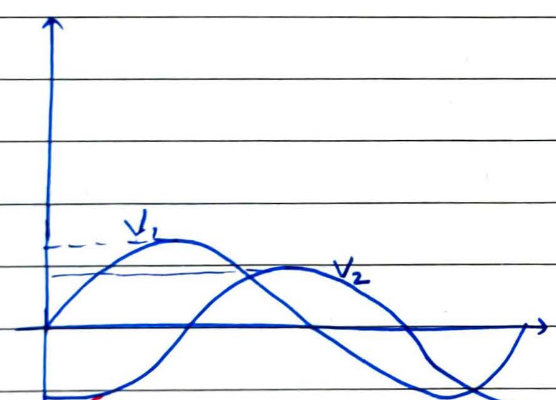
$i = V/R$

$\Rightarrow i = \frac{V_m}{R} \sin \omega t$

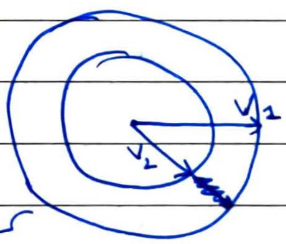


$\Rightarrow i = I_m \sin \omega t$

• consider



$\Rightarrow$



lagging behind and since its −ve ∴ it lags behind
ie in anticlockwise direction

→ IMPEDANCE

- Impedance is defined as

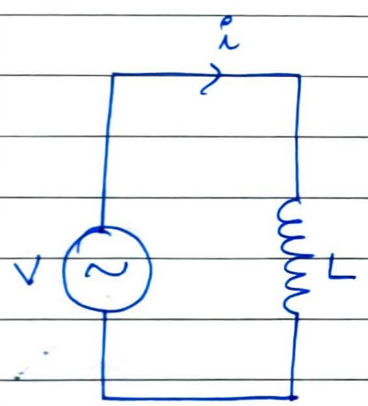
$$\boxed{Z = \frac{V}{I}}$$

- For resistor, $Z = \dfrac{V_m \sin\omega t}{I_m \sin\omega t}$

$$\Rightarrow \boxed{Z_R = R}$$

- Impedance can even be complex; what we used to calculate in class 12 was $|Z|$ ie its magnitude.

→ INDUCTOR IN AC CIRCUITS

- 

$V = L\dfrac{di}{dt}$ across inductor

$\Rightarrow L\dfrac{di}{dt} = V_m \sin\omega t$

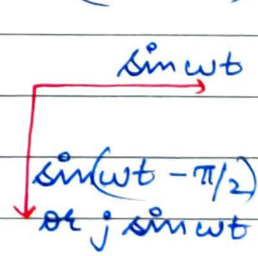
$\Rightarrow L\int di = V_m \int \sin\omega t \, dt$

$\Rightarrow Li = -V_m \cos\omega t$

$\Rightarrow Li = \dfrac{V_m}{\omega} \sin(\omega t - \pi/2)$ $\Rightarrow i = \dfrac{V_m}{\omega L} \sin(\omega t - \pi/2)$

- Now,

$Z = \dfrac{V}{I} = \dfrac{V_m \sin\omega t}{(V_m/\omega L) \sin(\omega t - \pi/2)}$



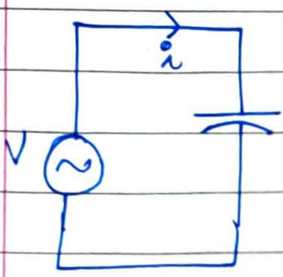
$\Rightarrow Z = \dfrac{\omega L \sin\omega t}{-j \sin\omega t}$ $\Rightarrow \boxed{Z_L = j\omega L}$

Impedance of inductor

- Here, we also define

$\boxed{X_L = \omega L}$ ie reactance of inductor

CAPACITOR IN AC CIRCUITS

- 

$$i = C\frac{dV}{dt}$$

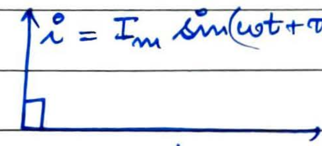
$$= C\frac{d(V_m \sin \omega t)}{dt}$$

$$\Rightarrow i = CV_m \omega \cos \omega t$$

$$\Rightarrow i = CV_m \omega \sin(\omega t + \pi/2)$$

$$\Rightarrow i = I_m \sin(\omega t + \pi/2)$$

- If represented as a phasor, we get



$$i = I_m \sin(\omega t + \pi/2) = I_m j \sin \omega t$$

$$V = V_m \sin \omega t$$

- $\therefore$ we have

$$i = C\omega V_m j \sin \omega t$$

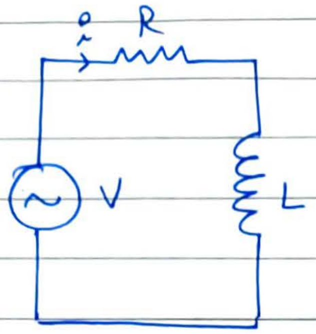
$$\therefore \quad Z = \frac{V}{I} = \frac{\cancel{V_m}\,\cancel{\sin \omega t}}{jC\omega \cancel{V_m}\,\cancel{\sin \omega t}}$$

$$\Rightarrow \boxed{Z_c = \frac{1}{jC\omega}}$$ ⇝ Impedance of capacitor

- We define

$$\boxed{X_c = \frac{1}{\omega C}}$$ ie reactance of capacitor.

RL circuit



Impedance of inductor $= jX_L$

$= j\omega L$

Total Impedance $Z = R + j\omega L$

i.e. $|Z| = \sqrt{R^2 + \omega^2 L^2}$

KVL $\equiv$ $V = Ri + L\,di/dt$

or we can directly write

$$V = Ri + ij\omega L$$

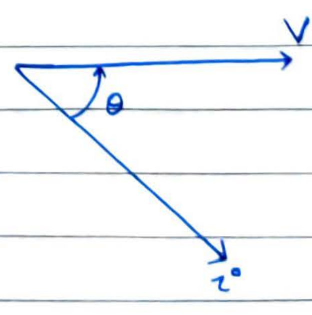
$$\Rightarrow i = \frac{V}{R + j\omega L}$$

$\therefore$ $\boxed{i = \frac{V}{R + j\omega L}}$ at any instant.

$$\Rightarrow i = \frac{V}{R^2 + \omega^2 L^2}(R - j\omega L)$$

$$\Rightarrow \boxed{i = \frac{V(R - j\omega L)}{R^2 + \omega^2 L^2}}$$

$\therefore$ we get a phasor diagram like



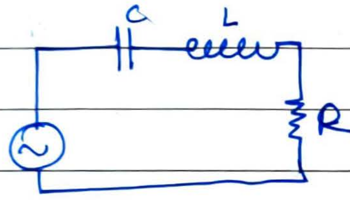
$\theta = \tan^{-1}\left(\frac{-\omega L}{R}\right)$

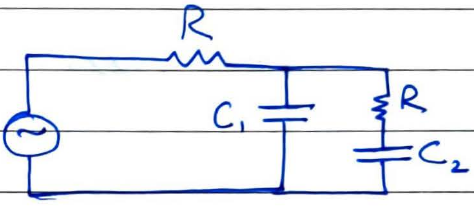
SECOND ORDER CIRCUITS →

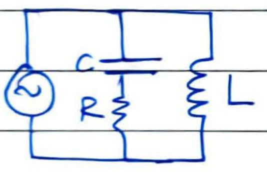
- First order circuits are characterised by first order diff. eq^n and similarly second order diff. eq^n represents second order circuits.

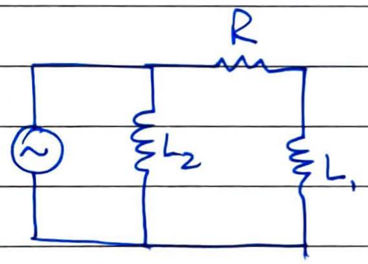
- First order circuits have one energy storage element (one capacitor or one inductor) but second order circuits have 2 storage elements.

- eg -

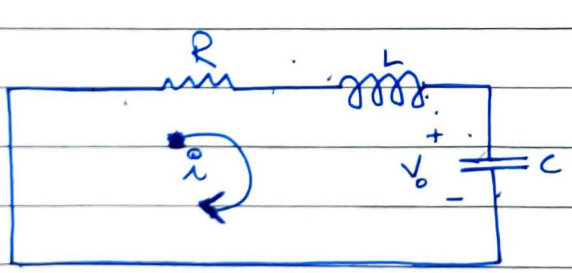








→ SOURCE FREE LCR circuits →



Let $i(0) = i_0$

And we can write

$$V_0 = \frac{1}{C}\int i\,dt$$

$\therefore$ writing KVL now

$$\frac{1}{C}\int i\,dt + L\frac{di}{dt} + iR = 0$$

Differentiating & dividing by L

$$\frac{i}{LC} + \frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} = 0$$

$$\therefore \boxed{\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}\, i = 0} \quad -(i)$$

- we have an initial condition of form

$$Ri(0) + L\frac{di(0)}{dt} + V_0(0) = 0$$

$$\Rightarrow \boxed{\frac{di(0)}{dt} = -\frac{1}{L}\left(Ri_0 + V_0(0)\right)}$$

- now to solve $-(i)$ we take an assumption (like 'Ansatz' in Physics)

$$\text{Let } i(t) = Ae^{st}$$

$\therefore$ Substituting in $-(i)$

$$Ae^{st}\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = 0$$

$$\Rightarrow s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$\Rightarrow s = \frac{-R/L \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2}$$

$$\Rightarrow s = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$\therefore \quad s_1 = \frac{-R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - 1/LC}$$

$$s_2 = \frac{-R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - 1/LC}$$

$\therefore$ We have two solutions of i, ie

$$i_1 = A_1 e^{s_1 t} \quad ; \quad i_2 = A_2 e^{s_2 t}$$

$\therefore$ we have a combined solution for i as

$$\boxed{i = A_1 e^{s_1 t} + A_2 e^{s_2 t}}$$

- We write

$$\left.\begin{aligned} s_1 &= -\alpha + \sqrt{\alpha^2 - \omega_0^2} \\ s_2 &= -\alpha - \sqrt{\alpha^2 - \omega_0^2} \end{aligned}\right\}$$ where $\alpha = R/2L$ = damping factor

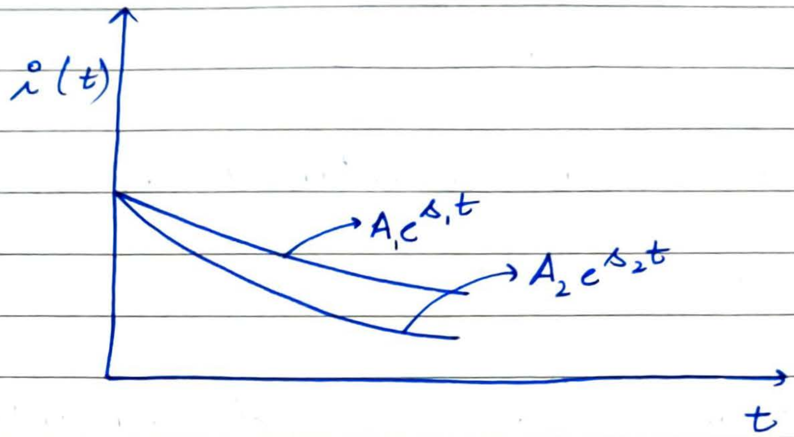
$\omega_0 = \frac{1}{\sqrt{LC}}$ = Undamped natural frequency

- Analysing values of α and ω_0, we could have 3 types of solutions of eq^n (i)

PTO

1. $\alpha > \omega_0$ ie $\frac{R}{2L} > \frac{1}{\sqrt{LC}}$ (OVERDAMPED SYSTEM)

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$



{Similar to overdamping in Physics}

2. $\alpha = \omega_0$, ie $s_1 = s_2 = -\alpha$ (CRITICAL DAMPING)

- We could write

$$i = A_1 e^{-\alpha t} + A_2 e^{-\alpha t} = A_3 e^{-\alpha t}$$

But note carefully that any second order eq^n has 2 initial conditions. (Here there would be some $i(0) = $ const. and $\left(\frac{di}{dt}\right)_{(t=0)} = $ const

But from above eq^n, we can see that we can get only one initial condition.
$\therefore$ this solution is somehow invalid.

- $\therefore$ To find the correct solution, we follow the following procedure

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = 0$$

$$\Rightarrow \frac{d}{dt}\underbrace{\left\{\frac{di}{dt} + \alpha i\right\}}_{f} + \alpha \underbrace{\left\{\frac{di}{dt} + \alpha i\right\}}_{f} = 0 \qquad \left(\because \alpha^2 = \omega_0^2\right)$$

$$\Rightarrow \frac{df}{dt} + \alpha f = 0$$

$$\Rightarrow f = A_1 e^{-\alpha t}$$

$$\Rightarrow \frac{di}{dt} + \alpha i = A_1 e^{-\alpha t}$$

$$\Rightarrow A_1 = e^{\alpha t}\frac{di}{dt} + e^{\alpha t}\alpha i$$

$$\Rightarrow \boxed{\cancel{A_1 = \frac{d}{dt}(i e^{\alpha t})}}$$

$$\Rightarrow A_1 = \frac{d}{dt}(i e^{\alpha t})$$

$$\Rightarrow \int d(i e^{\alpha t}) = \int A_1 dt$$

$$\Rightarrow i e^{\alpha t} = A_1 t + A_2$$

$$\Rightarrow \boxed{i = (A_1 t + A_2) e^{-\alpha t}}$$

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3. $\alpha < \omega_0$ ie $\frac{R}{2L} < \frac{1}{\sqrt{LC}}$ (UNDERDAMPED SYSTEM)

We can see

$$s_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)}$$

$$\therefore s_1 = -\alpha + j\omega_d$$

$$s_2 = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)}$$

$$\therefore s_2 = -\alpha - j\omega_d$$

$$i = A_1 e^{-\alpha + j\omega_d} + A_2 e^{-\alpha - j\omega_d}$$

$$\Rightarrow i = [A_1(\cos\omega_d + j\sin\omega_d) + A_2(\cos\omega_d - j\sin\omega_d)]e^{-\alpha}$$

$$\Rightarrow i = [(A_1 + A_2)\cos\omega_d + j(A_1 - A_2)\sin\omega_d]e^{-\alpha}$$

$$\Rightarrow \boxed{i = [B_1 \cos\omega_d + jB_2 \sin\omega_d]e^{-\alpha}}$$

where $B_1 = A_1 + A_2$; $B_2 = A_1 - A_2$