Midsem: MA 201 (Partial Differential Equation), 2017 **IIT Patna**

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Part 1 MUST be answered in a SINGLE place of the answer script only.

Part 1: (Marks: $8 \times 1 = 8$)

1. a) Fill the blank: Number of solutions of the PDE $u_x^2 + u_y^2 + 1 = 0$, where u is a real valued function, is ...... [1]

   b) True/False: The general integral of the PDE $u(xu_x - yu_y) = y^2 - x^2$ is $x^2 + y^2 + u^2 = f(xy)$, where f is an arbitrary function. [1]

   c) Pick the right one: $\int_0^y \int_{x-y+q}^{x+y-q} u(p,q)\,dp\,dq$ for a given nonzero function $u(p,q)$, will be the solution(s) of A. Laplace equation, B. Heat equation, C. Wave equation, D. Heat and Wave equations. [1]

   d) True/False: Consider the Laplace equation $u_{xx} + u_{yy} = 0$ for $(x,y) \in \Omega$. Then, u must have either maximum or minimum at all the points of the boundary $\partial\Omega$ of Ω. [1]

   e) Fill the blank: Maximum and minimum of Wave equation $u_{tt} = u_{xx}$, $-\infty < x < \infty$, $t > 0$, $u(x,0) = f(x)$, $u_t(x,0) = 0$, $-\infty < x < \infty$, will/can (pick the right one) be attained at .... [1]

   f) Fill the blank: Term by term differentiation of the Fourier series of the function $f(x) = x^2$, $x \in (-\pi, \pi)$ is valid/invalid (pick the correct one) because ...... [1]

   g) Pick the right one: The PDE $u_x - u_y = x^2 + y^2 + u^2$ is Linear/Semilinear/Quasilinear/Fully Nonlinear. [1]

   h) True/False: Gibbs phenomena only appears during the approximation of $f(x)$ by using the finite sum of its Fourier series. [1]

Part 2:

2. Write down the characteristic equations of the PDE: $2u_x + 3u_y + 8u = 0$ over the Cauchy data: $\Gamma : 3x - 2y = 1$ where $u(x,y) = e^{-4x}$. Find out a solution of this problem. [1+2]

3. Assume that $u(x,y)$ is a harmonic function (i.e., u satisfies $\Delta u = 0$) in a bounded domain Ω and continuous in $\overline{\Omega} = \Omega \cup \partial\Omega$ where the $\partial\Omega$ defines the boundary of Ω. Then, prove that u attains its maximum on the boundary $\partial\Omega$ of Ω. [4]

4. Mention the domain of dependence of

$$u_{tt} - c^2 u_{xx} = F(x,t), \quad -\infty < x < \infty,\ t > 0, \quad u(x,0) = u_t(x,0) = 0, \quad -\infty < x < \infty,$$

   of the solution $u(x,t)$ at the point (x_1, t_1) **by only drawing a figure.** Clearly mention it by pointing the domain of dependence inside the figure. [2]

5. Write down the solution of the following initial value problem $u_{tt} - 4u_{xx} = 2t$ with $u(x,0) = x^2$, $u_t(x,0) = 1$ based on Duhamel Principle. [2]

6. (Application of Fourier Transform): Using Fourier Integral Representation, show that [3]

$$\int_0^{\infty} \frac{\cos(wx)}{k^2+w^2}dw = \frac{\pi}{2k}e^{-kx}, \quad x>0,\ k>0.$$

7. Solve the following parabolic PDE by using *Fourier Transform*:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ x>0,\ t>0, \quad \frac{\partial u}{\partial x}(0,t)=0,\ u(x,0)=f(x).$$

Explain clearly, where did you use the condition $\frac{\partial u}{\partial x}(0,t)=0$ inside the derivation. How should one proceed to find out the continuously differentiable solution based on Fourier series method (using separation of variable) for the above problem (Only key idea is required). [2+1+1]

8. (Application of PDE) A thin rod of length $l$cm long, with insulated sides, has its ends A and B kept at a^oC and b^oC respectively until steady state conditions prevail. The temperature at A is then suddenly raised to c^0C and at the same time, at B is lowered to d^0C. Model this phenomena into a simple parabolic PDE (write only the governing equation by mentioning the significance of the unknowns and initial and boundary conditions) and then find out the temperature distribution $u(x,t)$ by solving it. For modeling purpose, take the end A as origin. [1+3]

Formulas:

1. Fourier transform of a function $f(x)$ is defined by

$$f(x) = \int_0^{\infty} (A(w)\cos(wx) + B(w)\sin(wx))dw,$$

where

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(s)\cos(ws)ds, \quad \text{and} \quad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(s)\sin(ws)ds.$$

2. Fourier Sine integral of a function $f(x)$ is defined by

$$f(x) = \int_0^{\infty} A(w)\sin(wx)dx, \quad \text{where} \quad A(w) = \frac{2}{\pi}\int_0^{\infty} f(s)\sin(ws)ds.$$

Similarly, Fourier Cosine integral of a function $f(x)$ is defined from the above formula where 'sin' function will be replaced by 'cos' function.

3. D Alembert's solution of the Wave equation

$$u_{tt} - c^2u_{xx} = 0, \quad -\infty < x < \infty,\ t>0, \quad u(x,0)=f(x),\ u_t(x,0)=g(x), \quad -\infty < x < \infty,$$

: is

$$u(x,t) = \frac{1}{2}[f(x+ct)+f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$