MA 201: Partial Differential Equations
Lecture - 1

▶What is a differential equation?

Recall n-th order ordinary differential equation (ODE):

$$F(x, y(x), y'(x), y''(x), \cdots, y^{(n)}(x)) = 0, \tag{1}$$

where $y'(x) = \frac{dy}{dx}, \ y''(x) = \frac{d^2y}{dx^2}, \cdots, y^{(n)}(x) = \frac{d^ny}{dx^n}$.

Facts:

- In an ODE, there is only one independent variable (Here x) so that all the derivatives appearing in the equation are ordinary derivatives of the unknown function $y(x)$.
- The order of an ODE is the order of the highest derivative that occurs in the equation.
- Equation (1) is linear if F is linear in $y, y', y'', \dots, y^{(n)}$, with the coefficients depending on the independent variable x.

Example

- $y''(x) + 3y'(x) + xy(x) = 0$ (second-order, linear)
- $y''(x) + 3y(x)y'(x) + xy(x) = 0$ (second-order, nonlinear)
- $y''(x) + 3y'(x) + xy^2(x) = 0$ (second-order, nonlinear)
- $y'''(x) + 5xy'(x) + \sin(x)y(x) = 0$ (third-order, linear)

Theorem (Picard's Theorem)

Let $R\ :\ |x - x_0| < a,\ \ |y - y_0| < b$ be a rectangle. Let $f(x, y)$ be continuous and bounded in R, i.e., there exists a number K such that

$$|f(x, y)| \leq K \ \ \forall (x, y) \in R.$$

Further, let f satisfy the Lipschitz condition with respect to y in R, i.e., there exists a number M such that

$$|f(x, y_2) - f(x, y_1)| \leq M|y_2 - y_1| \ \ \forall (x, y_1), (x, y_2) \in R. \tag{2}$$

Then, the IVP

$$y'(x) = f(x, y), \quad y(x_0) = y_0 \tag{3}$$

has a unique solution $y(x)$. This solution is defined for all x in the interval

$$|x - x_0| < \alpha, \quad \text{where} \quad \alpha = \min\{a, \frac{b}{K}\}.$$

Let's consider its generalization to a system of n first-order ODEs in n unknowns

$$\frac{dy_i(x)}{dx} = f_i(x, y_1, \ldots, y_n), \quad i = 1, \ldots, n, \tag{4}$$

satisfying the initial conditions

$$y_1(x_0) = y_1^0, \ldots, y_n(x_0) = y_n^0, \tag{5}$$

where $y_1^0, \ldots, y_n^0$ are the given initial values.

Theorem (Existence and uniqueness results)

Let Q be a box in $\mathbb{R}^{n+1}$ defined by

$$Q\ :\ |x-x_0|<a,\ \ |y_1-y_1^0|<b_1,\dots,|y_n-y_n^0|<b_n.$$

Let each of the functions $f_1,\dots,f_n$ be continuous and bounded in Q, and satisfy the following Lipschitz condition with respect to the variables $y_1,y_2,\dots,y_n$, i.e., there exists constants $L_1,\dots,L_n$ such that

$$|f(x,y_1^1,\dots,y_n^1)-f(x,y_1^2,\dots,y_n^2)|\le L_1|y_1^1-y_1^2|+\cdots+L_n|y_n^1-y_n^2|$$

*for all pairs of points $(x,y_1^1,\dots,y_n^1),(x,y_1^2,\dots,y_n^2)\in Q$.
Then there exists a unique set of functions $y_1(x),\dots,y_n(x)$ defined for x in some interval $|x-x_0|<h,\ \ 0<h<a$ such that $y_1(x),\dots,y_n(x)$ solve (4)-(5).*

Definition

A partial differential equation (PDE) for a function $u(x_1, x_2, \ldots, x_n)$ $(n \geq 2)$ is a relation of the form

$$F(x_1, x_2, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots, u_{x_1 x_1}, u_{x_1 x_2}, \ldots,) = 0, \tag{6}$$

where F is a given function of the independent variables $x_1, x_2, \ldots, x_n$; of the unknown function u and of a finite number of its partial derivatives.

The order of an equation:The order of a PDE is the order of the highest derivative appearing in the equation. If the highest derivative is of order m, then the equation is said to be of order m.

$$\begin{aligned} u_t - u_{xx} &= f(x,t) \;\; \text{(second-order equation)} \\ u_t + u_{xxx} + u_{xxxx} &= 0 \;\; \text{(fourth-order equation)} \end{aligned}$$

Definition (Classification)

- A PDE is said to be linear if F is linear in the unknown function u and its partial derivatives, with coefficients depending on the independent variables $x_1, x_2, \ldots, x_n$.
- A PDE of order m is said to be quasi-linear if it is linear in the derivatives of order m with coefficients that depend on $x_1, x_2, \ldots, x_n$ and the derivatives of order $< m$.
- A quasi-linear PDE of order m, where the coefficients of derivatives of order m are functions of the independent variables $x_1, \ldots, x_n$ alone is called a semi-linear PDE.
- A PDE of order m is called fully nonlinear if it is not linear in the derivatives of order m.

Example (Some well-known PDEs)

- The Laplace's equation in n dimensions:

$$\Delta u := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0 \text{ (second-order, linear, homogeneous)}$$

- The Poisson equation:

$$\Delta u = f \text{ (second-order, linear, nonhomogeneous)}$$

- The heat equation:

$$\frac{\partial u}{\partial t} - k\Delta u = 0 \ \ (k = \text{const.} > 0) \text{ (second-order, linear, homogeneous)}$$

- The wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0 \ \ (c = \text{const.} > 0) \text{ (second-order, linear, homogeneous)}$$

- The Transport equation:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \text{ (first-order, linear, homogeneous)}$$

- The Burger's equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \text{ (first-order, quasilinear, homogeneous)}$$

- Semi-linear equation

$$xuu_x + yuu_y = x^2 + y^2, \;\; x > 0, \; y > 0.$$

First-order PDEs: A first order PDE in two independent variables x, y and the dependent variable u can be written in the form

$$F(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = 0. \tag{7}$$

For convenience, set

$$p = \frac{\partial u}{\partial x}, \quad q = \frac{\partial u}{\partial y}.$$

Equation (7) then takes the form

$$F(x, y, u, p, q) = 0. \tag{8}$$

First-order PDEs arise in many applications, such as

- Transport of material in a fluid flow.
- Propagation of wave-fronts in optics.

- **Classification of first-order PDEs**
  - If (7) is of the form

$$\boxed{a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = c(x,y)u + d(x,y)}$$

    then it is called **linear** first-order PDE.
  - If (7) has the form

$$\boxed{a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = c(x,y,u)}$$

    then it is called **semilinear** because it is linear in the leading (highest-order) terms $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. However, it need not be linear in u.
  - If (7) has the form

$$\boxed{a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u)}$$

    then it is called **quasi-linear** PDE. Here the function F is linear in the derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ with the coefficients a, b and c depending on the independent variables x and y as well as on the unknown u.
  - If F is not linear in the derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, then (7) is said to be **nonlinear**.

$$\text{Linear PDE} \subsetneq \text{Semi-linear PDE} \subsetneq \text{Quasi-linear PDE} \subsetneq \text{PDE}$$

Example

- $xu_x + yu_y = u$ **(linear)**
- $xu_x + yu_y = u^2$ **(semi-linear)**
- $u_x + (x + y)u_y = xy$ **(linear)**
- $uu_x + u_y = 0$ **(quasi-linear)**
- $xu_x^2 + yu_y^2 = 2$ **(nonlinear)**

How first-order PDEs occur?

- The applications of conservation principles often yield first-order PDEs.
- **Two-parameter family of surfaces:** Let

$$f(x, y, u, a, b) = 0 \tag{9}$$

represent a two-parameter family of surfaces in $\mathbb{R}^3$, where a and b are arbitrary constants. Differentiating (9) with respect to x and y yields the relations

$$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial u} = 0, \tag{10}$$

$$\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial u} = 0. \tag{11}$$

Eliminating a and b from (9), (10) and (11), we get a relation of the form

$$F(x, y, u, p, q) = 0, \tag{12}$$

which is a first-order PDE for the unknown function u of two independent variables.

Example

The equation

$$x^2 + y^2 + (u - c)^2 = a^2, \tag{13}$$

where a and c are arbitrary constants, represents the set of all spheres whose centers lie on the u-axis. Differentiating (13) with respect to x, we obtain

$$\left(x + (u - c)\frac{\partial u}{\partial x}\right) = 0. \tag{14}$$

Differentiating (13) with respect to y to have

$$y + (u - c)\frac{\partial u}{\partial y} = 0. \tag{15}$$

Eliminating the arbitrary constant c from (14) and (15), we obtain the first-order PDE

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0. \tag{16}$$

- **Unknown function of known functions**
    - *Unknown function of a single known function*
      Let
      $$u = f(g), \tag{17}$$
      where f is an unknown function and g is a known function of two independent variables x and y.
      Differentiating (17) with respect to x and y yields the equations
      $$u_x = f'(g)g_x \tag{18}$$
      and
      $$u_y = f'(g)g_y, \tag{19}$$
      respectively. Eliminating $f'(g)$ from (18) and (19), we obtain
      $$g_y u_x - g_x u_y = 0,$$
      which is a first-order PDE for u.

Example

The surfaces described by an equation of the form

$$u = f(x^2 + y^2), \tag{20}$$

where f is an arbitrary function of a known function $g(x, y) = x^2 + y^2$. Differentiating (20) with respect to x and y, it follows that

$$u_x = 2xf'(g); \qquad u_y = 2yf'(g),$$

where $f'(g) = \frac{df}{dg}$. Eliminating $f'(g)$ from the above two equations, we obtain a first-order PDE

$$yu_x - xu_y = 0.$$

- *Unknown functions of two known functions*
  Let
  $$u = f(x - ay) + g(x + ay), \tag{21}$$
  where $a > 0$ is a constant. With $v(x, y) = x - ay$ and $w(x, y) = x + ay$, we write (21) as
  $$u = f(v) + g(w). \tag{22}$$
  Differentiating (22) w. r. t. x and y yields
  $$p = u_x = f'(x - ay) + g'(x + ay),$$
  $$q = u_y = -af'(x - ay) + ag'(x + ay)$$
  Eliminating $f'(v)$ and $g'(w)$, we get
  $$q_y = a^2 p_x.$$
  In terms of u, the above first-order PDE is the well-known wave equation
  $$u_{yy} = a^2 u_{xx}.$$