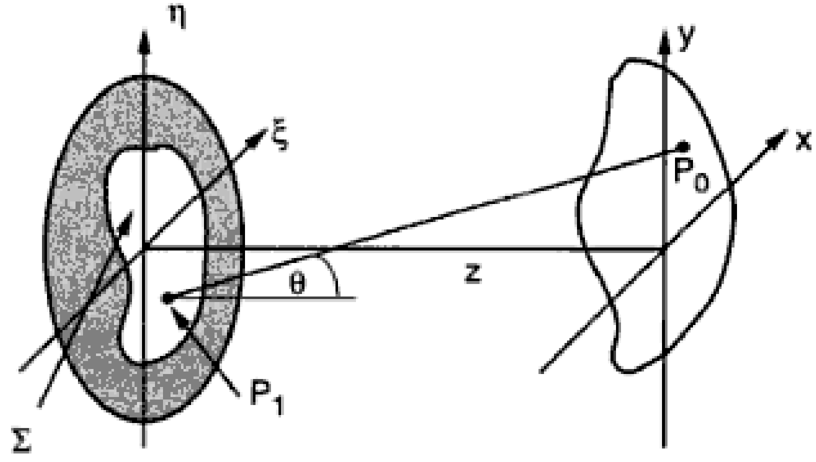
Huygens-Fresnel Principle

Diffracting aperture is assumed to lie in (ξ,η) plane, & is illuminated in + z direction.

Calculate wavefield across (x,y) plane, which is parallel to (ξ,η) plane & at normal distance z from it.

Huygens-Fresnel principle can be stated as



$$U(P_0) = \frac{1}{j\lambda} \iint_{\Sigma} U(P_1) \frac{\exp(jkr_{01})}{r_{01}} \cos\theta ds$$

where θ is the angle between outward normal $\hat{n}$ & $\vec{r}_{01}$ pointing from P_0 to P_1.

$$\cos\theta = \frac{z}{r_{01}}$$

$$U(P_0) = \frac{1}{j\lambda} \iint_{\Sigma} U(P_1) \frac{\exp(jkr_{01})}{r_{01}} \cos\theta ds$$

$$or, \qquad U(P_0) = \frac{z}{j\lambda} \iint_{\Sigma} U(P_1) \frac{\exp(jkr_{01})}{r_{01}^2} ds$$

$$\Rightarrow \qquad U(x,y)=\frac{z}{j\lambda}\iint_{\Sigma}U(\xi,\eta)\frac{\exp(jkr_{01})}{r_{01}^{2}}d\xi d\eta$$

$$r_{01}=\sqrt{z^{2}+(x-\xi)^{2}+(y-\eta)^{2}}$$

$$=\sqrt{z^{2}\left\{1+\frac{(x-\xi)^{2}}{z^{2}}+\frac{(y-\eta)^{2}}{z^{2}}\right\}}=z\sqrt{1+\frac{1}{z^{2}}\left\{(x-\xi)^{2}+(y-\eta)^{2}\right\}}$$

$$r_{01}=z\sqrt{1+\left(\frac{x-\xi}{z}\right)^{2}+\left(\frac{y-\eta}{z}\right)^{2}}$$

Fresnel Approximation

Let b be a number that is less than unity. Binomial expansion of square root,

$$\sqrt{1+b} = 1 + \frac{1}{2}b - \frac{1}{8}b^2 + \ldots$$

$$r_{01} = z\sqrt{1 + \left(\frac{x-\xi}{z}\right)^2 + \left(\frac{y-\eta}{z}\right)^2}$$

Retaining only first two terms of expansion,

$$r_{01} \approx z\left[1 + \frac{1}{2}\left(\frac{x-\xi}{z}\right)^2 + \frac{1}{2}\left(\frac{y-\eta}{z}\right)^2\right]$$

Question now arises as to whether we need to retain all the terms in approximation or whether only 1st term might suffice.

Answer depends on which of the several occurrences of r_{01} is being approximated.

For r_{01}^2 appearing in denominator, error introduced by dropping all terms but *z* is generally acceptably small. However, for r_{01} appearing in exponent, errors are much more critical.

1. They are multiplied by a very large number *k*, a typical value for which might be greater than 10^7 in visible region.
2. Phase changes of as little as a fraction of a radian can change value of exponential significantly.

For this reason, we retain both terms of binomial approximation in exponent.

$$\Rightarrow \qquad U(x,y) = \frac{e^{jkz}}{j\lambda z}\int\int_{-\infty}^{\infty} U(\xi,\eta)\exp\left\{ j\frac{k}{2z}\left[(x-\xi)^2 + (y-\eta)^2\right]\right\} d\xi d\eta \qquad (A)$$

This Eq. is seen to be a convolution,

$$U(x,y) = \int\int_{-\infty}^{\infty} U(\xi,\eta) h(x-\xi, y-\eta) d\xi d\eta$$

Where convolution kernel is,

$$h(x,y) = \frac{e^{jkz}}{j\lambda z}\exp\left[\frac{jk}{2z}\left(x^2+y^2\right)\right]$$

Another form of result (A) is found if term

$$\exp\left[\frac{jk}{2z}\left(x^2+y^2\right)\right]$$

is factored outside integral signs, yielding

$$\Rightarrow \qquad U(x,y)=\frac{e^{jkz}}{j\lambda z}e^{j\frac{k}{2z}(x^2+y^2)}\int\int_{-\infty}^{\infty}\left\{U(\xi,\eta)e^{j\frac{k}{2z}(\xi^2+\eta^2)}\right\}e^{-j\frac{2\pi}{\lambda z}(x\xi+y\eta)}d\xi d\eta \qquad (B)$$

This result is recognized as ***Fourier transform*** of product of complex field just to the right of aperture & a quadratic phase exponential.

Both results (A) & (B) are referred to as ***Fresnel Diffraction Integral***.

Positive vs. Negative Phases

It is common practice when using Fresnel approximation to replace expressions for spherical waves by quadratic phase-exponentials.

The question often arises as to whether sign of the phase should be positive or negative in a given expression.

Phasors chosen to rotate in clockwise direction, $exp(-j2\pi vt)$.

If we move in space in such a way as to intercept portions of a wavefield that were emitted *later* in time, the phasor will have advanced in clockwise direction, & therefore phase must become more *negative*.

If we move in space in such a way as to intercept portions of a wavefield that were emitted *earlier* in time, the phasor will have had time to rotate as far clockwise direction, & therefore phase must become more *positive*.

A spherical wave diverging from a point on z axis, observation being in an (x,y) plane that is normal to that axis, then movement away from origin always results in observation of portions of wavefront that were emitted earlier in time than that at the origin, since wave has had to propagate further to reach those points.

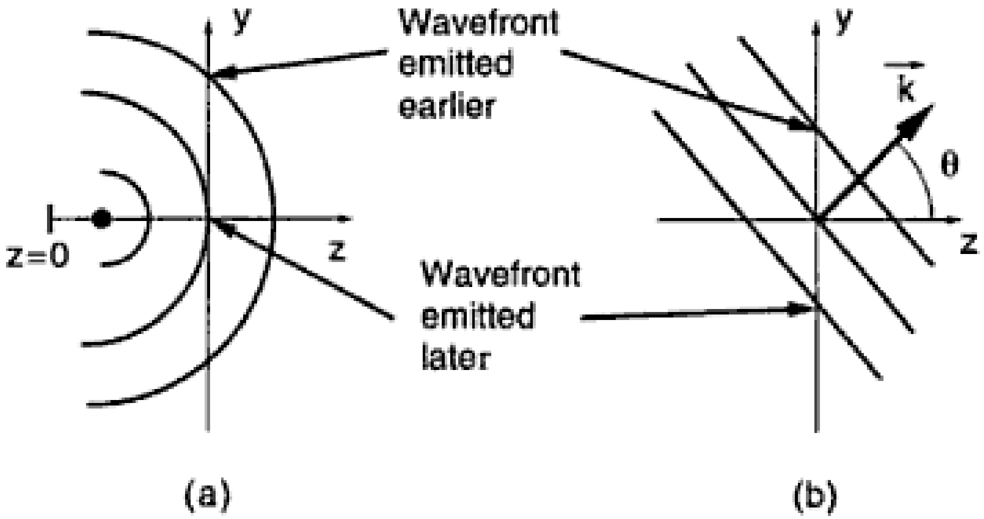
Contd…

For positive *z*, the expressions

$$\exp(jkr_{01})$$

$$\exp\left[j\frac{k}{2z}\left(x^2+y^2\right)\right]$$

represent a **diverging spherical wave** & a quadratic-phase approximation.



$$\exp(-jkr_{01})$$

$$\exp\left[-j\frac{k}{2z}\left(x^2+y^2\right)\right]$$

represent a **converging spherical wave** & a quadratic-phase approximation.

If *z* is a negative number, then interpretation must be reversed, since a negative sign is hidden in *z*.

For positive *α*, the expressions $\exp(j2\pi\alpha y)$
represents a plane wave with a wave vector in (*y*,*z*) plane.

Accuracy of Fresnel Approximation

In Fresnel approx., spherical secondary wavelets of Huygens-Fresnel principle have been replaced by wavelets with parabolic wavefronts. Accuracy of this approximation is determined by errors induced when terms higher than first order (linear in *b*) are dropped in binomial expansion.

A sufficient condition would be that the maximum phase change induced by dropping $b^2/8$ term be much less than 1 radian. This condition will be met if distance *z* satisfies,

$$z^3 >> \frac{\pi}{4\lambda}\left[(x-\xi)^2 + (y-\eta)^2\right]^2_{\max}$$

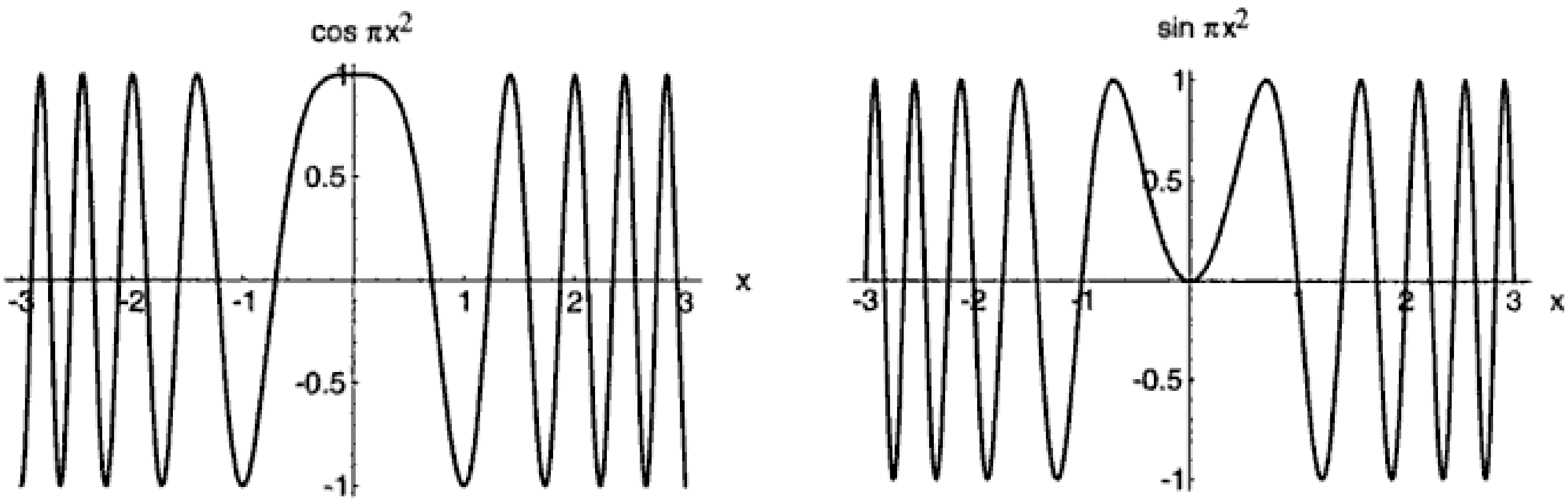
For a circular aperture of size 1 *cm*, a circular observation region of size 1 *cm*, & a wavelength of 0.5 *μm*, this condition would indicate that ***z* >> 25 *cm* for accuracy**. This condition is overly stringent & accuracy can be expected for much shorter distances.

Contd…

Expanding quadratic phase exponential into its real & imaginary parts, by dropping unit magnitude phasor e^{jkz},

$$\frac{1}{j\lambda z}\exp\left[j\frac{\pi}{\lambda z}\left(x^2+y^2\right)\right]=\frac{1}{j\lambda z}\left\{\cos\left[\frac{\pi}{\lambda z}\left(x^2+y^2\right)\right]+j\sin\left[\frac{\pi}{\lambda z}\left(x^2+y^2\right)\right]\right\}$$

Plot below shows one-dimensional quadratic phase cosine & sine functions.



Each of these functions has area $1/\sqrt{2}$. It can be shown that all of the unit area under 2-D quadratic phase exponential is contributed by 2-D sinusoidal term.

Contd…

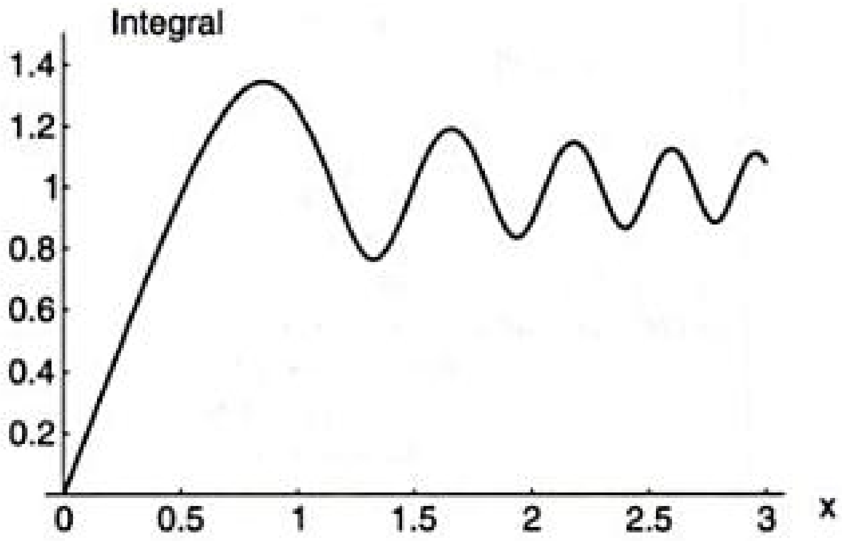
Magnitude of integral,

$$\left|\int_{-\infty}^{\infty}\exp(j\pi x^2)dx\right| = \left|\sqrt{2}C\left(\sqrt{2}X\right) + j\sqrt{2}S\left(\sqrt{2}X\right)\right|$$

Fresnel integrals,

$$C(z) = \int_0^z \cos\left(\frac{\pi t^2}{2}\right)dt$$

$$S(z) = \int_0^z \sin\left(\frac{\pi t^2}{2}\right)dt$$



Magnitude of integral of quadratic-phase exponential functions

Integral grows toward its asymptotic value of unity with increasing X. It first reaches unity at X = 0.5, & then oscillates about that value with diminishing fluctuations.

Conclusion: Major contributions to a convolution of this function with a second function that is smooth & slowly varying will come from range -2 < X < 2, due to the fact that outside this range rapid oscillations of integrand don’t yield a significant addition to total area.

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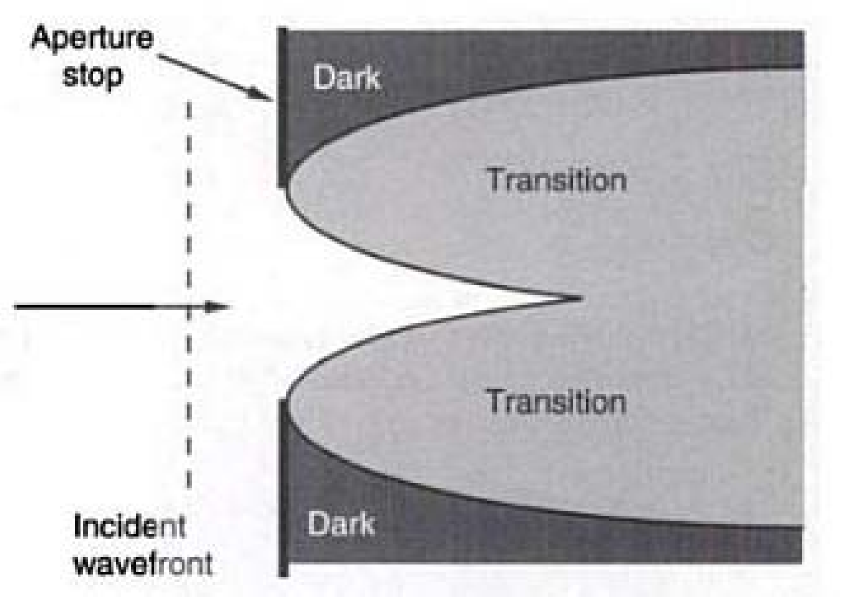
Contribution to convolution integral comes from a square in (ξ,η) plane, with width $4\sqrt{\lambda z}$ & centred on $(\xi = x,\ \eta = y)$. This square grows in size as distance z behind aperture increases.

When square lies entirely within open portion of aperture: Field observed at distance z is to a good approximation.

When square lies behind obstruction of aperture: Observation point lies in a region that is, to a good approximation, dark due to shadow of aperture.

When square bridges open & obstructed parts of aperture: Field observed is in transition region between light & dark.

For 1-D rectangular slit, boundaries between light region & transition region, & between dark region & transition region, can be shown to be **parabolas**.



Light, dark, & transition regions behind a rectangular slit aperture

Fresnel Approximation & Angular Spectrum

Transfer function of propagation through free space,

$$H(f_X, f_Y) = \begin{cases} \exp\left[j\dfrac{2\pi z}{\lambda}\sqrt{1-(\lambda f_X)^2-(\lambda f_Y)^2}\right] & \sqrt{f_X^2+f_Y^2} < \dfrac{1}{\lambda} \\ 0 & otherwise. \end{cases}$$

Fourier transforming the Fresnel diffraction impulse response, we find a transfer function valid for Fresnel diffraction,

$$H(f_X, f_Y) = \Im\left\{ \frac{e^{jkz}}{j\lambda z} \exp\left[\frac{j\pi}{\lambda z}\left(x^2+y^2\right)\right]\right\} = e^{jkz}\exp\left[-j\pi\lambda z\left(f_X^2+f_Y^2\right)\right]$$

Thus in Fresnel approximation, the general spatial phase dispersion representing propagation is reduced to a quadratic phase dispersion.

The factor e^{jkz} represents a constant phase delay suffered by all plane-wave components traveling between two parallel planes separated by normal distance *z*.

The 2nd term represents different phase delays suffered by plane-wave components traveling in different directions.

Contd…

Applying binomial expansion to the exponent,

$$\sqrt{1-(\lambda f_X)^2-(\lambda f_Y)^2} \approx 1-\frac{(\lambda f_X)^2}{2}-\frac{(\lambda f_Y)^2}{2}$$

which is valid provided

$$|\lambda f_X| << 1 \quad \& \quad |\lambda f_Y| << 1$$

Such restrictions on f_X & f_Y are simply restrictions to small angles.

From perspective of angular spectrum, Fresnel approximation is accurate provided only small angles of diffraction are involved. For this reason it is said that Fresnel approximations & paraxial approximation are equivalent.

Fraunhofer Approximation

$$U(x,y) = \frac{e^{jkz}}{j\lambda z} e^{j\frac{k}{2z}(x^2+y^2)} \int\int_{-\infty}^{\infty} \left\{ U(\xi,\eta) e^{j\frac{k}{2z}(\xi^2+\eta^2)} \right\} e^{-j\frac{2\pi}{\lambda z}(x\xi+y\eta)} d\xi d\eta \qquad (B)$$

In the region of Fresnel diffraction, observed field strength $U(x,y)$ can be found from a FT of product of aperture distribution $U(\xi,\eta)$ & a quadratic phase function $exp[j(k/2z)(\xi^2 + \eta^2)]$.

If in addition to Fresnel approximation the stronger (Fraunhofer) approximation

$$z >> \frac{k(\xi^2 + \eta^2)_{\max}}{2}$$

is satisfied, then quadratic phase factor in Eq. (B) is approximately unity over entire aperture, & observed field strength can be found directly from a Fourier transform of aperture itself.

Thus in the region of ***Fraunhofer diffraction*** (or in far field),

$$U(x,y) = \frac{e^{jkz} e^{j\frac{k}{2z}(x^2+y^2)}}{j\lambda z} \int\int_{-\infty}^{\infty} U(\xi,\eta) \exp\left[-j\frac{2\pi}{\lambda z}(x\xi + y\eta) \right] d\xi d\eta \qquad (C)$$

Contd…

Aside from multiplicative phase factors preceding the integral, Eq. (C) is simply Fourier transform of aperture distribution, evaluated at frequencies,

$$f_X = \frac{x}{\lambda z} \qquad \& \qquad f_Y = \frac{y}{\lambda z}$$

At optical frequencies, conditions required for validity of Fraunhofer approximation can be severe.

Ex. At a wavelength of 0.6 *μm* & an aperture width of 2.5 *cm*, the observation distance z must satisfy

$z \gg 1{,}600$ meters

An alternative, less stringent condition, known as "antenna designer's formula" states that for an aperture of linear dimension D, the Fraunhofer approximation will be valid provided

$$z > \frac{2D^2}{\lambda}$$

For this example, distance **z is still required to be larger than 2,000 meters**.

In addition, Fraunhofer diffraction patterns can be observed at distances much closer provided aperture is illuminated by a spherical wave converging toward observer, or if a positive lens is properly situated between observer & aperture.

Contd…

At first glance:

- there exists no transfer function that can be associated with Fraunhofer diffraction.
- secondary wavelets with parabolic surfaces (as implied by Fresnel approximation) no longer shift laterally in (x,y) plane with (ξ,η) particular point under consideration.
- when location of secondary source shifts, corresponding quadratic surface tilts in (x,y) plane by an amount that depends on location of secondary source.

Fraunhofer diffraction is only a special case of Fresnel diffraction, the transfer function remains valid throughout both Fresnel & Fraunhofer regimes.

Rectangular Aperture

Consider a rectangular aperture with an amplitude transmittance,

$$t_A(\xi,\eta) = rect\left(\frac{\xi}{2\omega_X}\right) rect\left(\frac{\eta}{2\omega_Y}\right)$$

Constants ω_X & ω_Y are half-widths of aperture in ξ & η directions. If aperture is illuminated by a unit-amplitude, normally incident, monochromatic plane wave, then field distribution across aperture is equal to transmittance function t_A. Fraunhofer diffraction pattern is,

$$U(x,y) = \frac{e^{jkz} e^{j\frac{k}{2z}(x^2+y^2)}}{j\lambda z} \Im\{U(\xi,\eta)\}\Big|_{f_X = x/\lambda z,\, f_Y = y/\lambda z}$$

$$\Im\{U(\xi,\eta)\} = A \sin c\left(\frac{2\omega_X x}{\lambda z}\right) \sin c\left(\frac{2\omega_Y y}{\lambda z}\right)$$

$$A = 4\omega_X \omega_Y, \quad area\ of\ aperture$$

$$U(x,y) = \frac{e^{jkz} e^{j\frac{k}{2z}(x^2+y^2)}}{j\lambda z} A \sin c\left(\frac{2\omega_X x}{\lambda z}\right) \sin c\left(\frac{2\omega_Y y}{\lambda z}\right)$$

$$I(x,y) = \frac{A^2}{\lambda^2 z^2} \sin c^2\left(\frac{2\omega_X x}{\lambda z}\right) \sin c^2\left(\frac{2\omega_Y y}{\lambda z}\right)$$

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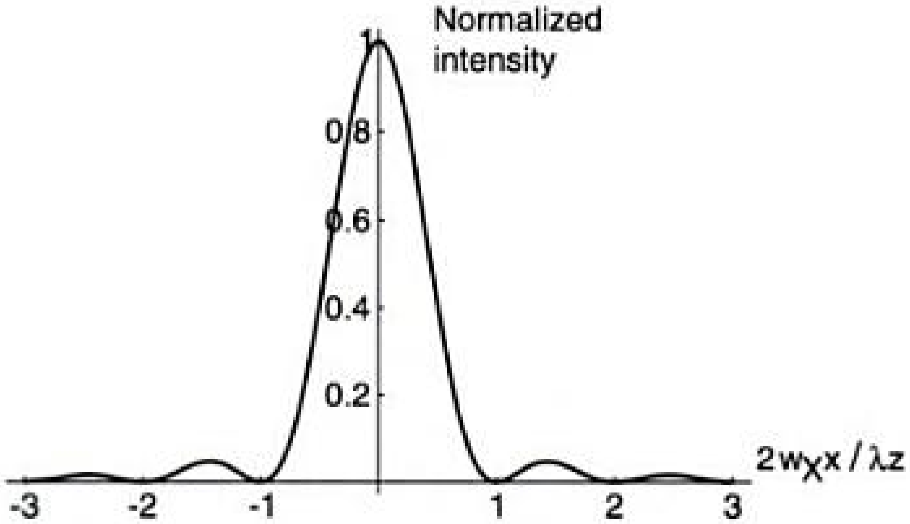
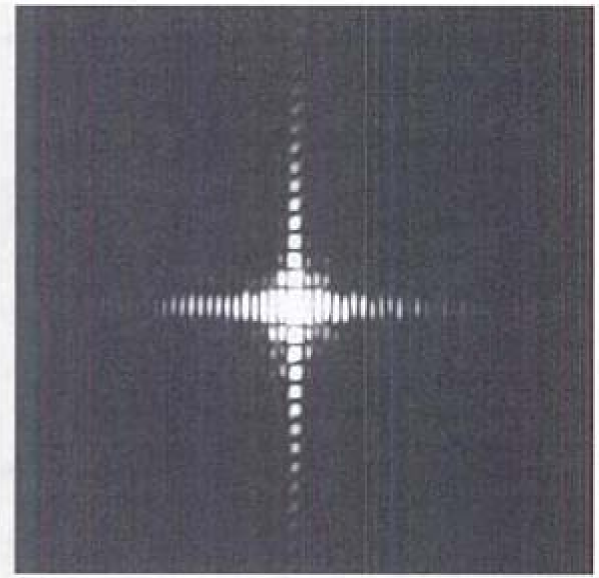


Cross section of Fraunhofer diffraction pattern of a rectangular aperture



Fraunhofer diffraction pattern of a rectangular aperture with a width ratio $\frac{\omega_X}{\omega_Y} = 2$

Width of main lobe (i.e., the distance between first two zeros) is,

$$\Delta x = \frac{\lambda z}{\omega_X}$$

Circular Aperture

Consider a circular diffracting aperture with radius of aperture be *ω*. If *q* is a radius coordinate in plane of aperture, then

$$t_A(q) = circ\left(\frac{q}{\omega}\right)$$

The circular symmetry of problem suggests that Fourier transform may be written as a Fourier-Bessel transform. Thus if *r* is radius coordinate in observation plane,

$$U(r) = \frac{e^{jkz}}{j\lambda z}\exp\left(j\frac{kr^2}{2z}\right)\mathrm{B}\{U(q)\}\Big|_{\rho=r/\lambda z}$$

$$q = \sqrt{\xi^2+\eta^2} \quad radius\ in\ aperture\ plane$$

$$\rho = \sqrt{f_X^2+f_Y^2} \quad radius\ in\ spatial\ frequency\ plane$$

For unit-amplitude, normally incident plane-wave illumination, field transmitted by aperture is equal to amplitude transmittance,

$$\mathrm{B}\left\{circ\left(\frac{q}{\omega}\right)\right\} = A\frac{J_1(2\pi\omega\rho)}{\pi\omega\rho}$$

$$A = \pi\omega^2$$

Contd…

Amplitude distribution in Fraunhofer diffraction pattern,

$$U(r) = e^{jkz} e^{j\frac{kr^2}{2z}} \frac{A}{j\lambda z}\left[2\frac{J_1(k\omega r/z)}{k\omega r/z}\right]$$

$$\Rightarrow \qquad I(r) = \left(\frac{A}{\lambda z}\right)^2 \left[2\frac{J_1(k\omega r/z)}{k\omega r/z}\right]^2$$

This intensity distribution is referred to as ***Airy pattern***, after G. B. Airy who first derived it.

Table shows values of Airy pattern at successive maxima & minima, from which it can be seen that width of the central lobe, measured along *x* or *y* axis is,

$$d = 1.22\frac{\lambda z}{\omega}$$

x	$\left[2\frac{J_1(\pi x)}{\pi x}\right]^2$	max, min
0	1	max
1.220	0	min
1.635	0.0175	max
2.233	0	min
2.679	0.0042	max
3.238	0	min
3.699	0.0016	max

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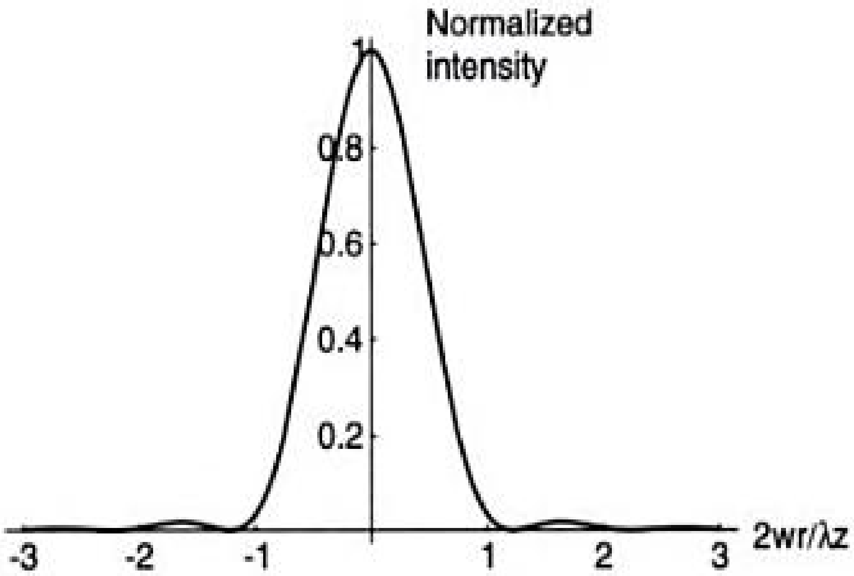
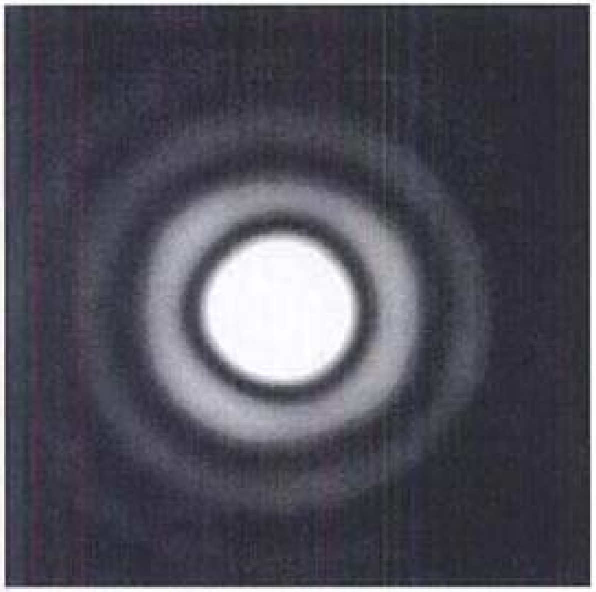


Cross-section of Fraunhofer diffraction pattern of a circular aperture



Fraunhofer diffraction pattern of a circular aperture