1 (a) Proton of kinetic energy 100 MeV.

$$m_p c^2 = 938.3 \text{ MeV}.$$

If $K.E. = T = \frac{p^2}{2m_p}$

$$\Rightarrow p = \sqrt{2m_p T}.$$

Now, $\lambda = \frac{h}{p} = 2\pi \frac{\hbar c}{pc} = \frac{2\pi \hbar c}{\sqrt{2m_p c^2 T}}$.

Now, $\hbar c = 197$ MeV-fm

$$\Rightarrow \lambda = \frac{2\pi \times 197}{\sqrt{2 \times 938.3 \times 100}} \text{ fm} \approx 2.857 \text{ fm}.$$

(Note: student may arrive at same result using SI units).

As this is comparable to size of nucleus, the wave properties play an important role in this case.

(b) A 100 gm bullet travelling at 1km/s.

$$\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{6.626 \times 10^{-34} \text{ Js}}{0.1 \text{ kg} \times 1000 \text{ m/s}}$$

$$= 6.626 \times 10^{-36} \text{ m}.$$

As this is negligible in comparison to the size of a typical bullet, the wave properties do not play an important role in this case.

2 (a) We have,

$\lambda_1 = 80\,nm, \quad T_1 = 11.390\,eV$

$\lambda_2 = 110\,nm, \quad T_2 = 7.154\,eV$

Now, $T_1 = \frac{hc}{\lambda_1} - W$ and $T_2 = \frac{hc}{\lambda_2} - W.$

$$\Rightarrow \quad T_1 - T_2 = \frac{hc(\lambda_2 - \lambda_1)}{\lambda_1 \lambda_2}.$$

$$\therefore \quad h = \frac{\lambda_1 \lambda_2 (T_1 - T_2)}{c(\lambda_2 - \lambda_1)}$$

$$= \frac{80 \times 110 \times 10^{-18} \times 4.236 \times 1.6 \times 10^{-19}}{3 \times 10^{8} \times 30 \times 10^{-9}}$$

$$= 6.626 \times 10^{-34}\,Js.$$

(b.) Work function

$$W = \frac{hc}{\lambda_1} - T_1 = \frac{6.626 \times 10^{-34} \times 3 \times 10^{8}}{80 \times 10^{-9}} - 11.390 \times 1.6 \times 10^{-19}\,J$$

$$= 24.8475 \times 10^{-19} - 18.224 \times 10^{-19}\,J$$

$$= 6.6235 \times 10^{-19}\,J \equiv 4.14\,eV.$$

Cut-off frequency $\nu_0 = \frac{W}{h} = \frac{6.6235 \times 10^{-19}}{6.626 \times 10^{-34}}\,Hz$

$$\approx 10^{15}\,Hz.$$

Cut-off wavelength $\lambda_0 = \frac{c}{\nu_0}$

$$= \frac{3 \times 10^{8}}{10^{15}}$$

$$= 300 \times 10^{-9}\,m$$

$$= 300\,nm.$$

3 (a.) $[x, p_x]$ $\qquad p_x = -i\hbar \frac{\partial}{\partial x}$

$\therefore [x, p_x]\Psi = [x, -i\hbar \frac{\partial}{\partial x}]\Psi = x(-i\hbar \frac{\partial}{\partial x})\Psi + i\hbar \frac{\partial}{\partial x}(x\Psi)$

$= -i\hbar x \frac{\partial \Psi}{\partial x} + i\hbar \frac{\partial x}{\partial x}\Psi + i\hbar x \frac{\partial}{\partial x}\Psi.$

$= i\hbar \Psi.$

$\Rightarrow [x, p_x] = i\hbar.$

(b.) $[x^2, p_x] = x[x, p_x] + [x, p_x]x = 2i\hbar x$
(using (a.))

(c.) $[x, p_x^2] = p_x[x, p_x] + [x, p_x]p_x = 2i\hbar p_x.$
(using (a.))

(d.) $[x^2, p_x^2] = x[x, p_x^2] + [x, p_x^2]x$

$= 2i\hbar x p_x + 2i\hbar p_x x$ (using (c.))

$= 2i\hbar (x p_x + p_x x)$

$= 2i\hbar (i\hbar + 2 p_x x)$

$= -2\hbar^2 + 4i\hbar p_x x.$

4. $\Psi(x,t) = \sin\left(\frac{\pi x}{a}\right) e^{\frac{iE_1 t}{\hbar}}$ for $-a \leq x \leq a$

$= 0$, otherwise.

Let, normalized wavefunction $\Psi_{norm}(x,t) = A\,\Psi(x,t)$

we have to determine A.

From normalization condition,

$$\int_{-\infty}^{\infty} |\Psi_{norm}(x,t)|^2 \, dx = 1.$$

$$\Rightarrow \quad |A|^2 \int_{-a}^{a} \sin^2\left(\frac{\pi x}{a}\right) dx = 1.$$

$$\Rightarrow \quad \frac{|A|^2}{2} \int_{-a}^{a} \left[1 - \cos\left(\frac{2\pi x}{a}\right)\right] dx = 1.$$

$$\Rightarrow \quad \frac{|A|^2}{2} \cdot 2a = 1. \qquad \therefore \quad A = \frac{1}{\sqrt{a}}.$$

$\therefore$ Normalized wavefunction

$$= \begin{cases} \frac{1}{\sqrt{a}} \sin\left(\frac{\pi x}{a}\right) e^{iE_1 t}, & -a \leq x \leq a \\ 0, & \text{otherwise} \end{cases}$$

5

We will assume that $I_1 > I_2 > I_3$.

Conservation of L^2 and E tells us that,

$$I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = L^2 \quad \text{--- (i)}$$

& $$I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = 2E \quad \text{--- (ii)}$$

are constants.

Eliminating ω_1 from (i) & (ii),

$$\Rightarrow I_2(I_2 - I_1)\omega_2^2 + I_3(I_3 - I_1)\omega_3^2 = L^2 - 2I_1E \quad \text{--- (iii)}$$

$\because$ $I_1 > I_2 > I_3$, both coeff. of ω_2^2 and ω_3^2 on L.h.s. of (iii) are negative.

$\therefore$ Multiplying throughout by -1, we get

$$A\omega_2^2 + B\omega_3^2 = C$$

$A, B > 0$ (hence, $C > 0$).

$\Rightarrow$ Ellipse in $\omega_2 - \omega_3$ plane.

Hence, ω_2 and ω_3 are bounded.

Similarly, if we eliminate ω_3 from eq. (i) and (ii), we obtain an ellipse in $\omega_1 - \omega_2$ plane.

However, if we eliminate ω_2, we obtain,

$$I_1(I_1 - I_2)\omega_1^2 + I_3(I_3 - I_2)\omega_3^2 = L^2 - 2I_2E.$$

The coeff. of ω_1^2 and ω_3^2 on L.h.s. are now of opposite signs.

$\therefore$ We have a hyperbola in $\omega_1 - \omega_3$ plane.

Thus, ω_1 and ω_3 are free to become large!

$\Rightarrow$ "Unstable" to perturbations.