Indian Institute of Technology Patna

MA-102 (Mathematics II)

B.Tech. I year (Spring Semester: 2011-12)

Mid Semester Examination

Maximum Marks: 30 Total Time: 2 Hours

Attempt all questions:

1. (a) Let $S = \{\alpha, \beta, \gamma\}$ and $T = \{\alpha, \beta, \alpha+\beta, \beta+\gamma\}$ be two subsets of a real vector space V. show that $L(S) = L(T)$.

   (b) Check whether the linear span of X-axis and the plane $x+y=0$ in $V_3(\mathbb{R})$ is a subspace of $V_3(\mathbb{R})$ or not. If yes, then find a basis of the above linear span.

   (c) Let S be a vector space of all $n \times n$ real symmetric matrices and T be a vector space of all $n \times n$ real skew-symmetric matrices. Prove that $dim(S) = \frac{n(n+1)}{2}$ and $dim(T) = \frac{n(n-1)}{2}$. Hence prove that the space $M_{n\times n}(\mathbb{R})$ is the direct sum of S and T.

   (d) Given $S_1 = \{(1,2,3), (0,1,2), (3,2,1)\}$ and $S_2 = \{(1,-2,3), (-1,1,-2), (1,-3,4)\}$. Determine the basis and dimension of $[S_1] \bigcap [S_2]$.

   (e) Is set $\{1, \sin x, \sin^2 x, \cos^2 x\}$ a basis of $V = C[-\pi, \pi]$? Justify your answer.

   [1 + 2 + 3 + 3 + 1]

2. (a)Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a map defined by $T(a,b,c) = (a+b+c, -a-c, b)$. Show that T is a linear transformation. Also find range space, null space, rank and nullity of T.

   (b) Let W be the subspace of $\mathbb{R}^6$ composed of all vectors $[a_1, ..., a_6]^t$ satisfying $\sum_{i=1}^{6} a_i = 0$. Does there exist a one-one mapping from W to $\mathbb{R}^4$?

   (c) Find a linear transformation, $T : \mathcal{P}_3[x] \to \mathbb{R}^3$, whose matrix representation is

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 1 \\ 5 & 4 & 1 & -1 \end{bmatrix}$$

   with respect to bases $\{1;\ 1+x^2;\ x+x^3;\ 1+x+x^2\}$ and $\{(1,0,1),\ (2,4,5),\ (0,0,1)\}$.

(d) Find the row-reduced echelon form and hence find rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}.$$

[3 + 2 + 2 + 3]

3. (a)Determine the conditions for which the system
$x + y + z = 1$
$x + 2y - z = b$
$5x + 7y + az = b^2$
admits
(i) a unique solution; (ii) no solution; and (iii) infinitely many solutions.

(b) Let v_0 be a particular solution of $AX = B$, and let W be the general solution of $AX = 0$. Then $U = v_0 + W = \{v_0 + w : w \in W\}$ is the general solution of $AX = B$. Give the geometrical interpretation of the above statement in $\mathbb{R}^3$.

(c) Diagonalize the matrix

$$A = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{bmatrix}.$$

(d) Let A and P be both $n \times n$ matrices and P be a nonsingular matrix. Then show that A and $P^{-1}AP$ have the same eigenvalues.

(e) State Cayley-Hamilton theorem and using this find A^{-1}, where

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}.$$

[3+1+3+1.5+1.5]