Dear Students:

we were discussing 2-dimensional random variables. So far, joint PMF, marginal PMFs, Conditional PMFs and some problems are taken care for discrete case similarly for Continuous case: joint PDF, marginal PDFs, conditional PDFs are discussed and illustrated through one example. Joint CDF for 2-dimensional case is also studied.

Next we are going to learn joint Expectation conditional expectation, conditional variance and also joint MGF will be discussed.

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joint Expectation of a 2-dimensional RV.

Let (X,Y) be a two dimensional RV and $g(X,Y)$ be a function of (X,Y). Then we have

$$E(g(X,Y)) = \sum_{x_i}\sum_{y_j} g(x_i, y_j)\, p_{X,Y}(x_i, y_j) \quad (x_i, y_j) \in R_{X \times Y}$$

when (X,Y) is a 2-dim. discrete RV.

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy,$$

when (X,Y) is a 2-dim. Continuous RV.

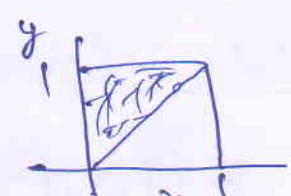
Here $g(X,Y)$ can be XY, $X+Y$, $\sqrt{XY}$, X^2+Y^2, e^{X+Y}, X, Y, and so on.

For example: If $g(X,Y) = XY$ then

$$E(XY) = \sum_{x_i}\sum_{y_j} x_i y_j\, p_{X,Y}(x_i, y_j)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x y\, f(x,y)\, dx\, dy.$$

Ex: Let $f_{X,Y}(x,y) = 2, \quad 0 \le x \le y \le 1$
$= 0, \quad$ otherwise.



Compute $E(XY)$ for this joint PDF.

$\Rightarrow$ $E(XY) = 2\int_{x=0}^{1}\int_{y=x}^{1} xy\, dy\, dx = 2\int_0^1 x \cdot \left(\frac{y^2}{2}\right)_x^1 dx$

$= \int_0^1 x(1-x^2)\, dx = \frac{1}{\ }$

Ex: $E(XY) = \sum_{y=0}^{1} \sum_{x=0}^{1} xy \; p_{X,Y}(x,y)$

X \ Y	0	1
0	1/8	3/8
1	2/8	2/8

$= 0 \cdot$ (Check!)

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Let us now define conditional Expectations.

As earlier assume that (X,Y) is a 2-dim rv with some probability distribution.

The conditional expectation of X given that $Y=y$ is given by

$$E[X \mid Y=y] \longrightarrow \sum_{x \in R_X} x_i \, p_{X|y}(x_i \mid y) \quad \text{(discrete case)}.$$

$$\downarrow$$

$$\int_{-\infty}^{\infty} x \cdot f_{X|y}(x \mid y)\, dx \quad \text{(Continuous case)}.$$

Similarly, conditional expectation of Y given that $X=x$ is

$$E[Y \mid X=x_i] = \sum_{y_j \in R_Y} y_j \, p_{Y|x_i}(y_j \mid x_i) \quad \text{(Discrete case)}$$

$$E(Y \mid X=x) = \int_{-\infty}^{\infty} y \, f_{y|x}(y \mid x)\, dy \quad \text{(continuous case)}$$

Note: In general if $g(x)$ & $h(y)$ are function then

$$E[g(X) \mid y] \longrightarrow \sum_{x_i \in R_X} g(x_i)\, p_{X_i|y}(x_i \mid y)$$

$$\downarrow$$

$$\int^{\infty} g(x)\, f(x \mid y)\, dx$$

$g(x)$ can be like $g(x) = X, X^2, |X|$, e^X, $\ln X$ and

$$E(g(Y)|x_i) \longrightarrow \sum_{y_j \in R_Y} g(y_j) \cdot p_{Y|x_i}(y_j|x_i)$$

$$\downarrow$$

$$\int_{-\infty}^{\infty} g(y)\, f_{Y|x}(y|x)\, dx.$$

Ex: $f_{X,Y}(x,y) = 2, \quad 0 \le x \le y \le 1$
$\qquad = 0, \quad$ otherwise.

Find $E(X|y)$ and $E(Y|x)$.

$\Rightarrow$ Recall from previous lectures that

$f_{X|y}(x|y) = \frac{1}{y}, \quad 0 < x < y$

$\therefore E(X|y) = \int_{-\infty}^{\infty} x \cdot f_{X|y}(x|y)\, dx$

$= \int_0^y \frac{x}{y}\, dx = \frac{y}{2}.$

$f_{Y|x}(y|x) = \frac{1}{1-x}, \quad x < y < 1$

$E(Y|x) = \int_x^1 y\, f_{Y|x}(y|x)\, dy$

$= \frac{1}{1-x} \cdot \int_x^1 y\, dy$

$= \frac{1+x}{2}.$

In place of $E(X|y)$ we can ask find $E(X|Y=1/2)$, $E(X|Y=1/7)$ and so on. $\rightarrow$ similarly here also.

A nice result about Conditional Expectation presented below.

<u>Theorem</u>: Let (X,Y) be jointly distributed R.V. Then

$$E(X) = E\{E(X|Y)\} \qquad \left[\begin{array}{l} \text{likewise} \\ E(Y) = E\,E(Y|X). \end{array} \right.$$

<u>Pf</u>: (for continuous case)

$$E[E(X|Y)] = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x\, f_{X|y}(x|y)\, dx \right\} f(y)\, dy$$

$$= \int_{-\infty}^{\infty} x \left\{ \int_{-\infty}^{\infty} f(x,y)\, dy \right\} dx = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = E(X)$$

Ex: verif $E(X) = E[E(X|Y)]$ for previous problem.

$\Rightarrow$ $f(x,y) = 2,\ 0<x<y<1.$

$f(x) = 2(1-x),\ 0\le x\le 1,$ | $f(y) = 2y,\ 0\le y\le 1$

$f_{X|Y}(x|y) = \frac{1}{y},\ 0<x<y,$ | $f_{Y|X}(y|x) = \frac{1}{1-x},\ x<y<1$

$\therefore\ E(X) = \frac{1}{3}.$

We just computed $E(X|Y) = \frac{Y}{2}.$

Now, $E[E(X|Y)] = E\left(\frac{Y}{2}\right) = \frac{1}{2}\int_0^1 y\cdot 2y\,dy$

$= \frac{1}{3}.$

$\therefore\ \boxed{E(X) = E\,E(X|Y) = 1/3}$

Note: Previous theorem is true in general also; that is, consider $g(x,y)$ then

$$E[g(X,Y)] = E\left[E\{g(X,Y)|Y\}\right]$$
$$\| $$
$$E\{E[g(X,Y)|X]\}$$

Let us look at Conditional Variance.

$$V(X|Y=y) = E(X^2|y) - (E(X|y))^2$$

$$V(Y|X=x) = E(Y^2|x) - (E(Y|x))^2$$

Ex: From previous problem.

$$V(X|y) = E(X^2|y) - (E(X|y))^2 = E(X^2|y) - \frac{y^2}{4}$$

Now $E(X^2|y) = \int_0^y x^2\cdot\frac{1}{y}\,dx = \frac{y^2}{3}.$

$\therefore\ V(X|y) = \frac{y^2}{\ } - \frac{y^2}{\ } \quad \frac{y^2}{\ }$