Bivariate Normal (Part II).

Dear Students

In the last class we introduced ~~the~~ two-dimensional normal distribution. Then marginal prob. density functions were described. Further both the conditional pdfs were obtained. A nice iterated expectation result was proved. Based on this covariance and correlation coeff^n were also evaluated.

Today we try to see some more properties of two dimensional normal dist^n.

First we compute joint MGF of two-dimensional normal dist^n. Recall the Def^n of joint MGF for a jointly distributed random variables (X,Y). It is

$$M_{X,Y}(t_1, t_2) = E\left(e^{t_1 X + t_2 Y}\right) \quad \text{--- (1)}$$

[If you have time try to simplify the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 x + t_2 y} f_{X,Y}(x,y)\,dx\,dy$$

to get the desired result where

$$f_{X,Y}(x,y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho).$$]

We, however, simplify Eq^n ① using the iterated expectation result. Thus we have

$$M_{X,Y}(t_1,t_2) = E\left(e^{t_1X+t_2Y}\right)$$

$$= E\left[E\left(e^{t_1X+t_2Y} \mid Y\right)\right]$$

$$= E\left[e^{t_2Y}\, E\left(e^{t_1X} \mid Y\right)\right] \quad ---②$$

Now what about $E\left(e^{t_1X} \mid Y\right)$??

This is like finding moment generating function (MGF) of the probability distribution $X|Y$. Under the given framework we have

$$X \mid y \sim N\left(\mu_1 + \rho\sigma_1\left(\frac{y-\mu_2}{\sigma_2}\right),\ \sigma_1^2(1-\rho^2)\right)$$

So, $E\left(e^{t_1X} \mid Y\right) = e^{\left[\mu_1 + \rho\sigma_1\left(\frac{y-\mu_2}{\sigma_2}\right)\right]t_1 + \frac{1}{2}\sigma_1^2(1-\rho^2)t_1^2}$ ③

[NOTE : If $X \sim N(\mu, \sigma^2)$ then $E\left(e^{tX}\right) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$]

Next utilize Equation ③ in Equation ② to get

$$M_{X,Y}(t_1,t_2) = E\left[e^{t_2Y}\left\{e^{\left[\mu_1 + \rho\sigma_1\left(\frac{Y-\mu_2}{\sigma_2}\right)\right]t_1 + \frac{1}{2}\sigma_1^2(1-\rho^2)t_1^2}\right\}\right]$$

(see now it is function of Y only).

$$M_{X,Y}(t_1,t_2) = e^{[\mu_1 t_1 - \rho\frac{\sigma_1}{\sigma_2}\mu_2 t_1 + \frac{1}{2}\sigma_1^2(1-\rho^2)t_1^2]} \cdot$$

$$E\left[e^{(t_2+\rho\frac{\sigma_1}{\sigma_2}t_1)Y}\right] \longrightarrow \text{MGF of } Y$$

$$= e^{[\mu_1 t_1 - \rho\frac{\sigma_1}{\sigma_2}\mu_2 t_1 + \frac{1}{2}\sigma_1^2(1-\rho^2)t_1^2]} \cdot M_Y\left(t_2+\rho\frac{\sigma_1}{\sigma_2}t_1\right)$$

(recall $Y \sim N(\mu_2, \sigma_2^2)$ $\therefore M_Y(t) = e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2}$)

$$= e^{\mu_1 t_1 - \rho\frac{\sigma_1}{\sigma_2}\mu_2 t_1 + \frac{1}{2}\sigma_1^2(1-\rho^2)t_1^2} \cdot$$

$$e^{\mu_2(t_2+\rho\frac{\sigma_1}{\sigma_2}t_1) + \frac{1}{2}\sigma_2^2(t_2+\rho\frac{\sigma_1}{\sigma_2}t_1)^2}$$

After simple algebraic manipulation of above two exponential terms we get the joint MGF as

$$M_{X,Y}(t_1,t_2) = e^{\mu_1 t_1 + \mu_2 t_2 + \frac{1}{2}\sigma_1^2 t_1^2 + \frac{1}{2}\sigma_2^2 t_2^2 + \rho\sigma_1\sigma_2 t_1 t_2}.$$

So what is the result -

If $(X,Y) \sim BVN(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,\rho)$ then

$$\boxed{M_{X,Y}(t_1,t_2) = e^{\mu_1 t_1 + \mu_2 t_2 + \frac{1}{2}\sigma_1^2 t_1^2 + \frac{1}{2}\sigma_2^2 t_2^2 + \rho\sigma_1\sigma_2 t_1 t_2}} \quad (4)$$

Another characterizing result for BVN

<u>Result</u>: Let $(X,Y) \sim BVN(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,\rho)$ then X and Y are independent if and only if $\rho = 0$.

Proof: If X and Y are independent then $\rho = 0$ (this is already known).

Consider the other case, that is, suppose $\rho = 0$ then show that X & Y independent.

When $\rho = 0$ joint MGF is written as

$$M_{X,Y}(t_1, t_2) = e^{\mu_1 t_1 + \mu_2 t_2 + \frac{1}{2}\sigma_1^2 t_1^2 + \frac{1}{2}\sigma_2^2 t_2^2 + \rho\sigma_1\sigma_2 t_1 t_2}$$

$$= e^{\mu_1 t_1 + \frac{1}{2}\sigma_1^2 t_1^2 + \mu_2 t_2 + \frac{1}{2}\sigma_2^2 t_2^2}$$

$$= e^{\mu_1 t_1 + \frac{1}{2}\sigma_1^2 t_1^2} \cdot e^{\mu_2 t_2 + \frac{1}{2}\sigma_2^2 t_2^2}$$

$$= M_X(t_1)\, M_Y(t_2).$$

Thus when $\rho = 0$ we see that joint MGF factors into marginal MGF. So X & Y independent.

[⊛ $\rho = 0 \Rightarrow f_{X,Y}(x, y) = f_X(x)\, f_Y(y)$
hence also X & Y independent. Check!]

Another result is presented below.

Result: (X, Y) is jointly distributed as $BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ if and only if for some given constant a and b,

$$aX + bY \sim N(a\mu_1 + b\mu_2,\ a^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\rho\sigma_1\sigma_2)$$

Proof: Part(i): Let us assume that

$$(X,Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho) \quad \text{——— } (**)$$

we need to show that $aX+bY \sim N(a\mu_1+b\mu_2, a^2\sigma_1^2+b^2\sigma_2^2 + 2ab\rho\sigma_1\sigma_2)$

We prove the required result using MGF technique. So let us compute MGF of variable of interest $aX+bY$.

$$M_{aX+bY}(t) = E\left[e^{(a\cancel{X}+bY)t}\right]$$

$$= E\left[e^{(at)X+(bt)Y}\right]$$

How to simplify last expectation. You have to use given information $(**)$ to simplify it. So to this end let us proceed as

$$M_{aX+bY}(t) = E\left[e^{(at)\cancel{X}+(bt)Y}\right]$$

$$= \underbrace{M_{X,Y}(at, bt)}$$

this is joint MGF of (X,Y) for which all information given in Eqn $(**)$. See also Equation ④ on page ③.

$M_{ax+by}(t) = M_{x,y}(at, bt)$

$= e^{\mu_1 at + bt\mu_2 + \frac{1}{2}\sigma_1^2 a^2 t^2 + \frac{1}{2}\sigma_2^2 b^2 t^2 + \rho\sigma_1\sigma_2 ab t^2}$

$= e^{(a\mu_1 + b\mu_2)t + \frac{1}{2}t^2(a^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\rho\sigma_1\sigma_2)}$

Make a guess for the last expression:

It is MGF of normal random variable with mean $a\mu_1 + b\mu_2$ and variance $a^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\rho\sigma_1\sigma_2$

Since MGF uniquely describe a given prob. distribution so

$ax + by \sim N(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\rho\sigma_1\sigma_2)$

(so part (i) is proved).

Let us prove other part:

part (ii) Now assume that

$ax + by \sim N(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\rho\sigma_1\sigma_2)$ — 5

and try to prove that (X, Y) is jointly distributed as $BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$.

so how to proceed ~~for~~ to prove part (ii) ??

In this case we try to comput joint MGF of desired two-dimensional rv (X,Y) using the information given in Equation (5).

so,

$$M_{X,Y}(t_1,t_2) = E\left(e^{t_1X+t_2Y}\right) \quad \left\{\begin{array}{l}\text{by Def}^n\text{ of}\\ \text{joint MGF}\end{array}\right\}.$$

how to compute this Expectation by using the given information in Eq^n (5).

$$M_{X,Y}(t_1,t_2) = E\left(e^{t_1X+t_2Y}\right)$$

$$= M_{t_1X+t_2Y}(1) \quad \left\{\begin{array}{l}\text{its like find MGF}\\ \text{of } t_1X+t_2Y \text{ at}\\ \text{parameter point } 1.\end{array}\right\}$$

(***)

[For your reference I provide MGF for Eqn (5)

$$M_{aX+bY}(t) = e^{(a\mu_1+b\mu_2)t+\frac{1}{2}t^2(a^2\sigma_1^2+b^2\sigma_2^2+2ab\rho\sigma_1\sigma_2)}$$]

Using this MGF, Eqn (***) is simplified as

$$M_{X,Y}(t_1,t_2) = M_{t_1X+t_2Y}(1)$$

$$= e^{t_1\mu_1+t_2\mu_2+\frac{1}{2}t_1^2\sigma_1^2+\frac{1}{2}t_2^2\sigma_2^2+\rho\sigma_1\sigma_2t_1t_2}$$

Can you recall it if not see Equation (4) page (3) this resembles with joint PDF of a BVN $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ distribution.

So we have shown that

$$\boxed{(X,Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)}.$$

Thus theorem is proved for both parts.

For Your Information:

We have the following result:

If $(X,Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ then $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$.

What About Converse, that is, if marginal distribution of X is normal and marginal distn of Y is normal, then 'Can we say (X,Y) is jointly distributed as two-dimensional normal'.

Answer is: In general it is not true.

One example is presented on the next page to support this.

Ex: ((X, Y) may not be bivariate normal still marginals can be normal).

Consider

$$f_{X,Y}(x,y) = \frac{1}{2}\left[\frac{1}{2\pi(1-\rho^2)^{1/2}} e^{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)} + \frac{1}{2\pi(1-\rho^2)^{1/2}} e^{-\frac{1}{2(1-\rho^2)}(x^2+2\rho xy+y^2)}\right]$$

$$-\infty < x < \infty$$
$$-\infty < y < \infty, \quad -1 < \rho < 1$$

Try to compute $f_X(x)$ and $f_Y(y)$ for this joint pdf to arrive at following answers.

$$X \sim N(0,1)$$
$$Y \sim N(0,1)$$

So marginal pdfs of X & Y both are one dimensional normal but joint pdf $f_{X,Y}(x,y)$ is not bivariate normal.