Indian Institute of Technology Patna
MA102: Mathematics II
End Semester Exam

Time: 3hrs Max. Marks: 50

Note: There are Eight questions in this paper. Answer all questions. Give precise and brief answer. Standard formulae may be used.

Q1. a. Define an inner product space and show that the vector space $V_2(\mathbb{R})$ is an inner product space under the inner product defined by

$$< \alpha, \beta >= a_1b_1 - a_2b_1 - a_1b_2 + 4a_2b_2,$$

where $\alpha = (a_1, a_2)$ and $\beta = (b_1, b_2) \in V_2(\mathbb{R})$. [3]

b. Apply Gram-Schmidt orthogonalization process to the vectors $\beta_1 = (1, 0, 1)$, $\beta_2 = (1, 0, -1)$, $\beta_3 = (0, 3, 4)$ to obtain an orthonormal basis for $V_3(\mathbb{R})$ with the standard inner product. [3]

c. Let f be the bilinear form on $V_2(\mathbb{R})$ defined by

$$f(\ (x_1, y_1),\ (x_2, y_2)\ ) = x_1y_1 + x_2y_2.$$

Find the matrix of f in the ordered basis $B = \{\ (1, -1), (1, 1)\ \}$ of $V_2(\mathbb{R})$. [2]

d. Is $\mathbf{4x^2 + 9y^2 + 2z^2 + 8yz + 6zx + 6xy}$ a positive definite? Justify. [2]

Q2 Solve the following differential equations: [1+3+3]

a. $\dfrac{dy}{dx} = e^{2x-y} + x^3e^{-y}$.

b. $\dfrac{dy}{dx} = \dfrac{x + 2y + 1}{2x + y - 1}$.

c. $x^2y''' + 4xy'' + 5y' = 0$.

Q3 (a) Reduce the following differential equation into **linear form** and hence solve it: [2]

$$\frac{dy}{dx} + \frac{y \ln y}{x - \ln y} = 0.$$

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(b) Show that the differential equation $(3y^2-x)+2y(y^2-3x)\frac{dy}{dx}=0$ admits an integrating factor which is a function of $(x+y^2)$. Hence solve the differential equation. [4]

Q4. a. Determine the value of m such that e^{mx} is a solution of following differential equation:

$$(2x+1)y''-4(x+1)y'+4y=0.$$

Hence find the general solution. [4]

b. Use operator method to solve $y''-2y'+y=xe^x\sin x$. [2]

Q5. Find **particular solution** of the following equations: [2+2]

a. $\frac{d^2y}{dx^2}-p^2y=\sinh px$, here sinh is hyperbolic sine function.

b. $x^2y''-3xy'+2y=\ln x$.

Q6. Find a **Frobenius series solution** of the following differential equation:

$$x^2y''+x(x-1)y'+(1-x)y=0.$$

Does there exist another LI Frobenius series solution in this case? If yes, find it. If no, then find another solution otherwise. [6]

Q7. a. Prove that [2]

$$\int_{-1}^{1}P_m(x)P_n(x)dx=0,\quad m\neq n.$$

b. Show that when p is an integer, then $J_{-p}(x)=(-1)^pJ_p(x)$. [1]

c. Prove that $J_0'(x)=-J_1(x)$. [2]

Q8. Solve the following system of differential equations:

$$\begin{cases}\frac{dx}{dt} &= -3x+y,\\ \frac{dy}{dt} &= x-3y.\end{cases}$$

Also determine the type of its critical point (origin) and sketch some of the solution trajectories. [6]
