Indian Institute of Technology Patna
MA201- (Partial Differential Equation) July-November 2019

Tutorial - 3 **Instructor: Dr. Pratibhamoy Das**

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1. Classify (Elliptic/Parabolic/Hyperbolic) the following second order PDEs:

   (i) $u_{xx}+4u_{xy}+4u_{yy}-12u_y+7u = x^2+y^2$, (ii) $(x+1)u_{xx}-2(x+2)u_{xy}+(x+3)u_{yy} = 0$, (iii) $yu_{xx} + (x + y)u_{xy} + xu_{yy} = 0$.

2. Reduce the following equations to canonical form:
   (i) $u_{xx} + 2u_{xy} - 3u_{yy} = 0$, (ii) $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$.

3. Reduce the following equations to canonical form and then solve:
   (i) $u_{xx} + 4u_{xy} + 3u_{yy} = 0$, (ii) $u_{xx} - 12u_{xy} + 9u_{yy} = e^{3x+2y}$,
   (iii) $u_{xx} + 2u_{xy} + u_{yy} = x^2 + 3\sin(x - 4y)$.

4. Find the D'Alembert solution of one-dimensional Wave equation with the following initial conditions:

   (i) $u(x, 0) = \sin x$, $u_t(x, 0) = 0$, (ii) $u(x, 0) = \sin x$, $u_t(x, 0) = \cos x$.

5. Find the solution of the Wave equation $u_{tt} = c^2u_{xx}$, $x \in \mathbb{R}$, $t > 0$, which satisfies the conditions $u(x, 0) = (1 + x^2)^{-1}$ and $u_t(x, 0) = \sin x$, for all $x \in \mathbb{R}$.

6. Solve the Heat diffusion problem:

$$\begin{cases} u_t = \alpha u_{xx}, & 0 < x < 1,\ t > 0, \\ u(0, t) = 0, & u(1, t) + u_x(1, t) = 0,\ t > 0, \\ u(x, 0) = f(x), & 0 < x < 1. \end{cases}$$

7. Consider a string which has been stretched to infinity in both directions with the initial displacement $\phi(x) = 1/(1+4x^2)$ and released from rest. Find its subsequent motion as a function of x and t.

8. Find the temperature $u(x,t)$ in a bar of length l which is perfectly insulated, also at both ends $x=0$ and $x=l$ such that $u_x(0,t)=u_x(l,t)=0$, and the initial temperature in the bar is $u(x,0)=f(x)$. Also, find the temperature in the bar, given $l=\pi$, $\alpha=1$, for (i) $f(x)=1$, (ii) $f(x)=x^2$.

9. A thin rod of length lcm long, with insulated sides, has its ends A and B kept at $a^o\,C$ and $b^o\,C$ respectively until steady state conditions prevail. The temperature at A is then suddenly raised to c^oC and at the same time, at B is lowered to d^oC. Find the temperature distribution $u(x,t)$ subsequently. For modeling purpose, take the end A as origin.

10. Assume that a thin membrane is stretched over a rectangular frame of length a, breadth b where the edges are held fixed and initial shape of the membrane is governed by the function $f(x,y)$. Now, the membrane is set to vibrate by displacing it vertically and releasing it, with the initial velocity $g(x,y)$. As you know that the vibration of the membrane satisfies the two dimensional Wave equation. There the governing PDE with initial and boundary conditions will be:

    PDE : $u_{tt}=c^2(u_{xx}+u_{yy}),\ \ 0<x<a,\ \ 0<y<b$ and $0<t<\infty$,
    BCs : $u(0,y,t)=0,\ \ u(a,y,t)=0,\ \ u(x,b,t)=0\quad 0<t<\infty$,
    ICs : $u(x,y,0)=f(x,y)$ and $u_t(x,y,0)=g(x,y),\ \ 0<x<a,\ \ 0<y<b$.

    Find the solution $u(x,y,t)$ using method of separation of variables.