Tutorial:

(Function of a variable).

EX(1): Let X be a random variable such that $X \sim \exp(\lambda)$. Consider the transformation $Y = X^{\frac{1}{\nu}}$. Find probability density function of the variable Y, where $\gamma > 0$.

Solution: Let us recall the fundamental result as stated below:

"Suppose that X is a continuous random variable with pdf $f_X(x)$. Let $Y = g(X)$ be a function of X such that $g(x)$ is either strictly increasing or strictly decreasing. Then pdf of Y is given by

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d g^{-1}(y)}{dy} \right|.$$

This is a very useful result for finding pdf of transformed variable under the given framework."

Let us solve the given problem.

We are given $X \sim \exp(\lambda)$. So we have

$$f_X(x) = \frac{1}{\lambda} e^{-x/\lambda}, \quad 0<x<\infty, \ \lambda>0$$

given transformation is $Y = g(X) = X^{1/\gamma}$

$$g'(x) = \frac{1}{\gamma} x^{\frac{1}{\gamma}-1} > 0$$

so given transformation is strictly increasing. we can apply fundamental result as stated earlier. The inverse function

$$x = g^{-1}(y) = y^{\gamma}..$$

$$\therefore \frac{d}{dy} g^{-1}(y) = \gamma y^{\gamma-1}$$

Thus we have

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| = f_X(y^{\gamma}) \cdot \gamma y^{\gamma-1}$$

$$= \frac{1}{\lambda} e^{-y^{\gamma}/\lambda} \cdot \gamma y^{\gamma-1}, \quad \underline{0<y<\infty}$$

$$\boxed{\therefore f_Y(y) = \frac{\gamma}{\lambda} y^{\gamma-1} e^{-y^{\gamma}/\lambda}, \quad 0<y<\infty, \ \gamma>0, \lambda>0}$$

[This is known as Weibull distribution (Y ~ Weibull(γ, λ)).]

This probability model is named after

Swedish Mathematician W. Weibull, who was the first person to use it as a prob. model for describing strength of materials.

This distribution is many different applications of life time analysis in Reliability theory.

Now try to solve following problems.

(ii) Let $X \sim \exp(\lambda)$. Consider the transformation $Y = \left(\frac{2X}{\lambda}\right)^{1/2}$, $\lambda > 0$. Find the pdf of random variable Y.

(Name of pdf of Y is Rayleigh distn useful in communication systems).

(iii) Let $X \sim \exp(1)$ then find pdf of $Y = \alpha - \gamma \log X$, $-\infty < \alpha < \infty$, $0 < \gamma < \infty$.

Your answer for pdf Y is Gumbel(α, γ) distribution.

Ex: Let X follows a gamma distribution $G(\frac{3}{2}, b)$ then find pdf of $Y = \left(\frac{X}{b}\right)^{1/2}$ where $b > 0$.

[pdf of Y is known as Maxwell distⁿ.]

Problem: Let $f_X(x) = 30x^2(1-x)^2$, $0 < x < 1$. Consider the transformation $Y = X^2$. Find pdf of Y.

$\Rightarrow$ $Y = X^2$ $\therefore g(x) = x^2$

$$g'(x) = 2x > 0 \quad \forall \; 0 < x < 1$$

So given transformation is ~~monot~~ strictly increasing. Then

$$g^{-1}(y) = \sqrt{y} \quad \therefore \frac{d}{dy} g^{-1}(y) = \frac{1}{2\sqrt{y}}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|, \quad 0 < y < 1$$

$$= f_X(\sqrt{y}) \frac{1}{2\sqrt{y}}, \quad 0 < y < 1$$

$$= 30\, \cancel{y}\, (1-\sqrt{y})^2 \cdot \frac{1}{2\sqrt{y}}, \quad 0 < y < 1$$

$$= 15\sqrt{y}\,(1-\sqrt{y})^2, \quad 0 < y < 1.$$

Ex: Let X has pdf given by

$$f_X(x) = \frac{1}{2} e^{-|x|}, \quad -\infty < x < \infty$$

consider the transformation $Y = |X|^3$, then find the pdf of Y.

Solution: here $g(x) = |x|^3, \quad -\infty < x < \infty$

$$= \begin{cases} x^3, & 0 < x < \infty \\ -x^3, & -\infty < x < 0 \end{cases}$$

consider two disjoint sets of x such as

$0 < x < \infty \Rightarrow 0 < y < \infty$

$Y = g(x) = x^3$

$g_1^{-1}(y) = y^{1/3}$

$\frac{d}{dy} g_1^{-1}(y) = \frac{1}{3} y^{-2/3}$

$-\infty < x < 0 \Rightarrow 0 < y < \infty$

$Y = g(x) = -x^3$

$g_2^{-1}(y) = -y^{1/3}$

$\frac{d}{dy} g_2^{-1}(y) = -\frac{1}{3} y^{-2/3}$

$$\therefore f_Y(y) = f_X(g_1^{-1}(y)) \left|\frac{d}{dy} g_1^{-1}(y)\right| + f_X(g_2^{-1}(y)) \left|\frac{d}{dy} g_2^{-1}(y)\right|$$

$$= f_X(y^{1/3}) \frac{1}{3} y^{-2/3} + f_X(-y^{1/3}) \cdot \frac{1}{3} y^{-2/3}; \quad 0 < y < \infty$$

$$= \frac{1}{2}\left[e^{-y^{1/3}} \frac{1}{3} y^{-2/3} + e^{-y^{1/3}} \frac{1}{3} y^{-2/3}\right], \quad 0 < y < \infty$$

$$= \frac{1}{3} e^{-y^{1/3}} y^{-2/3}, \quad 0 < y < \infty.$$

Ex: Let $f_X(x) = \frac{3}{8}(x+1)^2, \ -1<x<1.$

Consider the transformation $Y = 1-X^2$. Find pdf of Y.

Ex: Let X be distributed as exp (2) and consider the transformation $Y=(X-2)^2$. Find the pdf of Y.

Ex: Suppose that $f_X(x) = \begin{cases} 0.5\, a e^{-ax}, & x \geq 0 \\ 0.5\, a e^{ax}, & x<0 \end{cases}$

where $a>0$ is a known parameter. Verify $f_X(x)$ is a pdf. Find $P(X \leq x)$ and $P(|X| \leq x)$.

Ex: Let prob. mass func^n of X is given by

$$p_X(x) = \frac{1}{3}\left(\frac{2}{3}\right)^x, \quad x = 0,1,2,\ldots$$

Consider the func^n $Y = \frac{X}{1+X}$. Find PMF of Y.