We have

(1.) $$\ddot{x} + 2\Gamma\dot{x} + \omega_0^2 x = 0.$$

$$\Rightarrow \quad m\ddot{x} + m\omega_0^2 x = -2\Gamma m\dot{x}$$

Multiply both sides by $\dot{x}$

$$m\dot{x}\ddot{x} + m\omega_0^2 x\dot{x} = -2\Gamma m\dot{x}^2.$$

$$\Rightarrow \quad \frac{d}{dt}\left[\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2\right] = -2\Gamma m\dot{x}^2.$$

But $U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2.$

$$\therefore \quad \frac{dU}{dt} = -2\Gamma m\dot{x}^2.$$

Note:

Thus, the total energy goes on decreasing due to damping. Also, the rate of decrease is proportional to the square of velocity (when damping force $\propto$ velocity).

(2.) We have

$$m\ddot{x} = -kx - \beta\dot{x}$$

$$\Rightarrow -\beta\dot{x} = kx + m\ddot{x}$$

The total work done by the damping force by the time the mass m comes to $x = 0$ with $v = 0$ is therefore,

$$W = \int_{x_0, v_0}^{0,0} F_d \, dx = \int_{x_0, v_0}^{0,0} (-\beta\dot{x}) \, dx$$

$$= \int_{x_0, v_0}^{0} (kx + m\ddot{x}) \, dx$$

$$= \int_{x_0}^{0} kx \, dx + \int_{v_0}^{0} mv \frac{dv}{dx} \, dx$$

$$= -\frac{1}{2}kx_0^2 - \frac{1}{2}mv_0^2.$$

3. (a) Over damped case ($\Gamma > \omega_0$)

Mass crosses origin if

$$x(t) = e^{-\Gamma t}(Ae^{\lambda t} + Be^{-\lambda t}) = 0.$$

Background info. (done in class)

Note: $\ddot{x} + 2\Gamma\dot{x} + \omega_0^2 x = 0$

Trial soln. $e^{qx} \Rightarrow q^2 + 2\Gamma q + \omega_0^2 = 0.$

$q = -\Gamma \pm \sqrt{\Gamma^2 - \omega_0^2}.$

For over damped $\Gamma > \omega_0$; $\Gamma^2 - \omega_0^2 = \lambda^2.$

s.t., $\lambda < \Gamma.$

$$\Rightarrow e^{2\lambda t} = -\frac{B}{A}.$$

$$\therefore t = \left(\frac{1}{2\lambda}\right)\ln\left(-\frac{B}{A}\right).$$

Unique (only one) solution exists when $-\frac{B}{A} > 0.$

Critically damped case ($\Gamma = \omega_0$)

Mass crosses origin if,

$$e^{-\Gamma t}(A + Bt) = 0$$

$$\Rightarrow t = -\frac{A}{B}$$

Again there is at most one solution when $-\frac{A}{B} > 0.$

3. b. $\Gamma = \omega_0$

Initial conditions, $x(0) = x_0$.
$v(0) = v_0$.

$\therefore\ x(t) = e^{-\Gamma t}(A + Bt)$.

$\Rightarrow A = x_0$

& $-\Gamma(A) + B = v_0$. i.e., $B = v_0 + \Gamma x_0$.

Now, $x(t) = 0$ if $t = -\frac{A}{B} = -\left(\frac{x_0}{v_0 + \Gamma x_0}\right)$.

Mass crosses origin if $t > 0$

i.e., if $v_0 + \Gamma x_0 < 0$.

i.e., if $v_0 < -\Gamma x_0$.

$\therefore$ If $v_0 \geq -\Gamma x_0$, the mass doesnot cross the origin.

$\therefore$ Maximum speed desired

$= |v_0| = \Gamma x_0 = \omega_0 x_0$.