Indian Institute of Technology, Patna
MA102, B.Tech -I year
Spring Semester: 2015
(Mid Semester Examination)

Maximum Marks: 30 **Time: 2 Hours**

Note:

(i) Please check all pages and report the discrepancy, if any.

(ii) Attempt all questions.

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1. (i) Let $S = \{(x, y, z) \in \mathbb{R}^3 : 3x - 5y + z = 0, 4x + 5y = 0\}$. Show that S is a subspace of $\mathbb{R}^3$ and find a basis of S. $[1 + 2]$

   (ii) Determine the linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^2$ which maps the basis vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ of $\mathbb{R}^3$ to the vectors $(1, 1)$, $(2, 3)$ and $(3, 2)$ respectively.

   (a) Find $T(1, 1, 0)$ and $T(6, 0, -1)$.

   (b) Find the basis and dimension of $KerT$ and ImT.

   (b) Verify that T is not one-to-one, however onto. $[2.5 + 2 + 1]$

   (iii) Let V be an n dimensional vector space over a field F and let T be a linear transformation from V into V such that the range and the null space of T are identical. Prove that n is even. $[1.5]$

2. Let V be the set of all pairs (x, y) of real numbers, and let $\mathbb{R}$ be the field of real numbers. Define $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$, and $c(x, y) = (cx, 0)$. Is V, with these operations, a vector space over the field of real numbers? $[1]$

3. Determine the conditions for which the system $x + y + z = 1$, $x + 2y - z = b$, $5x + 7y + az = b^2$ admits of (i) only one solution, (ii) no solution and (iii) many solutions. $[1 + 1.5 + 1.5]$

4. Let P_n denotes the vector space of all real polynomials in t of degree $<$ n. Differentiation is a linear transformation from P_n to P_{n-1} over $\mathbb{R}$. Also integration defined by

$$\int (a_0 + a_1 t + ... + a_{n-2} t^{n-2}) = a_0 t + \frac{a_1}{2} t^2 + ... + \frac{a_{n-2}}{n-1} t^{n-1}$$

is a linear transformation from P_{n-1} to P_n over $\mathbb{R}$. Consider the differentiation transformation $f : P_4 \to P_3$ and the integration transformation $g : P_3 \to P_4$. Let X be the basis $\{1, t, t^2, t^3\}$ of P_4 and Y be the basis $\{1, t, t^2\}$ of P_3.

(81)

(i) Find the matrix of f with respect to X and Y. [1.5]

(ii) Find the matrix of g with respect to Y and X. [1.5]

5. Let A be a real skew-symmetric matrix of order n.

   (i) If n is odd, show that $det(A) = 0$ [1]

   (ii) If n is even, show that $det(A) \geq 0$ [1]

6. Which of the following statements are true, and which are false? Why?

   (i) Let $U = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix};: a + b = 0 \right\}$ and Let $W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix};: c + d = 0 \right\}$ be two subspaces of $\mathbb{R}_{2\times 2}$ ($\mathbb{R}_{2\times 2}$ denotes the set of all matrices of order 2×2). Then $dim(U + W)$ is equal to 2. [2]

   (ii) The polynomials $1 + t + t^2$, $2 - 3t + 4t^2$ and $1 - 9t + 5t^2$ form a linearly dependent set in P_3 (P_n denotes the vector space of all real polynomials in t of degree < n). [2]

   (iii) Let $m > n$. There cannot exist a linear transformation from F^n onto F^m. [2]

   (iv) Let a and b be two positive real numbers. Then the number of real eigen value of the matrix $A = \begin{pmatrix} a & 1 \\ 2 & b \end{pmatrix}$, is 1. [1.5]

   (v) Rotation of cartesian co-ordinate system with an angle θ is a linear transformation. [1.5]

   (vi) If A is an orthogonal matrix, then $detA$ is equal to -1. [1]