Mathematics I

Continuity

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Continuous Functions : Sequential Criterion

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$. Then f is continuous at $x_0 \in D$ if for every sequence $\{x_n\}$ in D converging to x_0, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Continuous Functions

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$. Then f is continuous at $x_0 \in D$ **iff** for every $\epsilon > 0$ there exists a $\delta > 0$ ($\delta = \delta(x_0, \epsilon)$) such that

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon.$$

Remark 1: Show that $f(x) = 2x^2 + 1$ is continuous over $\mathbb{R}$. Use both sequential criterion and $\epsilon - \delta$ definition. Observe that δ is function of both x_0 and ϵ.
Remark 2: Let $f(x)$ be defined as $f(x) = x^2 \sin \frac{1}{x}$, $x \in 0$ and $f(0) = 0$. Use $\epsilon - \delta$ definition to show that f is continuous on $\mathbb{R}$.
Remark 3: Show that $f(x) = \frac{1}{x^2}$ is continuous over $(0, \infty)$. Use $\epsilon - \delta$ definition. Observe that δ is function of both x_0 and ϵ.
Remark 4: Algebra of Continuous functions, Composition, some results e.g., Max-Min of continuous function on closed interval, Intermediate value theorem (zeros of a function), Monotonic Functions etc.
Remark 5: Continuity is property of function at a point.

Results on Continuity

Theorem

Let f be a real valued function with $dom(f) \subset \mathbb{R}$. If f is continuous at $x_0 \in dom(f)$, then $|f|$ and kf, $k \in \mathbb{R}$, are continuous at x_0.

Theorem

Let f and g be real valued functions that are continuous at x_0 in $\mathbb{R}$. Then

- $f + g$ is continuous at x_0.
- fg is continuous at x_0.
- f/g is continuous at x_0 if $g(x_0) \neq 0$.

Theorem

If f is continuous at x_0 and g is continuous at $f(x_0)$, then the composite function $g\, o\, f$ continuous at x_0.

Exercise

Let f and g be real valued functions which are continuous at x_0.

- Show that $\max\{f,g\}$ and $\min\{f,g\}$ are continuous at x_0.

  Hint: Use $\max\{f,g\} = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$, $\min\{f,g\} = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$ or $\min\{f,g\} = -\max\{-f,-g\}$.

Exercise

Let $f(x) = 1$ for rational numbers x and $f(x) = 0$ for irrational numbers. Show that f is discontinuous at every x in $\mathbb{R}$.

Exercise

Let $h(x) = x$ for rational numbers x and $h(x) = 0$ for irrational numbers. Show that h is continuous at $x = 0$ and at no other point.

Theorem

Let f be a continuous real valued function on the closed interval $[a, b]$ then f is a bounded function. Moreover f assumes it maximum and minimum values on $[a, b]$, that is, there exists $x_0, y_0 \in [a, b]$ such that $f(x_0) \leq f(x) \leq f(y_0)$ for all $x \in [a, b]$.

Theorem [Intermediate Value Theorem]

If f is a continuous real valued function on an interval I, then f has intermediate value property on I: Whenever $a, b \in I$, $a < b$ and y lies between $f(a)$ and $f(b)$ [i.e., $f(a) < y < f(b)$ or $f(b) < y < f(a)$] there exists at least one $x \in (a, b)$ such that $f(x) = y$.

Home Work

Exercise

Let f be continuous real valued function with domain (a, b). Show that if $f(r) = 0$ for each rational number $r \in (a, b)$, then $f(x) = 0$ for all $x \in (a, b)$.

Exercise

Let f and g be continuous real valued functions with domain (a, b) such that $f(r) = g(r)$ for each rational number $r \in (a, b)$, then prove that $f(x) = g(x)$ for all $x \in (a, b)$.

Exercise

For each rational number x write x as p/q where p and q are integers with no common factors and $q > 0$, and then define $f(x) = 1/q$. Also define $f(x) = 0$ for all $x \in \mathbb{R} \setminus \mathbb{Q}$. Thus $f(x) = 1$ for each integer. Show that f is continuous at each point of $\mathbb{R} \setminus \mathbb{Q}$ and discontinuous at each point of $\mathbb{Q}$.

Uniformly Continuous Functions

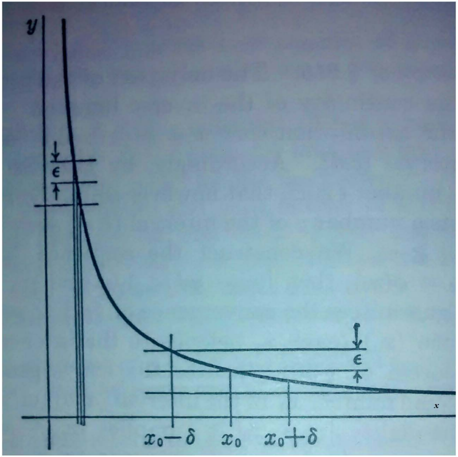
Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$. Then f is uniformly continuous on D **iff** for every $\epsilon > 0$ there exists a $\delta > 0$ $(\delta = \delta(\epsilon))$ such that for any $x', x'' \in D$

$$|x' - x''| < \delta \Rightarrow |f(x') - f(x'')| < \epsilon.$$

Example

1. $1/x^2$ is continuous on $(0, \infty)$ but not uniformly continuous on $(0, \infty)$.
2. $1/x^2$ is uniformly continuous on $[a, \infty),\ a > 0$.
3. $2x^2 + 1$ is uniformly continuous on $[-M, M] \subset \mathbb{R}$, where $M > 0$ and finite.

Example



Continuity vs Uniform Continuity

1. Continuity is property of a function at a point where as uniform continuity is defined on a set.
2. Order of occurrence of the point, ϵ and δ.

   In Continuity we have the point x_0, ϵ and then $\delta = \delta(x_0, \epsilon)$.
   In Uniform continuity we have the positive number ϵ, then $\delta = \delta(\epsilon)$, then the points x', x''.

Results on Uniform Continuity

Theorem

If f is continuous on a closed and bounded interval $[a,b]$, then f is uniformly continuous on $[a,b]$.

Theorem

If f is uniformly continuous on a set S, and $\{s_n\}$ is a Cauchy Sequence then $\{f(s_n)\}$ is a Cauchy Sequence.

Ex. Show that $\frac{1}{x^2}$ is not uniform continuous in $(0,1)$.

Theorem

Let f be a continuous function on an interval I [**I** ***may be bounded or unbounded***]. Let I^o be an interval obtained by removing from I any end points that happen to be in I. If f is differentiable on I^o and if f' is bounded on I^o, then f is uniformly continuous on I.

Use the above theorem and discuss uniform continuity of the following:
Ex. Take $1/x^2$ on $[a, \infty)$ where $a > 0$.
Ex. Consider $\sin x$ over $\mathbb{R}$.
Ex. Consider $x + 1$ over $\mathbb{R}$.

This result is not discussed in class interested students may read

Definition : Extension of a function

We say that a function $\widetilde{f}$ is an extension of f if

$$dom(f) \subset dom(\widetilde{f}) \text{ and } f(x) = \widetilde{f}(x) \text{ for all } x \in dom(f).$$

Theorem

A real valued function f on (a, b) is uniformly continuous on (a, b) if and only if it can be extended to a continuous function $\widetilde{f}$ on $[a, b]$. Ex. Take $f(x) = \frac{\sin x}{x}$ on $\left(0, \frac{1}{\pi}\right)$.

Exercise

Let f be continuous function on $[0, \infty)$. Prove that if f is uniformly continuous on $[k, \infty)$ for some k, then f is uniformly continuous on $[0, \infty)$.

Exercise : Uniform Continuity

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at zero and satisfies the following conditions:

$$f(0) = 0,\ f(x_1 + x_2) \le f(x_1) + f(x_2),\ \forall x_1, x_2 \in \mathbb{R}.$$

Prove that if f is uniformly continuous on $\mathbb{R}$.

Solution: Discussed in class.

Differentiation

Review

1. Differentiability
2. Rolle's theorem, Mean Value Theorem
3. Derivative Test for monotonic function
4. Convexity and Concavity
5. L'Hospital Rule

Generalized (Cauchy's) Mean Value Theorem

Let $f \& g$ be continuous on $[a, b]$ and differentiable on (a, b) and assume that $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists c in (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. Hint: Construct $h(x) = \frac{f(b)-f(a)}{g(b)-g(a)} (g(x) - g(a) - f(x) + f(a))$

Exercise

Let $f \& g$ be two functions continuous on $[a, b]$ and differentiable on (a, b) and let $f(a) = f(b) = 0$. Show that there exists a point $x \in (a, b)$ such that $g'(x)f(x) + f'(x) = 0$.

Exercise

Let f be continuous on $[a, b]$ and differentiable on (a, b) and let $f^2(a) - f^2(b) = b^2 - a^2$ then equation $f'(x)f(x) = x$ has at least one root in (a, b).

Inverse Function Theorem

Let $f(x)$ be a $1-1$ function defined on some open interval (a,b) such that $f(a,b)=(c,d)$, where (c,d) is some open interval. Let f be differentiable at $x_0 \in (a,b)$ such that $f'(x_0) \neq 0$. Then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)}.$$

Example: Let $f(x) = \sin x$ on $[-\pi/2, \pi/2]$. Discuss.