Transformation in Two-Dimensional Case

Dear Students,

Today we are going to discuss transformation in two-dimensional case.

The corresponding problem goes as follows:
" Given a jointly distributed random variable (X,Y) ~~suppose~~ our interest is not in studying properties of (X,Y), but we want to learn probabilistic behavior of some function of (X,Y), say $g(X,Y)$ where $g(X,Y)$ can be like

$$g(X,Y) = X+Y,\ X-Y,\ \frac{X}{Y},\ XY,\ \sqrt{X^2+Y^2}$$

$$\max(X,Y),\ \min(X,Y)$$

and so on.

Thus in this lecture we try to obtain probability distribution of

" One function of two variables"

Results are discussed for Continuous RVs.

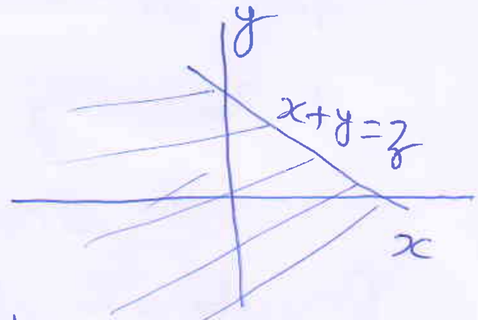
Before we move further please revise the following formula commonly known as Leibnitz formula of "differentiation under Integral Sign".

Now we state the first result.

<u>Result(1)</u>: Let (X, Y) be jointly distributed continuous random variable with joint PDF $f_{X,Y}(x, y)$, $-\infty < x < \infty$, $-\infty < y < \infty$. Consider the transformation $Z = X + Y$ (one function of two rvs X & Y) and then find the pdf of this rv Z.

<u>Answer</u>: [<u>Method</u>: We try to compute the CDF of variable of interest Z using given information. Because Z is continuous, we differentiate the CDF to get the required prob. density function.] So let us proceed to derive the desired result.

Cumulative Distribution Function of $Z = X + Y$ is



$$F_Z(z) = P(Z \le z) = P(X + Y \le z)$$

$$= P(X \le z - y)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{X,Y}(x, y)\, dx\, dy$$

so

$$F_Z(z) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{z-y} f_{X,Y}(x,y)\, dx \right] dy \quad \text{--- (1)}$$

How to obtain the corresponding PDF. Infact it is given by $\boxed{f_Z(z) = \frac{d}{dz} F_Z(z)}$ — (2)

In order to differentiate (1) wrt z we take help of Leibnitz formula given below.

If $F(t) = \int_{a(t)}^{b(t)} f(x,t)\, dx$ then

$$\frac{d}{dt} F(t) = \frac{d\, b(t)}{dt} \cdot f(b(t), t) - \frac{d\, a(t)}{dt} f(a(t), t) + \int_{a(t)}^{b(t)} \left[\frac{\partial}{\partial t} f(x,t) \right] dx$$

From Equation (2) [using the above formula]

$$f_Z(z) = \frac{d}{dz} \left\{ \int_{-\infty}^{\infty} \left[\int_{-\infty}^{z-y} f_{X,Y}(x,y)\, dx \right] dy \right\}$$

$$= \int_{-\infty}^{\infty} \frac{\partial}{\partial z} \left[\int_{-\infty}^{z-y} f_{X,Y}(x,y)\, dx \right] dy$$

$$= \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial z}(z-y)\, f_{X,Y}(z-y, y) - 0 + \int_{-\infty}^{z-y} \left[\frac{\partial}{\partial z} f_{X,Y}(x,y) \right] dx \right] dy$$

$$= \int_{-\infty}^{\infty} \left[f_{X,Y}(z-y, y) - 0 + 0 \right] dy$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(z-y, y)\, dy.$$

Thus pdf of $Z = X+Y$ is given by

$$\boxed{f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(z-y, y)\, dy} \quad \text{— (3)}$$

Can you guess Eqn (3) is the same as

$$\boxed{f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\, dx} \quad \text{— (4)}$$

Special Case: In the above result if we have X and Y as independent rvs then Eqn (3) becomes

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\, dy$$

Similarly you can update (4) also.

(*) Result (1) is derived by assuming $-\infty < x < \infty$, $-\infty < y < \infty$.

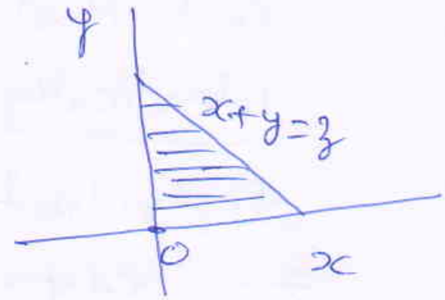
Suppose that range of (X, Y) is as $0 < x < \infty$, $0 < y < \infty$. Then what is the pdf of Z.

So what is the problem at our hand??

See next page

Remark: Consider the transformation $Z = X+Y$.
find pdf of Z where $X>0$, $Y>0$

In this Case



$$F_Z(z) = P(Z \le z) = P(X+Y \le z)$$

$$= P(X \le z - Y)$$

$$= \int_{y=0}^{z} \left[\int_{x=0}^{z-y} f_{X,Y}(x,y)\, dx \right] dy$$

Then pdf of Z is

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz} \int_0^z \left[\int_0^{z-y} f_{X,Y}(x,y)\, dx \right] dy$$

$$= \int_0^z \frac{\partial}{\partial z} \left[\int_0^{z-y} f_{X,Y}(x,y)\, dx \right] dy$$

$$= \int_0^z \left[\frac{\partial (z-y)}{\partial z} \cdot f_{X,Y}(z-y, y) - 0 + \int_0^{z-y} \left(\frac{\partial}{\partial z} f(x,y) \right) dx \right] dy$$

$$= \int_0^z \left[f_{X,Y}(z-y, y) - 0 + 0 \right] dy$$

$$= \int_0^z f_{X,Y}(z-y, y)\, dy$$

Again If we know that X & Y independent then above formula becomes

$$\boxed{f_Z(z) = \int_0^z f_X(z-y)\, f_Y(y)\, dy} \quad \text{— (5)}$$

Ex: let (X, Y) be jointly distributed random variables such that both X and Y are independent and identically distributed (iid) as exp(1) distribution. Consider the transformation variable Z.

$Z = X+Y$. find pdf of this variable Z.

$$\boxed{\begin{array}{c} X \sim \exp(\beta) \\ f_X(x) = \frac{1}{\beta} e^{-x/\beta} \\ x > 0 \\ \beta > 0 \end{array}}$$

Solution: $X, Y \overset{iid}{\sim} \exp(1)$

so $f_X(x) = e^{-x}, \; 0 < x < \infty$

$f_Y(y) = e^{-y}, \; 0 < y < \infty$

Kindly refer to the formula (5).

$$f_Z(z) = \int_0^z f_X(z-y)\, f_Y(y)\, dy$$

$$= \int_0^z e^{-(z-y)} \cdot I(0 < z-y < \infty) \cdot e^{-y}\, I(0 < y < \infty)\, dy$$

$$\boxed{\text{Note: Indicator function } I_A(x) = \begin{cases} 0, & x \notin A \\ 1, & x \in A \end{cases}}$$

$$= e^{-z} \int_0^z I(0 < z-y < \infty)\, I(0 < y < \infty)\, dy$$

$$= e^{-z} \int_0^z I(y < z)\, I(y > 0)\, dy = e^{-z} \int_0^z dy$$

$$= z e^{-z}.$$

$$\therefore \boxed{f_Z(z) = z e^{-z}, \; 0 < z < \infty}$$

Name this distn: $Z \sim G(2,1)$. ~~so what is the~~

So what is result we get here. It goes like "If $X, Y \overset{iid}{\sim} \exp(1)$ then their sum $(X+Y)$ follows gamma $G(2,1)$ distribution"!

Let us see one more example

<u>Ex</u>: Let $X, Y \overset{iid}{\sim} U(0,1)$. Take $Z = X+Y$. Find pdf of Z.

<u>Solution</u>:
$$f_X(x) = 1, \quad 0<x<1$$
$$f_Y(y) = 1, \quad 0<y<1$$

$$f_Z(z) = \int_0^z f_X(z-y)\, f_Y(y)\, dy$$
$$= \int_0^z 1 \cdot I(0<z-y<1) \cdot 1 \cdot I(0<y<1)\, dy$$
$$= \int_0^z I(z-1<y<z)\, I(0<y<1)\, dy$$
$$= \int_{\max(0,z-1)}^{\min(z,1)} dy$$
$$= \min(z,1) - \max(0, z-1)$$
$$= \begin{cases} z-0, & 0<z<1 \\ 1-(z-1), & 1<z<2 \end{cases} = \begin{cases} z, & 0<z<1 \\ 2-z, & 1<z<2. \end{cases}$$