Covariance and Correlation Coefficient

Dear students,

In the last session, we have covered joint maf and Independence.

In this section, we will study covariance between two random variables and associated correlation.

Covariance Between two random variables

Let (X, Y) be jointly distributed random variables then covariance between X & Y is defined as

$$Cov(X,Y) = E[(X - EX)(Y - EY)]$$

$$= E(XY) - E(X)E(Y)$$

$$\boxed{\therefore Cov(X,Y) = E(XY) - E(X)E(Y)}$$

The variance of a single random variable looked upon as a measure of dispersion of the distribution of the r.v. Similar motivation is given to the effect that $cov(X,Y)$ may be thought of as a measure of the degree to which X and Y tend to ~~~~ increase or decrease, simultaneously.

~~That is, when Cov(X,Y) > 0~~

~~X and Y~~

That is, when X and Y increase or decrease together then $\text{Cov}(X,Y) > 0$.

If they move in opposite direction, i.e, if one increase then other decrease and vice versa then $\text{Cov}(X,Y) < 0$.

* We can say that $\text{Cov}(X,Y)$ is a measure of joint variability of X and Y.

* $\text{Cov}(X,Y)$ is any real number.

Ex: Find $\text{Cov}(X,Y)$ when $f_{X,Y}(x,y) = 1, \quad 0 < x < 1, \; x < y < x+1$

Solution: Note that $\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$

Now $E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \, f_{X,Y}(x,y)\, dx\, dy$

$$= \int_{x=0}^{1} x \int_{y=x}^{x+1} y \, dy \, dx = \int_0^1 x \cdot \left[\frac{y^2}{2}\right]_x^{x+1} dx$$

$$= \frac{1}{2}\int_0^1 x\left[(x+1)^2 - x^2\right] dx = \frac{1}{2}\int_0^1 x(2x+1)\, dx$$

$$= 7/12.$$

Next we compute $E(X)$ and $E(Y)$, for that we need $f_X(x)$ and $f_Y(y)$ respectively.

Now, $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$

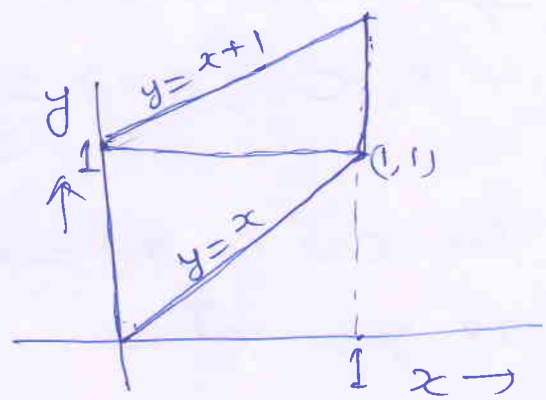
$$= \int_{y=x}^{x+1} dy = 1,$$

$\boxed{\therefore f_X(x) = 1,\ 0<x<1}$ $\therefore E(X) = \int_0^1 x\,dx = \frac{1}{2}.$

Marginal pdf of Y is given by

$$f_Y(y) = \begin{cases} \int_{x=0}^{y} f(x,y)\,dx, & 0<y<1 \\ \int_{x=y-1}^{1} f(x,y)\,dx, & 1<y<2 \end{cases}$$

$$= \begin{cases} \int_0^y dx, & 0<y<1 \\ \int_{y-1}^{1} dx, & 1<y<2 \end{cases}$$



$$\boxed{f_Y(y) = \begin{cases} y, & 0<y<1 \\ 2-y, & 1<y<2 \end{cases}}$$

$E(Y) = \int_0^1 y\cdot y\,dy + \int_1^2 y(2-y)\,dy$

$= 1$ (After simplification)

$\therefore Cov(X,Y) = E(XY) - E(X)E(Y)$

$$= \frac{7}{12} - \frac{1}{2}\cdot 1 = \frac{1}{12}.$$

(*) Here $Cov(X,Y) > 0$ so X & Y tend to increase or decrease together.

Ex: Find $Cov(X,Y)$ for the given joint pmf of (X,Y).

$Y \backslash X$	-1	0	1
0	$1/16$	$1/16$	$1/16$
1	$1/16$	$1/16$	$2/16$
2	$2/16$	$1/16$	$6/16$

$\Rightarrow$ $Cov(X,Y) = E(XY) - E(X)E(Y)$

Hint: $E(XY) = \sum_{y=0}^{2} \sum_{x=-1}^{1} xy \, p_{X,Y}(x,y)$.

Properties

1. If X and Y are independent rvs then $Cov(X,Y) = 0$. (Converse may not be true)

Ex: Let $X \sim N(0,1)$ and $Y = X^2$. Let us compute $Cov(X,Y)$.

$\Rightarrow$ Note that $E(X) = 0$. $E(XY) = E(X^3) = 0$.

$\therefore$ $Cov(X,Y) = E(XY) - E(X) \cdot E(Y)$
$= 0 - 0 \cdot E(Y) = 0$

So $Cov(X,Y) = 0$ but X & Y are not independent as $Y = X^2$.

2. $Cov(aX+b, cY+d) = ac \, Cov(X,Y)$ where a, b, c, d are given constants.

Note that covariance is not affected by translation.

A nice formula:

(*) $V(aX \pm bY) = a^2 V(X) + b^2 V(Y) \pm 2ab\, Cov(X,Y)$

In particular if X and Y independent then

$$V(aX + bY) = a^2 V(X) + b^2 V(Y)$$

Furthermore $V(X+Y) = V(X) + V(Y)$, X, Y independent.

this is true for n variables also

$$V(X_1 + X_2 + \cdots + X_n) = V(X_1) + V(X_2) + \cdots + V(X_n)$$

if $X_1, X_2, \ldots, X_n$ all are independent.

Keeping in the mind covariance is any real number we next define correlation coefficient between X and Y as follows:

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y},$$

$\sigma_X \rightarrow$ standard deviation of X.

$\sigma_Y \rightarrow$ s.d. of Y.

Properties: (i) $-1 \leq \rho_{X,Y} \leq 1$

(ii) $\rho_{X,Y} = 1$ if and only if $P(Y = aX + b) = 1$, $a > 0$, $b \in \mathbb{R}$

(iii) $\rho_{X,Y} = -1$ if and only if $P(Y = aX + b) = 1$, $a < 0$, $b \in \mathbb{R}$.

Proof: Note that $V\left(\frac{X}{\sigma_X} \pm \frac{Y}{\sigma_Y}\right) \geq 0$

Now $V\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}\right) \geq 0 \Rightarrow \frac{V(X)}{\sigma_X^2} + \frac{V(Y)}{\sigma_Y^2} + 2\frac{Cov(X,Y)}{\sigma_X \sigma_Y} \geq 0$

This implies that

$$2+2\rho_{x,y} \geq 0 \Rightarrow \rho_{x,y} \geq -1 \quad \text{— } (A')$$

Similarly $V\left(\frac{X}{\sigma_x} - \frac{Y}{\sigma_y}\right) \geq 0 \Rightarrow 2-2\rho_{x,y} \geq 0$

$$\Rightarrow \rho_{x,y} \leq 1 \quad \text{— } (A'')$$

Thus from (A') & (A'') we get that

$$\boxed{-1 \leq \rho_{x,y} \leq 1}$$

~~Lets~~ Let us prove (ii)

$$\rho_{x,y} = 1 \Rightarrow V\left(\frac{X}{\sigma_x} - \frac{Y}{\sigma_y}\right) = 0$$

$$\Rightarrow P\left(\frac{X}{\sigma_x} - \frac{Y}{\sigma_y} = c\right) = 1 \text{ for some constant } c.$$

$$\Rightarrow P\left(Y = \frac{\sigma_Y}{\sigma_X} X - c\sigma_Y\right) = 1$$

$$\Rightarrow P(Y = aX + b) = 1, \quad a = \frac{\sigma_Y}{\sigma_X} > 0, \quad b = -c\sigma_Y$$

For (iii) $\rho_{x,y} = -1 \Rightarrow V\left(\frac{X}{\sigma_x} + \frac{Y}{\sigma_Y}\right) = 0$

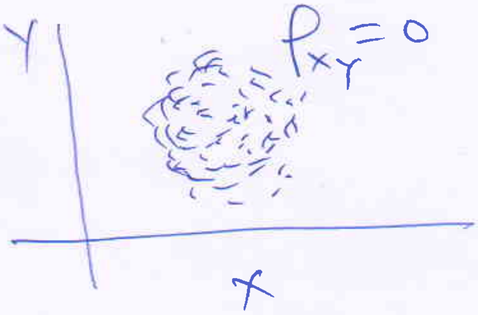
$$\Rightarrow P\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y} = c\right) = 1 \Rightarrow P\left(Y = -\frac{\sigma_Y}{\sigma_X} X + c\sigma_Y\right) = 1$$

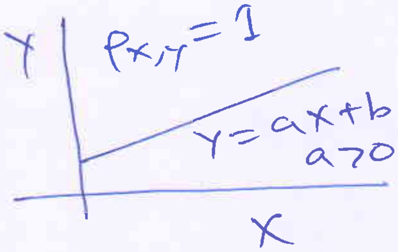
$$\Rightarrow P(Y = aX + b) = 1, \quad a = -\frac{\sigma_Y}{\sigma_X} < 0$$

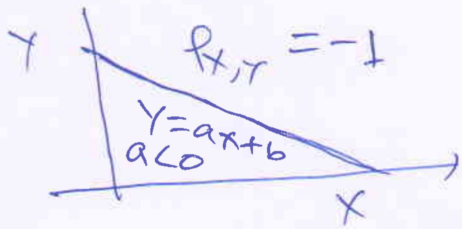
$$b = c\sigma_Y.$$

~~Thus~~ Thus result is proved.

Thus $\rho_{X,Y}$ measures the strength of the linear relationship between X and Y. Also look at visuals given below.







Ex: find $\rho_{X,Y}$ for $f_{X,Y} = 1$, $\begin{matrix} 0 < x < 1 \\ x < y < x+1 \end{matrix}$ (previous problem).

$$\Rightarrow \rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} \quad \left\{ \begin{array}{l} \text{we already computed} \\ Cov(X,Y) = \frac{1}{12}. \end{array} \right.$$

we need to σ_X & σ_Y also.

$f_X(x) = 1, \quad 0 < x < 1$

$E(X) = 1/2$

$E(X^2) = 1/3$

$V(X) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

$\sigma_X = \sqrt{1/12}$

$f_Y(y) = \begin{cases} y, & 0 < y < 1 \\ 2-y, & 1 < y < 2 \end{cases}$

$V(Y) = E(Y^2) - (EY)^2$

$= \frac{131}{144}.$

$\therefore \rho_{X,Y} = \frac{1/12}{\sqrt{\frac{1}{12}}\sqrt{\frac{131}{144}}} = \sqrt{\frac{12}{131}}$

Check!

Ex: Find $\rho_{X,Y}$ when $f_{X,Y}(x,y) = 2,\ 0 \le x \le y \le 1$.

Ex: Find $\rho_{X,Y}$ for discrete case example.