Third Edition

CHAPTER

3

MECHANICS OF MATERIALS

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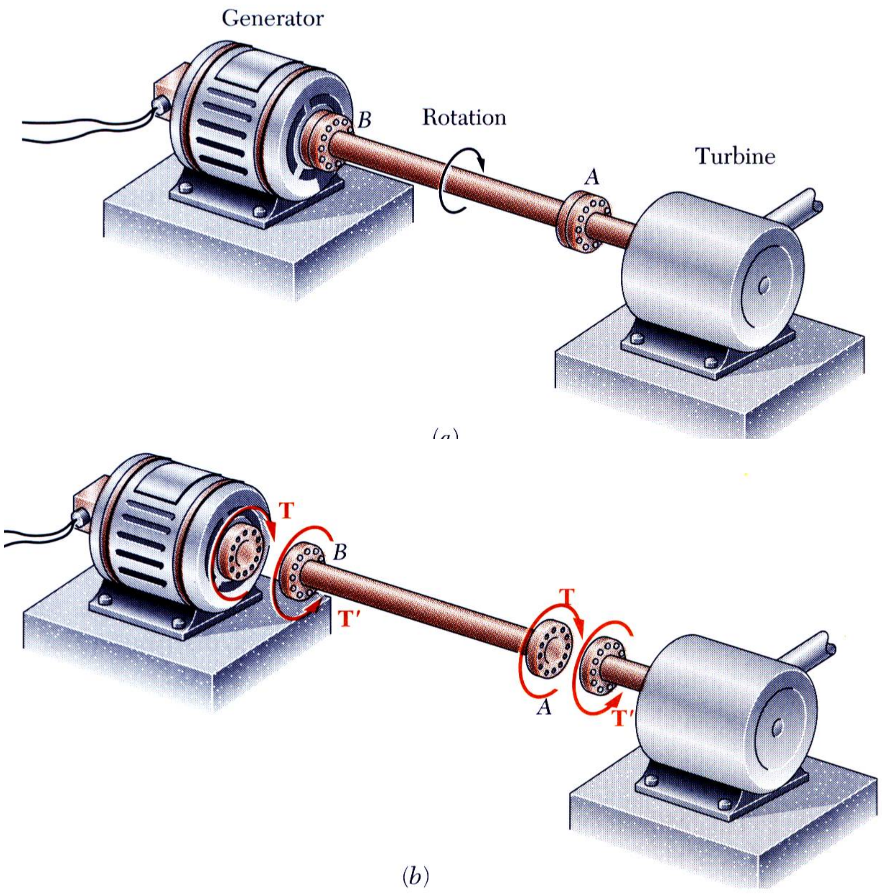
Torsion

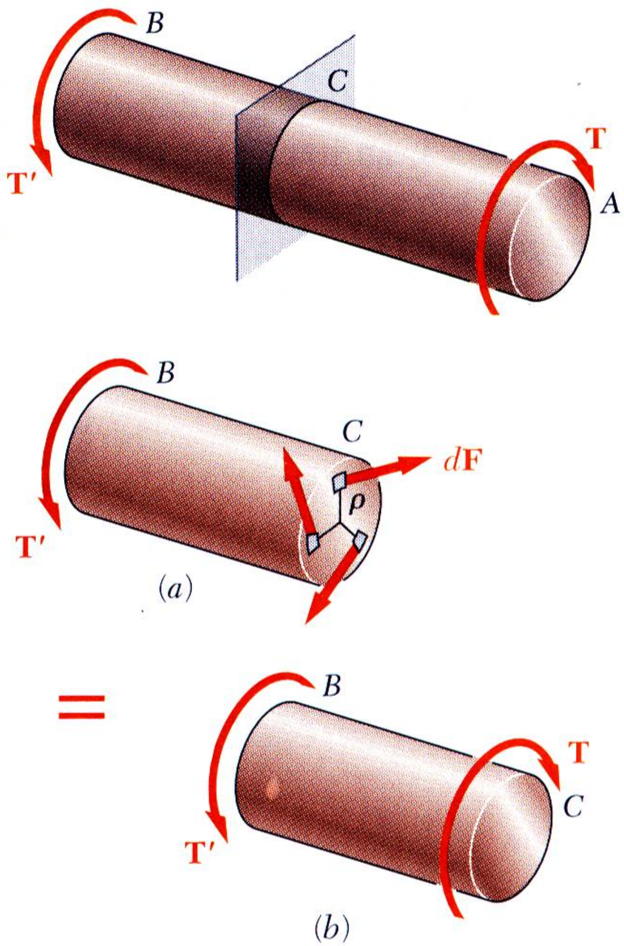
Contents

Torsional Loads on Circular Shafts



- Interested in stresses and strains of circular shafts subjected to twisting couples or *torques*
- Turbine exerts torque *T* on the shaft
- Shaft transmits the torque to the generator
- Generator creates an equal and opposite torque *T*'

Net Torque Due to Internal Stresses

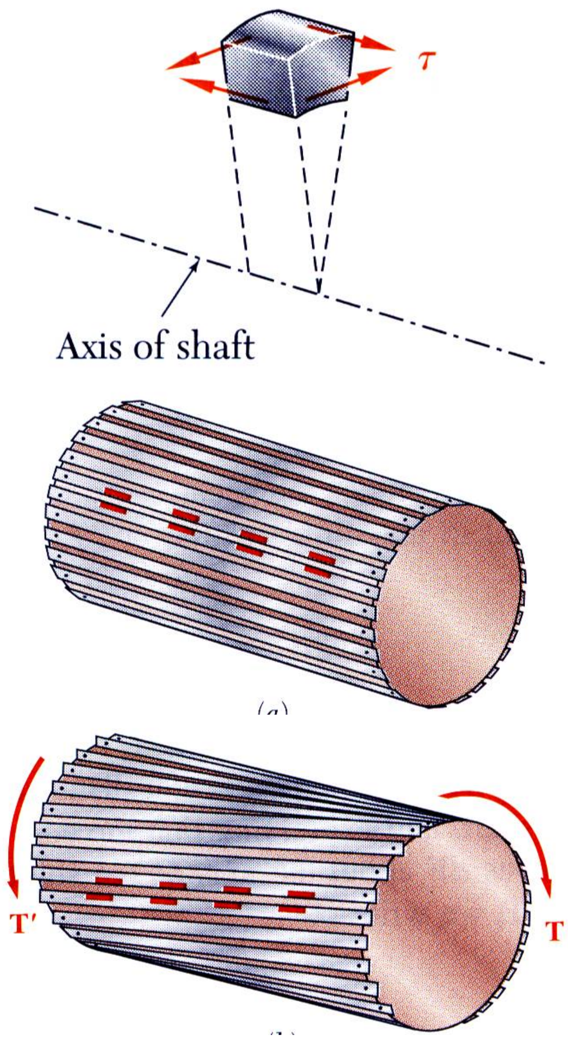


- Net of the internal shearing stresses is an internal torque, equal and opposite to the applied torque,

$$T = \int \rho \, dF = \int \rho (\tau \, dA)$$

- Although the net torque due to the shearing stresses is known, the distribution of the stresses is not
- Distribution of shearing stresses is statically indeterminate – must consider shaft deformations
- Unlike the normal stress due to axial loads, the distribution of shearing stresses due to torsional loads can not be assumed uniform.

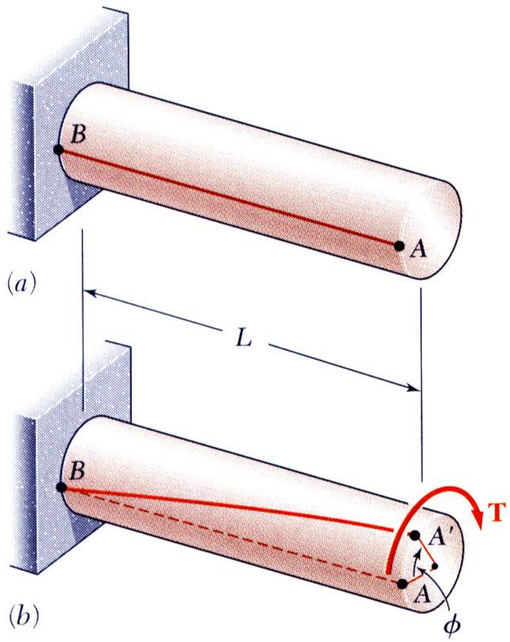
Axial Shear Components

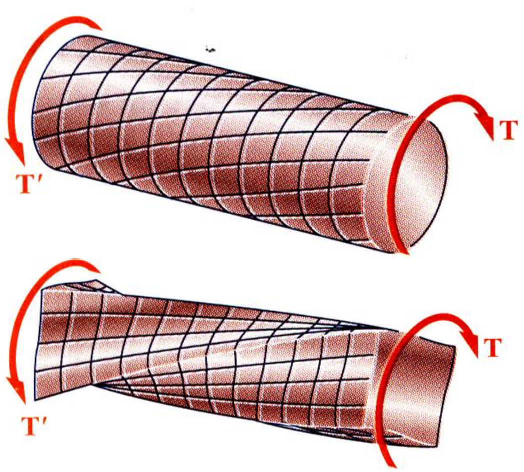


- Torque applied to shaft produces shearing stresses on the faces perpendicular to the axis.
- Conditions of equilibrium require the existence of equal stresses on the faces of the two planes containing the axis of the shaft
- The existence of the axial shear components is demonstrated by considering a shaft made up of axial slats.

  The slats slide with respect to each other when equal and opposite torques are applied to the ends of the shaft.

Shaft Deformations





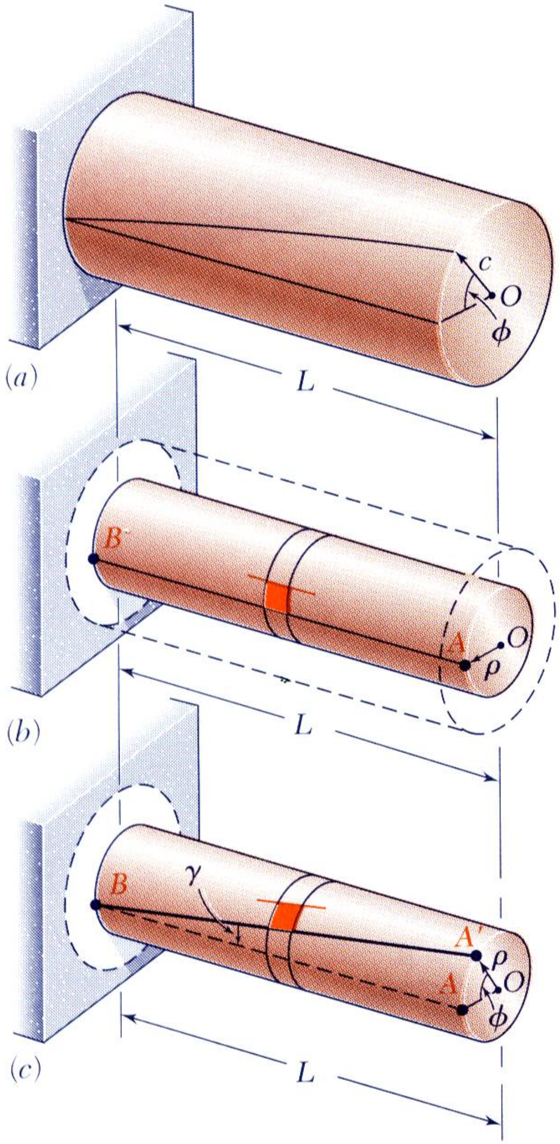
- From observation, the angle of twist of the shaft is proportional to the applied torque and to the shaft length.

$$\phi \propto T$$

$$\phi \propto L$$

- When subjected to torsion, every cross-section of a circular shaft remains plane and undistorted.
- Cross-sections for hollow and solid circular shafts remain plain and undistorted because a circular shaft is axisymmetric.
- Cross-sections of noncircular (non-axisymmetric) shafts are distorted when subjected to torsion.

Shearing Strain



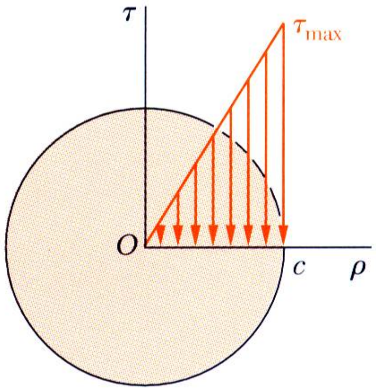
- Consider an interior section of the shaft. As a torsional load is applied, an element on the interior cylinder deforms into a rhombus.
- Since the ends of the element remain planar, the shear strain is equal to angle of twist.
- It follows that

$$L\gamma = \rho\phi \quad \text{or} \quad \gamma = \frac{\rho\phi}{L}$$

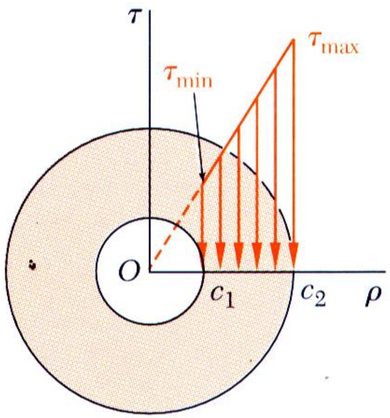
- Shear strain is proportional to twist and radius

$$\gamma_{\max} = \frac{c\phi}{L} \quad \text{and} \quad \gamma = \frac{\rho}{c}\gamma_{\max}$$

Stresses in Elastic Range



$$J = \tfrac{1}{2}\pi c^4$$



$$J = \tfrac{1}{2}\pi\left(c_2^4 - c_1^4\right)$$

- Multiplying the previous equation by the shear modulus,

  $$G\gamma = \frac{\rho}{c} G\gamma_{max}$$

  From Hooke's Law, $\tau = G\gamma$, so

  $$\tau = \frac{\rho}{c}\tau_{max}$$

  The shearing stress varies linearly with the radial position in the section.

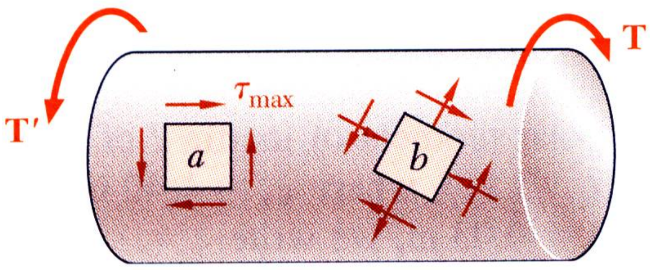
- Recall that the sum of the moments from the internal stress distribution is equal to the torque on the shaft at the section,

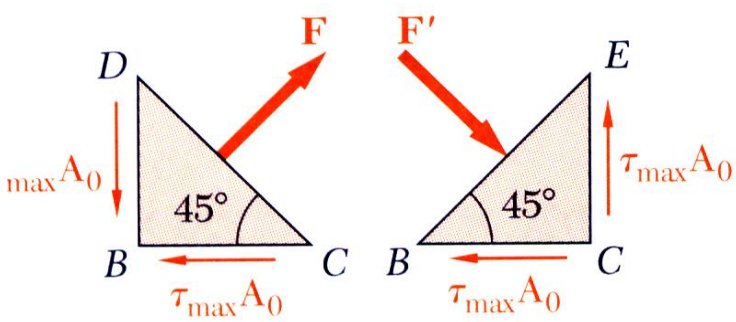
  $$T = \int \rho\tau \, dA = \frac{\tau_{max}}{c}\int \rho^2 \, dA = \frac{\tau_{max}}{c} J$$

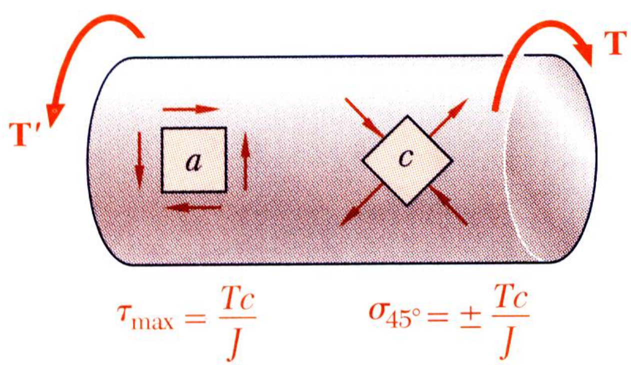
- The results are known as the elastic torsion formulas,

  $$\tau_{max} = \frac{Tc}{J} \quad \text{and} \quad \tau = \frac{T\rho}{J}$$

Normal Stresses







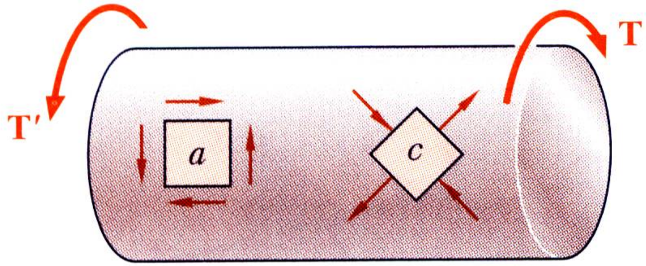
- Elements with faces parallel and perpendicular to the shaft axis are subjected to shear stresses only. Normal stresses, shearing stresses or a combination of both may be found for other orientations.
- Consider an element at 45° to the shaft axis,

$$F = 2(\tau_{max} A_0)\cos 45 = \tau_{max} A_0 \sqrt{2}$$

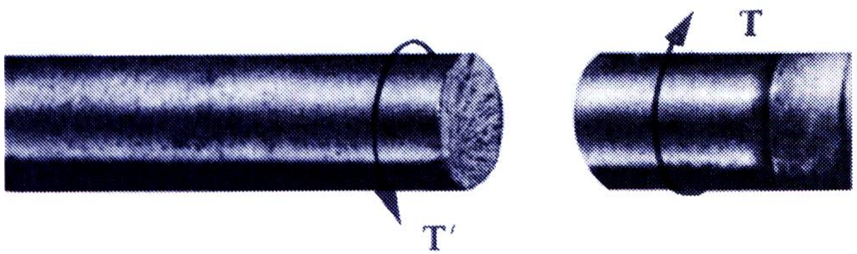
$$\sigma_{45^o} = \frac{F}{A} = \frac{\tau_{max} A_0 \sqrt{2}}{A_0 \sqrt{2}} = \tau_{max}$$

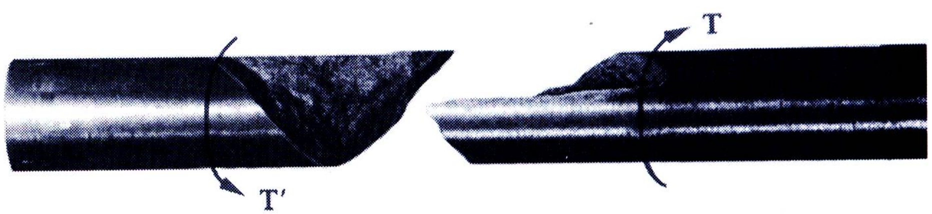
- Element *a* is in pure shear.
- Element *c* is subjected to a tensile stress on two faces and compressive stress on the other two.
- Note that all stresses for elements *a* and *c* have the same magnitude

Torsional Failure Modes



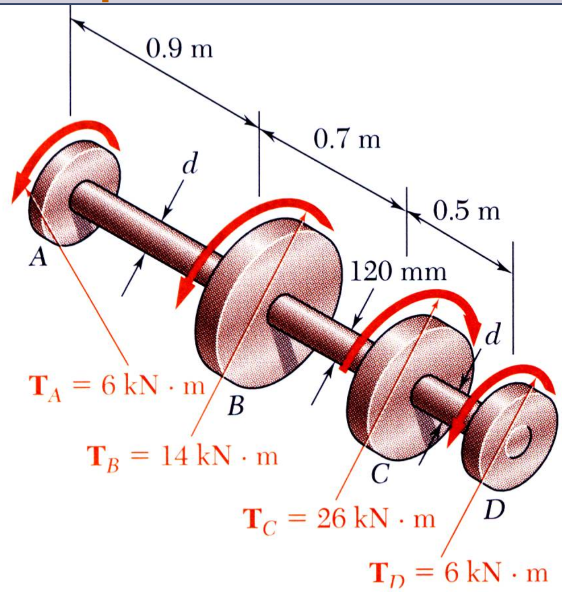
$$\tau_{\max} = \frac{Tc}{J} \qquad \sigma_{45^\circ} = \pm \frac{Tc}{J}$$





- Ductile materials generally fail in shear. Brittle materials are weaker in tension than shear.
- When subjected to torsion, a ductile specimen breaks along a plane of maximum shear, i.e., a plane perpendicular to the shaft axis.
- When subjected to torsion, a brittle specimen breaks along planes perpendicular to the direction in which tension is a maximum, i.e., along surfaces at 45° to the shaft axis.

Sample Problem 3.1



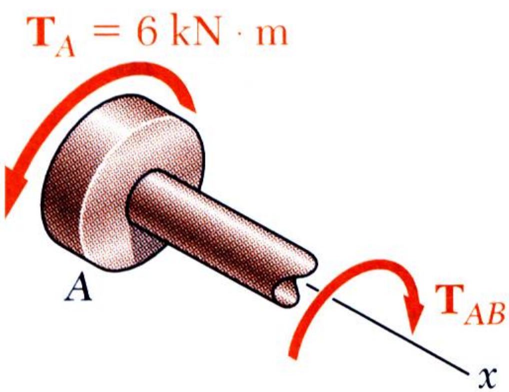
Shaft *BC* is hollow with inner and outer diameters of 90 mm and 120 mm, respectively. Shafts *AB* and *CD* are solid of diameter *d*. For the loading shown, determine (*a*) the minimum and maximum shearing stress in shaft *BC*, (*b*) the required diameter *d* of shafts *AB* and *CD* if the allowable shearing stress in these shafts is 65 MPa.

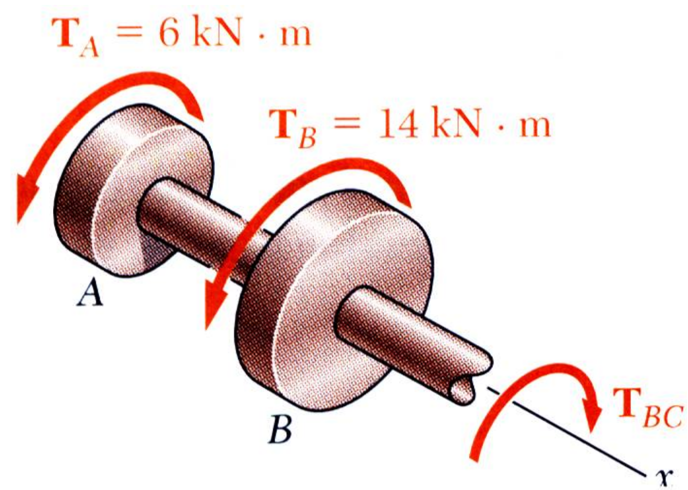
SOLUTION:

- Cut sections through shafts *AB* and *BC* and perform static equilibrium analysis to find torque loadings
- Apply elastic torsion formulas to find minimum and maximum stress on shaft *BC*
- Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter

Sample Problem 3.1

- Cut sections through shafts *AB* and *BC* and perform static equilibrium analysis to find torque loadings





$$\sum M_x = 0 = (6\,\text{kN}\cdot\text{m}) - T_{AB}$$

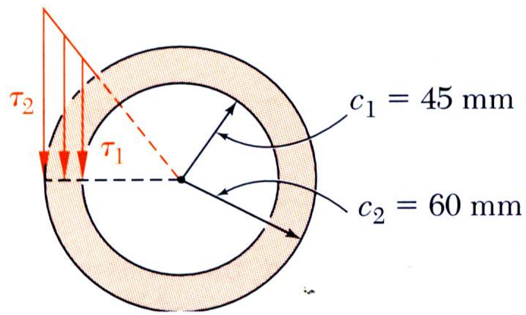
$$T_{AB} = 6\,\text{kN}\cdot\text{m} = T_{CD}$$

$$\sum M_x = 0 = (6\,\text{kN}\cdot\text{m}) + (14\,\text{kN}\cdot\text{m}) - T_{BC}$$

$$T_{BC} = 20\,\text{kN}\cdot\text{m}$$

Sample Problem 3.1

- Apply elastic torsion formulas to find minimum and maximum stress on shaft *BC*



$$J = \frac{\pi}{2}\left(c_2^4 - c_1^4\right) = \frac{\pi}{2}\left[(0.060)^4 - (0.045)^4\right]$$

$$= 13.92 \times 10^{-6}\,\mathrm{m}^4$$

$$\tau_{max} = \tau_2 = \frac{T_{BC}c_2}{J} = \frac{(20\,\mathrm{kN\cdot m})(0.060\,\mathrm{m})}{13.92 \times 10^{-6}\,\mathrm{m}^4}$$

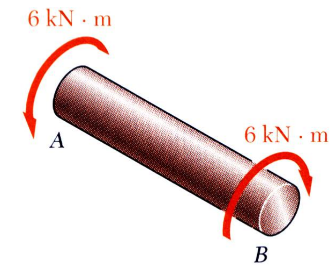
$$= 86.2\,\mathrm{MPa}$$

$$\frac{\tau_{min}}{\tau_{max}} = \frac{c_1}{c_2} \qquad \frac{\tau_{min}}{86.2\,\mathrm{MPa}} = \frac{45\,\mathrm{mm}}{60\,\mathrm{mm}}$$

$$\tau_{min} = 64.7\,\mathrm{MPa}$$

$$\boxed{\begin{aligned}\tau_{max} &= 86.2\,\mathrm{MPa}\\ \tau_{min} &= 64.7\,\mathrm{MPa}\end{aligned}}$$

- Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter

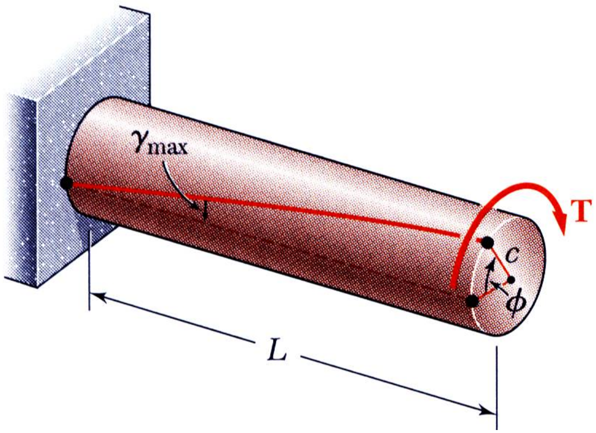


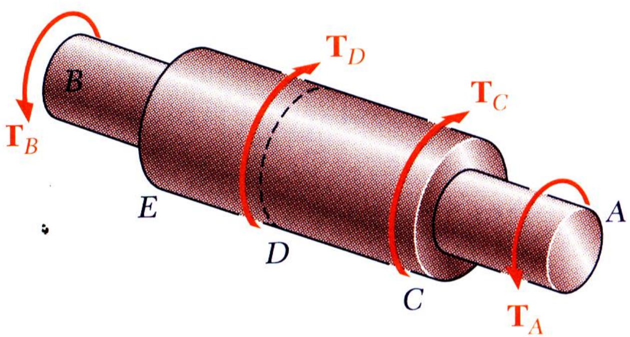
$$\tau_{max} = \frac{Tc}{J} = \frac{Tc}{\frac{\pi}{2}c^4} \qquad 65\,MPa = \frac{6\,\mathrm{kN\cdot m}}{\frac{\pi}{2}c^3}$$

$$c = 38.9 \times 10^{-3}\,\mathrm{m}$$

$$\boxed{d = 2c = 77.8\,\mathrm{mm}}$$

Angle of Twist in Elastic Range





- Recall that the angle of twist and maximum shearing strain are related,

$$\gamma_{max} = \frac{c\phi}{L}$$

- In the elastic range, the shearing strain and shear are related by Hooke's Law,

$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{Tc}{JG}$$

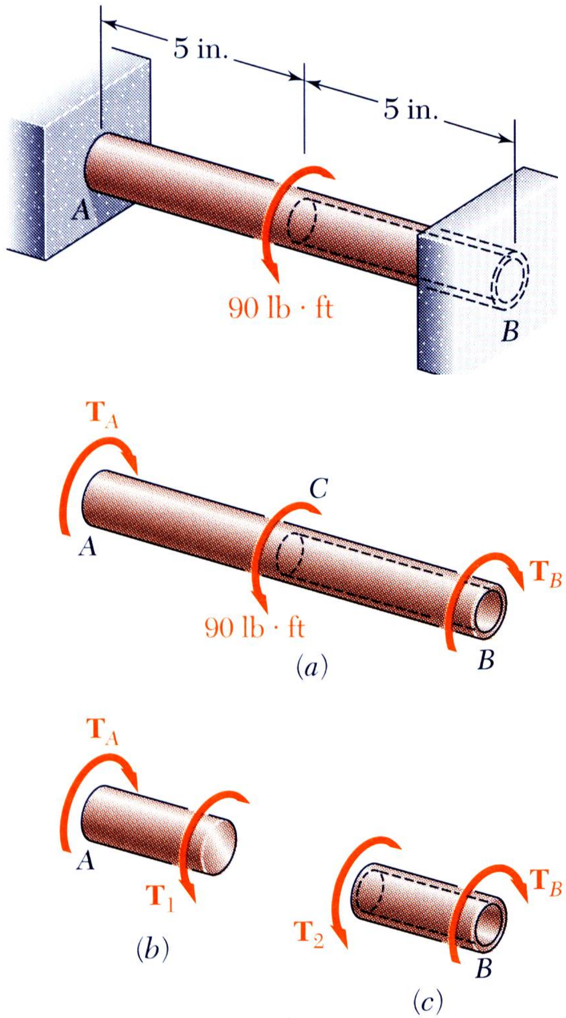
- Equating the expressions for shearing strain and solving for the angle of twist,

$$\phi = \frac{TL}{JG}$$

- If the torsional loading or shaft cross-section changes along the length, the angle of rotation is found as the sum of segment rotations

$$\phi = \sum_i \frac{T_i L_i}{J_i G_i}$$

Statically Indeterminate Shafts



- Given the shaft dimensions and the applied torque, we would like to find the torque reactions at A and B.
- From a free-body analysis of the shaft,

$$T_A + T_B = 90\,\text{lb}\cdot\text{ft}$$

which is not sufficient to find the end torques. The problem is statically indeterminate.

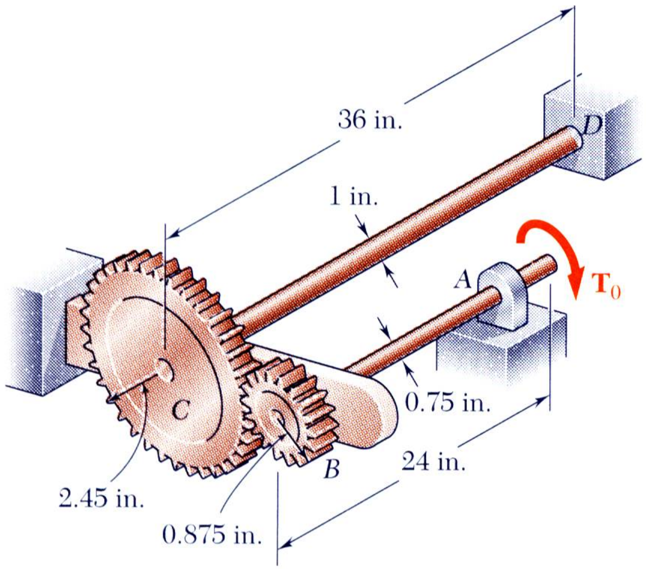
- Divide the shaft into two components which must have compatible deformations,

$$\phi = \phi_1 + \phi_2 = \frac{T_A L_1}{J_1 G} - \frac{T_B L_2}{J_2 G} = 0 \qquad T_B = \frac{L_1 J_2}{L_2 J_1} T_A$$

- Substitute into the original equilibrium equation,

$$T_A + \frac{L_1 J_2}{L_2 J_1} T_A = 90\,\text{lb}\cdot\text{ft}$$

Sample Problem 3.4



Two solid steel shafts are connected by gears. Knowing that for each shaft $G = 11.2 \times 10^6$ psi and that the allowable shearing stress is 8 ksi, determine (*a*) the largest torque T_0 that may be applied to the end of shaft *AB*, (*b*) the corresponding angle through which end *A* of shaft *AB* rotates.

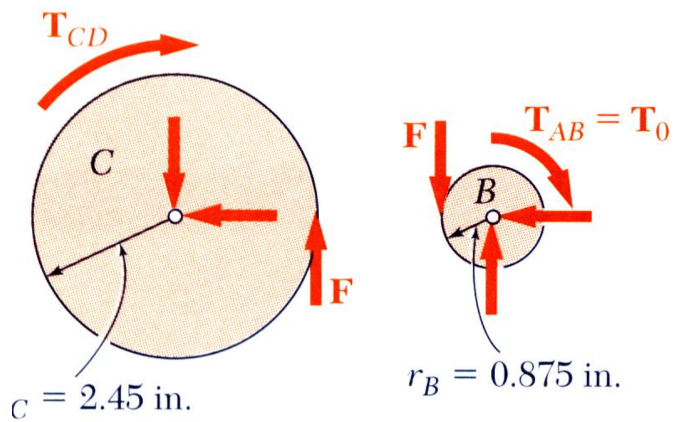
SOLUTION:

- Apply a static equilibrium analysis on the two shafts to find a relationship between T_{CD} and T_0
- Apply a kinematic analysis to relate the angular rotations of the gears
- Find the maximum allowable torque on each shaft – choose the smallest
- Find the corresponding angle of twist for each shaft and the net angular rotation of end *A*

Sample Problem 3.4

SOLUTION:

- Apply a static equilibrium analysis on the two shafts to find a relationship between T_{CD} and T_0

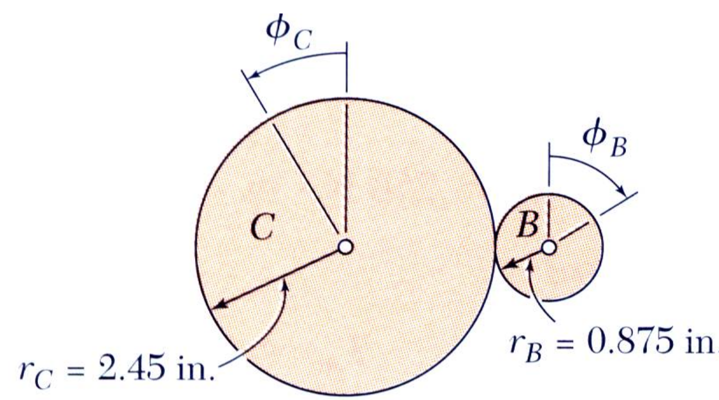


$$\sum M_B = 0 = F(0.875\text{in.}) - T_0$$

$$\sum M_C = 0 = F(2.45\text{in.}) - T_{CD}$$

$$T_{CD} = 2.8 T_0$$

- Apply a kinematic analysis to relate the angular rotations of the gears



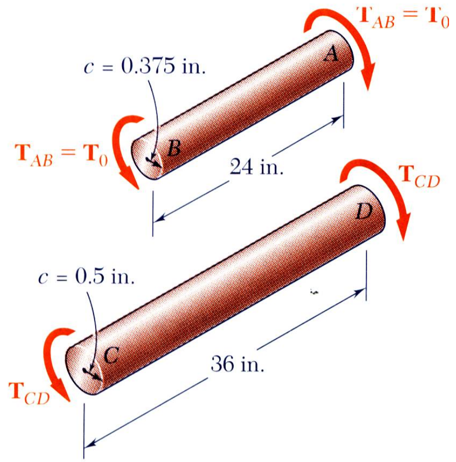
$$r_B \phi_B = r_C \phi_C$$

$$\phi_B = \frac{r_C}{r_B}\phi_C = \frac{2.45\text{in.}}{0.875\text{in.}}\phi_C$$

$$\phi_B = 2.8\phi_C$$

Sample Problem 3.4

- Find the T_0 for the maximum allowable torque on each shaft – choose the smallest



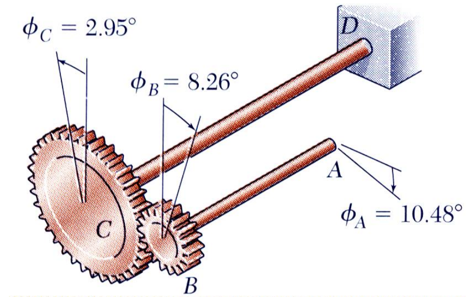
$$\tau_{max} = \frac{T_{AB}c}{J_{AB}} \quad 8000\,psi = \frac{T_0(0.375\,\text{in.})}{\frac{\pi}{2}(0.375\,\text{in.})^4}$$

$$T_0 = 663\,\text{lb}\cdot\text{in.}$$

$$\tau_{max} = \frac{T_{CD}c}{J_{CD}} \quad 8000\,psi = \frac{2.8T_0(0.5\,\text{in.})}{\frac{\pi}{2}(0.5\,\text{in.})^4}$$

$$T_0 = 561\,\text{lb}\cdot\text{in.} \qquad \boxed{T_0 = 561\,\text{lb}\cdot\text{in}}$$

- Find the corresponding angle of twist for each shaft and the net angular rotation of end A



$$\phi_{A/B} = \frac{T_{AB}L}{J_{AB}G} = \frac{(561\,\text{lb}\cdot\text{in.})(24\,in.)}{\frac{\pi}{2}(0.375\,\text{in.})^4(11.2\times10^6\,\text{psi})}$$

$$= 0.387\,\text{rad} = 2.22^{o}$$

$$\phi_{C/D} = \frac{T_{CD}L}{J_{CD}G} = \frac{2.8(561\,\text{lb}\cdot\text{in.})(24\,in.)}{\frac{\pi}{2}(0.5\,\text{in.})^4(11.2\times10^6\,\text{psi})}$$

$$= 0.514\,\text{rad} = 2.95^{o}$$

$$\phi_B = 2.8\phi_C = 2.8(2.95^{o}) = 8.26^{o}$$

$$\phi_A = \phi_B + \phi_{A/B} = 8.26^{o} + 2.22^{o} \qquad \boxed{\phi_A = 10.48^{o}}$$

Design of Transmission Shafts

- Principal transmission shaft performance specifications are:
  - power
  - speed

- Designer must select shaft material and cross-section to meet performance specifications without exceeding allowable shearing stress.

- Determine torque applied to shaft at specified power and speed,

$$P = T\omega = 2\pi f T$$

$$T = \frac{P}{\omega} = \frac{P}{2\pi f}$$

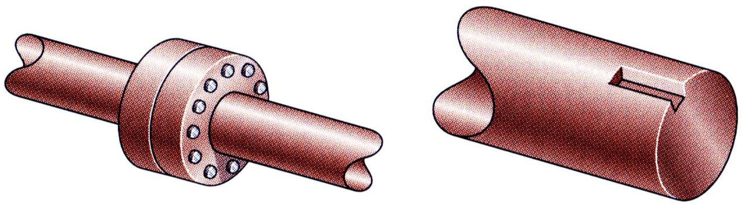
- Find shaft cross-section which will not exceed the maximum allowable shearing stress,

$$\tau_{\max} = \frac{Tc}{J}$$

$$\frac{J}{c} = \frac{\pi}{2} c^3 = \frac{T}{\tau_{\max}} \quad \text{(solid shafts)}$$

$$\frac{J}{c_2} = \frac{\pi}{2c_2}\left(c_2^4 - c_1^4\right) = \frac{T}{\tau_{\max}} \quad \text{(hollow shafts)}$$

Stress Concentrations



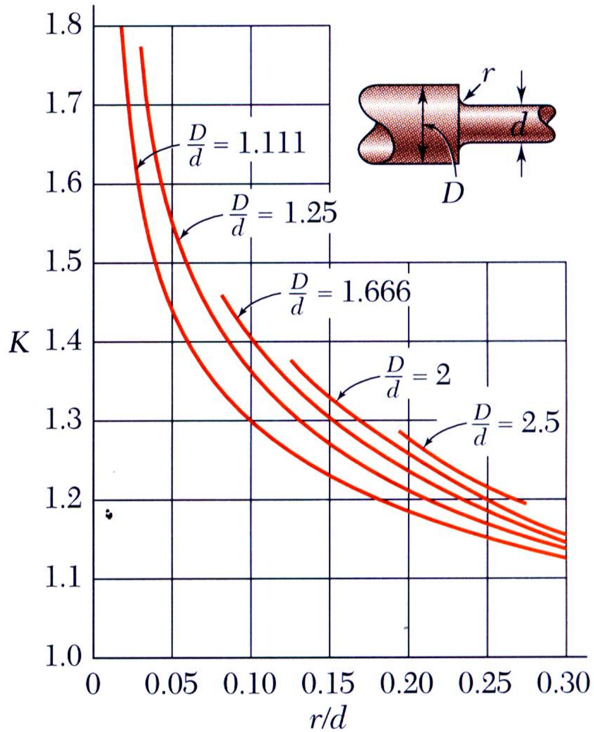


Fig. 3.32 Stress-concentration factors for fillets in circular shafts.†

- The derivation of the torsion formula,

$$\tau_{\max} = \frac{Tc}{J}$$

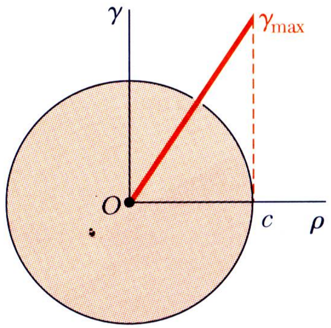
  assumed a circular shaft with uniform cross-section loaded through rigid end plates.

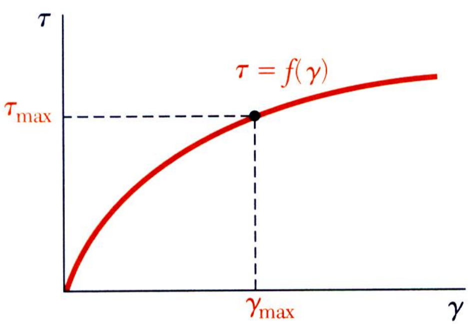
- The use of flange couplings, gears and pulleys attached to shafts by keys in keyways, and cross-section discontinuities can cause stress concentrations

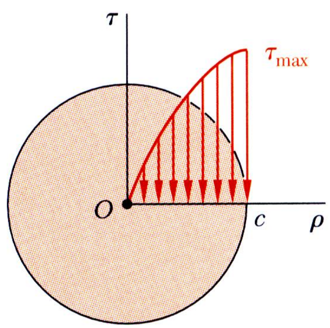
- Experimental or numerically determined concentration factors are applied as

$$\tau_{\max} = K\frac{Tc}{J}$$

Plastic Deformations







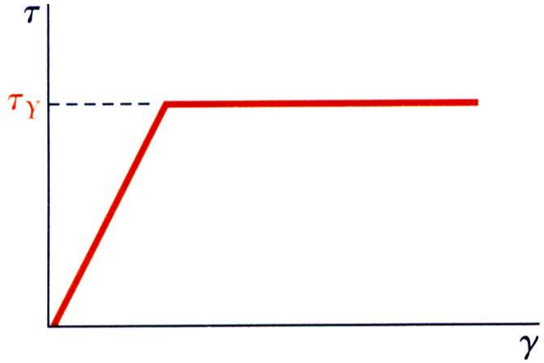
- With the assumption of a linearly elastic material,

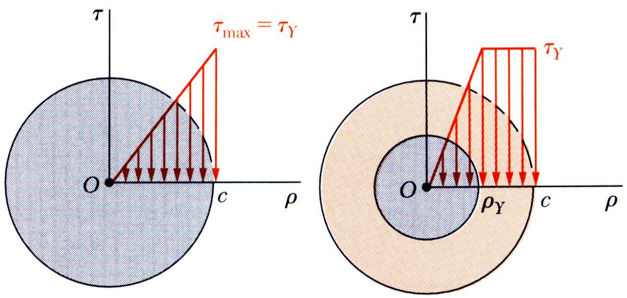
$$\tau_{\max} = \frac{Tc}{J}$$

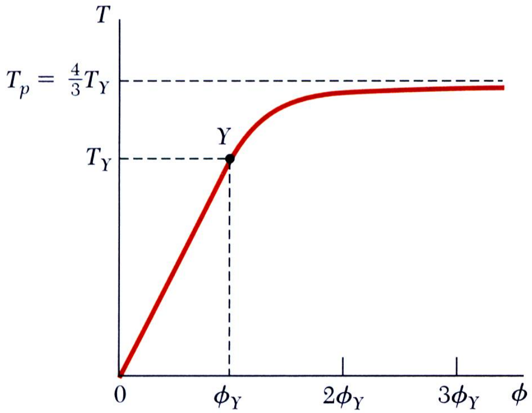
- If the yield strength is exceeded or the material has a nonlinear shearing-stress-strain curve, this expression does not hold.
- Shearing strain varies linearly regardless of material properties. Application of shearing-stress-strain curve allows determination of stress distribution.
- The integral of the moments from the internal stress distribution is equal to the torque on the shaft at the section,

$$T = \int_0^c \rho\tau(2\pi\rho\, d\rho) = 2\pi\int_0^c \rho^2\tau\, d\rho$$

Elastoplastic Materials







- At the *maximum elastic torque*,

$$T_Y = \frac{J}{c}\tau_Y = \tfrac{1}{2}\pi c^3 \tau_Y \qquad \phi_Y = \frac{L\gamma_Y}{c}$$

- As the torque is increased, a plastic region ($\tau = \tau_Y$) develops around an elastic core ($\tau = \frac{\rho}{\rho_Y}\tau_Y$)

$$\rho_Y = \frac{L\gamma_Y}{\phi}$$

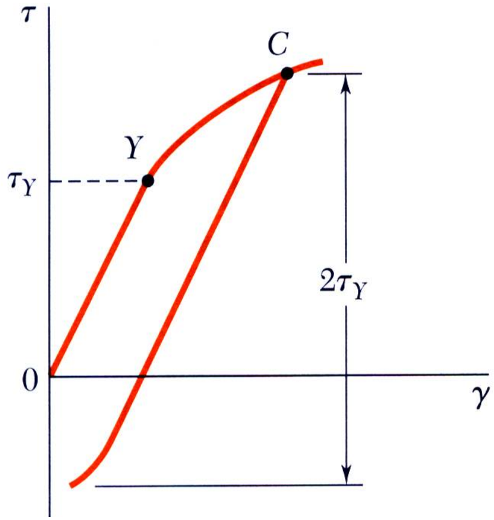
$$T = \tfrac{2}{3}\pi c^3 \tau_Y \left(1 - \frac{1}{4}\frac{\rho_Y^3}{c^3}\right) = \tfrac{4}{3}T_Y \left(1 - \frac{1}{4}\frac{\rho_Y^3}{c^3}\right)$$

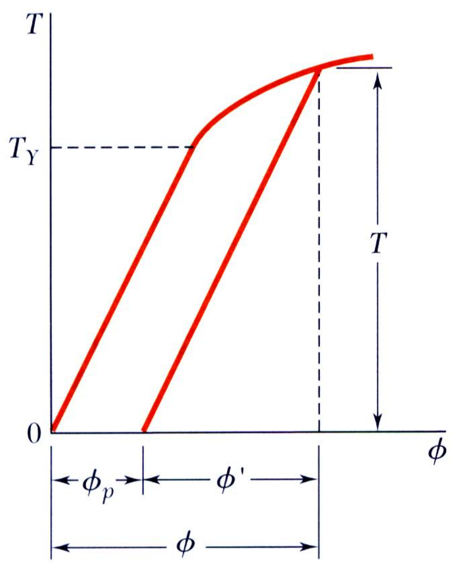
$$T = \tfrac{4}{3}T_Y \left(1 - \frac{1}{4}\frac{\phi_Y^3}{\phi^3}\right)$$

- As $\rho_Y \to 0$, the torque approaches a limiting value,

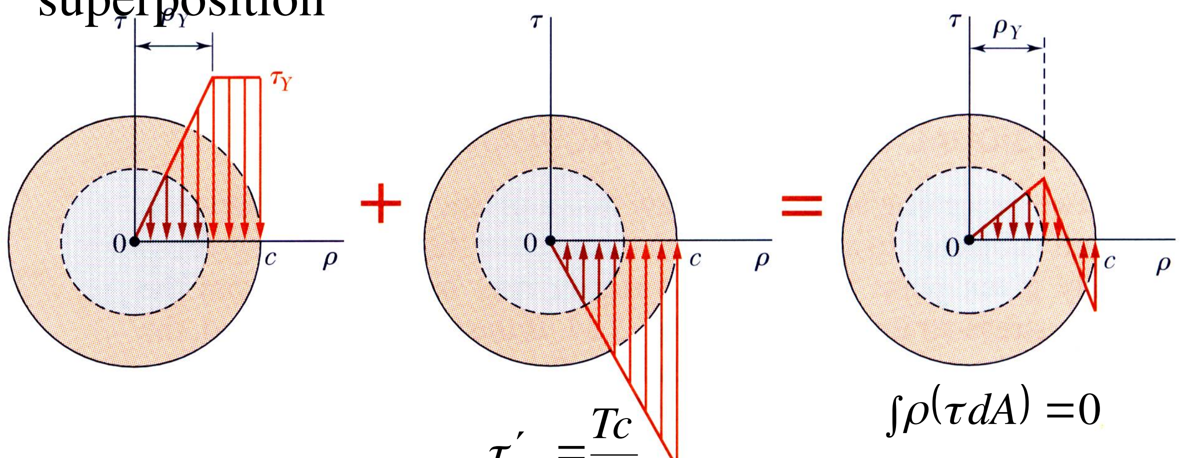
$$T_P = \tfrac{4}{3}T_Y = \textit{plastic torque}$$

Residual Stresses





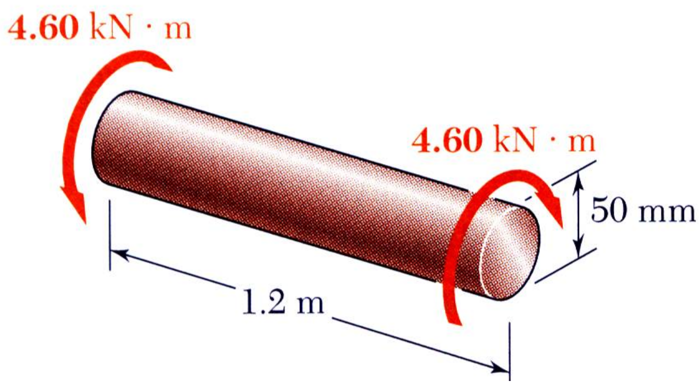
- Plastic region develops in a shaft when subjected to a large enough torque
- When the torque is removed, the reduction of stress and strain at each point takes place along a straight line to a generally non-zero residual stress
- On a T-ϕ curve, the shaft unloads along a straight line to an angle greater than zero
- Residual stresses found from principle of superposition



$$\tau'_m = \frac{Tc}{J}$$

$$\int \rho(\tau \, dA) = 0$$

Example 3.08/3.09



A solid circular shaft is subjected to a torque $T = 4.6\ \mathrm{kN \cdot m}$ at each end. Assuming that the shaft is made of an elastoplastic material with $\tau_Y = 150\ \mathrm{MPa}$ and $G = 77\ \mathrm{GPa}$ determine (a) the radius of the elastic core, (b) the angle of twist of the shaft. When the torque is removed, determine (c) the permanent twist, (d) the distribution of residual stresses.

SOLUTION:

- Solve Eq. (3.32) for ρ_Y/c and evaluate the elastic core radius
- Solve Eq. (3.36) for the angle of twist
- Evaluate Eq. (3.16) for the angle which the shaft untwists when the torque is removed. The permanent twist is the difference between the angles of twist and untwist
- Find the residual stress distribution by a superposition of the stress due to twisting and untwisting the shaft

Example 3.08/3.09

- Solve Eq. (3.32) for ρ_Y/c and evaluate the elastic core radius

$$T = \tfrac{4}{3}T_Y\left(1 - \frac{1}{4}\frac{\rho_Y^3}{c^3}\right) \Rightarrow \frac{\rho_Y}{c} = \left(4 - 3\frac{T}{T_Y}\right)^{1/3}$$

$$J = \tfrac{1}{2}\pi c^4 = \tfrac{1}{2}\pi\left(25\times10^{-3}\,\mathrm{m}\right)$$

$$= 614\times10^{-9}\,\mathrm{m}^4$$

$$\tau_Y = \frac{T_Y c}{J} \Rightarrow T_Y = \frac{\tau_Y J}{c}$$

$$T_Y = \frac{\left(150\times10^6\,\mathrm{Pa}\right)\left(614\times10^{-9}\,\mathrm{m}^4\right)}{25\times10^{-3}\,\mathrm{m}}$$

$$= 3.68\,\mathrm{kN\cdot m}$$

$$\frac{\rho_Y}{c} = \left(4 - 3\frac{4.6}{3.68}\right)^{1/3} = 0.630$$

$$\boxed{\rho_Y = 15.8\,\mathrm{mm}}$$

- Solve Eq. (3.36) for the angle of twist

$$\frac{\phi}{\phi_Y} = \frac{\rho_Y}{c} \Rightarrow \phi = \frac{\phi_Y}{\rho_Y/c}$$

$$\phi_Y = \frac{T_Y L}{JG} = \frac{\left(3.68\times10^3\,\mathrm{N}\right)\left(1.2\,\mathrm{m}\right)}{\left(614\times10^{-9}\,\mathrm{m}^4\right)\left(77\times10\,\mathrm{Pa}\right)}$$

$$\phi_Y = 93.4\times10^{-3}\,\mathrm{rad}$$

$$\phi = \frac{93.4\times10^{-3}\,\mathrm{rad}}{0.630} = 148.3\times10^{-3}\,\mathrm{rad} = 8.50^{\circ}$$

$$\boxed{\phi = 8.50^{\circ}}$$

Example 3.08/3.09

- Evaluate Eq. (3.16) for the angle which the shaft untwists when the torque is removed. The permanent twist is the difference between the angles of twist and untwist

$$\phi' = \frac{TL}{JG}$$

$$= \frac{\left(4.6\times10^{3}\,\mathrm{N\cdot m}\right)(1.2\,\mathrm{m})}{\left(6.14\times10^{9}\,\mathrm{m}^4\right)\left(77\times10^{9}\,\mathrm{Pa}\right)}$$

$$= 116.8\times10^{-3}\,\mathrm{rad}$$

$$\varphi_p = \phi - \phi'$$

$$= \left(116.8\times10^{-3} - 116.8\times10^{-3}\right)\mathrm{rad}$$

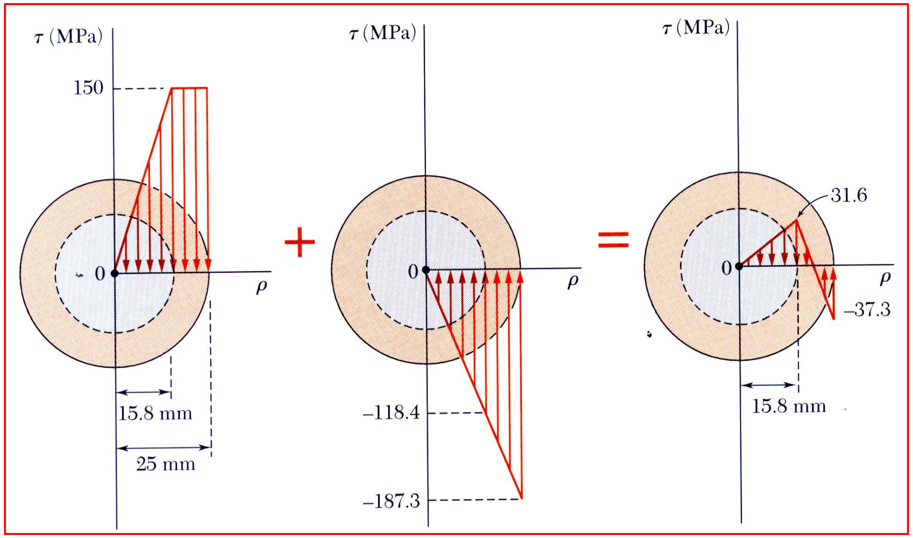
$$= 1.81^{\mathrm{o}}$$

$$\boxed{\phi_p = 1.81^{\mathrm{o}}}$$

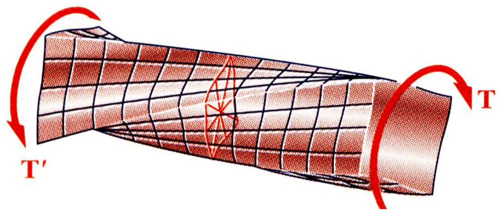
- Find the residual stress distribution by a superposition of the stress due to twisting and untwisting the shaft

$$\tau'_{\max} = \frac{Tc}{J} = \frac{\left(4.6\times10^{3}\,\mathrm{N\cdot m}\right)\left(25\times10^{-3}\,\mathrm{m}\right)}{614\times10^{-9}\,\mathrm{m}^4}$$

$$= 187.3\,\mathrm{MPa}$$



Torsion of Noncircular Members



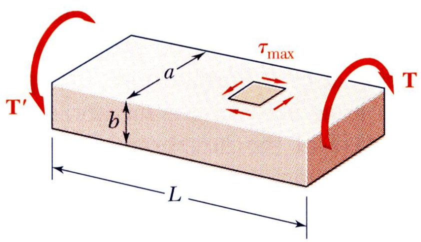
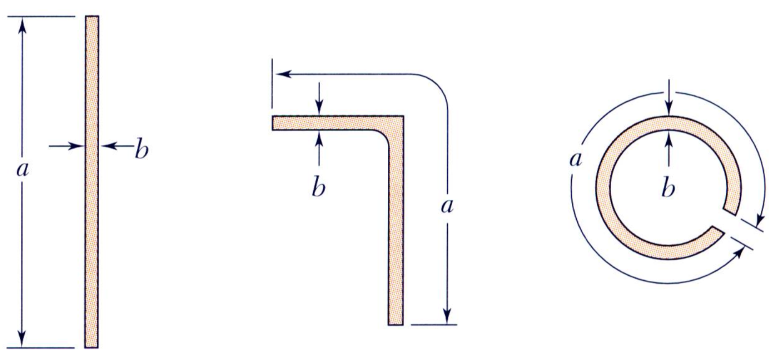


TABLE 3.1. Coefficients for Rectangular Bars in Torsion

a/b	c_1	c_2
1.0	0.208	0.1406
1.2	0.219	0.1661
1.5	0.231	0.1958
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333



- Previous torsion formulas are valid for axisymmetric or circular shafts

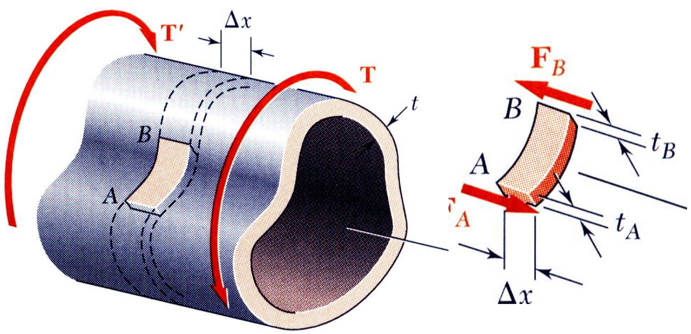
- Planar cross-sections of noncircular shafts do not remain planar and stress and strain distribution do not vary linearly

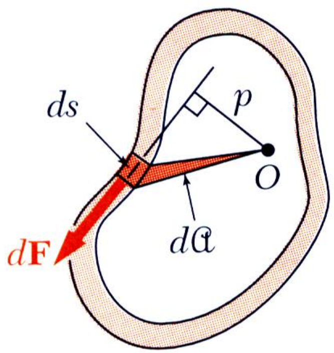
- For uniform rectangular cross-sections,

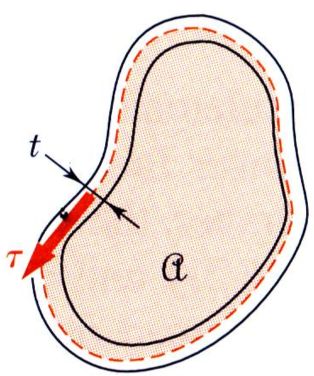
$$\tau_{max} = \frac{T}{c_1 ab^2} \qquad \phi = \frac{TL}{c_2 ab^3 G}$$

- At large values of *a/b*, the maximum shear stress and angle of twist for other open sections are the same as a rectangular bar.

Thin-Walled Hollow Shafts







- Summing forces in the x-direction on *AB*,

$$\sum F_x = 0 = \tau_A (t_A \Delta x) - \tau_B (t_B \Delta x)$$

$$\tau_A t_A = \tau_B t_B = \tau t = q = \text{shear flow}$$

shear stress varies inversely with thickness

- Compute the shaft torque from the integral of the moments due to shear stress

$$dM_0 = p\,dF = p\tau(t\,ds) = q(p\,ds) = 2q\,dA$$

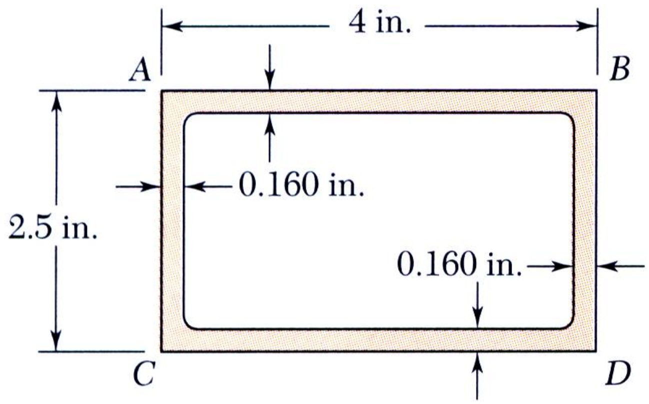
$$T = \oint dM_0 = \oint 2q\,dA = 2qA$$

$$\tau = \frac{T}{2tA}$$

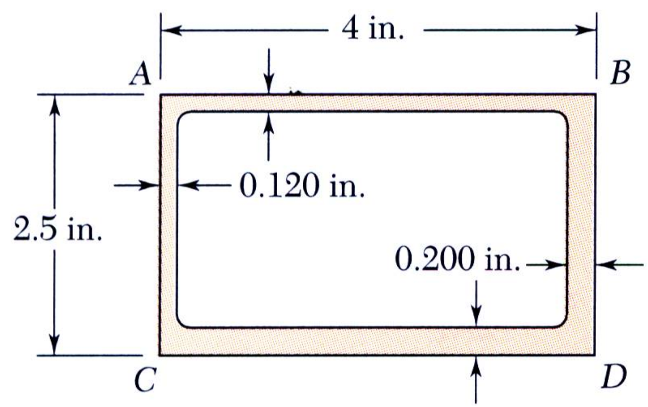
- Angle of twist (from Chapt 11)

$$\phi = \frac{TL}{4A^2G} \oint \frac{ds}{t}$$

Example 3.10



(a)



(b)

Extruded aluminum tubing with a rectangular cross-section has a torque loading of 24 kip-in. Determine the shearing stress in each of the four walls with (a) uniform wall thickness of 0.160 in. and wall thicknesses of (b) 0.120 in. on *AB* and *CD* and 0.200 in. on *CD* and *BD*.

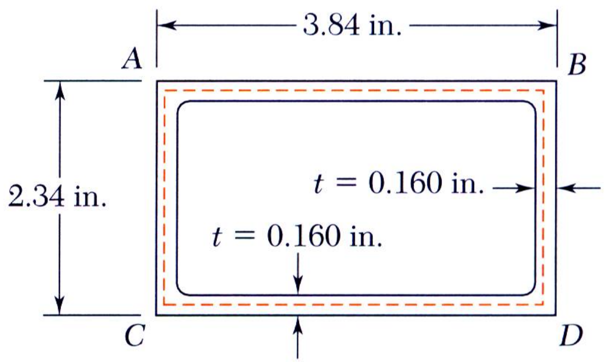
SOLUTION:

- Determine the shear flow through the tubing walls
- Find the corresponding shearing stress with each wall thickness

Example 3.10

SOLUTION:

- Determine the shear flow through the tubing walls



$$A = (3.84\,\text{in.})(2.34\,\text{in.}) = 8.986\,\text{in.}^2$$

$$q = \frac{T}{2A} = \frac{24\,\text{kip-in.}}{2(8.986\,\text{in.}^2)} = 1.335\frac{\text{kip}}{\text{in.}}$$

- Find the corresponding shearing stress with each wall thickness

with a uniform wall thickness,

$$\tau = \frac{q}{t} = \frac{1.335\,\text{kip/in.}}{0.160\,\text{in.}}$$

$$\boxed{\tau = 8.34\,\text{ksi}}$$

with a variable wall thickness

$$\tau_{AB} = \tau_{AC} = \frac{1.335\,\text{kip/in.}}{0.120\,\text{in.}}$$

$$\boxed{\tau_{AB} = \tau_{BC} = 11.13\,\text{ksi}}$$

$$\tau_{BD} = \tau_{CD} = \frac{1.335\,\text{kip/in.}}{0.200\,\text{in.}}$$

$$\boxed{\tau_{BC} = \tau_{CD} = 6.68\,\text{ksi}}$$