गणित विभाग, भारतीय प्रौद्योगिकी संस्थान पटना
DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY PATNA

MA-101, Mid Semester Examination September, 2013

Time: 2 hrs Max Marks: 30

Attempt all questions. Write brief and precise solutions to each question.

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(1) Let $X = (x_n)$ be a sequence defined by $x_1 = 1$, $x_2 = 2$ and $x_n = \frac{1}{2}(x_{n-2}+x_{n-1})$ for $n > 2$. Prove that X is a Cauchy sequence in $\mathbb{R}$ and then find the limit. [4]

(2) Use Gauss's test to find the values of k for which the series

$$\sum_{n=1}^{\infty}\left(\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}\right)^k$$

is convergent and divergent. [4]

(3) Use $\epsilon - \delta$ definition of limit of a function to show that $\lim_{x\to 1}\frac{x^2-x+1}{x+1} = \frac{1}{2}$. [2]

(4) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and $f(r) = 0$ for every rational number r. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$. [2]

(5) (a) Show that the polynomial $p(x) = x^4 + 7x^3 - 9$ has at least two real roots. [1]

(b) Find the points of discontinuity in following function and classify them:
$f(x) = x^2$ if $x < -2$; $= 2x + 3$ if $-2 \leq x < 0$; $= |x - 1|$ if $0 \leq x \leq 2$ [2]

(6) Prove or disprove that $f(x) = \sin\frac{1}{x}$ is uniformly continuous on $(0, \infty)$. [2]

(7) Let $f(x) = x\sqrt{8 - x^2}$. Then

(a) Find the intervals on which the function f is increasing or decreasing. [2]

(b) Identify the function's local extreme values, if any, mentioning where they are taken on. [2]

(c) Which, if any, of the extreme values are absolute? [1]

(8) Use L'Hôpital's rule to evaluate $\lim_{x\to 0+}\frac{\ln\sin x}{\ln x}$, where both numerator and denominator are defined on $(0, \pi)$. [2]

(9) Prove that if $a > 0$ then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$. [1.5]

(10) Let $a, b \in \mathbb{R}$. Show that if $a \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}$ then $a \leq b$. [1.5]

(11) Give examples of functions f and g such that f and g do not have limits at a point c, but such that both $f + g$ and fg have limits at c. [3]