3) Given, $f(z) = \begin{cases} e^{-z^{-4}}, & z \neq 0 \\ 0, & z = 0. \end{cases}$

put $z = re^{i\theta}$.

Then $f(z) = e^{-(re^{i\theta})^{-4}}$

$= e^{-r^{-4}e^{-4i\theta}}$

$= e^{-r^{-4}(\cos 4\theta - i\sin 4\theta)}$

$= e^{-r^{-4}\cos 4\theta}\left[\cos(r^{-4}\sin 4\theta) + i\sin(r^{-4}\sin 4\theta)\right]$

$= u + iv,$

where, $u = e^{-r^{-4}\cos 4\theta} \cdot \cos(r^{-4}\sin 4\theta)$

& $v = e^{-r^{-4}\cos\theta} \cdot \sin(r^{-4}\sin 4\theta)$.

Then find the partial derivatives of u & v with respect to r & θ and verify the C-R equations: $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ & $\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$.

Let $a_n = (2\pi n i)^{1/4}$, $n \geq 0$.

Then $a_n \to \infty$ as $n \to \infty$. $\Rightarrow$ $\frac{1}{a_n} \to 0$ as $n \to \infty$.

If $f(z)$ is continuous at $z=0$, then it must be $f(\frac{1}{a_n}) \to f(0) = 0$ as $n \to \infty$.

But $f(\frac{1}{a_n}) = e^{-(\frac{1}{a_n})^{-4}} = e^{-a_n^4} = e^{-2n\pi i}$

$= 1.$

$\therefore$ $f(\frac{1}{a_n})$ does not tend to '0' as $n \to \infty$.

So, $f(z)$ is not continuous at $z=0$ and hence not differentiable. Therefore, $f(z)$ is not analytic at $z=0$.

(completed)

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2) Let $f(z) = u + iv$,

Then necessary condition for $f(z)$ to be analytic is – the partial derivatives u_x, u_y, v_x, v_y are exist and satisfy C-R equation, i.e, $u_x = v_y$

& $u_y = -v_x$

Sufficient condition :- The partial derivatives u_x, u_y, v_x, v_y exist and continuous, and also satisfy C-R equation.

If possible, let there exist an analytic function whose real part is $u = x^2+y^2$.

Then u must satisfies Laplace's equation.

But, here $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2+2 = 4 \neq 0$.

– which is a contradiction.

∴ There does not exist any analytic function whose real part is $u = x^2+y^2$.

(complete)

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5). By the assumption, u must be harmonic function. Then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

We know, $x = \frac{z + \bar{z}}{2}$ $y = \frac{z - \bar{z}}{2i}$

$\therefore \frac{\partial x}{\partial z} = \frac{1}{2} = \frac{\partial x}{\partial \bar{z}}$, $\frac{\partial y}{\partial z} = \frac{1}{2i}$, $\frac{\partial y}{\partial \bar{z}} = -\frac{1}{2i} = \frac{i}{2}$

Also, we have,

$$\frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}}$$

$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} \right).$$

$$\Rightarrow \frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial x}{\partial z} + i \frac{\partial^2 u}{\partial y^2} \cdot \frac{\partial y}{\partial z} \right)$$

$$= \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$= \frac{1}{4} \cdot 0 = 0.$$

(proved)

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