Tutorial

Ex: Let (X, Y, Z) be jointly distributed continuous random variables with probability density function $f_{X,Y,Z}(x,y,z) = K\,x\,y\,e^{-\left(\frac{x+y+z}{3}\right)}$; $0<x<\infty$, $0<y<\infty$; $0<z<\infty$. Find K and then compute the probability $P(X<Y<Z)$.

Sol^n: Since $f_{X,Y,Z}(x,y,z)$ is a joint pdf of (X,Y,Z). So we must have

$$\int_0^\infty \int_0^\infty \int_0^\infty f_{X,Y,Z}(x,y,z)\,dx\,dy\,dz = 1$$

That is,

$$K \int_0^\infty \int_0^\infty \int_0^\infty x\,y\,e^{-\left(\frac{x+y+z}{3}\right)}\,dx\,dy\,dz = 1$$

$$\Rightarrow K\left(\int_0^\infty e^{-z/3}\,dz\right)\left(\int_0^\infty y\,e^{-y/3}\,dy\right)\left(\int_0^\infty x\,e^{-x/3}\,dx\right) = 1$$

$$\Rightarrow K \cdot 3 \cdot 9 \cdot 9 \cdot = 1$$

$$\Rightarrow \boxed{K = \frac{1}{243}}$$

Let us compute $P(X<Y<Z)$.

$$P(X<Y<Z) = \iiint\limits_{\{(x,y,z):\, x<y<z,\ x>0,\, y>0,\, z>0\}} f_{X,Y,Z}(x,y,z)\,dx\,dy\,dz$$

$$= \int_{z=0}^{\infty}\int_{y=0}^{z}\int_{x=0}^{y} xy\, e^{-\left(\frac{x+y+z}{3}\right)}\,dx\,dy\,dz$$

$$= \frac{7}{108}.$$ (After some simplifications).

(You can fix the limits of this triple integral in different ways. Try one from your side.).

Ex: Let X and Y have the joint probability mass function (PMF) $p_{X,Y}(x,y) = c\,\lambda^x \mu^y$, $x \geq 1, y \geq 1$, $0 < \lambda, \mu < 1$. Find the value of c. Find $P(X>Y)$. Find $P(X=Y)$.

Sol^n: This example is related to a 2-dimensional discrete random variable.

Since $p_{X,Y}(x,y)$ is the joint PMF of (X,Y) and so we must have

$$\sum_{x\in R_X}\sum_{y\in R_Y} p_{X,Y}(x,y) = 1$$

That is

$$c \cdot \sum_{x=1}^{\infty} \sum_{y=1}^{\infty} \lambda^x \mu^y = 1$$

$$\Rightarrow c \cdot \left(\sum_{x=1}^{\infty} \lambda^x\right) \cdot \left(\sum_{y=1}^{\infty} \mu^y\right) = 1$$

$$= c \cdot \frac{\lambda}{1-\lambda} \cdot \frac{\mu}{1-\mu} = 1$$

$$\Rightarrow \boxed{c = \frac{(1-\lambda)(1-\mu)}{\lambda \mu}}$$

Let us compute $P(X > Y)$.

$$P(X > Y) = \sum_{y=1}^{\infty} \sum_{x=y+1}^{\infty} p_{X,Y}(x,y)$$

$$= c \cdot \sum_{y=1}^{\infty} \sum_{x=y+1}^{\infty} \lambda^x \mu^y$$

$$= c \left(\sum_{y=1}^{\infty} \mu^y\right) \left(\sum_{x=y+1}^{\infty} \lambda^x\right)$$

$$= c \sum_{y=1}^{\infty} \mu^y \cdot \frac{\lambda^{y+1}}{1-\lambda} = \left(\frac{c\lambda}{1-\lambda}\right) \cdot \sum_{y=1}^{\infty} (\mu\lambda)^y$$

$$= \left(\frac{c\lambda}{1-\lambda}\right) \cdot \frac{\mu\lambda}{1-\mu\lambda} = \underbrace{\frac{(1-\lambda)(1-\mu)}{\lambda\mu}}_{\text{value of } c} \cdot \frac{\lambda}{1-\lambda} \cdot \frac{\mu\lambda}{1-\mu\lambda}$$

$$= \frac{\lambda(1-\mu)}{(1-\lambda\mu)}.$$

finally $P(X=Y)$ is computed as

$$P(X=Y) = c\sum_{x=1}^{\infty} \lambda^x \mu^x = c\cdot\sum_{x=1}^{\infty}(\lambda\mu)^x$$

$$= c\cdot\frac{\lambda\mu}{1-\lambda\mu} = \frac{(1-\lambda)(1-\mu)}{1-\lambda\mu}.$$

__Ex__: Let X and Y be two independent random variables distributed as $X \sim N(0,1)$ and $Y \sim \chi^2(n)$ respectively. Define

$$T = \frac{X}{\sqrt{Y/n}}.$$

The prob. distribution of T is known as the student's t - distribution. We denote the pdf of T as $\boxed{T \sim t_n}$ where n is the only parameter taking values as $n=1,2,3,\cdots$

__Sol^n__: We obtain the pdf of T using multivariate jacobian formula. In this formula dimension of random variables and no of transformations given must match. For this problem (X,Y) is a 2-dimensional RV, and only one transformation $T = X/\sqrt{Y/n}$ is given.

So how to proceed ??

We can consider one auxilary transformation say $U = Y$ and then try to compute joint pdf of (T, U). From this joint pdf we can easily get pdf of our variable of interest which is T here.

Now first note the given information.

$X \sim N(0,1)$, $Y \sim \chi^2(n)$, X & Y independent

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty.$$

$$f_Y(y) = \frac{y^{\frac{n}{2}-1} e^{-y/2}}{2^{\frac{n}{2}} \Gamma\frac{n}{2}}, \quad 0 < y < \infty$$

n is a positive integer.

So joint pdf of (X, Y) (given the independence) is

$$f_{X,Y}(x, y) = f_X(x) f_Y(y).$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot \frac{y^{n/2-1} e^{-y/2}}{2^{n/2} \Gamma\frac{n}{2}}, \quad \begin{array}{l} -\infty < x < \infty \\ 0 < y < \infty \end{array}$$

Next the given transformations are

$$T = \frac{X}{\sqrt{Y/n}}, \quad U = Y.$$

This transformation is one-one. Let us find inverse functions as

$$x = h_1(t,u) = t\sqrt{\frac{u}{n}}$$

$$y = h_2(t,u) = u.$$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial u} \end{vmatrix} = \begin{vmatrix} \sqrt{\frac{u}{n}} & \frac{t}{2\sqrt{un}} \\ 0 & 1 \end{vmatrix} = \sqrt{\frac{u}{n}}$$

range of (U,T): $\left.\begin{matrix} -\infty < x < \infty \\ 0 < y < \infty \end{matrix}\right\} \Rightarrow \begin{matrix} -\infty < t < \infty \\ 0 < u < \infty \end{matrix}$

Finally joint pdf of (T,U) is given by

$$f_{T,U}(t,u) = f_{X,Y}(h_1(t,u), h_2(t,u)) \cdot |J|$$

$$= f_X(h_1(t,u))\, f_Y(h_2(t,u))\, |J|$$

$$= f_X\left(t\sqrt{\frac{u}{n}}\right) f_Y(u) \sqrt{\frac{u}{n}}, \quad \begin{matrix} -\infty < t < \infty \\ 0 < u < \infty \end{matrix}$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{t^2 u}{2n}} \cdot \frac{u^{\frac{n}{2}-1} e^{-u/2}}{2^{n/2}\,\Gamma\frac{n}{2}} \cdot \sqrt{\frac{u}{n}}$$

$$= \frac{1}{\sqrt{2\pi n}\; 2^{n/2}\,\Gamma\frac{n}{2}} \cdot u^{\frac{n-1}{2}}\, e^{-\frac{u}{2}\left(1+\frac{t^2}{n}\right)}$$

So joint pdf of (T, U) is

$$\boxed{f_{T,U}(t,u) = \frac{1}{\sqrt{2\pi n}\, 2^{n/2} \Gamma\frac{n}{2}} u^{\frac{n-1}{2}} e^{-\frac{u}{2}\left(1+\frac{t^2}{n}\right)} \qquad \begin{array}{l} -\infty < t < \infty \\ 0 < u < \infty \end{array}}$$

But note that our variable of interest is T and we wanted pdf of this variable.

So,

$$f_T(t) = \int_{-\infty}^{\infty} f_{T,U}(t,u)\, du$$

$$= \int_0^{\infty} f_{T,U}(t,u)\, du$$

$$= \frac{1}{\sqrt{2\pi n}\, 2^{n/2} \Gamma\frac{n}{2}} \int_0^{\infty} u^{\frac{n-1}{2}} e^{-\frac{u}{2}\left(1+\frac{t^2}{n}\right)} du$$

$$\boxed{\begin{array}{l} \text{put } \frac{u}{2}\left(1+\frac{t^2}{n}\right) = z \\ \Rightarrow du = \frac{2}{\left(1+\frac{t^2}{n}\right)} dz \end{array}}$$

$$= \frac{1}{\sqrt{2\pi n}\, 2^{n/2} \Gamma\frac{n}{2}} \int_0^{\infty} \left[\frac{2z}{1+\frac{t^2}{n}}\right]^{\frac{n-1}{2}} e^{-z} \cdot \left(\frac{2}{1+\frac{t^2}{n}}\right) dz$$

$$= \frac{2^{\frac{n+1}{2}}}{\sqrt{2\pi n}\, 2^{n/2} \Gamma\frac{n}{2} \left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} \cdot \int_0^{\infty} z^{\frac{n-1}{2}} e^{-z} dz$$

$$f_T(t) = \frac{1}{\sqrt{\pi n}} \cdot \frac{1}{\Gamma\frac{n}{2}} \cdot \frac{1}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} \cdot \Gamma\frac{n+1}{2}$$

$\therefore$ PDF of Students' T random variable is

$$\boxed{f_T(t) = \frac{\Gamma\frac{n+1}{2}}{\sqrt{\pi n}\,\Gamma\frac{n}{2}} \cdot \frac{1}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} \; ; \; -\infty<t<\infty, \; n=1,2,3,\ldots}$$

* T-dist^n is very useful in statistics. It is symmetric about the point $t=0$. Its mean, median and mode are all same and it is zero. Furthere $V(T) = \frac{n}{n-2}$, $n>2$. From CLT, as $n \to \infty$, $T \sim N(0,1)$.

Ex: Let (X, Y, Z) be iid random variables with common pdf as $\exp(1)$. Define $U = X+Y+Z$, $V = \frac{X+Y}{X+Y+Z}$, $W = \frac{X}{X+Y}$. Find joint pdf of (U, V, W). Furthere find corresponding marginal probability density functions of U, V and W respectively.

<u>Ex</u>: Let $X_1, X_2 \overset{iid}{\sim} N(0,1)$ distribution. Find pdf of $\frac{(X_1 - X_2)^2}{2}$.

<u>Ex</u>: Let $X, Y \overset{iid}{\sim} U(0,1)$ distribution. Consider the transformation $U = X+Y$, $V = X-Y$. Find joint pdf of (U,V). Also compute marginal pdfs.

<u>Ex</u>: Let X and Y be independently distributed random variables such that $X \sim \chi^2(m)$ and $Y \sim \chi^2(n)$ where $m, \& n$ are positive integers. Define $F = \frac{X/m}{Y/n}$. Then F is said to have fisher distⁿ (F- distⁿ). Find pdf of this variable. Denote it as $\boxed{F \sim F(m,n)}$.

(Note: Like t- distⁿ, F distⁿ has found wide applications in statistical literature.)

Thank you!