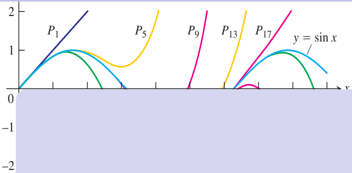


Mathematics I

Series

Amit K. Verma

Department of Mathematics
IIT Patna

Infinite Series, n^{th} Term, Partial Sum, Converges, Sum

1. **Consider infinite series**

$$\mathbf{a_1 + a_2 + a_3 + \cdots + a_n + \cdots .}$$

2. **The sequence $\{s_n\}$ defined by $s_n = \sum_{k=1}^{n} a_k$ is the sequence of partial sums of the series, the number s_n being the n^{th} partial sum.**
3. **If the sequence of partial sums converges to a limit L, we say that the series converges and that its sum is L.**
4. **In this case, we also write**

$$\mathbf{a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k = L.}$$

5. **If the sequence of partial sums of the series does not converge, we say that the series diverges.**

Combining Series

THEOREM

If $\sum \mathbf{a_n}$ and $\sum \mathbf{b_n}$ are convergent series, then

1. Sum Rule: $\sum(\mathbf{a_n} + \mathbf{b_n}) = \sum \mathbf{a_n} + \sum \mathbf{b_n} = \mathbf{A} + \mathbf{B}$.
2. Difference Rule: $\sum(\mathbf{a_n} - \mathbf{b_n}) = \sum \mathbf{a_n} - \sum \mathbf{b_n} = \mathbf{A} - \mathbf{B}$.
3. Constant Multiple Rule: $\sum \mathbf{k a_n} = \mathbf{k} \sum \mathbf{a_n} = \mathbf{kA}$.

Corollary

As corollaries of above Theorem, we have

1. Every nonzero constant multiple of a divergent series diverges.
2. If $\sum \mathbf{a_n}$ converges and $\sum \mathbf{b_n}$ diverges, then $\sum(\mathbf{a_n} + \mathbf{b_n})$ and $\sum(\mathbf{a_n} - \mathbf{b_n})$ both diverge. **CAUTION**

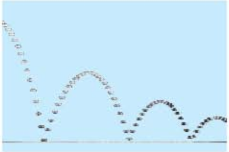
Adding or Deleting Terms

We can add a finite number of terms to a series or delete a finite number of terms without altering the series' convergence or divergence, although in the case of convergence this will usually change the sum.

Reindexing

As long as we preserve the order of its terms, we can reindex any series without altering its convergence. To raise the starting value of the index h units, replace the n in the formula for a_n by $n - h$.

Known Examples



1. **Geometric Series** Consider the geometric series

$$\mathbf{a + ar + ar^2 + \cdots + ar^{n-1} + \cdots.}$$

2. $|\mathbf{r}| < \mathbf{1}$, geometric series is Convergent and it converges to $\frac{\mathbf{a}}{\mathbf{1-r}}$.
3. if $|\mathbf{r}| \geq \mathbf{1}$ then geometric series is Divergent.
4. **A Nongeometric but Telescoping Series** $\displaystyle\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{1}}{\mathbf{n(n+1)}} = \mathbf{1}.$

THEOREM

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Caution:

Converse of this theorem is not true (Condition $\lim_{n\to\infty} a_n = 0$ Necessary but not sufficient) that is if $\lim_{n\to\infty} a_n = 0$ no conclusion can be drawn about the convergence of the series $\sum_{n=1}^{\infty} a_n$. For Example $\sum \left(\frac{1}{n}\right)$

Corollary

If $\lim_{n\to\infty} a_n$ fails to exist or is different from zero, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Does $\sum a_n$ converge?

1. In this section and the next two, **we study series that do not have negative terms**.
2. The reason for this restriction is that the partial sums of these series form **nondecreasing sequences** and
3. nondecreasing sequences that are bounded from above always converge (Theorem 6, Section 11.1, Thomas Calculus).
4. To show that a series of nonnegative terms converges, we need only show that its **partial sums are bounded from above**.

Suppose that $\sum_{\mathbf{n=1}}^{\infty} \mathbf{a_n}$ is an infinite series with $\mathbf{a_n \geq 0}$ for all $\mathbf{n}$. Then sequence of partial sums $\mathbf{s_n}$ satisfies $\mathbf{s_1 \leq s_2 \leq \cdots \leq s_n \leq s_{n+1} \leq \cdots}$.

Corollary of Theorem

A series $\sum_{n=1}^{\infty} a_n$ of **nonnegative terms** **converges** **if and only if** its partial sums are bounded from above.

The Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is divergent, but this does not follow from the n^{th}-Term Test. The n^{th} term $1/n$ does go to zero, but the series still diverges. The reason it diverges is because there is no upper bound for its partial sums.

Given $\sum \mathbf{a_n}$.

$$\Downarrow$$

Once $\lim_{n\to\infty} \mathbf{a_n} = \mathbf{0}$ then Further test is needed.

$$\Downarrow$$

For that we assume that $\mathbf{a_n} \geq \mathbf{0}$ so that $\{\mathbf{s_n}\}$ is Nondecreasing and as soon as $\{\mathbf{s_n}\}$ is bounded above we are done.

$$\Downarrow$$

As in that case

$$\lim_{n\to\infty} \mathbf{s_n} = \textbf{Finite}$$

gives the sum of the series and also indicates that $\sum \mathbf{a_n}$ is convergent.

Improper Integration

$\int_a^\infty \mathbf{f(x)dx} = \textbf{Finite}$ then Integral is said to be **convergent** and if $\int_a^\infty \mathbf{f(x)dx} = \infty$ (**or** $-\infty$ **or does not exist**) then integral is said to be **divergent** (a is some positive number).

Cauchy Criterion

$\sum a_n$ satisfies Cauchy's criterion if its sequence $\{s_n\}$ of partial sums is a Cauchy Sequence, i.e., for each $\epsilon > 0$ there a positive integer N such that

$$m, n > N \Rightarrow |s_n - s_m| < \epsilon;$$

or for each $\epsilon > 0$ there a positive integer N such that

$$n \geq m > N \Rightarrow |s_n - s_{m-1}| < \epsilon;$$

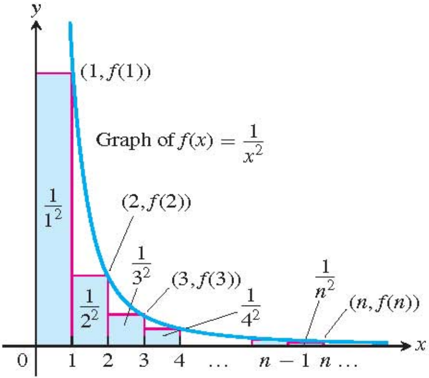
or for each $\epsilon > 0$ there a positive integer N such that

$$n \geq m > N \Rightarrow |\sum_{k=m}^{n} a_k| < \epsilon.$$

The Integral Test

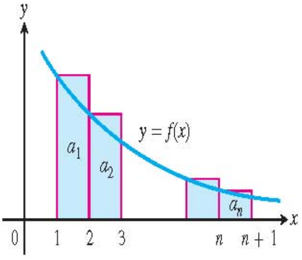
EXAMPLE : Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots .$$

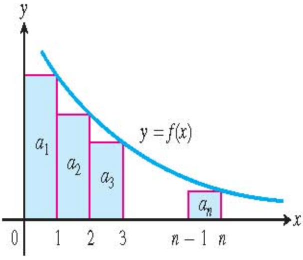


Caution The series and integral need not have the same value in the convergent case, e.g., we compute $\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$ but $\sum_{n=1}^{\infty} \left(\frac{1}{n^2}\right) = \frac{\pi^2}{6}$.

Geometrical Interpretation of Integral Test



(a)



(b)

$$\int_1^{n+1} f(x)dx \le a_1 + a_2 + \cdots + a_n \le a_1 + \int_1^{n} f(x)dx.$$

THEOREM: The Integral Test

1. Let $\{\mathbf{a_n}\}$ be a sequence of positive terms.
2. Suppose that $\mathbf{a_n} = \mathbf{f}(\mathbf{n})$ where,
3. $\mathbf{f}$ is a continuous, positive, decreasing function of $\mathbf{x}$ for all $\mathbf{x} \geq \mathbf{N}$ ($\mathbf{N}$ a positive integer).
4. Then the series

$$\sum_{\mathbf{n}=\mathbf{N}}^{\infty} \mathbf{a_n}$$

and the integral

$$\int_{\mathbf{N}}^{\infty} \mathbf{f}(\mathbf{x})\mathbf{dx}$$

both converge or **both diverge**.

Convergence and Divergence of the p-series

The **p**-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{n^p} + \cdots$$

(p a real constant) converges if $p > 1$, and diverges $p \leq 1$.

The p-series with $p = 1$ is the harmonic series. The p-Series Test shows that the harmonic series is just **barely divergent**; if we increase p to 1.000000001, for instance, the series converges!

Comparison Tests

1. The Comparison Tests
2. Limit Comparison Tests

The Comparison Test

THEOREM: The Comparison Test

Let $\sum \mathbf{a_n}$ be a series of **non-negative** terms. Then,

(a) $\sum \mathbf{a_n}$ converges if there is a convergent series $\sum \mathbf{c_n}$ with $\mathbf{a_n} \leq \mathbf{c_n}$ for all $\mathbf{n} > \mathbf{N}$, for some integer $\mathbf{N}$.

(b) $\sum \mathbf{a_n}$ diverges if there is a divergent series $\sum \mathbf{d_n}$ with $\mathbf{a_n} \geq \mathbf{d_n}$ for all $\mathbf{n} > \mathbf{N}$, for some integer $\mathbf{N}$.

The Limit Comparison Test

We now introduce a comparison test that is particularly useful for series in which is $\mathbf{a_n}$ is a rational function of $\mathbf{n}$.

THEOREM: Limit Comparison Test

Suppose that $\mathbf{a_n > 0}$, $\mathbf{b_n > 0}$ and $\mathbf{c_n > 0}$ for all $\mathbf{n \geq N}$ (N an integer).

1. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

The Ratio Test

THEOREM : The Ratio Test

Let $\sum \mathbf{a_n}$ be a series with positive terms and suppose that

$$\lim_{\mathbf{n}\to\infty} \frac{\mathbf{a_{n+1}}}{\mathbf{a_n}} = \rho.$$

Then

(a) the series converges if $\rho < \mathbf{1}$,

(b) the series diverges if $\rho > \mathbf{1}$ or ρ is infinite,

(c) the test is inconclusive if $\rho = \mathbf{1}$.

The Root Test

THEOREM : The Root Test

Let $\sum \mathbf{a_n}$ be a series with $\mathbf{a_n} \geq \mathbf{0}$ for $\mathbf{n} \geq \mathbf{N}$, suppose that

$$\lim_{\mathbf{n}\to\infty} \sqrt[n]{\mathbf{a_n}} = \rho.$$

Then

(a) the series converges if $\rho < \mathbf{1}$,

(b) the series diverges if $\rho > \mathbf{1}$ or ρ is infinite,

(c) the test is inconclusive if $\rho = \mathbf{1}$.

1. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ **Integral Test**, **Conv.** for $p > 1$ and **Div.** for $p \leq 1$.
2. $\sum_{n=1}^{\infty} \frac{3}{n + \sqrt{n}}$ **DCT**, **Div.**
3. $\sum_{n=1}^{\infty} \frac{1}{n!}$ **DCT**, **Conv.**
4. $\sum_{n=1}^{\infty} \frac{2n+1}{n^2 + 3n - 2}$ **LCT (1)**, $b_n = (1/n)$ **Div.**
5. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$ **LCT (3)**, $b_n = (1/n)$ **Div.**
6. $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$ **LCT (2)**, $b_n = (1/n^2)$ **Conv.**

1. $\sum_{n=1}^{\infty} \frac{a^n}{n!}$ Ratio, Conv.
2. $\sum_{n=1}^{\infty} \frac{n^{10}}{n!}$ Ratio, Conv.
3. $\sum_{n=1}^{\infty} \frac{n^n}{(2^n)^2}$ Root, Div.
4. $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$ Root, Conv.

Problem

Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{\frac{3}{2}}}$.

Solution: Given $a_n = \frac{(\ln n)^2}{n^{\frac{3}{2}}}$. Let $b_n = \frac{1}{n^a}$. Consider

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(\ln n)^2}{n^{\frac{3}{2}-a}}$. Under what condition on a we can apply L'Hopital rule: $a < (3/2)$. Converting in to functions and applying L'Hopital rule we get

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2}{((3/2)-a)^2 n^{(3/2)-a}} = 0, \quad a < (3/2).$$

But to conclude from LCT the series $\sum b_n$ should be convergent. For that we must have $1 < a$. So for this exercise any value of a between 1 and $3/2$ will work. For example b_n can be chosen as $b_n = \frac{1}{n^{1.25}}$. Final answer:

$\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{\frac{3}{2}}}$ is **Convergent from LCT (2)**.

Alternating Series

A series in which the terms are alternately positive and negative is an alternating series.

Examples:

1. $\mathbf{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots + \frac{(-1)^{n+1}}{n} + \cdots}$. Convergent (we will see very soon.)
2. $\mathbf{-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \frac{(-1)^n 4}{2^n} + \cdots}$. Convergent (being a Geometric Series)
3. $\mathbf{1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n+1} n + \cdots}$. Divergent (nth term test) (What is the important point to be observed here???)

THEOREM : The Alternating Series Test (Leibnitz's Theorem)

The series

$$\sum_{\mathbf{n=1}}^{\infty}(-1)^{\mathbf{n+1}}\mathbf{u_n} = \mathbf{u_1} - \mathbf{u_2} + \mathbf{u_3} - \mathbf{u_4} + \cdots$$

converges if all three of the following conditions are satisfied:

AS1. The $\mathbf{u_n}$'s are all positive.

AS2. $\mathbf{u_n} \geq \mathbf{u_{n+1}}$ for all for $\mathbf{n} \geq \mathbf{N}$, some integer $\mathbf{N}$.

AS3. $\mathbf{u_n} \to \mathbf{0}$, i.e., $\lim_{\mathbf{n}\to\infty} \mathbf{u_n} = \mathbf{0}$.

Example: $\sum(-1)^{n+1}\frac{1}{n}$ is convergent Leibnitz's Theorem.

Important Result

For a sequence $\{\mathbf{a_n}\}$ the terms of even index are denoted by $\mathbf{a_{2k}}$ and the terms of odd index by $\mathbf{a_{2k+1}}$. Prove that if $\mathbf{a_{2k}} \to \mathbf{L}$ and $\mathbf{a_{2k+1}} \to \mathbf{L}$ then $\mathbf{a_n} \to \mathbf{L}$.

Consider that n is even (i.e., $n = 2m$) let us write the partial sum

$$s_{2m} = (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2m-1} - u_{2m}).$$

Using (AS2) we have $s_{2m+2} \geq s_{2m}$ ($\{s_{2m}\}$ is Non-decreasing Sequence). Also

$$s_{2m} = u_1 - \{(u_2 - u_3) + (u_4 - u_5) + \cdots + (u_{2m-2} - u_{2m-1}) + u_{2m}\}$$

gives $s_{2m} \leq u_1$. Thus $\{s_{2m}\}$ is nondecreasing and bounded above by u_1 it must have a limit say L. That is

$$\lim_{m\to\infty} s_{2m} = L.$$

If n is odd say $n = 2m + 1$. Then $s_{2m+1} = s_{2m} + u_{2m+1}$. By **AS3** we have

$$\lim_{m\to\infty} s_{2m+1} = \lim_{m\to\infty} s_{2m} = L.$$

Therefore by (Section 11.1, Exercise 119 Thomas Calculus 11th Edition) we have $\lim_{n\to\infty} s_n = L$.

DEFINITION : Absolutely Convergent

A series $\sum \mathbf{a_n}$ converges absolutely (is absolutely convergent) if the corresponding series of absolute values, $\sum |\mathbf{a_n}|$, converges. Example: $\sum(-1)^{n+1}\frac{1}{n^2}$ is absolutely convergent.

DEFINITION : Conditionally Convergent

A series that converges but does not converge absolutely converges conditionally. Example: $\sum(-1)^{n+1}\frac{1}{n}$ is conditionally convergent.

THEOREM : The Absolute Convergence Test

If $\sum_{\mathbf{n=1}}^{\infty} |\mathbf{a_n}|$ converges, then $\sum_{\mathbf{n=1}}^{\infty} \mathbf{a_n}$ converges.[a]

[a] **CAUTION We can rephrase above Theorem to say that every absolutely convergent series converges. However, the converse statement is false:** Many convergent series do not converge absolutely such as the alternating harmonic series $\sum(-1)^{n+1}\frac{1}{n}$.

Proof of Absolute Convergence Test

Proof.

$$\sum_{n=1}^{\infty} \mathbf{a_n} = \sum_{n=1}^{\infty} (\mathbf{a_n} + |\mathbf{a_n}| - |\mathbf{a_n}|) = \sum_{n=1}^{\infty} (\mathbf{a_n} + |\mathbf{a_n}|) - \sum_{n=1}^{\infty} |\mathbf{a_n}| = \mathbf{S_1} - \mathbf{S_2}(say).$$

The Series $\mathbf{S_2} = \sum_{n=1}^{\infty} |\mathbf{a_n}|$ is given to be convergent. So if we can establish the convergence of $\mathbf{S_1} = \sum_{n=1}^{\infty} (\mathbf{a_n} + |\mathbf{a_n}|)$ we are done.
Consider

$$-|\mathbf{a_n}| \leq \mathbf{a_n} \leq |\mathbf{a_n}| \quad \Rightarrow \quad 0 \leq \mathbf{a_n} + |\mathbf{a_n}| \leq \mathbf{2}|\mathbf{a_n}|.$$

By Direct Comparision Test we achieve the convergence of $\sum_{\mathbf{n=1}}^{\infty}(\mathbf{a_n} + |\mathbf{a_n}|)$.
Which completes the proof of the theorem.

Exercise

1. $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{10}\right)^{n}$. **Divergent by n-th term test**
2. $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{10^{n}}{n^{10}}\right)$. **Divergent by n-th term test**
3. $\sum_{n=2}^{\infty}(-1)^{n+1}\left(\frac{1}{\ln n}\right)$. **Convergent by Alternating Series Test**
4. $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{\ln n}{n}\right)$. **Convergent by Alternating Series Test**
5. $\sum_{n=1}^{\infty}(-1)^{n}\frac{\sin n}{n^{2}}$. **Absolute Convergent**
6. $\sum_{n=2}^{\infty}(-1)^{n}\frac{1}{\ln n^{3}}$. **Conditionally Convergent**

Home Work : Discuss convergence of the following.

1. $\sum a_n$ where $a_{n+1} = \frac{n}{n+1} a_n$, $a_1 = 3$.
2. $\sum a_n$ where $a_{n+1} = \frac{n+\ln n}{n+10} a_n$, $a_1 = 1/2$.
3. $\sum e^{-\alpha n} n^{\beta}$.

Dirichlet Test

Let $a_n \geq 0$, $a_n \geq a_{n+1}$ and $\lim_{n\to\infty} a_n = 0$. Suppose there exists a number M such that $\left|\sum_{n=1}^{N} b_n\right| \leq M$ for all N (partial sums of $\sum b_n$ are bounded). Then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Apply Dirichlet's test to discuss the convergence of the following.

1. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n}$.
2. $\sum_{n=1}^{\infty} \frac{\sin n}{n}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$.

Cauchy's Condensation Test

Given series $\sum a_n$. If a_n is decreasing sequence of positive numbers, then $\sum a_n$ converges if and only if $\sum 2^n a_{2^n}$ converges.

1. $\sum \frac{1}{n^p}$. For $p > 0$, $\frac{1}{n^p}$ is positive and monotonically decreasing hence the Cauchy's condensation test is applicable.
2. $\sum \frac{1}{n \ln n}$. Verify conditions and apply Cauchy's condensation test.
3. $\sum \frac{1}{n \ln n \ln \ln n}$. Verify conditions and apply Cauchy's condensation test.

Power Series

DEFINITIONS Power Series, Center, Coefficients

A power series about $\mathbf{x = 0}$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots .$$

A power series about $\mathbf{x = a}$ is a series of the form

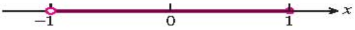
$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + \cdots ,$$

in which the center $\mathbf{a}$ and the coefficients $c_0, c_1, c_2, \cdots, c_n, \cdots$ are constants.

- What is the meaning of the convergence of a Power Series?
- For what values of $\mathbf{x}$ does a power series converge? (Use Mathematica)

EXAMPLE : Testing for Convergence Using the Ratio Test

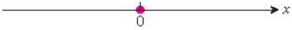
1. $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots.$$



2. $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots.$$

3. $$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \cdots.$$

4. $$\sum_{n=0}^{\infty} n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots.$$



THEOREM : The Convergence Theorem for Power Series

Consider the power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

1. If it converges for $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$.
2. If the series diverges for $x = d$, then it diverges for all x with $|x| > |d|$.

Remark The power series $\sum c_n (x - a)^n$ behaves in one of three possible ways.

1. It might converge only at $x = a$ or
2. converge everywhere, or
3. converge on some interval of radius R centered at $x = a$.

COROLLARY

The convergence of the series $\sum \mathbf{c_n}(\mathbf{x}-\mathbf{a})^{\mathbf{n}}$ is described by one of the following three possibilities:

1. There is a positive number $\mathbf{R}$ such that the series diverges for $\mathbf{x}$ with $|\mathbf{x}-\mathbf{a}| > \mathbf{R}$ but converges absolutely for $\mathbf{x}$ with $|\mathbf{x}-\mathbf{a}| < \mathbf{R}$. The series may or may not converge at either of the endpoints $\mathbf{x} = \mathbf{a} - \mathbf{R}$ and $\mathbf{x} = \mathbf{a} + \mathbf{R}$.
2. The series converges absolutely for every $\mathbf{x}$ ($\mathbf{R} = \infty$).
3. The series converges at $\mathbf{x} = \mathbf{a}$ and diverges elsewhere ($\mathbf{R} = \mathbf{0}$).

Radius of Convergence

- $\mathbf{R}$ is called the **radius of convergence** of the power series
- and the interval of radius $\mathbf{R}$ centered at $\mathbf{x} = \mathbf{a}$ is called the **interval of convergence**.
- The interval of convergence may be open, closed, or half-open, depending on the particular series.
- At points $\mathbf{x}$ with $|\mathbf{x} - \mathbf{a}| < \mathbf{R}$ the **series converges absolutely**.
- If the series converges for all values of $\mathbf{x}$, we say its radius of **convergence is infinite**.
- If it converges only at $\mathbf{x} = \mathbf{a}$ we say its **radius of convergence** is **zero**.
- **Note that a power series always converges for the center. So, our aim is to determine for what non-zero values of $\mathbf{x} - \mathbf{a}$, or values of $\mathbf{x} \neq \mathbf{a}$, it can converge.**

Problem

Discuss the convergence of the following

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n}.$$

Solution:

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| < 1 \Rightarrow \lim_{n\to\infty} \left| \frac{(x+2)^{n+1}}{n+1} \cdot \frac{n}{(x+2)^n} \right| < 1 \Rightarrow |x+2| < 1.$$

Which gives $-3 < x < -1$. At $x = -3$ we have $\sum_{n=1}^{\infty} \frac{1}{n}$ which is Divergent (Harmonic Series). At $x = -1$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is convergent by Alternating Series Test but does not converge absolutely.

- Radius is 1. Interval of absolute convergence $(-3, -1)$.
- Interval of convergence $(-3, -1]$.
- The Series is conditionally convergent at $x = -1$.

Taylor's Theorem

If f and its first n derivatives $f', f'', \cdots, f^{(n)}$ are continuous on the closed interval between a and b, and $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^n(a)}{n!}(b-a)^n + \frac{f^{n+1}(c)}{(n+1)!}(b-a)^{n+1}. \quad (1)$$

Taylor's Theorem is generalization of the Mean Value Theorem.