# MA 201, Mathematics III, July-November 2016, Fourier Transform (Contd.)

Lecture 17

# Fourier Transform

For a given function $f$, the improper integral

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{i\sigma t} \left[ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} f(\tau) e^{-i\sigma\tau} \, d\tau \right] d\sigma = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{i\sigma t} g(\sigma) d\sigma$$

is called Fourier integral of $f$.

If $f$ is continuous at $t$, we have

$$f(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{i\sigma t} g(\sigma) d\sigma.$$

The function $g$ is called *Fourier transform* of a function $f$ and it is denoted by $\mathcal{F}\{f(t)\}$

$$\mathcal{F}\{f(t)\} = g(\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\sigma\tau} f(\tau) \, d\tau. \tag{1}$$

The *inverse Fourier transform* is defined as

$$f(t) = \mathcal{F}^{-1}\{g(\sigma)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\sigma) e^{i\sigma t} \, d\sigma. \tag{2}$$

# Fourier cosine transform

For a even function $f$ defined in $(-\infty, \infty)$, the Fourier integral

$$\begin{aligned}
f(t) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{i\sigma t} \left[ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} f(\tau) e^{-i\sigma\tau}\, d\tau \right] d\sigma \quad \text{(Basic definition)} \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{i\sigma t} g_c(\sigma) d\sigma \\
&= \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(\tau) \cos\sigma\tau \cos\sigma t\, d\tau\, d\sigma \quad \text{(Special case)} \\
&= \int_0^{\infty} \sqrt{\frac{2}{\pi}} \cos\sigma t \left[ \int_0^{\infty} \sqrt{\frac{2}{\pi}} f(\tau) \cos\sigma\tau d\tau \right] d\sigma \\
&= \int_0^{\infty} \sqrt{\frac{2}{\pi}} \cos\sigma t g_c(\sigma) d\sigma
\end{aligned}$$

We can define the *Fourier cosine transform* of $f$ as

$$\begin{aligned}
\mathcal{F}_c\{f(t)\} = g_c(\sigma) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\sigma\tau} f(\tau)\, d\tau \quad \text{(From Basic definition)} \\
&= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(\tau) \cos\sigma\tau\, d\tau. \quad \text{(From Special case)}
\end{aligned}$$

The *inverse Fourier cosine transform* is defined as

$$\begin{aligned}
\mathcal{F}_c^{-1}\{g_c(\sigma)\} = f(t) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} g_c(\sigma) \cos\sigma t\, d\sigma. \quad \text{(From Special case)} \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{i\sigma t} g_c(\sigma) d\sigma. \quad \text{(From Basic definition)}
\end{aligned}$$

# Fourier sine transform

For a odd function $f$ defined in $(-\infty, \infty)$, the Fourier integral

$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{i\sigma t}\left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} f(\tau) e^{-i\sigma\tau}\, d\tau\right] d\sigma \quad \text{(Basic definition)} \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{i\sigma t} g_s(\sigma) d\sigma \\ &= \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} f(\tau)\sin\sigma\tau \sin\sigma t\; d\tau\; d\sigma \quad \text{(Special case)} \\ &= \int_0^{\infty} \sqrt{\frac{2}{\pi}} \sin\sigma t \left[\int_0^{\infty} \sqrt{\frac{2}{\pi}} f(\tau)\sin\sigma\tau d\tau\right] d\sigma \\ &= \int_0^{\infty} \sqrt{\frac{2}{\pi}} \sin\sigma t g_s(\sigma) d\sigma \end{aligned}$$

We can define the *Fourier sine transform* of $f$ as

$$\begin{aligned} \mathcal{F}_s\{f(t)\} = g_s(\sigma) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\sigma\tau} f(\tau)\, d\tau \quad \text{(From Basic definition)} \\ &= \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(\tau)\sin\sigma\tau\; d\tau. \quad \text{(From Special case)} \end{aligned}$$

The *inverse Fourier sine transform* is defined as

$$\begin{aligned} \mathcal{F}_s^{-1}\{g_s(\sigma)\} = f(t) &= \sqrt{\frac{2}{\pi}}\int_0^{\infty} g_s(\sigma)\sin\sigma t\; d\sigma. \quad \text{(From Special case)} \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{i\sigma t} g_s(\sigma) d\sigma. \quad \text{(From Basic definition)} \end{aligned}$$

# Example

**Consider $f(t) = e^{-t},\ t > 0$**

Justify that

(a) $f$ has Fourier sine and cosine transformations.

(b) Find Fourier cosine transformation of $f$.

(c) Use part (b) to evaluate following improper integral

$$\int_0^\infty \frac{\cos \sigma t}{\sigma^2 + 1} d\sigma, \quad t > 0.$$

**(a) Use even and odd extensions of $f$ in $(-\infty, \infty)$.**

**(b) Suppose $f_e$ is the even extension of $f$ in $(-\infty, \infty)$. Then**

$$\begin{aligned}\mathcal{F}_c\{f_e(t)\} = g_c(\sigma) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\sigma\tau} f_e(\tau)\, d\tau \ \ (\text{From Basic definition}) \\ &= \frac{1}{\sqrt{2\pi}} \frac{2}{1+\sigma^2}\end{aligned}$$

# Example (Contd.)

**(c) Finally, use inverse Fourier cosine transform to have**

$$\begin{aligned} f_e(t) = \mathcal{F}_c^{-1}\{g_c(\sigma)\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty g_c(\sigma)\cos\sigma t\, d\sigma. \quad \text{(From Special case)} \\ &= \int_0^\infty \frac{2}{\pi}\frac{\cos\sigma t}{\sigma^2+1} d\sigma. \quad \text{(From part (b))} \end{aligned}$$

**What about following examples?**

$$\begin{aligned} (a) &\int_0^\infty \frac{\sigma^3 \sin\sigma t}{\sigma^4+4} d\sigma, \quad t>0. \\ (b) &\int_0^\infty \frac{\sin\sigma t}{\sigma} d\sigma, \quad t>0. \end{aligned}$$

**Here, $f$ is not given explicitly.**

# Convolution

## Convolution Theorem

**Theorem**

*If $h(t)$ is the response obtained by convolving the function $f_1(t)$ with $f_2(t)$, then*

$$\mathcal{F}\{h(t)\} = \mathcal{F}\{f_1 * f_2\} = \mathcal{F}\{f_1(t)\}\mathcal{F}\{f_2(t)\}.$$

**In other words,**

we can say that the Fourier transform of the convolution of $f_1(t)$ and $f_2(t)$ is the product of the Fourier transforms of $f_1(t)$ and $f_2(t)$.

**Recall: The convolution of two functions $f_1(t)$ and $f_2(t)$, $-\infty < t < \infty$, is defined as**

$$f_1(t) * f_2(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau.$$

# Convolution

**Proof:** By definition

$$\begin{aligned}\mathcal{F}\{h(t)\} = \mathcal{F}\{f_1 * f_2\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(f_1(t) * f_2(t))e^{-i\sigma t}\,dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)\,d\tau\right]e^{-i\sigma t}\,dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f_1(\tau)\left[\int_{-\infty}^{\infty} f_2(t-\tau)e^{-i\sigma t}\,dt\right]d\tau.\end{aligned}$$

Putting $t - \tau = \omega$ in the inner integral

$$\begin{aligned}\mathcal{F}\{h(t)\} = \mathcal{F}\{f_1 * f_2\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f_1(\tau)\left[\int_{-\infty}^{\infty} f_2(\omega)e^{-i\sigma(\omega+\tau)}\,d\omega\right]d\tau \\ &= \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(\tau)e^{-i\sigma\tau}\,d\tau\right)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_2(\omega)e^{-i\sigma\omega}\,d\omega\right) \\ &= \mathcal{F}\{f_1\}\mathcal{F}\{f_2\}.\end{aligned}$$

This is known as the *convolution integral*.

# Convolution

**Example:**

A system has the impulse response given by $f_1(t)$ and is subjected to the rectangular pulse given by $f_2(t)$ as follows:

$$
\begin{aligned}
f_1(t) &= \begin{cases} 0, & -\infty < t \leq 0, \\ ae^{-at}, & 0 < t < \infty,\ a > 0 \end{cases} \\
f_2(t) &= \begin{cases} 0, & -\infty < t \leq -1, \\ 1, & -1 < t \leq 1, \\ 0, & 1 < t < \infty. \end{cases}
\end{aligned}
$$

Find the output time function $h(t)$ by convolving $f_1(t)$ with $f_2(t)$.

# Convolution

**Solution:**

The Fourier transform of $f_1(t)$:

$$\begin{aligned}
\mathcal{F}\{f_1(t)\} &= g_1(\sigma) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(t) e^{-i\sigma t}\, dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} a e^{-at}\, e^{-i\sigma t}\, dt \\
&= \frac{a}{\sqrt{2\pi}} \left| \frac{e^{-(a+i\sigma)t}}{-(a+i\sigma)} \right|_0^{\infty} \\
&= \frac{a}{\sqrt{2\pi}(a+i\sigma)}.
\end{aligned}$$

The Fourier transform of $f_2(t)$:

$$\begin{aligned}
\mathcal{F}\{f_2(t)\} &= g_2(\sigma) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_2(t) e^{-i\sigma t}\, dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\sigma t}\, dt \\
&= \frac{1}{\sqrt{2\pi}} \left| \frac{e^{-(i\sigma)t}}{-i\sigma} \right|_{-1}^{1} \\
&= \frac{1}{\sqrt{2\pi}} \frac{e^{i\sigma} - e^{-i\sigma}}{i\sigma} \\
&= \sqrt{\frac{2}{\pi}} \frac{\sin\sigma}{\sigma}.
\end{aligned}$$

# Convolution

**Therefore**

$$\mathcal{F}\{f_1(t)\}\mathcal{F}\{f_2(t)\} = \frac{1}{\pi}\frac{a\sin\sigma}{\sigma(a+i\sigma)}$$

**According to convolution theorem:**

$$\mathcal{F}\{h(t)\} = \mathcal{F}\{f_1 * f_2\} = \mathcal{F}\{f_1(t)\}\mathcal{F}\{f_2(t)\}.$$

**Hence**

$$h(t) = \mathcal{F}^{-1}[\mathcal{F}\{f_1(t)\}\mathcal{F}\{f_2(t)\}] = \mathcal{F}^{-1}\left[\frac{1}{\pi}\,\frac{a\sin\sigma}{\sigma(a+i\sigma)}\right]$$

**which will give us**

$$h(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\pi}\frac{a\sin\sigma}{\sigma(a+i\sigma)}e^{i\sigma t}\,d\sigma$$

which is the output time function.

# Application of Fourier Transform to PDEs

**As of now we have defined all variants of Fourier transform**

for a function of one variable only.

**It is obvious that if we want to use Fourier transforms to solve PDEs,**

we need to define Fourier transform for a function of at least two variables.

**Based on the definitions discussed earlier for a function of one variable,**

let us define Fourier transform of a function of two variables $x$ and $t$ and those of their partial derivatives

so that we can at least solve one-dimensional transient IBVPs governed by a PDE.

# Application of Fourier Transform to PDEs

**Fourier transform:**

The Fourier transform of a function $U(x,t)$ with respect to $x$ is defined as

$$\mathcal{F}\{U(x,t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\sigma x} U(x,t)\, dx = \hat{U}(\sigma,t). \tag{3}$$

**Inverse Fourier transform:**

The inverse Fourier transform $U(x,t)$ of $\hat{U}(\sigma,t)$ is defined as

$$U(x,t) = \mathcal{F}^{-1}\{\hat{U}(\sigma,t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\sigma x} \hat{U}(\sigma,t)\, d\sigma.$$

**Under the assumption that $U$ and $\frac{\partial U}{\partial x}$ vanish as $x \to \pm\infty$, we obtain the following results of transforms of the partial derivatives:**

$$\begin{aligned}
\mathcal{F}\left\{\frac{\partial}{\partial x} U(x,t)\right\} &= i\sigma \hat{U}(\sigma,t), \\
\mathcal{F}\left\{\frac{\partial^2}{\partial x^2} U(x,t)\right\} &= (i\sigma)^2 \hat{U}(\sigma,t), \\
\mathcal{F}\left\{\frac{\partial}{\partial t} U(x,t)\right\} &= \frac{d}{dt} \hat{U}(\sigma,t), \\
\mathcal{F}\left\{\frac{\partial^2}{\partial t^2} U(x,t)\right\} &= \frac{d^2}{dt^2} \hat{U}(\sigma,t).
\end{aligned}$$

# Application of Fourier Transform to PDEs

**Result I:**

$$\mathcal{F}\left\{\frac{\partial}{\partial x}U(x,t)\right\} = \mathrm{i}\sigma\hat{U}(\sigma,t). \tag{4}$$

**Proof:**

$$\begin{aligned}
\mathcal{F}\left\{\frac{\partial}{\partial x}U(x,t)\right\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma x}\frac{\partial U}{\partial x}dx \\
&= \frac{1}{\sqrt{2\pi}}\left\{[e^{-\mathrm{i}\sigma x}U(x,t)]_{-\infty}^{\infty} + \mathrm{i}\sigma\int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma x}U(x,t)dx\right\} \\
&= \mathrm{i}\sigma\hat{U}(\sigma,t).
\end{aligned}$$

□

**Result II:**

$$\mathcal{F}\left\{\frac{\partial^2}{\partial x^2}U(x,t)\right\} = (\mathrm{i}\sigma)^2\hat{U}(\sigma,t). \tag{5}$$

**Proof:**

$$\begin{aligned}
\mathcal{F}\left\{\frac{\partial^2}{\partial x^2}U(x,t)\right\} &= \frac{1}{\sqrt{2\pi}}\left\{\int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma x}\frac{\partial^2 U}{\partial x^2}dx\right\} \\
&= \frac{1}{\sqrt{2\pi}}\left\{[e^{-\mathrm{i}\sigma x}\frac{\partial U}{\partial x}]_{-\infty}^{\infty} + \mathrm{i}\sigma\int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma x}\frac{\partial U}{\partial x}dx\right\} \\
&= \frac{1}{\sqrt{2\pi}}\left\{\mathrm{i}\sigma([e^{-\mathrm{i}\sigma x}U(x,t)]_{-\infty}^{\infty} + \mathrm{i}\sigma\int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma x}U(x,t)dx)\right\} \\
&= (\mathrm{i}\sigma)^2\hat{U}(\sigma,t).
\end{aligned}$$

□

# Application of Fourier Transform to PDEs

**Result III:**

$$\mathcal{F}\left\{\frac{\partial}{\partial t}U(x,t)\right\} = \frac{d}{dt}\hat{U}(\sigma,t). \tag{6}$$

**Proof:**

$$\begin{aligned}
\mathcal{F}\left\{\frac{\partial}{\partial t}U(x,t)\right\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma x}\frac{\partial U}{\partial t}dx \\
&= \frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma x}U(x,t)dx \\
&= \frac{1}{\sqrt{2\pi}}\frac{d}{dt}\int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma x}U(x,t)dx \\
&= \frac{d}{dt}\hat{U}(\sigma,t).
\end{aligned}$$

□

**Proceeding in a similar manner as above, Result 4 as given below can be obtained:**

$$\mathcal{F}\left\{\frac{\partial^2}{\partial t^2}U(x,t)\right\} = \frac{d^2}{dt^2}\hat{U}(\sigma,t). \tag{7}$$

□

The use of Fourier transform to solve partial differential equations is best described by examples.

# Application of Fourier Transform to PDEs

**Example A**:

An infinitely long string extending in $-\infty < x < \infty$ under uniform tension is displaced into the curve $y = f(x)$ and let go from rest with velocity $g(x)$. To find the displacement $U(x,t)$ at any point at any subsequent time.

**Solution:**

The boundary value problem is

$$\begin{aligned} \frac{\partial^2 U}{\partial t^2} &= c^2 \frac{\partial^2 U}{\partial x^2}, \ -\infty < x < \infty, \ t > 0, && (8) \\ U(x,0) &= f(x) \ \text{(initial displacement)}, && (9) \\ \frac{\partial U}{\partial t}(x,0) &= g(x) \ \text{(initial velocity)}. && (10) \end{aligned}$$

Taking Fourier transform on both sides of PDE (8),

$$\frac{d^2}{dt^2}\hat{U}(\sigma,t) = -c^2\sigma^2\hat{U}(\sigma,t), \tag{11}$$

# Application of Fourier Transform to PDEs

**(11) can be written in standard form as**

$$\frac{d^2}{dt^2}\hat{U}(\sigma,t) + c^2\sigma^2\hat{U}(\sigma,t) = 0. \tag{12}$$

**On solving, we get**

$$\hat{U}(\sigma,t) = A(\sigma)\cos(c\sigma t) + B(\sigma)\sin(c\sigma t). \tag{13}$$

**Taking Fourier transforms on the initial conditions (9) and (10),**

$$\hat{U}(\sigma,0) = \hat{f}(\sigma), \tag{14}$$

$$\frac{d}{dt}\hat{U}(\sigma,0) = \hat{g}(\sigma), \tag{15}$$

where $\hat{f}(\sigma)$ and $\hat{g}(\sigma)$, are, respectively the Fourier transform of $f(x)$ and $g(x)$.

**Using the initial conditions, $A(\sigma)$ and $B(\sigma)$ can be obtained as:**

$$\begin{aligned}\hat{U}(\sigma,0) &= A(\sigma) = \hat{f}(\sigma),\\ \frac{d}{dt}\hat{U}(\sigma,0) &= c\sigma B(\sigma) = \hat{g}(\sigma).\end{aligned}$$

# Application of Fourier Transform to PDEs

**Now**

$$\hat{U}(\sigma,t) = \hat{f}(\sigma)\cos(c\sigma t) + \frac{1}{c\,\sigma}\,\hat{g}(\sigma)\sin(c\sigma t). \tag{16}$$

**To get the solution, we use the inverse Fourier transform to obtain**

$$U(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\hat{f}(\sigma)\cos(c\sigma t) + \frac{1}{c\,\sigma}\,\hat{g}(\sigma)\sin(c\sigma t)\right] e^{i\sigma x}\,d\sigma. \tag{17}$$

Formula (17) gives us the solution of the wave boundary value problem in the form of an integral involving the Fourier transform of the initial displacement and velocity.

# Application of Fourier Transform to PDEs

We summarize the Fourier transform method as follows:

**Step 1:** Take Fourier transform on the given boundary value problem in $U(x,t)$ and get an ordinary differential equation in $\hat{U}(\sigma,t)$ in the variable $t$.

**Step 2:** Solve the ordinary differential equation and find $\hat{U}(\sigma,t)$.

**Step 3:** Take inverse Fourier transform to get $U(x,t)$

# Application of Fourier Transform to PDEs

**Example B:**

Consider the heat conduction in an infinite rod with thermal diffusivity $\alpha$ with initial temperature distribution $f(x)$. To find the temperature distribution $U(x,t)$ at any point at any subsequent time.

**Solution:**

The boundary value problem is

$$\begin{aligned} \frac{\partial U}{\partial t} &= \alpha \frac{\partial^2 U}{\partial x^2}, \;\; -\infty < x < \infty, \; t > 0, && (18) \\ U(x,0) &= f(x) \;\; \text{(initial temperature distribution)}. && (19) \end{aligned}$$

Taking Fourier transform on both sides of PDE (18), we observe

$$\hat{U}(\sigma, t) = \hat{f}(\sigma) e^{-\alpha\sigma^2 t}, \tag{20}$$

where $\hat{f}(\sigma)$ is the Fourier transform of $f(x)$.

We use the inverse Fourier transform to obtain

$$U(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\sigma) e^{-\alpha\sigma^2 t} e^{i\sigma x} \, d\sigma. \tag{21}$$

# Application of Fourier Transform to PDEs

**Example C: Semi-infinite rod with Dirichlet condition**

**Mathematical Model**

The boundary value problem is the following:

$$\begin{aligned} \frac{\partial U}{\partial t} &= \alpha \frac{\partial^2 U}{\partial x^2}, \;\; x > 0, \; t > 0, && (22) \\ U(x,0) &= f(x), \; t > 0 \;\; U(0,t) = U_0, \; x > 0. && (23) \end{aligned}$$

**Example D: Semi-infinite rod with Neumann condition**

Consider the same equation as in the previous example subject to the boundary conditions

$$\frac{\partial U}{\partial x}(0,t) = 0, \;\; U(x,0) = f(x)$$

The Fourier sine and cosine transforms can be employed to solve a partial differential equation when the range of the spatial variable extends from 0 to $\infty$.

If the boundary condition is in terms of some value of $U(0,t)$ (i.e. Dirichlet boundary condition), then sine transform is to be used.

When the boundary condition is in terms of some value of $\frac{\partial U}{\partial x}(0,t)$ (i.e. Neumann boundary condition), then cosine transform is to be used.

# Application of Fourier Transform to PDEs

The *Fourier sine transform* of a function $U(x,t)$ with respect to $x$ is defined as

$$\mathcal{F}_s\{U(x,t)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty U(x,t)\sin\sigma x\,dx = \hat{U}_s(\sigma,t). \tag{24}$$

Under the assumption that $U$ and $\frac{\partial U}{\partial x}$ vanish as $x \to \infty$,

$$\mathcal{F}_s\left\{\frac{\partial U(x,t)}{\partial t}\right\} = \frac{d}{dt}\hat{U}_s(\sigma,t), \tag{25}$$

$$\mathcal{F}_s\left\{\frac{\partial^2 U(x,t)}{\partial t^2}\right\} = \frac{d^2}{dt^2}\hat{U}_s(\sigma,t), \tag{26}$$

$$\mathcal{F}_s\left\{\frac{\partial^2 U(x,t)}{\partial x^2}\right\} = \sqrt{\frac{2}{\pi}}\sigma U(0,t) - \sigma^2\hat{U}_s(\sigma,t). \tag{27}$$

# Application of Fourier Transform to PDEs

The *Fourier cosine transform* of a function $U(x,t)$ with respect to $x$ is defined as

$$\mathcal{F}_c\{U(x,t)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty U(x,t)\cos\sigma x \, dx = \hat{U}_c(\sigma,t). \tag{28}$$

Under the assumption that $U$ and $\frac{\partial U}{\partial x}$ vanish as $x \to \infty$,

$$\mathcal{F}_c\left\{\frac{\partial U(x,t)}{\partial t}\right\} = \frac{d}{dt}\hat{U}_c(\sigma,t), \tag{29}$$

$$\mathcal{F}_c\left\{\frac{\partial^2 U(x,t)}{\partial t^2}\right\} = \frac{d^2}{dt^2}\hat{U}_c(\sigma,t), \tag{30}$$

$$\mathcal{F}_c\left\{\frac{\partial^2 U(x,t)}{\partial x^2}\right\} = -\sqrt{\frac{2}{\pi}}U_x(0,t) - \sigma^2\hat{U}_c(\sigma,t). \tag{31}$$

# Application of Fourier Transform to PDEs

**Example C:**

If $U(x,t)$ is the temperature at time $t$ and $\alpha$ the thermal diffusivity of a semi-infinite metal bar, find the temperature distribution in the bar at any point at any subsequent time if the initial temperature distribution is given as $f(x)$ and the boundary is kept at $U_0$ degrees.

**Solution:**

The boundary value problem is the following:

$$\begin{aligned} \frac{\partial U}{\partial t} &= \alpha \frac{\partial^2 U}{\partial x^2},\ \ x>0,\ t>0, && (32) \\ U(x,0) &= f(x),\ t>0\ \ U(0,t)=U_0,\ x>0. && (33) \end{aligned}$$

The boundary condition suggests that we need to use Fourier sine transform. Taking the transform on (32),

$$\frac{d}{dt}\hat{U}_s(\sigma,t) = \alpha[\sqrt{\frac{2}{\pi}}\sigma U(0,t) - \sigma^2 \hat{U}_s(\sigma,t)]. \tag{34}$$

Using the boundary condition,

$$\frac{d}{dt}\hat{U}_s(\sigma,t) + \alpha\sigma^2\hat{U}_s(\sigma,t) = \alpha\sqrt{\frac{2}{\pi}}\sigma U_0. \tag{35}$$

# Application of Fourier Transform to PDEs

## On solving

$$\hat{U}_s(\sigma,t) = A(\sigma)e^{-\alpha\sigma^2 t} + \frac{1}{\sigma}\sqrt{\frac{2}{\pi}}U_0. \tag{36}$$

## Using the initial condition,

$$\hat{f}_s(\sigma) = A(\sigma) + \sqrt{\frac{2}{\pi}}\frac{U_0}{\sigma},$$

where $\hat{f}_s(\sigma)$ is the Fourier sine transform of $f(x)$. that is,

$$A(\sigma) = \hat{f}_s(\sigma) - \sqrt{\frac{2}{\pi}}\frac{U_0}{\sigma}$$

## Now $\hat{U}_s(\sigma,t)$ is

$$\begin{aligned}\hat{U}_s(\sigma,t) &= (\hat{f}_s(\sigma) - \sqrt{\frac{2}{\pi}}\frac{U_0}{\sigma})e^{-\alpha\sigma^2 t} + \sqrt{\frac{2}{\pi}}\frac{U_0}{\sigma} \\ &= \hat{f}_s e^{-\alpha\sigma^2 t} + \sqrt{\frac{2}{\pi}}\frac{U_0}{\sigma}(1 - e^{-\alpha\sigma^2 t}) \end{aligned} \tag{37}$$

## The inversion gives

$$U(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \left[\hat{f}_s e^{-\alpha\sigma^2 t} + \sqrt{\frac{2}{\pi}}\frac{U_0}{\sigma}(1 - e^{-\alpha\sigma^2 t})\right]\sin\sigma x \, d\sigma. \tag{38}$$

# Application of Fourier Transform to PDEs

Similarly, we can try the vibration problem in a semi-infinite string, $0 < x < \infty$.

In this case, there will be a second-order ordinary differential equation due to the presence of the term $\frac{\partial^2 U}{\partial t^2}$.

The two initial conditions $U(x, 0)$ and $\frac{\partial U}{\partial t}(x, 0)$ will help us in determining the coefficients of $\hat{U}_c(\sigma, t)$ or $\hat{U}_s(\sigma, t)$ depending on the boundary condition at $x = 0$.