8 The IBVP [illegible]

$$u_t = \alpha u_{xx}$$ [illegible]

BC: $u_x(0,t) = 0 = u_x(l,t)$ | [illegible]

IC: $u(x,0) = f(x)$

Here the boundary conditions are not Dirichlet type, but they are Neumann type. By separation of variables

$$u(x,t) = X(x)T(t)$$

The eqn becomes

$$\frac{T'}{\alpha T} = \frac{X''}{X} = K = -\lambda^2$$

Since the BCs are homogeneous, only for negative values of k, we will get nontrivial solutions conforming to the boundary conditions. Hence the PDE is converted to the following ODEs:

$$X'' + \lambda^2 X = 0, \quad T' + \alpha\lambda^2 T = 0$$

The solutions are:

$$X(x) = A\cos ax + B\sin ax$$

$$T(t) = C\,e^{-\alpha a^2 t}$$

$$u(x,t) = (A\cos ax + B\sin ax)\,e^{-\alpha a^2 t}$$

$$u_x = (-Aa\sin ax + Ba\cos ax)\,e^{-\alpha a^2 t}$$

First BC $\Rightarrow B = 0$

2nd BC $\Rightarrow \sin al = 0 \Rightarrow a_n = \frac{n\pi}{l}, \; n = 0, 1, 2, \ldots$

Corresponding → $\lambda_0 = 0$

$u_0(x,t) = \frac{A_0}{2}$

$$u_n(x,t) = A_n \cos\frac{n\pi}{l}x\, e^{-\frac{\alpha n^2\pi^2}{l^2}t}, \; n = 0, 1, 2, \ldots$$

$$\therefore u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}\, e^{-\frac{\alpha n^2\pi^2 t}{l^2}} \quad \text{[Fourier series]}$$

Using the IC:

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}$$

$$\Rightarrow A_0 = \frac{2}{l}\int_0^l f(x)\,dx, \quad A_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx.$$

(i) $f(x) = 1, \quad l = \pi, \; \alpha = 1.$

$$\therefore u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos nx\, e^{-\alpha n^2 t}$$

$$A_0 = \frac{2}{\pi}\int_0^{\pi} dx = 2.$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} \cos nx\,dx = \frac{2}{\pi}\left|\sin nx\right|_0^{\pi} = 0$$

$$\therefore u(x,t) = 1.$$

(ii) $f(x) = x^2.$

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos nx\, e^{-\alpha n^2 t}.$$

$$A_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\cdot\frac{\pi^3}{3} = \frac{2\pi^2}{3}$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx = \frac{4}{n^2}(-1)^n$$

$$\therefore u(x,t) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx\, e^{-\alpha n^2 t}$$

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~~(15) Sistem of difform~~

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$u_t = \alpha u_{xx}$, $0 < x < 1$, $t > 0$

$u(0,t) = 0$, $u(1,t) + u_x(1,t) = 0$,

$u(x,0) = f(x)$.

Solⁿ: The solⁿ is:

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-\alpha\lambda^2 t}$$

BC $u(0,t) = 0 \Rightarrow A = 0$

BC $u(1,t) + u_x(1,t) = 0 \Rightarrow (B\sin\lambda + \lambda B\cos\lambda)\, e^{-\alpha\lambda^2 t} = 0$

$\Rightarrow B(\sin\lambda + \lambda\cos\lambda) = 0$

$\Rightarrow \sin\lambda + \lambda\cos\lambda = 0$

$\Rightarrow \lambda = -\tan\lambda$. ——— (1)

This equation cannot be solved analytically But the graphs of the functions λ and $-\tan\lambda$ shows that the eqⁿ has infinitely many positive solutions λ_1, λ_2 - - These values λ_n are the eigenvalues.

$$\therefore u(x,t) = \sum_{n=1}^{\infty} B_n \sin\lambda_n x\, e^{-\alpha\lambda_n^2 t}$$

where λ_n are solutions of (1).

IC $u(x,0) = f(x) \Rightarrow$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\lambda_n x.$$

$$\therefore B_n = \frac{2}{l}\int_0^l f(x)\sin\lambda_n x\, dx$$

(7)

This is a D'Alembert's problem with $u_t(x,0) = 0 = \psi$

$$u(x,0) = \varphi(x) = \frac{1}{1+4x^2}$$

The D'Alembert solⁿ. is

$$u(x,t) = \frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

~~where~~

$$= \frac{1}{2}\left[\frac{1}{1+4(x+ct)^2} + \frac{1}{1+4(x-ct)^2}\right]$$

$$= \frac{1}{2}\,\frac{\left[1+4(x-ct)^2 + 1 + 4(x+ct)^2\right]}{\left[1+4(x+ct)^2\right]\left[1+4(x-ct)^2\right]}$$

$$= \frac{1}{2}\,\frac{\left[2 + 8(x^2+c^2t^2)\right]}{\left[1+4(x+ct)^2\right]\left[1+4(x-ct)^2\right]}$$

$$= \frac{1+4c^2t^2+4x^2}{\left[1+4(x+ct)^2\right]\left[1+4(x-ct)^2\right]}$$

We have to solve the following IBVP :

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = 0 \quad , \quad u_t(x,0) = u_0 \\ u(0,t) = 0 \quad , \quad u(l,t) = 0 \end{cases}$$