Mid-Semester Examination, MA102
Indian Institute of Technology Patna
23 February 2011

*All nine problems are compulsory. Problems 1, 2 and 7 carry four marks and all other problems carry three marks. Total marks are 30 and total time is 120 minuets.*

1. Let $W_1$ be the subspace of $\mathbb{R}^4$ generated by the set

$$S_1 = \{(1,1,0,-1),(1,2,3,0),(2,3,3,-1)\}$$

and $W_2$ be the subspace of $\mathbb{R}^4$ generated by the set

$$S_2 = \{(1,2,2,-2),(2,3,2,-3),(1,3,4,-3)\}.$$

Then find (i) $\dim(W_1 + W_2)$, (ii) $\dim(W_1 \cap W_2)$.

2. Let $S_n(F)$ and $SS_n(F)$ be the spaces of symmetric and skew-symmetric matrices of order $n \times n$ over the field $F$ respectively. Prove that $M_n(F)$, the space of all square matrices of order $n$ over the field $F$ is the direct sum of $S_n(F)$ and $SS_n(F)$. Find the dimensions of $S_n(F)$ and $SS_n(F)$.

3. Prove that the row-equivalent matrices have the same row space.

4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a function defined by

$$T(a,b,c) = (a-b+2c, 2a+b, -a-2b+2c).$$

Assuming that $T$ is a linear transformation find the range space of $T$, rank $T$ and nullity $T$.

5. Let $V$ and $W$ be finite dimensional vector spaces over the field $F$ such that $\dim V = \dim W$. If $T$ is a linear transformation from $V$ into $W$, then prove that
$T$ is non-singular $\Leftrightarrow$ if $\{x_1, x_2, \ldots, x_n\}$ is a linearly independent subset of $V$ then $\{Tx_1, Tx_2, \ldots, Tx_n\}$ is a linearly independent subset of $W$.

6. Using the relation between $\det A$ and the eigen values of $A$, where the matrix $A$ is given below, can you conclude that the matrix $A$ is invertible? If it is invertible find its inverse by Gauss-Jordan method.

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

7. Examine whether the matrix $A$ over the field of real numbers, where

$$A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$$

is diagonalizable. If so, obtain the matrices $P$ and $D$ such that $P^{-1}AP = D$.

8. (i) Prove that similar matrices have the same characteristic polynomial.

   (ii) Let $A$ be an $n \times n$ invertible matrix. Explain how can you find the inverse of $A$ using the Cayley-Hamilton theorem?

9. Let $P_3[x]$ be the inner product space of polynomials of degree less than or equal to three over the field of real numbers with the inner product

$$(f, g) = \int_0^1 f(x)g(x)dx.$$

Apply the Gram-Schmidt process on the basis $B = \{1, x, x^2, x^3\}$ to obtain the orthogonal basis of $P_3[x]$.