MA 201, Mathematics III, July-November 2016, Fourier Integral and Transforms

Lecture 16

Fourier Integral

We already know that Fourier series and orthogonal expansions can be used as tools

to solve boundary value problems over bounded regions such as intervals, rectangles, disks and cylinders.

But we can understand that the modelling of certain physical phenomena will give rise naturally

to boundary value problems over unbounded regions.

For example,

to describe the temperature distribution in a very long insulated wire, we can suppose that the length of the wire is infinite, which gives rise to a boundary value problem over an infinite line.

Fourier Integral

To solve this type of problem we will generalize the notion of Fourier series by developing Fourier transform.

While Fourier transform will be suitable to solve problems over infinite domains,

Fourier sine and Fourier cosine transform will be helpful for problems with semi-infinite domains.

A periodic function can be adequately represented by a Fourier series expansion satisfying certain conditions.

In many problems of physical interest the impressed force is non-periodic rather than periodic.

The non-periodic function can be obtained when the period of a function goes to infinity.

Fourier Integral

Recall that a periodic function $f_p(t)$ has a Fourier series in the interval $(-L, L)$:

$$f_p(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos\frac{n\pi t}{L} + B_n \sin\frac{n\pi t}{L} \right]$$

where

$$\begin{aligned} A_n &= \frac{1}{L}\int_{-L}^{L} f_p(t)\cos\frac{n\pi t}{L}\,dt, \quad n = 0, 1, 2, 3, \ldots, \\ B_n &= \frac{1}{L}\int_{-L}^{L} f_p(t)\sin\frac{n\pi t}{L}\,dt, \quad n = 1, 2, 3, \ldots. \end{aligned}$$

We know

$$\begin{aligned} \cos\left(\frac{n\pi t}{L}\right) &= \frac{1}{2}[\exp(\mathrm{i}n\pi t/L) + \exp(-\mathrm{i}n\pi t/L)], \\ \sin\left(\frac{n\pi t}{L}\right) &= \frac{1}{2\mathrm{i}}[\exp(\mathrm{i}n\pi t/L) - \exp(-\mathrm{i}n\pi t/L)] \end{aligned}$$

Fourier Integral

Putting back these in $f_p(t), A_n, B_n$:

$$\begin{aligned} f_p(t) &= \frac{A_0}{2} + \sum_{n=1}^{\infty}\left[A_n\left(\frac{1}{2}[\exp(\mathrm{i}n\pi t/L) + \exp(-\mathrm{i}n\pi t/L)]\right)\right. \\ &+ \left. B_n\left(\frac{1}{2\mathrm{i}}[\exp(\mathrm{i}n\pi t/L) - \exp(-\mathrm{i}n\pi t/L)]\right)\right] \end{aligned}$$

From the expressions of A_n and B_n and writing

$$C_n = \frac{1}{2}\left(A_n - \mathrm{i}B_n\right),\ \ \bar{C_n} = \frac{1}{2}\left(A_n + \mathrm{i}B_n\right) = C_{-n},\ \ n = 0,\ 1,\ 2, \ldots$$

We obtain Fourier series for $f_p(t)$ in complex form as

$$f_p(t) = \sum_{n=-\infty}^{\infty} C_n e^{\mathrm{i}n\pi t/L}, \tag{1}$$

where

$$C_n = \frac{1}{2L}\int_{-L}^{L} f_p(\tau)e^{-\mathrm{i}n\pi\tau/L}\, d\tau,\ \ n = 0, \pm 1, \pm 2, \ldots \tag{2}$$

Fourier Integral

Substituting (2) into (1):

$$f_p(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-L}^{L} f_p(\tau) e^{-in\pi\tau/L} \, d\tau \right] e^{in\pi t/L} \left(\frac{\pi}{L} \right) \tag{3}$$

Define the frequency of the general term by

$$\sigma_n = \frac{n\pi}{L}, \tag{4}$$

and the difference in frequencies between successive terms by

$$\Delta\sigma = \sigma_{n+1} - \sigma_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}. \tag{5}$$

Then (3) reduces to

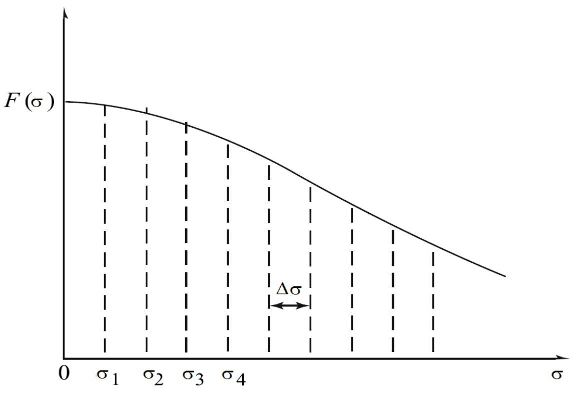
$$\begin{aligned} f_p(t) &= \sum_{n=-\infty}^{\infty} \left[\frac{e^{i\sigma_n t}}{2\pi} \int_{-L}^{L} f_p(\tau) e^{-i\sigma_n \tau} \, d\tau \right] \Delta\sigma \\ &= \sum_{n=-\infty}^{\infty} F(\sigma_n) \Delta\sigma, \quad F(\sigma) = \left[\frac{e^{i\sigma t}}{2\pi} \int_{-L}^{L} f_p(\tau) e^{-i\sigma\tau} \, d\tau \right]. \end{aligned} \tag{6}$$

Fourier Integral

If we plot $F(\sigma)$ against σ, we can clearly see that the sum

$$\sum_{n=-\infty}^{\infty} F(\sigma_n)\Delta\sigma \tag{7}$$

is an approximation to the area under the curve $y = F(\sigma)$



Fourier Integral

Thus as, $L \to \infty$, (6) can be written as

$$\begin{aligned} \lim_{L\to\infty} f_p(t) = f(t) &= \lim_{L\to\infty} \sum_{n=-\infty}^{\infty} F(\sigma_n)\Delta\sigma \\ &= \int_{-\infty}^{\infty} F(\sigma)d\sigma = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} e^{i\sigma t} \int_{-\infty}^{\infty} f(\tau)e^{-i\sigma\tau}\, d\tau\right] d\sigma, \quad (8) \end{aligned}$$

which is known as the complex form of Fourier integral.

In fact, Fourier integral is actually a valid representation of the non-periodic function $f(t)$ provided that

(a) in every finite interval $f(t)$ is defined and is piecewise continuous.
(b) the improper integral $\int_{-\infty}^{\infty} |f(t)|dt$ exists.

More precisely,

$$\frac{1}{2}(f(t^+) + f(t^-)) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} e^{i\sigma t} \int_{-\infty}^{\infty} f(\tau)e^{-i\sigma\tau}\, d\tau\right] d\sigma,$$

Fourier Integral

Now proceed to obtain the trigonometric representation of the Fourier integral:

Equation (8) can, after writing $e^{-i\sigma(\tau-t)} = \cos\sigma(\tau-t) - i\sin\sigma(\tau-t)$, be expressed as

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)\Big(\int_{-\infty}^{\infty} e^{-i\sigma(\tau-t)}d\sigma\Big)d\tau \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)\Big(\int_{-\infty}^{\infty} \cos\sigma(\tau-t)d\sigma\Big)d\tau \end{aligned}$$

(Since $\sin\sigma(\tau-t)$ is an odd function of σ)

Also note that $\cos\sigma(\tau-t)$ is an even function of σ, therefore

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)\Big(\int_{-\infty}^{\infty} e^{-i\sigma(\tau-t)}d\sigma\Big)d\tau \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tau)\Big(\int_{0}^{\infty} \cos\sigma(\tau-t)d\sigma\Big)d\tau. \qquad (9) \end{aligned}$$

This way the integral has a representation with sine and cosine terms only.

Fourier Integral

Fourier cosine and sine integrals:

Expanding $\cos\sigma(\tau - t)$ in (9)

$$
\begin{aligned}
f(t) &= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^0 f(\tau)\cos\sigma\tau\cos\sigma t \, d\tau + \int_0^\infty f(\tau)\cos\sigma\tau\cos\sigma t \, d\tau\right] d\sigma \\
&+ \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^0 f(\tau)\sin\sigma\tau\sin\sigma t \, d\tau + \int_0^\infty f(\tau)\sin\sigma\tau\sin\sigma t \, d\tau\right] d\sigma. \quad (10)
\end{aligned}
$$

Changing τ to $-\tau$ in the first and third integrals:

$$
\begin{aligned}
f(t) &= \frac{1}{\pi}\int_0^\infty \left[\int_0^\infty f(-\tau)\cos\sigma\tau\cos\sigma t \, d\tau + \int_0^\infty f(\tau)\cos\sigma\tau\cos\sigma t \, d\tau\right] d\sigma \\
&+ \frac{1}{\pi}\int_0^\infty \left[\int_0^\infty f(-\tau)(-\sin\sigma\tau)\sin\sigma t \, d\tau + \int_0^\infty f(\tau)\sin\sigma\tau\sin\sigma t \, d\tau\right] d\sigma. \quad (11)
\end{aligned}
$$

Fourier Integral

if $f(-\tau) = f(\tau)$, i.e., if f is an even function of τ,

(11) becomes

$$f(t) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(\tau) \cos \sigma\tau \cos \sigma t \, d\tau \, d\sigma. \tag{12}$$

Equation (12) is called the *Fourier cosine integral* of f

and is analogous to the half-range cosine expansion of an even periodic function.

Fourier Integral

If $f(-\tau) = -f(\tau)$, i.e., when f is an odd function,

equation (11) becomes

$$f(t) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(\tau) \sin \sigma\tau \sin \sigma t \, d\tau \, d\sigma. \tag{13}$$

Equation (13) is called the *Fourier sine integral* of f

and is analogous to the half-range sine expansion of an odd periodic function.

With these definitions of Fourier integrals,

we are in a position to define Fourier transforms including sine and cosine transforms.

Fourier Transform

The Fourier transform method is a powerful device for solving partial differential equations.

Its importance stems from its ability to handle a large variety of problems.

Choosing the appropriate transform is a crucial step in implementing the method.

The choice is suggested by the type of region and the type of the boundary conditions.

Integral Transform

The integral transform of a function $F(t)$ is defined by

$$\mathcal{I}\{F(t)\} = \int_a^b K(\sigma,\tau)F(\tau)d\tau.$$

Here the values of a and b can be finite or infinite and $K(\sigma,\tau)$ is called the kernel of the transform.

As the kernel changes, we get different types of integral transforms.

The selection of transform depends on the coordinates chosen, which is influenced by the type of the problem.

Complex Fourier Transform

Recalling equation (8):

$$f(t) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} e^{i\sigma t} \int_{-\infty}^{\infty} f(\tau) e^{-i\sigma\tau} \, d\tau \right] d\sigma,$$

we are now in a position to define the complex Fourier transform pair.

The *Fourier transform* of a function $f \in L^1(\mathbb{R})$ is defined by

$$\mathcal{F}\{f(t)\} = g(\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\sigma\tau} f(\tau) \, d\tau. \tag{14}$$

The *inverse Fourier transform* is defined as

$$f(t) = \mathcal{F}^{-1}\{g(\sigma)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\sigma) e^{i\sigma t} \, d\sigma. \tag{15}$$

Here $g(\sigma)$ is in the frequency domain and $f(t)$ in the time domain.

Fourier cosine transform

Recalling equation (12):

$$f(t) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(\tau) \cos \sigma\tau \cos \sigma t \, d\tau \, d\sigma,$$

we can define the *Fourier cosine transform* of f as

$$\mathcal{F}_c\{f(t)\} = g_c(\sigma) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(\tau) \cos \sigma\tau \, d\tau. \tag{16}$$

The *inverse Fourier cosine transform* is defined as

$$f(t) = \mathcal{F}_c^{-1}\{g_c(\sigma)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty g_c(\sigma) \cos \sigma t \, d\sigma. \tag{17}$$

It is obvious that this transform pair exists for even $f(t)$ only.

Fourier sine transform

Recalling equation (13):

$$f(t) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(\tau) \sin \sigma\tau \sin \sigma t \; d\tau \; d\sigma,$$

we can define the *Fourier sine transform* of f as

$$\mathcal{F}_s\{f(t)\} = g_s(\sigma) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(\tau) \sin \sigma\tau \; d\tau. \tag{18}$$

The *inverse Fourier sine transform* is defined as

$$f(t) = \mathcal{F}_s^{-1}\{g_s(\sigma)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty g_s(\sigma) \sin \sigma t \; d\sigma. \tag{19}$$

It is obvious that this transform pair exists for odd $f(t)$ only.

Some Properties of Fourier transform

Linear Property:

If $\mathcal{F}\{f_1(t)\} = g_1(\sigma), \mathcal{F}\{f_2(t)\} = g_2(\sigma)$, then

$$\mathcal{F}\{c_1 f_1(t) \pm c_2 f_2(t)\} = c_1 \mathcal{F}\{f_1(t)\} \pm c_2 \mathcal{F}\{f_2(t)\} = c_1 g_1(\sigma) \pm c_2 g_2(\sigma),$$

where c_1 and c_2 are constants.

Theorem I:

If $\mathcal{F}\{f(t)\} = g(\sigma)$, then

$$\mathcal{F}\{f(t-a)\} = e^{-\mathrm{i}\sigma a} g(\sigma)$$

Proof:

By definition,

$$\begin{aligned} \mathcal{F}\{f(t-a)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma\tau} f(\tau - a) d\tau \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma(\xi + a)} f(\xi) d\xi, \quad \text{by taking } \tau - a = \xi \\ &= e^{-\mathrm{i}\sigma a} g(\sigma) \end{aligned}$$

□

This is known as shifting property.

Some Properties of Fourier transform

Theorem II:

If $\mathcal{F}\{f(t)\} = g(\sigma)$, then

$$\mathcal{F}\{f(at)\} = \frac{1}{a}g(\sigma/a)$$

Proof:

By definition,

$$\begin{aligned}\mathcal{F}\{f(at)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma\tau} f(a\tau)d\tau \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\mathrm{i}(\sigma/a)\xi} f(\xi)d\xi/a, \quad \text{by taking } a\tau = \xi \\ &= \frac{1}{a}g(\sigma/a)\end{aligned}$$

$\square$

This is known as scaling property.

Some Properties of Fourier transform

Theorem III:

If $\mathcal{F}\{f(t)\} = g(\sigma)$, then

$$\mathcal{F}\{e^{\mathrm{i}at} f(t)\} = g(\sigma - a)$$

Proof:

By definition,

$$\begin{aligned}
\mathcal{F}\{e^{\mathrm{i}at} f(t)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma\tau} e^{\mathrm{i}a\tau} f(\tau) d\tau \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}(\sigma - a)\tau} f(\tau) d\tau \\
&= g(\sigma - a)
\end{aligned}$$

$\square$

This is known as translation property.

Some Properties of Fourier transform

Theorem III:

If $\mathcal{F}\{f(t)\} = g(\sigma)$, $f(t)$ is continuously differentiable and $f(t) \to 0$ as $|t| \to \infty$, then

$$\mathcal{F}\{f'(t)\} = \mathrm{i}\sigma g(\sigma).$$

Proof:

By definition,

$$\begin{aligned} \mathcal{F}\{f'(t)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma\tau} f'(\tau) d\tau \\ &= \frac{1}{\sqrt{2\pi}} \{[f(\tau)e^{-\mathrm{i}\sigma\tau}]_{-\infty}^{\infty} + i\sigma \int_{-\infty}^{\infty} e^{-\mathrm{i}\sigma\tau} f(\tau) d\tau\} \\ &= \mathrm{i}\sigma g(\sigma). \end{aligned}$$

□

Theorem IV (Extension of Theorem III):

If $f(t)$ is continuously n-times differentiable and $f^{(k)}(t) \to 0$ as $|t| \to \infty$ for $k = 1, 2, \ldots, (n-1)$, then the Fourier transform of the n-th derivative of $f(t)$ is given by

$$\mathcal{F}\{f^{(n)}(t)\} = (\mathrm{i}\sigma)^n \mathcal{F}\{f(t)\} = (\mathrm{i}\sigma)^n g(\sigma).$$

□

Some examples of Fourier integrals

Example

Find the Fourier integral representation of the following non-periodic function

$$f(t) = \begin{cases} e^{at}, & t < 0, \\ e^{-at}, & t > 0. \end{cases} \quad a > 0$$

Solution :-The Fourier transform of $f(t)$:

$$\begin{aligned} \mathcal{F}\{f(t)\} &= g(\sigma) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau) e^{-i\sigma\tau} \, d\tau \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(a-i\sigma)\tau} \, d\tau + \int_{0}^{\infty} e^{-(a+i\sigma)\tau} \, d\tau \\ &= \frac{1}{\sqrt{2\pi}} \frac{2a}{a^2+\sigma^2}. \end{aligned}$$

Taking the inverse

$$\begin{aligned} f(t) = \mathcal{F}^{-1}\{g(\sigma)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\sigma) e^{i\sigma t} \, d\sigma \\ &= \frac{2a}{\pi} \int_{0}^{\infty} \frac{\cos \sigma t}{a^2+\sigma^2} \, d\sigma. \end{aligned}$$

Convolution

A convolution is an integral that expresses the amount of overlap of one function f_2 as it is shifted over another function f_1.

It therefore "blends" one function with another.

In other words, the output which produces a third function can be viewed as a modified version of one of the original functions.

Definition:

The convolution of two functions $f_1(t)$ and $f_2(t)$, $-\infty < t < \infty$, is defined as

$$f_1(t)*f_2(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(t-\xi) f_2(\xi)\, d\xi = f_2(t)*f_1(t),$$

provided the integral exists for each t.

This is known as the convolution integral.

Convolution: Some algebraic properties

Property I (Commutative):

$$f_1 * f_2 = f_2 * f_1$$

Property II (Associative):

$$f_1 * (f_2 * f_3) = (f_1 * f_2) * f_3$$

Property III (Distributive):

$$(\alpha f_1 + \beta f_2) * f_3 = \alpha(f_1 * f_3) + \beta(f_2 * f_3)$$