Indian Institute of Technology Patna
End Semester Examination

MA201 (July-November 2017)

Time: 3 hours | Total Marks: 50

Instructions: There are 13 questions and the credit for each question is mentioned at the end. Solutions should be written clearly. Please keep the answers to the first three questions together.

1. Find the Fourier series of the periodic function defined by:

$$f(x) = \begin{cases} 0, & -\pi \le x < 0, \\ x, & 0 \le x < \pi. \end{cases}$$

Hence, prove that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ... = \frac{\pi^2}{8}$. [2 + 2]

2. Find the integral surface of $xz_x + yz_y = z$ over $\Gamma : x_0 = t^2, y_0 = t + 1, z_0 - t$. [3]

3. (a) Pick the correct one/s: If the number of real characteristics of a PDE $au_{xx} + bu_{xy} + cu_{yy} = 0$ are less than two, then the PDE can be a) Hyperbolic, b) Parabolic c) Elliptic, d) can be none of the Parabolic/Hyperboic/Elliptic type. $[1\frac{1}{2}]$

   (b) True/False: For a harmonic function $u \in C^2(\Omega)$ such that $\Delta u = 0$ defined in $\Omega = \{(x, y) | x^2 + y^2 < 1\}$, one can not have $u_x(x_0, y_0) = 0, u_y(x_0, y_0) = 0$ where (x_0, y_0) is an interior point of Ω. One line idea in favor of your answer is required. $[1 + \frac{1}{2}]$

4. State whether the following statements are true or false: [+1/-1]

   (a) A point z_0 is a boundary point of a set $S \leq \mathbb{C}$ if there exists a neighborhood of z_0 which contains points in S and points not in S.

   (b) The function $f(z) = \frac{\sin z}{z^6}$ has a pole of order 6.

   (c) If C is a simple closed contour lying inside an annular region R and a function f is analytic throughout R, then $\int_C f(z)dz = 0$.

   (d) The power series $\sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n}}$ diverges at some point on its circle of convergence.

5. Determine whether the following statements are true or false. Give explanation for your answers. [2 each]

   (a) There exists an entire function f such that $f^{(3)}(z) = \frac{1}{z^2+3}$.

   (b) The function $f(z) = \frac{1}{1-\cos(\frac{\pi}{2z})}$ has only one isolated singularity.

(c) The function $f(x+iy) = xy + iy$ is analytic at $z = i$.

6. Let f be a non-constant entire function. Which of the following properties are possible for each $z \in \mathbb{C}$:

   i) $\Re f(z) = \Im f(z)$; ii) The function $\overline{f(z)}$ is entire; iii) $f(z) \neq 0$.

   Give reasons/examples in support of your answer. [1+1+1]

7. Show that $\left|\int_C (e^z - \bar{z})dz\right| \leq 96$, where C is the positively oriented boundary of the triangle with vertices on the points $0, 3i$ and -4. [3]

8. Identify the isolated singularities of the following functions, classify it as removable, pole or essential. If it is pole, find its order. Also find the residue at each singularity:

   i) $f(z) = \frac{\sin z}{z(z-2\pi)}$, ii) $f(z) = \frac{z^2-3z+2}{(z-1)^2(z-3)^3}$, iii) $\frac{1-\cosh z}{z^3}$. [1 + 2 + 1]

9. i) Show that if f is analytic within and on a simple closed contour C and z_0 is not on C, then

$$\int_C \frac{f'(z)}{z-z_0}dz = \int_C \frac{f(z)}{(z-z_0)^2}dz.$$

   ii) Evaluate the integral $\int_C \frac{\cos z}{z(z^2+8)}dz$ using Cauchy integral formula, where C is the positively oriented boundary of the square with vertices $2+2i, -2+2i, -2-2i$ and $2-2i$. [3+2]

10. Let C be the unit circle $z = e^{i\theta}$ $(-\pi \leq \theta \leq \pi)$. First show that for any real constant a, $\int_C \frac{e^{az}}{z}dz = 2\pi i$. Then, write this integral in terms of θ to derive the formula $\int_0^\pi e^{a\cos\theta}\cos(a\sin\theta)d\theta = \pi$. [5]

11. Evaluate the integral $\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz$, where C denote the positively oriented circle $|z| = 4$. [3]

12. Find the Laurent series, centered at the point $z_0 = 0$, of the function $f(z) = \frac{1}{3-4z+z^2}$, that converges in a domain including $z = 2$. [3]

13. Let the function $f(z) = u(x,y) + iv(x,y)$ be analytic in some domain D. Prove that the function $U(x,y) = e^{u(x,y)}\cos(v(x,y))$ and $V(x,y) = e^{u(x,y)}\sin(v(x,y))$ are harmonic in D and $V(x,y)$ is a harmonic conjugate of $U(x,y)$. [4]
