Indian Institute of Technology Patna
MA201: Mathematics III
End Semester Exam (24-11-2014)

Time: 3hrs   Max. Marks: 50

Note: There are total **SIX** questions. Answer all questions. Give precise and brief answer. Standard formulae may be used.

Que 1. Answer all parts of this question at one place. **[1x10]**

(a.) Sketch the region in xy-plane where the p.d.e. $u_{xx}+2xu_{xy}-(y^2-1)u_{yy}=0$ is elliptic.

(b.) Find Fourier Transform of $f(x)=xe^{-x^2/2}$. Given $\mathcal{F}(e^{-x^2/2})=e^{-w^2/2}$.

(c.) The function $u(x,t)=\phi(x-ct)+\psi(x+ct)$ is general solution of 1D- wave equation $u_{tt}=c^2u_{xx}$. Give interpretation of function $\phi(x-ct)$.

(d.) Let u be harmonic in $\Omega=\{(x,y): x^2+y^2<1\}$, and $u(x,y)=1+3x$ for $(x,y)\in\partial\Omega$. Without solving, determine $\max_{(x,y)\in\Omega} u$ and $\min_{(x,y)\in\Omega} u$.

(e.) Obtain a first order p.d.e. by elimination of constants for the surface $z=ax+by+\frac{1}{a+b}$.

(f.) What transformation function ξ and η should be selected for equation $yu_{xx}+(x+y)u_{xy}+xu_{yy}=0$ to change in canonical form.

(g.) Is following equation homogeneous or nonhomogeneous: $u_{xx}+u_{yy}=x$.

(h.) Solve the heat equation:

$$u_t=u_{xx},\quad 0<x<1,\quad t>0,$$
$$u(x,0)=0,\quad 0<x<1,$$
$$u(0,t)=0,\quad u(1,t)=0, \forall t.$$

(i.) Is equation $e^z=-2$ solvable? If yes, find a solution. Is your solution unique?

(j.) Show that the set of points satisfying $|z-2i|+|z+2i|=8$ represent ellipse. Also find the foci of the ellipse.

Que 2. **(a.)** Let $u(x,y)$ and $v(x,y)$ be real and imaginary parts of a complex function

$$f(z)=\begin{cases}\dfrac{\bar{z}^2}{z}, & z\neq 0\\ 0, & z=0.\end{cases}$$

Show that Cauchy-Riemann equations are satisfied at the point $z=(0,0)$. Further show that $f(z)$ is not differentiable at $z=(0,0)$. **[3+2]**

(b.) Evaluate the integral $\int_C \frac{e^z}{z(z-1)^2}\,dz$ where C is the circle $|z|=2$. **[3]**

Que 3. **(a.)** Find the Fourier Series expansion of the following function:

$$f(x) = 2x + 1, \quad -\pi < x < \pi, \quad \text{and} \quad f(x + 2\pi) = f(x), \quad \forall x,$$

and discuss its convergence in $\mathbb{R}$. [4]

(b.) Using Fourier Transform, solve the Heat equation: [4]

$$\begin{array}{ll} DE : u_t = k u_{xx}, & -\infty < x < \infty, \; t > 0, \\ IC : u(x,0) = f(x), & -\infty < x < \infty, \\ u \text{ and } u_x \to 0 & \text{as } |x| \to \infty. \end{array}$$

Given that $\mathcal{F}(e^{-ax^2}) = \frac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}}$

Que 4. **(a.)** Discuss in each of following cases, the existence of solution of the p.d.e. $p + q = z^2$ which contains the initial curve:

(i) $\Gamma : x = t, \; y = 0, \; z = t^2,$

(ii) $\Gamma : x = t, \; y = t, \; z = -1,$

(iii) $\Gamma : x = t, \; y = t, \; z = -1/t.$

[6]

(b.) Solve the PDE: $z(x + y)p + z(x - y)q = x^2 + y^2$, with Cauchy data $z = 0$ on $y = 2x$.

[4]

Que 5. Use Duhamel to solve:

[6]

$u_{tt} = u_{xx} + t \sin \pi x, \;\; 0 < x < 1, \; t > 0,$
IC: $u(x,0) = \sin \pi x, \;\; u_t(x,0) = 2 \sin \pi x + 4 \sin 3\pi x, \;\; 0 < x < 1,$
BCs: $u(0,t) = 0, \; u(1,t) = 0, \; t > 0.$

Que 6. **(a.)** Determine the solution of following wave equation for the **semi-infinite range**:

$u_{tt} = 4u_{xx}, \;\; 0 < x < \infty, \; t > 0,$
IC: $u(x,0) = |\sin x|, \; u_t(x,0) = 0, \;\; 0 < x < \infty,$
BC: $u(0,t) = 0, \; t > 0.$

[3]

(b.) Solve the Laplace equation:

$u_{xx} + u_{yy} = 0, \;\; 0 < x < 1, \;\; 0 < y < 2,$
$u(x,0) = x, \; u(x,2) = 0, \; 0 < x < 1,$
$u(0,y) = 0, \;\; u(1,y) = 0, \; 0 < y < 2.$

[5]

Good Luck
