MA101

INTEGRATION IN VECTOR FIELDS

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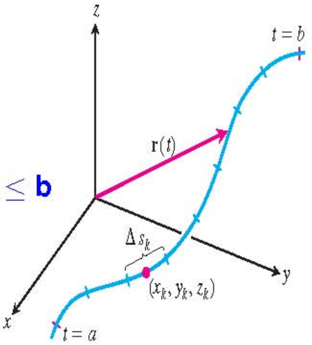
Suppose that $\mathbf{f(x,y,z)}$ is a real-valued function we wish to integrate over the curve

$$\mathbf{C} : \vec{r}(\mathbf{t}) = \mathbf{g(t)}\hat{\mathbf{i}} + \mathbf{h(t)}\hat{\mathbf{j}} + \mathbf{k(t)}\hat{\mathbf{k}}, \quad \mathbf{a} \leq \mathbf{t} \leq \mathbf{b}$$

lying within the domain of f.



$$\mathbf{S_n} = \sum_{\mathbf{k}=1}^{\mathbf{n}} \mathbf{f(x_k, y_k, z_k)\Delta s_k}.$$

$$\lim_{\mathbf{n}\to\infty} \mathbf{S_n} = \int_{\mathbf{C}} \mathbf{f(x,y,z)\,ds} = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f(x(t),y(t),z(t))\,|v(t)|\,dt}.$$

Line integrals have the useful property that if a curve C is made by joining a finite number of curves $C_1, C_2, \cdots, C_n$ end to end, then the integral of a function over C is the sum of the integrals over the curves that make it up:

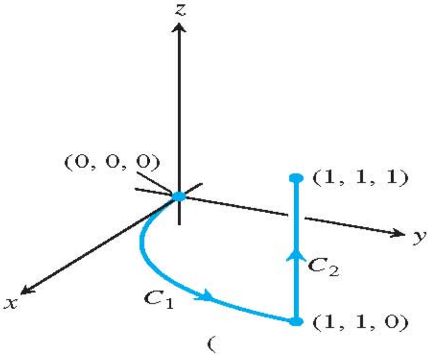
$$\int_{\mathbf{C}} \mathbf{f}\,\mathbf{ds} = \int_{\mathbf{C_1}} \mathbf{f}\,\mathbf{ds} + \int_{\mathbf{C_2}} \mathbf{f}\,\mathbf{ds} + \cdots + \int_{\mathbf{C_n}} \mathbf{f}\,\mathbf{ds}.$$

16.1 Problem 15(a)

Integrate $f(x,y,z) = x + \sqrt{y} - z^2$ over the path (as shown in adjacent figure) from $(0,0,0)$ to $(1,1,1)$, given by

$$C_1 : r(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 1,$$

$$C_2 : r(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 1.$$



Gradient Field

The gradient field of a differentiable function $f(x, y, z)$ is the field of gradient vectors

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Work Done by a Force over a Curve in Space

Suppose that the vector field $F = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + P(x,y,z)\mathbf{k}$ represents a force throughout a region in space (it might be the **force of gravity** or an **electromagnetic force of some kind**) and that $\overrightarrow{r}(t) = g(t)\widehat{\mathbf{i}} + h(t)\widehat{\mathbf{j}} + k(t)\widehat{\mathbf{k}}, \quad a \le t \le b,$ is a smooth curve in the region. The work done by a force F (defined above) over a smooth curve $\overrightarrow{\mathbf{r}}(t)$ from a to b is

$$W = \int_{t=a}^{t=b} \mathbf{F} \cdot \mathbf{T}\, ds.$$

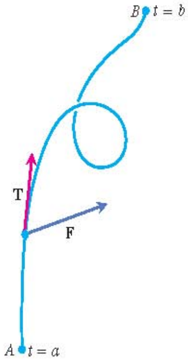


FIGURE 16.16 The work done by a force $\mathbf{F}$ is the line integral of the scalar component $\mathbf{F} \cdot \mathbf{T}$ over the smooth curve from A to B.

Other ways to write Work Integral

$$\begin{aligned} W &= \int_a^b \mathbf{F}\cdot\mathbf{T}\,ds, && (1)\\ &= \int_a^b \mathbf{F}\cdot d\mathbf{r}, && (2)\\ &= \int_a^b \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt, && (3)\\ &= \int_a^b \left(M\frac{dx}{dt}+N\frac{dy}{dt}+P\frac{dz}{dt}\right)\,dt. && (4)\\ &= \int_a^b M\,dx+N\,dy+P\,dz. && (5) \end{aligned}$$

Evaluating a Work Integral

To evaluate the work integral along a smooth curve $r(t)$, take these steps:

- Evaluate $\mathbf{F}$ on the curve as a function of the parameter t.
- Find $\frac{d\mathbf{r}}{dt}$.
- Integrate $\mathbf{F} \cdot \mathbf{dr}/\mathbf{dt}$ from $t = a$ to $t = b$.

16.2 Problem 10

Find the work done by force $\mathbf{F} = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ from $(0,0,0)$ to $(1,1,1)$ over each of the following paths

- The straight line path $C_1 : \mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k},\ 0 \le t \le 1.$
- The curved path $C_2 : \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k},\ 0 \le t \le 1.$
- The path $C_3 \cup\ C_4$ consisting of the line segment from $(0,0,0)$ to $(1,1,0)$ followed by the segment from $(1,1,0)$ to $(1,1,1)$.

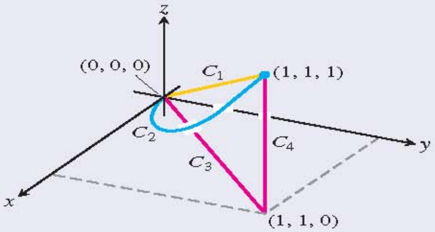


FIGURE 16.21 The paths from (0, 0, 0) to (1, 1, 1).

Flow Integrals and Circulation for Velocity Fields

If $\mathbf{r}(\mathbf{t})$ is a smooth curve in the domain of a continuous **velocity field F**, the **flow along the curve** from $t = a$ to $t = b$ is

$$Flow = \int_a^b \mathbf{F} \cdot \mathbf{T}\, \mathbf{ds}.$$

The integral in this case is called a **flow integral**. **If the curve is a closed loop, the flow is called the circulation around the curve.**

Problem 27, Page 16.2

Find the circulation of the field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ around and across the closed semicircular path that consists of the semicircular arch $\mathbf{r}_1(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}, \quad 0 \leq t \leq \pi$ followed by the line segment $\mathbf{r}_2(t) = t\mathbf{i}, \quad -a \leq t \leq a$.

Flux Across a Closed Curve in the Plane

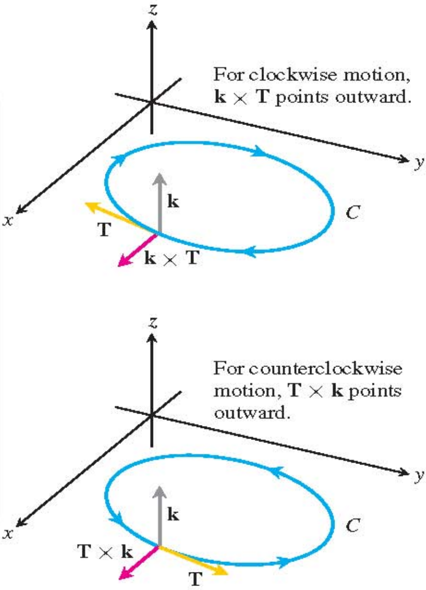
If C is a smooth closed curve in the domain of a continuous vector field in the plane $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ and if $\mathbf{n}$ is the outward-pointing unit normal vector on C, the **flux** of $\mathbf{F}$ **across** C is

$$\textit{Flux of } \mathbf{F} \textit{ across } C = \int_C \mathbf{F} \cdot \mathbf{n}\, ds.$$

Calculating Flux Across a Smooth Closed Plane Curve

$$\text{Flux of } \mathbf{F} = M\mathbf{i} + N\mathbf{j} \text{ across } C$$
$$= \int_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C M\,dy - N\,dx.$$

The integral can be evaluated from any smooth parametrization $x = g(t)$, $y = h(t)$ for $a \leq t \leq b$, that traces C counterclockwise exactly once.



- If **F** is force field. Then
$$\int_a^b \mathbf{F} \cdot \mathbf{T}\, ds$$
represents **work done**.
- If **F** is velocity field. Then
$$\int_a^b \mathbf{F} \cdot \mathbf{T}\, ds$$
represents **flow along** the curve.
- If **F** is velocity field and the curve C is closed. Then
$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$$
represents **circulation along the closed curve** C.
- If **F** is vector field. Then
$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$$
represents **flux across the closed curve** C.

Problem 27, Page 16.2

Find the flux of the field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ around and across the closed semicircular path that consists of the semicircular arch $\mathbf{r}_1(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}, \quad 0 \leq t \leq \pi$ followed by the line segment $\mathbf{r}_2(t) = t\mathbf{i}, \quad -a \leq t \leq a.$

Problem 42, Page 16.2, Zero circulation

Let C be the ellipse in which the plane $2x + 3y - z = 0$ meets the cylinder $x^2 + y^2 = 12$. Show without evaluating either line integral directly, that the circulation of the field $F = x\hat{i} + y\hat{j} + z\hat{k}$ around C in either direction is zero.

MA101

Path Independence, Potential Functions, and Conservative Fields

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If A and B are two points in an open region D in space, the work $\int \mathbf{F} \cdot d\mathbf{r}$. done in moving a particle from A to B by a field $\mathbf{F}$ defined on D usually depends on the path taken. For some special fields, however, the integral's value is the same for all paths from A to B.

Path Independence, Conservative Field

Let $\mathbf{F}$ be a field defined on an open region D in space, and suppose that for any two points A and B in D the work $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ done in moving from A to B is the same over all paths from A to B. Then the integral is **path independent in D** and the field $\mathbf{F}$ is **conservative on D**.

Remark

The word conservative comes from physics, where it refers to fields in which the principle of conservation of energy holds (it does, in conservative fields).

Under differentiability conditions normally met in practice, a field $\mathbf{F}$ is conservative if and only if it is the gradient field of a scalar function f; that is, if and only if $\mathbf{F} = \nabla f$ for some f. The function f then has a special name "Potential Function".

Potential Function

If $\mathbf{F}$ is a field defined on D and $\mathbf{F} = \nabla f$ for some scalar function f on D, then f is called a potential function for $\mathbf{F}$.

Examples

1. An electric potential is a scalar function whose gradient field is an electric field.
2. A gravitational potential is a scalar function whose gradient field is a gravitational field, and so on.

Fundamental Theorem of Calculus formula

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f}'(\mathbf{x})\mathbf{dx} = \mathbf{f}(\mathbf{b}) - \mathbf{f}(\mathbf{a}).$$

Analogue of Fundamental Theorem of Calculus formula in Higher Dimension

Let f be a potential function for a field $\mathbf{F}$, i.e., $\mathbf{F} = \nabla f$ then we can evaluate all the work integrals in the domain of $\mathbf{F}$ over any path between A and B by

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B \nabla \mathbf{f} \cdot d\mathbf{r} = f(B) - f(A).$$

Theorem 1 The Fundamental Theorem of Line Integrals

1. Let $\mathbf{F} = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + P(x,y,z)\mathbf{k}$ be a **vector field whose components are continuous** throughout an open **connected region** D in space. Then there exists a **differentiable function** f such that

$$\mathbf{F} = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

   if and only if for all points A and B in D the value of $\int_A^B \mathbf{F}\cdot\, d\mathbf{r}$ is independent of the path joining A to B in D.

2. If the integral is independent of the path from A to B, its value is

$$\int_A^B \mathbf{F}\cdot\, d\mathbf{r} = f(B) - f(A).$$

Theorem 2 Closed-Loop Property of Conservative Fields

The following statements are equivalent.

1. $\int \mathbf{F} \cdot \, d\mathbf{r} = 0$ around every closed loop in D.
2. The field $\mathbf{F}$ is conservative on D.

The following diagram summarizes the results of Theorems 1 and 2.

$$\mathbf{F} = \nabla f \text{ on } D.$$

$$\Updownarrow$$

$$\mathbf{F} \text{ is conservative on } D.$$

$$\Updownarrow$$

$$\oint \mathbf{F} \cdot \, d\mathbf{r} = 0 \text{ on any closed path in } D.$$

The test for being conservative is the following. Keep in mind our assumption that the domain of F is **connected and simply connected**.

Component Test for Conservative Fields

Let $\mathbf{F} = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + P(x,y,z)\mathbf{k}$ be a field whose component functions have continuous first partial derivatives. Then, **F** is conservative **if and only if**

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}. \qquad (1)$$ [a]

[a] The second half of the proof, that Equations (1) imply that **F** is conservative, is a consequence of Stokes' Theorem, taken up in Section 16.7, and requires our assumption that the domain of **F** be **simply connected**.

Finding a Potential Function

Once we know that **F** is conservative, we usually want to find a potential function f for **F**. This requires solving the equation $\mathbf{F} = \nabla f$ or

$$M\mathbf{i} + N\mathbf{j} + P\mathbf{k} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

for f. We accomplish this by integrating the three equations

$$M = \frac{\partial f}{\partial x}, \quad N = \frac{\partial f}{\partial y}, \quad P = \frac{\partial f}{\partial z}.$$

Potential functions is not Unique.

16.3

Problem 7

Verify whether potential functions exists, then find a potential function f for field

$$F = 2x\hat{i} + 3y\hat{j} + 4z\hat{k}.$$

Answer: $f(x, y, z) = x^2 + (3/2)y^2 + 2z^2 + c.$

Problem 8

Verify whether potential functions exists, then find a potential function f for field

$$F = (2xy - z^2)\hat{i} + (x^2 + 2yz)\hat{j} + (y^2 - 2zx)\hat{k}.$$

Answer: $f(x, y, z) = x^2y - xz^2 + y^2z + c.$

Exact Differential Forms

It is often convenient to express work and circulation integrals in the "differential" form

$$\int_A^B Mdx + Ndy + Pdz.$$

Such integrals are relatively easy to evaluate if $Mdx + Ndy + Pdz$ is the total differential of a function f, i.e.,

$$Mdx + Ndy + Pdz = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

For then

$$\int_A^B Mdx+Ndy+Pdz = \int_A^B \frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz = \int_A^B \nabla f \cdot d\mathbf{r} = f(B)-f(A).$$

Definitions: Exact Differential Form

Any expression $M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz$ is a differential form. A differential form is exact on a domain D in space if

$$M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df$$

for some scalar function f throughout D.

Component Test for Exactness of $M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz$

The differential form $M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz$ is exact if and only if

$$M = \frac{\partial f}{\partial x}, \quad N = \frac{\partial f}{\partial y}, \quad P = \frac{\partial f}{\partial z}.$$

This is equivalent to saying that the field $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative.

16.3

Problem 18

Find a potential function and evaluate the integral.

$$\int_{(0,2,1)}^{(1,\pi/2,2)} 2\cos y \; dx + \left(\frac{1}{y} - 2x \sin y\right) \; dy + \frac{1}{z} \; dz.$$