Indian Institute of Technology Patna
MA201: Mathematics III
End Semester Exam (22-11-2016)

Time: 3hrs **Max. Marks: 50**

<u>**Note:**</u> There are total **10** questions. Answer all questions. Give precise and brief answer. Standard formulae may be used. Some formulae given at the end. **Do not write anything on the question paper.**

Que 1. Answer all parts of this question at one place. [1x8]

(a.) Is the function $f(x) = \begin{cases} 0, & -1 < x < 0, \\ x^2, & 0 < x < 1/2, \\ \frac{1}{1-x}, & 1/2 < x < 1. \end{cases}$ piecewise continuous in [-1,1]? Justify your answer.

(b.) Will Gibbs Phenomenon be observed for the Fourier Series of the 2π periodic function $f(x) = x^2,\ x \in [-\pi, \pi]$? Justify.

(c.) If $g(-x) = \overline{f(x)}$ then show that $\overline{F(w)} = G(w)$. Here $F(w) = \mathcal{F}(f(x))$ and $G(w) = \mathcal{F}(g(x))$

(d.) If $\mathcal{F}(f(x)) = F(w)$ then find $\mathcal{F}(\int\limits_0^x f(u)du)$.

(e.) Classify the following pde as linear/semilinear/quasilinear or nonlinear: $pq = z^2$.

(f.) Does pde $xp + yq = z$ have a solution passing through a curve $\Gamma : x_0 = t^2,\ y_0 = t+1,\ z_0 = t$?

(g.) Obtain a first order pde for the surface $z = x + ax^2y^2 + b^4$.

(h.) Solution of the Dirichlet problem $\nabla^2 u = 0$ on domain $\Omega \subset \mathbb{R}^2$ and $u = 0$ on $\partial\Omega$ is given as ....

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Que 2. **a)** Obtain Fourier Series for the function $f(x) = |x|,\ f(x + 2\pi) = f(x), \forall x$. Discuss the convergence of the Fourier Series obtained. [3]

b) Using Fourier Integral show that, [3]

$$\int\limits_0^\infty \frac{w^3 \sin xw}{w^4 + 4} dw = \frac{\pi}{2} e^{-x} \cos x, \quad x > 0.$$

Que 3. **a)** Find the Fourier Transform of the function $f(x) = xe^{-\frac{(x-2)^2}{2}}$. [2]

b) Using Fourier Transform solve the wave equation: [3]

$$\begin{aligned} DE: u_{tt} &= c^2 u_{xx}, \quad -\infty < x < \infty,\ t > 0, \\ ICs: u(x,0) &= e^{-|x|}, \quad -\infty < x < \infty, \\ u_t(x,0) &= 0, \quad -\infty < x < \infty, \\ u \text{ and } u_x &\to 0 \quad \text{as } x \to \infty. \end{aligned}$$

Que 4. **a)** Obtain general solution of the PDE: $(x+y-z)p - (x+y+z)q = 2z$, for the initial data $z = 1$ on line $y = x$. [3]

b) Classify the PDE $u_{xx} + x^2 u_{yy} = 0,\ x \neq 0$. Obtain the corresponding canonical form. [4]

Que 5. a) Solve the following nonhomogeneous wave equation: $u_{tt} = c^2 u_{xx} + x(t-1)$, $x \in \mathbb{R}$, $t > 0$ with ICs $u(x,0) = 0$ and $u_t(x,0) = 0$ $x \in \mathbb{R}$. [3]

b) Use method of separation of variables to solve heat equation $u_t = u_{xx}$, with IC, $u(x,0) = 4\sin \pi x + \sin 3\pi x - 2\sin 5\pi x$, $0 < x < 1$, and BCs, $u(0,t) = 0 = u(1,t)$. *Do not use direct formula.* [4]

Que 6. Either Prove or Disprove (by an example):
Let u be harmonic in $\Omega = \{(x,y) : \frac{x^2}{4} + \frac{y^2}{9} < 1\}$, and $u(x,y) = 3 + x$ for $(x,y) \in \partial\Omega$, then the function $u(x,y) > 0$ $\forall (x,y) \in \Omega$. [2]

Que 7. Either Prove or Disprove (by an example):

a) Identities on real line also hold on complex plane. [2]

b) If f is a continuous function in $\mathbb{C}$ and satisfies $f(z) = f(2z)$ for all $z \in \mathbb{C}$, then f is also a differentiable function. [2]

Que 8. Develop a complex transformation, $z \to w$ (i.e., complex mapping $w = f(z)$) that can achieve a rotation of $\pi/3$ and an expansion of 2, both about the point $z_1 = (1+i)$.[2]

Que 9. By using the contour integral in complex analysis, evaluate $\int_0^\infty \frac{\cos(mx)}{x^2 + a^2} dx$ where $m > 0$ and $a > 0$. [5]

Que 10. Using Rouches theorem, find the number of roots of the equation $z^9 - 2z^6 + z^2 - 8z - 2 = 0$ lying in $|z| < 1$. [4]

Important Formulae:

- The second order general PDE : $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$ can be transformed using $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$ into following canonical form $\overline{A}u_{\xi\xi} + \overline{B}u_{\xi\eta} + \overline{C}u_{\eta\eta} + \overline{D}u_\xi + \overline{E}u_\eta + \overline{F}u = \overline{G}$ where

$\overline{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$
$\overline{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$
$\overline{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$
$\overline{D} = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y$
$\overline{E} = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y$
$\overline{F} = F$, $\overline{G} = G$.

- Fourier Transform of $f(x)$, $\mathcal{F}(f(x)) = F(w) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x)e^{-iwx}dx$
- $\mathcal{F}(e^{-ax^2}) = \frac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}}$.
- Fourier Sine Integral $f(x) = \int\limits_0^\infty B(w)\sin wx\, dw$, where $B(w) = \frac{2}{\pi}\int\limits_0^\infty f(v)\sin wv\, dv$

Good Luck

ROLL NUMBER: