**Department of Mathematics**
**Indian Institute of Technology Patna**
**MA-102 : Mathematics- II (Linear Algebra)**
**(Spring Semester: 2018-2019)**

**Tutorial- 5** **Instructor: Dr. Om Prakash**

1. Let $V = C[0,1]$ be the space of all continuous function on the interval $[0,1]$. For any $f, g \in V$, define
$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt.$$
Let $f(t) = t, g(t) = e^t$. Compute $\langle f, g\rangle$, $\| f \|, \| g \|$, and $\| f + g \|$. Then verify both the Cauchy-Schwarz inequality and triangle inequality.

2. Let $V$ be an inner product space. Then for any two vectors $x, y \in V$ are orthogonal if and only if $\| x + y \|^2 = \| x \|^2 + \| y \|^2$.

3. Prove that if $V$ is an inner product space, then $| \langle x, y\rangle | = \| x \| . \| y \|$ if and only if one of the vectors $x$ or $y$ is a multiple of the other.

4. Let $T$ be a linear operator on an inner product space $V$, and suppose that $\| T(x) \| = \| x \|$ for all $x$. Prove that $T$ is one-to-one.

5. Apply Gram-Schmidt process to obtain an orthonormal set:
   (a) $\{(-1, 0, 1), (1, -1, 0), (0, 0, 1)\}$ in $\mathbb{R}^3$.
   (b) $\{1, p_1(t) = t, p_2(t) = t^2\}$ of $\mathcal{P}_2$ with the inner product
   $$\langle p, q\rangle = \int_0^1 p(t)q(t)dt.$$
   (c) $\{(1, 1, 1, 1), (0, 2, 0, 2), (-1, 1, 3 - 1)\}$ in $\mathbb{R}^4$.

6. In each of the following parts, find the orthogonal projection of the given vector on the given subspace $W$ of the inner product space $V$.
   (a) $V = \mathbb{R}^2, u = (2, 6)$ and $W = \{(x, y) \mid y = 4x\}$.
   (b) $V = \mathbb{R}^3, u = (2, 1, 3)$ and $W = \{(x, y, z) \mid x + 3y - 2z = 0\}$.
   (c) $V = P(\mathbb{R})$ with the inner product
   $$\langle f, g\rangle = \int_0^1 f(t)g(t)dt,$$
   $h(x) = 4 + 3x - 2x^2$ and $W = P_1(\mathbb{R})$.

7. Let $\beta$ is a basis for a subsapce $W$ of an inner product space $V$, and let $z \in V$. Prove that $z \in W^{\perp}$ if and only if $\langle z, v\rangle = 0$ for every $v \in \beta$.

8. For each of the following inner product space $V$ (over $\mathbb{F}$) and linear transformations $g : V \to \mathbb{F}$, find a vector $y$ such that $g(x) = \langle x, y \rangle$ for all $x \in V$.

   (a) $V = \mathbb{R}^3, g(a_1, a_2, a_3) = a_1 - 2a_2 + 4a_3$.

   (b) $V = \mathbb{C}^2, g(a_1, a_2) = a_1 - 2a_2$.

   (c) $V = P_2(\mathbb{R})$ with
   $$\langle f, g \rangle = \int_0^1 f(t)g(t)dt,$$
   $g(f) = f(0) + f'(1)$.

9. What is the associated matrix of the quadratic form $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(X) = x^2 - 3xy + 4y^2$.

10. Find the bilinear form of $\mathbb{R}^4$ whose associated quadratic forms

    (a) $x_1x_2$

    (b) $x_1x_3 + x_4^2$

    (c) $2x_1x_2 - x_3x_4$

    (d) $x_1^2 - 5x_2x_3 + x_4^2$.

11. Let $V$ be $n$-dimensional vector space over a field $\mathbb{F}$ and $f : V \to \mathbb{F}$ be a function. Assume $g : V \times V \to \mathbb{F}$ define by $g(u, v) = f(u + v) - f(u) - f(v)$ be a bilinear function. Also, assume that $f(au) = a^2 f(u)$, for $u \in V, a \in \mathbb{F}$. Show that $f$ is a quadratic form and find the bilinear form which it comes.

12. Let $T$ be a linear operator on a real inner product space $V$, and define $H : V \times V \to \mathbb{R}$ by $H(x, y) = \langle x, T(y) \rangle$ for all $x, y \in V$.

    (a) Prove that $H$ is bilinear form.

    (b) Prove that $H$ is symmetric if and only if $T$ is self-adjoint.

    (c) What properties must $T$ have for $H$ to be an inner product on $V$.

13. For each of the given quadratic forms $K$ on a real inner product space $V$, find the symmetric bilinear forms $H$ such that $K(x) = H(x, x)$ for $x \in V$. Then find an orthonormal basis $\beta$ for $V$ such that $\psi_\beta(H)$ is a diagonal matrix.

    (a) $K : \mathbb{R}^2 \to \mathbb{R}$ defined by
    $$K \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2x_1^2 + 4x_1x_2 + x_2^2$$

    (b) $K : \mathbb{R}^3 \to \mathbb{R}$ defined by
    $$K \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 3x_1^2 - 2x_1x_3 + 3x_2^2 + 3x_3^2$$

14. $Q$ is the quadratic form on $M_{2\times2}(\mathbb{R})$ defined by $Q(A) = det(A)$. Find an orthogonal basis for $M_{2\times2}(\mathbb{R})$.

15. $Q$ is the quadratic form on $\mathbb{R}^3$ defined by $Q(x_1, x_2, x_3) = x_1x_2 + x_2x_3 + x_3x_1$. Find an orthogonal basis for $\mathbb{R}^3$.