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# Median Sensitivity Proofs

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**Definition 1.** *The sample median of a database  $X = (x_1, \dots, x_n)$  is given by*

$$f(X) = \frac{\tilde{x}_l + \tilde{x}_u}{2}$$

*where  $l = \lfloor \frac{n+1}{2} \rfloor$  and  $u = \lceil \frac{n+1}{2} \rceil$  and  $\tilde{X} = (\tilde{x}_1, \dots, \tilde{x}_n)$  is the sorted version of  $X$ .*

## 1 NEIGHBORING DEFINITION: CHANGE ONE

### 1.1 $\ell_1$ -sensitivity

**Theorem 1.** *Let the database  $X$  have elements bounded above by  $M$  and below by  $m$ . Then the  $\ell_1$ -global sensitivity in the change-one model of the median is*

$$\Delta f(\cdot) = \begin{cases} \frac{M-m}{2}, & \text{if } n \equiv 0 \pmod{2} \\ M - m & \text{if } n \equiv 1 \pmod{2} \end{cases}$$

*Proof.* First consider the case where  $n$  is such that  $n \equiv 0 \pmod{2}$ . Then our worst-case scenario is that exactly half of the data elements in  $X$  are  $m$  and the other half are  $M$ . In this case,  $f(X) = \frac{m+M}{2}$ .

Now consider an  $X'$  which is identical to  $X$  but with one element switched from  $m$  to  $M$  WLOG.<sup>1</sup> Then we have that  $f(X') = M$ . Because we are looking at our worst-case pairing of  $X, X'$ , we know that

$$\Delta f(\cdot) = \max_{X, X'} |f(X') - f(X)| = \left| M - \frac{m+M}{2} \right| = \frac{M-m}{2}.$$

Now consider the case where  $n$  is such that  $n \equiv 1 \pmod{2}$ . Then our worst-case scenario is that  $\frac{n-1}{2}$  of the elements of  $X$  are  $m$ ,  $\frac{n-1}{2}$  of the elements of  $X$  are  $M$ , and the remaining

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<sup>1</sup>The result holds if it is switched from  $M$  to  $m$ .

element of  $X$  is  $m$  (WLOG). In this setting,  $f(X) = m$ . Let  $X'$  be identical to  $x$  but with one element switched from  $m$  to  $M$ . Then  $f(X') = M$  and we have that

$$\Delta f(\cdot) = \max_{X, X'} |f(X') - f(X)| = |M - m|.$$

□

## 1.2 $\ell_2$ -sensitivity

**Theorem 2.** *Let the database  $X$  have elements bounded above by  $M$  and below by  $m$ . Then the global  $\ell_2$ -sensitivity in the change-one model of the median is*

$$\Delta f(\cdot) = \begin{cases} \left(\frac{M-m}{2}\right)^2, & \text{if } n \equiv 0 \pmod{2} \\ (M-m)^2 & \text{if } n \equiv 1 \pmod{2} \end{cases}$$

*Proof.* The logic follows exactly from the proof of the  $\ell_1$  sensitivity, just with the norm in the last line of each statement switched from 1 to 2. □

## 2 NEIGHBORING DEFINITION: ADD/DROP ONE

### 2.1 $\ell_1$ -sensitivity

### 2.2 $\ell_2$ -sensitivity