Sum Sensitivity Proofs

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Definition 1. *A sum query on database x of size n is defined to be*

$$s(x) = \sum_{i=1}^{n} x_i.$$

1 Neighboring Definition: Change One

1.1 ℓ_1-sensitivity

Theorem 1. *Say the space of datapoints $\mathcal{X}$ is bounded above by M and bounded below by m. Then s over $\mathcal{X}^n$ has ℓ_1-sensitivity bounded above by $M - m$.*

Proof. Say X and X' are neighboring databases which differ at data-point x_j. Then

$$\begin{aligned}
\Delta s &= \max_{X,X'} \left| s(X) - s(X)' \right| \\
&= \max_{X,X'} \left| \left(\sum_{\{i \in [n] | i \neq j\}} x_i \right) + x_j - \left(\sum_{\{i \in [n] | i \neq j\}} x'_i \right) + x'_j \right| \\
&= \max_{X,X'} \left| x_j - x'_j \right| \\
&\leq M - m.
\end{aligned}$$

□

1.2 ℓ_2-sensitivity

Theorem 2. *Say the space of datapoints $\mathcal{X}$ is bounded above by M and bounded below by m. Then s over $\mathcal{X}^n$ has ℓ_1-sensitivity bounded above by $(M - m)^2$.*

Proof. Say X and X' are neighboring databases which differ only at index j. Then

$$\begin{aligned}
\Delta\bar{X} &= \max_{X,X'}(s(X) - s(X)')^2 \\
&= \max_{X,X'}\left(\left(\left(\sum_{i\in[n]|i\neq j} x_i\right) + x_j - \left(\sum_{i\in[n]|i\neq j} x_i'\right) - x_j'\right)^2\right. \\
&= \max_{X,X'}(x_j - x_j')^2 \\
&\leq (M-m)^2.
\end{aligned}$$

□

2 Neighboring Definition: Add/Drop One

2.1 ℓ_1-sensitivity

Theorem 3. *Say the space of datapoints $\mathcal{X}$ is bounded above by M and bounded below by m. Then s has ℓ_1-sensitivity bounded above by* $\max(|m|, |M|)$.

Proof. For notational ease, let n always refer to the size of database x. We must consider both adding and removing an element from x. First, consider adding a point:

Let $X' = X \cup x$. Without loss of generality, assume the point added is the $(n+1)^{\text{th}}$ element of database x'. Note that

$$\begin{aligned}
\left|s(X) - s(X')\right| &= \left|\sum_{i=1}^{n} x_i - \sum_{i=1}^{n+1} x_i\right| \\
&= \left|\sum_{i=1}^{n} x_i - \left(\sum_{i=1}^{n} x_i\right) - x\right| \\
&= x \\
&\leq \max(|m|, |M|).
\end{aligned}$$

Second, consider removing a point:
Let $X' = X \backslash \{x\}$. Without loss of generality assume that the point subtracted is the n^{th} element of database x.

$$\begin{aligned}
\left|s(X) - s(X')\right| &= \left|\sum_{i=1}^{n} x_i - \sum_{i=1}^{n-1} x_i\right| \\
&= \left|\left(\sum_{i=1}^{n-1} x_i\right) + x - \sum_{i=1}^{n-1} x_i\right| \\
&= x \\
&\leq \max(|m|, |M|).
\end{aligned}$$

□

2.2 ℓ_2-sensitivity

Theorem 4. *Say the space of datapoints $\mathcal{X}$ is bounded above by M and bounded below by m. Then s has ℓ_2-sensitivity bounded above by* $\max(m^2, M^2)$.

Proof. For notational ease, let n always refer to the size of database x. We must consider both adding and removing an element from x. First, consider adding a point:

Let $X' = X \cup x$. Without loss of generality, assume the point added is the $(n+1)^{\text{th}}$ element of database x'. Note that

$$\begin{aligned}
\big(s(X) - s(X')\big)^2 &= \left(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n+1} x_i\right)^2 \\
&= \left(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i - x\right)^2 \\
&= x^2 \\
&\leq \max(m^2, M^2).
\end{aligned}$$

Second, consider removing a point:
Let $X' = X \backslash \{x\}$. Without loss of generality assume that the point subtracted is the n^{th} element of database x. Then,

$$\begin{aligned}
\big(s(X) - s(X')\big)^2 &= \left(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n-1} x_i\right)^2 \\
&= \left(\sum_{i=1}^{n-1} x_i + x - \sum_{i=1}^{n-1} x_i\right)^2 \\
&= x^2 \\
&\leq \max(m^2, M^2).
\end{aligned}$$

$\square$