Harvard Privacy Tools Project

Mean Sensitivity Proofs

March 17, 2020

Definition 1. *The sample mean of database X of size n is*

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

1 Neighboring Definition: Change One

1.1 ℓ_1-sensitivity

Theorem 1. *Say database X has size n and is bounded above by M and bounded below by m. Then $\bar{X}$ has ℓ_1-sensitivity bounded above by*

$$\frac{M-m}{n}.$$

Proof. Say X and X' are neighboring databases which differ at data-point x_j. Then

$$\begin{aligned}
\Delta\bar{X} &= \max_{X,X'} \left|\bar{X} - \bar{X}'\right| \\
&= \max_{X,X'} \frac{1}{n}\left|\left(\sum_{\{i\in[n] \mid i\neq j\}} x_i\right) + x_j - \left(\sum_{\{i\in[n] \mid i\neq j\}} x'_i\right) + x'_j\right| \\
&= \max_{X,X'} \frac{1}{n}\left|x_j - x'_j\right| \\
&\leq \frac{M-m}{n}.
\end{aligned}$$

□

1.2 ℓ_2-sensitivity

Theorem 2. *Say database X has size n and is bounded above by M and bounded below by m. Then $\bar{X}$ has ℓ_2-sensitivity bounded above by*

$$\left(\frac{M-m}{n}\right)^2.$$

Proof. Say X and X' are neighboring databases which differ only at index j. Then

$$\begin{aligned}
\Delta\bar{X} &= \max_{X,X'}(\bar{X}-\bar{X}')^2 \\
&= \max_{X,X'}\frac{1}{n^2}\left(\left(\sum_{i\in[n]|i\neq j} x_i\right)+x_j-\left(\sum_{i\in[n]|i\neq j} x_i'\right)-x_j'\right)^2 \\
&= \max_{X,X'}\frac{1}{n^2}(x_j-x_j')^2 \\
&\leq \frac{(M-m)^2}{n^2} \\
&= \left(\frac{M-m}{n}\right)^2
\end{aligned}$$

□

2 Neighboring Definition: Add/Drop One

2.1 ℓ_1-sensitivity

Theorem 3. *Say database X has size $n \geq 2$ and has elements bounded above by M and bounded below by m. Then $\bar{X}$ has ℓ_1-sensitivity bounded above by*

$$\frac{M-m}{n}.$$

Proof. For notational ease, let n always refer to the size of database x. We must consider both adding and removing an element from x. First, consider adding a point:

Let $X' = X \cup x$. Without loss of generality, assume the point added is the $(n+1)^{\text{th}}$ element of database x'. Note that

$$\begin{aligned}
\left|\bar{X}-\bar{X}'\right| &= \left|\frac{1}{n}\sum_{i=1}^{n} x_i - \frac{1}{n+1}\sum_{i=1}^{n+1} x_i\right| \\
&= \left|\left(\frac{1}{n}-\frac{1}{n+1}\right)\sum_{i=1}^{n} x_i - \frac{x}{n+1}\right| \\
&= \frac{1}{n+1}\left|\frac{1}{n}\sum_{i=1}^{n} x_i - x\right| \\
&\leq \frac{|M-m|}{n+1}.
\end{aligned}$$

Second, consider removing a point:
Let $X' = X\backslash\{x\}$. Without loss of generality assume that the point subtracted is the n^{th}

element of database x.

$$\begin{aligned}
\left|\bar{X}-\bar{X}'\right| &= \left|\frac{1}{n-1}\sum_{i=1}^{n-1}x_i - \frac{1}{n}\sum_{i=1}^{n}x_i\right| \\
&= \left|\left(\frac{1}{n-1}-\frac{1}{n}\right)\sum_{i=1}^{n-1}x_i - \frac{x}{n}\right| \\
&= \frac{1}{n}\left|\frac{1}{n-1}\sum_{i=1}^{n-1}x_i - x\right| \\
&\leq \frac{|M-m|}{n}.
\end{aligned}$$

Then, since $\forall n > 0$,

$$\frac{1}{n+1} < \frac{1}{n},$$

the sensitivity of the mean in general is bound from above by

$$\frac{M-m}{n}.$$

□

2.2 ℓ_2-sensitivity

Theorem 4. *Say database X has size $n \geq 2$ and has elements bounded above by M and bounded below by m. Then $\bar{X}$ has ℓ_2-sensitivity bounded above by*

$$\left(\frac{M-m}{n}\right)^2.$$

Proof. We must consider both adding and removing an element from X.

Adding an element:
Let $X' = X \cup x'_{n+1}$. Then,

$$\begin{aligned}
\Delta\bar{X} &= \max_{X,X'}(\bar{X}-\bar{X}')^2 \\
&= \max_{X,X'}\left(\frac{1}{n}\sum_{i=1}^{n}x_i - \frac{1}{n+1}\sum_{i=1}^{n+1}x'_i\right)^2 \\
&= \max_{X,X'}\left(\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right) - \left(\frac{1}{n+1}\sum_{i=1}^{n}x'_i\right) - \frac{x'_{n+1}}{n+1}\right)^2 \\
&= \max_{X,X'}\left(\frac{\left(\sum_{i=1}^{n}x_i\right) - nx'_{n+1}}{n(n+1)}\right)^2 \\
&= \left(\frac{nM-nm}{n(n+1)}\right)^2 \\
&= \left(\frac{M-m}{n+1}\right)^2.
\end{aligned}$$

Removing an element:
Let $X' = X \setminus \{x_n\}$. Then,

$$\begin{aligned}
\Delta \bar{X} &= \max_{X,X'} (\bar{X} - \bar{X}')^2 \\
&= \max_{X,X'} \left(\frac{1}{n} \sum_{i=1}^{n} x_i - \frac{1}{n-1} \sum_{i=1}^{n-1} x_i' \right)^2 \\
&= \max_{X,X'} \left(\left(\frac{1}{n} \sum_{i=1}^{n-1} x_i \right) + \frac{x_n}{n} - \left(\frac{1}{n-1} \sum_{i=1}^{n-1} x_i' \right) \right)^2 \\
&= \max_{X,X'} \left(\frac{(n-1)x_n - \sum_{i=1}^{n-1} x_i}{n(n-1)} \right)^2 \\
&= \left(\frac{(n-1)M - (n-1)m}{n(n-1)} \right)^2 \\
&= \left(\frac{M - m}{n} \right)^2 .
\end{aligned}$$

□