Covariance Sensitivity Proofs

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1 Preliminaries

Definition 1. *Let X be a matrix of values and X_i indicate the i^{th} column of the matrix. Denote the sample mean of column X_i as $\bar{X}_i$, and let n be the size of X_i. Then the covariance matrix of X has ij^{th} element*

$$\frac{1}{n-1}\sum_{k=1}^{n}(x_{ki}-\bar{X}_i)(x_{kj}-\bar{X}_j).$$

Lemma 1. *Let X be a matrix of values and X_i indicate the i^{th} column of the matrix. Denote the sample mean of column X_i as $\bar{X}_i$, and let n be the size of X_i. Then, $\forall i$,*

$$\sum_{j=1}^{n}(x_{ji}-\bar{X}_i) = 0.$$

Proof.

$$\begin{aligned}
\sum_{j=1}^{n}(x_{ji}-\bar{X}_i) &= \sum_{j=1}^{n}x_{ji} - n\bar{X}_i,\\
&= \sum_{j=1}^{n}x_{ji} - n\left(\frac{1}{n}\sum_{j=1}^{n}x_{ji}\right),\\
&= 0.
\end{aligned}$$

□

Lemma 2. *Let X be a matrix and let*

$$f_{ij}(X) = \sum_{k=1}^{n}(x_{ki}-\bar{X}_i)(x_{kj}-\bar{X}_j).$$

Note that this is equivalent to the ij^{th} element of the sample covariance matrix for X, without the normalization by $n-1$. Consider the matrix X' equal to X with a single row

Y added, so that $X_i' = X_i \cup \{y_i\}$. Say X_i has size n. Let $\bar{X}_i$, $\bar{X}_j$, $\bar{X}_i'$, and $\bar{X}_j'$ be the sample means of X_i, X_j, X_i' and X_j' respectively. Then,

$$f_{ij}(X') = f_{ij}(X) + n(\bar{X}_i - \bar{X}_i')(\bar{X}_j - \bar{X}_j') + (y_i - \bar{X}_i)(y_j - \bar{X}_j).$$

Proof. Note that

$$\begin{aligned}
f_{ij}(X') &= \sum_{k=1}^{n+1}(x_{ki}' - \bar{X}_i')(x_{kj}' - \bar{X}_j'),\\
&= \sum_{k=1}^{n}(x_{ki} - \bar{X}_i')(x_{kj} - \bar{X}_j') + (y_i - \bar{X}_i')(y_j - \bar{X}_j'),\\
&= \sum_{k=1}^{n}\left((x_{ki} - \bar{X}_i) + (\bar{X}_i - \bar{X}_i')\right)\left((x_{kj} - \bar{X}_j) + (\bar{X}_j - \bar{X}_j')\right) + (y_i - \bar{X}_i')(y_j - \bar{X}_j'),\\
&= \sum_{k=1}^{n}(x_{ki} - \bar{X}_i)(x_{kj} - \bar{X}_j) + (\bar{X}_j - \bar{X}_j')\sum_{k=1}^{n}(x_{ki} - \bar{X}_i) + (\bar{X}_i - \bar{X}_i')\sum_{k=1}^{n}(x_{kj} - \bar{X}_j),\\
&\quad + \sum_{k=1}^{n}(\bar{X}_i - \bar{X}_i')(\bar{X}_j - \bar{X}_j') + (y_i - \bar{X}_i')(y_j - \bar{X}_j'),\\
&= f_{ij}(X) + n(\bar{X}_i - \bar{X}_i')(\bar{X}_j - \bar{X}_j') + (y_i - \bar{X}_i')(y_j - \bar{X}_j'),
\end{aligned}$$

where the cancellation of the second and third terms in the second-to-last line is due to Lemma 1. □

Lemma 3. *Let X be a matrix of values and X_i indicate the i^{th} column of the matrix. Let X_i have size n and consider the matrix X' equal to X with a single row Y added, so that $X_i' = X_i \cup \{y_i\}$. Say that the space of datapoints $\mathcal{X}_i$ that the elements of X_i' are drawn from is bounded above by M_i and bounded below by m_i. Let $\bar{X}_i$, $\bar{X}_j$, $\bar{X}_i'$, and $\bar{X}_j'$ be the sample means of X_i, X_j, X_i' and X_j' respectively. Then,*

$$n\left|(\bar{X}_i - \bar{X}_i')(\bar{X}_j - \bar{X}_j')\right| \leq \frac{n}{(n+1)^2}(M_i - m_i)(M_j - m_j).$$

Proof. Note that

$$\begin{aligned}
n\left|(\bar{X}_i - \bar{X}_i')(\bar{X}_j - \bar{X}_j')\right| &= n\left|\left(\frac{1}{n}\sum_{k=1}^{n}x_{ki} - \frac{1}{n+1}\sum_{k=1}^{n+1}x_{ki}'\right)\left(\frac{1}{n}\sum_{k=1}^{n}x_{kj} - \frac{1}{n+1}\sum_{k=1}^{n+1}x_{kj}'\right)\right|,\\
&= n\left|\left(\left(\frac{1}{n} - \frac{1}{n+1}\right)\sum_{k=1}^{n}x_{ki} - \frac{y_i}{n+1}\right)\left(\left(\frac{1}{n} - \frac{1}{n+1}\right)\sum_{k=1}^{n}x_{kj} - \frac{y_j}{n+1}\right)\right|,\\
&= n\left|\left(\frac{1}{n(n+1)}\sum_{k=1}^{n}x_{ki} - \frac{y_i}{n+1}\right)\left(\frac{1}{n(n+1)}\sum_{k=1}^{n}x_{kj} - \frac{y_j}{n+1}\right)\right|,\\
&= \frac{n}{(n+1)^2}\left|\left(\frac{1}{n}\sum_{k=1}^{n}x_{ki} - \frac{y_i}{n+1}\right)\left(\frac{1}{n}\sum_{k=1}^{n}x_{kj} - \frac{y_j}{n+1}\right)\right|,\\
&\leq \frac{n}{(n+1)^2}(M_i - m_i)(M_j - m_j).
\end{aligned}$$

□

Lemma 4. *Let X be a matrix of values and X_i indicate the i^{th} column of the matrix. Let X_i have size n and consider the matrix X' equal to X with a single row Y added, so that $X_i' = X_i \cup \{y_i\}$. Say that the space of datapoints $\mathcal{X}_i$ that the elements of X_i' are drawn from is bounded above by M_i and bounded below by m_i. Let $\bar{X}_i$, $\bar{X}_j$, $\bar{X}_i'$, and $\bar{X}_j'$ be the sample means of X_i, X_j, X_i' and X_j' respectively. Then,*

$$\left|(y_i - \bar{X}_i')(y_j - \bar{X}_j')\right| \leq \frac{n^2}{(n+1)^2}(M_i - m_i)(M_j - m_j).$$

Proof. Note that

$$\begin{aligned}
\left|(y_i - \bar{X}_i')(y_j - \bar{X}_j')\right| &= \left|\left(y_i - \frac{y_i + n\bar{X}_i}{n+1}\right)\left(y_j - \frac{y_j + n\bar{X}_j}{n+1}\right)\right|, \\
&= \frac{1}{(n+1)^2}\left|\left((n+1)y_i - y_i - n\bar{X}_i\right)\left((n+1)y_j - y_j - n\bar{X}_j\right)\right|, \\
&= \frac{n^2}{(n+1)^2}\left|(y_i - \bar{X}_i)(y_j - \bar{X}_j)\right|, \\
&\leq \frac{n^2}{(n+1)^2}(M_i - m_i)(M_j - m_j).
\end{aligned}$$

□

2 Neighboring Definition: Add/Drop One

2.1 ℓ_1-sensitivity

Theorem 1. *Let X be a matrix of values and let X_i indicate the i^{th} column of the matrix. Let*

$$f_{ij}(X) = \sum_{k=1}^{n}(x_{ki} - \bar{X}_i)(x_{kj} - \bar{X}_j).$$

Say that the space of datapoints $\mathcal{X}_i$ that X_i is drawn from is bounded above by M_i and bounded below by m_i. Then the ℓ_1-sensitivity in the add/drop-one model of $f(\cdot)$ is bounded above by

$$\frac{n}{(n+1)}(M_i - m_i)(M_j - m_j).$$

Proof. We must consider both adding and removing a row from X.
Adding a row:
Let $X_i' = X_i \cup \{y_i\}$. Then, from Lemma 2,

$$\begin{aligned}
\left|f_{ij}(X') - f_{ij}(X)\right| &= \left|n(\bar{X}_i - \bar{X}_i')(\bar{X}_j - \bar{X}_j') + (y_i - \bar{X}_i)(y_j - \bar{X}_j)\right| \\
&\leq n\left|(\bar{X}_i - \bar{X}_i')(\bar{X}_j - \bar{X}_j')\right| + \left|(y_i - \bar{X}_i)(y_j - \bar{X}_j)\right| \\
&\leq \frac{n}{(n+1)^2}(M_i - m_i)(M_j - m_j) + \frac{n^2}{(n+1)^2}(M_i - m_i)(M_j - m_j) \\
&\qquad \text{(By Lemmas 3 and 4)} \\
&= \frac{n}{n+1}(M_i - m_i)(M_j - m_j). \qquad (2.1)
\end{aligned}$$

Removing a row:
Let Y be the last row of X, and let $X_i' = X_i \setminus \{y_i\}$. Note that Lemma 2 can be rewritten in this setting by parametrizing n as $n - 1$ and swapping X and X' in its expression:

$$f_{ij}(X) = f_{ij}(X') + (n-1)(\bar{X}_i' - \bar{X}_i)(\bar{X}_j' - \bar{X}_j) + (y_i - \bar{X}_i')(y_j - \bar{X}_j').$$

Lemmas 3 and 4 may be rewritten with the same reparametrization:

$$(n-1)\left|(\bar{X}_i' - \bar{X}_i)(\bar{X}_j' - \bar{X}_j)\right| \leq \frac{n-1}{n^2}(M_i - m_i)(M_j - m_j),$$

and

$$\left|(y_i - \bar{X}_i')(y_j - \bar{X}_j')\right| \leq \frac{(n-1)^2}{n^2}(M_i - m_i)(M_j - m_j).$$

Then,

$$\begin{aligned}
\left|f_{ij}(X) - f_{ij}(X')\right| &= \left|(n-1)(\bar{X}_i' - \bar{X}_i)(\bar{X}_j' - \bar{X}_j) + (y_i - \bar{X}_i')(y_j - \bar{X}_j')\right|, \\
&\leq \left|(n-1)(\bar{X}_i' - \bar{X}_i)(\bar{X}_j' - \bar{X}_j)\right| + \left|(y_i - \bar{X}_i')(y_j - \bar{X}_j')\right|, \\
&\leq \frac{n-1}{n^2}(M_i - m_i)(M_j - m_j) + \frac{(n-1)^2}{n^2}(M_i - m_i)(M_j - m_j), \\
&= \frac{n-1}{n}(M_i - m_i)(M_j - m_j). \qquad (2.2)
\end{aligned}$$

Note that for any $n \geq 1$,

$$\frac{n}{n+1} > \frac{n-1}{n}. \qquad (2.3)$$

So, the worst-case bound always occurs in the "add-one" case, and in general the ℓ_1 sensitivity of $f(\cdot)$ is bounded by

$$\frac{n}{n+1}(M_i - m_i)(M_j - m_j).$$

□

Corollary 1. *Let $X \leftarrow \mathcal{X}$ where $\mathcal{X}_i$ is bounded above by M_i and bounded below by m_i. Then the ℓ_1-sensitivity in the add/drop-one model of the ij^{th} element of a sample covariance matrix for X is bounded above by*

$$\frac{n}{n^2-1}(M_i - m_i)(M_j - m_j).$$

Proof. Note that the sample covariance of X is equal to $f(x)/(n-1)$, and that $(n-1)(n+1) = n^2 - 1$. □

2.2 ℓ_2-sensitivity

Theorem 2. *Let $X \leftarrow \mathcal{X}$ where $\mathcal{X}_i$ is bounded above by M_i and bounded below by m_i. Then the ℓ_2-sensitivity in the add/drop-one model of the ij^{th} element of a sample covariance matrix for X is bounded above by*

$$\left(\frac{n}{n^2-1}(M_i - m_i)(M_j - m_j)\right)^2.$$

Proof. This follows from the bounds in Equations 2.1 and 2.2 and the inequality in Equation 2.3, and a renormalization by $n-1$ from the definition of covariance as in the proof of Corollary 1. □

3 Neighboring Definition: Change One

3.1 ℓ_1-sensitivity

Theorem 3. *Let X be a matrix of values and let X_i indicate the i^{th} column of the matrix. Let*

$$f_{ij}(X) = \sum_{k=1}^{n}(x_{ki} - \bar{X}_i)(x_{kj} - \bar{X}_j).$$

Say that the space of datapoints $\mathcal{X}_i$ that X_i is drawn from is bounded above by M_i and bounded below by m_i. Then the ℓ_1-sensitivity in the change-one model of $f(\cdot)$ is bounded above by

$$\frac{2(n-1)}{n}(M_i - m_i)(M_j - m_j).$$

Proof. Recall from Equation 2.1 that

$$\left|f_{ij}(X) - f_{ij}(X')\right| \leq \frac{n}{(n+1)}(M_i - m_i)(M_j - m_j).$$

and

$$\left|f_{ij}(X) - f_{ij}(X'')\right| \leq \frac{n}{(n+1)}(M_i - m_i)(M_j - m_j).$$

Reparametrizing these equations so that n is the size of X' and X'' gives that

$$\left|f_{ij}(X) - f_{ij}(X')\right| \leq \frac{n-1}{n}(M_i - m_i)(M_j - m_j).$$

and

$$\left|f_{ij}(X) - f_{ij}(X'')\right| \leq \frac{n-1}{n}(M_i - m_i)(M_j - m_j).$$

It then follows from the triangle inequality that

$$\left|f_{ij}(X') - f_{ij}(X'')\right| \leq \frac{2(n-1)}{n}(M_i - m_i)(M_j - m_j).$$

□

Corollary 2. *The ℓ_1-sensitivity in the change-one model of sample covariance is bounded above by*

$$\frac{2}{n}(M_i - m_i)(M_j - m_j).$$

Proof. Note that the sample covariance of X is equal to $f(x)/(n-1)$. □

3.2 ℓ_2-sensitivity

Theorem 4. *The ℓ_2-sensitivity in the change-one model of sample covariance is bounded above by*

$$\left(\frac{2}{n}(M_i - m_i)(M_j - m_j)\right)^2.$$

Proof. This follows from the bounds in the proof of Theorem 3 and a renormalization by $n-1$ by the definition of covariance (as in the proof of Corollary 2). □