
Mean Sensitivity Proofs

March 9, 2020

Definition 1. *The sample mean of database X of size n is*

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n x_i$$

1 NEIGHBORING DEFINITION: CHANGE ONE

1.1 ℓ_1 -sensitivity

Theorem 1. *Say database X has size n and is bounded above by M and bounded below by m . Then \bar{X} has ℓ_1 -sensitivity bounded above by*

$$\frac{M - m}{n}.$$

Proof. Say X and X' are neighboring databases which differ at data-point x_j . Then

$$\begin{aligned} \Delta \bar{X} &= \max_{X, X'} |\bar{X} - \bar{X}'| \\ &= \max_{X, X'} \frac{1}{n} \left| \left(\sum_{\{i \in [n] | i \neq j\}} x_i \right) + x_j - \left(\sum_{\{i \in [n] | i \neq j\}} x'_i \right) + x'_j \right| \\ &= \max_{X, X'} \frac{1}{n} |x_j - x'_j| \\ &\leq \frac{M - m}{n}. \end{aligned}$$

□

1.2 ℓ_2 -sensitivity

Theorem 2. *Say database X has size n and is bounded above by M and bounded below by m . Then \bar{X} has ℓ_2 -sensitivity bounded above by*

Proof. Say X and X' are neighboring databases which differ only at index j . Then

$$\begin{aligned}
\Delta \bar{X} &= \max_{X, X'} (\bar{X} - \bar{X}')^2 \\
&= \max_{X, X'} \frac{1}{n^2} \left(\left(\sum_{i \in [n] | i \neq j} x_i \right) + x_j - \left(\sum_{i \in [n] | i \neq j} x'_i \right) - x'_j \right)^2 \\
&= \max_{X, X'} \frac{1}{n^2} (x_j - x'_j)^2 \\
&\leq \frac{(M - m)^2}{n^2} \\
&= \left(\frac{M - m}{n} \right)^2
\end{aligned}$$

□

2 NEIGHBORING DEFINITION: ADD/DROP ONE

2.1 ℓ_1 -sensitivity

Theorem 3. Say database X has size $n \geq 2$ and has elements bounded above by M and bounded below by m . Then \bar{X} has ℓ_1 -sensitivity bounded above by

$$\frac{M - m}{n}.$$

Proof. WLOG assume point being added/subtracted is x_n .

Adding a point: $X' = X \cup x$

$$\begin{aligned}
|\bar{X} - \bar{X}'| &= \left| \frac{1}{n} \sum_{i=1}^n x_i - \frac{1}{n+1} \sum_{i=1}^{n+1} x_i \right| \\
&= \left| \left(\frac{1}{n} - \frac{1}{n+1} \right) \sum_{i=1}^n x_i - \frac{x}{n+1} \right| \\
&= \frac{1}{n+1} \left| \frac{1}{n} \sum_{i=1}^n x_i - x \right| \\
&\leq \frac{|M - m|}{n+1}
\end{aligned}$$

Taking a point away: $X' = X \setminus \{x\}$

$$\begin{aligned}
|\bar{X} - \bar{X}'| &= \left| \frac{1}{n-1} \sum_{i=1}^{n-1} x_i - \frac{1}{n} \sum_{i=1}^n x_i \right| \\
&= \left| \left(\frac{1}{n-1} - \frac{1}{n} \right) \sum_{i=1}^{n-1} x_i - \frac{x}{n} \right| \\
&= \frac{1}{n} \left| \frac{1}{n-1} \sum_{i=1}^{n-1} x_i - x \right| \\
&\leq \frac{|M - m|}{n}
\end{aligned}$$

□

2.2 ℓ_2 -sensitivity

Theorem 4. Say database X has size $n \geq 2$ and has elements bounded above by M and bounded below by m . Then \bar{X} has ℓ_2 -sensitivity bounded above by

$$\left(\frac{M - m}{n} \right)^2.$$

Proof. We must consider both adding and removing an element from X .

Adding an element:

Let $X' = X \cup x'_{n+1}$. Then,

$$\begin{aligned} \Delta \bar{X} &= \max_{X, X'} (\bar{X} - \bar{X}')^2 \\ &= \max_{X, X'} \left(\frac{1}{n} \sum_{i=1}^n x_i - \frac{1}{n+1} \sum_{i=1}^{n+1} x'_i \right)^2 \\ &= \max_{X, X'} \left(\left(\frac{1}{n} \sum_{i=1}^n x_i \right) - \left(\frac{1}{n+1} \sum_{i=1}^n x'_i \right) - \frac{x'_{n+1}}{n+1} \right)^2 \\ &= \max_{X, X'} \left(\frac{(\sum_{i=1}^n x_i) - nx'_{n+1}}{n(n+1)} \right)^2 \\ &= \left(\frac{nM - nm}{n(n+1)} \right)^2 \\ &= \left(\frac{M - m}{n+1} \right)^2. \end{aligned}$$

Removing an element:

Let $X' = X \setminus \{x_n\}$. Then,

$$\begin{aligned} \Delta \bar{X} &= \max_{X, X'} (\bar{X} - \bar{X}')^2 \\ &= \max_{X, X'} \left(\frac{1}{n} \sum_{i=1}^n x_i - \frac{1}{n-1} \sum_{i=1}^{n-1} x'_i \right)^2 \\ &= \max_{X, X'} \left(\left(\frac{1}{n} \sum_{i=1}^{n-1} x_i \right) + \frac{x_n}{n} - \left(\frac{1}{n-1} \sum_{i=1}^{n-1} x'_i \right) \right)^2 \\ &= \max_{X, X'} \left(\frac{(n-1)x_n - \sum_{i=1}^{n-1} x'_i}{n(n-1)} \right)^2 \\ &= \left(\frac{(n-1)M - (n-1)m}{n(n-1)} \right)^2 \\ &= \left(\frac{M - m}{n} \right)^2. \end{aligned}$$

□