Median Sensitivity Proofs

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Definition 1. *The sample median of a database* $X = (x_1, \ldots, x_n)$ *is given by*

$$f(X) = \frac{\tilde{x}_l + \tilde{x}_u}{2}$$

where $l = \lfloor \frac{n+1}{2} \rfloor$ *and* $u = \lceil \frac{n+1}{2} \rceil$ *and* $\tilde{X} = (\tilde{x}_1, \ldots, \tilde{x}_n)$ *is the sorted version of* X.

1 Neighboring Definition: Change One

1.1 ℓ_1-sensitivity

Theorem 1. *Let the database* X *have elements bounded above by* M *and below by* m. *Then the global sensitivity of the median is*

$$\Delta f(\cdot) = \begin{cases} \frac{M-m}{2}, & \textit{if } n \equiv 0 \pmod 2 \\ M - m & \textit{if } n \equiv 1 \pmod 2 \end{cases}$$

Proof. First consider the case where n is such that $n \equiv 0 \pmod 2$. Then our worst-case scenario is that exactly half of the data elements in X are m and the other half are M. In this case, $f(X) = \frac{m+M}{2}$.

Now consider an X' which is identical to X but with one element switched from m to M WLOG.[1] Then we have that $f(X') = M$. Because we are looking at our worst-case pairing of X, X', we know that

$$\Delta f(\cdot) = \max_{X,X'} |f(X') - f(X)| = \left|M - \frac{m+M}{2}\right| = \frac{M-m}{2}.$$

Now consider the case where n is such that $n \equiv 1 \pmod 2$. Then our worst-case scenario is that $\frac{n-1}{2}$ of the elements of X are m, $\frac{n-1}{2}$ of the elements of X are M, and the remaining

[1] The result holds if it is switched from M to m.

element of X is m (WLOG). In this setting, $f(X) = m$. Let X' be identical to x but with one element switched from m to M. Then $f(X') = M$ and we have that

$$\Delta f(\cdot) = \max_{X,X'} |f(X') - f(X)| = |M - m|.$$

$\square$

1.2 ℓ_2-sensitivity

Theorem 2. *Let the database X have elements bounded above by M and below by m. Then the global sensitivity of the median is*

$$\Delta f(\cdot) = \begin{cases} \left(\frac{M-m}{2}\right)^2, & \textit{if } n \equiv 0 \pmod 2 \\ (M-m)^2 & \textit{if } n \equiv 1 \pmod 2 \end{cases}$$

Proof. The logic follows exactly from the proof of the ℓ_1 sensitivity, just with the norm in the last line of each statement switched from 1 to 2. $\square$

2 Neighboring Definition: Add/Drop One

2.1 ℓ_1-sensitivity

2.2 ℓ_2-sensitivity