
Sum Sensitivity Proofs

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Definition 1. A sum query on database x of size n is defined to be

$$s(x) = \sum_{i=1}^n x_i.$$

1 NEIGHBORING DEFINITION: CHANGE ONE

1.1 ℓ_1 -sensitivity

Theorem 1. Say the space of datapoints \mathcal{X} is bounded above by M and bounded below by m . Then s over \mathcal{X}^n has ℓ_1 -sensitivity in the change-one model bounded above by $M - m$.

Proof. Say X and X' are neighboring databases which differ at data-point x_j . Then

$$\begin{aligned} \Delta s &= \max_{X, X'} |s(X) - s(X')| \\ &= \max_{X, X'} \left| \left(\sum_{\{i \in [n] | i \neq j\}} x_i \right) + x_j - \left(\sum_{\{i \in [n] | i \neq j\}} x'_i \right) + x'_j \right| \\ &= \max_{X, X'} |x_j - x'_j| \\ &\leq M - m. \end{aligned}$$

□

1.2 ℓ_2 -sensitivity

Theorem 2. Say the space of datapoints \mathcal{X} is bounded above by M and bounded below by m . Then s over \mathcal{X}^n has ℓ_1 -sensitivity in the change-one model bounded above by $(M - m)^2$.

Proof. Say X and X' are neighboring databases which differ only at index j . Then

$$\begin{aligned}
\Delta \bar{X} &= \max_{X, X'} (s(X) - s(X'))^2 \\
&= \max_{X, X'} \left(\left(\sum_{i \in [n] | i \neq j} x_i \right) + x_j - \left(\sum_{i \in [n] | i \neq j} x'_i \right) - x'_j \right)^2 \\
&= \max_{X, X'} (x_j - x'_j)^2 \\
&\leq (M - m)^2.
\end{aligned}$$

□

2 NEIGHBORING DEFINITION: ADD/DROP ONE

2.1 ℓ_1 -sensitivity

Theorem 3. *Say the space of datapoints \mathcal{X} is bounded above by M and bounded below by m . Then s has ℓ_1 -sensitivity in the add/drop-one model bounded above by $\max(|m|, |M|)$.*

Proof. For notational ease, let n always refer to the size of database x . We must consider both adding and removing an element from x . First, consider adding a point:

Let $X' = X \cup x$. Without loss of generality, assume the point added is the $(n+1)^{\text{th}}$ element of database x' . Note that

$$\begin{aligned}
|s(X) - s(X')| &= \left| \sum_{i=1}^n x_i - \sum_{i=1}^{n+1} x_i \right| \\
&= \left| \sum_{i=1}^n x_i - \left(\sum_{i=1}^n x_i \right) - x \right| \\
&= x \\
&\leq \max(|m|, |M|).
\end{aligned}$$

Second, consider removing a point:

Let $X' = X \setminus \{x\}$. Without loss of generality assume that the point subtracted is the n^{th} element of database x .

$$\begin{aligned}
|s(X) - s(X')| &= \left| \sum_{i=1}^n x_i - \sum_{i=1}^{n-1} x_i \right| \\
&= \left| \left(\sum_{i=1}^{n-1} x_i \right) + x - \sum_{i=1}^{n-1} x_i \right| \\
&= x \\
&\leq \max(|m|, |M|).
\end{aligned}$$

□

2.2 ℓ_2 -sensitivity

Theorem 4. *Say the space of datapoints \mathcal{X} is bounded above by M and bounded below by m . Then s has ℓ_2 -sensitivity in the add/drop-one model bounded above by $\max(m^2, M^2)$.*

Proof. For notational ease, let n always refer to the size of database x . We must consider both adding and removing an element from x . First, consider adding a point:

Let $X' = X \cup x$. Without loss of generality, assume the point added is the $(n+1)^{\text{th}}$ element of database x' . Note that

$$\begin{aligned} (s(X) - s(X'))^2 &= \left(\sum_{i=1}^n x_i - \sum_{i=1}^{n+1} x_i \right)^2 \\ &= \left(\sum_{i=1}^n x_i - \sum_{i=1}^n x_i - x \right)^2 \\ &= x^2 \\ &\leq \max(m^2, M^2). \end{aligned}$$

Second, consider removing a point:

Let $X' = X \setminus \{x\}$. Without loss of generality assume that the point subtracted is the n^{th} element of database x . Then,

$$\begin{aligned} (s(X) - s(X'))^2 &= \left(\sum_{i=1}^n x_i - \sum_{i=1}^{n-1} x_i \right)^2 \\ &= \left(\sum_{i=1}^{n-1} x_i + x - \sum_{i=1}^{n-1} x_i \right)^2 \\ &= x^2 \\ &\leq \max(m^2, M^2). \end{aligned}$$

□