Documentation of R Programs for "Measuring the Natural Rate of Interest After COVID-19"*

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This note documents the R code used for the estimation of the natural rate of interest, natural rate of output, and its trend growth rate for the United States, Canada, and the Euro Area, presented in "Measuring the Natural Rate of Interest After COVID-19" (Holston, Laubach, Williams 2023; henceforth HLW). The COVID-adjusted HLW model presented here builds on the model in Holston, Laubach, and Williams (2017). While we provide all input data series used in the estimation, this guide also catalogues steps to download and prepare the data for reference. The code documented here corresponds to the FRBNY Staff Report published in June 2023.[1]

1 Code Layout and Directory Structure

For each country, there is one main R file, *run.hlw.XX.R* (for *XX* set to *us*, *ca*, or *ea*), which does the following:

1. Imports data to be used in the HLW estimation, available on the FRBNY r^* website[2];
2. Defines the sample period, constraints, and variables to be used throughout the estimation, including specifications of the COVID-adjusted model;
3. Runs the three-stage HLW estimation;
4. Saves output.

This file calls multiple R functions and files, each of which are described in this guide. To run the code without modification, use the following structure:

1. A subdirectory titled "inputData" should contain the provided input data;

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[1] `https://www.newyorkfed.org/medialibrary/media/research/staff_reports/sr1063.pdf`

[2] `https://www.newyorkfed.org/research/policy/rstar`

2. An empty subdirectory titled "output" should be created and will populate with estimates and model output.
3. Optional: create a subdirectory titled "Rpackages" to use as the library location for downloaded packages.

Input data for the United States, Canada, and the Euro Area will be published quarterly with the current HLW estimates on the FRBNY website. Please save this Excel file in your *inputData* directory. For reference, we are using R Version 4.2.1 at the time of release.

2 Raw Data

A description of the data we require can be found in the Data Appendix to the paper. For each economy, we require data for real GDP, inflation, and the short-term nominal interest rate, as well as a procedure to compute inflation expectations, which are necessary to calculate the ex ante real short-term interest rate. The inflation measure is the annualized quarterly growth rate of the specified consumer price series. Interest rates are expressed on a 365-day annualized basis. As of 2020:Q1, we also use a COVID-19 indicator variable. We provide the input data on a quarterly basis, published alongside the quarterly updates of our estimates. For reference, the full list of data sources is included here.

2.1 United States

We use real GDP and core PCE data published by the Bureau of Economic Analysis. The short-term interest rate is the annualized nominal federal funds rate, available from the Board of Governors. Because the federal funds rate frequently fell below the discount rate prior to 1965, we use the Federal Reserve Bank of New York's discount rate, part of the IMF's International Financial Statistics Yearbooks (IFS), prior to 1965. All US data can be downloaded from the St. Louis Fed's Federal Reserve Economic Data (FRED) website.

FRED Mnemonics:

- Real GDP: GDPC1
- Core PCE: PCEPILFE
- Federal Funds Rate: FEDFUNDS
- FRBNY Discount Rate: INTDSRUSM193N

2.2 Canada

We use real GDP from the IMF's IFS. The short-term nominal interest rate is the Bank of Canada's target for the overnight rate, taken as the end-of-period value for each month and aggregated to quarterly frequency. Prior to May 2001, we use the Bank of Canada's bank rate as the short-term interest rate. We use the BoC's core consumer price index to construct our inflation series. Prior to 1984, we use CPI containing all items. With the exception of GDP, all data is from Statistics Canada.

Mnemonics:

- Real GDP: IFS series "Gross Domestic Product, Real, Seasonally adjusted, Index"
- Core CPI: v41690926 (Table 326-0022); Source: Statistics Canada
- CPI: v41690914 (Table 326-0022); v41690973 (Table 326-0020); Source: Statistics Canada
- Bank Rate: v122530 (Table 176-0043); Source: Statistics Canada
- Target Rate: v39079 (Table 176-0048); Source: Statistics Canada

2.3 Euro Area

All data is from the ECB's Area-Wide Model, which is available from the Euro Area Business Cycle Network (Fagan et al. 2001). We use the core price index beginning in 1988 and the total price index prior; because data availability is longer for the non-seasonally adjusted series, we use those and seasonally adjust them. The nominal short-term interest rate is the three-month rate.

Mnemonics:

- Real GDP: YER
- Price Index: HICP (not seasonally adjusted)
- Core Price Index: HEX
- Nominal Short-term Rate: STN

The AWM was previously released annually, and we update each series using data from the ECB's Statistical Data Warehouse. At the time of publication of HLW (2017), we used AWM data through 2015:Q4 and updated the series using ECB data for the first three quarters of 2016. As of May 2020, the last update to the AWM was in 2017; we now use the ECB's Statistical Data Warehouse each quarter.

ECB Statistical Data Warehouse Mnemonics:

- Real GDP: MNA.Q.Y.I8.W2.S1.S1.B.B1GQ._Z._Z._Z.EUR.LR.N
- Core Price Index: ICP.M.U2.N.XE0000.4.INX

- Nominal Short-term Rate: FM.Q.U2.EUR.RT.MM.EURIBOR3MD_.HSTA

2.4 COVID-19 Indicator Variable

We use a COVID indicator variable equal to the quarterly average of the COVID-19 Stringency Index from the Oxford COVID-19 Government Response Tracker (OxCGRT) for each country or region (Hale et al., 2021). We use the national weighted average of the stringency indices for vaccinated and unvaccinated populations. For the Euro Area, we use a GDP-weighted stringency index with 2019 GDP weights. As the OxCGRT project suspended data collection at the end of 2022, we assume each indicator variable declines linearly beginning in 2023:Q1, reaching zero in 2024:Q4. The COVID indicator is set equal to zero up to and including 2019:Q4 in all economies.

3 Basic Functions used Throughout HLW Programs

In the accompanying set of code, these functions are stored in *utilities.R*.

Function: *shiftQuarter*
Description: This function takes in a (year, quarter) date in time series format and a shift number, and returns the (year, quarter) date corresponding to the shift. Positive values of shift produce leads and negative values of shift produce lags. For example, entering 2014q1 with a shift of -1 would return 2013q4. Entering 2014q1 with a shift of 1 would return 2014q2. In each case, the first argument of the function must be entered as a two-element vector, where the first element corresponds to the year and the second element corresponds to the quarter. For example, 2014q1 must be entered as "c(2014, 1)".

Function: *shiftMonth*
Description: This function takes in a (year, month) date in time series format and a shift number, and returns the (year, month) date corresponding to the shift. Positive values of shift produce leads and negative values of shift produce lags. For example, entering 2014m1 with a shift of -1 would return 2013m12. Entering 2014m1 with a shift of 1 would return 2014m2. In each case, the first argument of the function must be entered as a two-element vector, where the first element corresponds to the year and the second element corresponds to the month. This function is analogous to shiftQuarter().

Function: *getFRED*
Description: This function downloads data from FRED. It returns quarterly data. User must provide the FRED url. If using the provided input data, this function is not necessary.

Function: *splice*
Description: This function splices two series, with the series s2 beginning at splice.date and extended back using the growth rate at the splice.date times series s1. The freq argument accepts two values - 'quarterly' and 'monthly' - but it could be modified to take more.

Function: *gradient*
Description: This function computes the gradient of a function f given a vector input x.

4 R Packages

The "tis" package is used to manage time series data. The "mFilter" package contains the hpfilter() function. We use the "nloptr" package for optimization. The "openxlsx" package is used to read from and write to Excel.

5 Estimation

The results reported in Holston, Laubach, and Williams (2023) are based on our estimation method described in Sections 2.1 and 3 of the paper and in HLW (2017). The estimation proceeds in sequential steps through three stages, each of which is implemented in an R program. These and all other R programs are described in this section.

5.1 Main Estimation Program: *run.hlw.estimation.R*

The function *run.hlw.estimation.R* is called by *run.hlw.XX.R* once for each economy. It takes as inputs the key variables for the given economy: log output, inflation, and the real and nominal short-term interest rates, as well as the constraints on a_r and b_y, specifications for the COVID-adjusted model, and initialization of the state vector and covariance matrix, as defined in *run.hlw.XX.R*. It calls the programs *rstar.stageX.R* to run the three stages of the HLW estimation. Additionally, it calls the programs *median.unbiased.estimator.stageX.R* to obtain the signal-to-noise ratios λ_g and λ_z.

The programs *unpack.parameters.stageX.R* set up coefficient matrices for the corresponding state-space models for the given parameter vectors. In all stages, we impose the constraint $b_y \geq 0.025$. In stages 2 and 3, we impose $a_r \leq -0.0025$. These constraints are labeled as a.r.constraint and b.y.constraint, respectively, in the code.

5.2 The Stage 1, 2, and 3 COVID-adjusted State-Space Models

This section presents the COVID-adjusted state-space models; that is, the models estimated in HLW (2023). See Section 3 and Appendix A1 for a description of COVID-related and other technical modifications to the HLW (2017) model. For reference, the next section presents the HLW (2017) state-space models. The following section documents the corresponding R programs. Notation matches that of Hamilton (1994) and is also used in the R programs. All of the state-space models can be cast in the form:

$$\begin{aligned} \mathbf{y}_t &= \mathbf{A}' \cdot \mathbf{x}_t + \mathbf{H}' \cdot \xi_t + \epsilon_t & (1) \\ \xi_t &= \mathbf{F} \cdot \xi_{t-1} + \eta_t & (2) \end{aligned}$$

Here, $\mathbf{y}_t$ is a vector of contemporaneous endogenous variables, while $\mathbf{x}_t$ is a vector of exogenous and lagged exogenous variables. ξ_t is vector of unobserved states. In the HLW (2017) model, the vectors of stochastic disturbances ϵ_t and η_t are assumed to be Gaussian and mututally uncorrelated, with mean zero and covariance matrices $\mathbf{R}$ and $\mathbf{Q}$, respectively. The covariance matrix $\mathbf{R}$ is always assumed to be diagonal. In the HLW (2023) model, the covariance matrix $\mathbf{R}_t$ is time-varying.

For each model, there is a corresponding vector of parameters to be estimated by maximum likelihood. Because maximum likelihood estimates of the innovations to g and z, σ_g and σ_z, are likely to be biased towards zero (see Section 2.2 of HLW for explanation), we use Stock and Watson's (1998) mediun unbiased estimator to obtain estimates of two ratios, $\lambda_g \equiv \frac{\sigma_g}{\sigma_{y^*}}$ and $\lambda_z \equiv \frac{a_r \sigma_z}{\sigma_{\tilde{y}}}$. We impose these ratios when estimating the remaining model parameters by maximum likelihood. Departures from the HLW (2017) model related to the COVID pandemic are highlighted in red, while other modifications are highlighted in blue.

5.3 The COVID-Adjusted State-Space Models

5.3.1 The COVID-adjusted Stage 1 Model

The first-stage model, which corresponds to the *rstar.stage1.R* program, can be represented by the following matrices:

$$\begin{aligned} \mathbf{y}_t &= [y_t, \pi_t]' \\ \mathbf{x}_t &= [y_{t-1}, y_{t-2}, \pi_{t-1}, \pi_{t-2,4}, d_t, d_{t-1}, d_{t-2}]' \\ \xi_t &= [y_t^*, y_{t-1}^*, y_{t-2}^*]' \end{aligned}$$

$$\mathbf{H}' = \begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} \\ 0 & -b_y & 0 \end{bmatrix}, \quad \mathbf{A}' = \begin{bmatrix} a_{y,1} & a_{y,2} & 0 & 0 & \phi & -\phi a_{y,1} & -\phi a_{y,2} \\ b_y & 0 & b_\pi & 1-b_\pi & 0 & -\phi b_y & 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \sigma_{y^*}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{R}_t = \begin{bmatrix} (\kappa_t \sigma_{\tilde{y}})^2 & 0 \\ 0 & (\kappa_t \sigma_\pi)^2 \end{bmatrix}$$

The vector of parameters to be estimated by maximum likelihood is as follows:

$$\theta_1 = \left[a_{y,1}, a_{y,2}, b_\pi, b_y, g, \sigma_{\tilde{y}}, \sigma_\pi, \sigma_{y*}, \phi, \kappa_{2020Q2-Q4}, \kappa_{2021}, \kappa_{2022}\right]$$

5.3.2 The COVID-adjusted Stage 2 Model

The second-stage model, which corresponds to the *rstar.stage2.R* program, can be represented by the following matrices:

$$\begin{aligned}
\mathbf{y}_t &= [y_t, \pi_t]' \\
\mathbf{x}_t &= [y_{t-1}, y_{t-2}, r_{t-1}, r_{t-2}, \pi_{t-1}, \pi_{t-2,4}, 1, d_t, d_{t-1}, d_{t-2}]' \\
\xi_t &= [y_t^*, y_{t-1}^*, y_{t-2}^*, g_t, g_{t-1}, g_{t-2}]'
\end{aligned}$$

$$\mathbf{H}' = \begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} & 0 & \frac{a_g}{2} & \frac{a_g}{2} \\ 0 & -b_y & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}' = \begin{bmatrix} a_{y,1} & a_{y,2} & \frac{a_r}{2} & \frac{a_r}{2} & 0 & 0 & a_0 & \phi & -\phi a_{y,1} & -\phi a_{y,2} \\ b_y & 0 & 0 & 0 & b_\pi & 1-b_\pi & 0 & 0 & -\phi b_y & 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \sigma_{y^*}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\lambda_g \sigma_{y^*})^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{R}_t = \begin{bmatrix} (\kappa_t \sigma_{\tilde{y}})^2 & 0 \\ 0 & (\kappa_t \sigma_\pi)^2 \end{bmatrix}$$

The vector of parameters to be estimated by maximum likelihood is as follows:

$$\theta_2 = \left[a_{y,1}, a_{y,2}, a_r, a_0, a_g, b_\pi, b_y, \sigma_{\tilde{y}}, \sigma_\pi, \sigma_{y*}, \phi, \kappa_{2020Q2-Q4}, \kappa_{2021}, \kappa_{2022}\right]$$

5.3.3 The COVID-adjusted Stage 3 Model

The third-stage model, which corresponds to the *rstar.stage3.R* program, can be represented by the following matrices:

$$\begin{aligned}
\mathbf{y}_t &= [y_t, \pi_t]' \\
\mathbf{x}_t &= [y_{t-1}, y_{t-2}, r_{t-1}, r_{t-2}, \pi_{t-1}, \pi_{t-2,4}, d_t, d_{t-1}, d_{t-2}]' \\
\xi_t &= \left[y_t^*, y_{t-1}^*, y_{t-2}^*, g_t, g_{t-1}, g_{t-2}, z_t, z_{t-1}, z_{t-2}\right]'
\end{aligned}$$

$$\mathbf{H}' = \begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} & 0 & -4c\cdot\frac{a_r}{2} & -4c\cdot\frac{a_r}{2} & 0 & \frac{-a_r}{2} & \frac{-a_r}{2} \\ 0 & -b_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}' = \begin{bmatrix} a_{y,1} & a_{y,2} & \frac{a_r}{2} & \frac{a_r}{2} & 0 & 0 & \phi & -\phi a_{y,1} & -\phi a_{y,2} \\ b_y & 0 & 0 & 0 & b_\pi & 1-b_\pi & 0 & -\phi b_y & 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1&0&0&1&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \sigma_{y^*}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&(\lambda_g\sigma_{y^*})^2&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&\left(\frac{\lambda_z\sigma_{\tilde{y}}}{a_r}\right)^2&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \end{bmatrix},$$

$$\mathbf{R}_t = \begin{bmatrix} (\kappa_t\sigma_{\tilde{y}})^2 & 0 \\ 0 & (\kappa_t\sigma_\pi)^2 \end{bmatrix}$$

The vector of parameters to be estimated by maximum likelihood is as follows:

$$\theta_3 = \left[a_{y,1}, a_{y,2}, a_r, b_\pi, b_y, \sigma_{\tilde{y}}, \sigma_\pi, \sigma_{y*}, \phi, c, \kappa_{2020Q2-Q4}, \kappa_{2021}, \kappa_{2022}\right]$$

The law of motion for the natural rate of interest is $r_t^* = c \cdot g_t + z_t$.

5.4 The HLW (2017) Stage 1, 2, and 3 State-Space Models

This section includes HLW (2017) models for reference.

5.4.1 The Stage 1 Model

The first-stage model, which corresponds to the *rstar.stage1.R* program, can be represented by the following matrices:

$$\begin{aligned} \mathbf{y}_t &= [y_t, \pi_t]' \\ \mathbf{x}_t &= [y_{t-1}, y_{t-2}, \pi_{t-1}, \pi_{t-2,4}]' \\ \xi_t &= \left[y_t^*, y_{t-1}^*, y_{t-2}^*\right]' \end{aligned}$$

$$\mathbf{H}' = \begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} \\ 0 & -b_y & 0 \end{bmatrix}, \quad \mathbf{A}' = \begin{bmatrix} a_{y,1} & a_{y,2} & 0 & 0 \\ b_y & 0 & b_\pi & 1-b_\pi \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \sigma^2_{y^*} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \sigma^2_{\tilde{y}} & 0 \\ 0 & \sigma^2_\pi \end{bmatrix}$$

The vector of parameters to be estimated by maximum likelihood is as follows:

$$\theta_1 = \left[a_{y,1}, a_{y,2}, b_\pi, b_y, g, \sigma_{\tilde{y}}, \sigma_\pi, \sigma_{y*}\right]$$

5.4.2 The Stage 2 Model

The second-stage model, which corresponds to the *rstar.stage2.R* program, can be represented by the following matrices:

$$\begin{aligned} \mathbf{y}_t &= [y_t, \pi_t]' \\ \mathbf{x}_t &= [y_{t-1}, y_{t-2}, r_{t-1}, r_{t-2}, \pi_{t-1}, \pi_{t-2,4}, 1]' \\ \xi_t &= \left[y^*_t, y^*_{t-1}, y^*_{t-2}, g_{t-1}\right]' \end{aligned}$$

$$\mathbf{H}' = \begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} & a_g \\ 0 & -b_y & 0 & 0 \end{bmatrix}, \quad \mathbf{A}' = \begin{bmatrix} a_{y,1} & a_{y,2} & \frac{a_r}{2} & \frac{a_r}{2} & 0 & 0 & a_0 \\ b_y & 0 & 0 & 0 & b_\pi & 1-b_\pi & 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \sigma^2_{y^*} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\lambda_g \sigma_{y^*})^2 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \sigma^2_{\tilde{y}} & 0 \\ 0 & \sigma^2_\pi \end{bmatrix}$$

The vector of parameters to be estimated by maximum likelihood is as follows:

$$\theta_2 = \left[a_{y,1}, a_{y,2}, a_r, a_0, a_g, b_\pi, b_y, \sigma_{\tilde{y}}, \sigma_\pi, \sigma_{y*}\right]$$

5.4.3 The Stage 3 Model

The third-stage model, which corresponds to the *rstar.stage3.R* program, can be represented by the following matrices:

$$\begin{aligned} \mathbf{y}_t &= [y_t, \pi_t]' \\ \mathbf{x}_t &= [y_{t-1}, y_{t-2}, r_{t-1}, r_{t-2}, \pi_{t-1}, \pi_{t-2,4}]' \\ \xi_t &= \left[y^*_t, y^*_{t-1}, y^*_{t-2}, g_{t-1}, g_{t-2}, z_{t-1}, z_{t-2}\right]' \end{aligned}$$

$$
\mathbf{H}' = \begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} & \frac{-a_r}{2} & \frac{-a_r}{2} & \frac{-a_r}{2} & \frac{-a_r}{2} \\ 0 & -b_y & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}' = \begin{bmatrix} a_{y,1} & a_{y,2} & \frac{a_r}{2} & \frac{a_r}{2} & 0 & 0 \\ b_y & 0 & 0 & 0 & b_\pi & 1-b_\pi \end{bmatrix}
$$

$$
\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \left(1+\lambda_g^2\right)\sigma_{y^*}^2 & 0 & 0 & (\lambda_g \sigma_{y^*})^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (\lambda_g \sigma_{y^*})^2 & 0 & 0 & (\lambda_g \sigma_{y^*})^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{\lambda_z \sigma_{\tilde{y}}}{a_r}\right)^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
\mathbf{R} = \begin{bmatrix} \sigma_{\tilde{y}}^2 & 0 \\ 0 & \sigma_\pi^2 \end{bmatrix}
$$

The vector of parameters to be estimated by maximum likelihood is as follows:

$$
\theta_3 = \left[a_{y,1}, a_{y,2}, a_r, b_\pi, b_y, \sigma_{\tilde{y}}, \sigma_\pi, \sigma_{y*}\right]
$$

The law of motion for the natural rate of interest in HLW (2017) is $r_t^* = g_t + z_t$.

5.5 R Programs to Run the State-Space Models

The programs *rstar.stageX.R* run the models in stages 1-3 of the HLW estimation.

5.6 R Programs for Median Unbiased Estimators

The function *median.unbiased.estimator.stage1.R* computes the exponential Wald statistic of Andrews and Ploberger (1994) for a structural break with unknown break date from the first difference of the preliminary estimate of the natural rate of output from the stage 1 model to obtain the median unbiased estimate of λ_g.

The function *median.unbiased.estimator.stage2.R* applies the exponential Wald test for an intercept shift in the IS equation at an unknown date to obtain the median unbiased estimate of λ_z, taking as input estimates from the stage 2 model.

5.7 Kalman Filter Programs

Within the program *kalman.states.R*, the function kalman.states() calls kalman.states.filtered() and kalman.states.smoothed() to apply the Kalman filter and smoother. It takes as input the coefficient matrices for the given state-space model as well as the conditional expectation and covariance matrix of the initial state, xi.tm1tm1 $\left(\xi_{t-1|t-1}\right)$ and P.tm1tm1 $\left(P_{t-1|t-1}\right)$, respectively.

kalman.states.wrapper.R is a wrapper function for *kalman.states.R* that specifies inputs based on the estimation stage.

5.8 Log Likelihood Programs

The function *kalman.log.likelihood.R* takes as input the coefficient matrices of the given state-space model and the conditional expectation and covariance matrix of the initial state and returns the log likelihood value and a vector with the log likelihood at each time t. *log.likelihood.wrapper.R* is a wrapper function for *kalman.log.likelihood.R* that specifies inputs based on the estimation stage.

5.9 Standard Error Program

The function *kalman.standard.errors.R* computes confidence intervals and corresponding standard errors for the estimates of the states using Hamilton's (1986) Monte Carlo procedure that accounts for both filter and parameter uncertainty. See footnote 7 in HLW.

5.10 Miscellaneous Programs

The function *calculate.covariance.R* calculates the covariance matrix of the initial state from the gradients of the likelihood function. The function *format.output.R* generates a dataframe to be written to a CSV containing one-sided estimates, parameter values, standard errors, and other statistics of interest.

6 Run Estimation for Each Economy

For each economy, the file *run.hlw.xx.R* reads in the provided data, runs the HLW estimation by calling *run.hlw.estimation.R*, and saves the one-sided estimates and a spreadsheet of output. We do not include the two-sided estimates in the output CSV, but they are returned by *run.hlw.estimation.R* and stored as *two.sided.est.XX*.

The following variables defined in *run.hlw.xx.R* determine the model specification:

- **Sample dates:** Set *sample.start* and *sample.end* corresponding to the input data.
- **Initialization of state vector:** Set *xi.00.stageX* to a vector of values for the initial states, or set as *NA* (default) to initializing following the HLW procedure. One can also set the initial covariance matrix for each stage, *P.00.stageX*.
- **Constraints:** Set *a.r.constraint* and *b.y.constraint* to impose bounds on the slopes of the IS and Phillips curve equations, respectively.

- **COVID indicator parameter (ϕ):** Set *fix.phi* equal to a numeric value in order to fix the parameter ϕ (e.g. at $\phi = 0$ to impose no role for the COVID supply shock). Set the value to *NA* (default) to estimate ϕ. Note that while the code is set up to accommodate any numeric value in *fix.phi*, the procedure to obtain the initial guess of the parameter vector for maximum likelihood estimation implicitly assumes $\phi = 0$ when ϕ is not estimated.
- **Time-varying volatility specification:** Set the flag *use.kappa* to *TRUE* when introducing time-varying volatility. Set *kappa.inputs* corresponding to the instructions in *run.hlw.xx.R* to specify the variance scale parameters.

7 References

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