

# 1 LW's (2003) State-Space Model (SSM) Form

LW03 use the following standard State-Space Form (SSF):

$$\begin{aligned}\text{Measurement : } \mathbf{y}_t &= \mathbf{A}\mathbf{x}_t + \mathbf{H}\boldsymbol{\xi}_t + \mathbf{R}^{1/2}\boldsymbol{\varepsilon}_t^y \\ \text{State : } \boldsymbol{\xi}_t &= \mathbf{F}\boldsymbol{\xi}_{t-1} + \mathbf{Q}^{1/2}\boldsymbol{\varepsilon}_t^\xi\end{aligned}$$

where

$$\begin{aligned}\mathbf{y}_t &= \begin{bmatrix} y_t & \pi_t \end{bmatrix}', \\ \mathbf{x}_t &= \begin{bmatrix} y_{t-1} & y_{t-2} & r_{t-1} & r_{t-2} & \pi_{t-1} & \pi_{t-2,4} & \pi_{t-5,8} & (\pi_{t-1}^0 - \pi_{t-1}) & (\pi_t^m - \pi_t) \end{bmatrix}', \\ \boldsymbol{\xi}_t &= \begin{bmatrix} y_t^* & y_{t-1}^* & y_{t-2}^* & g_{t-1} & g_{t-2} & z_{t-1} & z_{t-2} \end{bmatrix}', \\ \mathbf{A} &= \begin{bmatrix} a_1 & a_2 & \frac{a_3}{2} & \frac{a_3}{2} & 0 & 0 & 0 & 0 & 0 \\ b_3 & 0 & 0 & 0 & b_1 & b_2 & (1 - b_1 - b_2) & b_4 & b_5 \end{bmatrix}', \\ \mathbf{H} &= \begin{bmatrix} 1 & -a_1 & -a_2 & -4c\frac{a_3}{2} & -4c\frac{a_3}{2} & -\frac{a_3}{2} & -\frac{a_3}{2} \\ 0 & -b_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.\end{aligned}$$

**Note:** I follow the notation used in the documentation file: 'LW\_Code\_Guide.pdf' included in the zip file: 'LW\_replication.zip' that contains the replication code of LW03 and which is available from the NYFED website at: [https://www.newyorkfed.org/medialibrary/media/research/economists/williams/data/LW\\_replication.zip](https://www.newyorkfed.org/medialibrary/media/research/economists/williams/data/LW_replication.zip).

Also, in the construction of  $r_t^*$ , trend growth  $g_t$  is annualized, but not in the state equations for  $g_t$ . That is, in the matrices above, the entries in  $\mathbf{H}(1, 4 : 5)$  corresponding to trend growth are multiplied by 4 (as well as  $c$ ) in the code that performs the estimation (see `unpack.parameters.stage3.R` in `LW_replication.zip` files, line 24, which reads):

```
H[1,4:5] <- -parameters[9]*parameters[3]*2 ## c, a_3 (annualized).
```

Finally, in 'LW\_Code\_Guide.pdf', the standard deviations of the shocks (and presumably) the shocks themselves are denoted by  $\begin{bmatrix} \sigma_{\tilde{y}} & \sigma_{\pi} & \sigma_{y^*} & \sigma_g & \sigma_z \end{bmatrix}'$  with no particular order given, but with the only sensible order based on the SSM equations being:  $\begin{bmatrix} \varepsilon_t^y & \varepsilon_t^\pi & \varepsilon_t^{y^*} & \varepsilon_t^g & \varepsilon_t^z \end{bmatrix}'$ . I will thus use this notation and order in the equations below that use and describe their SSF, and will use the labelling  $\{\sigma_i\}_{i=1}^5$  later on when referring back to the published paper's numerical values and notation, and in the construction of the SSM used for shock recovery.

## 1.1 SSM of LW03

obs0

The measurement equation is:

$$\mathbf{y}_t = \mathbf{A}\mathbf{x}_t + \mathbf{H}\xi_t + \mathbf{R}^{1/2}\boldsymbol{\varepsilon}_t^y$$

$$\underbrace{\begin{bmatrix} y_t \\ \pi_t \end{bmatrix}}_{\mathbf{y}_t} = \underbrace{\begin{bmatrix} a_1 & a_2 & \frac{a_3}{2} & \frac{a_3}{2} & 0 & 0 & 0 & 0 & 0 \\ b_3 & 0 & 0 & 0 & b_1 & b_2 & (1-b_1-b_2) & b_4 & b_5 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} y_{t-1} \\ y_{t-2} \\ r_{t-1} \\ r_{t-2} \\ \pi_{t-1} \\ \pi_{t-2,4} \\ \pi_{t-5,8} \\ (\pi_{t-1}^0 - \pi_{t-1}) \\ (\pi_t^m - \pi_t) \end{bmatrix}}_{\mathbf{x}_t} \quad (1a)$$

$$+ \underbrace{\begin{bmatrix} 1 & -a_1 & -a_2 & -4c\frac{a_3}{2} & -4c\frac{a_3}{2} & -\frac{a_3}{2} & -\frac{a_3}{2} \\ 0 & -b_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ g_{t-2} \\ z_{t-1} \\ z_{t-2} \end{bmatrix}}_{\xi_t} + \underbrace{\begin{bmatrix} \sigma_{\tilde{y}} & 0 \\ 0 & \sigma_{\pi} \end{bmatrix}}_{\mathbf{R}^{1/2}} \underbrace{\begin{bmatrix} \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^{\pi} \end{bmatrix}}_{\boldsymbol{\varepsilon}_t^y}. \quad (1b)$$

The state equation is:

$$\xi_t = \mathbf{F}\xi_{t-1} + \mathbf{Q}^{1/2}\boldsymbol{\varepsilon}_t^{\xi}$$

$$\underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ g_{t-2} \\ z_{t-1} \\ z_{t-2} \end{bmatrix}}_{\xi_t} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{F}} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ y_{t-3}^* \\ g_{t-2} \\ g_{t-3} \\ z_{t-2} \\ z_{t-3} \end{bmatrix}}_{\xi_{t-1}} + \underbrace{\begin{bmatrix} \sigma_{y^*} & \sigma_g & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sigma_g & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_z \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{Q}^{1/2}} \underbrace{\begin{bmatrix} \varepsilon_t^{y^*} \\ \varepsilon_{t-1}^g \\ \varepsilon_{t-1}^z \end{bmatrix}}_{\boldsymbol{\varepsilon}_t^{\xi}} \quad (2) \quad \text{state1}$$

Expanding the relations in (1) and (2) and re-arranging yields:

$$\begin{bmatrix} y_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} y_t^* + a_1 (y_{t-1} - y_{t-1}^*) + a_2 (y_{t-2} - y_{t-2}^*) \\ + \frac{1}{2} a_3 ([r_{t-1} - 4c g_{t-1} - z_{t-1}] + [r_{t-2} - 4c g_{t-2} - z_{t-2}]) + \sigma_{\tilde{y}} \tilde{\varepsilon}_t^{\tilde{y}} \\ b_3 (y_{t-1} - y_{t-1}^*) + b_1 \pi_{t-1} + b_2 \pi_{t-2,4} + (1 - b_1 - b_2) \pi_{t-5,8} \\ + b_4 (\pi_{t-1}^0 - \pi_{t-1}) + b_5 (\pi_t^m - \pi_t) + \sigma_{\pi} \varepsilon_t^{\pi} \end{bmatrix} \quad (3) \quad \text{lwa}$$

$$\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ g_{t-2} \\ z_{t-1} \\ z_{t-2} \end{bmatrix} = \begin{bmatrix} y_{t-1}^* + \overbrace{g_{t-2} + \sigma_g \varepsilon_{t-1}^g}^{g_{t-1}} + \sigma_{y^*} \varepsilon_t^{y^*} \\ y_{t-1}^* \\ y_{t-2}^* \\ g_{t-2} + \sigma_g \varepsilon_{t-1}^g \\ g_{t-2} \\ z_{t-2} + \sigma_z \varepsilon_{t-1}^z \\ z_{t-2} \end{bmatrix} \Rightarrow \begin{bmatrix} \Delta y_t^* \\ \Delta g_{t-1} \\ \Delta z_{t-1} \end{bmatrix} = \begin{bmatrix} g_{t-1} + \sigma_{y^*} \varepsilon_t^{y^*} \\ \sigma_g \varepsilon_{t-1}^g \\ \sigma_z \varepsilon_{t-1}^z \end{bmatrix} \quad (4) \quad \text{lwb}$$

and with  $r_t^* = 4c g_t + z_t$ , we get:

$$\begin{aligned} \Delta r_t^* &= 4c \overbrace{\Delta g_t}^{\sigma_g \varepsilon_t^g} + \overbrace{\Delta z_t}^{\sigma_z \varepsilon_t^z} \\ &= 4c \sigma_g \varepsilon_t^g + \sigma_z \varepsilon_t^z. \end{aligned} \quad (5) \quad \text{drstar}$$

## 2 Shock recovery SSM

### 2.1 SSM with lagged states

SSM Kurz's (2018) SSM has the following general form:

$$\text{Measurement : } Z_t = D_1 X_t + D_2 X_{t-1} + R \varepsilon_t \quad (6a) \quad \text{ssm1}$$

$$\text{State : } X_t = A X_{t-1} + C \varepsilon_t, \quad (6b) \quad \text{ssm2}$$

where  $\varepsilon_t \sim MN(0, I_m)$ ,  $D_1, D_2, A, R$  are conformable system matrices,  $Z_t$  the observed variable and  $X_t$  the latent state variable.

### 2.2 LW03 equations re-ordered and numbered shocks

LW03's SSM equations in (3) and (4), with the re-labelling of shocks according to Table 1 (page 1065) of their paper yields:  $[\sigma_1(\tilde{y}) \quad \sigma_2(\pi) \quad \sigma_3(z) \quad \sigma_4(y^*) \quad \sigma_5(g)]'$  (Note  $\sigma_3 \Rightarrow \sigma_z$ ). The resulting SSM equations are then:

$$y_t = y_t^* + \sum_{i=1}^2 a_i (y_{t-i} - y_{t-i}^*) + \frac{1}{2} a_3 \sum_{i=1}^2 (r_{t-i} - r_{t-i}^*) + \sigma_1 \varepsilon_{1t} \quad (7a) \quad \text{LW03a}$$

$$\pi_t = b_3 (y_{t-1} - y_{t-1}^*) + b_1 \pi_{t-1} + b_2 \pi_{t-2,4} + (1 - b_1 - b_2) \pi_{t-5,8} + b_4 (\pi_{t-1}^0 - \pi_{t-1}) + b_5 (\pi_t^m - \pi_t) + \sigma_2 \varepsilon_{2t} \quad (7b)$$

$$\Delta z_t = \sigma_3 \varepsilon_{3t} \quad (7c) \quad \text{LW03c}$$

$$\Delta y_t^* = g_{t-1} + \sigma_4 \varepsilon_{4t} \quad (7d) \quad \text{LW03d}$$

$$\Delta g_t = \sigma_5 \varepsilon_{5t} \quad (7e) \quad \text{LW03e}$$

with

$$\Delta r_t^* = 4c\sigma_5 \varepsilon_{5t} + \sigma_3 \varepsilon_{3t}. \quad (7f) \quad \text{LW03f}$$

## 2.3 LW03 SSM for shock recovery

To assess recovery, re-write the model in ‘*shock recovery*’ form. That is, collect all observables in  $Z_t$ , and all shocks (and other state variables) in state vector  $X_t$ . The relevant equations from (7) (incorporating (7e)) for the ‘*shock recovery*’ SSM are:

$$\text{Measurement : } Z_{1t} = y_t^* - a_1 y_{t-1}^* - a_2 y_{t-2}^* - \frac{1}{2} a_3 (r_{t-1}^* + r_{t-2}^*) + \sigma_1 \varepsilon_{1t} \quad (8a)$$

$$Z_{2t} = -b_3 y_{t-1}^* + \sigma_2 \varepsilon_{2t} \quad (8b)$$

$$\text{State : } \Delta y_t^* = g_{t-1} + \sigma_4 \varepsilon_{4t} \quad (8c)$$

$$\Delta g_t = \sigma_5 \varepsilon_{5t} \quad (8d)$$

$$\Delta r_t^* = 4c\sigma_5 \varepsilon_{5t} + \sigma_3 \varepsilon_{3t}, \quad (8e)$$

with the observables  $Z_t$  in the measurement equations defined as:

$$Z_{1t} = y_t - \sum_{i=1}^2 a_i y_{t-i} - \frac{1}{2} a_3 \sum_{i=1}^2 r_{t-i}$$

$$Z_{2t} = \pi_t - b_1 \pi_{t-1} - b_2 \pi_{t-2,4} - (1 - b_1 - b_2) \pi_{t-5,8} - b_4 (\pi_{t-1}^0 - \pi_{t-1}) - b_5 (\pi_t^m - \pi_t) - b_3 y_{t-1}.$$

The ‘shock recovery’ SSF corresponding to (8) is then:

Measurement :  $Z_t = D_1 X_t + D_2 X_{t-1} + R \varepsilon_t$

$$\underbrace{\begin{bmatrix} Z_{1t} \\ Z_{2t} \end{bmatrix}}_{Z_t} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & -b_3 & 0 & 0 & 0 & 0 & \sigma_2 & 0 & 0 & 0 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{X_t} \quad (9a)$$

$$+ \underbrace{\begin{bmatrix} -a_1 & -a_2 & 0 & -\frac{a_3}{2} & -\frac{a_3}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{4t-1} \\ \varepsilon_{5t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\mathbf{0}_{2 \times 5}}_R \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{\varepsilon_t} \quad (9b)$$

State :  $X_t = A X_{t-1} + C \varepsilon_t$ ,

$$\underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_A \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{4t-1} \\ \varepsilon_{5t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & \sigma_4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_5 \\ 0 & 0 & \sigma_3 & 0 & 4c\sigma_5 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_C \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{\varepsilon_t} \quad (9c)$$

### 2.3.1 Correlation between true change in natural rate and estimate

The correlation between the true and estimated  $\Delta r_t^*$  from the SSM can be constructed from the relation:

$$\rho = 0.5 \frac{\text{Var}(\Delta r_t^*) + \text{Var}(E_T \Delta r_t^*) - \phi}{\sigma(\Delta r_t^*)\sigma(E_T \Delta r_t^*)}, \quad (10)$$

where  $\text{Var}(\Delta r_t^*) = 4^2 c^2 \sigma_5^2 + \sigma_3^2$ ,  $\sigma(\Delta r_t^*) = \sqrt{\text{Var}(\Delta r_t^*)}$ , and  $\text{Var}(E_T \Delta r_t^*)$  can be computed from simulating from the true model, applying the Kalman Filter and Smoother to get  $E_T \Delta r_t^*$  and then computing the sample variance of  $E_T \Delta r_t^*$  as an estimate of  $\text{Var}(E_T \Delta r_t^*)$ .

To obtain  $\phi$ , add  $\Delta r_t^*$  to the state-vector  $X_t$  and augment the remaining matrices to be conformable. The required  $\phi$  term is then the entry of  $\text{diag}(P_{t|T}^*)$  that corresponds to  $\Delta r_t^*$ , which will be the very last element.

The augmented SSF of (9) is then:

Measurement :  $Z_t = D_1 X_t + D_2 X_{t-1} + R \varepsilon_t$

$$\underbrace{\begin{bmatrix} Z_{1t} \\ Z_{2t} \end{bmatrix}}_{Z_t} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \sigma_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -b_3 & 0 & 0 & 0 & 0 & \sigma_2 & 0 & 0 & 0 & 0 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \\ \Delta r_t^* \end{bmatrix}}_{X_t} \quad (11)$$

$$+ \underbrace{\begin{bmatrix} -a_1 & -a_2 & 0 & -\frac{a_3}{2} & -\frac{a_3}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{4t-1} \\ \varepsilon_{5t-1} \\ \Delta r_{t-1}^* \end{bmatrix}}_{X_{t-1}} + \underbrace{\mathbf{0}_{2 \times 5}}_R \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{\varepsilon_t} \quad (12)$$

State :  $X_t = A X_{t-1} + C \varepsilon_t,$

$$\begin{aligned}
\underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \\ \Delta r_t^* \end{bmatrix}}_{X_t} &= \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_A \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{4t-1} \\ \varepsilon_{5t-1} \\ \Delta r_{t-1}^* \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & \sigma_4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_5 \\ 0 & 0 & \sigma_3 & 0 & 4c\sigma_5 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \sigma_3 & 0 & 4c\sigma_5 \end{bmatrix}}_C \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{\varepsilon_t}
\end{aligned}
\tag{13}$$