1 HP97

The Hodrick and Prescott (1997, HP) Filter can be expressed in State Space Form (SSF) using
HP0 the following an Unobserved Component (UC) model structure:

$$y_t = y_t^* + y_t^c \tag{1a}$$ HP0a

$$\Delta^2 y_t^* = \varepsilon_{1t} \tag{1b}$$ HP0b

$$y_t^c = \phi \varepsilon_{2t}, \tag{1c}$$ HP0c

where y_t is (100 times) the log of GDP, and ε_{1t} and ε_{2t} are $N(0,1)$. The standard deviation ϕ is the (square root of the) smoothing parameter, commonly set to 40 for quarterly macroeconomic data, implying a value of 'λ' of 1600.

The '*numbered shock*' to '*named shock*' mapping is:

$$\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} = \begin{bmatrix} \varepsilon_t^* \\ \varepsilon_t^c \end{bmatrix}, \tag{2}$$

where ε_t^* is the trend (or permanent) shock, and ε_t^c is the cycle (or transitory) shock.

2 Shock recovery

2.1 State Space Models with lagged states

SSM Kurz (2018) adopts the following general SSF with lagged states in the measurement:

$$\text{Measurement}: \quad Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t \tag{3a}$$ ssm1

$$\text{State}: \quad X_t = A X_{t-1} + C\varepsilon_t, \tag{3b}$$ ssm2

where $\varepsilon_t \sim MN(0, I_m)$, D_1, D_2, A, R are C are conformable system matrices, Z_t the observed variable and X_t the latent state variable.

2.2 HP97 in '*shock recovery*' SSF

To assess shock recovery, write the model in (1) in '*shock recovery*' SSF by collecting all observable variables in Z_t and all shocks (and other latent state variables) in X_t. Differencing y_t and y_t^c twice, and re-arranging the relations in (1) then yields:

$$\begin{aligned} \Delta^2 y_t &= \Delta^2 y_t^* + \Delta^2 y_t^c \\ &= \varepsilon_{1t} + \phi \Delta^2 \varepsilon_{2t} \\ &= \varepsilon_{1t} + \phi \varepsilon_{2t} - 2\phi \varepsilon_{2t-1} + \phi \varepsilon_{2t-2}, \end{aligned} \tag{4}$$ Z

where $\Delta^2 y_t$ is the only observed variable.

The Measurement and State equations of the '*shock recovery*' SSF corresponding to the
KOSSM relations in (4) are then given by:

$$\begin{aligned}
\text{Measurement}: \quad Z_t &= D_1 X_t + D_2 X_{t-1} + R\varepsilon_t \\
&= \varepsilon_{1t} + \phi\varepsilon_{2t} - 2\phi\varepsilon_{2t-1} + \phi\varepsilon_{2t-2}
\end{aligned} \tag{5a}$$

$$Z_t = \underbrace{\begin{bmatrix} 1 & \phi & 0 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{2t-1} \end{bmatrix}}_{X_t} + \underbrace{\begin{bmatrix} 0 & -2\phi & \phi \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{2t-2} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}} \tag{5b}$$

$$\text{State}: \quad X_t = AX_{t-1} + C\varepsilon_t,$$

$$\underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{2t-1} \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{2t-2} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}}_{\varepsilon_t}. \tag{5c}$$

2.3 Shock Identities

Kalman Filter estimates of the permanent and transitory shocks ε_{1t} and ε_{2t} contained in $E_t X_t = E_t \begin{bmatrix} \varepsilon_{1t} & \varepsilon_{2t} & \varepsilon_{2t-1} \end{bmatrix}'$ are linked by the identity:

$$E_t \varepsilon_{2t} = \phi E_t \varepsilon_{1t},$$

and Kalman Smoother estimates $E_T X_t = E_T \begin{bmatrix} \varepsilon_{1t} & \varepsilon_{2t} & \varepsilon_{2t-1} \end{bmatrix}'$ give the identity:

$$\Delta^2 E_T \varepsilon_{1t} = \frac{1}{\phi^2} E_T \varepsilon_{2t-2}. \tag{6}$$ KS

Since $\varepsilon_{1t} = \Delta^2 y_t^*$ and $\varepsilon_{2t} = \frac{1}{\phi^2} y_t^c$, this implies that the output from the standard HP–Filter will give the identity:

$$\begin{aligned}
\Delta^4 y_t^* &= \frac{1}{\phi^2} y_{t-2}^c \\
\Delta^4 \text{HP–trend}_t &= \frac{1}{\phi^2} \text{HP–cycle}_{t-2}.
\end{aligned}$$

Indeed, running a regression of Δ^4HP–trend$_t$ on HP–cycle$_{t-2}$ (without an intercept) yields a regression coefficient of $0.000625 = 1/1600$ when applied to US–GDP data that was HP–Filtered with the smoothing parameter set to $\lambda = 1600 = 40^2$. The regression fit is perfect, yielding an R^2 of 1 and a residual sum of squares of exactly 0.

3 Alternative way to assess shock recovery without Kurz

TBA