1 HLW's (2023) post COVID19 State-Space Model (SSM) Form

HLW23 (post COVID19) use the same standard State-Space Form (SSF) as in HLW17:

$$\begin{aligned} \text{Measurement}: \quad & \mathbf{y}_t = \mathbf{A}\mathbf{x}_t + \mathbf{H}\boldsymbol{\xi}_t + \boldsymbol{R}_t^{1/2}\boldsymbol{\varepsilon}_t^{\mathbf{y}} \\ \text{State}: \quad & \boldsymbol{\xi}_t = \mathbf{F}\boldsymbol{\xi}_{t-1} + \boldsymbol{Q}^{1/2}\boldsymbol{\varepsilon}_t^{\xi} \end{aligned}$$

but where the states and exogenous variables have been modified:

$$\begin{aligned}
\mathbf{y}_t &= \begin{bmatrix} y_t & \pi_t \end{bmatrix}', \\
\mathbf{x}_t &= \begin{bmatrix} y_{t-1} & y_{t-2} & r_{t-1} & r_{t-2} & \pi_{t-1} & \pi_{t-2,4} & d_t & d_{t-1} & d_{t-2} \end{bmatrix}', \\
\boldsymbol{\xi}_t &= \begin{bmatrix} y_t^* & y_{t-1}^* & y_{t-2}^* & g_t & g_{t-1} & g_{t-2} & z_t & z_{t-1} & z_{t-2} \end{bmatrix}', \\
\mathbf{A} &= \begin{bmatrix} a_{y,1} & a_{y,2} & \frac{a_r}{2} & \frac{a_r}{2} & 0 & 0 & \phi & -\phi a_{y,1} & -\phi a_{y,2} \\ b_y & 0 & 0 & 0 & b_\pi & 1-b_\pi & 0 & -\phi & 0 \end{bmatrix}, \\
\mathbf{H} &= \begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} & 0 & -4c\frac{a_r}{2} & -4c\frac{a_r}{2} & 0 & -\frac{a_r}{2} & -\frac{a_r}{2} \\ 0 & -b_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}$$

to incorporate dummy variables in the measurement equation and a time-varying R_t covariance matrix, and some correction I pointed out in the formulation of the state equation (see pages $7-8$ in 'HLW_Replication_Code_Guide.pdf' and pages $43-44$ in '*Measuring the Natural Rate of Interest after COVID-19*'.

Note: This follows the notation used in the documentation file: 'HLW_Replication_Code_Guide.pdf' included in the zip file: 'HLW_2023_Replication_Code.zip' that contains the replication code of HLW23 and which is available from the NYFED website at: https://www.newyorkfed.org/medialibrary/media/research/economists/williams/data/HLW_Code.zip.

Compared to HLW17, the constant c is estimated again and dummies are added to account for the COVID19 years.

In the construction of r_t^*, trend growth g_t is again annualized, but not in the state equations for g_t. That is, in the matrices above, the entries in $\mathbf{H}(1,4:5)$ corresponding to trend growth are multiplied by 4 (as well as c) in the code that performs the estimation (see `unpack.parameters.stage3.R` in `HLW_replication.zip` files, line 30, which reads):

`H[1, 5:6] <- -parameters[param.num["c"]]*parameters[param.num["a_r"]]*2`.

The standard deviations of the shocks are denoted by $\begin{bmatrix} \sigma_{\tilde{y}} & \sigma_\pi & \sigma_{y^*} & \sigma_g & \sigma_z \end{bmatrix}'$ in the documentation in 'HLW_Code_Guide.pdf'.

1.1 SSM of HLW23

obs0 The measurement equation is:

$$\mathbf{y}_t = \mathbf{A}\mathbf{x}_t + \mathbf{H}\xi_t + \boldsymbol{R}_t^{1/2}\boldsymbol{\varepsilon}_t^{\mathbf{y}}$$

$$\underbrace{\begin{bmatrix} y_t \\ \pi_t \end{bmatrix}}_{\mathbf{y}_t} = \underbrace{\begin{bmatrix} a_{y,1} & a_{y,2} & \frac{a_r}{2} & \frac{a_r}{2} & 0 & 0 & \phi & -\phi a_{y,1} & -\phi a_{y,2} \\ b_y & 0 & 0 & 0 & b_\pi & 1-b_\pi & 0 & -\phi b_y & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} y_{t-1} \\ y_{t-2} \\ r_{t-1} \\ r_{t-2} \\ \pi_{t-1} \\ \pi_{t-2,4} \\ d_t \\ d_{t-1} \\ d_{t-2} \end{bmatrix}}_{\mathbf{x}_t} \tag{1a}$$

$$+ \underbrace{\begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} & 0 & -4c\frac{a_r}{2} & -4c\frac{a_r}{2} & 0 & -\frac{a_r}{2} & -\frac{a_r}{2} \\ 0 & -b_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_t \\ g_{t-1} \\ g_{t-2} \\ z_t \\ z_{t-1} \\ z_{t-2} \end{bmatrix}}_{\xi_t} + \underbrace{\begin{bmatrix} \kappa_t\sigma_{\tilde{y}} & 0 \\ 0 & \kappa_t\sigma_\pi \end{bmatrix}}_{\boldsymbol{R}_t^{1/2}} \underbrace{\begin{bmatrix} \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^{\pi} \end{bmatrix}}_{\boldsymbol{\varepsilon}_t^{\mathbf{y}}}. \tag{1b}$$

The state equation is:

$$\boldsymbol{\xi}_t = \mathbf{F}\boldsymbol{\xi}_{t-1} + \boldsymbol{Q}^{1/2}\boldsymbol{\varepsilon}_t^{\xi}$$

$$\underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_t \\ g_{t-1} \\ g_{t-2} \\ z_t \\ z_{t-1} \\ z_{t-2} \end{bmatrix}}_{\xi_t} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{F}} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ y_{t-3}^* \\ g_{t-1} \\ g_{t-2} \\ g_{t-3} \\ z_{t-1} \\ z_{t-2} \\ z_{t-3} \end{bmatrix}}_{\xi_{t-1}} + \underbrace{\begin{bmatrix} \sigma_{y^*} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sigma_g & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\boldsymbol{Q}^{1/2}} \underbrace{\begin{bmatrix} \varepsilon_t^{y^*} \\ \varepsilon_t^{g} \\ \varepsilon_t^{z} \end{bmatrix}}_{\boldsymbol{\varepsilon}_t^{\xi}} \tag{2}$$ state1

Expanding the relations in (1) and (2) and re-arranging yields:

$$\begin{bmatrix} y_t \\ \\ \pi_t \end{bmatrix} = \begin{bmatrix} y_t^* + \phi d_t + a_{y,1}\overbrace{(y_{t-1} - y_{t-1}^* - \phi d_{t-1})}^{\tilde{y}_{t-1,COVID}} + a_{y,2}\overbrace{(y_{t-2} - y_{t-2}^* - \phi d_{t-2})}^{\tilde{y}_{t-2,COVID}} \\ +\frac{1}{2}a_r\left([r_{t-1} - 4cg_{t-1} - z_{t-1}] + [r_{t-2} - 4cg_{t-2} - z_{t-2}]\right) + \kappa_t \sigma_{\tilde{y}} \varepsilon_t^{\tilde{y}} \\ b_y\overbrace{(y_{t-1} - y_{t-1}^* - \phi d_{t-1})}^{\tilde{y}_{t-1,COVID}} + b_\pi \pi_{t-1} + (1 - b_\pi 1)\, \pi_{t-2,4} + \kappa_t \sigma_\pi \kappa_t \varepsilon_t^\pi \end{bmatrix} \tag{3}$$

lwa

$$\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_t \\ g_{t-1} \\ g_{t-2} \\ z_t \\ z_{t-1} \\ z_{t-2} \end{bmatrix} = \begin{bmatrix} y_{t-1}^* + g_{t-1} + \sigma_{y^*}\varepsilon_t^{y^*} \\ y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} + \sigma_g \varepsilon_t^g \\ g_{t-1} \\ g_{t-2} \\ z_{t-1} + \sigma_z \varepsilon_t^z \\ z_{t-1} \\ z_{t-2} \end{bmatrix} \Rightarrow \begin{bmatrix} \Delta y_t^* \\ \Delta g_t \\ \Delta z_t \end{bmatrix} = \begin{bmatrix} g_{t-1} + \sigma_{y^*}\varepsilon_t^{y^*} \\ \sigma_g \varepsilon_t^g \\ \sigma_z \varepsilon_t^z \end{bmatrix} \tag{4}$$

lwb

and with $r_t^* = 4cg_t + z_t$, we get:

$$\begin{aligned} \Delta r_t^* &= 4\,\overbrace{c\Delta g_t}^{\sigma_g \varepsilon_t^g} + \overbrace{\Delta z_t}^{\sigma_z \varepsilon_t^z} \\ &= 4c\sigma_g \varepsilon_t^g + \sigma_z \varepsilon_t^z. \end{aligned} \tag{5}$$

drstar

Note here that the timing mismatch has been resolved now so that the equations read as expected, that is, without any lagged values in ε_t^ξ in (2).

The COVID-adjusted natural rate of output is defined in equation 16 of their paper:

$$\tilde{y}_{t,COVID} = 100(y_t - y_{t,COVID}^*) = 100\left(y_t - y_t^*\right) - \phi d_t.$$

2 Shock recovery SSM

2.1 SSM with lagged states

SSM Kurz's (2018) SSM has the following general from:

$$\text{Measurement}: \quad Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t \tag{6a}$$

ssm1

$$\text{State}: \quad X_t = AX_{t-1} + C\varepsilon_t, \tag{6b}$$

ssm2

where $\varepsilon_t \sim MN(0, I_m)$, D_1, D_2, A, R are C are conformable system matrices, Z_t the observed variable and X_t the latent state variable.

2.2 HLW23 equations

Following the same format as before, HLW23's SSM equations in (3) and (4) gives the following SSM equations: LW03

$$
\begin{aligned}
y_t &= y_t^* + \phi d_t + \sum_{i=1}^{2} a_{y,i}\left(y_{t-i} - y_{t-i}^* - \phi d_{t-i}\right) + \tfrac{1}{2} a_r \sum_{i=1}^{2}\left(r_{t-i} - r_{t-i}^*\right) + \kappa_t \sigma_{\tilde{y}} \varepsilon_t^{\tilde{y}} && \text{(7a)} \\
\pi_t &= b_y(y_{t-1} - y_{t-1}^* - \phi d_{t-1}) + b_\pi \pi_{t-1} + (1 - b_\pi)\pi_{t-2,4} + \kappa_t \sigma_\pi \varepsilon_t^\pi && \text{(7b)} \\
\Delta z_t &= \sigma_z \varepsilon_t^z && \text{(7c)} \\
\Delta y_t^* &= g_{t-1} + \sigma_{y^*} \varepsilon_t^{y^*} && \text{(7d)} \\
\Delta g_t &= \sigma_g \varepsilon_t^g && \text{(7e)}
\end{aligned}
$$

LW03a LW03c LW03d LW03e

with

$$
\Delta r_t^* = 4c\sigma_g \varepsilon_t^g + \sigma_z \varepsilon_t^z. \tag{7f}
$$

LW03f

where the numbered shock to named shock mapping is:

$$
\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix} = \begin{bmatrix} \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^\pi \\ \varepsilon_t^z \\ \varepsilon_t^{y^*} \\ \varepsilon_t^g \end{bmatrix}. \tag{8}
$$

2.3 HLW23 SSM for shock recovery

To assess recovery, re-write the model in '*shock recovery*' form. That is, collect all observables in Z_t, and all shocks (and other state variables) in state vector X_t. The relevant equations from (7) (incorporating (7f)) for the '*shock recovery*' SSM are (note the inclusion of c in (9e)): ssm0

$$
\begin{aligned}
\text{Measurement :} \quad Z_{1t} &= y_t^* - a_{y,1} y_{t-1}^* - a_{y,2} y_{t-2}^* - \tfrac{1}{2} a_r \left(r_{t-1}^* + r_{t-2}^*\right) + \kappa_t \sigma_{\tilde{y}} \varepsilon_t^{\tilde{y}} && \text{(9a)} \\
Z_{2t} &= -b_y y_{t-1}^* + \kappa_t \sigma_\pi \varepsilon_t^\pi && \text{(9b)} \\
\text{State :} \quad \Delta y_t^* &= g_{t-1} + \sigma_{y^*} \varepsilon_t^{y^*} && \text{(9c)} \\
\Delta g_t &= \sigma_g \varepsilon_t^g && \text{(9d)} \\
\Delta r_t^* &= 4c\sigma_g \varepsilon_t^g + \sigma_z \varepsilon_t^z, && \text{(9e)}
\end{aligned}
$$

drstar2

ssm0 with the observables Z_t in the measurement equations defined as:

$$
\begin{aligned}
Z_{1t} &= y_t - \sum_{i=1}^{2}(a_{y,i} y_{t-i} - \phi d_{t-i}) - \tfrac{1}{2} a_r \sum_{i=1}^{2} r_{t-i} && \text{(10a)} \\
Z_{2t} &= \pi_t - b_\pi \pi_{t-1} - (1 - b_\pi)\pi_{t-2,4} - (b_y y_{t-1} - \phi d_{t-1}). && \text{(10b)}
\end{aligned}
$$

So these are the same as before for the HLW17 model, with the dummy variables only impacting the observable part of the Z_t measurement in (10) and the only change being the inclusion of the 3 different possibilities for κ_t: $\kappa_{20220Q2-Q4}, \kappa_{2021}, \kappa_{2022}$ and 1 for the baseline, but with the 2023 estimates of all other parameters.

KOSSM The '*shock recovery*' SSF corresponding to (9) is then:

$$\text{Measurement}: \quad Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t$$

$$\underbrace{\begin{bmatrix} Z_{1t} \\ Z_{2t} \end{bmatrix}}_{Z_t} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \kappa_t \sigma_{\tilde{y}} & 0 & 0 & 0 & 0 \\ 0 & -b_y & 0 & 0 & 0 & 0 & \kappa_t \sigma_\pi & 0 & 0 & 0 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^{\pi} \\ \varepsilon_t^{z} \\ \varepsilon_t^{y^*} \\ \varepsilon_t^{g} \end{bmatrix}}_{X_t} \tag{11a}$$

$$+ \underbrace{\begin{bmatrix} -a_{y,1} & -a_{y,2} & 0 & -\frac{a_r}{2} & -\frac{a_r}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ \varepsilon_{t-1}^{\tilde{y}} \\ \varepsilon_{t-1}^{\pi} \\ \varepsilon_{t-1}^{z} \\ \varepsilon_{t-1}^{y^*} \\ \varepsilon_{t-1}^{g} \end{bmatrix}}_{X_{t-1}} + \underbrace{\mathbf{0}_{2\times 5}}_{R} \underbrace{\begin{bmatrix} \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^{\pi} \\ \varepsilon_t^{z} \\ \varepsilon_t^{y^*} \\ \varepsilon_t^{g} \end{bmatrix}}_{\varepsilon_t} \tag{11b}$$

$$\text{State}: \quad X_t = AX_{t-1} + C\varepsilon_t,$$

$$\underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^{\pi} \\ \varepsilon_t^{z} \\ \varepsilon_t^{y^*} \\ \varepsilon_t^{g} \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ \varepsilon_{t-1}^{\tilde{y}} \\ \varepsilon_{t-1}^{\pi} \\ \varepsilon_{t-1}^{z} \\ \varepsilon_{t-1}^{y^*} \\ \varepsilon_{t-1}^{g} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & \sigma_{y^*} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_g \\ 0 & 0 & \sigma_z & 0 & 4c\sigma_g \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^{\pi} \\ \varepsilon_t^{z} \\ \varepsilon_t^{y^*} \\ \varepsilon_t^{g} \end{bmatrix}}_{\varepsilon_t} \tag{11c}$$

2.3.1 Correlation between true change in natural rate and estimate

The correlation between the true and estimated Δr_t^* from the SSM can be constructed from the relation:

$$\rho = 0.5 \frac{\mathrm{Var}(\Delta r_t^*) + \mathrm{Var}(E_T \Delta r_t^*) - \phi}{\sigma(\Delta r_t^*)\sigma(E_T \Delta r_t^*)}, \tag{12}$$

where $\mathrm{Var}(\Delta r_t^*) = 4^2 c^2 \sigma_5^2 + \sigma_3^2$, $\sigma(\Delta r_t^*) = \sqrt{\mathrm{Var}(\Delta r_t^*)}$, and $\mathrm{Var}(E_T \Delta r_t^*)$ can be computed from simulating from the true model, applying the Kalman Filter and Smoother to get $E_T \Delta r_t^*$ and then computing the sample variance of $E_T \Delta r_t^*$ as an estimate of $\mathrm{Var}(E_T \Delta r_t^*)$.

To obtain ϕ, add Δr_t^* to the state-vector X_t and augment the remaining matrices to be conformable. The required ϕ term is then the entry of $\mathrm{diag}(P_{t|T}^*)$ that corresponds to Δr_t^*, which will be the very last element.

The augmented SSF of (11) is then:

$$\text{Measurement}: \quad Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t$$

$$\underbrace{\begin{bmatrix} Z_{1t} \\ Z_{2t} \end{bmatrix}}_{Z_t} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \kappa_t \sigma_{\tilde{y}} & 0 & 0 & 0 & 0 & 0 \\ 0 & -b_y & 0 & 0 & 0 & 0 & \kappa_t \sigma_\pi & 0 & 0 & 0 & 0 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^{\pi} \\ \varepsilon_t^{z} \\ \varepsilon_t^{y^*} \\ \varepsilon_t^{g} \\ \Delta r_t^* \end{bmatrix}}_{X_t} \tag{13}$$

$$+ \underbrace{\begin{bmatrix} -a_{y,1} & -a_{y,2} & 0 & -\frac{a_r}{2} & -\frac{a_r}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ \varepsilon_{t-1}^{\tilde{y}} \\ \varepsilon_{t-1}^{\pi} \\ \varepsilon_{t-1}^{z} \\ \varepsilon_{t-1}^{y^*} \\ \varepsilon_{t-1}^{g} \\ \Delta r_{t-1}^* \end{bmatrix}}_{X_{t-1}} + \underbrace{\mathbf{0}_{2\times 5}}_{R} \underbrace{\begin{bmatrix} \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^{\pi} \\ \varepsilon_t^{z} \\ \varepsilon_t^{y^*} \\ \varepsilon_t^{g} \end{bmatrix}}_{\varepsilon_t} \tag{14}$$

$$\text{State}: \quad X_t = AX_{t-1} + C\varepsilon_t,$$

$$
\underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^{\pi} \\ \varepsilon_t^{z} \\ \varepsilon_t^{y^*} \\ \varepsilon_t^{g} \\ \color{red}{\Delta r_t^*} \end{bmatrix}}_{X_t}
=
\underbrace{\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \color{red}{0} \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \color{red}{0} \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \color{red}{0} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \color{red}{0} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \color{red}{0} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \color{red}{0} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \color{red}{0} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \color{red}{0} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \color{red}{0} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \color{red}{0} \\
\color{red}{0} & \color{red}{0} & \color{red}{0} & \color{red}{0} & \color{red}{0} & \color{red}{0} & \color{red}{0} & \color{red}{0} & \color{red}{0} & \color{red}{0} & \color{red}{0}
\end{bmatrix}}_{A}
\underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ \varepsilon_{t-1}^{\tilde{y}} \\ \varepsilon_{t-1}^{\pi} \\ \varepsilon_{t-1}^{z} \\ \varepsilon_{t-1}^{y^*} \\ \varepsilon_{t-1}^{g} \\ \color{red}{\Delta r_{t-1}^*} \end{bmatrix}}_{X_{t-1}}
+
\underbrace{\begin{bmatrix}
0 & 0 & 0 & \sigma_{y^*} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sigma_g \\
0 & 0 & \sigma_z & 0 & 4c\sigma_g \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\color{red}{0} & \color{red}{0} & \color{red}{\sigma_z} & \color{red}{0} & \color{red}{4c\sigma_g}
\end{bmatrix}}_{C}
\underbrace{\begin{bmatrix} \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^{\pi} \\ \varepsilon_t^{z} \\ \varepsilon_t^{y^*} \\ \varepsilon_t^{g} \end{bmatrix}}_{\varepsilon_t}
\tag{15}
$$