1 LW's (2003) State-Space Model (SSM) Form

LW03 use the following standard State-Space Form (SSF):

$$\begin{aligned}\text{Measurement}: \quad \mathbf{y}_t &= \mathbf{A}\mathbf{x}_t + \mathbf{H}\boldsymbol{\xi}_t + \boldsymbol{R}^{1/2}\varepsilon_t^{\mathbf{y}} \\ \text{State}: \quad \boldsymbol{\xi}_t &= \mathbf{F}\boldsymbol{\xi}_{t-1} + \boldsymbol{Q}^{1/2}\varepsilon_t^{\xi}\end{aligned}$$

where

$$\begin{aligned}
\mathbf{y}_t &= \begin{bmatrix} y_t & \pi_t \end{bmatrix}', \\
\mathbf{x}_t &= \begin{bmatrix} y_{t-1} & y_{t-2} & r_{t-1} & r_{t-2} & \pi_{t-1} & \pi_{t-2,4} & \pi_{t-5,8} & (\pi_{t-1}^0 - \pi_{t-1}) & (\pi_t^m - \pi_t) \end{bmatrix}', \\
\boldsymbol{\xi}_t &= \begin{bmatrix} y_t^* & y_{t-1}^* & y_{t-2}^* & g_{t-1} & g_{t-2} & z_{t-1} & z_{t-2} \end{bmatrix}', \\
\mathbf{A} &= \begin{bmatrix} a_1 & a_2 & \frac{a_3}{2} & \frac{a_3}{2} & 0 & 0 & 0 & 0 & 0 \\ b_3 & 0 & 0 & 0 & b_1 & b_2 & (1-b_1-b_2) & b_4 & b_5 \end{bmatrix}, \\
\mathbf{H} &= \begin{bmatrix} 1 & -a_1 & -a_2 & -4c\frac{a_3}{2} & -4c\frac{a_3}{2} & -\frac{a_3}{2} & -\frac{a_3}{2} \\ 0 & -b_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}$$

Note: I follow the notation used in the documentation file: '`LW_Code_Guide.pdf`' included in the zip file: '`LW_replication.zip`' that contains the replication code of LW03 and which is available from the NYFED website at: `https://www.newyorkfed.org/medialibrary/media/research/economists/williams/data/LW_replication.zip`.

Also, in the construction of r_t^*, trend growth g_t is annualized, but not in the state equations for g_t. That is, in the matrices above, the entries in $\mathbf{H}(1, 4:5)$ corresponding to trend growth are multiplied by 4 (as well as c) in the code that performs the estimation (see `unpack.parameters.stage3.R` in `LW_replication.zip` files, line 24, which reads):

```
H[1,4:5] <- -parameters[9]*parameters[3]*2 ## c, a_3 (annualized).
```

Finally, in '`LW_Code_Guide.pdf`', the standard deviations of the shocks (and presumably) the shocks themselves are denoted by $\begin{bmatrix} \sigma_{\tilde{y}} & \sigma_\pi & \sigma_{y^*} & \sigma_g & \sigma_z \end{bmatrix}'$ with no particular order given, but with the only sensible order based on the SSM equations being: $\begin{bmatrix} \varepsilon_t^y & \varepsilon_t^\pi & \varepsilon_t^{y^*} & \varepsilon_t^g & \varepsilon_t^z \end{bmatrix}'$. I will thus use this notation and order in the equations below that use and describe their SSF, and will use the labelling $\{\sigma_i\}_{i=1}^5$ later on when referring back to the published paper's numerical values and notation, and in the construction of the SSM used for shock recovery.

1.1 SSM of LW03

obs0 The measurement equation is:

$$\mathbf{y}_t = \mathbf{A}\mathbf{x}_t + \mathbf{H}\xi_t + \boldsymbol{R}^{1/2}\varepsilon_t^{\mathbf{y}}$$

$$\underbrace{\begin{bmatrix} y_t \\ \pi_t \end{bmatrix}}_{\mathbf{y}_t} = \underbrace{\begin{bmatrix} a_1 & a_2 & \frac{a_3}{2} & \frac{a_3}{2} & 0 & 0 & 0 & 0 & 0 \\ b_3 & 0 & 0 & 0 & b_1 & b_2 & (1-b_1-b_2) & b_4 & b_5 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} y_{t-1} \\ y_{t-2} \\ r_{t-1} \\ r_{t-2} \\ \pi_{t-1} \\ \pi_{t-2,4} \\ \pi_{t-5,8} \\ (\pi_{t-1}^0 - \pi_{t-1}) \\ (\pi_t^m - \pi_t) \end{bmatrix}}_{\mathbf{x}_t} \tag{1a}$$

$$+ \underbrace{\begin{bmatrix} 1 & -a_1 & -a_2 & -4c\frac{a_3}{2} & -4c\frac{a_3}{2} & -\frac{a_3}{2} & -\frac{a_3}{2} \\ 0 & -b_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ g_{t-2} \\ z_{t-1} \\ z_{t-2} \end{bmatrix}}_{\xi_t} + \underbrace{\begin{bmatrix} \sigma_{\tilde{y}} & 0 \\ 0 & \sigma_\pi \end{bmatrix}}_{\boldsymbol{R}^{1/2}} \underbrace{\begin{bmatrix} \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^{\pi} \end{bmatrix}}_{\varepsilon_t^{\mathbf{y}}}. \tag{1b}$$

The state equation is:

$$\boldsymbol{\xi}_t = \mathbf{F}\boldsymbol{\xi}_{t-1} + \boldsymbol{Q}^{1/2}\varepsilon_t^{\xi}$$

$$\underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ g_{t-2} \\ z_{t-1} \\ z_{t-2} \end{bmatrix}}_{\xi_t} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{F}} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ y_{t-3}^* \\ g_{t-2} \\ g_{t-3} \\ z_{t-2} \\ z_{t-3} \end{bmatrix}}_{\xi_{t-1}} + \underbrace{\begin{bmatrix} \sigma_{y^*} & \sigma_g & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sigma_g & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_z \\ 0 & 0 & 0 \end{bmatrix}}_{\boldsymbol{Q}^{1/2}} \underbrace{\begin{bmatrix} \varepsilon_t^{y^*} \\ \varepsilon_{t-1}^{g} \\ \varepsilon_{t-1}^{z} \end{bmatrix}}_{\varepsilon_t^{\xi}} \tag{2}$$ state1

Expanding the relations in (1) and (2) and re-arranging yields:

$$\begin{bmatrix} y_t \\ \\ \\ \pi_t \end{bmatrix} = \begin{bmatrix} y_t^* + a_1\left(y_{t-1} - y_{t-1}^*\right) + a_2\left(y_{t-2} - y_{t-2}^*\right) \\ +\frac{1}{2}a_3\left(\left[r_{t-1} - 4cg_{t-1} - z_{t-1}\right] + \left[r_{t-2} - 4cg_{t-2} - z_{t-2}\right]\right) + \sigma_{\tilde{y}}\varepsilon_t^{\tilde{y}} \\ \\ b_3\left(y_{t-1} - y_{t-1}^*\right) + b_1\pi_{t-1} + b_2\pi_{t-2,4} + (1 - b_1 - b_2)\,\pi_{t-5,8} \\ +b_4\left(\pi_{t-1}^0 - \pi_{t-1}\right) + b_5\left(\pi_t^m - \pi_t\right) + \sigma_\pi\varepsilon_t^\pi \end{bmatrix} \tag{3}$$

lwa

$$\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ g_{t-2} \\ z_{t-1} \\ z_{t-2} \end{bmatrix} = \begin{bmatrix} y_{t-1}^* + \overbrace{g_{t-2} + \sigma_g\varepsilon_{t-1}^g}^{g_{t-1}} + \sigma_{y^*}\varepsilon_t^{y^*} \\ y_{t-1}^* \\ y_{t-2}^* \\ g_{t-2} + \sigma_g\varepsilon_{t-1}^g \\ g_{t-2} \\ z_{t-2} + \sigma_z\varepsilon_{t-1}^z \\ z_{t-2} \end{bmatrix} \Rightarrow \begin{bmatrix} \Delta y_t^* \\ \Delta g_{t-1} \\ \Delta z_{t-1} \end{bmatrix} = \begin{bmatrix} g_{t-1} + \sigma_{y^*}\varepsilon_t^{y^*} \\ \sigma_g\varepsilon_{t-1}^g \\ \sigma_z\varepsilon_{t-1}^z \end{bmatrix} \tag{4}$$

lwb

and with $r_t^* = 4cg_t + z_t$, we get:

$$\begin{aligned} \Delta r_t^* &= 4c\,\overbrace{\Delta g_t}^{\sigma_g\varepsilon_t^g} + \overbrace{\Delta z_t}^{\sigma_z\varepsilon_t^z} \\ &= 4c\sigma_g\varepsilon_t^g + \sigma_z\varepsilon_t^z. \end{aligned} \tag{5}$$

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2 Shock recovery SSM

2.1 SSM with lagged states

SSM

Kurz's (2018) SSM has the following general from:

$$\begin{aligned} \text{Measurement}: \quad Z_t &= D_1X_t + D_2X_{t-1} + R\varepsilon_t \qquad \text{(6a)} \\ \text{State}: \quad X_t &= AX_{t-1} + C\varepsilon_t, \qquad \text{(6b)} \end{aligned}$$

ssm1 ssm2

where $\varepsilon_t \sim MN(0, I_m)$, D_1, D_2, A, R are C are conformable system matrices, Z_t the observed variable and X_t the latent state variable.

2.2 LW03 equations re-ordered and numbered shocks

LW03's SSM equations in (3) and (4), with the re-labelling of shocks according to Table 1 (page 1065) of their paper yields: $\begin{bmatrix}\sigma_1(\tilde{y}) & \sigma_2(\pi) & \sigma_3(z) & \sigma_4(y^*) & \sigma_5(g)\end{bmatrix}'$ (Note $\sigma_3 \Rightarrow \sigma_z$). The resulting SSM equations are then:

LW03

$$y_t = y_t^* + \sum_{i=1}^{2} a_i \left(y_{t-i} - y_{t-i}^*\right) + \tfrac{1}{2} a_3 \sum_{i=1}^{2} \left(r_{t-i} - r_{t-i}^*\right) + \sigma_1 \varepsilon_{1t} \tag{7a}$$ LW03a

$$\begin{aligned}\pi_t &= b_3 \left(y_{t-1} - y_{t-1}^*\right) + b_1 \pi_{t-1} + b_2 \pi_{t-2,4} + (1 - b_1 - b_2)\, \pi_{t-5,8} \\ &\quad + b_4 (\pi_{t-1}^0 - \pi_{t-1}) + b_5 \left(\pi_t^m - \pi_t\right) + \sigma_2 \varepsilon_{2t}\end{aligned} \tag{7b}$$

$$\Delta z_t = \sigma_3 \varepsilon_{3t} \tag{7c}$$ LW03c

$$\Delta y_t^* = g_{t-1} + \sigma_4 \varepsilon_{4t} \tag{7d}$$ LW03d

$$\Delta g_t = \sigma_5 \varepsilon_{5t} \tag{7e}$$ LW03e

with

$$\Delta r_t^* = 4c\sigma_5 \varepsilon_{5t} + \sigma_3 \varepsilon_{3t}. \tag{7f}$$ LW03f

2.3 LW03 SSM for shock recovery

To assess recovery, re-write the model in '*shock recovery*' form. That is, collect all observables in Z_t, and all shocks (and other state variables) in state vector X_t. The relevant equations from (7) (incorporating (7e)) for the '*shock recovery*' SSM are:

ssm0

$$\begin{aligned}
\text{Measurement :}\quad Z_{1t} &= y_t^* - a_1 y_{t-1}^* - a_2 y_{t-2}^* - \tfrac{1}{2} a_3 \left(r_{t-1}^* + r_{t-2}^*\right) + \sigma_1 \varepsilon_{1t} && \text{(8a)} \\
Z_{2t} &= -b_3 y_{t-1}^* + \sigma_2 \varepsilon_{2t} && \text{(8b)} \\
\text{State :}\quad \Delta y_t^* &= g_{t-1} + \sigma_4 \varepsilon_{4t} && \text{(8c)} \\
\Delta g_t &= \sigma_5 \varepsilon_{5t} && \text{(8d)} \\
\Delta r_t^* &= 4c\sigma_5 \varepsilon_{5t} + \sigma_3 \varepsilon_{3t}, && \text{(8e)}
\end{aligned}$$

with the observables Z_t in the measurement equations defined as:

$$\begin{aligned}
Z_{1t} &= y_t - \sum_{i=1}^{2} a_i y_{t-i} - \tfrac{1}{2} a_3 \sum_{i=1}^{2} r_{t-i} \\
Z_{2t} &= \pi_t - b_1 \pi_{t-1} - b_2 \pi_{t-2,4} - (1 - b_1 - b_2)\, \pi_{t-5,8} \\
&\quad - b_4 (\pi_{t-1}^0 - \pi_{t-1}) - b_5 \left(\pi_t^m - \pi_t\right) - b_3 y_{t-1}.
\end{aligned}$$

KOSSM The '*shock recovery*' SSF corresponding to (8) is then:

$$\text{Measurement}: \quad Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t$$

$$\underbrace{\begin{bmatrix} Z_{1t} \\ Z_{2t} \end{bmatrix}}_{Z_t} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & -b_3 & 0 & 0 & 0 & 0 & \sigma_2 & 0 & 0 & 0 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{X_t} \tag{9a}$$

$$+ \underbrace{\begin{bmatrix} -a_1 & -a_2 & 0 & -\frac{a_3}{2} & -\frac{a_3}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{4t-1} \\ \varepsilon_{5t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\mathbf{0}_{2\times 5}}_{R} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{\varepsilon_t} \tag{9b}$$

$$\text{State}: \quad X_t = AX_{t-1} + C\varepsilon_t,$$

$$\underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{4t-1} \\ \varepsilon_{5t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & \sigma_4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_5 \\ 0 & 0 & \sigma_3 & 0 & 4c\sigma_5 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{\varepsilon_t} \tag{9c}$$

2.3.1 Correlation between true change in natural rate and estimate

The correlation between the true and estimated Δr_t^* from the SSM can be constructed from the relation:

$$\rho = 0.5\frac{\text{Var}(\Delta r_t^*) + \text{Var}(E_T \Delta r_t^*) - \phi}{\sigma(\Delta r_t^*)\sigma(E_T \Delta r_t^*)}, \quad (10)$$

where $\text{Var}(\Delta r_t^*) = 4^2 c^2 \sigma_5^2 + \sigma_3^2$, $\sigma(\Delta r_t^*) = \sqrt{\text{Var}(\Delta r_t^*)}$, and $\text{Var}(E_T \Delta r_t^*)$ can be computed from simulating from the true model, applying the Kalman Filter and Smoother to get $E_T \Delta r_t^*$ and then computing the sample variance of $E_T \Delta r_t^*$ as an estimate of $\text{Var}(E_T \Delta r_t^*)$.

To obtain ϕ, add Δr_t^* to the state-vector X_t and augment the remaining matrices to be conformable. The required ϕ term is then the entry of $\text{diag}(P_{t|T}^*)$ that corresponds to Δr_t^*, which will be the very last element.

The augmented SSF of (9) is then:

$$\text{Measurement}: \quad Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t$$

$$\underbrace{\begin{bmatrix} Z_{1t} \\ Z_{2t} \end{bmatrix}}_{Z_t} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \sigma_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -b_3 & 0 & 0 & 0 & 0 & \sigma_2 & 0 & 0 & 0 & 0 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \\ \Delta r_t^* \end{bmatrix}}_{X_t} \quad (11)$$

$$+ \underbrace{\begin{bmatrix} -a_1 & -a_2 & 0 & -\frac{a_3}{2} & -\frac{a_3}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{4t-1} \\ \varepsilon_{5t-1} \\ \Delta r_{t-1}^* \end{bmatrix}}_{X_{t-1}} + \underbrace{\mathbf{0}_{2\times 5}}_{R} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{\varepsilon_t} \quad (12)$$

$$\text{State}: \quad X_t = AX_{t-1} + C\varepsilon_t,$$

$$
\underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \\ \Delta r_t^* \end{bmatrix}}_{X_t}
=
\underbrace{\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}}_{A}
\underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{4t-1} \\ \varepsilon_{5t-1} \\ \Delta r_{t-1}^* \end{bmatrix}}_{X_{t-1}}
+
\underbrace{\begin{bmatrix}
0 & 0 & 0 & \sigma_4 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sigma_5 \\
0 & 0 & \sigma_3 & 0 & 4c\sigma_5 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & \sigma_3 & 0 & 4c\sigma_5
\end{bmatrix}}_{C}
\underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{\varepsilon_t}
\tag{13}
$$