1 Clark87

The Clark (1987) Unobserved Component (UC) model is a generalisation of the HP–Filter (a local linear trend model) that can be expressed in State Space Form (SSF) as:

$$
\begin{aligned}
y_t &= y_t^* + \tilde{y}_t && \text{(1a)}\\
\Delta y_t^* &= g_{t-1} + \sigma_1 \varepsilon_{1t} && \text{(1b)}\\
\Delta g_t &= \sigma_2 \varepsilon_{2t} && \text{(1c)}\\
a(L)\tilde{y}_t &= \sigma_3 \varepsilon_{3t}, && \text{(1d)}
\end{aligned}
$$

where y_t is (100 times) the log of GDP, and the shocks $\{\varepsilon_{it}\}_{i=1}^3$ are mutually uncorrelated *i.i.d.* $N(0,1)$, with standard deviations $\{\sigma_i\}_{i=1}^3$, and $a(L)$ is a lag polynomial commonly assumed to be a stable AR(2), ie., $a(L) = (1 - a_1 L - a_2 L^2)$, with the roots of $a(L)$ being outside the unit circle. The cycle $\tilde{y}_t$ is allowed to be serially correlated. There are 3 shocks in the model.

The '*numbered shock*' to '*named shock*' mapping is:

$$
\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix} = \begin{bmatrix} \varepsilon_t^{y^*} \\ \varepsilon_t^{g} \\ \varepsilon_t^{\tilde{y}} \end{bmatrix}, \tag{2}
$$

where $\varepsilon_t^{y^*}$, ε_t^{g}, and $\varepsilon_t^{\tilde{y}}$ are the trend (permanent), trend growth and cycle shocks, respectively.

2 Shock recovery

2.1 State Space Models with lagged states

Kurz's (2018) SSM has the following general from:

$$
\begin{aligned}
\text{Measurement}: \quad Z_t &= D_1 X_t + D_2 X_{t-1} + R\varepsilon_t && \text{(3a)}\\
\text{State}: \quad X_t &= A X_{t-1} + C\varepsilon_t, && \text{(3b)}
\end{aligned}
$$

where $\varepsilon_t \sim MN(0_m, I_m)$, D_1, D_2, A, R are C are conformable system matrices, Z_t the observed variable and X_t the latent state variable, and m is the number of shocks $\{\varepsilon_{it}\}_{i=1}^2$.

2.2 Clark87 in '*shock recovery*' SSF

To assess shock recovery, write the model in (1) in '*shock recovery*' SSF by collecting all observable variables in Z_t and all shocks (and other latent state variables) in X_t. Differencing y_t and y_t^c twice, and re-arranging the relations in (1) then yields:

$$
\begin{aligned}
\Delta^2 y_t &= \Delta^2 y_t^* + \Delta^2 \tilde{y}_t \\
&= \sigma_1 \Delta\varepsilon_{1t} + \Delta g_{t-1} + \Delta^2 \tilde{y}_t
\end{aligned}
$$

$$= \sigma_1 \Delta\varepsilon_{1t} + \sigma_2 \varepsilon_{2t-1} + a(L)^{-1} \sigma_3 \Delta^2 \varepsilon_{3t}$$
$$\Leftrightarrow a(L)\Delta^2 y_t = \sigma_1 a(L)\Delta\varepsilon_{1t} + \sigma_2 a(L)\varepsilon_{2t-1} + \sigma_3 \Delta^2 \varepsilon_{3t} \tag{4}$$

where $a(L)\Delta^2 y_t$ is the only observed variable. Re-writing (4) in more convenient form for the SSF yields:

$$\begin{aligned} \underbrace{a(L)\Delta^2 y_t}_{Z_t} &= a(L)\sigma_1 \Delta\varepsilon_{1t} + a(L)\sigma_2 \varepsilon_{2t-1} + \sigma_3 \Delta^2 \varepsilon_{3t} \\ &= \sigma_1 \Delta\varepsilon_{1t} - a_1 \sigma_1 \Delta\varepsilon_{1t-1} - a_2 \sigma_1 \Delta\varepsilon_{1t-2} + \sigma_2 \varepsilon_{2t-1} - a_1 \sigma_2 \varepsilon_{2t-2} \\ &- a_2 \sigma_2 \varepsilon_{2t-3} + \sigma_3 \Delta\varepsilon_{3t} - \sigma_3 \Delta\varepsilon_{3t-1}. \end{aligned} \tag{5}$$

The Measurement and State equations of the *'shock recovery'* SSF corresponding to the relations in (5) are then given by:

$$\text{Measurement}: \quad Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t \tag{6a}$$

$$Z_t = \underbrace{\begin{bmatrix} 0 & 0 & 0 & \sigma_1 & -a_1\sigma_1 & 0 & 0 & \sigma_3 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \Delta\varepsilon_{1t} \\ \Delta\varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{2t-2} \\ \Delta\varepsilon_{3t} \end{bmatrix}}_{X_t} \tag{6b}$$

$$+ \underbrace{\begin{bmatrix} 0 & \sigma_2 & 0 & 0 & -a_2\sigma_1 & -a_1\sigma_2 & -a_2\sigma_2 & -\sigma_3 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \Delta\varepsilon_{1t-1} \\ \Delta\varepsilon_{1t-2} \\ \varepsilon_{2t-2} \\ \varepsilon_{2t-3} \\ \Delta\varepsilon_{3t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}}_{\varepsilon_t}$$

$$\text{State}: \quad X_t = AX_{t-1} + C\varepsilon_t \tag{6c}$$

$$\underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \Delta\varepsilon_{1t} \\ \Delta\varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{2t-2} \\ \Delta\varepsilon_{3t} \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \Delta\varepsilon_{1t-1} \\ \Delta\varepsilon_{1t-2} \\ \varepsilon_{2t-2} \\ \varepsilon_{2t-3} \\ \Delta\varepsilon_{3t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}}_{\varepsilon_t}. \tag{6d}$$

2.3 Shock recovery

The diagonal of the steady-state variance/covariance matrix of the smoothed and filtered states X_t denoted by $P^*_{t|T}$ and $P^*_{t|t}$, respectively, are:

Shocks	$P^*_{t\|T}$	$P^*_{t\|t}$
ε_{1t}	0.5469	0.5989
ε_{2t}	0.9870	1.0000
ε_{3t}	0.4661	0.5153

, (7)

indicating that the trend growth shock $\varepsilon_{2t} = \varepsilon^g_t$ cannot be recovered ($P^* \approx 1$), while the cycle shock $\varepsilon_{3t} = \varepsilon^{\tilde{y}}_t$ and the trend shock $\varepsilon_{1t} = \varepsilon^{y^*}_t$ have $P^* \approx 0.5$, suggesting that there are recovery difficulties. Note that $P^*_{t|t} = 1$, which implies that Kalman filtered estimates of ε_{2t} are exactly zero for all t. This can be seen from the entries on the left hand side in (8) (smoothed estimates are shown on the right).

$E_t\varepsilon_{1t}$	$E_t\varepsilon_{2t}$	$E_t\varepsilon_{3t}$	$E_T\varepsilon_{1t}$	$E_T\varepsilon_{2t}$	$E_T\varepsilon_{3t}$
−0.7088	0	−0.7792	0.3098	−0.0160	0.0344
−0.8039	0	−0.8837	−0.5269	0.0043	−0.5442
−0.2554	0	−0.2808	−0.1336	0.0094	0.1118
0.0179	0	0.0196	0.6757	−0.0166	0.4942
−0.5488	0	−0.6032	−0.2535	−0.0068	−0.4492
−0.3244	0	−0.3566	−0.3713	0.0075	−0.2660
0.1304	0	0.1434	0.2703	−0.0029	0.3115
−0.2821	0	−0.3101	−0.6689	0.0228	0.0549

. (8)

In Figure 1, simulated states and estimated Kalman smoothed states are plotted for the two shocks of interest.

The correlation between the true (simulated) and estimated Kalman smoothed shocks can be analyzed by simply computing $\text{Corr}(X_t, \hat{X}_{t|T})$, where $X_t = \begin{bmatrix}\varepsilon_{1t} & \varepsilon_{2t} & \varepsilon_{3t}\end{bmatrix}'$ and $\hat{X}_{t|T} = E_T X_t = E_T \begin{bmatrix}\varepsilon_{1t} & \varepsilon_{2t} & \varepsilon_{2t-1}\end{bmatrix}'$, yielding:

Shocks	$\text{Corr}(X_t, \hat{X}_{t\|T})$
ε_{1t}	0.6736
ε_{2t}	0.1184
ε_{3t}	0.7304

. (9)

As can be seen from (9), the estimated ε_{1t} and ε_{3t} are rather weakly correlated (0.6736 and 0.7304) with the true value, while the estimated ε_{2t} shock is nearly uncorrelated (0.1184). These facts can also be seen from Figure 1 below.

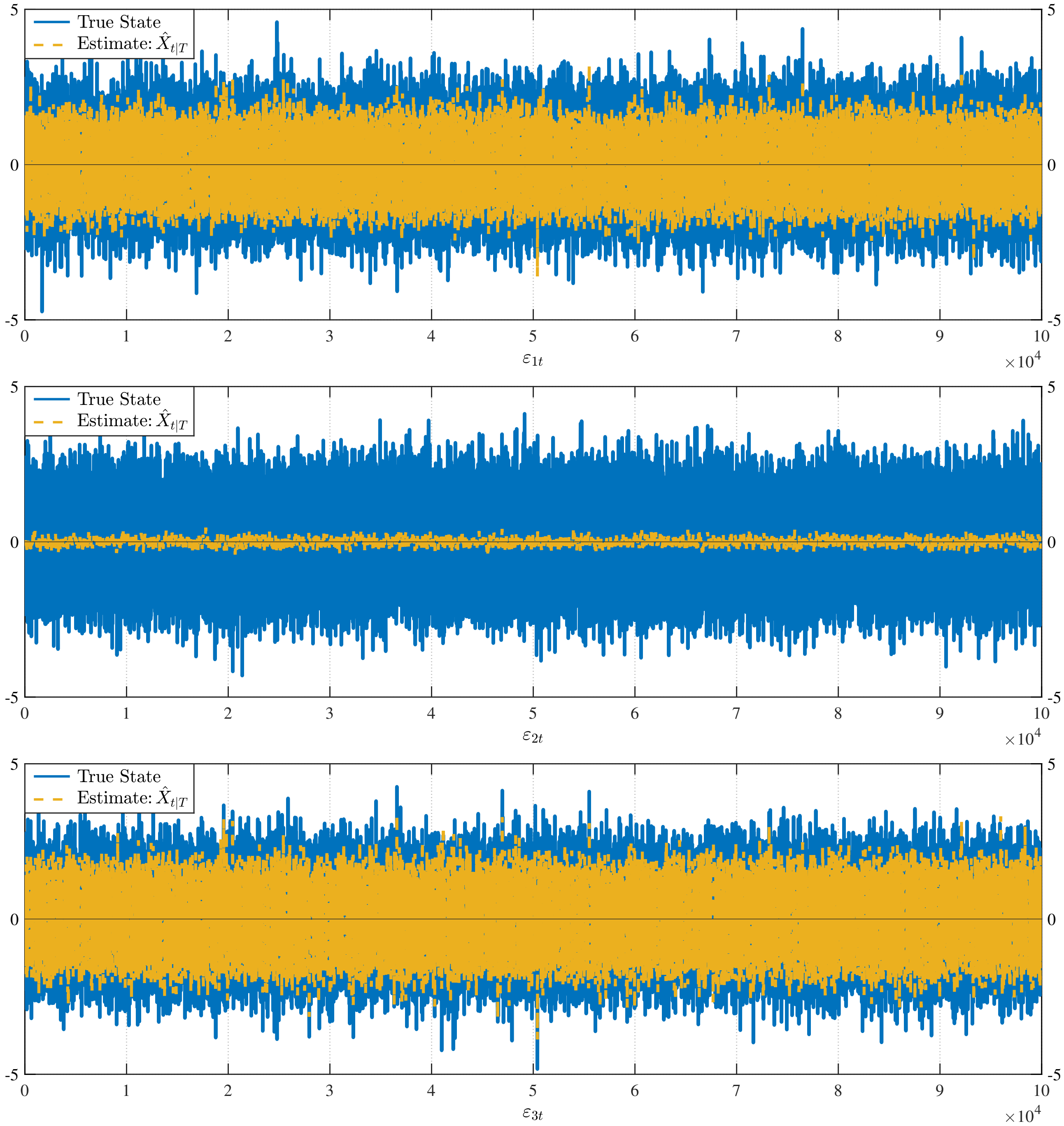


Figure 1: Comparison of true shocks and Kalman Smoothed estimates $\varepsilon_{t|T}$.

2.4 Shock Identities

Similar to the HP-Filter, the Kalman filtered estimates of the trend and cycle shocks ε_{1t} and ε_{3t} are linked by the identity:

$$E_t \varepsilon_{1t} = 0.909694 E_t \varepsilon_{3t}, \tag{10}$$

and the corresponding Kalman Smoother estimates are linked by the *dynamic* identity:

$$\Delta^2 E_T \varepsilon_{1t} = \frac{1}{\phi} E_T \varepsilon_{2t-2}. \tag{11}$$

The filtered and smoothed estimates of the contemporaneous correlations are, respectively:

$\text{Corr}(\hat{X}_{t\|t}, \hat{X}_{t\|t})$		
Shocks	ε_{1t}	ε_{2t}
ε_{1t}	1.0000	1.0000
ε_{2t}	1.0000	1.0000

and

$\text{Corr}(\hat{X}_{t\|T}, \hat{X}_{t\|T})$		
Shocks	ε_{1t}	ε_{2t}
ε_{1t}	1.0000	−0.1907
ε_{2t}	−0.1907	1.0000

.

Note that ε_{1t} and ε_{2t} were generated as *i.i.d.* $N(0,1)$ and mutually uncorrelated.

Running the shock recovery code `Clark87.m` we get the following steady-state diagonal entries:

Shocks	$P^*_{t\|T}$	$P^*_{t\|t}$
ε_{1t}	0.5469	0.5989
ε_{2t}	0.9870	1.0000
ε_{3t}	0.4661	0.5153

The second shock ε_{2t} (corresponding to ε_t^g, ie., trend growth) is not recoverable.

Looking at (**??**), we see that ε_{2t} enters with a lag into the measurement equation. Following the same logic that Adrian used, we should then get only $\hat{\varepsilon}_{2t-1|t}$ from the Kalman Filter.

Below I plot estimates of the Filtered and Smoothed shocks from the model fitted to US data, where shocks from the SSM in (1) are constructed in line with the equations listed there, ie., as $\eta_{2t|t} = \sigma_2 \varepsilon_{2t} = \Delta g_t$ for the filtered and smoothed alternatives.

2.5 Maximum Likelihood estimation

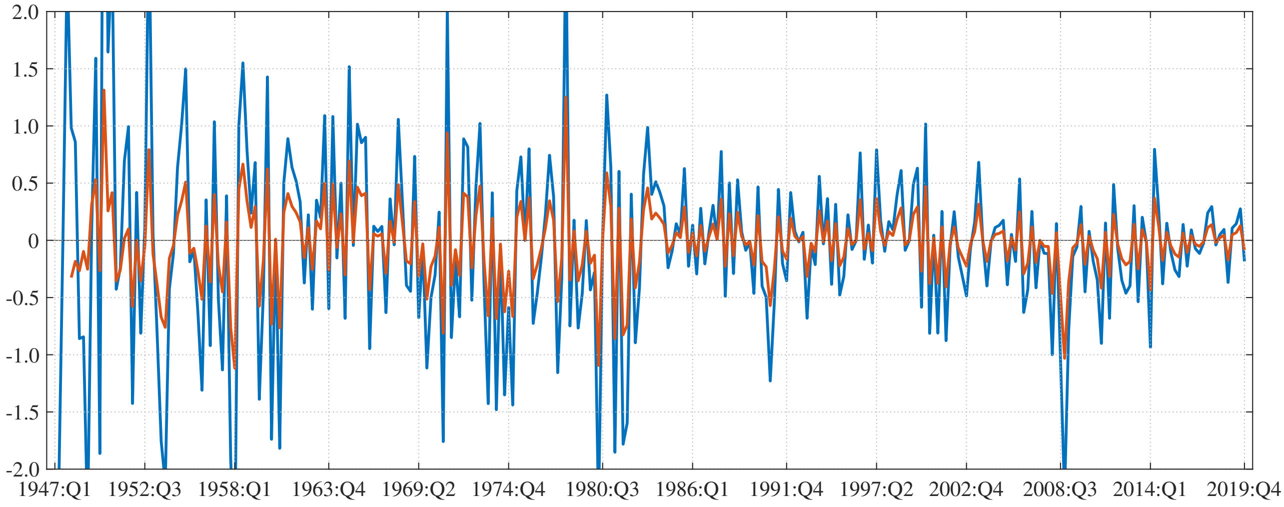
of the model in (1)

The (standard) SSM for ML estimation is:

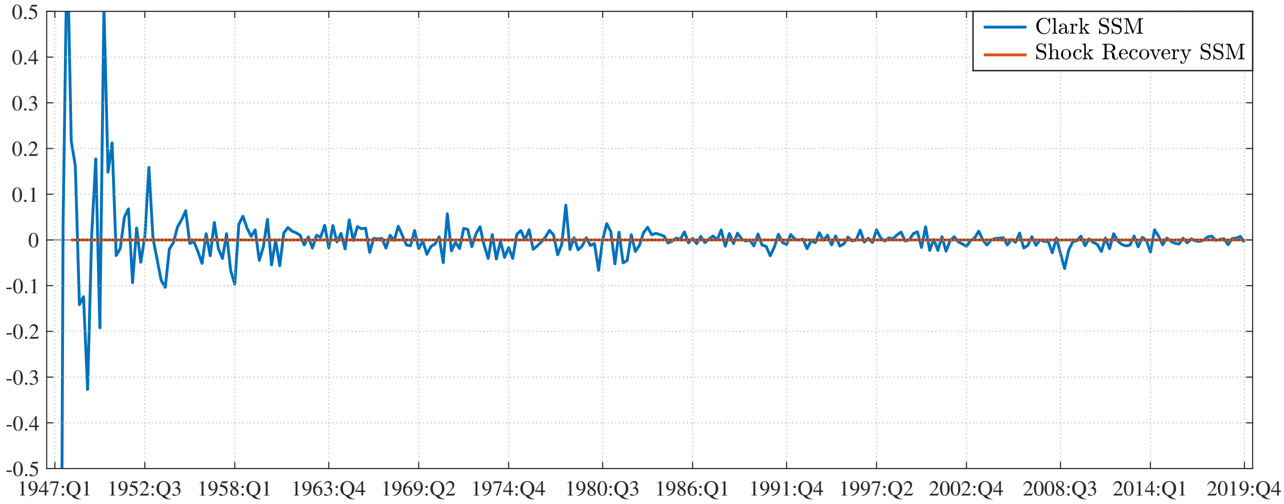
$$y_t = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_t^* \\ g_t \\ \tilde{y}_t \\ \tilde{y}_{t-1} \end{bmatrix} + 0\varepsilon_t \tag{12}$$

$$\begin{bmatrix} y_t^* \\ g_t \\ \tilde{y}_t \\ \tilde{y}_{t-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1}^* \\ g_{t-1} \\ \tilde{y}_{t-1} \\ \tilde{y}_{t-2} \end{bmatrix} + \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}. \tag{13}$$

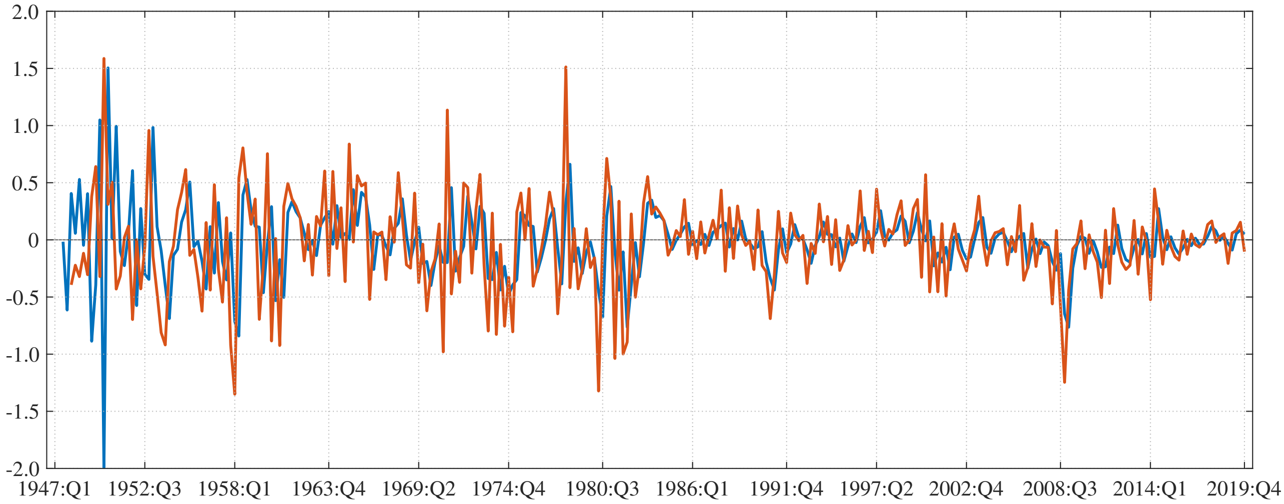
The code estimates Clark's 87 model on US GDP data from 1947:Q1 to 2019:Q4. A plot of the smoothed and filtered estimates is shown below.



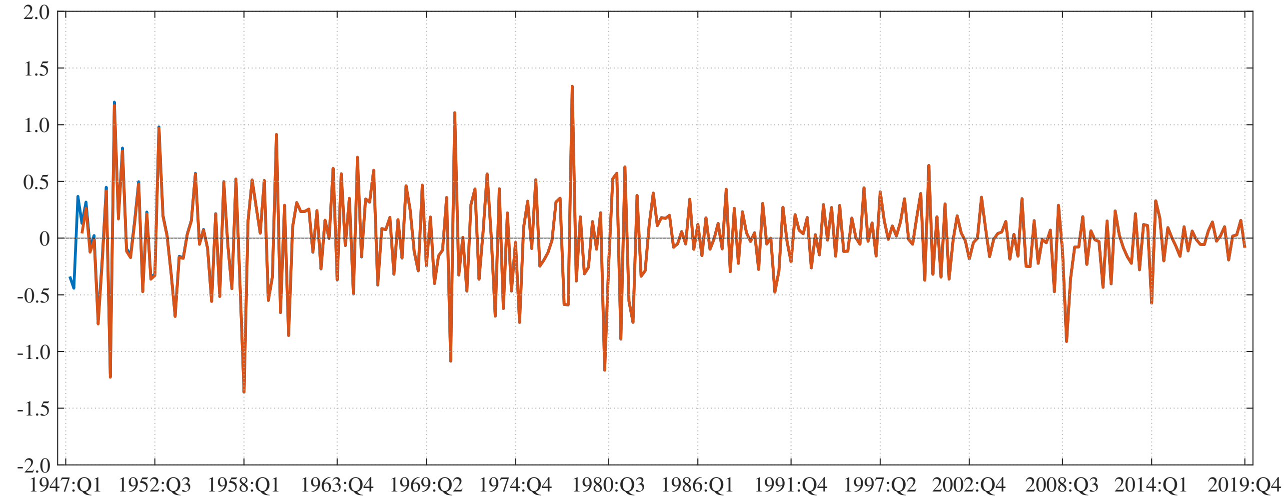
Filtered $\eta_1 = \sigma_1 \varepsilon_1$



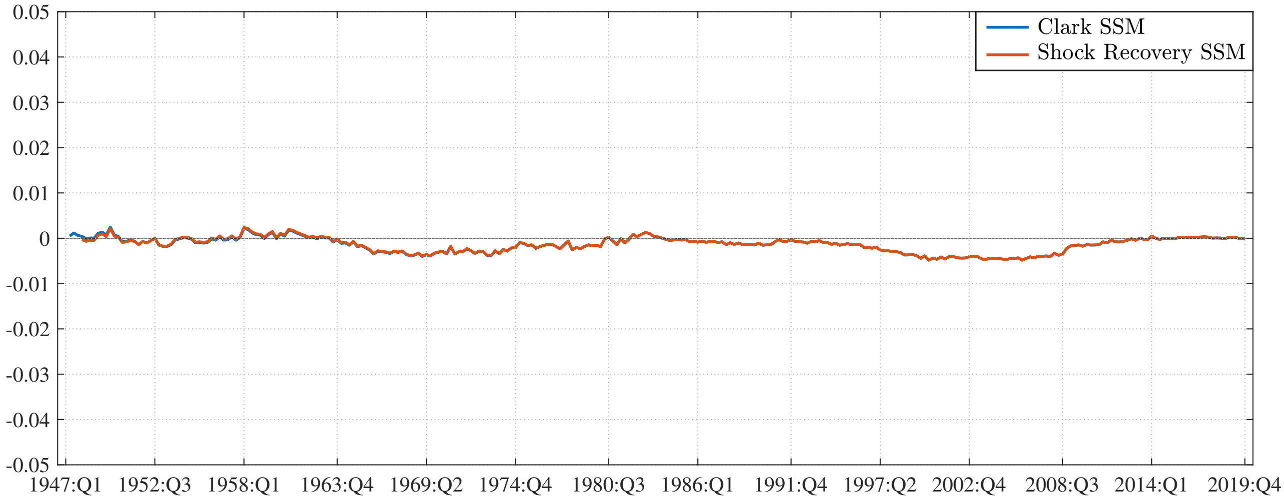
Filtered $\eta_2 = \sigma_2 \varepsilon_2$



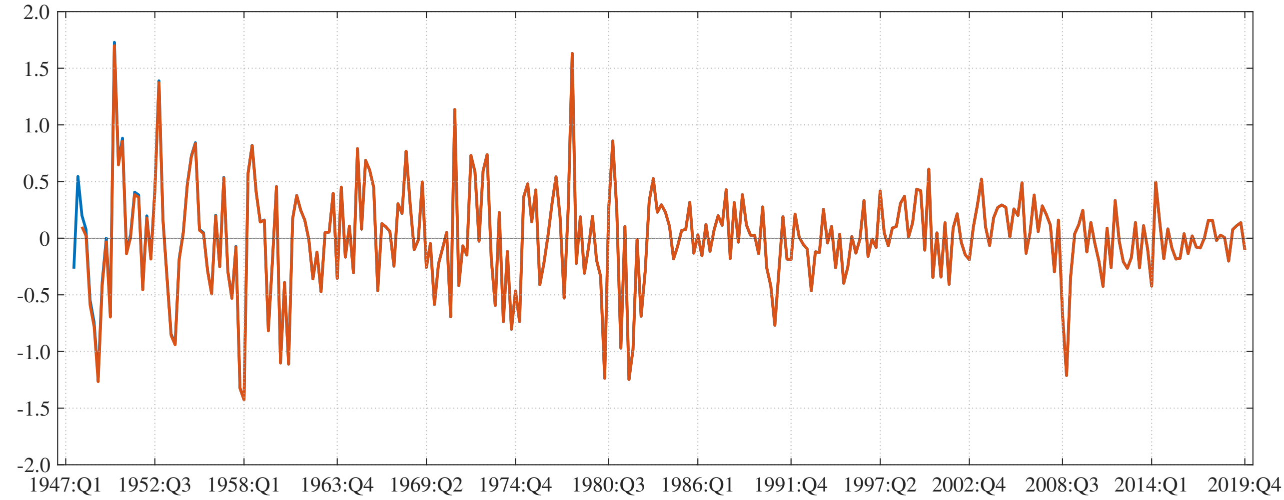
Filtered $\eta_3 = \sigma_3 \varepsilon_3$



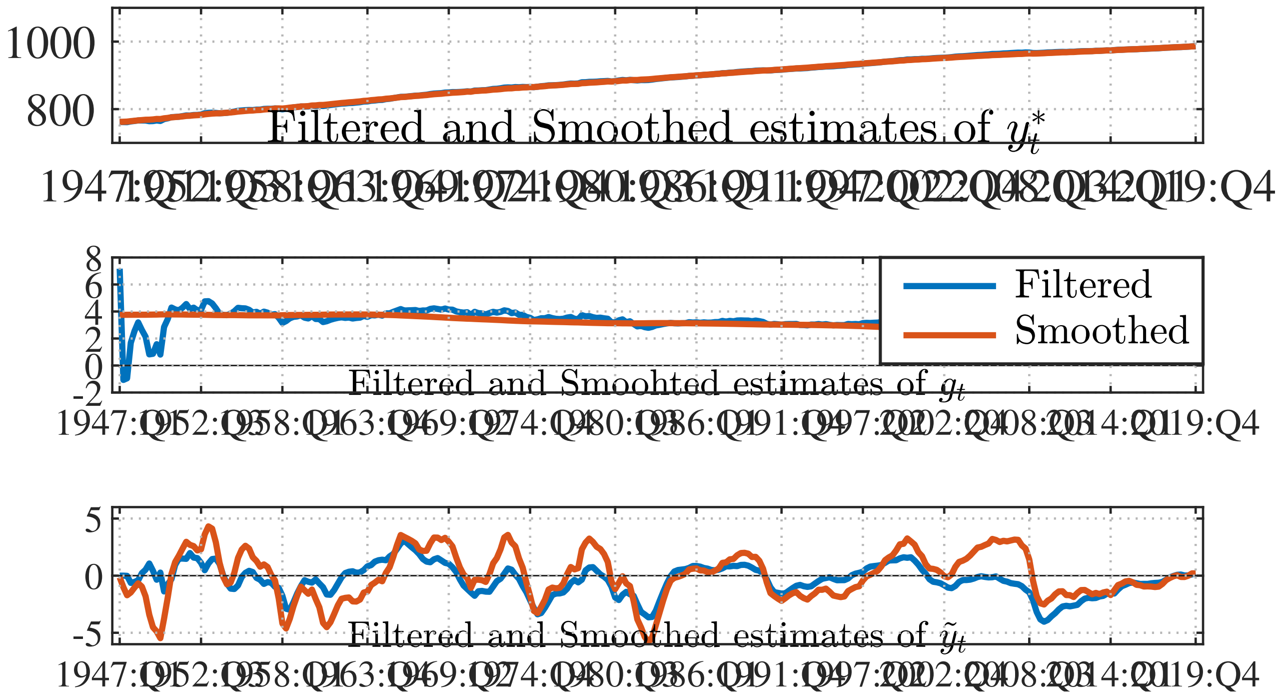
Smoothed $\eta_1 = \sigma_1 \varepsilon_1$



Smoothed $\eta_2 = \sigma_2 \varepsilon_2$



Smoothed $\eta_3 = \sigma_3 \varepsilon_3$


Filtered and Smoothed estimates of y_t^*
Filtered
Smoothed
Filtered and Smoohted estimates of g_t
Filtered and Smoothed estimates of $\tilde{y}_t$