1 Clark87

The Clark (1987) Unobserved Component (UC) model is a generalisation of the HP–Filter (a local linear trend model) that can be expressed in State Space Form (SSF) as:

$$\begin{aligned} y_t &= y_t^* + \tilde{y}_t && \text{(1a)} \\ \Delta y_t^* &= g_{t-1} + \sigma_1 \varepsilon_{1t} && \text{(1b)} \\ \Delta g_t &= \sigma_2 \varepsilon_{2t} && \text{(1c)} \\ a(L)\tilde{y}_t &= \sigma_3 \varepsilon_{3t}, && \text{(1d)} \end{aligned}$$

where y_t is (100 times) the log of GDP, and the shocks $\{\varepsilon_{it}\}_{i=1}^3$ are mutually uncorrelated $i.i.d.$ $N(0,1)$, with standard deviations $\{\sigma_i\}_{i=1}^3$, and $a(L)$ is a lag polynomial commonly assumed to be a stable AR(2), ie., $a(L) = (1 - a_1 L - a_2 L^2)$, with the roots of $a(L)$ being outside the unit circle. The cycle $\tilde{y}_t$ is allowed to be serially correlated. There are 3 shocks in the model.

The '*numbered shock*' to '*named shock*' mapping is:

$$\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix} = \begin{bmatrix} \varepsilon_t^{y^*} \\ \varepsilon_t^{g} \\ \varepsilon_t^{\tilde{y}} \end{bmatrix}, \tag{2}$$

where $\varepsilon_t^{y^*}$, ε_t^{g}, and $\varepsilon_t^{\tilde{y}}$ are the trend (permanent), trend growth and cycle shocks, respectively.

2 Shock recovery

2.1 State Space Models with lagged states

Kurz's (2018) SSM has the following general from:

$$\begin{aligned} \text{Measurement}: \quad Z_t &= D_1 X_t + D_2 X_{t-1} + R\varepsilon_t && \text{(3a)} \\ \text{State}: \quad X_t &= A X_{t-1} + C\varepsilon_t, && \text{(3b)} \end{aligned}$$

where $\varepsilon_t \sim MN(0_m, I_m)$, D_1, D_2, A, R are C are conformable system matrices, Z_t the observed variable and X_t the latent state variable, and m is the number of shocks $\{\varepsilon_{it}\}_{i=1}^m$.

2.2 Clark87 in '*shock recovery*' SSF

To assess shock recovery, write the model in (1) in '*shock recovery*' SSF by collecting all observable variables in Z_t and all shocks (and other latent state variables) in X_t. Differencing y_t and y_t^c twice, and re-arranging the relations in (1) then yields:

$$\begin{aligned} \Delta^2 y_t &= \Delta^2 y_t^* + \Delta^2 \tilde{y}_t \\ &= \sigma_1 \Delta\varepsilon_{1t} + \Delta g_{t-1} + \Delta^2 \tilde{y}_t \end{aligned}$$

$$
\begin{aligned}
&= \sigma_1 \Delta\varepsilon_{1t} + \sigma_2 \varepsilon_{2t-1} + a(L)^{-1}\sigma_3 \Delta^2 \varepsilon_{3t} \\
\Leftrightarrow a(L)\Delta^2 y_t &= \sigma_1 a(L)\Delta\varepsilon_{1t} + \sigma_2 a(L)\varepsilon_{2t-1} + \sigma_3 \Delta^2 \varepsilon_{3t}
\end{aligned} \tag{4}
$$

where $a(L)\Delta^2 y_t$ is the only observed variable. Re-writing (4) in more convenient form for the SSF yields:

$$
\begin{aligned}
\underbrace{a(L)\Delta^2 y_t}_{Z_t} &= a(L)\sigma_1 \Delta\varepsilon_{1t} + a(L)\sigma_2 \varepsilon_{2t-1} + \sigma_3 \Delta^2 \varepsilon_{3t} \\
&= \sigma_1 \Delta\varepsilon_{1t} - a_1\sigma_1 \Delta\varepsilon_{1t-1} - a_2\sigma_1 \Delta\varepsilon_{1t-2} + \sigma_2 \varepsilon_{2t-1} - a_1\sigma_2 \varepsilon_{2t-2} \\
&- \ a_2\sigma_2 \varepsilon_{2t-3} + \sigma_3 \Delta\varepsilon_{3t} - \sigma_3 \Delta\varepsilon_{3t-1}.
\end{aligned} \tag{5}
$$

The Measurement and State equations of the *'shock recovery'* SSF corresponding to the relations in (5) are then given by:

$$
\text{Measurement}: \quad Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t \tag{6a}
$$

$$
Z_t = \underbrace{\begin{bmatrix} 0 & 0 & 0 & \sigma_1 & -a_1\sigma_1 & 0 & 0 & \sigma_3 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \Delta\varepsilon_{1t} \\ \Delta\varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{2t-2} \\ \Delta\varepsilon_{3t} \end{bmatrix}}_{X_t} \tag{6b}
$$

$$
+ \underbrace{\begin{bmatrix} 0 & \sigma_2 & 0 & 0 & -a_2\sigma_1 & -a_1\sigma_2 & -a_2\sigma_2 & -\sigma_3 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \Delta\varepsilon_{1t-1} \\ \Delta\varepsilon_{1t-2} \\ \varepsilon_{2t-2} \\ \varepsilon_{2t-3} \\ \Delta\varepsilon_{3t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}}_{\varepsilon_t}
$$

$$
\text{State}: \quad X_t = AX_{t-1} + C\varepsilon_t \tag{6c}
$$

$$
\underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \Delta\varepsilon_{1t} \\ \Delta\varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{2t-2} \\ \Delta\varepsilon_{3t} \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \Delta\varepsilon_{1t-1} \\ \Delta\varepsilon_{1t-2} \\ \varepsilon_{2t-2} \\ \varepsilon_{2t-3} \\ \Delta\varepsilon_{3t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}}_{\varepsilon_t}. \tag{6d}
$$

2.3 Shock recovery

The diagonal of the steady-state variance/covariance matrix of the smoothed and filtered states X_t denoted by $P^*_{t|T}$ and $P^*_{t|t}$, respectively, are:

$$\begin{array}{ccc} \hline \text{Shocks} & P^*_{t|T} & P^*_{t|t} \\ \hline \varepsilon_{1t} & 0.5469 & 0.5989 \\ \varepsilon_{2t} & 0.9870 & 1.0000 \\ \varepsilon_{3t} & 0.4661 & 0.5153 \\ \hline \end{array}, \tag{7}$$

indicating that the trend growth shock $\varepsilon_{2t} = \varepsilon_t^g$ cannot be recovered ($P^* \approx 1$), while the cycle shock $\varepsilon_{3t} = \varepsilon_t^{\tilde{y}}$ and the trend shock $\varepsilon_{1t} = \varepsilon_t^{y^*}$ have $P^* \approx 0.5$, suggesting that there are recovery difficulties. Note that $P^*_{t|t} = 1$ implies that Kalman filtered estimates of ε_{2t} are *exactly* zero for all t. This can be seen from the entries on the left hand side of (8) which shows filtered estimates of shocks (smoothed estimates are shown on the right hand side).

$$\begin{array}{rrrrrr} \hline & \text{Filtered} & & & \text{Smoothed} & \\ E_t\varepsilon_{1t} & E_t\varepsilon_{2t} & E_t\varepsilon_{3t} & E_T\varepsilon_{1t} & E_T\varepsilon_{2t} & E_T\varepsilon_{3t} \\ \hline -0.7088 & 0 & -0.7792 & 0.3098 & -0.0160 & 0.0344 \\ -0.8039 & 0 & -0.8837 & -0.5269 & 0.0043 & -0.5442 \\ -0.2554 & 0 & -0.2808 & -0.1336 & 0.0094 & 0.1118 \\ 0.0179 & 0 & 0.0196 & 0.6757 & -0.0166 & 0.4942 \\ -0.5488 & 0 & -0.6032 & -0.2535 & -0.0068 & -0.4492 \\ -0.3244 & 0 & -0.3566 & -0.3713 & 0.0075 & -0.2660 \\ 0.1304 & 0 & 0.1434 & 0.2703 & -0.0029 & 0.3115 \\ -0.2821 & 0 & -0.3101 & -0.6689 & 0.0228 & 0.0549 \\ \hline \end{array}. \tag{8}$$

In Figure 1, simulated states and corresponding Kalman smoothed estimates are plotted for the shocks of interest.

The correlation between the true (simulated) and estimated Kalman smoothed shocks can be analyzed by simply computing $\mathrm{Corr}(X_t, \hat{X}_{t|T})$, where $X_t = \begin{bmatrix} \varepsilon_{1t} & \varepsilon_{2t} & \varepsilon_{3t} \end{bmatrix}'$ and $\hat{X}_{t|T} = E_T X_t = E_T \begin{bmatrix} \varepsilon_{1t} & \varepsilon_{2t} & \varepsilon_{2t-1} \end{bmatrix}'$, which yields:

$$\begin{array}{cc} \hline \text{Shocks} & \mathrm{Corr}(X_t, \hat{X}_{t|T}) \\ \hline \varepsilon_{1t} & 0.6736 \\ \varepsilon_{2t} & 0.1184 \\ \varepsilon_{3t} & 0.7304 \\ \hline \end{array}. \tag{9}$$

From (9) one can see that the estimated ε_{1t} and ε_{3t} shocks are rather '*weakly*' correlated with the true values, 0.6736 and 0.7304, respectively. The estimated ε_{2t} shock is nearly uncorrelated (0.1184) with the true value. These facts can also be seen from Figure 1 below.

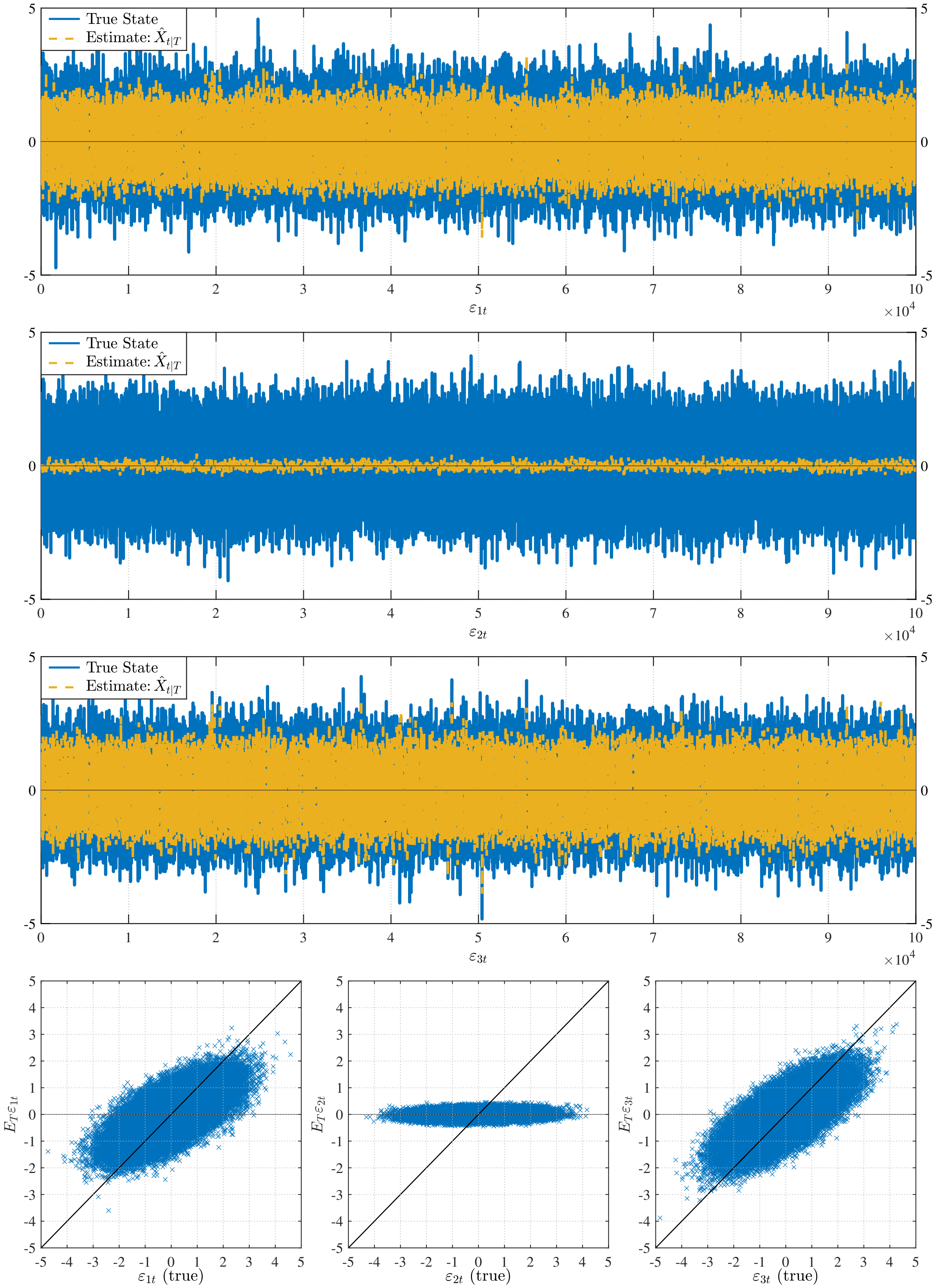


Figure 1: Comparison of true shocks and Kalman Smoothed estimates $\varepsilon_{t|T}$.

2.4 Shock Identities

As was the case for the HP-Filter, the Kalman filtered estimates of the trend and cycle shocks ε_{1t} and ε_{3t} are linked by the (filter) identity:

$$E_t\varepsilon_{1t} = 0.909694 E_t\varepsilon_{3t}. \tag{10}$$

Kalman smoothed estimates give the following *dynamic* identities:

$$\Delta^2 E_T\varepsilon_{2t} = -0.038486 \Delta E_T\varepsilon_{1t}, \tag{11}$$

and also:

$$\Delta^2 E_T\varepsilon_{3t} = -1.935746 \Delta E_T\varepsilon_{1t-1} + 1.659427 E_T\varepsilon_{3t-1} - 1.760937 E_T\varepsilon_{3t-2} \tag{12}$$

$$\Delta^2 E_T\varepsilon_{3t} = 50.297162 \Delta E_T\varepsilon_{2t-1} + 1.659427 E_T\varepsilon_{3t-1} - 1.760937 E_T\varepsilon_{3t-2}. \tag{13}$$

The contemporaneous correlations of the shocks from the Kalman filtered and smoothed estimates are, respectively:

$\mathrm{Corr}(\hat{X}_{t|t}, \hat{X}_{t|t})$

Shocks	ε_{1t}	ε_{2t}	ε_{3t}
ε_{1t}	1.0000	NaN	1.0000
ε_{2t}	NaN	NaN	NaN
ε_{3t}	1.0000	NaN	1.0000

and

$\mathrm{Corr}(\hat{X}_{t|T}, \hat{X}_{t|T})$

Shocks	ε_{1t}	ε_{2t}	ε_{3t}
ε_{1t}	1.0000	−0.1104	1.0000
ε_{2t}	−0.1104	1.0000	−0.1446
ε_{3t}	0.8403	−0.1446	1.0000

.

Note that the $\{\varepsilon_{it}\}_{i=1}^{3}$ were generated as mutually uncorrelated *i.i.d.* $N(0,1)$ processes, while their '*large sample*' estimates are correlated. The code `Clark87.m` replicates the output summarized here.

2.5 Maximum Likelihood estimation

The (standard) SSF for ML estimation for the model in (1) is:

$$y_t = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_t^* \\ g_t \\ \tilde{y}_t \\ \tilde{y}_{t-1} \end{bmatrix} + 0\varepsilon_t \tag{14}$$

$$\begin{bmatrix} y_t^* \\ g_t \\ \tilde{y}_t \\ \tilde{y}_{t-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1}^* \\ g_{t-1} \\ \tilde{y}_{t-1} \\ \tilde{y}_{t-2} \end{bmatrix} + \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}. \tag{15}$$

Below are estimates and plots of Clark's 87 model fitted to U.S. GDP data from 1947:Q2 to 2019:Q4.

	Fminunc	Stderr	InitVals
AR(1)	1.51023433	0.09768302	1.16620820
AR(2)	-0.56787952	0.10215079	-0.37268536
sigma_y*	0.54396738	0.09693249	1.00000000
sigma_g	-0.02093523	0.01124424	0.25796221
sigma_y~	0.59796738	0.10479322	0.77438264
Log-Like	-384.71939454	NaN	-446.20409194

Table 1: Clark (1987) model MLE parameter estimates for the U.S. from 1947:Q2 to 2019:Q4.

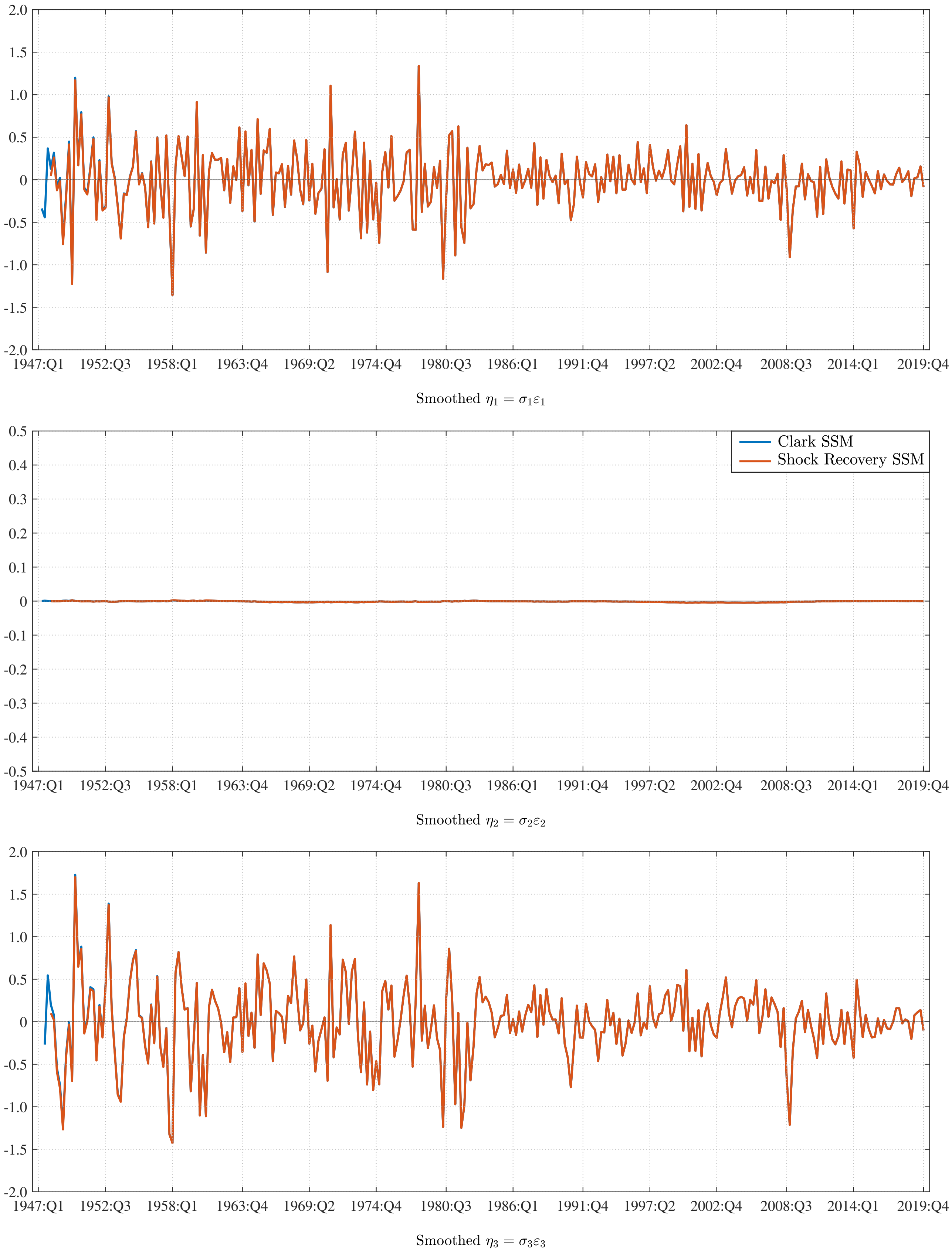
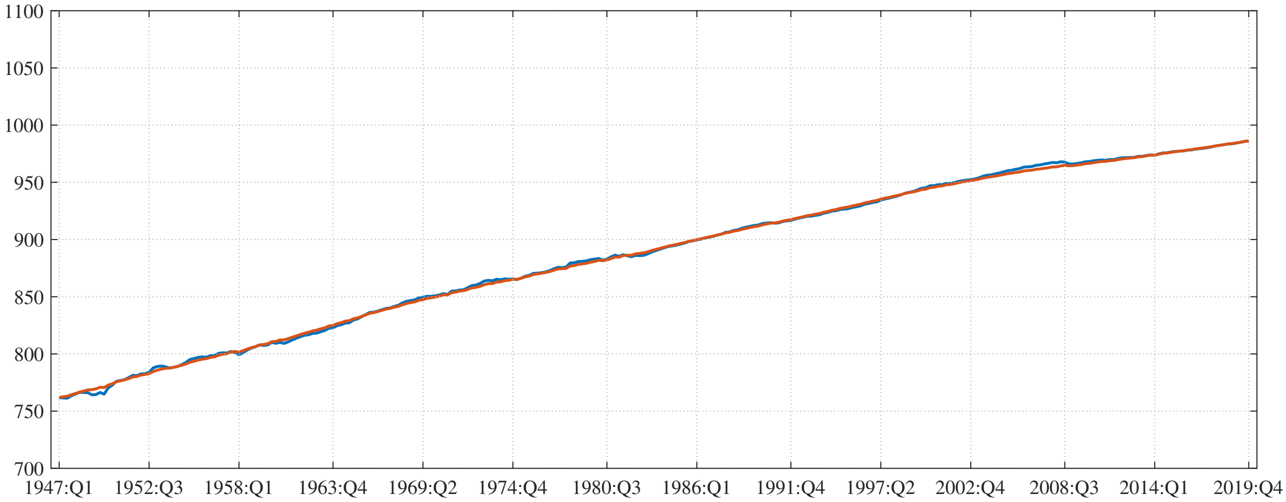
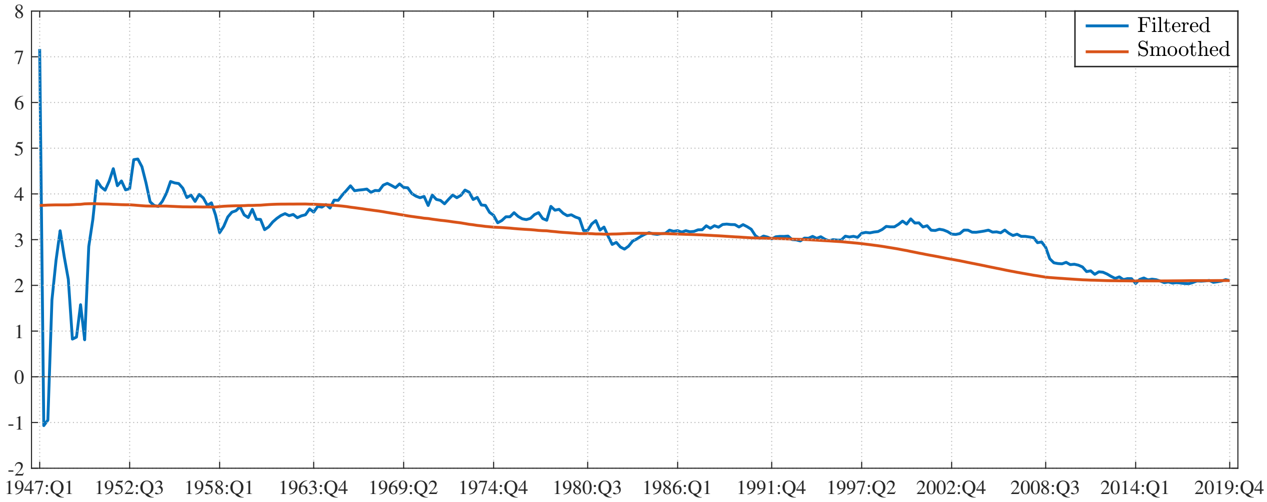


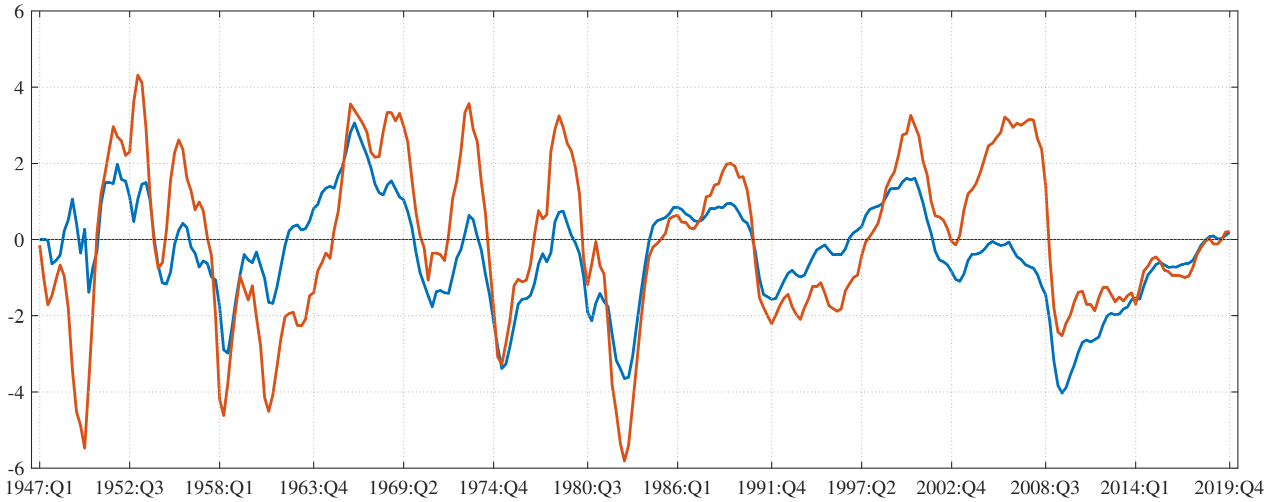
Figure 2: Smoother estimates of scaled shocks $\eta_{it} = \sigma_i \varepsilon_{it}, \forall i = 1, 2, 3$.



Filtered and Smoothed estimates of y_t^*



Filtered and Smoohted estimates of g_t



Filtered and Smoothed estimates of $\tilde{y}_t$

Figure 3: Filtered and Smoothed estimates trend, trend growth and cycle.