1 MR17

McCririck and Rees (2017, MR17) is effectively an extension of LW03's model, adding an equation for Okun's law. To keep these two models comparable, the notation and number labelling of shocks is kept as in LW03. Following the description of the model on page 16 in MR17 (Appendix A: Estimating the Model), the model takes the form:

$$\tilde{y}_t = a_{y,1}\tilde{y}_{t-1} + a_{y,2}\tilde{y}_{t-2} - \frac{a_r}{2}\sum_{i=1}^{2}(r_{t-i} - r^*_{t-i}) + \sigma_1\varepsilon_{1t} \tag{1}$$ MR1

$$\pi_t = (1-\beta_1)\pi^e_t + \frac{\beta_1}{3}\sum_{i=1}^{3}\pi_{t-i} + \beta_2(u_{t-1} - u^*_{t-1}) + \sigma_2\varepsilon_{2t} \tag{2}$$ MR2

$$\Delta z_t = \sigma_3\varepsilon_{3t}, \tag{3}$$ MR3

$$\Delta y^*_t = g_t + \sigma_4\varepsilon_{4t} \qquad \text{(MR17 use } g_t \text{ in paper, we use } g_{t-1} \text{ as in LW03 in SSM below)} \tag{4}$$ MR4

$$\Delta g_t = \sigma_5\varepsilon_{5t} \tag{5}$$ MR5

$$\Delta u^*_t = \sigma_6\varepsilon_{6t} \tag{6}$$ MR6

$$u_t = u^*_t + \beta(.4\tilde{y}_t + .3\tilde{y}_{t-1} + .2\tilde{y}_{t-2} + .1\tilde{y}_{t-3}) + \sigma_7\varepsilon_{7t} \tag{7}$$ MR7

where $\tilde{y}_t = (y_t - y^*_t)$ and $r^*_t = 4g_t + z_t$, and the numbered shock to named shock mapping is:

$$\begin{bmatrix}\varepsilon_{1t}\\ \varepsilon_{2t}\\ \varepsilon_{3t}\\ \varepsilon_{4t}\\ \varepsilon_{5t}\\ \varepsilon_{6t}\\ \varepsilon_{7t}\end{bmatrix} = \begin{bmatrix}\varepsilon^{\tilde{y}}_t\\ \varepsilon^{\pi}_t\\ \varepsilon^{z}_t\\ \varepsilon^{y^*}_t\\ \varepsilon^{g}_t\\ \varepsilon^{u^*}_t\\ \varepsilon^{u}_t\end{bmatrix}. \tag{8}$$

2 Shock recovery SSM

2.1 SSM with lagged states

SSM Kurz's (2018) SSM has the following general from:

$$\text{Measurement}: \quad Z_t = D_1X_t + D_2X_{t-1} + R\varepsilon_t \tag{9a}$$ ssm1

$$\text{State}: \quad X_t = AX_{t-1} + C\varepsilon_t, \tag{9b}$$ ssm2

where $\varepsilon_t \sim MN(0, I_m)$, D_1, D_2, A, R are C are conformable system matrices, Z_t the observed variable and X_t the latent state variable.

2.2 MR17 SSM for shock recovery

To assess recovery, re-write the model in '*shock recovery*' form. That is, collect all observables
ssm0 in Z_t, and all shocks (and other state variables) in state vector X_t.

$$
\begin{aligned}
\text{Measurement :} \quad Z_{1t} &= y_t^* - a_{y,1} y_{t-1}^* - a_{y,2} y_{t-2}^* - \frac{a_r}{2}\left(r_{t-1}^* + r_{t-2}^*\right) + \sigma_1 \varepsilon_{1t} && \text{(10a)}\\
Z_{2t} &= -\beta_2 u_{t-1}^* + \sigma_2 \varepsilon_{2t} && \text{(10b)}\\
Z_{3t} &= u_t^* - \beta(.4 y_t^* + .3 y_{t-1}^* + .2 y_{t-2}^* + .1 y_{t-3}^*) + \sigma_7 \varepsilon_{7t} && \text{(10c)}\\
\text{State :} \quad \Delta y_t^* &= g_{t-1} + \sigma_4 \varepsilon_{4t} && \text{(10d)}\\
\Delta g_t &= \sigma_5 \varepsilon_{5t} && \text{(10e)}\\
\Delta u_t^* &= \sigma_6 \varepsilon_{6t}, && \text{(10f)}\\
\Delta r_t^* &= 4\sigma_5 \varepsilon_{5t} + \sigma_3 \varepsilon_{3t}, && \text{(10g)}
\end{aligned}
$$
drstar2

ssm0 with the observables Z_t in the measurement equations defined as:

$$
\begin{aligned}
Z_{1t} &= y_t - \left(\sum_{i=1}^{2} (a_{y,i} y_{t-i}) - \frac{a_r}{2} \sum_{i=1}^{2} r_{t-i} \right) && \text{(11a)}\\
Z_{2t} &= \pi_t - \left((1-\beta_1)\pi_t^e + \frac{\beta_1}{3} \sum_{i=1}^{3} \pi_{t-i} + \beta_2 u_{t-1} \right) && \text{(11b)}\\
Z_{3t} &= u_t - \beta(.4 y_t + .3 y_{t-1} + .2 y_{t-2} + .1 y_{t-3}). && \text{(11c)}
\end{aligned}
$$

KOSSM The '*shock recovery*' SSF corresponding to (10) is then:

$$
\text{Measurement :} \quad Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t
$$

$$
\underbrace{\begin{bmatrix} Z_{1t} \\ Z_{2t} \\ Z_{3t} \end{bmatrix}}_{Z_t} = \underbrace{\begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} & 0 & 0 & 0 & 0 & \sigma_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_2 & 0 & 0 & 0 & 0 & 0 \\ -.4\beta & -.3\beta & -.2\beta & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_7 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ u_t^* \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \\ \varepsilon_{6t} \\ \varepsilon_{7t} \end{bmatrix}}_{X_t} \tag{12a}
$$

$$
+\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{a_r}{2} & -\frac{a_r}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\beta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -.1\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} y^*_{t-1} \\ y^*_{t-2} \\ y^*_{t-3} \\ g_{t-1} \\ r^*_{t-1} \\ r^*_{t-2} \\ u^*_{t-1} \\ \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{4t-1} \\ \varepsilon_{5t-1} \\ \varepsilon_{6t-1} \\ \varepsilon_{7t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\mathbf{0}_{2\times 5}}_{R} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \\ \varepsilon_{6t} \\ \varepsilon_{7t} \end{bmatrix}}_{\varepsilon_t} \tag{12b}
$$

State : $X_t = AX_{t-1} + C\varepsilon_t$,

$$
\underbrace{\begin{bmatrix} y^*_t \\ y^*_{t-1} \\ y^*_{t-2} \\ g_t \\ r^*_t \\ r^*_{t-1} \\ u^*_t \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \\ \varepsilon_{6t} \\ \varepsilon_{7t} \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y^*_{t-1} \\ y^*_{t-2} \\ y^*_{t-3} \\ g_{t-1} \\ r^*_{t-1} \\ r^*_{t-2} \\ u^*_{t-1} \\ \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{4t-1} \\ \varepsilon_{5t-1} \\ \varepsilon_{6t-1} \\ \varepsilon_{7t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & \sigma_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_5 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & 4\sigma_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_6 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \\ \varepsilon_{6t} \\ \varepsilon_{7t} \end{bmatrix}}_{\varepsilon_t} \tag{12c}
$$

2.2.1 Correlation between true change in natural rate and estimate

The correlation between the true and estimated Δr^*_t from the SSM can be constructed from the relation:

$$
\rho = 0.5\frac{\mathrm{Var}(\Delta r^*_t) + \mathrm{Var}(E_T \Delta r^*_t) - \phi}{\sigma(\Delta r^*_t)\sigma(E_T \Delta r^*_t)}, \tag{13}
$$

where $\text{Var}(\Delta r_t^*) = 4^2 c^2 \sigma_5^2 + \sigma_3^2$, $\sigma(\Delta r_t^*) = \sqrt{\text{Var}(\Delta r_t^*)}$, and $\text{Var}(E_T \Delta r_t^*)$ can be computed from simulating from the true model, applying the Kalman Filter and Smoother to get $E_T \Delta r_t^*$ and then computing the sample variance of $E_T \Delta r_t^*$ as an estimate of $\text{Var}(E_T \Delta r_t^*)$.

To obtain ϕ, add Δr_t^* to the state-vector X_t and augment the remaining matrices to be conformable. The required ϕ term is then the entry of $\text{diag}(P_{t|T}^*)$ that corresponds to Δr_t^*, which will be the very last element (see also LW03.pdf how this is done).