1 Kurz (2018) SSM with lagged states

SSM Kurz's SSM is:

$$\begin{aligned} \text{Observation}: \quad & Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t && \text{(1a)} \quad \texttt{ssm1} \\ \text{State}: \quad & X_t = A X_{t-1} + C\varepsilon_t, && \text{(1b)} \quad \texttt{ssm2} \end{aligned}$$

and $\varepsilon_t \sim MN(0, I_m)$, where R and C are covariance matrices.

1.1 LW (2003)

$$\begin{aligned} \alpha(L)\tilde{y}_t &= \alpha_r(L)(r_t - r_t^*) + \sigma_1\varepsilon_{1t} && (2) \quad \texttt{lw1} \\ B(L)\pi_t &= b_I(\pi_t^I - \pi_t) + b_o(\pi_{t-1}^o - \pi_{t-1}) + b_y(y_{t-1} - y_{t-1}^*) + \sigma_2\varepsilon_{2t} && (3) \quad \texttt{lw2} \\ \Delta z_t &= \sigma_3\varepsilon_{3t} && (4) \quad \texttt{lw3} \\ \Delta y_t^* &= g_{t-1} + \sigma_4\varepsilon_{4t} && (5) \quad \texttt{lw4} \\ \Delta g_t &= \sigma_5\varepsilon_{5t} && (6) \quad \texttt{lw5} \\ r_t^* &= c g_t + z_t && (7) \quad \texttt{lw6} \end{aligned}$$

where $\tilde{y}_t = (y_t - y_t^*)$.

Express LW model in SSF with all observables in Z_t and remaining latent states in the state vector X_t, where the observables Z_t are:

$$\begin{aligned} Z_{1t} &= y_t - \alpha_1 y_{t-1} - \alpha_2 y_{t-2} + \frac{a_r}{2}(r_{t-1} + r_{t-2}) \\ Z_{2t} &= B(L)\pi_t - b_I(\pi_t^I - \pi_t) - b_o(\pi_{t-1}^o - \pi_{t-1}) - b_y y_{t-1} \end{aligned}$$

Then the measurement equations is:

$$\begin{aligned} Z_{1t} &= y_t^* - \alpha_1 y_{t-1}^* - \alpha_2 y_{t-2}^* - \frac{a_r}{2}(r_{t-1}^* + r_{t-2}^*) + \sigma_1\varepsilon_{1t}, && (8) \quad \texttt{z1} \\ Z_{2t} &= b_y y_{t-1}^* + \sigma_2\varepsilon_{2t} && (9) \quad \texttt{z2} \end{aligned}$$

The state equation is:

$$\begin{aligned} \Delta y_t^* &= g_{t-1} + \sigma_4\varepsilon_{4t} && (10) \quad \texttt{x1} \\ \Delta g_t &= \sigma_5\varepsilon_{5t} && (11) \quad \texttt{x2} \\ \Delta r_t^* &= c\sigma_5\varepsilon_{5t} + \sigma_3\varepsilon_{3t}. && (12) \quad \texttt{x3} \end{aligned}$$

and state vector is $X_t = \begin{bmatrix} y_t^* & y_{t-1}^* & g_t & r_t^* & r_{t-1}^* & \varepsilon_t & \varepsilon_{2t} & \varepsilon_{3t} & \varepsilon_{4t} & \varepsilon_{5t} \end{bmatrix}'$.

Written out, the full SSM is:

$$\underbrace{\begin{bmatrix} Z_{1t} \\ Z_{2t} \end{bmatrix}}_{Z_t} = \underbrace{\begin{bmatrix} 1 & -\alpha_1 & 0 & 0 & -\frac{a_r}{2} & \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & b_y & 0 & 0 & 0 & 0 & \sigma_2 & 0 & 0 & 0 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{X_t} + \underbrace{\begin{bmatrix} 0 & -\alpha_2 & 0 & 0 & -\frac{a_r}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{4t-1} \\ \varepsilon_{5t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{\varepsilon_t}$$

$$\underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{4t-1} \\ \varepsilon_{5t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & \sigma_4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_5 \\ 0 & 0 & \sigma_3 & 0 & c\sigma_5 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{\varepsilon_t}$$

The parameter values from LW(2003) are:

$$\begin{aligned}
&\sigma_1 = 0.387,\ \sigma_2 = \ \ 0.731,\ \sigma_3 = 0.323,\ \sigma_4 = 0.605,\ \ \sigma_5 = 0.102 \\
&\alpha_1 = 1.51,\ \ \ \alpha_2 = -0.57,\ \ b_y = 0.043,\ \alpha_r = -0.098,\ c = 1.068.
\end{aligned}$$