1 SSMs with lagged states following Kurz (2018)

SSM Kurz's SSM takes the following genereal from:

$$
\begin{aligned}
\text{Measurement}: \quad & Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t && \text{(1a)} \quad \texttt{ssm1}\\
\text{State}: \quad & X_t = A X_{t-1} + C\varepsilon_t, && \text{(1b)} \quad \texttt{ssm2}
\end{aligned}
$$

where $\varepsilon_t \sim MN(0, I_m)$, D_1, D_2, A, R are C are conformable system matrices, Z_t the observed variable and X_t the latent state.

2 HP97

The HP-Filter of Hodrick and Prescott (1997) can be expressed as an SSM following a UC
HP0 model structure as:

$$
\begin{aligned}
y_t &= y_t^* + y_t^c && \text{(2a)} \quad \texttt{HP0a}\\
y_t^c &= \phi \varepsilon_t^c && \text{(2b)} \quad \texttt{HP0b}\\
\Delta^2 y_t^* &= \varepsilon_t^*, && \text{(2c)} \quad \texttt{HP0c}
\end{aligned}
$$

where y_t is (generally 100 times) the log of GDP, and ε_t^c and ε_t^* are $N(0,1)$. The standard deviation ϕ is the (square root of the) smoothing parameter, generally set to 40, implying a value of 'λ' of 1600.

To assess recovery, re-write the model in'*shock recovery*' state-space from (SSF). That is, collect all observables in Z_t and all shocks (and other state variables) in X_t. The relation in (2) can be written as:

$$
\begin{aligned}
\Delta^2 y_t &= \Delta^2 y_t^* + \Delta^2 y_t^c\\
&= \varepsilon_t^* + \phi \Delta^2 \varepsilon_t^c\\
Z_t &= \varepsilon_t^* + \phi \varepsilon_t^c - 2\phi \varepsilon_{t-1}^c + \phi \varepsilon_{t-2}^c, && \text{(3)} \quad \texttt{Z}
\end{aligned}
$$

where $Z_t = \Delta^2 y_t$ is the observed variable.

KOSSM The Measurement and State equations corresponding to (1) are:

$$
Z_t = \underbrace{\begin{bmatrix} 1 & \phi & -2\phi \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} \varepsilon_t^* \\ \varepsilon_t^c \\ \varepsilon_{t-1}^c \end{bmatrix}}_{X_t} + \underbrace{\begin{bmatrix} 0 & 0 & \phi \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} \varepsilon_{t-1}^* \\ \varepsilon_{t-1}^c \\ \varepsilon_{t-2}^c \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} \varepsilon_t^* \\ \varepsilon_t^c \end{bmatrix}}_{\varepsilon_t} \tag{4a}
$$

$$
\underbrace{\begin{bmatrix} \varepsilon_t^* \\ \varepsilon_t^c \\ \varepsilon_{t-1}^c \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \varepsilon_t^* \\ \varepsilon_t^c \\ \varepsilon_{t-1}^c \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \varepsilon_t^* \\ \varepsilon_t^c \end{bmatrix}}_{\varepsilon_t} \tag{4b}
$$

The Kalman Filter and Smoother based steady-state $\mathrm{diag}(P)$ are, respectively:

$$\mathrm{diag}(P^*_{t|t}) = \begin{bmatrix} 0.9995 \\ 0.2006 \\ 0.1608 \end{bmatrix} \text{ and } \mathrm{diag}(P^*_{t|T}) = \begin{bmatrix} 0.9995 \\ 0.2006 \\ 0.1608 \end{bmatrix}. \tag{5}$$