1 MR17 with Monetary policy rule

Adding a Monetary Policy (MP) rule to McCririck and Rees (2017, MR17) yields to:

$$
\begin{aligned}
\tilde{y}_t &= a_{y,1}\tilde{y}_{t-1} + a_{y,2}\tilde{y}_{t-2} - \frac{a_r}{2}\sum_{i=1}^{2}(r_{t-i} - r^*_{t-i}) + \sigma_1\varepsilon_{1t} && (1) \quad \texttt{MR1} \\
\pi_t &= (1-\beta_1)\pi^e_t + \frac{\beta_1}{3}\sum_{i=1}^{3}\pi_{t-i} + \beta_2(u_{t-1} - u^*_{t-1}) + \sigma_2\varepsilon_{2t} && (2) \quad \texttt{MR2} \\
\Delta z_t &= \sigma_3\varepsilon_{3t}, && (3) \quad \texttt{MR3} \\
\Delta y^*_t &= g_t + \sigma_4\varepsilon_{4t} \qquad \text{(MR17 use } g_t \text{ in paper, we use } g_{t-1} \text{ as in LW03 in SSM below)} && (4) \quad \texttt{MR4} \\
\Delta g_t &= \sigma_5\varepsilon_{5t} && (5) \quad \texttt{MR5} \\
\Delta u^*_t &= \sigma_6\varepsilon_{6t} && (6) \quad \texttt{MR6} \\
u_t &= u^*_t + \beta(.4\tilde{y}_t + .3\tilde{y}_{t-1} + .2\tilde{y}_{t-2} + .1\tilde{y}_{t-3}) + \sigma_7\varepsilon_{7t} && (7) \quad \texttt{MR7} \\
r_t &= (1-b_1)(r_{t-1} - \Delta\pi_t) + b_1\left[r^*_t - \bar{\pi} - 2(u_t - u^*_t)\right] - \Delta_2 u_t + \sigma_8\varepsilon_{8t}, && (8) \quad \texttt{MR8}
\end{aligned}
$$

where $\tilde{y}_t = (y_t - y^*_t)$ and $r^*_t = 4g_t + z_t$ as before, and (8) is obtained from the MARTIN policy model of the Reserve Bank of Australia (Ballantyne *et al.*, 2020), with nominal rule:

$$
i_t = (1-b_1)i_{t-1} + b_1(r^*_t + 2\pi_t - \bar{\pi} - 2(u_t - u^*_t)) - \Delta_2 u_t + \sigma_8\varepsilon_{8t}, \qquad (9) \quad \texttt{eq:NR}
$$

where $\bar{\pi}$ denotes the inflation target (there is a 2 missing in front of π_t in our equation see below eq. 36 from Ballantyne *et al.*, 2020). The parameters in (8) and (9) are: $b_1 = .3$ and $\sigma_8 = 1.19$ (where is 1.19 taken from???). These are from the equation below (page 237):

$$
\begin{aligned}
NCR_t = 0.7 \times NCR_{t-1} + 0.3 \\
\times \left[RSTAR_t + 2 \times \left(\frac{PTM_t}{PTM_{t-4}} \times 100 - 100 \right) \right. \\
\left. -\bar{\pi} - 2 \times LURGAP_t \right] - \Delta_2 LUR_t + \varepsilon_{ncr,t},
\end{aligned}
$$

TIM: Why use $r_t = i_t + \pi_t$ and not $r_t = i_t + \pi^e_t$, $\Rightarrow$ inconsistent within same model? Also not sure if the equations are correct... but for shock recovery, all that matters is r^*_t and u^*_t

The numbered shock to named shock mapping is:

$$
\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \\ \varepsilon_{6t} \\ \varepsilon_{7t} \\ \varepsilon_{8t} \end{bmatrix} = \begin{bmatrix} \varepsilon^{\tilde{y}}_t \\ \varepsilon^{\pi}_t \\ \varepsilon^{z}_t \\ \varepsilon^{y^*}_t \\ \varepsilon^{g}_t \\ \varepsilon^{u^*}_t \\ \varepsilon^{u}_t \\ \varepsilon^{r}_t \end{bmatrix}. \qquad (10)
$$

2 Shock recovery SSM

2.1 SSM with lagged states

SSM Kurz's (2018) SSM has the following general from:

$$\text{Measurement}: \quad Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t \tag{11a}$$ ssm1

$$\text{State}: \quad X_t = AX_{t-1} + C\varepsilon_t, \tag{11b}$$ ssm2

where $\varepsilon_t \sim MN(0, I_m)$, D_1, D_2, A, R are C are conformable system matrices, Z_t the observed variable and X_t the latent state variable.

2.2 MR17 SSM for shock recovery

To assess recovery, re-write the model in '*shock recovery*' form. That is, collect all observables
ssm0 in Z_t, and all shocks (and other state variables) in state vector X_t.

$$\text{Measurement}: \quad Z_{1t} = y_t^* - a_{y,1}y_{t-1}^* - a_{y,2}y_{t-2}^* - \frac{a_r}{2}\left(r_{t-1}^* + r_{t-2}^*\right) + \sigma_1\varepsilon_{1t} \tag{12a}$$

$$Z_{2t} = -\beta_2 u_{t-1}^* + \sigma_2\varepsilon_{2t} \tag{12b}$$

$$Z_{3t} = u_t^* - \beta(.4y_t^* + .3y_{t-1}^* + .2y_{t-2}^* + .1y_{t-3}^*) + \sigma_7\varepsilon_{7t} \tag{12c}$$

$$Z_{4t} = b_1 r_t^* + 2b_1 u_t^* + \sigma_8\varepsilon_{7t} \tag{12d}$$

$$\text{State}: \quad \Delta y_t^* = g_{t-1} + \sigma_4\varepsilon_{4t} \tag{12e}$$

$$\Delta g_t = \sigma_5\varepsilon_{5t} \tag{12f}$$

$$\Delta u_t^* = \sigma_6\varepsilon_{6t}, \tag{12g}$$ drstar2

$$\Delta r_t^* = 4\sigma_5\varepsilon_{5t} + \sigma_3\varepsilon_{3t}, \tag{12h}$$

ssm0 with the observables Z_t in the measurement equations defined as:

$$Z_{1t} = y_t - \left(\sum_{i=1}^{2}(a_{y,i}y_{t-i}) - \frac{a_r}{2}\sum_{i=1}^{2} r_{t-i}\right) \tag{13a}$$

$$Z_{2t} = \pi_t - \left((1-\beta_1)\pi_t^e + \frac{\beta_1}{3}\sum_{i=1}^{3}\pi_{t-i} + \beta_2 u_{t-1}\right) \tag{13b}$$

$$Z_{3t} = u_t - \beta(.4y_t + .3y_{t-1} + .2y_{t-2} + .1y_{t-3}) \tag{13c}$$

$$Z_{4t} = r_t - (1-b_1)(r_{t-1} - \Delta\pi_t) - b_1\bar{\pi} + \Delta_2 u_t + 2b_1 u_t \tag{13d}$$

KOSSM The '*shock recovery*' SSF corresponding to (12) is then:

Measurement : $\quad Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t$

$$\underbrace{\begin{bmatrix} Z_{1t} \\ Z_{2t} \\ Z_{3t} \\ Z_{4t} \end{bmatrix}}_{Z_t} = \underbrace{\begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} & 0 & 0 & 0 & 0 & \sigma_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -.4\beta & -.3\beta & -.2\beta & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_7 & 0 \\ 0 & 0 & 0 & 0 & b_1 & 0 & 2b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_8 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ u_t^* \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \\ \varepsilon_{6t} \\ \varepsilon_{7t} \\ \varepsilon_{8t} \end{bmatrix}}_{X_t} \tag{14a}$$

$$+ \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{a_r}{2} & -\frac{a_r}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\beta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -.1\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ y_{t-3}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ u_{t-1}^* \\ \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{4t-1} \\ \varepsilon_{5t-1} \\ \varepsilon_{6t-1} \\ \varepsilon_{7t-1} \\ \varepsilon_{8t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\mathbf{0}_{2\times 5}}_{R} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \\ \varepsilon_{6t} \\ \varepsilon_{7t} \\ \varepsilon_{8t} \end{bmatrix}}_{\varepsilon_t} \tag{14b}$$

State : $\quad X_t = AX_{t-1} + C\varepsilon_t,$

$$
\underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ u_t^* \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \\ \varepsilon_{6t} \\ \varepsilon_{7t} \\ \varepsilon_{8t} \end{bmatrix}}_{X_t}
=
\underbrace{\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}}_{A}
\underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ y_{t-3}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ u_{t-1}^* \\ \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{4t-1} \\ \varepsilon_{5t-1} \\ \varepsilon_{6t-1} \\ \varepsilon_{7t-1} \\ \varepsilon_{8t-1} \end{bmatrix}}_{X_{t-1}}
+
\underbrace{\begin{bmatrix}
0 & 0 & 0 & \sigma_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sigma_5 & 0 & 0 & 0 \\
0 & 0 & \sigma_3 & 0 & 4\sigma_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sigma_6 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}}_{C}
\underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \\ \varepsilon_{6t} \\ \varepsilon_{7t} \\ \varepsilon_{8t} \end{bmatrix}}_{\varepsilon_t}
\tag{14c}
$$

2.2.1 Correlation between true change in natural rate and estimate

The correlation between the true and estimated Δr_t^* from the SSM can be constructed from the relation:

$$
\rho = 0.5 \frac{\mathrm{Var}(\Delta r_t^*) + \mathrm{Var}(E_T \Delta r_t^*) - \phi}{\sigma(\Delta r_t^*)\sigma(E_T \Delta r_t^*)}, \tag{15}
$$

where $\mathrm{Var}(\Delta r_t^*) = 4^2 c^2 \sigma_5^2 + \sigma_3^2$, $\sigma(\Delta r_t^*) = \sqrt{\mathrm{Var}(\Delta r_t^*)}$, and $\mathrm{Var}(E_T \Delta r_t^*)$ can be computed from simulating from the true model, applying the Kalman Filter and Smoother to get $E_T \Delta r_t^*$ and then computing the sample variance of $E_T \Delta r_t^*$ as an estimate of $\mathrm{Var}(E_T \Delta r_t^*)$.

To obtain ϕ, add Δr_t^* to the state-vector X_t and augment the remaining matrices to be conformable. The required ϕ term is then the entry of $\mathrm{diag}(P_{t|T}^*)$ that corresponds to Δr_t^*, which will be the very last element (see also LW03.pdf how this is done).

2.3 Check i_t to r_t conversion

Ballantyne et al., 2020 equation 36 is something like

$$i_t = Ai_{t-1} + b_1\left[r_t^* + 2\pi_t - \bar{\pi} - 2\left(u_t - u_t^*\right)\right] - \Delta_2 u_t + \sigma_8\varepsilon_t$$

where $A = (1 - b_1)$. With $i_t = r_t + \pi_t$ (not inflation expectations as in the model), we get:

$$\begin{aligned}
i_t &= Ai_{t-1} + b_1 2\pi_t + \underbrace{b_1\left[r_t^* - \bar{\pi} - 2\left(u_t - u_t^*\right)\right] - \Delta_2 u_t + \sigma_8\varepsilon_t}_{OT = \text{ other terms}} \\
&= Ai_{t-1} + b_1 2\pi_t + OT \\
\Leftrightarrow (r_t + \pi_t) &= A(r_{t-1} + \pi_{t-1}) + 2b_1\pi_t + OT \\
r_t &= Ar_{t-1} + A\pi_{t-1} + 2b_1\pi_t - \pi_t + OT \\
&= Ar_{t-1} + (b_1 - 1)\pi_t + A\pi_{t-1} + b_1\pi_t + OT \\
&= Ar_{t-1} - (1 - b_1)\pi_t + A\pi_{t-1} + b_1\pi_t + OT \\
&= Ar_{t-1} - A\pi_t + A\pi_{t-1} + b_1\pi_t + OT \\
&= Ar_{t-1} - A\Delta\pi_t + b_1\pi_t + OT \\
&= A(r_{t-1} - \Delta\pi_t) + b_1\pi_t + b_1\left[r_t^* - \bar{\pi} - 2\left(u_t - u_t^*\right)\right] - \Delta_2 u_t + \sigma_8\varepsilon_t \\
&= A(r_{t-1} - \Delta\pi_t) + b_1\left[r_t^* + (\pi_t - \bar{\pi}) - 2\left(u_t - u_t^*\right)\right] - \Delta_2 u_t + \sigma_8\varepsilon_t \\
r_t &= (1 - b_1)(r_{t-1} - \Delta\pi_t) + b_1\left[r_t^* + (\pi_t - \bar{\pi}) - 2\left(u_t - u_t^*\right)\right] - \Delta_2 u_t + \sigma_8\varepsilon_t \qquad (16)
\end{aligned}$$

$\Rightarrow$ If the additional 2 in front of $\left(\frac{PTM_t}{PTM_{t-4}} \times 100 - 100\right)$ is intentional.