

1 HLW's (2023) post COVID19 State-Space Model (SSM) Form

HLW23 (post COVID19) use the same standard State-Space Form (SSF) as in HLW17:

$$\begin{aligned}\text{Measurement : } \mathbf{y}_t &= \mathbf{A}\mathbf{x}_t + \mathbf{H}\boldsymbol{\xi}_t + \mathbf{R}_t^{1/2}\boldsymbol{\varepsilon}_t^y \\ \text{State : } \boldsymbol{\xi}_t &= \mathbf{F}\boldsymbol{\xi}_{t-1} + \mathbf{Q}^{1/2}\boldsymbol{\varepsilon}_t^\xi\end{aligned}$$

but where the states and exogenous variables have been modified:

$$\begin{aligned}\mathbf{y}_t &= \begin{bmatrix} y_t & \pi_t \end{bmatrix}', \\ \mathbf{x}_t &= \begin{bmatrix} y_{t-1} & y_{t-2} & r_{t-1} & r_{t-2} & \pi_{t-1} & \pi_{t-2,4} & d_t & d_{t-1} & d_{t-2} \end{bmatrix}', \\ \boldsymbol{\xi}_t &= \begin{bmatrix} y_t^* & y_{t-1}^* & y_{t-2}^* & g_t & g_{t-1} & g_{t-2} & z_t & z_{t-1} & z_{t-2} \end{bmatrix}', \\ \mathbf{A} &= \begin{bmatrix} a_{y,1} & a_{y,2} & \frac{a_r}{2} & \frac{a_r}{2} & 0 & 0 & \phi & -\phi a_{y,1} & -\phi a_{y,2} \\ b_y & 0 & 0 & 0 & b_\pi & 1-b_\pi & 0 & -\phi & 0 \end{bmatrix}', \\ \mathbf{H} &= \begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} & 0 & -4c\frac{a_r}{2} & -4c\frac{a_r}{2} & 0 & -\frac{a_r}{2} & -\frac{a_r}{2} \\ 0 & -b_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.\end{aligned}$$

to incorporate dummy variables in the measurement equation and a time-varying R_t covariance matrix, and some correction I pointed out in the formulation of the state equation (see pages 7 – 8 in ‘HLW_Replication_Code_Guide.pdf’ and pages 43 – 44 in ‘Measuring the Natural Rate of Interest after COVID-19’.

Note: This follows the notation used in the documentation file: ‘HLW_Replication_Code_Guide.pdf’ included in the zip file: ‘HLW_2023_Replication_Code.zip’ that contains the replication code of HLW23 and which is available from the NYFED website at: https://www.newyorkfed.org/medialibrary/media/research/economists/williams/data/HLW_Code.zip.

Compared to HLW17, the constant c is estimated again and dummies are added to account for the COVID19 years.

In the construction of r_t^* , trend growth g_t is again annualized, but not in the state equations for g_t . That is, in the matrices above, the entries in $\mathbf{H}(1, 4 : 5)$ corresponding to trend growth are multiplied by 4 (as well as c) in the code that performs the estimation (see `unpack.parameters.stage3.R` in `HLW_replication.zip` files, line 30, which reads):

```
H[1, 5:6] <- -parameters[param.num["c"]]*parameters[param.num["a_r"]]*2.
```

The standard deviations of the shocks are denoted by $\begin{bmatrix} \sigma_{\tilde{y}} & \sigma_\pi & \sigma_{y^*} & \sigma_g & \sigma_z \end{bmatrix}'$ in the documentation in ‘HLW_Code_Guide.pdf’.

1.1 SSM of HLW23

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The measurement equation is:

$$\mathbf{y}_t = \mathbf{A}\mathbf{x}_t + \mathbf{H}\xi_t + \mathbf{R}_t^{1/2}\boldsymbol{\varepsilon}_t^y$$

$$\underbrace{\begin{bmatrix} y_t \\ \pi_t \end{bmatrix}}_{\mathbf{y}_t} = \underbrace{\begin{bmatrix} a_{y,1} & a_{y,2} & \frac{a_r}{2} & \frac{a_r}{2} & 0 & 0 & \phi & -\phi a_{y,1} & -\phi a_{y,2} \\ b_y & 0 & 0 & 0 & b_\pi & 1-b_\pi & 0 & -\phi b_y & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} y_{t-1} \\ y_{t-2} \\ r_{t-1} \\ r_{t-2} \\ \pi_{t-1} \\ \pi_{t-2,4} \\ d_t \\ d_{t-1} \\ d_{t-2} \end{bmatrix}}_{\mathbf{x}_t} \quad (1a)$$

$$+ \underbrace{\begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} & 0 & -4c\frac{a_r}{2} & -4c\frac{a_r}{2} & 0 & -\frac{a_r}{2} & -\frac{a_r}{2} \\ 0 & -b_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_t \\ g_{t-1} \\ g_{t-2} \\ z_t \\ z_{t-1} \\ z_{t-2} \end{bmatrix}}_{\boldsymbol{\xi}_t} + \underbrace{\begin{bmatrix} \kappa_t \sigma_{\tilde{y}} & 0 \\ 0 & \kappa_t \sigma_\pi \end{bmatrix}}_{\mathbf{R}_t^{1/2}} \underbrace{\begin{bmatrix} \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^\pi \end{bmatrix}}_{\boldsymbol{\varepsilon}_t^y}. \quad (1b)$$

The state equation is:

$$\boldsymbol{\xi}_t = \mathbf{F}\boldsymbol{\xi}_{t-1} + \mathbf{Q}^{1/2}\boldsymbol{\varepsilon}_t^\xi$$

$$\underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_t \\ g_{t-1} \\ g_{t-2} \\ z_t \\ z_{t-1} \\ z_{t-2} \end{bmatrix}}_{\boldsymbol{\xi}_t} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{F}} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ y_{t-3}^* \\ g_{t-1} \\ g_{t-2} \\ g_{t-3} \\ z_{t-1} \\ z_{t-2} \\ z_{t-3} \end{bmatrix}}_{\boldsymbol{\xi}_{t-1}} + \underbrace{\begin{bmatrix} \sigma_{y^*} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sigma_g & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{Q}^{1/2}} \underbrace{\begin{bmatrix} \varepsilon_t^{y^*} \\ \varepsilon_t^g \\ \varepsilon_t^z \end{bmatrix}}_{\boldsymbol{\varepsilon}_t^\xi} \quad (2) \text{ state1}$$

Expanding the relations in (1) and (2) and re-arranging yields:

$$\begin{bmatrix} y_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} y_t^* + \phi d_t + a_{y,1} \overbrace{(y_{t-1} - y_{t-1}^* - \phi d_{t-1})}^{\tilde{y}_{t-1,COVID}} + a_{y,2} \overbrace{(y_{t-2} - y_{t-2}^* - \phi d_{t-2})}^{\tilde{y}_{t-2,COVID}} \\ + \frac{1}{2} a_r ([r_{t-1} - 4cg_{t-1} - z_{t-1}] + [r_{t-2} - 4cg_{t-2} - z_{t-2}]) + \kappa_t \sigma_{\tilde{y}} \tilde{y}_t^y \\ b_y \overbrace{(y_{t-1} - y_{t-1}^* - \phi d_{t-1})}^{\tilde{y}_{t-1,COVID}} + b_\pi \pi_{t-1} + (1 - b_\pi) \pi_{t-2,4} + \kappa_t \sigma_\pi \kappa_t \varepsilon_t^\pi \end{bmatrix} \quad (3) \quad \text{lw a}$$

$$\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_t \\ g_{t-1} \\ g_{t-2} \\ z_t \\ z_{t-1} \\ z_{t-2} \end{bmatrix} = \begin{bmatrix} y_{t-1}^* + g_{t-1} + \sigma_{y^*} \varepsilon_t^{y^*} \\ y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} + \sigma_g \varepsilon_t^g \\ g_{t-1} \\ g_{t-2} \\ z_{t-1} + \sigma_z \varepsilon_t^z \\ z_{t-1} \\ z_{t-2} \end{bmatrix} \Rightarrow \begin{bmatrix} \Delta y_t^* \\ \Delta g_{t-1} \\ \Delta z_{t-1} \end{bmatrix} = \begin{bmatrix} g_{t-1} + \sigma_{y^*} \varepsilon_t^{y^*} \\ \sigma_g \varepsilon_t^g \\ \sigma_z \varepsilon_t^z \end{bmatrix} \quad (4) \quad \text{lw b}$$

and with $r_t^* = 4cg_t + z_t$, we get:

$$\begin{aligned} \Delta r_t^* &= 4c \overbrace{\Delta g_t}^{\sigma_g \varepsilon_t^g} + \overbrace{\Delta z_t}^{\sigma_z \varepsilon_t^z} \\ &= 4c \sigma_g \varepsilon_t^g + \sigma_z \varepsilon_t^z. \end{aligned} \quad (5) \quad \text{drstar}$$

Note here that the timing mismatch has been resolved now so that the equations read as expected, that is, without any lagged values in ε_t^ξ in (2).

The COVID-adjusted natural rate of output is defined in equation 16 of their paper:

$$\tilde{y}_{t,COVID} = 100(y_t - y_{t,COVID}^*) = 100(y_t - y_t^*) - \phi d_t.$$

2 Shock recovery SSM

2.1 SSM with lagged states

SSM Kurz's (2018) SSM has the following general from:

$$\text{Measurement : } Z_t = D_1 X_t + D_2 X_{t-1} + R \varepsilon_t \quad (6a) \quad \text{ssm1}$$

$$\text{State : } X_t = A X_{t-1} + C \varepsilon_t, \quad (6b) \quad \text{ssm2}$$

where $\varepsilon_t \sim MN(0, I_m)$, D_1, D_2, A, R are C conformable system matrices, Z_t the observed variable and X_t the latent state variable.

2.2 HLW23 equations

Following the same format as before, HLW23's SSM equations in (3) and (4) gives the following SSM equations:

$$y_t = y_t^* + \phi d_t + \sum_{i=1}^2 a_{y,i} (y_{t-i} - y_{t-i}^* - \phi d_{t-i}) + \frac{1}{2} a_r \sum_{i=1}^2 (r_{t-i} - r_{t-i}^*) + \kappa_t \sigma_{\tilde{y}} \varepsilon_t^{\tilde{y}} \quad (7a) \quad \text{LW03a}$$

$$\pi_t = b_y (y_{t-1} - y_{t-1}^* - \phi d_{t-1}) + b_\pi \pi_{t-1} + (1 - b_\pi) \pi_{t-2,4} + \kappa_t \sigma_\pi \varepsilon_t^\pi \quad (7b)$$

$$\Delta z_t = \sigma_z \varepsilon_t^z \quad (7c) \quad \text{LW03c}$$

$$\Delta y_t^* = g_{t-1} + \sigma_{y^*} \varepsilon_t^{y^*} \quad (7d) \quad \text{LW03d}$$

$$\Delta g_t = \sigma_g \varepsilon_t^g \quad (7e) \quad \text{LW03e}$$

with

$$\Delta r_t^* = 4c \sigma_g \varepsilon_t^g + \sigma_z \varepsilon_t^z. \quad (7f) \quad \text{LW03f}$$

where the numbered shock to named shock mapping is:

$$\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix} = \begin{bmatrix} \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^\pi \\ \varepsilon_t^z \\ \varepsilon_t^{y^*} \\ \varepsilon_t^g \end{bmatrix}. \quad (8)$$

2.3 HLW23 SSM for shock recovery

To assess recovery, re-write the model in '*shock recovery*' form. That is, collect all observables in Z_t , and all shocks (and other state variables) in state vector X_t . The relevant equations from (7) (incorporating (7f)) for the '*shock recovery*' SSM are (note the inclusion of c in (9e)):

$$\text{Measurement : } Z_{1t} = y_t^* - a_{y,1} y_{t-1}^* - a_{y,2} y_{t-2}^* - \frac{1}{2} a_r (r_{t-1}^* + r_{t-2}^*) + \kappa_t \sigma_{\tilde{y}} \varepsilon_t^{\tilde{y}} \quad (9a)$$

$$Z_{2t} = -b_y y_{t-1}^* + \kappa_t \sigma_\pi \varepsilon_t^\pi \quad (9b)$$

$$\text{State : } \Delta y_t^* = g_{t-1} + \sigma_{y^*} \varepsilon_t^{y^*} \quad (9c)$$

$$\Delta g_t = \sigma_g \varepsilon_t^g \quad (9d)$$

$$\Delta r_t^* = 4c \sigma_g \varepsilon_t^g + \sigma_z \varepsilon_t^z, \quad (9e) \quad \text{drstar2}$$

with the observables Z_t in the measurement equations defined as:

$$Z_{1t} = y_t - \sum_{i=1}^2 (a_{y,i} y_{t-i} - \phi d_{t-i}) - \frac{1}{2} a_r \sum_{i=1}^2 r_{t-i} \quad (10a)$$

$$Z_{2t} = \pi_t - b_\pi \pi_{t-1} - (1 - b_\pi) \pi_{t-2,4} - (b_y y_{t-1} - \phi d_{t-1}). \quad (10b)$$

So these are the same as before for the HLW17 model, with the dummy variables only impacting the observable part of the Z_t measurement in (10) and the only change being the inclusion of the 3 different possibilities for κ_t : $\kappa_{2022Q2-Q4}$, κ_{2021} , κ_{2022} and 1 for the baseline, but with the 2023 estimates of all other parameters.

The 'shock recovery' SSF corresponding to (9) is then:

Measurement : $Z_t = D_1 X_t + D_2 X_{t-1} + R \varepsilon_t$

$$\underbrace{\begin{bmatrix} Z_{1t} \\ Z_{2t} \end{bmatrix}}_{Z_t} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \kappa_t \sigma_{\tilde{y}} & 0 & 0 & 0 & 0 \\ 0 & -b_y & 0 & 0 & 0 & 0 & \kappa_t \sigma_{\pi} & 0 & 0 & 0 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ \tilde{y}_t \\ \varepsilon_t^\pi \\ \varepsilon_t^z \\ \varepsilon_t^{y^*} \\ \varepsilon_t^g \end{bmatrix}}_{X_t} \quad (11a)$$

$$+ \underbrace{\begin{bmatrix} -a_{y,1} & -a_{y,2} & 0 & -\frac{a_r}{2} & -\frac{a_r}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ \tilde{y}_{t-1} \\ \varepsilon_{t-1}^\pi \\ \varepsilon_{t-1}^z \\ \varepsilon_{t-1}^{y^*} \\ \varepsilon_{t-1}^g \end{bmatrix}}_{X_{t-1}} + \underbrace{\mathbf{0}_{2 \times 5}}_R \underbrace{\begin{bmatrix} \tilde{y}_t \\ \varepsilon_t^\pi \\ \varepsilon_t^z \\ \varepsilon_t^{y^*} \\ \varepsilon_t^g \end{bmatrix}}_{\varepsilon_t} \quad (11b)$$

State : $X_t = A X_{t-1} + C \varepsilon_t$,

$$\underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ \tilde{y}_t \\ \varepsilon_t^\pi \\ \varepsilon_t^z \\ \varepsilon_t^{y^*} \\ \varepsilon_t^g \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_A \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ \tilde{y}_{t-1} \\ \varepsilon_{t-1}^\pi \\ \varepsilon_{t-1}^z \\ \varepsilon_{t-1}^{y^*} \\ \varepsilon_{t-1}^g \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & \sigma_{y^*} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_g \\ 0 & 0 & \sigma_z & 0 & 4c\sigma_g \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_C \underbrace{\begin{bmatrix} \tilde{y}_t \\ \varepsilon_t^\pi \\ \varepsilon_t^z \\ \varepsilon_t^{y^*} \\ \varepsilon_t^g \end{bmatrix}}_{\varepsilon_t} \quad (11c)$$

2.3.1 Correlation between true change in natural rate and estimate

The correlation between the true and estimated Δr_t^* from the SSM can be constructed from the relation:

$$\rho = 0.5 \frac{\text{Var}(\Delta r_t^*) + \text{Var}(E_T \Delta r_t^*) - \phi}{\sigma(\Delta r_t^*)\sigma(E_T \Delta r_t^*)}, \quad (12)$$

where $\text{Var}(\Delta r_t^*) = 4^2 c^2 \sigma_5^2 + \sigma_3^2$, $\sigma(\Delta r_t^*) = \sqrt{\text{Var}(\Delta r_t^*)}$, and $\text{Var}(E_T \Delta r_t^*)$ can be computed from simulating from the true model, applying the Kalman Filter and Smoother to get $E_T \Delta r_t^*$ and then computing the sample variance of $E_T \Delta r_t^*$ as an estimate of $\text{Var}(E_T \Delta r_t^*)$.

To obtain ϕ , add Δr_t^* to the state-vector X_t and augment the remaining matrices to be conformable. The required ϕ term is then the entry of $\text{diag}(P_{t|T}^*)$ that corresponds to Δr_t^* , which will be the very last element.

The augmented SSF of (11) is then:

$$\text{Measurement : } Z_t = D_1 X_t + D_2 X_{t-1} + R \varepsilon_t$$

$$\begin{aligned} \underbrace{\begin{bmatrix} Z_{1t} \\ Z_{2t} \end{bmatrix}}_{Z_t} &= \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \kappa_t \sigma_{\tilde{y}} & 0 & 0 & 0 & 0 & 0 \\ 0 & -b_y & 0 & 0 & 0 & 0 & \kappa_t \sigma_{\pi} & 0 & 0 & 0 & 0 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ g_t^* \\ r_t^* \\ r_{t-1}^* \\ \tilde{y}_t^* \\ \varepsilon_t^\pi \\ \varepsilon_t^\pi \\ \varepsilon_t^z \\ \varepsilon_t^{y^*} \\ \varepsilon_t^g \\ \Delta r_t^* \end{bmatrix}}_{X_t} \\ &+ \underbrace{\begin{bmatrix} -a_{y,1} & -a_{y,2} & 0 & -\frac{a_r}{2} & -\frac{a_r}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1}^* \\ r_{t-1}^* \\ r_{t-2}^* \\ \tilde{y}_{t-1}^* \\ \varepsilon_{t-1}^\pi \\ \varepsilon_{t-1}^\pi \\ \varepsilon_{t-1}^z \\ \varepsilon_{t-1}^{y^*} \\ \varepsilon_{t-1}^g \\ \Delta r_{t-1}^* \end{bmatrix}}_{X_{t-1}} + \underbrace{\mathbf{0}_{2 \times 5}}_R \underbrace{\begin{bmatrix} \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^\pi \\ \varepsilon_t^z \\ \varepsilon_t^{y^*} \\ \varepsilon_t^g \end{bmatrix}}_{\varepsilon_t} \end{aligned} \quad (13)$$

$$\text{State : } X_t = A X_{t-1} + C \varepsilon_t,$$

$$\begin{aligned}
\underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^\pi \\ \varepsilon_t^z \\ \varepsilon_t^{y^*} \\ \varepsilon_t^g \\ \Delta r_t^* \end{bmatrix}}_{X_t} &= \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_A \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ \varepsilon_{t-1}^{\tilde{y}} \\ \varepsilon_{t-1}^\pi \\ \varepsilon_{t-1}^z \\ \varepsilon_{t-1}^{y^*} \\ \varepsilon_{t-1}^g \\ \Delta r_{t-1}^* \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & \sigma_{y^*} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_g \\ 0 & 0 & \sigma_z & 0 & 4c\sigma_g \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \sigma_z & 0 & 4c\sigma_g \end{bmatrix}}_C \underbrace{\begin{bmatrix} \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^\pi \\ \varepsilon_t^z \\ \varepsilon_t^{y^*} \\ \varepsilon_t^g \end{bmatrix}}_{\varepsilon_t}
\end{aligned}
\tag{15}$$