1 LW's (2003) State-Space Model (SSM) Form

LW03 use the following standard State-Space Form (SSF) (see their `LW_Code_Guide.pdf` file `LW_replication.zip` from the NYFED website):

$$\text{Measurement}: \quad \mathbf{y}_t = \mathbf{A}\mathbf{x}_t + \mathbf{H}\boldsymbol{\xi}_t + R^{1/2}\boldsymbol{\varepsilon}_t^{\mathbf{y}}$$

$$\text{State}: \quad \boldsymbol{\xi}_t = \mathbf{F}\boldsymbol{\xi}_{t-1} + Q^{1/2}\boldsymbol{\varepsilon}_t^{\boldsymbol{\xi}}$$

where

$$\mathbf{y}_t = \begin{bmatrix} y_t & \pi_t \end{bmatrix}',$$

$$\mathbf{x}_t = \begin{bmatrix} y_{t-1} & y_{t-2} & r_{t-1} & r_{t-2} & \pi_{t-1} & \pi_{t-2,4} & \pi_{t-5,8} & (\pi_{t-1}^0 - \pi_{t-1}) & (\pi_t^m - \pi_t) \end{bmatrix}',$$

$$\boldsymbol{\xi}_t = \begin{bmatrix} y_t^* & y_{t-1}^* & y_{t-2}^* & g_{t-1} & g_{t-2} & z_{t-1} & z_{t-2} \end{bmatrix}',$$

$$\mathbf{A} = \begin{bmatrix} a_1 & a_2 & \frac{a_3}{2} & \frac{a_3}{2} & 0 & 0 & 0 & 0 & 0 \\ b_3 & 0 & 0 & 0 & b_1 & b_2 & (1-b_1-b_2) & b_4 & b_5 \end{bmatrix},$$

$$\mathbf{H} = \begin{bmatrix} 1 & -a_1 & -a_2 & -4c\frac{a_3}{2} & -4c\frac{a_3}{2} & -\frac{a_3}{2} & -\frac{a_3}{2} \\ 0 & -b_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note: I follow the notation used in the documentation file: 'LW_Code_Guide.pdf' included in the zip file: 'LW_replication.zip' that contains the replication code of LW03 and which is available from the NYFED website at: https://www.newyorkfed.org/medialibrary/media/research/economists/williams/data/LW_replication.zip. Also, in the construction of r_t^*, trend growth g_t is annualized, but not in the state equations for g_t. That is, in the matrices above, the entries in $\mathbf{H}(1,4:5)$ corresponding to trend growth are multiplied by 4 (as well as c) in the code that performs the estimation (see `unpack.parameters.stage3.R` in `LW_replication.zip` files, line 24, which reads):

```
H[1,4:5] <- -parameters[9]*parameters[3]*2 ## c, a_3 (annualized).
```

Finally, in 'LW_Code_Guide.pdf', the standard deviations of the shocks (and presumably) the shocks themselves are denoted by $\begin{bmatrix} \sigma_{\tilde{y}} & \sigma_{\pi} & \sigma_{y^*} & \sigma_g & \sigma_z \end{bmatrix}'$ with no particular order given, but with the only sensible order based on the SSM equations being: $\begin{bmatrix} \varepsilon_t^y & \varepsilon_t^{\pi} & \varepsilon_t^{y^*} & \varepsilon_t^g & \varepsilon_t^z \end{bmatrix}'$. I will thus use this notation and order in the equations below that use and describe their SSF, and will use the labelling $\{\sigma_i\}_{i=1}^5$ later on when referring back to the published paper's numerical values and notation, and in the construction of the SSM used for shock recovery.

1.1 SSM of LW03

The measurement equation is:

$$\mathbf{y}_t = \mathbf{A}\mathbf{x}_t + \mathbf{H}\xi_t + \mathbf{R}^{1/2}\varepsilon_t^{\mathbf{y}}$$

$$
\underbrace{\begin{bmatrix} y_t \\ \pi_t \end{bmatrix}}_{\mathbf{y}_t} = \underbrace{\begin{bmatrix} a_1 & a_2 & \frac{a_3}{2} & \frac{a_3}{2} & 0 & 0 & 0 & 0 & 0 \\ b_3 & 0 & 0 & 0 & b_1 & b_2 & (1-b_1-b_2) & b_4 & b_5 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} y_{t-1} \\ y_{t-2} \\ r_{t-1} \\ r_{t-2} \\ \pi_{t-1} \\ \pi_{t-2,4} \\ \pi_{t-5,8} \\ \left(\pi_{t-1}^0 - \pi_{t-1}\right) \\ \left(\pi_t^m - \pi_t\right) \end{bmatrix}}_{\mathbf{x}_t} \tag{1a}
$$

$$
+ \underbrace{\begin{bmatrix} 1 & -a_1 & -a_2 & -4c\frac{a_3}{2} & -4c\frac{a_3}{2} & -\frac{a_3}{2} & -\frac{a_3}{2} \\ 0 & -b_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ g_{t-2} \\ z_{t-1} \\ z_{t-2} \end{bmatrix}}_{\xi_t} + \underbrace{\begin{bmatrix} \sigma_{\tilde{y}} & 0 \\ 0 & \sigma_{\pi} \end{bmatrix}}_{\mathbf{R}^{1/2}} \underbrace{\begin{bmatrix} \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^{\pi} \end{bmatrix}}_{\varepsilon_t^{\mathbf{y}}}. \tag{1b}
$$

The state equation is:

$$\xi_t = \mathbf{F}\xi_{t-1} + \mathbf{Q}^{1/2}\varepsilon_t^{\xi}$$

$$
\underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ g_{t-2} \\ z_{t-1} \\ z_{t-2} \end{bmatrix}}_{\xi_t} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{F}} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ y_{t-3}^* \\ g_{t-2} \\ g_{t-3} \\ z_{t-2} \\ z_{t-3} \end{bmatrix}}_{\xi_{t-1}} + \underbrace{\begin{bmatrix} \sigma_{y^*} & \sigma_g & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sigma_g & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_z \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{Q}^{1/2}} \underbrace{\begin{bmatrix} \varepsilon_t^{y^*} \\ \varepsilon_{t-1}^{g} \\ \varepsilon_{t-1}^{z} \end{bmatrix}}_{\varepsilon_t^{\xi}} \tag{2}
$$

Expanding the relations in (1) and (2) and re-arranging yields:

$$
\begin{bmatrix} y_t \\ \\ \\ \pi_t \end{bmatrix} =
\begin{bmatrix}
y_t^* + a_1\left(y_{t-1}-y_{t-1}^*\right) + a_2\left(y_{t-2}-y_{t-2}^*\right) \\
+\frac{1}{2}a_3\left([r_{t-1}-4cg_{t-1}-z_{t-1}] + [r_{t-2}-4cg_{t-2}-z_{t-2}]\right) + \sigma_{\tilde{y}}\varepsilon_t^{\tilde{y}} \\
\\
b_3\left(y_{t-1}-y_{t-1}^*\right) + b_1\pi_{t-1} + b_2\pi_{t-2,4} + (1-b_1-b_2)\pi_{t-5,8} \\
+b_4\left(\pi_{t-1}^0-\pi_{t-1}\right) + b_5\left(\pi_t^m-\pi_t\right) + \sigma_\pi\varepsilon_t^\pi
\end{bmatrix}
\tag{3} \quad \texttt{lwa}
$$

$$
\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ g_{t-2} \\ z_{t-1} \\ z_{t-2} \end{bmatrix} =
\begin{bmatrix}
y_{t-1}^* + \overbrace{g_{t-2}+\sigma_g\varepsilon_{t-1}^g}^{g_{t-1}} + \sigma_{y^*}\varepsilon_t^{y^*} \\
y_{t-1}^* \\
y_{t-2}^* \\
g_{t-2}+\sigma_g\varepsilon_{t-1}^g \\
g_{t-2} \\
z_{t-2}+\sigma_z\varepsilon_{t-1}^z \\
z_{t-2}
\end{bmatrix}
\Rightarrow
\begin{bmatrix} \Delta y_t^* \\ \Delta g_{t-1} \\ \Delta z_{t-1} \end{bmatrix} =
\begin{bmatrix} g_{t-1}+\sigma_{y^*}\varepsilon_t^{y^*} \\ \sigma_g\varepsilon_{t-1}^g \\ \sigma_z\varepsilon_{t-1}^z \end{bmatrix}
\tag{4} \quad \texttt{lwb}
$$

and with $r_t^* = 4cg_t + z_t$, we get:

$$
\begin{aligned}
\Delta r_t^* &= 4c\,\overbrace{\Delta g_t}^{\sigma_g\varepsilon_t^g} + \overbrace{\Delta z_t}^{\sigma_z\varepsilon_t^z} \\
&= 4c\sigma_g\varepsilon_t^g + \sigma_z\varepsilon_t^z.
\end{aligned}
\tag{5} \quad \texttt{drstar}
$$

2 Shock recovery SSM

2.1 SSM with lagged states

Kurz's (2018) SSM has the following general from:

$$
\begin{aligned}
\text{Measurement}: \quad & Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t & \text{(6a)} \quad \texttt{ssm1} \\
\text{State}: \quad & X_t = A X_{t-1} + C\varepsilon_t, & \text{(6b)} \quad \texttt{ssm2}
\end{aligned}
$$

where $\varepsilon_t \sim MN(0, I_m)$, D_1, D_2, A, R are C are conformable system matrices, Z_t the observed variable and X_t the latent state variable.

2.2 LW03 equations re-ordered and numbered shocks

LW03's SSM equations in (3) and (4), with the re-labelling of shocks according to Table 1 (page 1065) of their paper yields: $\begin{bmatrix} \sigma_1(\tilde{y}) & \sigma_2(\pi) & \sigma_3(z) & \sigma_4(y^*) & \sigma_5(g) \end{bmatrix}'$ (Note $\sigma_3 \Rightarrow \sigma_z$). The resulting SSM equations are then:

$$y_t = y_t^* + \sum_{i=1}^{2} a_i \left(y_{t-i} - y_{t-i}^* \right) + \tfrac{1}{2} a_3 \sum_{i=1}^{2} \left(r_{t-i} - r_{t-i}^* \right) + \sigma_1 \varepsilon_{1t} \tag{7a}$$

$$\pi_t = b_3 \left(y_{t-1} - y_{t-1}^* \right) + b_1 \pi_{t-1} + b_2 \pi_{t-2,4} + (1 - b_1 - b_2)\, \pi_{t-5,8}$$
$$+ \, b_4 (\pi_{t-1}^0 - \pi_{t-1}) + b_5 \left(\pi_t^m - \pi_t \right) + \sigma_2 \varepsilon_{2t} \tag{7b}$$

$$\Delta z_t = \sigma_3 \varepsilon_{3t} \tag{7c}$$

$$\Delta y_t^* = g_{t-1} + \sigma_4 \varepsilon_{4t} \tag{7d}$$

$$\Delta g_t = \sigma_5 \varepsilon_{5t} \tag{7e}$$

with

$$\Delta r_t^* = 4 c \sigma_5 \varepsilon_{5t} + \sigma_3 \varepsilon_{3t}. \tag{7f}$$

2.3 LW03 SSM for shock recovery

To assess recovery, re-write the model in '*shock recovery*' form. That is, collect all observables in Z_t, and all shocks (and other state variables) in state vector X_t. The relevant equations from (7) (incorporating (7e)) for the '*shock recovery*' SSM are:

$$\text{Measurement}: \quad Z_{1t} = y_t^* - a_1 y_{t-1}^* - a_2 y_{t-2}^* - \tfrac{1}{2} a_3 \left(r_{t-1}^* + r_{t-2}^* \right) + \sigma_1 \varepsilon_{1t} \tag{8a}$$

$$Z_{2t} = -b_3 y_{t-1}^* + \sigma_2 \varepsilon_{2t} \tag{8b}$$

$$\text{State}: \quad \Delta y_t^* = g_{t-1} + \sigma_4 \varepsilon_{4t} \tag{8c}$$

$$\Delta g_t = \sigma_5 \varepsilon_{5t} \tag{8d}$$

$$\Delta r_t^* = 4 c \sigma_5 \varepsilon_{5t} + \sigma_3 \varepsilon_{3t}, \tag{8e}$$

with the observables Z_t in the measurement equations defined as:

$$Z_{1t} = y_t - \sum_{i=1}^{2} a_i y_{t-i} - \tfrac{1}{2} a_3 \sum_{i=1}^{2} r_{t-i}$$

$$Z_{2t} = \pi_t - b_1 \pi_{t-1} - b_2 \pi_{t-2,4} - (1 - b_1 - b_2)\, \pi_{t-5,8}$$
$$- \, b_4 (\pi_{t-1}^0 - \pi_{t-1}) - b_5 \left(\pi_t^m - \pi_t \right) - b_3 y_{t-1}.$$

The *'shock recovery'* SSF corresponding to (8) is then:

Measurement : $\quad Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t$

$$
\underbrace{\begin{bmatrix} Z_{1t} \\ Z_{2t} \end{bmatrix}}_{Z_t} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & -b_3 & 0 & 0 & 0 & 0 & \sigma_2 & 0 & 0 & 0 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{X_t} \tag{9a}
$$

$$
+ \underbrace{\begin{bmatrix} -a_1 & -a_2 & 0 & -\frac{a_3}{2} & -\frac{a_3}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{4t-1} \\ \varepsilon_{5t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\mathbf{0}_{2\times 5}}_{R} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{\varepsilon_t}
$$

(9b)

State : $\quad X_t = A X_{t-1} + C\varepsilon_t,$

$$
\underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{4t-1} \\ \varepsilon_{5t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & \sigma_4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_5 \\ 0 & 0 & \sigma_3 & 0 & 4c\sigma_5 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{\varepsilon_t}
$$

(9c)

5

2.3.1 Steady-state MSE

The Kalman Filter and Smoother based steady-state $\text{diag}(P)$ are, respectively:

$$\text{diag}(P^*_{t|t}) = \begin{bmatrix} 0.7479 \\ 0.0319 \\ 1.0000 \\ 0.3840 \\ 1.0000 \end{bmatrix} \text{ and } \text{diag}(P^*_{t|T}) = \begin{bmatrix} 0.6952 \\ 0.0146 \\ 0.9749 \\ 0.3353 \\ 0.9800 \end{bmatrix} \text{ for entries: } \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix} \Rightarrow \begin{bmatrix} \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^{\pi} \\ \varepsilon_t^{z} \\ \varepsilon_t^{y^*} \\ \varepsilon_t^{g} \end{bmatrix}. \quad (10)$$

2.3.2 Correlation between true states and smoothed estimates

$\text{Corr}(X_t, a_{t|T})$	ε_{1t}	ε_{2t}	ε_{3t}	ε_{4t}	ε_{5t}
ε_{1t}	0.5616	−0.0043	−0.1154	0.4555	−0.0907
ε_{2t}	0.0073	0.9923	−0.0359	0.0273	0.0444
ε_{3t}	−0.0234	−0.0057	0.1482	−0.0071	0.0728
ε_{4t}	0.6660	0.0231	−0.0744	0.8159	−0.1189
ε_{5t}	−0.0255	0.0029	0.0683	−0.0349	0.1351

2.3.3 Correlation between true change in natural rate and estimate

To examine the correlation between the true and estimated Δr_t^* from the SSM, we can add Δr_t^* to the state-vector X_t and augment the remaining matrices to be conformable. The augmented SSF corresponding to (9) is then:

Measurement : $Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t$

$$\underbrace{\begin{bmatrix} Z_{1t} \\ Z_{2t} \end{bmatrix}}_{Z_t} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \sigma_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -b_3 & 0 & 0 & 0 & 0 & \sigma_2 & 0 & 0 & 0 & 0 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \\ \Delta r_t^* \end{bmatrix}}_{X_t} \quad (11)$$

$$
+ \underbrace{\begin{bmatrix} -a_1 & -a_2 & 0 & -\frac{a_3}{2} & -\frac{a_3}{2} & 0 & 0 & 0 & 0 & 0 & {\color{red}0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & {\color{red}0} \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} y^*_{t-1} \\ y^*_{t-2} \\ g_{t-1} \\ r^*_{t-1} \\ r^*_{t-2} \\ \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{4t-1} \\ \varepsilon_{5t-1} \\ {\color{red}\Delta r^*_{t-1}} \end{bmatrix}}_{X_{t-1}} + \underbrace{\mathbf{0}_{2\times 5}}_{R} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{\varepsilon_t}
$$

$$\text{(12)}$$

State : $\quad X_t = A X_{t-1} + C \varepsilon_t,$

$$
\underbrace{\begin{bmatrix} y^*_t \\ y^*_{t-1} \\ g_t \\ r^*_t \\ r^*_{t-1} \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \\ {\color{red}\Delta r^*_t} \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & {\color{red}0} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & {\color{red}0} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & {\color{red}0} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & {\color{red}0} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & {\color{red}0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & {\color{red}0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & {\color{red}0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & {\color{red}0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & {\color{red}0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & {\color{red}0} \\ {\color{red}0} & {\color{red}0} & {\color{red}0} & {\color{red}0} & {\color{red}0} & {\color{red}0} & {\color{red}0} & {\color{red}0} & {\color{red}0} & {\color{red}0} & {\color{red}0} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y^*_{t-1} \\ y^*_{t-2} \\ g_{t-1} \\ r^*_{t-1} \\ r^*_{t-2} \\ \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{4t-1} \\ \varepsilon_{5t-1} \\ {\color{red}\Delta r^*_{t-1}} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & \sigma_4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_5 \\ 0 & 0 & \sigma_3 & 0 & 4c\sigma_5 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ {\color{red}0} & {\color{red}0} & {\color{red}\sigma_3} & {\color{red}0} & {\color{red}4c\sigma_5} \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{\varepsilon_t}
$$

$$\text{(13)}$$