

# 1 MR17

McCririck and Rees (2017, MR17) is effectively an extension of LW03's model, adding an equation for Okun's law. To keep these two models comparable, the notation and number labelling of shocks is kept as in LW03. Following the description of the model on page 16 in MR17 (Appendix A: Estimating the Model), the model takes the form:

$$\tilde{y}_t = a_{y,1}\tilde{y}_{t-1} + a_{y,2}\tilde{y}_{t-2} - \frac{a_r}{2} \sum_{i=1}^2 (r_{t-i} - r_{t-i}^*) + \sigma_1 \varepsilon_{1t} \quad (1) \quad \text{MR1}$$

$$\pi_t = (1 - \beta_1)\pi_t^e + \frac{\beta_1}{3} \sum_{i=1}^3 \pi_{t-i} + \beta_2(u_{t-1} - u_{t-1}^*) + \sigma_2 \varepsilon_{2t} \quad (2) \quad \text{MR2}$$

$$\Delta z_t = \sigma_3 \varepsilon_{3t}, \quad (3) \quad \text{MR3}$$

$$\Delta y_t^* = g_t + \sigma_4 \varepsilon_{4t} \quad (\text{MR17 use } g_t \text{ in paper, we use } g_{t-1} \text{ as in LW03 in SSM below}) \quad (4) \quad \text{MR4}$$

$$\Delta g_t = \sigma_5 \varepsilon_{5t} \quad (5) \quad \text{MR5}$$

$$\Delta u_t^* = \sigma_6 \varepsilon_{6t} \quad (6) \quad \text{MR6}$$

$$u_t = u_t^* + \beta(.4\tilde{y}_t + .3\tilde{y}_{t-1} + .2\tilde{y}_{t-2} + .1\tilde{y}_{t-3}) + \sigma_7 \varepsilon_{7t} \quad (7) \quad \text{MR7}$$

where  $\tilde{y}_t = (y_t - y_t^*)$  and  $r_t^* = 4g_t + z_t$ , and the numbered shock to named shock mapping is:

$$\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \\ \varepsilon_{6t} \\ \varepsilon_{7t} \end{bmatrix} = \begin{bmatrix} \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^{\pi} \\ \varepsilon_t^z \\ \varepsilon_t^{y^*} \\ \varepsilon_t^g \\ \varepsilon_t^{u^*} \\ \varepsilon_t^u \end{bmatrix}. \quad (8)$$

## 2 Shock recovery SSM

### 2.1 SSM with lagged states

SSM Kurz's (2018) SSM has the following general form:

$$\text{Measurement : } Z_t = D_1 X_t + D_2 X_{t-1} + R \varepsilon_t \quad (9a) \quad \text{ssm1}$$

$$\text{State : } X_t = A X_{t-1} + C \varepsilon_t, \quad (9b) \quad \text{ssm2}$$

where  $\varepsilon_t \sim MN(0, I_m)$ ,  $D_1, D_2, A, R$  are  $C$  conformable system matrices,  $Z_t$  the observed variable and  $X_t$  the latent state variable.

## 2.2 MR17 SSM for shock recovery

To assess recovery, re-write the model in ‘*shock recovery*’ form. That is, collect all observables in  $Z_t$ , and all shocks (and other state variables) in state vector  $X_t$ .

$$\text{Measurement : } Z_{1t} = y_t^* - a_{y,1}y_{t-1}^* - a_{y,2}y_{t-2}^* - \frac{a_r}{2}(r_{t-1}^* + r_{t-2}^*) + \sigma_1\varepsilon_{1t} \quad (10a)$$

$$Z_{2t} = -\beta_2 u_{t-1}^* + \sigma_2\varepsilon_{2t} \quad (10b)$$

$$Z_{3t} = u_t^* - \beta(.4y_t^* + .3y_{t-1}^* + .2y_{t-2}^* + .1y_{t-3}^*) + \sigma_7\varepsilon_{7t} \quad (10c)$$

$$\text{State : } \Delta y_t^* = g_{t-1} + \sigma_4\varepsilon_{4t} \quad (10d)$$

$$\Delta g_t = \sigma_5\varepsilon_{5t} \quad (10e)$$

$$\Delta u_t^* = \sigma_6\varepsilon_{6t}, \quad (10f) \quad \text{drstar2}$$

$$\Delta r_t^* = 4\sigma_5\varepsilon_{5t} + \sigma_3\varepsilon_{3t}, \quad (10g)$$

with the observables  $Z_t$  in the measurement equations defined as:

$$Z_{1t} = y_t - \left( \sum_{i=1}^2 (a_{y,i}y_{t-i}) - \frac{a_r}{2} \sum_{i=1}^2 r_{t-i} \right) \quad (11a)$$

$$Z_{2t} = \pi_t - \left( (1 - \beta_1)\pi_t^e + \frac{\beta_1}{3} \sum_{i=1}^3 \pi_{t-i} + \beta_2 u_{t-1} \right) \quad (11b)$$

$$Z_{3t} = u_t - \beta(.4y_t + .3y_{t-1} + .2y_{t-2} + .1y_{t-3}). \quad (11c)$$

The ‘*shock recovery*’ SSF corresponding to (10) is then:

$$\text{Measurement : } Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t$$

$$\underbrace{\begin{bmatrix} Z_{1t} \\ Z_{2t} \\ Z_{3t} \end{bmatrix}}_{Z_t} = \underbrace{\begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} & 0 & 0 & 0 & 0 & \sigma_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_2 & 0 & 0 & 0 & 0 & 0 \\ -.4\beta & -.3\beta & -.2\beta & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_7 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ u_t^* \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \\ \varepsilon_{6t} \\ \varepsilon_{7t} \end{bmatrix}}_{X_t} \quad (12a)$$

$$\begin{aligned}
& + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{a_r}{2} & -\frac{a_r}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\beta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -.1\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ y_{t-3}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ u_{t-1}^* \\ \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{4t-1} \\ \varepsilon_{5t-1} \\ \varepsilon_{6t-1} \\ \varepsilon_{7t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\mathbf{0}_{2 \times 5}}_R \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \\ \varepsilon_{6t} \\ \varepsilon_{7t} \end{bmatrix}}_{\varepsilon_t} \\
\end{aligned} \tag{12b}$$

State :  $X_t = AX_{t-1} + C\varepsilon_t$ ,

$$\begin{aligned}
\underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ u_t^* \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \\ \varepsilon_{6t} \\ \varepsilon_{7t} \end{bmatrix}}_{X_t} &= \underbrace{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_A \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ y_{t-3}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ u_{t-1}^* \\ \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{4t-1} \\ \varepsilon_{5t-1} \\ \varepsilon_{6t-1} \\ \varepsilon_{7t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & \sigma_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_5 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & 4\sigma_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_6 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_C \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \\ \varepsilon_{6t} \\ \varepsilon_{7t} \end{bmatrix}}_{\varepsilon_t} \\
\end{aligned} \tag{12c}$$

### 2.2.1 Correlation between true change in natural rate and estimate

The correlation between the true and estimated  $\Delta r_t^*$  from the SSM can be constructed from the relation:

$$\rho = 0.5 \frac{\text{Var}(\Delta r_t^*) + \text{Var}(E_T \Delta r_t^*) - \phi}{\sigma(\Delta r_t^*) \sigma(E_T \Delta r_t^*)}, \tag{13}$$

where  $\text{Var}(\Delta r_t^*) = 4^2 c^2 \sigma_5^2 + \sigma_3^2$ ,  $\sigma(\Delta r_t^*) = \sqrt{\text{Var}(\Delta r_t^*)}$ , and  $\text{Var}(E_T \Delta r_t^*)$  can be computed from simulating from the true model, applying the Kalman Filter and Smoother to get  $E_T \Delta r_t^*$  and then computing the sample variance of  $E_T \Delta r_t^*$  as an estimate of  $\text{Var}(E_T \Delta r_t^*)$ .

To obtain  $\phi$ , add  $\Delta r_t^*$  to the state-vector  $X_t$  and augment the remaining matrices to be conformable. The required  $\phi$  term is then the entry of  $\text{diag}(P_{t|T}^*)$  that corresponds to  $\Delta r_t^*$ , which will be the very last element (see also LW03.pdf how this is done).