

1 MR17 with Monetary policy rule

Adding a Monetary Policy (MP) rule to McCririck and Rees (2017, MR17) yields to:

$$\tilde{y}_t = a_{y,1}\tilde{y}_{t-1} + a_{y,2}\tilde{y}_{t-2} - \frac{a_r}{2} \sum_{i=1}^2 (r_{t-i} - r_{t-i}^*) + \sigma_1 \varepsilon_{1t} \quad (1) \quad \text{MR1}$$

$$\pi_t = (1 - \beta_1)\pi_t^e + \frac{\beta_1}{3} \sum_{i=1}^3 \pi_{t-i} + \beta_2(u_{t-1} - u_{t-1}^*) + \sigma_2 \varepsilon_{2t} \quad (2) \quad \text{MR2}$$

$$\Delta z_t = \sigma_3 \varepsilon_{3t}, \quad (3) \quad \text{MR3}$$

$$\Delta y_t^* = g_t + \sigma_4 \varepsilon_{4t} \quad (\text{MR17 use } g_t \text{ in paper, we use } g_{t-1} \text{ as in LW03 in SSM below}) \quad (4) \quad \text{MR4}$$

$$\Delta g_t = \sigma_5 \varepsilon_{5t} \quad (5) \quad \text{MR5}$$

$$\Delta u_t^* = \sigma_6 \varepsilon_{6t} \quad (6) \quad \text{MR6}$$

$$u_t = u_t^* + \beta(.4\tilde{y}_t + .3\tilde{y}_{t-1} + .2\tilde{y}_{t-2} + .1\tilde{y}_{t-3}) + \sigma_7 \varepsilon_{7t} \quad (7) \quad \text{MR7}$$

$$r_t = (1 - b_1)(r_{t-1} - \Delta\pi_t) + b_1[r_t^* - \bar{\pi} - 2(u_t - u_t^*)] - \Delta_2 u_t + \sigma_8 \varepsilon_{8t}, \quad (8) \quad \text{MR8}$$

where $\tilde{y}_t = (y_t - y_t^*)$ and $r_t^* = 4g_t + z_t$ as before, and (8) is obtained from the MARTIN policy model of the Reserve Bank of Australia (Ballantyne *et al.*, 2020), with nominal rule:

$$i_t = (1 - b_1)i_{t-1} + b_1(r_t^* + 2\pi_t - \bar{\pi} - 2(u_t - u_t^*)) - \Delta_2 u_t + \sigma_8 \varepsilon_{8t}, \quad (9) \quad \text{eq:NR}$$

where $\bar{\pi}$ denotes the inflation target (there is a 2 missing in front of π_t in our equation see below eq. 36 from Ballantyne *et al.*, 2020). The parameters in (8) and (9) are: $b_1 = .3$ and $\sigma_8 = 1.19$ (where is 1.19 taken from???). These are from the equation below (page 237):

$$\begin{aligned} NCR_t &= 0.7 \times NCR_{t-1} + 0.3 \\ &\times \left[RSTAR_t + 2 \times \left(\frac{PTM_t}{PTM_{t-4}} \times 100 - 100 \right) \right. \\ &\quad \left. - \bar{\pi} - 2 \times LURGAP_t \right] - \Delta_2 LUR_t + \varepsilon_{ncr,t}, \end{aligned}$$

TIM: Why use $r_t = i_t + \pi_t$ and not $r_t = i_t + \pi_t^e$, \Rightarrow inconsistent within same model? Also not sure if the equations are correct... but for shock recovery, all that matters is r_t^* and u_t^*

The numbered shock to named shock mapping is:

$$\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \\ \varepsilon_{6t} \\ \varepsilon_{7t} \\ \varepsilon_{8t} \end{bmatrix} = \begin{bmatrix} \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^{\pi} \\ \varepsilon_t^z \\ \varepsilon_t^{y^*} \\ \varepsilon_t^g \\ \varepsilon_t^{u^*} \\ \varepsilon_t^u \\ \varepsilon_t^r \end{bmatrix}. \quad (10)$$

2 Shock recovery SSM

2.1 SSM with lagged states

SSM Kurz's (2018) SSM has the following general form:

$$\text{Measurement : } Z_t = D_1 X_t + D_2 X_{t-1} + R \varepsilon_t \quad (11a) \quad \text{ssm1}$$

$$\text{State : } X_t = A X_{t-1} + C \varepsilon_t, \quad (11b) \quad \text{ssm2}$$

where $\varepsilon_t \sim MN(0, I_m)$, D_1, D_2, A, R are C conformable system matrices, Z_t the observed variable and X_t the latent state variable.

2.2 MR17 SSM for shock recovery

ssm0 To assess recovery, re-write the model in '*shock recovery*' form. That is, collect all observables in Z_t , and all shocks (and other state variables) in state vector X_t .

$$\text{Measurement : } Z_{1t} = y_t^* - a_{y,1} y_{t-1}^* - a_{y,2} y_{t-2}^* - \frac{a_r}{2} (r_{t-1}^* + r_{t-2}^*) + \sigma_1 \varepsilon_{1t} \quad (12a)$$

$$Z_{2t} = -\beta_2 u_{t-1}^* + \sigma_2 \varepsilon_{2t} \quad (12b)$$

$$Z_{3t} = u_t^* - \beta(.4y_t^* + .3y_{t-1}^* + .2y_{t-2}^* + .1y_{t-3}^*) + \sigma_7 \varepsilon_{7t} \quad (12c)$$

$$Z_{4t} = b_1 r_t^* + 2b_1 u_t^* + \sigma_8 \varepsilon_{7t} \quad (12d)$$

$$\text{State : } \Delta y_t^* = g_{t-1} + \sigma_4 \varepsilon_{4t} \quad (12e)$$

$$\Delta g_t = \sigma_5 \varepsilon_{5t} \quad (12f)$$

$$\Delta u_t^* = \sigma_6 \varepsilon_{6t}, \quad (12g) \quad \text{drstar2}$$

$$\Delta r_t^* = 4\sigma_5 \varepsilon_{5t} + \sigma_3 \varepsilon_{3t}, \quad (12h)$$

ssm0 with the observables Z_t in the measurement equations defined as:

$$Z_{1t} = y_t - \left(\sum_{i=1}^2 (a_{y,i} y_{t-i}) - \frac{a_r}{2} \sum_{i=1}^2 r_{t-i} \right) \quad (13a)$$

$$Z_{2t} = \pi_t - \left((1 - \beta_1) \pi_t^e + \frac{\beta_1}{3} \sum_{i=1}^3 \pi_{t-i} + \beta_2 u_{t-1} \right) \quad (13b)$$

$$Z_{3t} = u_t - \beta(.4y_t + .3y_{t-1} + .2y_{t-2} + .1y_{t-3}) \quad (13c)$$

$$Z_{4t} = r_t - (1 - b_1)(r_{t-1} - \Delta \pi_t) - b_1 \bar{\pi} + \Delta_2 u_t + 2b_1 u_t \quad (13d)$$

KOSSM The '*shock recovery*' SSF corresponding to (12) is then:

Measurement : $Z_t = D_1 X_t + D_2 X_{t-1} + R \varepsilon_t$

$$\underbrace{\begin{bmatrix} Z_{1t} \\ Z_{2t} \\ Z_{3t} \\ Z_{4t} \end{bmatrix}}_{Z_t} = \underbrace{\begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} & 0 & 0 & 0 & 0 & \sigma_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -.4\beta & -.3\beta & -.2\beta & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_7 & 0 \\ 0 & 0 & 0 & 0 & b_1 & 0 & 2b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_8 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ u_t^* \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \\ \varepsilon_{6t} \\ \varepsilon_{7t} \\ \varepsilon_{8t} \end{bmatrix}}_{X_t} \quad (14a)$$

$$+ \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{a_r}{2} & -\frac{a_r}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\beta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -.1\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ y_{t-3}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ u_{t-1}^* \\ \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{4t-1} \\ \varepsilon_{5t-1} \\ \varepsilon_{6t-1} \\ \varepsilon_{7t-1} \\ \varepsilon_{8t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\mathbf{0}_{2 \times 5}}_R \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \\ \varepsilon_{6t} \\ \varepsilon_{7t} \\ \varepsilon_{8t} \end{bmatrix}}_{\varepsilon_t}$$

(14b)

State : $X_t = AX_{t-1} + C\varepsilon_t,$

$$\begin{aligned}
\underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ u_t^* \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \\ \varepsilon_{6t} \\ \varepsilon_{7t} \\ \varepsilon_{8t} \end{bmatrix}}_{X_t} &= \underbrace{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_A \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ y_{t-3}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ u_{t-1}^* \\ \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{4t-1} \\ \varepsilon_{5t-1} \\ \varepsilon_{6t-1} \\ \varepsilon_{7t-1} \\ \varepsilon_{8t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & \sigma_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_5 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & 4\sigma_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_6 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_C \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \\ \varepsilon_{6t} \\ \varepsilon_{7t} \\ \varepsilon_{8t} \end{bmatrix}}_{\varepsilon_t}
\end{aligned} \tag{14c}$$

2.2.1 Correlation between true change in natural rate and estimate

The correlation between the true and estimated Δr_t^* from the SSM can be constructed from the relation:

$$\rho = 0.5 \frac{\text{Var}(\Delta r_t^*) + \text{Var}(E_T \Delta r_t^*) - \phi}{\sigma(\Delta r_t^*) \sigma(E_T \Delta r_t^*)}, \tag{15}$$

where $\text{Var}(\Delta r_t^*) = 4^2 c^2 \sigma_5^2 + \sigma_3^2$, $\sigma(\Delta r_t^*) = \sqrt{\text{Var}(\Delta r_t^*)}$, and $\text{Var}(E_T \Delta r_t^*)$ can be computed from simulating from the true model, applying the Kalman Filter and Smoother to get $E_T \Delta r_t^*$ and then computing the sample variance of $E_T \Delta r_t^*$ as an estimate of $\text{Var}(E_T \Delta r_t^*)$.

To obtain ϕ , add Δr_t^* to the state-vector X_t and augment the remaining matrices to be conformable. The required ϕ term is then the entry of $\text{diag}(P_{t|T}^*)$ that corresponds to Δr_t^* , which will be the very last element (see also LW03.pdf how this is done).

2.3 Check i_t to r_t conversion

Ballantyne et al., 2020 equation 36 is something like

$$i_t = Ai_{t-1} + b_1 [r_t^* + 2\pi_t - \bar{\pi} - 2(u_t - u_t^*)] - \Delta_2 u_t + \sigma_8 \varepsilon_t$$

where $A = (1 - b_1)$. With $i_t = r_t + \pi_t$ (not inflation expectations as in the model), we get:

$$\begin{aligned} i_t &= Ai_{t-1} + b_1 2\pi_t + \underbrace{b_1 [r_t^* - \bar{\pi} - 2(u_t - u_t^*)]}_{OT = \text{other terms}} - \Delta_2 u_t + \sigma_8 \varepsilon_t \\ &= Ai_{t-1} + b_1 2\pi_t + OT \\ \Leftrightarrow (r_t + \pi_t) &= A(r_{t-1} + \pi_{t-1}) + 2b_1 \pi_t + OT \\ r_t &= Ar_{t-1} + A\pi_{t-1} + 2b_1 \pi_t - \pi_t + OT \\ &= Ar_{t-1} + (b_1 - 1)\pi_t + A\pi_{t-1} + b_1 \pi_t + OT \\ &= Ar_{t-1} - (1 - b_1)\pi_t + A\pi_{t-1} + b_1 \pi_t + OT \\ &= Ar_{t-1} - A\pi_t + A\pi_{t-1} + b_1 \pi_t + OT \\ &= Ar_{t-1} - A\Delta\pi_t + b_1 \pi_t + OT \\ &= A(r_{t-1} - \Delta\pi_t) + b_1 \pi_t + b_1 [r_t^* - \bar{\pi} - 2(u_t - u_t^*)] - \Delta_2 u_t + \sigma_8 \varepsilon_t \\ &= A(r_{t-1} - \Delta\pi_t) + b_1 [r_t^* + (\pi_t - \bar{\pi}) - 2(u_t - u_t^*)] - \Delta_2 u_t + \sigma_8 \varepsilon_t \\ r_t &= (1 - b_1)(r_{t-1} - \Delta\pi_t) + b_1 [r_t^* + (\pi_t - \bar{\pi}) - 2(u_t - u_t^*)] - \Delta_2 u_t + \sigma_8 \varepsilon_t \quad (16) \end{aligned}$$

\Rightarrow If the additional 2 in front of $\left(\frac{PTM_t}{PTM_{t-4}} \times 100 - 100\right)$ is intentional.