1 AP23

1.1 SSM model with two lags

SSM Kurz's (2018) SSM has the following general from:

$$\begin{aligned} \text{Measurement}: \quad Z_t &= D_1 X_t + D_2 X_{t-1} + R\varepsilon_t && \text{(1a)} \quad \text{ssm1} \\ \text{State}: \quad X_t &= AX_{t-1} + C\varepsilon_t, && \text{(1b)} \quad \text{ssm2} \end{aligned}$$

where $\varepsilon_t \sim MN(0, I_m)$, D_1, D_2, A, R are C are conformable system matrices, Z_t the observed variable and X_t the latent state variable.

1.2 Modified (standard) SSM

clark0 The simple model takes the form:

$$\begin{aligned} y_t &= g_{t-1} + \varepsilon_{1t} && \text{(2a)} \\ \Delta g_t &= \varepsilon_{2t} && \text{(2b)} \end{aligned}$$

where the shocks $\{\varepsilon_{it}\}_{i=1}^{2}$ are assumed to be *i.i.d.* standard normal and mutually uncorrelated, with unit standard deviation. The modified (standard) SSM with shocks down the bottom of state vector:

$$y_t = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} g_t \\ g_{t-1} \\ \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} + \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} \tag{3}$$

$$\begin{bmatrix} g_t \\ g_{t-1} \\ \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} g_{t-1} \\ g_{t-2} \\ \varepsilon_{1t-1} \\ \varepsilon_{2t-2} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}. \tag{4}$$

Running the shock recovery code `AP23.m` we get the steady-state values of:

Shocks	$P^*_{t\|T}$	$P^*_{t\|t}$
ε_{1t}	0.4472	0.6180
ε_{2t}	0.5528	1.0000

(5) Pstar0

and the first 8 filtered and smoothed shocks of state vector $X_t = \begin{bmatrix} g_t & g_{t-1} & \varepsilon_{1t} & \varepsilon_{2t} \end{bmatrix}'$ are as follows:

Filtered				Smoothed			
$\hat{g}_{t\|t}$	$\hat{g}_{t-1\|t}$	$\hat{\varepsilon}_{1t\|t}$	$\hat{\varepsilon}_{2t\|t}$	$\hat{g}_{t\|T}$	$\hat{g}_{t-1\|T}$	$\hat{\varepsilon}_{1t\|T}$	$\hat{\varepsilon}_{2t\|T}$
0	0	0	0	−0.2221	−0.0740	0.0740	−0.1481
−0.6210	−0.6210	−0.4140	0	0.4427	−0.2221	−0.8129	0.6648
0.2962	0.2962	0.5732	0	0.6807	0.4427	0.4267	0.2381
0.8620	0.8620	0.3503	0	0.3873	0.6807	0.5315	−0.2935
0.6175	0.6175	−0.1511	0	0.0147	0.3873	0.0791	−0.3726
0.0192	0.0192	−0.3698	0	0.0074	0.0147	−0.3653	−0.0073
0.3997	0.3997	0.2352	0	−0.6274	0.0074	0.6275	−0.6348
−0.6234	−0.6234	−0.6323	0	−0.6340	−0.6274	−0.6283	−0.0065

From the table above it is clear that the filtered estimates imply: $\hat{g}_{t|t} \equiv \hat{g}_{t|t-1}$ and $\hat{\varepsilon}_{2t|t} = 0$ as you proposed. Below uses the shock recovery SSM form as a check.

1.3 Shock recovery SSM

Collect all observables in $Z_t = \Delta y_t$ and all shocks (and other state variables) in X_t to yield:

$$\Delta y_t = \varepsilon_{2t-1} + \Delta\varepsilon_{1t} \tag{6}$$

which can then be written in SSF as:

$$\text{State}: \quad X_t = AX_{t-1} + C\varepsilon_t \tag{7}$$

$$\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{1t-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{1t-2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}. \tag{8}$$

$$\text{Measurement}: \quad Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t$$

$$Z_t = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{1t-1} \end{bmatrix} + \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{1t-2} \end{bmatrix} + 0 \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}$$

From the shock recovery SSM, the steady-state diagonal entries are:

Shocks	$P^*_{t\|T}$	$P^*_{t\|t}$
ε_{1t}	0.4472	0.6180
ε_{2t}	0.5528	1.0000

(9)

and thus identical to the ones in (5).

As above, the first 8 filtered and smoothed shocks of (shock-recovery) state vector $X_t = \begin{bmatrix} \varepsilon_{1t} & \varepsilon_{2t} & \varepsilon_{1t-1} \end{bmatrix}'$ are as follows:

Filtered			Smoothed		
$\hat{\varepsilon}_{1t\|t}$	$\hat{\varepsilon}_{2t\|t}$	$\hat{\varepsilon}_{1t-1\|t}$	$\hat{\varepsilon}_{1t\|T}$	$\hat{\varepsilon}_{2t\|T}$	$\hat{\varepsilon}_{1t-1\|T}$
0	0	0	−0.0768	0.1151	−0.0384
−0.1536	0	0.1024	−0.6014	0.7165	−0.0768
0.6041	0	−0.5311	0.4214	0.2951	−0.6014
−0.0176	0	0.6150	−0.5239	0.8191	0.4214
0.7235	0	−0.4648	0.5689	0.2501	−0.5239
−0.0546	0	0.7573	−0.5477	0.7978	0.5689
0.8556	0	−0.5835	0.9492	−0.1514	−0.5477
0.2709	0	0.6882	0.9542	−1.1056	0.9492

Consistent with the $P^*_{t|t} = 1$ entry for ε_{2t} and as previously discussed, the filtered $\hat{\varepsilon}_{2t|t}$ are exactly zero for all t.