

# 1 HP97

The Filter of Hodrick and Prescott (1997, HP-Filter) can be expressed as an SSM following a UC model structure as:

HP0

$$y_t = y_t^* + y_t^c \quad (1a) \quad \text{HP0a}$$

$$\Delta^2 y_t^* = \varepsilon_{1t} \quad (1b) \quad \text{HP0b}$$

$$y_t^c = \phi \varepsilon_{2t}, \quad (1c) \quad \text{HP0c}$$

where  $y_t$  is (generally 100 times) the log of GDP, and  $\varepsilon_{1t}$  and  $\varepsilon_{2t}$  are  $N(0, 1)$ . The standard deviation  $\phi$  is the (square root of the) smoothing parameter, generally set to 40, implying a value of ' $\lambda$ ' of 1600.

## 2 Shock recovery SSM

### 2.1 SSM with lagged states

Kurz's (2018) SSM has the following general form:

SSM

$$\text{Measurement : } Z_t = D_1 X_t + D_2 X_{t-1} + R \varepsilon_t \quad (2a) \quad \text{ssm1}$$

$$\text{State : } X_t = A X_{t-1} + C \varepsilon_t, \quad (2b) \quad \text{ssm2}$$

where  $\varepsilon_t \sim MN(0, I_m)$ ,  $D_1, D_2, A, R$  are  $C$  conformable system matrices,  $Z_t$  the observed variable and  $X_t$  the latent state variable.

### 2.2 HP97 SSM for shock recovery

To assess recovery, re-write the model in (1) in '*shock recovery*' State Space Form (SSF). That is, collect all observables in  $Z_t$  and all shocks (and other state variables) in  $X_t$  to yield:

$$\begin{aligned} \Delta^2 y_t &= \Delta^2 y_t^* + \Delta^2 y_t^c \\ &= \varepsilon_{1t} + \phi \Delta^2 \varepsilon_{2t} \\ &= \varepsilon_{1t} + \phi \varepsilon_{2t} - 2\phi \varepsilon_{2t-1} + \phi \varepsilon_{2t-2}, \end{aligned} \quad (3) \quad \text{z}$$

where  $\Delta^2 y_t = Z_t$  is the only observed variable.

Note: The estimates of the shocks from the Kalman Filter  $E_t X_t = E_t \begin{bmatrix} \varepsilon_t & \varepsilon_{2t} & \varepsilon_{2t-1} \end{bmatrix}'$  will be linked by the identity:

$$E_t \varepsilon_{2t} = \phi E_t \varepsilon_{1t}$$

and from the Kalman Smoother  $E_T X_t = E_T \begin{bmatrix} \varepsilon_t & \varepsilon_{2t} & \varepsilon_{2t-1} \end{bmatrix}'$  by the identity:

$$\Delta^2 E_T \varepsilon_{1t} = \frac{1}{\phi} E_T \varepsilon_{2t-2}. \quad (4) \quad \text{KS}$$

With  $\varepsilon_{1t} = \Delta^2 y_t^*$  and  $\varepsilon_{2t} = \frac{1}{\phi} y_t^c$ , this means that the output from the standard HP-Filter will give the identity:

$$\Delta^4 y_t^* = \frac{1}{\phi} y_{t-2}^c$$

$$\Delta^4 \text{HP-trend}_t = \frac{1}{\phi^2} \text{HP-cycle}_{t-2}.$$

Indeed, running a regression of  $\Delta^4 \text{HP-trend}_t$  on  $\text{HP-cycle}_{t-2}$  (without an intercept) yields indeed a regression coefficient of  $0.000625 = 1/1600$  when applied to US-GDP data that was HP-Filtered with the smoothing parameter set to  $\lambda = 1600 = 40^2$ . The regression fit is perfect, with an  $R^2$  of 1 and a residual sum of squares of 0.

The Measurement and State equations corresponding to (3) are:

Measurement :  $Z_t = D_1 X_t + D_2 X_{t-1} + R \varepsilon_t$

$$= \varepsilon_{1t} + \phi \varepsilon_{2t} - 2\phi \varepsilon_{2t-1} + \phi \varepsilon_{2t-2} \quad (5a)$$

$$Z_t = \underbrace{\begin{bmatrix} 1 & \phi & 0 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{2t-1} \end{bmatrix}}_{X_t} + \underbrace{\begin{bmatrix} 0 & -2\phi & \phi \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{2t-2} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 \end{bmatrix}}_R \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}} \quad (5b)$$

State :  $X_t = A X_{t-1} + C \varepsilon_t,$

$$\underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{2t-1} \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_A \underbrace{\begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{2t-2} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_C \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}}_{\varepsilon_t}. \quad (5c)$$