1 Clark87

The UC model of Clark (1987) is a generalisation of the HP–Filter (a local linear trend model)
clark0 that can be expressed as an SSM taking the following form:

$$y_t = y_t^* + \tilde{y}_t \tag{1a}$$
$$\Delta y_t^* = g_{t-1} + \sigma_1 \varepsilon_{1t} \tag{1b}$$
$$\Delta g_t = \sigma_2 \varepsilon_{2t} \tag{1c}$$
$$a(L)\tilde{y}_t = \sigma_3 \varepsilon_{3t}, \tag{1d}$$

where the shocks $\{\varepsilon_{it}\}_{i=1}^{3}$ are assumed to be *i.i.d*. standard normal and mutually uncorrelated, with standard deviation σ_i and $a(L)$ is commonly assumed to be a stable AR(2), so that $a(L) = (1 - a_1 L - a_2 L^2)$. The only observable is y_t (generally 100 times) the log of real GDP and with the cycle (denoted by $\tilde{y}_t$) now allowed to be serially correlated, following a stationary AR(2) process. There are 3 shocks in the model.

The '*numbered*' shock to '*named*' shock mapping is:

$$\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix} = \begin{bmatrix} \varepsilon_t^{y^*} \\ \varepsilon_t^{g} \\ \varepsilon_t^{\tilde{y}} \end{bmatrix}. \tag{2}$$

2 Shock recovery SSM

2.1 SSM with lagged states

SSM Kurz's (2018) SSM has the following general from:

$$\text{Measurement}: \quad Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t \tag{3a}$$ ssm1
$$\text{State}: \quad X_t = AX_{t-1} + C\varepsilon_t, \tag{3b}$$ ssm2

where $\varepsilon_t \sim MN(0, I_m)$, D_1, D_2, A, R are C are conformable system matrices, Z_t the observed variable and X_t the latent state variable.

2.2 Clark87 SSM for shock recovery

To assess recovery, re-write the model in (1) in '*shock recovery*' State Space Form (SSF). That is, collect all observables in Z_t and all shocks (and other state variables) in X_t to yield:

$$\begin{aligned}
\Delta^2 y_t &= \Delta^2 y_t^* + \Delta^2 \tilde{y}_t \\
&= \sigma_1 \Delta\varepsilon_{1t} + \Delta g_{t-1} + \Delta^2 \tilde{y}_t \\
&= \sigma_1 \Delta\varepsilon_{1t} + \sigma_2 \varepsilon_{2t-1} + a(L)^{-1} \Delta^2 \sigma_3 \varepsilon_{3t} \\
\Leftrightarrow a(L)\Delta^2 y_t &= \sigma_1 a(L)\Delta\varepsilon_{1t} + \sigma_2 a(L)\varepsilon_{2t-1} + \sigma_3 \Delta^2 \varepsilon_{3t}
\end{aligned}$$

$$
\begin{aligned}
Z_t = {} & \sigma_1 \left[\varepsilon_{1t} - (2+a_1)\,\varepsilon_{1t-1} + (1+2a_1-a_2)\,\varepsilon_{1t-2} + (2a_2-a_1)\,\varepsilon_{1t-3} - a_2\varepsilon_{1t-4}\right] \\
& + \sigma_2(\varepsilon_{2t-1} - a_1\varepsilon_{2t-2} - a_2\varepsilon_{2t-3}) \\
& + \sigma_3(\varepsilon_{3t} - 2\varepsilon_{3t-1} + \varepsilon_{3t-2}),
\end{aligned} \tag{4}
$$

Z

where I have made use of the fact:

$$
\begin{aligned}
a(L)\Delta &= (1 - a_1 L - a_2 L^2)(1-L)^2 \\
&= 1 - (2+a_1)\,L + (1+2a_1-a_2)\,L^2 + (2a_2-a_1)\,L^3 - a_2 L^4,
\end{aligned} \tag{5}
$$

expaL

and $Z_t = a(L)\Delta^2 y_t$ is the only observed variable.

The Measurement and State equations corresponding to (4) are:

KOSSM

$$
\text{Measurement}: \quad Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t
$$

$$
Z_t = \underbrace{\begin{bmatrix} \sigma_1 & 0 & \sigma_3 & -\sigma_1(2+a_1) & \sigma_1(1+2a_1-a_2) & \sigma_1(2a_2-a_1) & 0 & 0 & 0 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{1t-1} \\ \varepsilon_{1t-2} \\ \varepsilon_{1t-3} \\ \varepsilon_{2t-1} \\ \varepsilon_{2t-2} \\ \varepsilon_{3t-1} \end{bmatrix}}_{X_t} \tag{6a}
$$

$$
+ \underbrace{\begin{bmatrix} 0 & \sigma_2 & -2\sigma_3 & 0 & 0 & -\sigma_1 a_2 & -\sigma_2 a_1 & -\sigma_2 a_2 & \sigma_3 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{1t-2} \\ \varepsilon_{1t-3} \\ \varepsilon_{1t-4} \\ \varepsilon_{2t-2} \\ \varepsilon_{2t-3} \\ \varepsilon_{3t-2} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}}_{\varepsilon_t} \tag{6b}
$$

$$
\text{State}: \quad X_t = AX_{t-1} + C\varepsilon_t,
$$

$$\underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{1t-1} \\ \varepsilon_{1t-2} \\ \varepsilon_{1t-3} \\ \varepsilon_{2t-1} \\ \varepsilon_{2t-2} \\ \varepsilon_{3t-1} \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{1t-2} \\ \varepsilon_{1t-3} \\ \varepsilon_{1t-4} \\ \varepsilon_{2t-2} \\ \varepsilon_{2t-3} \\ \varepsilon_{3t-2} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}}_{\varepsilon_t}. \tag{6c}$$