1 SSMs with lagged states following Kurz (2018)

SSM Kurz's SSM takes the following general from:

$$\begin{aligned} \text{Measurement}: \quad & Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t && \text{(1a)} \quad \texttt{ssm1} \\ \text{State}: \quad & X_t = A X_{t-1} + C\varepsilon_t, && \text{(1b)} \quad \texttt{ssm2} \end{aligned}$$

where $\varepsilon_t \sim MN(0, I_m)$, D_1, D_2, A, R are C are conformable system matrices, Z_t the observed variable and X_t the latent state.

2 HP97

The HP-Filter of Hodrick and Prescott (1997) can be expressed as an SSM following a UC
HP0 model structure as:

$$\begin{aligned} y_t &= y_t^* + y_t^c && \text{(2a)} \quad \texttt{HP0a} \\ \Delta^2 y_t^* &= \varepsilon_{1t} && \text{(2b)} \quad \texttt{HP0b} \\ y_t^c &= \phi\varepsilon_{2t}, && \text{(2c)} \quad \texttt{HP0c} \end{aligned}$$

where y_t is (generally 100 times) the log of GDP, and ε_{1t} and ε_{2t} are $N(0, 1)$. The standard deviation ϕ is the (square root of the) smoothing parameter, generally set to 40, implying a value of 'λ' of 1600.

To assess recovery, re-write the model in'*shock recovery*' state-space from (SSF). That is, collect all observables in Z_t and all shocks (and other state variables) in X_t. The relation in (2) can be written as:

$$\begin{aligned} \Delta^2 y_t &= \Delta^2 y_t^* + \Delta^2 y_t^c \\ &= \varepsilon_{1t} + \phi\Delta^2\varepsilon_{2t} \\ Z_t &= \varepsilon_{1t} + \phi\varepsilon_{2t} - 2\phi\varepsilon_{2t-1} + \phi\varepsilon_{2t-2}, && \text{(3)} \quad \texttt{Z} \end{aligned}$$

where $Z_t = \Delta^2 y_t$ is the observed variable.

KOSSM The Measurement and State equations corresponding to (1) are:

$$Z_t = \underbrace{\begin{bmatrix} 1 & \phi & -2\phi \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{2t-1} \end{bmatrix}}_{X_t} + \underbrace{\begin{bmatrix} 0 & 0 & \phi \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{2t-2} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}}_{\varepsilon_t} \tag{4a}$$

$$\underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{2t-1} \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{2t-2} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}}_{\varepsilon_t} \tag{4b}$$

The Kalman Filter and Smoother based steady-state $\mathrm{diag}(P)$ are, respectively:

$$\mathrm{diag}(P^*_{t|t}) = \begin{bmatrix} 0.9995 \\ 0.2006 \\ 0.1608 \end{bmatrix} \text{ and } \mathrm{diag}(P^*_{t|T}) = \begin{bmatrix} 0.9439 \\ 0.0561 \\ 0.0561 \end{bmatrix}. \tag{5}$$