1 HP97

The Filter of Hodrick and Prescott (1997, HP-Filter) can be expressed as an SSM following a
HP0 UC model structure as:

$$
\begin{aligned}
y_t &= y_t^* + y_t^c && \text{(1a)} \quad \texttt{HP0a} \\
\Delta^2 y_t^* &= \varepsilon_{1t} && \text{(1b)} \quad \texttt{HP0b} \\
y_t^c &= \phi \varepsilon_{2t}, && \text{(1c)} \quad \texttt{HP0c}
\end{aligned}
$$

where y_t is (generally 100 times) the log of GDP, and ε_{1t} and ε_{2t} are $N(0,1)$. The standard deviation ϕ is the (square root of the) smoothing parameter, generally set to 40, implying a value of 'λ' of 1600.

The number shock to named shock mapping is:

$$
\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} = \begin{bmatrix} \varepsilon_t^* \\ \varepsilon_t^c \end{bmatrix}. \tag{2}
$$

2 Shock recovery SSM

2.1 SSM with lagged states

SSM Kurz's (2018) SSM has the following general from:

$$
\begin{aligned}
\text{Measurement :} \quad Z_t &= D_1 X_t + D_2 X_{t-1} + R\varepsilon_t && \text{(3a)} \quad \texttt{ssm1} \\
\text{State :} \quad X_t &= A X_{t-1} + C\varepsilon_t, && \text{(3b)} \quad \texttt{ssm2}
\end{aligned}
$$

where $\varepsilon_t \sim MN(0, I_m)$, D_1, D_2, A, R are C are conformable system matrices, Z_t the observed variable and X_t the latent state variable.

2.2 HP97 SSM for shock recovery

To assess recovery, re-write the model in (1) in '*shock recovery*' State Space Form (SSF). That is, collect all observables in Z_t and all shocks (and other state variables) in X_t to yield:

$$
\begin{aligned}
\Delta^2 y_t &= \Delta^2 y_t^* + \Delta^2 y_t^c \\
&= \varepsilon_{1t} + \phi \Delta^2 \varepsilon_{2t} \\
&= \varepsilon_{1t} + \phi \varepsilon_{2t} - 2\phi \varepsilon_{2t-1} + \phi \varepsilon_{2t-2}, && \text{(4)} \quad \texttt{Z}
\end{aligned}
$$

where $\Delta^2 y_t = Z_t$ is the only observed variable.

Note: The estimates of the shocks from the Kalman Filter $E_t X_t = E_t \begin{bmatrix} \varepsilon_t & \varepsilon_{2t} & \varepsilon_{2t-1} \end{bmatrix}'$ will be linked by the identity:

$$
E_t \varepsilon_{2t} = \phi E_t \varepsilon_{1t}
$$

and from the Kalman Smoother $E_T X_t = E_T \begin{bmatrix} \varepsilon_t & \varepsilon_{2t} & \varepsilon_{2t-1} \end{bmatrix}'$ by the identity:

$$\Delta^2 E_T \varepsilon_{1t} = \frac{1}{\phi} E_T \varepsilon_{2t-2}. \tag{5}$$ KS

With $\varepsilon_{1t} = \Delta^2 y_t^*$ and $\varepsilon_{2t} = \frac{1}{\phi} y_t^c$, this means that the output from the standard HP–Filter will give the identity:

$$\begin{aligned} \Delta^4 y_t^* &= \frac{1}{\phi} y_{t-2}^c \\ \Delta^4 \text{HP–trend}_t &= \frac{1}{\phi^2} \text{HP–cycle}_{t-2}. \end{aligned}$$

Indeed, running a regression of Δ^4HP–trend$_t$ on HP–cycle$_{t-2}$ (without an intercept) yields indeed a regression coefficient of $0.000625 = 1/1600$ when applied to US–GDP data that was HP–Filtered with the smoothing parameter set to $\lambda = 1600 = 40^2$. The regression fit is perfect, with an R^2 of 1 and a residual sum of squares of 0.

KOSSM The Measurement and State equations corresponding to (4) are:

$$\begin{aligned} \text{Measurement :} \quad Z_t &= D_1 X_t + D_2 X_{t-1} + R\varepsilon_t \\ &= \varepsilon_{1t} + \phi\varepsilon_{2t} - 2\phi\varepsilon_{2t-1} + \phi\varepsilon_{2t-2} \end{aligned} \tag{6a}$$

$$Z_t = \underbrace{\begin{bmatrix} 1 & \phi & 0 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{2t-1} \end{bmatrix}}_{X_t} + \underbrace{\begin{bmatrix} 0 & -2\phi & \phi \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{2t-2} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}}_{} \tag{6b}$$

$$\text{State :} \quad X_t = AX_{t-1} + C\varepsilon_t,$$

$$\underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{2t-1} \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{2t-2} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}}_{\varepsilon_t}. \tag{6c}$$