Can we Recover a Varying Natural Rate of Interest and the NAIRU?

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January 25, 2022

1 Introduction

Recovering a time varying NAIRU and a neutral interest rate has been the concern of much research in central banks over the past decade. The Reserve Bank of Australia and the Treasury have been involved in this work. McCririck and Rees (2017) (MR) is an example and it uses a model consisting of the following equations

$$
\begin{aligned}
\tilde{y}_t &= \alpha_1 \tilde{y}_{t-1} + \alpha_2 \tilde{y}_{t-2} - \frac{a_r}{2} \sum_{i=1}^{2} (r_{t-i} - r^*_{t-i}) + \sigma_1 \varepsilon_{1t} \\
u_t &= u^*_t + \beta(.4\tilde{y}_t + .3\tilde{y}_{t-1} + .2\tilde{y}_{t-2} + .1\tilde{y}_{t-3}) + \sigma_2 \varepsilon_{2t} \\
\pi_t &= (1-\beta_1)\pi^e_t + \frac{\beta_1}{3} \sum_{i=1}^{3} \pi_{t-i} + \beta_2 (u_{t-1} - u^*_{t-1}) + \sigma_3 \varepsilon_{3t} \\
y^*_t &= y^*_{t-1} + g_t + \sigma_4 \varepsilon_{4t} = y^*_{t-1} + g_{t-1} + \sigma_5 \varepsilon_{5t} + \sigma_4 \varepsilon_{4t} \\
\Delta g_t &= \sigma_5 \varepsilon_{5t} \\
\Delta u^*_t &= \sigma_6 \varepsilon_{6t} \\
r^*_t &= 4g_t + z_t = 4g_{t-1} + 4\sigma_5 \varepsilon_{5t} + z_{t-1} + \sigma_7 \varepsilon_{7t} \\
\Delta z_t &= \sigma_7 \varepsilon_{7t}
\end{aligned}
$$

where $\tilde{y}_t = y_t - y^*_t$, is an output gap, y^*_t is potential gdp; r_t is a real interest rate, r^*_t the neutral rate; u_t is unemployment rate and u^*_t the NAIRU; π_t is inflation and π^e_t is measured expected inflation. There are evolving processes for potential GDP, the NAIRU and the neutral rate. Here there are 7 shocks whose standard deviations are σ_j and they are $\sigma_j \varepsilon_{jt}$ where ε_{jt} have variances of unity. This type of model is similar to Holsten et al (2017) and has been widely used. Buncic (2020) has pointed out issues with the parameter estimation of Holsten et al (2017) but, in MR, the estimation is just MLE with some priors. The estimated standard deviations from MR are

$$\begin{aligned}
\sigma_1 &= .37, \sigma_2 = .07, \sigma_3 = .8, \sigma_4 = .54 \\
\sigma_5 &= .05, \sigma_6 = .15, \sigma_7 = .34
\end{aligned}$$

Now define the states

$$\begin{aligned}
\psi_{1t} &= \tilde{y}_t, \psi_{2t} = u_t^*, \psi_{3t} = r_t^*, \psi_{4t} = \tilde{y}_{t-1}, \psi_{5t} = \tilde{y}_{t-2} \\
\psi_{6t} &= r_{t-1}^*, \psi_{7t} = y_t^*, \psi_{8t} = g_t, \psi_{9t} = z_t, \{\psi_{9+j,t} = \varepsilon_{jt}\}_{j=1}^{7}
\end{aligned}$$

and the observable (data) on the three variables

$$\begin{aligned}
z_{1t} &= y_t \\
z_{2t} &= u_t \\
z_{3t} &= \pi_t - (1-\beta_1)\pi_t^e - \frac{\beta_1}{3}\sum_{i=1}^{3}\pi_{t-i} - \beta_2 u_{t-1}.
\end{aligned}$$

The variable π_t^e is observed expectations which has been constructed and MR take it as exogenous to their system. Therefore there are 3 observables and 7 shocks and we know that we cannot recover this number. At best we can find three shocks. The work below aims at finding what these might be.

To begin it is useful to think about having one observable and two shocks

$$\begin{aligned}
z_t &= \varepsilon_{1t} + \varepsilon_{2t} \\
&= \begin{bmatrix} 1 & 1 \end{bmatrix} \varepsilon_t \\
&= G\varepsilon_t
\end{aligned}$$

Then it should be apparent that ε_t cannot be recovered from z_t as G is not square and so we can't form an inverse. One might say that one should use a generalized inverse. This finds the estimated $\tilde{\varepsilon}_t$ that is closest to ε_t using a quadratic norm. In this case the g inverse is

$$\begin{bmatrix} .5 \\ .5 \end{bmatrix},$$

and so $\tilde{\varepsilon}_{1t} = .5z_t = .5(\varepsilon_{1t} + \varepsilon_{2t})$ and $\tilde{\varepsilon}_{1t}$ does not equal ε_{1t} but captures other shocks. To get some idea of the difference between the estimate and the assumed shock consider looking at the index $\delta = var(\ \tilde{\varepsilon}_{1t}$ -$\varepsilon_{1t}) = var(\frac{1}{2}(\varepsilon_{1t} + \varepsilon_{2t})) = .5$. If $\delta = 0$ we recover ε_{1t}. Essentially all we do in Pagan and Robinson is to show how one can recover things like δ for a broad range of models.

Consider $\delta = var(\tilde{\varepsilon}_t - \varepsilon_t)$. This is

$$\delta = var(\tilde{\varepsilon}_t) - 2cov(\tilde{\varepsilon}_t, \varepsilon_t) + var(\varepsilon_t)$$

The last term is unity and so

$$\delta = var(\tilde{\varepsilon}_t) - 2\rho \times std(\tilde{\varepsilon}_t) + 1,$$

where ρ is the correlation between $\tilde{\varepsilon}_t$ and ε_t. Consequently

$$\begin{aligned} 1 - \delta &= 2\rho \times std(\tilde{\varepsilon}_t) - var(\tilde{\varepsilon}_t) \\ &= std(\tilde{\varepsilon}_t)(2\rho - std(\tilde{\varepsilon}_t)) \end{aligned}$$

When $\delta = 0$ we have $std(\tilde{\varepsilon}_t) = 1$ as $\tilde{\varepsilon}_t = \varepsilon_t$ and so $\rho = 0$ i.e. there is no recovery of the shock. Things are more complex when δ is away from zero. When $\delta = 1$ $\rho = \frac{std(\tilde{\varepsilon}_t)}{2}$ and so the correlation depends on the $std(\tilde{\varepsilon}_t)$.

Returning to the MR model we need to work out the connections between states and observables. These are

$$\begin{aligned} z_{1t} &= \psi_{1t} + \psi_{7t} && (1) \\ z_{2t} &= \psi_{2t} + \beta(.4\psi_{1t} + .3\psi_{4t} + .2\psi_{5t} + .1\psi_{5t-1}) + \sigma_2\psi_{11,t} && (2) \\ z_{3t} &= \beta_2\psi_{2t-1} + \sigma_3\psi_{12,t} && (3) \\ & && (4) \end{aligned}$$

Finally the relations between the states and shocks are

$$
\begin{aligned}
\psi_{1t} &= \alpha_1\psi_{1,t-1} + \alpha_2\psi_{4,t-1} + \frac{a_r}{2}\sum_{i=1}^{2}(\psi_{3t-1} + \psi_{6t-1}) + \sigma_1\varepsilon_{1t} \\
\psi_{2t} &= \psi_{2t-1} + \sigma_6\varepsilon_{6t} \\
\psi_{3t} &= 4\psi_{8t-1} + \psi_{9t-1} + 4\varepsilon_{5t} + \sigma_7\varepsilon_{7t} \\
\psi_{4t} &= \psi_{1t-1} \\
\psi_{5t} &= \psi_{4t-1} \\
\psi_{6t} &= \psi_{3t-1} \\
\psi_{7t} &= \psi_{7t-1} + \psi_{8t-1} + \sigma_5\varepsilon_{5t} + \sigma_4\varepsilon_{4t} \\
\psi_{8t} &= \psi_{8t-1} + \sigma\varepsilon_{5t} \\
\psi_{9t} &= \psi_{9t-1} + \sigma_7\varepsilon_{7t} \\
\psi_{9+j,t} &= \varepsilon_{jt},\ j = 1, .., 7
\end{aligned}
$$

These equations can then be placed in Nimark's (2015) SSF which allows for lagged states in the observation equation. This was set out in Pagan and Robinson (2021) to look at recovery of shocks and the analysis below uses tools from their paper.

The SSF is estimated using the Kalman filter and smoother and one can get estimates of the shocks from it. For the filtered ones the estimate of ε_{it} is based on data up to and including t. For smoothed estimates all the data is used. The Kalman filter computes these recursively using a generalized i.e. leaving aside the lags, it expresses the estimated shocks Kz_t where K is the gain of the Kalman filter and this only has rank equal to the number of observables (as in the simple example above).

A matrix that is computed as part of the Kalman equations is P, often called the covariance matrix of the states. Actually it is the covariance matrix of the difference between them and their estimated values. So selecting the part of $P \;\; -P^*-$ that corresponds to the shocks, we take the diagonal elements of it and these give us the equivalent of δ above for each shock. There are two versions of P^* depending on whether we compute the estimate of the shocks using data up to the current position or all of it. In the latter case there is the Kalman smoother for this SSF, set out in Kurz (2018).

Using the parameters in MR we find that the diagonal elements of $P^*(t|t)$ and $P^*(t|T)$ are

$$
P^*(t|t) = \begin{bmatrix} .66 & .9 & .05 & .34 & .99 & .55 & 1.0 \end{bmatrix}
$$

and

$$P^*(t|T) = \left[\begin{array}{ccccccc} .55 & .76 & .03 & .24 & .95 & .48 & .98 \end{array} \right].$$

A value of zero is needed for shocks to be recoverable. The $P^*(t|t)$ are the values for whether one can recover the shocks from the filtered shocks and $P^*(t|T)$ for whether that can be done using the smoothed shocks. It is clear from the latter that the trend growth (ε_{5t}) and "unexplained r*" shocks (ε_{7t}) definitely cannot be recovered and they constitute the neutral rate shock. Using the standard deviations of the smoothed shocks we find that the implied correlations for these two shocks with the true values are .09 and .1. For the NAIRU shock (ε_{4t}) the correlation is .6. So the NAIRU is far better estimated than the neutral rate.

We might ask why the 7th element of the $P^*(t|t)$ is zero? The reason is that the filtered shock is zero. To explain that we note that the seventh shock only affects the data via the IS curve through r^*_{t-1}, so that one cannot estimate the shock at time t using just past and present data. One needs future data and that is what the smoother provides. This was discussed in Pagan and Robinson in the context of a model by Leeper et al.(2013). So $var(\tilde{\varepsilon}_{7t|t} - \varepsilon_{7t}) = 1$ when the filtered shock is used.

The consequences of the Kalmah gain being rank deficient is that estimated shocks are correlated. In the case of filtered shocks combinations of them are zero. For example we find

$$\begin{aligned} \tilde{\varepsilon}_{1t|t} &= .69\tilde{\varepsilon}_{4t|t} + .63\tilde{\varepsilon}_{6t|t} \\ \tilde{\varepsilon}_{4t|t} &= \tilde{\varepsilon}_{5t|t}, \end{aligned}$$

and so the the filtered potential gdp shock ($\tilde{\varepsilon}_{4t|t}$) and the trend growth shock ($\tilde{\varepsilon}_{5t|t}$) have a correlation of unity, while the filtered demand shock ($\tilde{\varepsilon}_{1t|t}$) and the trend growth shock have a correlation of -.46. Smoothed shocks have more complex dependence as it involves dependence between the current shocks and their lagged values. For example in this case

$$\tilde{\varepsilon}_{1t|T} = -\tilde{\varepsilon}_{1t-1|T} - 61(\Delta\tilde{\varepsilon}_{6t-1|T} + \Delta\tilde{\varepsilon}_{5t-1|T}) + 3.8\tilde{\varepsilon}_{3t|T} - 5.7\tilde{\varepsilon}_{4t-2|T}$$

So it makes them both correlated and serially correlated e.g. $\tilde{\varepsilon}_{1t|T}$ has a first order autocorrelation of .8.

Thus the shocks estimated from the data are very different from what is assumed about them. It is for this reason that techniques such as variable and variance decompositions do not apply as they are constructed from the estimated shocks and they do not have the properties needed for their correct computation, One also sees from above that even if ε_{jt} are uncorrelated and serially correlated with one another the estimated variants will not be so it is hard to check properties that have been assumed for the ε_{jt}

2 References

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