1 Clark87

The UC model of Clark (1987) is a generalisation of the HP–Filter (a local linear trend model)
clark0 that can be expressed in SSF ~~as an SSM~~ taking the following form:

$$y_t = y_t^* + \tilde{y}_t \tag{1a}$$
$$\Delta y_t^* = g_{t-1} + \sigma_1 \varepsilon_{1t} \tag{1b}$$
$$\Delta g_t = \sigma_2 \varepsilon_{2t} \tag{1c}$$
$$a(L)\tilde{y}_t = \sigma_3 \varepsilon_{3t}, \tag{1d}$$

where the shocks $\{\varepsilon_{it}\}_{i=1}^{3}$ are assumed to be *i.i.d.* $N(0,1)$ ~~standard normal~~ and mutually uncorrelated, with standard deviation $\{\sigma_i\}_{i=1}^{3}$ and $a(L)$ is commonly assumed to be a stable AR(2), so that $a(L) = (1 - a_1 L - a_2 L^2)$. The only observable is y_t (~~generally~~ 100 times) the log of real GDP and with the cycle (denoted by $\tilde{y}_t$) now allowed to be serially correlated, following a stationary AR(2) process. There are 3 shocks in the model.

The '*numbered*' shock to '*named*' shock mapping is:

$$\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix} = \begin{bmatrix} \varepsilon_t^{y^*} \\ \varepsilon_t^{g} \\ \varepsilon_t^{\tilde{y}} \end{bmatrix}. \tag{2}$$

SSF corresponding (to 1)

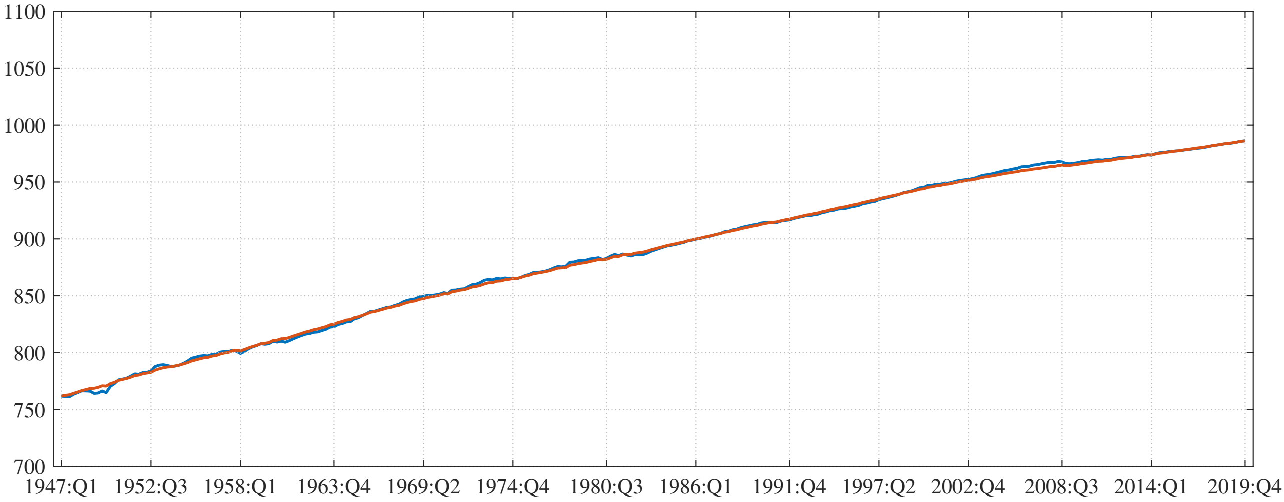
The ~~(standard)~~ SSM for ML estimation is:

$$y_t = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_t^* \\ g_t \\ \tilde{y}_t \\ \tilde{y}_{t-1} \end{bmatrix} + 0\varepsilon_t \tag{3}$$

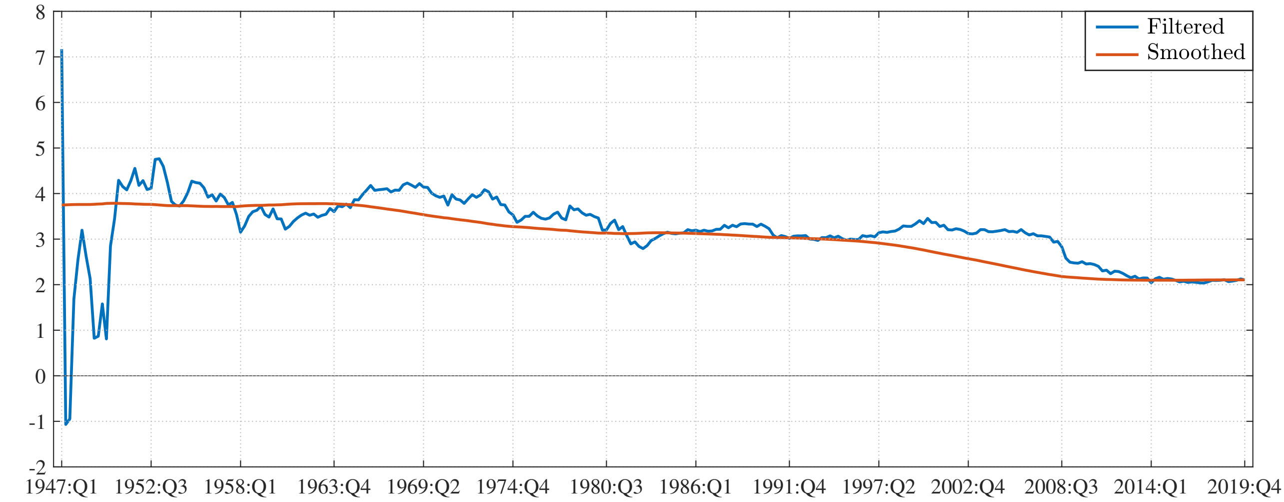
$$\begin{bmatrix} y_t^* \\ g_t \\ \tilde{y}_t \\ \tilde{y}_{t-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1}^* \\ g_{t-1} \\ \tilde{y}_{t-1} \\ \tilde{y}_{t-2} \end{bmatrix} + \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}. \tag{4}$$

in where

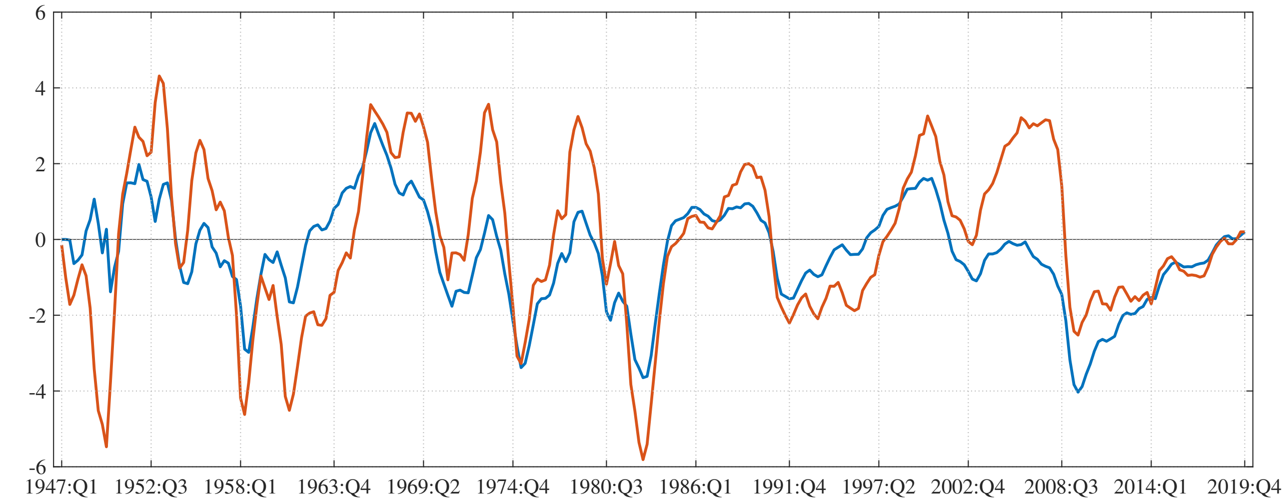
The code estimates Clark's 87 model on US GDP data from 1947:Q1 to 2019:Q4. A plot of the smoothed and filtered estimates is shown below.



Filtered and Smoothed estimates of y_t^*



Filtered and Smoohted estimates of g_t



Filtered and Smoothed estimates of $\tilde{y}_t$

fig:
KFS

2 Shock recovery SSM

2.1 SSM with lagged states

SSM Kurz's (2018) SSM has the following general from:

$$\begin{aligned} \text{Measurement}: \quad & Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t && \text{(5a)} \\ \text{State}: \quad & X_t = AX_{t-1} + C\varepsilon_t, && \text{(5b)} \end{aligned}$$

ssm1 ssm2

where $\varepsilon_t \sim MN(0, I_m)$, D_1, D_2, A, R are C are conformable system matrices, Z_t the observed variable and X_t the latent state variable.

2.2 Clark87 SSM for shock recovery

To assess recovery, re-write the model in (1) in '*shock recovery*' State Space Form (SSF). That is, collect all observables in Z_t and all shocks (and other state variables) in X_t to yield:

$$\begin{aligned} \Delta^2 y_t &= \Delta^2 y_t^* + \Delta^2 \tilde{y}_t && (6) \\ &= \sigma_1 \Delta\varepsilon_{1t} + \Delta g_{t-1} + \Delta^2 \tilde{y}_t \\ &= \sigma_1 \Delta\varepsilon_{1t} + \sigma_2 \varepsilon_{2t-1} + a(L)^{-1} \sigma_3 \Delta^2 \varepsilon_{3t} && (7) \\ \Leftrightarrow a(L)\Delta^2 y_t &= \sigma_1 a(L)\Delta\varepsilon_{1t} + \sigma_2 a(L)\varepsilon_{2t-1} + \sigma_3 \Delta^2 \varepsilon_{3t} \end{aligned}$$

2diff e2lag

where $Z_t = a(L)\Delta^2 y_t$ is the only observed variable.

Re-writing these in more convenient form for the SSF yields:

$$\begin{aligned} \underbrace{a(L)\Delta^2 y_t}_{Z_t} &= a(L)\sigma_1 \Delta\varepsilon_{1t} + a(L)\sigma_2 \varepsilon_{2t-1} + \sigma_3 \Delta^2 \varepsilon_{3t} && (8) \\ &= \sigma_1 \Delta\varepsilon_{1t} - a_1 \sigma_1 \Delta\varepsilon_{1t-1} - a_2 \sigma_1 \Delta\varepsilon_{1t-2} \\ &+ \sigma_2 \varepsilon_{2t-1} - a_1 \sigma_2 \varepsilon_{2t-2} - a_2 \sigma_2 \varepsilon_{2t-3} \\ &+ \sigma_3 \Delta\varepsilon_{3t} - \sigma_3 \Delta\varepsilon_{3t-1}, \end{aligned}$$

eqZ

which can then be written in SSF as:

$$\text{State}: \quad X_t = AX_{t-1} + C\varepsilon_t \qquad (9)$$

$$\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \Delta\varepsilon_{1t} \\ \Delta\varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{2t-2} \\ \Delta\varepsilon_{3t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \Delta\varepsilon_{1t-1} \\ \Delta\varepsilon_{1t-2} \\ \varepsilon_{2t-2} \\ \varepsilon_{2t-3} \\ \Delta\varepsilon_{3t-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}. \qquad (10)$$

$$\text{Measurement}: \quad Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t$$

$$Z_t = \begin{bmatrix} 0 & 0 & 0 & \sigma_1 & -a_1\sigma_1 & 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \Delta\varepsilon_{1t} \\ \Delta\varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{2t-2} \\ \Delta\varepsilon_{3t} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & \sigma_2 & 0 & 0 & -a_2\sigma_1 & -a_1\sigma_2 & -a_2\sigma_2 & -\sigma_3 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \Delta\varepsilon_{1t-1} \\ \Delta\varepsilon_{1t-2} \\ \varepsilon_{2t-2} \\ \varepsilon_{2t-3} \\ \Delta\varepsilon_{3t-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}$$

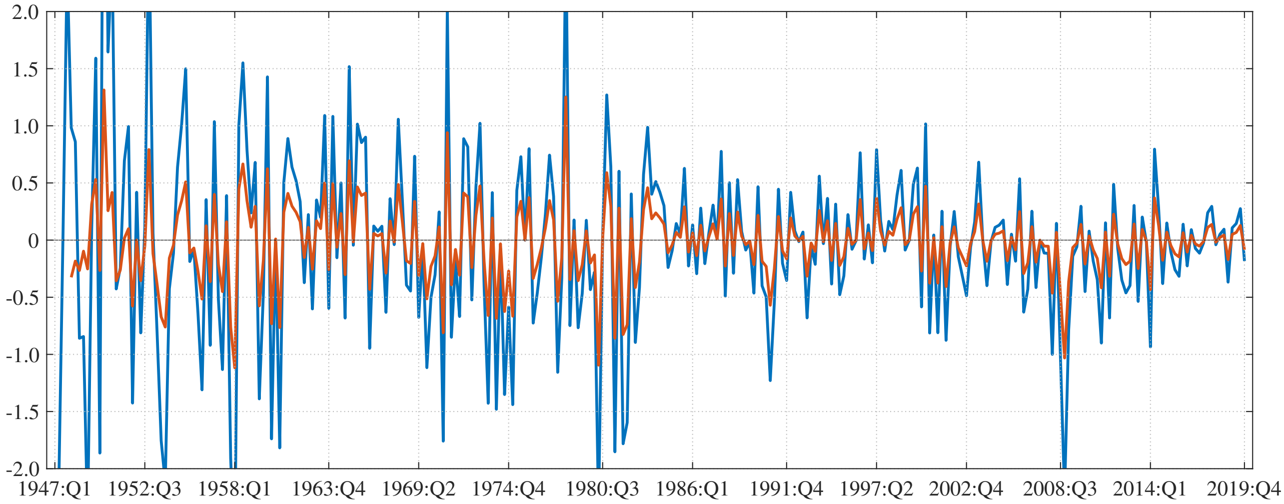
Running the shock recovery code `Clark87.m` we get the following steady-state diagonal entries:

Shocks	$P^*_{t\|T}$	$P^*_{t\|t}$
ε_{1t}	0.5469	0.5989
ε_{2t}	0.9870	1.0000
ε_{3t}	0.4661	0.5153

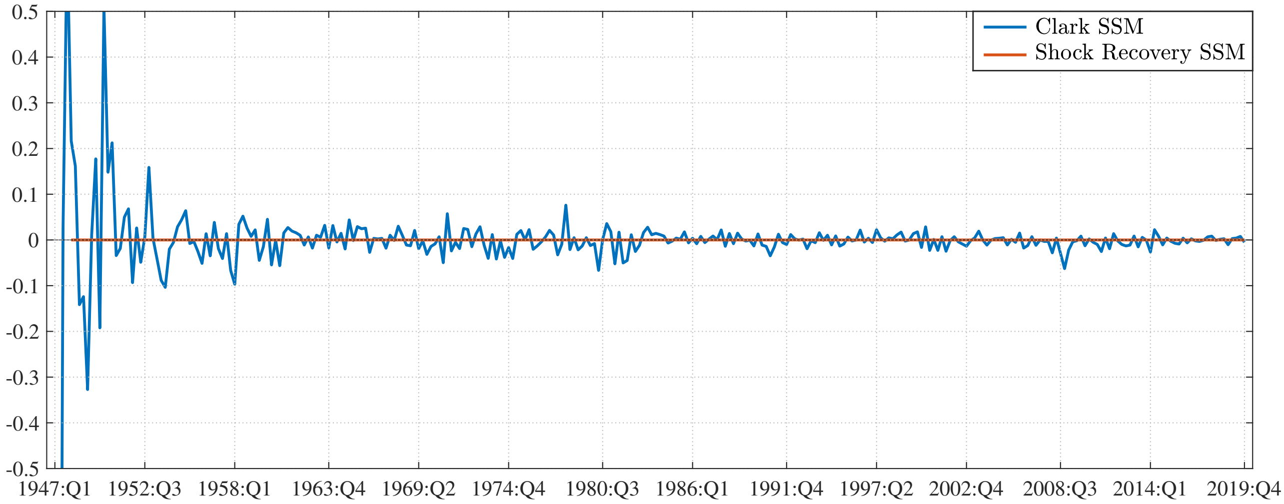
The second shock ε_{2t} (corresponding to ε_t^g, ie., trend growth) is not recoverable.

Looking at (7), we see that ε_{2t} enters with a lag into the measurement equation. Following the same logic that Adrian used, we should then get only $\hat{\varepsilon}_{2t-1|t}$ from the Kalman Filter.

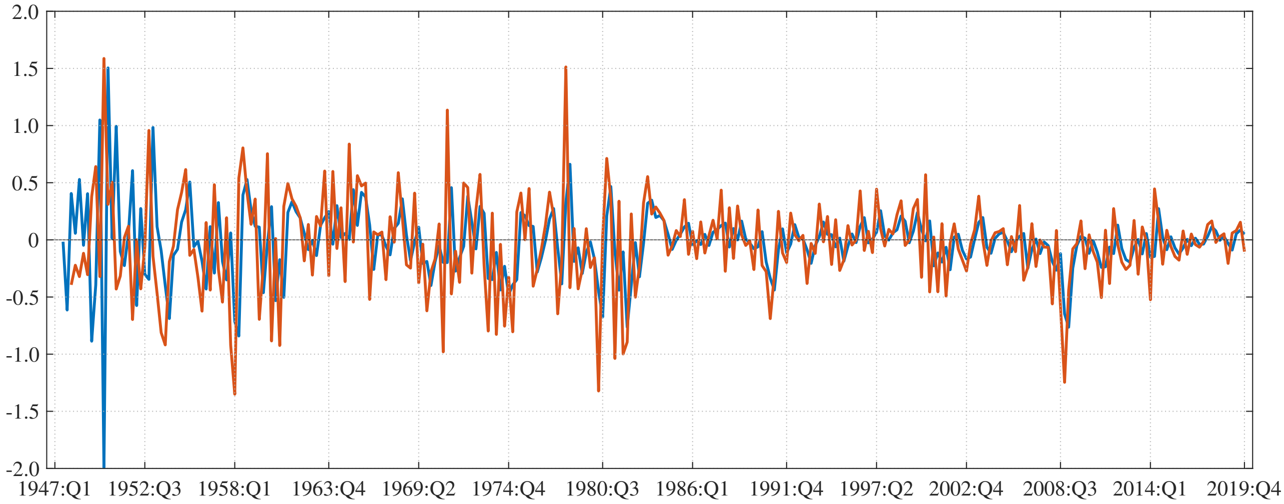
Below I plot estimates of the Filtered and Smoothed shocks from the model fitted to US data, where shocks from the SSM in (1) are constructed in line with the equations listed there, ie., as $\eta_{2t|t} = \sigma_2\varepsilon_{2t} = \Delta g_t$ for the filtered and smoothed alternatives.



Filtered $\eta_1 = \sigma_1 \varepsilon_1$



Filtered $\eta_2 = \sigma_2 \varepsilon_2$



Filtered $\eta_3 = \sigma_3 \varepsilon_3$

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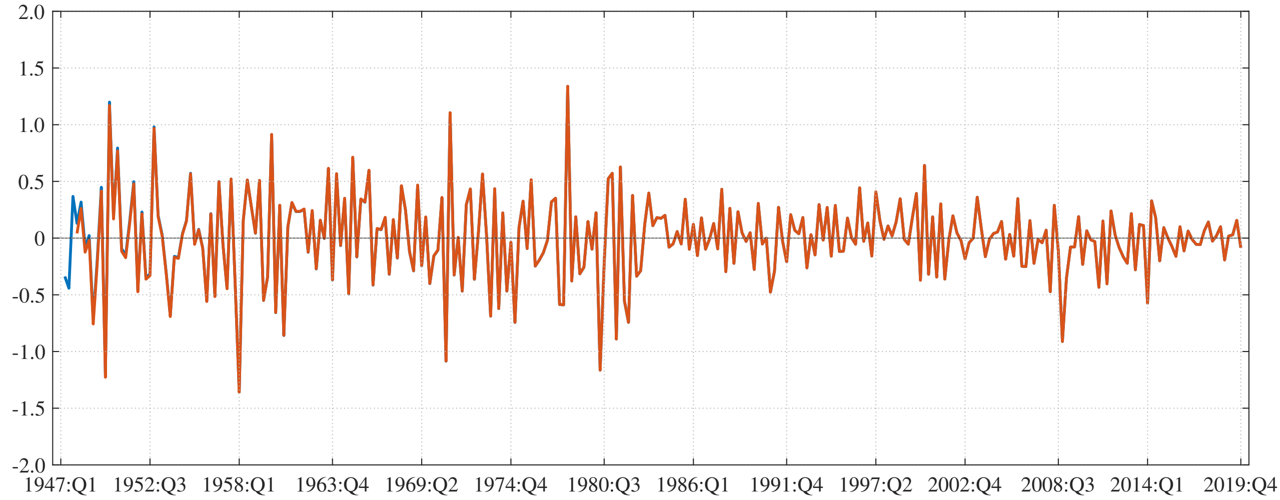
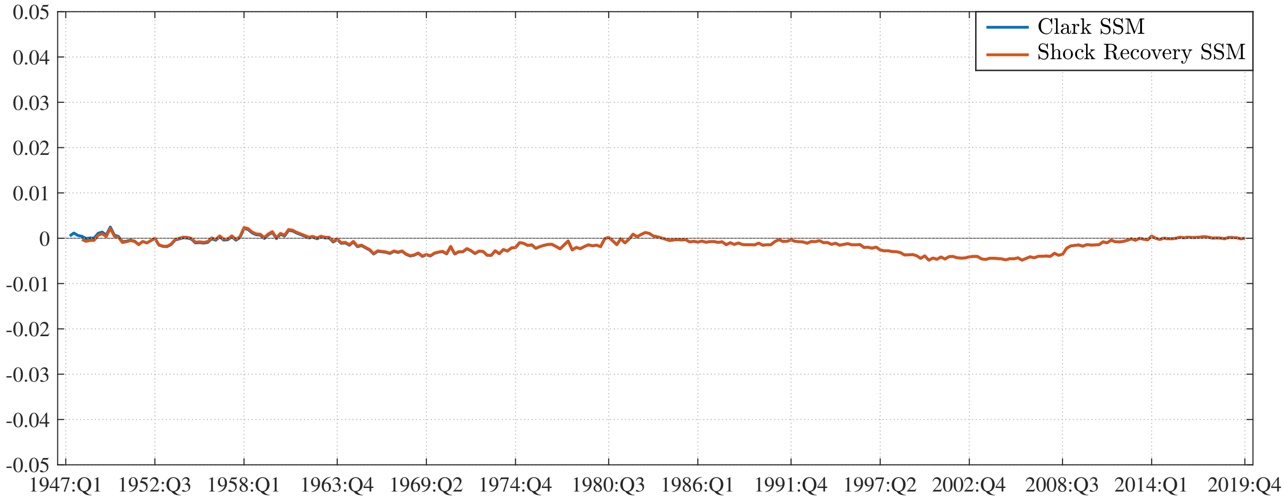
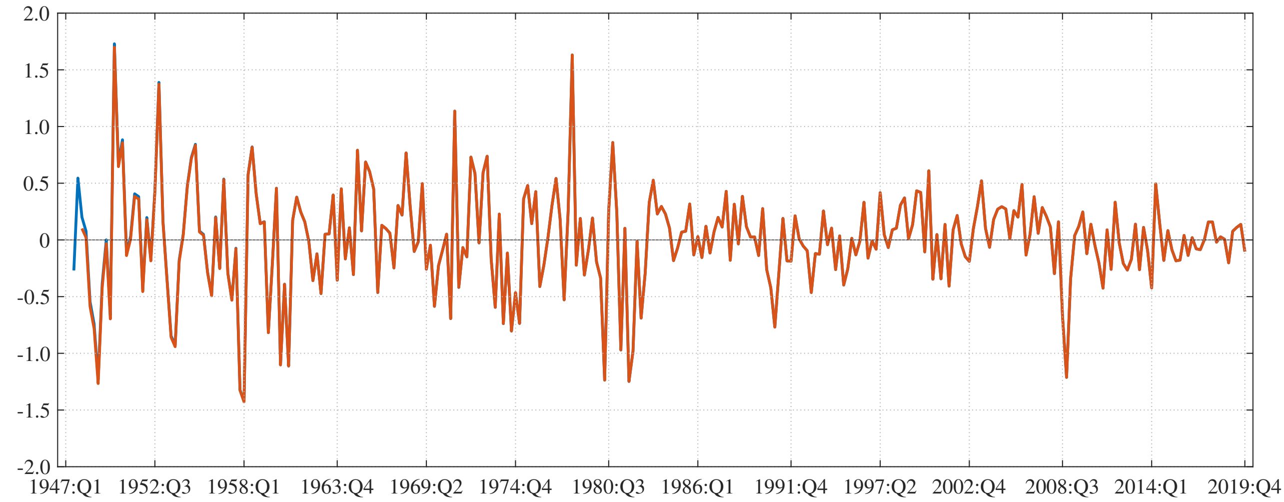


Smoothed $\eta_1 = \sigma_1 \varepsilon_1$



Smoothed $\eta_2 = \sigma_2 \varepsilon_2$



Smoothed $\eta_3 = \sigma_3 \varepsilon_3$

fig:KS