# 1 HLW (2023) - after Covid SSF

HLW use Hamilton' (1994) SSF (see page 31 of their 2023 paper):

$$
\begin{aligned}
\mathbf{y}_t &= \mathbf{A}\mathbf{x}_t + \mathbf{H}\xi_t + \boldsymbol{R}^{1/2}\boldsymbol{\varepsilon}_t^{\mathbf{y}} \\
\boldsymbol{\xi}_t &= \mathbf{F}\boldsymbol{\xi}_{t-1} + \boldsymbol{Q}^{1/2}\boldsymbol{\varepsilon}_t^{\boldsymbol{\xi}}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{y}_t &= \begin{bmatrix} y_t & \pi_t \end{bmatrix}', \; \mathbf{x}_t = \begin{bmatrix} y_{t-1} & y_{t-2} & r_{t-1} & r_{t-2} & \pi_{t-1} & \pi_{t-2,4} & d_t & d_{t-1} & d_{t-2} \end{bmatrix}', \\
\boldsymbol{\xi}_t &= \begin{bmatrix} y_t^* & y_{t-1}^* & y_{t-2}^* & g_t & g_{t-1} & g_{t-2} & z_t & z_{t-1} & z_{t-2} \end{bmatrix}', \\
\mathbf{A} &= \begin{bmatrix} a_{y,1} & a_{y,2} & \frac{a_r}{2} & \frac{a_r}{2} & 0 & 0 & \phi & -\phi a_{y,1} & -\phi a_{y,2} \\ b_y & 0 & 0 & 0 & b_\pi & (1-b_\pi) & 0 & -\phi b_y & 0 \end{bmatrix}, \\
\mathbf{H} &= \begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} & 0 & -4c\frac{a_r}{2} & -4c\frac{a_r}{2} & 0 & -\frac{a_r}{2} & -\frac{a_r}{2} \\ 0 & -b_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

obs0 Observation is:

$$
\mathbf{y}_t = \mathbf{A}\mathbf{x}_t + \mathbf{H}\xi_t + \boldsymbol{R}^{1/2}\boldsymbol{\varepsilon}_t^{\mathbf{y}}
$$

$$
\underbrace{\begin{bmatrix} y_t \\ \pi_t \end{bmatrix}}_{\mathbf{y}_t} = \underbrace{\begin{bmatrix} a_{y,1} & a_{y,2} & \frac{a_r}{2} & \frac{a_r}{2} & 0 & 0 & \phi & -\phi a_{y,1} & -\phi a_{y,2} \\ b_y & 0 & 0 & 0 & b_\pi & (1-b_\pi) & 0 & -\phi b_y & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} y_{t-1} \\ y_{t-2} \\ r_{t-1} \\ r_{t-2} \\ \pi_{t-1} \\ \pi_{t-2,4} \\ d_t \\ d_{t-1} \\ d_{t-2} \end{bmatrix}}_{\mathbf{x}_t} \tag{1a}
$$

$$
+ \underbrace{\begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} & 0 & -4c\frac{a_r}{2} & -4c\frac{a_r}{2} & 0 & -\frac{a_r}{2} & -\frac{a_r}{2} \\ 0 & -b_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_t \\ g_{t-1} \\ g_{t-2} \\ z_t \\ z_{t-1} \\ z_{t-2} \end{bmatrix}}_{\boldsymbol{\xi}_t} + \underbrace{\begin{bmatrix} \kappa_t \sigma_{\tilde{y}} & 0 \\ 0 & \kappa_t \sigma_\pi \end{bmatrix}}_{\boldsymbol{R}^{1/2}} \underbrace{\begin{bmatrix} \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^{\pi} \end{bmatrix}}_{\boldsymbol{\varepsilon}_t^{\mathbf{y}}}. \tag{1b}
$$

Using lag operators

$$
\begin{aligned}
a_y(L) &= (1 - a_{y,1}L - a_{y,2}L^2) \\
a_r(L) &= \frac{a_r}{2}(L + L^2) \\
b_\pi(L) &= (1 - b_\pi L - (1 - b_\pi)(L^2 + L^3 + L^4)),
\end{aligned}
$$

obs1 and with $\tilde{y}_t = (y_t - y_t^*)$ and $r_t^* = (c4g_t + z_t)$, the observation equation in (1) becomes:

$$
a_y(L)\tilde{y}_t = a_r(L)[r_t - \overbrace{(4cg_t + z_t)}^{r_t^*}] + \phi a_y(L)d_t + \kappa_t \sigma_{\tilde{y}} \varepsilon_t^{\tilde{y}} \tag{2a}
$$

$$
b_\pi(L)\pi_t = b_y \tilde{y}_{t-1} - \phi b_y d_{t-1} + \kappa_t \sigma_\pi \varepsilon_t^\pi. \tag{2b}
$$

state0 Defining $\sigma_g = \lambda_g \sigma_{y^*}$ and $\sigma_z = \lambda_z \frac{\sigma_{\tilde{y}}}{a_r}$, the state relations are:

$$
\boldsymbol{\xi}_t = \mathbf{F}\xi_{t-1} + \boldsymbol{Q}^{1/2}\boldsymbol{\varepsilon}_t^{\xi}
$$

$$
\underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_t \\ g_{t-1} \\ g_{t-2} \\ z_t \\ z_{t-1} \\ z_{t-2} \end{bmatrix}}_{\xi_t} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{F}} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ y_{t-3}^* \\ g_{t-1} \\ g_{t-2} \\ g_{t-3} \\ z_{t-1} \\ z_{t-2} \\ z_{t-3} \end{bmatrix}}_{\xi_{t-1}} + \underbrace{\begin{bmatrix} \sigma_{y^*} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sigma_g & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\boldsymbol{Q}^{1/2}} \underbrace{\begin{bmatrix} \varepsilon_t^{y^*} \\ \varepsilon_t^g \\ \varepsilon_t^z \end{bmatrix}}_{\varepsilon_t^\xi}, \tag{3a}
$$

which after expanding and removing identities can be simplified to:

$$
\Delta y_t^* = g_{t-1} + \sigma_{y^*} \varepsilon_t^{y^*} \tag{3b}
$$

$$
\Delta g_t = \sigma_g \varepsilon_t^g \tag{3c}
$$

$$
\Delta z_t = \sigma_z \varepsilon_t^z. \tag{3d}
$$

HLW Equations (2) and (3) (with $r_t^* = (c4g_t + z_t)$ and $\tilde{y}_t = (y_t - y_t^*)$) can be written '*compactly*' as:

$$
a_y(L)\tilde{y}_t = a_r(L)[r_t - r_t^*] + \phi a_y(L)d_t + \kappa_t \sigma_{\tilde{y}} \varepsilon_t^{\tilde{y}} \tag{4a}
$$

$$
b_\pi(L)\pi_t = b_y \tilde{y}_{t-1} - \phi b_y d_{t-1} + \kappa_t \sigma_\pi \varepsilon_t^\pi \tag{4b}
$$

$$
\Delta y_t^* = g_{t-1} + \sigma_{y^*} \varepsilon_t^{y^*} \tag{4c}
$$

$$
\Delta g_t = \sigma_g \varepsilon_t^g \tag{4d}
$$

$$
\Delta z_t = \sigma_z \varepsilon_t^z \tag{4e}
$$

## 2 Kurz SSF

We now need to express HLW's SSF above in the format required to analyze recoverability, that is, collect all the shocks of the model in (4) in the state vector (together with other required latent variables). Using Kurz's SSF defined as:

Kurz0

$$\begin{aligned} \text{Observation}: \quad Z_t &= D_1 X_t + D_2 X_{t-1} + R\varepsilon_t && \text{(5a)} \\ \text{State}: \quad X_t &= AX_{t-1} + C\varepsilon_t, && \text{(5b)} \end{aligned}$$

A

where $\varepsilon_t \sim MN(0, I_m)$, the **LHS (**observables) vector $Z_t$ is:

$$\begin{aligned} a_y(L)y_t - a_r(L)r_t - \phi a_y(L)d_t &= Z_{1t} && \text{(6a)} \\ b_\pi(L)\pi_t - b_y y_{t-1} + \phi b_y d_{t-1} &= Z_{2t} && \text{(6b)} \end{aligned}$$

and all remaining latent variables and shocks on the **RHS** are:

$$\begin{aligned} Z_{1t} &= a_y(L)y_t^* - a_r(L)r_t^* + \kappa_t \sigma_{\tilde{y}} \varepsilon_t^{\tilde{y}} && \text{(6c)} \\ Z_{2t} &= b_y y_{t-1}^* + \kappa_t \sigma_\pi \varepsilon_t^\pi, && \text{(6d)} \end{aligned}$$

AA

with state equations:

$$\begin{aligned} \Delta y_t^* &= g_{t-1} + \sigma_{y^*} \varepsilon_t^{y^*} && \text{(7a)} \\ \Delta g_t &= \sigma_g \varepsilon_t^g && \text{(7b)} \\ \Delta r_t^* &= c4\sigma_g \varepsilon_t^g + \sigma_z \varepsilon_t^z. && \text{(7c)} \end{aligned}$$

Given state vector $X_t = \begin{bmatrix} y_t^* & y_{t-1}^* & g_t & r_t^* & r_{t-1}^* & \varepsilon_t^{\tilde{y}} & \varepsilon_t^\pi & \varepsilon_t^{y^*} & \varepsilon_t^g & \varepsilon_t^z \end{bmatrix}'$, the SSF of (7) is:

$$\underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^\pi \\ \varepsilon_t^{y^*} \\ \varepsilon_t^g \\ \varepsilon_t^z \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ \varepsilon_{t-1}^{\tilde{y}} \\ \varepsilon_{t-1}^\pi \\ \varepsilon_{t-1}^{y^*} \\ \varepsilon_{t-1}^g \\ \varepsilon_{t-1}^z \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 & \sigma_{y^*} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_g & 0 \\ 0 & 0 & 0 & c4\sigma_g & \sigma_z \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^\pi \\ \varepsilon_t^{y^*} \\ \varepsilon_t^g \\ \varepsilon_t^z \end{bmatrix}}_{\varepsilon_t}$$

$$\underbrace{\begin{bmatrix} Z_{1t} \\ Z_{2t} \end{bmatrix}}_{Z_t} = \underbrace{\begin{bmatrix} 1 & -a_{y,1} & 0 & 0 & -\frac{a_r}{2} & \kappa_t\sigma_{\tilde{y}} & 0 & 0 & 0 & 0 \\ 0 & b_y & 0 & 0 & 0 & 0 & \kappa_t\sigma_\pi & 0 & 0 & 0 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^{\pi} \\ \varepsilon_t^{y^*} \\ \varepsilon_t^{g} \\ \varepsilon_t^{z} \end{bmatrix}}_{X_t}$$

$$+ \underbrace{\begin{bmatrix} 0 & -a_{y,2} & 0 & 0 & -\frac{a_r}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ \varepsilon_{t-1}^{\tilde{y}} \\ \varepsilon_{t-1}^{\pi} \\ \varepsilon_{t-1}^{y^*} \\ \varepsilon_{t-1}^{g} \\ \varepsilon_{t-1}^{z} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^{\pi} \\ \varepsilon_t^{y^*} \\ \varepsilon_t^{g} \\ \varepsilon_t^{z} \end{bmatrix}}_{\varepsilon_t}$$

The parameter values from HLW(2023, p 17, Table 1, taken from their .xlsx file) are:

$$\begin{aligned} \sigma_{\tilde{y}} &= 0.4516 & \sigma_\pi &= 0.7873 & \sigma_{y^*} &= 0.5000 & \sigma_g &= 0.1453/4 & \sigma_z &= 0.1181 \\ a_{y,1} &= 1.3872 & a_{y,2} &= -0.4507 & a_r &= -0.0790 & b_\pi &= 0.6800 & b_y &= 0.0733 \\ \kappa_{2020Q2-Q4} &= 9.0326 & \kappa_{2021} &= 1.7908 & \kappa_{2022} &= 1.6760 & c &= 1.1283 & \phi &= -0.0854 \end{aligned}$$

with the "signal-to-noise" parameter estimates being:

$$\begin{aligned} \lambda_g &= 0.0727 \\ \lambda_z &= 0.0207. \end{aligned}$$