1 Clark87

The UC model of Clark (1987) is a generalisation of the HP–Filter (a local linear trend model) clark0 that can be expressed as an SSM taking the following form:

$$
\begin{aligned}
y_t &= y_t^* + \tilde{y}_t && \text{(1a)}\\
\Delta y_t^* &= g_{t-1} + \sigma_1 \varepsilon_{1t} && \text{(1b)}\\
\Delta g_t &= \sigma_2 \varepsilon_{2t} && \text{(1c)}\\
a(L)\tilde{y}_t &= \sigma_3 \varepsilon_{3t}, && \text{(1d)}
\end{aligned}
$$

where the shocks $\{\varepsilon_{it}\}_{i=1}^{3}$ are assumed to be *i.i.d.* standard normal and mutually uncorrelated, with standard deviation σ_i and $a(L)$ is commonly assumed to be a stable AR(2), so that $a(L) = (1 - a_1 L - a_2 L^2)$. The only observable is y_t (generally 100 times) the log of real GDP and with the cycle (denoted by $\tilde{y}_t$) now allowed to be serially correlated, following a stationary AR(2) process. There are 3 shocks in the model.

The '*numbered*' shock to '*named*' shock mapping is:

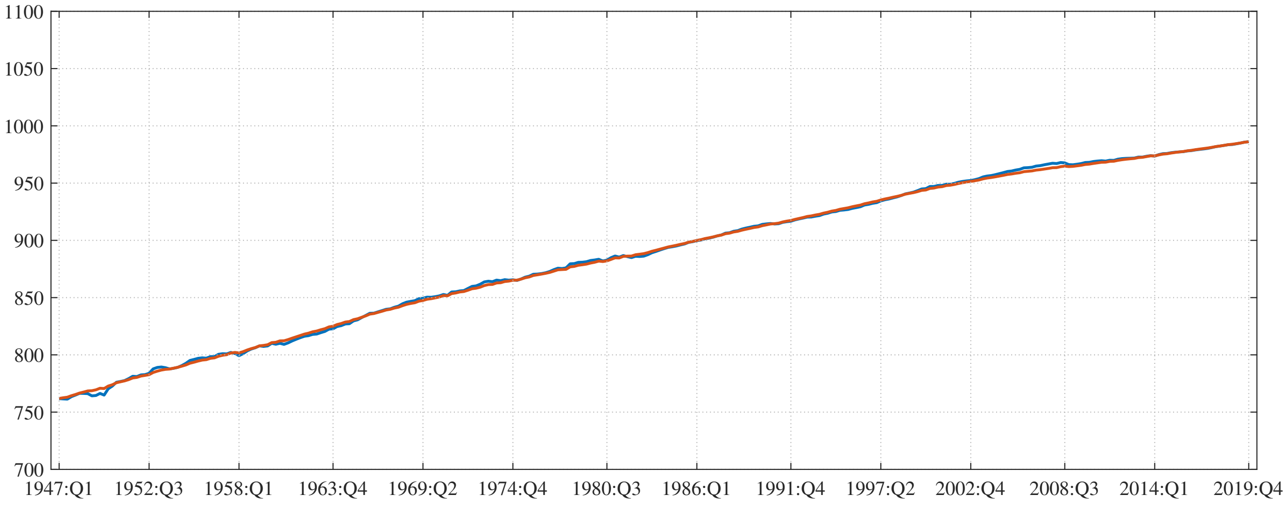
$$
\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix} = \begin{bmatrix} \varepsilon_t^{y^*} \\ \varepsilon_t^{g} \\ \varepsilon_t^{\tilde{y}} \end{bmatrix}. \tag{2}
$$

The (standard) SSM for ML estimation is:

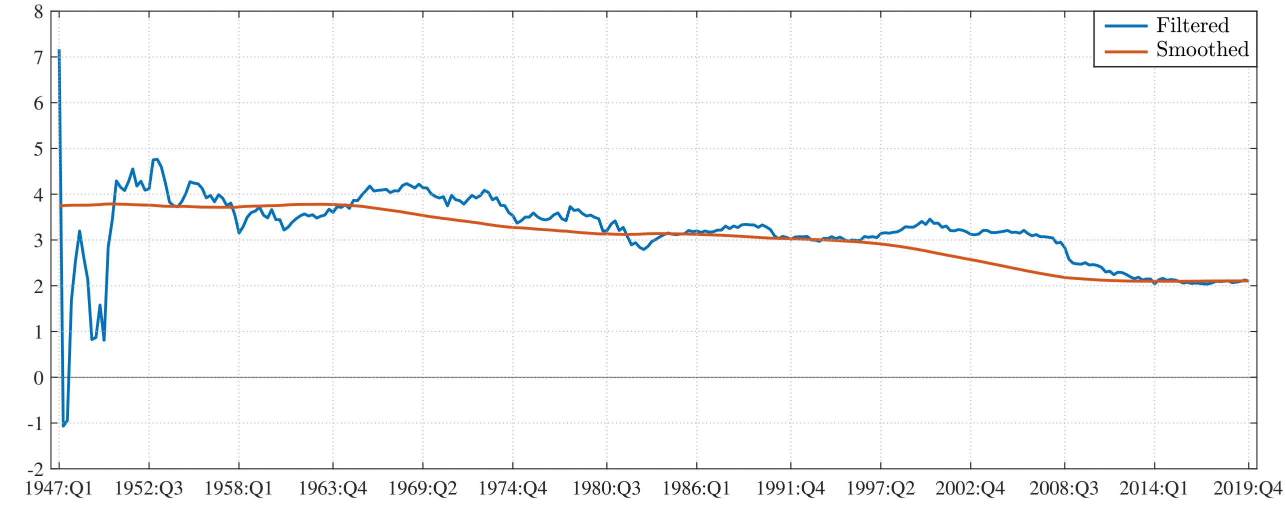
$$
y_t = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_t^* \\ g_t \\ \tilde{y}_t \\ \tilde{y}_{t-1} \end{bmatrix} + 0\varepsilon_t \tag{3}
$$

$$
\begin{bmatrix} y_t^* \\ g_t \\ \tilde{y}_t \\ \tilde{y}_{t-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1}^* \\ g_{t-1} \\ \tilde{y}_{t-1} \\ \tilde{y}_{t-2} \end{bmatrix} + \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}. \tag{4}
$$

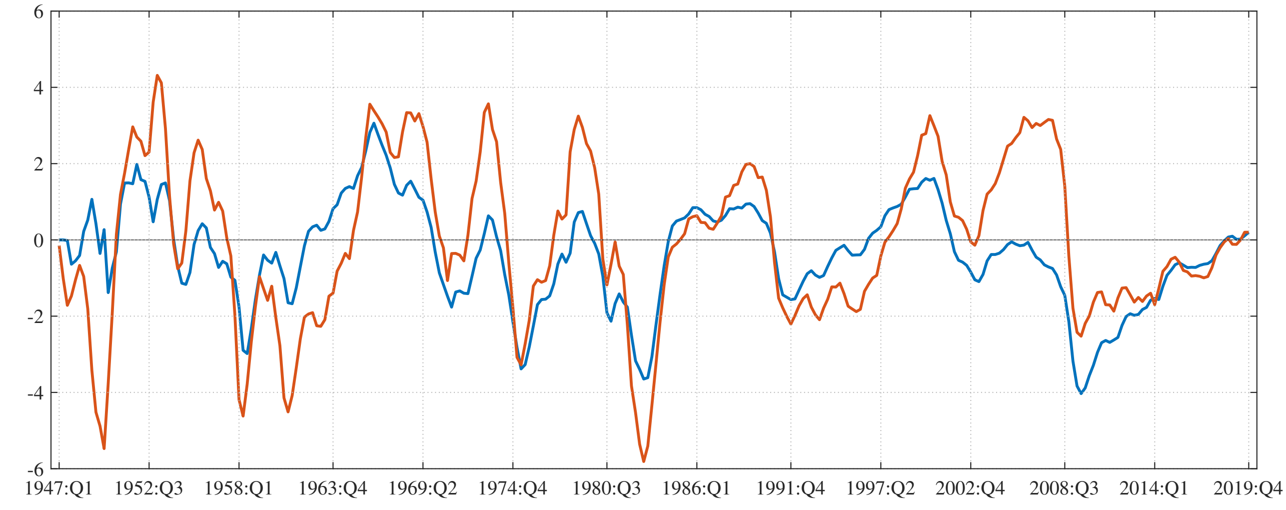
The code estimates Clark's 87 model on US GDP data from 1947:Q1 to 2019:Q4. A plot of the smoothed and filtered estimates is shown below.



Filtered and Smoothed estimates of y_t^*



Filtered and Smoohted estimates of g_t



Filtered and Smoothed estimates of $\tilde{y}_t$

fig:
xRet

2 Shock recovery SSM

2.1 SSM with lagged states

SSM Kurz's (2018) SSM has the following general from:

$$
\begin{aligned}
\text{Measurement}: \quad & Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t && \text{(5a)} \quad \texttt{ssm1} \\
\text{State}: \quad & X_t = AX_{t-1} + C\varepsilon_t, && \text{(5b)} \quad \texttt{ssm2}
\end{aligned}
$$

where $\varepsilon_t \sim MN(0, I_m)$, D_1, D_2, A, R are C are conformable system matrices, Z_t the observed variable and X_t the latent state variable.

2.2 Clark87 SSM for shock recovery

To assess recovery, re-write the model in (1) in '*shock recovery*' State Space Form (SSF). That is, collect all observables in Z_t and all shocks (and other state variables) in X_t to yield:

$$
\begin{aligned}
\Delta^2 y_t &= \Delta^2 y_t^* + \Delta^2 \tilde{y}_t \\
&= \sigma_1 \Delta\varepsilon_{1t} + \Delta g_{t-1} + \Delta^2 \tilde{y}_t \\
&= \sigma_1 \Delta\varepsilon_{1t} + \sigma_2 \varepsilon_{2t-1} + a(L)^{-1}\Delta^2 \sigma_3 \varepsilon_{3t} \\
\Leftrightarrow a(L)\Delta^2 y_t &= \sigma_1 a(L)\Delta\varepsilon_{1t} + \sigma_2 a(L)\varepsilon_{2t-1} + \sigma_3 \Delta^2 \varepsilon_{3t} \\
Z_t &= \sigma_1 \left[\varepsilon_{1t} - (2+a_1)\,\varepsilon_{1t-1} + (1+2a_1-a_2)\,\varepsilon_{1t-2} + (2a_2-a_1)\,\varepsilon_{1t-3} - a_2\varepsilon_{1t-4}\right] \\
&+ \sigma_2(\varepsilon_{2t-1} - a_1\varepsilon_{2t-2} - a_2\varepsilon_{2t-3}) \qquad \text{(6)} \quad \texttt{Z} \\
&+ \sigma_3(\varepsilon_{3t} - 2\varepsilon_{3t-1} + \varepsilon_{3t-2}),
\end{aligned}
$$

where I have made use of the fact:

$$
\begin{aligned}
a(L)\Delta^2 &= (1 - a_1 L - a_2 L^2)(1-L)^2 \\
&= 1 - (2+a_1)\,L + (1+2a_1-a_2)\,L^2 + (2a_2-a_1)\,L^3 - a_2 L^4, \qquad \text{(7)} \quad \texttt{expaL}
\end{aligned}
$$

and $Z_t = a(L)\Delta^2 y_t$ is the only observed variable.

The Measurement and State equations corresponding to (6) are:

KOSSM

$$
\text{Measurement}: \quad Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t
$$

$$Z_t = \underbrace{\begin{bmatrix} \sigma_1 & 0 & \sigma_3 & -\sigma_1(2+a_1) & \sigma_1(1+2a_1-a_2) & \sigma_1(2a_2-a_1) & 0 & 0 & 0 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{1t-1} \\ \varepsilon_{1t-2} \\ \varepsilon_{1t-3} \\ \varepsilon_{2t-1} \\ \varepsilon_{2t-2} \\ \varepsilon_{3t-1} \end{bmatrix}}_{X_t} \tag{8a}$$

$$+ \underbrace{\begin{bmatrix} 0 & \sigma_2 & -2\sigma_3 & 0 & 0 & -\sigma_1 a_2 & -\sigma_2 a_1 & -\sigma_2 a_2 & \sigma_3 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{1t-2} \\ \varepsilon_{1t-3} \\ \varepsilon_{1t-4} \\ \varepsilon_{2t-2} \\ \varepsilon_{2t-3} \\ \varepsilon_{3t-2} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}}_{\varepsilon_t} \tag{8b}$$

State : $\quad X_t = AX_{t-1} + C\varepsilon_t,$

$$\underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{1t-1} \\ \varepsilon_{1t-2} \\ \varepsilon_{1t-3} \\ \varepsilon_{2t-1} \\ \varepsilon_{2t-2} \\ \varepsilon_{3t-1} \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&1&0&0&0&0&0&0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{1t-2} \\ \varepsilon_{1t-3} \\ \varepsilon_{1t-4} \\ \varepsilon_{2t-2} \\ \varepsilon_{2t-3} \\ \varepsilon_{3t-2} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \\ 0&0&0 \\ 0&0&0 \\ 0&0&0 \\ 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}}_{\varepsilon_t}. \tag{8c}$$

2.3 Alternative formulation

Try to reduce the state vector X_t by using changes for the error terms that are not required to be expanded. Starting again from:

$$\Delta^2 y_t = \Delta^2 y_t^* + \Delta^2 \tilde{y}_t \tag{9}$$

$$\Delta^2 y_t^* = \sigma_2 \varepsilon_{2t-1} + \sigma_1 \Delta \varepsilon_{1t} \tag{10}$$

$$\Delta^2 \tilde{y}_t = a(L)^{-1} \sigma_3 \Delta^2 \varepsilon_{3t}, \tag{11}$$

we get (but we do not expand differenced terms of not needed):

$$\begin{aligned}
\Delta^2 y_t &= \Delta^2 y_t^* + \Delta^2 \tilde{y}_t \\
&= \sigma_2 \varepsilon_{2t-1} + \sigma_1 \Delta \varepsilon_{1t} + a(L)^{-1} \sigma_3 \Delta^2 \varepsilon_{3t} \\
\Leftrightarrow \underbrace{a(L)\Delta^2 y_t}_{Z_t} &= \sigma_2 a(L) \varepsilon_{2t-1} + \sigma_1 a(L) \Delta \varepsilon_{1t} + \sigma_3 \Delta^2 \varepsilon_{3t} \\
Z_t &= \sigma_1 a(L) \Delta \varepsilon_{1t} + \sigma_2 a(L) \varepsilon_{2t-1} + \sigma_3 \Delta^2 \varepsilon_{3t} \\
&= \sigma_1 \Delta \varepsilon_{1t} - a_1 \sigma_1 \Delta \varepsilon_{1t-1} - a_2 \sigma_1 \Delta \varepsilon_{1t-2} \\
&+ \sigma_2 \varepsilon_{2t-1} - a_1 \sigma_2 \varepsilon_{2t-2} - a_2 \sigma_2 \varepsilon_{2t-3} \\
&+ \sigma_3 \Delta \varepsilon_{3t} - \sigma_3 \Delta \varepsilon_{3t-1}
\end{aligned} \tag{12}$$

$$\text{Measurement :} \quad Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t$$

$$Z_t = \underbrace{\begin{bmatrix} \sigma_1 & 0 & 0 & -a_1\sigma_2 & -a_1\sigma_1 & \sigma_3 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{2t-2} \\ \Delta\varepsilon_{1t-1} \\ \Delta\varepsilon_{3t} \end{bmatrix}}_{X_t} \tag{13}$$

$$+ \underbrace{\begin{bmatrix} -\sigma_1 & \sigma_2 & 0 & -a_2\sigma_2 & -a_2\sigma_1 & -\sigma_3 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{2t-3} \\ \Delta\varepsilon_{1t-2} \\ \Delta\varepsilon_{3t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}}_{\varepsilon_t} \tag{14}$$

$$\text{State :} \quad X_t = AX_{t-1} + C\varepsilon_t \tag{15}$$

$$
\underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{2t-2} \\ \Delta\varepsilon_{1t-1} \\ \Delta\varepsilon_{3t} \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{2t-3} \\ \Delta\varepsilon_{1t-2} \\ \Delta\varepsilon_{3t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}}_{\varepsilon_t}. \tag{16}
$$

$\varepsilon_{1t-1} - \varepsilon_{1t-2}$

$$
\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{2t-2} \\ \varepsilon_{1t-1} \\ \varepsilon_{1t-2} \\ \Delta\varepsilon_{3t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{2t-3} \\ \varepsilon_{1t-2} \\ \varepsilon_{1t-3} \\ \Delta\varepsilon_{3t-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}
$$

$\varepsilon_{1t-1} - \varepsilon_{1t-2}$

$-\varepsilon_{t-2}$

2.4 Alternative 2: lower number of states

$$\Delta^2 y_t = \Delta^2 y_t^* + \Delta^2 \tilde{y}_t \tag{17}$$

$$\Delta^2 y_t = \underbrace{\sigma_2 \varepsilon_{2t-1} + \sigma_1 \Delta \varepsilon_{1t}}_{\Delta^2 y_t^*} + \underbrace{a(L)^{-1} \sigma_3 \Delta^2 \varepsilon_{3t}}_{\Delta^2 \tilde{y}_t}, \quad \text{since we have} \tag{18}$$

$$\Delta g_t = \sigma_2 \varepsilon_{2t} \tag{19}$$

$$\Delta^2 y_t^* = \Delta g_{t-1} + \sigma_1 \Delta \varepsilon_{1t} \tag{20}$$

$$= \sigma_2 \varepsilon_{2t-1} + \sigma_1 \Delta \varepsilon_{1t} \tag{21}$$

$$\Delta^2 \tilde{y}_t = a(L)^{-1} \sigma_3 \Delta^2 \varepsilon_{3t} \tag{22}$$

These give then the Observed to shock relations:

$$\begin{aligned}
\underbrace{a(L)\Delta^2 y_t}_{Z_t} &= a(L)\sigma_1 \Delta \varepsilon_{1t} + a(L)\sigma_2 \varepsilon_{2t-1} + \sigma_3 \Delta^2 \varepsilon_{3t} \\
&= \sigma_1 \Delta \varepsilon_{1t} - a_1 \sigma_1 \Delta \varepsilon_{1t-1} - a_2 \sigma_1 \Delta \varepsilon_{1t-2} \\
&+ \sigma_2 \varepsilon_{2t-1} - a_1 \sigma_2 \varepsilon_{2t-2} - a_2 \sigma_2 \varepsilon_{2t-3} \\
&+ \sigma_3 \Delta \varepsilon_{3t} - \sigma_3 \Delta \varepsilon_{3t-1}
\end{aligned}$$

$$\text{State}: \quad X_t = A X_{t-1} + C \varepsilon_t \tag{23}$$

$$\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \Delta\varepsilon_{1t} \\ \Delta\varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{2t-2} \\ \Delta\varepsilon_{3t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \Delta\varepsilon_{1t-1} \\ \Delta\varepsilon_{1t-2} \\ \varepsilon_{2t-2} \\ \varepsilon_{2t-3} \\ \Delta\varepsilon_{3t-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}. \tag{24}$$

$$\text{Measurement}: \quad Z_t = D_1 X_t + D_2 X_{t-1} + R \varepsilon_t$$

$$Z_t = \begin{bmatrix} 0 & 0 & 0 & \sigma_1 & -a_1\sigma_1 & 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \Delta\varepsilon_{1t} \\ \Delta\varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{2t-2} \\ \Delta\varepsilon_{3t} \end{bmatrix} + \begin{bmatrix} 0 & \sigma_2 & 0 & 0 & -a_2\sigma_1 & -a_1\sigma_2 & -a_2\sigma_2 & -\sigma_3 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \Delta\varepsilon_{1t-1} \\ \Delta\varepsilon_{1t-2} \\ \varepsilon_{2t-2} \\ \varepsilon_{2t-3} \\ \Delta\varepsilon_{3t-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix}$$