

# 1 Clark87

clark0

The UC model of Clark (1987) is a generalisation of the HP-Filter (a local linear trend model) that can be expressed as an SSM taking the following form:

$$y_t = y_t^* + \tilde{y}_t \quad (1a)$$

$$\Delta y_t^* = g_{t-1} + \sigma_1 \varepsilon_{1t} \quad (1b)$$

$$\Delta g_t = \sigma_2 \varepsilon_{2t} \quad (1c)$$

$$a(L)\tilde{y}_t = \sigma_3 \varepsilon_{3t}, \quad (1d)$$

where the shocks  $\{\varepsilon_{it}\}_{i=1}^3$  are assumed to be *i.i.d.* standard normal and mutually uncorrelated, with standard deviation  $\sigma_i$  and  $a(L)$  is commonly assumed to be a stable AR(2), so that  $a(L) = (1 - a_1 L - a_2 L^2)$ . The only observable is  $y_t$  (generally 100 times) the log of real GDP and with the cycle (denoted by  $\tilde{y}_t$ ) now allowed to be serially correlated, following a stationary AR(2) process. There are 3 shocks in the model.

The ‘*numbered*’ shock to ‘*named*’ shock mapping is:

$$\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix} = \begin{bmatrix} \varepsilon_t^{y^*} \\ \varepsilon_t^g \\ \varepsilon_t^{\tilde{y}} \end{bmatrix}. \quad (2)$$

The (standard) SSM for ML estimation is:

$$y_t = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_t^* \\ g_t \\ \tilde{y}_t \\ \tilde{y}_{t-1} \end{bmatrix} + 0\varepsilon_t$$

$$\begin{bmatrix} y_t^* \\ g_t \\ \tilde{y}_t \\ \tilde{y}_{t-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1}^* \\ g_{t-1} \\ \tilde{y}_{t-1} \\ \tilde{y}_{t-2} \end{bmatrix} + \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}.$$

The code estimates Clark’s 87 model to use GDP data from 1947:Q1 to 2019:Q4.

## 2 Shock recovery SSM

### 2.1 SSM with lagged states

SSM Kurz's (2018) SSM has the following general form:

$$\text{Measurement : } Z_t = D_1 X_t + D_2 X_{t-1} + R \varepsilon_t \quad (3a) \quad \text{ssm1}$$

$$\text{State : } X_t = A X_{t-1} + C \varepsilon_t, \quad (3b) \quad \text{ssm2}$$

where  $\varepsilon_t \sim MN(0, I_m)$ ,  $D_1, D_2, A, R$  are  $C$  conformable system matrices,  $Z_t$  the observed variable and  $X_t$  the latent state variable.

### 2.2 Clark87 SSM for shock recovery

To assess recovery, re-write the model in (1) in 'shock recovery' State Space Form (SSF). That is, collect all observables in  $Z_t$  and all shocks (and other state variables) in  $X_t$  to yield:

$$\begin{aligned} \Delta^2 y_t &= \Delta^2 y_t^* + \Delta^2 \tilde{y}_t \\ &= \sigma_1 \Delta \varepsilon_{1t} + \Delta g_{t-1} + \Delta^2 \tilde{y}_t \\ &= \sigma_1 \Delta \varepsilon_{1t} + \sigma_2 \varepsilon_{2t-1} + a(L)^{-1} \Delta^2 \sigma_3 \varepsilon_{3t} \\ \Leftrightarrow a(L) \Delta^2 y_t &= \sigma_1 a(L) \Delta \varepsilon_{1t} + \sigma_2 a(L) \varepsilon_{2t-1} + \sigma_3 \Delta^2 \varepsilon_{3t} \\ Z_t &= \sigma_1 [\varepsilon_{1t} - (2 + a_1) \varepsilon_{1t-1} + (1 + 2a_1 - a_2) \varepsilon_{1t-2} + (2a_2 - a_1) \varepsilon_{1t-3} - a_2 \varepsilon_{1t-4}] \\ &\quad + \sigma_2 (\varepsilon_{2t-1} - a_1 \varepsilon_{2t-2} - a_2 \varepsilon_{2t-3}) \\ &\quad + \sigma_3 (\varepsilon_{3t} - 2\varepsilon_{3t-1} + \varepsilon_{3t-2}), \end{aligned} \quad (4) \quad z$$

where I have made use of the fact:

$$\begin{aligned} a(L) \Delta^2 &= (1 - a_1 L - a_2 L^2)(1 - L)^2 \\ &= 1 - (2 + a_1) L + (1 + 2a_1 - a_2) L^2 + (2a_2 - a_1) L^3 - a_2 L^4, \end{aligned} \quad (5) \quad \text{expaL}$$

and  $Z_t = a(L) \Delta^2 y_t$  is the only observed variable.

The Measurement and State equations corresponding to (4) are:

KOSSM

$$\text{Measurement : } Z_t = D_1 X_t + D_2 X_{t-1} + R \varepsilon_t$$

$$Z_t = \underbrace{\begin{bmatrix} \sigma_1 & 0 & \sigma_3 & -\sigma_1(2+a_1) & \sigma_1(1+2a_1-a_2) & \sigma_1(2a_2-a_1) & 0 & 0 & 0 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{1t-1} \\ \varepsilon_{1t-2} \\ \varepsilon_{1t-3} \\ \varepsilon_{2t-1} \\ \varepsilon_{2t-2} \\ \varepsilon_{3t-1} \end{bmatrix}}_{X_t} \quad (6a)$$

$$+ \underbrace{\begin{bmatrix} 0 & \sigma_2 & -2\sigma_3 & 0 & 0 & -\sigma_1 a_2 & -\sigma_2 a_1 & -\sigma_2 a_2 & \sigma_3 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{1t-2} \\ \varepsilon_{1t-3} \\ \varepsilon_{1t-4} \\ \varepsilon_{2t-2} \\ \varepsilon_{2t-3} \\ \varepsilon_{3t-2} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}}_R \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}}_{\varepsilon_t} \quad (6b)$$

State :  $X_t = AX_{t-1} + C\varepsilon_t$ ,

$$\underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{1t-1} \\ \varepsilon_{1t-2} \\ \varepsilon_{1t-3} \\ \varepsilon_{2t-1} \\ \varepsilon_{2t-2} \\ \varepsilon_{3t-1} \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_A \underbrace{\begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{1t-2} \\ \varepsilon_{1t-3} \\ \varepsilon_{1t-4} \\ \varepsilon_{2t-2} \\ \varepsilon_{2t-3} \\ \varepsilon_{3t-2} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_C \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}}_{\varepsilon_t}. \quad (6c)$$

## 2.3 Alternative formulation

Try to reduce the state vector  $X_t$  by using changes for the error terms that are not required to be expanded. Starting again from:

$$\Delta^2 y_t = \Delta^2 y_t^* + \Delta^2 \tilde{y}_t \quad (7)$$

$$\Delta^2 y_t^* = \sigma_2 \varepsilon_{2t-1} + \sigma_1 \Delta \varepsilon_{1t} \quad (8)$$

$$\Delta^2 \tilde{y}_t = a(L)^{-1} \sigma_3 \Delta^2 \varepsilon_{3t}, \quad (9)$$

we get (but we do not expand differenced terms if not needed):

$$\begin{aligned} \Delta^2 y_t &= \Delta^2 y_t^* + \Delta^2 \tilde{y}_t \\ &= \sigma_2 \varepsilon_{2t-1} + \sigma_1 \Delta \varepsilon_{1t} + a(L)^{-1} \sigma_3 \Delta^2 \varepsilon_{3t} \\ \Leftrightarrow \underbrace{a(L) \Delta^2 y_t}_{Z_t} &= \sigma_2 a(L) \varepsilon_{2t-1} + \sigma_1 a(L) \Delta \varepsilon_{1t} + \sigma_3 \Delta^2 \varepsilon_{3t} \\ Z_t &= \sigma_1 a(L) \Delta \varepsilon_{1t} + \sigma_2 a(L) \varepsilon_{2t-1} + \sigma_3 \Delta^2 \varepsilon_{3t} \\ &= \sigma_1 \Delta \varepsilon_{1t} - a_1 \sigma_1 \Delta \varepsilon_{1t-1} - a_2 \sigma_1 \Delta \varepsilon_{1t-2} \\ &\quad + \sigma_2 \varepsilon_{2t-1} - a_1 \sigma_2 \varepsilon_{2t-2} - a_2 \sigma_2 \varepsilon_{2t-3} \\ &\quad + \sigma_3 \Delta \varepsilon_{3t} - \sigma_3 \Delta \varepsilon_{3t-1} \end{aligned} \quad (10)$$

Measurement :  $Z_t = D_1 X_t + D_2 X_{t-1} + R \varepsilon_t$

$$Z_t = \underbrace{\begin{bmatrix} \sigma_1 & 0 & 0 & -a_1 \sigma_2 & -a_1 \sigma_1 & \sigma_3 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{2t-2} \\ \Delta \varepsilon_{1t-1} \\ \Delta \varepsilon_{3t} \end{bmatrix}}_{X_t} \quad (11)$$

$$\begin{aligned} &+ \underbrace{\begin{bmatrix} -\sigma_1 & \sigma_2 & 0 & -a_2 \sigma_2 & -a_2 \sigma_1 & -\sigma_3 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{2t-3} \\ \Delta \varepsilon_{1t-2} \\ \Delta \varepsilon_{3t-1} \end{bmatrix}}_{X_{t-1}} \\ &+ \underbrace{\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}}_R \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}}_{\varepsilon_t} \end{aligned} \quad (12)$$

State :  $X_t = A X_{t-1} + C \varepsilon_t \quad (13)$

$$\begin{aligned}
& \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{2t-2} \\ \Delta\varepsilon_{1t-1} \\ \Delta\varepsilon_{3t} \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}}_A \underbrace{\begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{2t-3} \\ \Delta\varepsilon_{1t-2} \\ \Delta\varepsilon_{3t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_C \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}}_{\varepsilon_t}. \quad (14)
\end{aligned}$$

$$\varepsilon_{1t-1} - \varepsilon_{1t-2}$$

$$\begin{aligned}
& \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{2t-2} \\ \varepsilon_{1t-1} \\ \varepsilon_{1t-2} \\ \Delta\varepsilon_{3t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{2t-3} \\ \varepsilon_{1t-2} \\ \varepsilon_{1t-3} \\ \Delta\varepsilon_{3t-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}
\end{aligned}$$

$$\varepsilon_{1t-1} - \varepsilon_{1t-2}$$

$$-\varepsilon_{t-2}$$

??correct now:

$$\begin{aligned}
& \begin{bmatrix} \sigma_1 & 0 & 0 & -a_1\sigma_2 & -a_1\sigma_1 & \sigma_3 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{2t-2} \\ \Delta\varepsilon_{1t-1} \\ \Delta\varepsilon_{3t} \end{bmatrix} + \begin{bmatrix} -\sigma_1 & \sigma_2 & 0 & -a_2\sigma_2 & -a_2\sigma_1 & -\sigma_3 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{2t-3} \\ \Delta\varepsilon_{1t-2} \\ \Delta\varepsilon_{3t-1} \end{bmatrix} = \sigma_2\varepsilon_{2t-1} - \\
& \sigma_1\varepsilon_{t-1} + \sigma_1\varepsilon_t + \Delta\sigma_3\varepsilon_{3t} - \Delta\sigma_3\varepsilon_{3t-1} - \sigma_2a_1\varepsilon_{2t-2} - \sigma_2a_2\varepsilon_{2t-3} - \Delta\sigma_1a_1\varepsilon_{t-1} - \Delta\sigma_1a_2\varepsilon_{t-2}
\end{aligned}$$