1 LW(2003) Structural Model and standard SSM Form

LW use the following standard State-Space From (SSF) (see their `LW_Code_Guide.pdf` file `LW_replication.zip` from the NYFED website):

$$\begin{aligned} \text{Measurement}: \quad & \mathbf{y}_t = \mathbf{A}\mathbf{x}_t + \mathbf{H}\boldsymbol{\xi}_t + \boldsymbol{R}^{1/2}\boldsymbol{\varepsilon}_t^{\mathbf{y}} \\ \text{State}: \quad & \boldsymbol{\xi}_t = \mathbf{F}\boldsymbol{\xi}_{t-1} + \boldsymbol{Q}^{1/2}\boldsymbol{\varepsilon}_t^{\boldsymbol{\xi}} \end{aligned}$$

where

$$\begin{aligned} \mathbf{y}_t &= \begin{bmatrix} y_t & \pi_t \end{bmatrix}', \\ \mathbf{x}_t &= \begin{bmatrix} y_{t-1} & y_{t-2} & r_{t-1} & r_{t-2} & \pi_{t-1} & \pi_{t-2,4} & \pi_{t-5,8} & (\pi^0_{t-1} - \pi_{t-1}) & (\pi^m_t - \pi_t) \end{bmatrix}', \\ \boldsymbol{\xi}_t &= \begin{bmatrix} y^*_t & y^*_{t-1} & y^*_{t-2} & g_{t-1} & g_{t-2} & z_{t-1} & z_{t-2} \end{bmatrix}', \\ \mathbf{A} &= \begin{bmatrix} a_1 & a_2 & \frac{a_3}{2} & \frac{a_3}{2} & 0 & 0 & 0 & 0 & 0 \\ b_3 & 0 & 0 & 0 & b_1 & b_2 & (1-b_1-b_2) & b_4 & b_5 \end{bmatrix}, \\ \mathbf{H} &= \begin{bmatrix} 1 & -a_1 & -a_2 & -4c\frac{a_3}{2} & -4c\frac{a_3}{2} & -\frac{a_3}{2} & -\frac{a_3}{2} \\ 0 & -b_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Note: $\mathbf{H}(1,4:5)$ entries corresponding to trend growth cg_t are *'annualized'* (multiplied by 4) in the code that performs the estimation (see `unpack.parameters.stage3.R` in `LW_replication.zip` files, line 24, which reads: `H[1,4:5] <- -parameters[9]*parameters[3]*2 ## c, a_3 (annualized)`).

obs0 The measurement equation is:

$$\mathbf{y}_t = \mathbf{A}\mathbf{x}_t + \mathbf{H}\xi_t + \boldsymbol{R}^{1/2}\boldsymbol{\varepsilon}_t^{\mathbf{y}}$$

$$\begin{bmatrix} y_t \\ \pi_t \end{bmatrix} = \underbrace{\begin{bmatrix} a_1 & a_2 & \frac{a_3}{2} & \frac{a_3}{2} & 0 & 0 & 0 & 0 & 0 \\ b_3 & 0 & 0 & 0 & b_1 & b_2 & (1-b_1-b_2) & b_4 & b_5 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ r_{t-1} \\ r_{t-2} \\ \pi_{t-1} \\ \pi_{t-2,4} \\ \pi_{t-5,8} \\ (\pi^0_{t-1} - \pi_{t-1}) \\ (\pi^m_t - \pi_t) \end{bmatrix} \tag{1a}$$

$$+ \underbrace{\begin{bmatrix} 1 & -a_1 & -a_2 & -4c\frac{a_3}{2} & -4c\frac{a_3}{2} & -\frac{a_3}{2} & -\frac{a_3}{2} \\ 0 & -b_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{H}} \begin{bmatrix} y^*_t \\ y^*_{t-1} \\ y^*_{t-2} \\ g_{t-1} \\ g_{t-2} \\ z_{t-1} \\ z_{t-2} \end{bmatrix} + \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}. \tag{1b}$$

The state equation is:

$$\boldsymbol{\xi}_t = \mathbf{F}\boldsymbol{\xi}_{t-1} + \boldsymbol{Q}^{1/2}\varepsilon_t^{\xi}$$

$$\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ g_{t-2} \\ z_{t-1} \\ z_{t-2} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{F}} \begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ y_{t-3}^* \\ g_{t-2} \\ g_{t-3} \\ z_{t-2} \\ z_{t-3} \end{bmatrix} + \begin{bmatrix} \sigma_3 & \sigma_4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sigma_4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{3t} \\ \varepsilon_{4t-1} \\ \varepsilon_{5t-1} \end{bmatrix} \tag{2}$$ state1

Expanding the relations in (1) and (2) and re-arranging yields:

$$\begin{bmatrix} y_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} y_t^* + a_1\left(y_{t-1} - y_{t-1}^*\right) + a_2\left(y_{t-2} - y_{t-2}^*\right) + \frac{1}{2}a_3\left([r_{t-1} - 4cg_{t-1} - z_{t-1}] + [r_{t-2} - 4cg_{t-2} - z_{t-2}]\right) + \sigma_1\varepsilon_{1t} \\ b_3\left(y_{t-1} - y_{t-1}^*\right) + b_1\pi_{t-1} + b_2\pi_{t-2,4} + (1 - b_1 - b_2)\,\pi_{t-5,8} + b_4\left(\pi_{t-1}^0 - \pi_{t-1}\right) + b_5\left(\pi_t^m - \pi_t\right) + \sigma_2\varepsilon_{2t} \end{bmatrix}$$

$$\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ g_{t-2} \\ z_{t-1} \\ z_{t-2} \end{bmatrix} = \begin{bmatrix} y_{t-1}^* + \overbrace{g_{t-2} + \sigma_4\varepsilon_{4t-1}}^{g_{t-1}} + \sigma_3\varepsilon_{3t} \\ y_{t-1}^* \\ y_{t-2}^* \\ g_{t-2} + \sigma_4\varepsilon_{4t-1} \\ g_{t-2} \\ z_{t-2} + \sigma_5\varepsilon_{5t-1} \\ z_{t-2} \end{bmatrix}, \text{ where the shock labels are: } \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix} = \begin{bmatrix} \varepsilon_t^{\tilde{y}} \\ \varepsilon_t^{\pi} \\ \varepsilon_t^{y^*} \\ \varepsilon_t^{g} \\ \varepsilon_t^{z} \end{bmatrix}.$$

2 Shock recovery SSM

2.1 SSM with lagged states

SSM Kurz's (2018) SSM has the following general from:

$$\text{Measurement}: \quad Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t \tag{3a}$$ ssm1

$$\text{State}: \quad X_t = AX_{t-1} + C\varepsilon_t, \tag{3b}$$ ssm2

where $\varepsilon_t \sim MN(0, I_m)$, D_1, D_2, A, R are C are conformable system matrices, Z_t the observed variable and X_t the latent state variable.

2.2 LW03 equations

LW03's SSM written out equation by equation is:

LW03

$$y_t = y_t^* + a_1\left(y_{t-1} - y_{t-1}^*\right) + a_2\left(y_{t-2} - y_{t-2}^*\right) + \tfrac{1}{2}a_3\left(\left[r_{t-1} - r_{t-1}^*\right] + \left[r_{t-2} - r_{t-2}^*\right]\right) + \sigma_1\varepsilon_{1t} \tag{4a}$$
LW03a

$$\pi_t = b_3\left(y_{t-1} - y_{t-1}^*\right) + b_1\pi_{t-1} + b_2\pi_{t-2,4} + (1 - b_1 - b_2)\,\pi_{t-5,8} + b_4(\pi_{t-1}^0 - \pi_{t-1}) + b_5\left(\pi_t^m - \pi_t\right) + \sigma_2\varepsilon_{2t} \tag{4b}$$
LW03b

$$y_t^* = y_{t-1}^* + g_{t-1} + \sigma_3\varepsilon_{3t} \tag{4c}$$
LW03c

$$g_{t-1} = g_{t-2} + \sigma_4\varepsilon_{4t-1} \tag{4d}$$
LW03d

$$z_{t-1} = z_{t-2} + \sigma_5\varepsilon_{5t-1}, \tag{4e}$$
LW03e

where $r_t^* = 4cg_t + z_t$. Note that this can be written as:

$$\begin{aligned} r_t^* &= 4cg_t + z_t \\ \Leftrightarrow \Delta r_t^* &= 4c\,\underbrace{\Delta g_t}_{\sigma_4\varepsilon_{4t}} + \underbrace{\Delta z_t}_{\sigma_5\varepsilon_{5t}} \\ \Delta r_t^* &= 4c\sigma_4\varepsilon_{4t} + \sigma_5\varepsilon_{5t}. \end{aligned} \tag{5}$$
Drstar

2.3 LW03 SSM for shock recovery

To assess recovery, re-write the model in '*shock recovery*' form. That is, collect all observables in Z_t, and all shocks (and other state variables) in state vector X_t. The relations in (4), incorporating (5) then become:

ssm0

$$\begin{aligned} \text{Measurement}: \quad Z_{1t} &= y_t^* - a_1 y_{t-1}^* - a_2 y_{t-2}^* - \tfrac{1}{2}a_3\left(r_{t-1}^* + r_{t-2}^*\right) + \sigma_1\varepsilon_{1t} && \text{(6a)} \\ Z_{2t} &= -b_3 y_{t-1}^* + \sigma_2\varepsilon_{2t} && \text{(6b)} \\ \text{State}: \quad \Delta y_t^* &= g_{t-1} + \sigma_3\varepsilon_{3t} && \text{(6c)} \\ \Delta g_t &= \sigma_4\varepsilon_{4t} && \text{(6d)} \\ \Delta r_t^* &= 4c\sigma_4\varepsilon_{4t} + \sigma_5\varepsilon_{5t}, && \text{(6e)} \end{aligned}$$

with the observables Z_t in the measurement equations defined as:

$$\begin{aligned} Z_{1t} &= y_t - a_1 y_{t-1} - a_2 y_{t-2} - \tfrac{1}{2}a_3(r_{t-1} + r_{t-2}) \\ Z_{2t} &= \pi_t - b_1\pi_{t-1} - b_2\pi_{t-2,4} - (1 - b_1 - b_2)\,\pi_{t-5,8} - b_4(\pi_{t-1}^0 - \pi_{t-1}) - b_5\left(\pi_t^m - \pi_t\right) - b_3 y_{t-1}. \end{aligned}$$

The '*shock recovery*' SSF corresponding to (6) is then:

Measurement : $Z_t = D_1 X_t + D_2 X_{t-1} + R\varepsilon_t$

$$\underbrace{\begin{bmatrix} Z_{1t} \\ Z_{2t} \end{bmatrix}}_{Z_t} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & -b_3 & 0 & 0 & 0 & 0 & \sigma_2 & 0 & 0 & 0 \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{X_t} \tag{7a}$$

$$+ \underbrace{\begin{bmatrix} -a_1 & -a_2 & 0 & -\frac{a_3}{2} & -\frac{a_3}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{4t-1} \\ \varepsilon_{5t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\mathbf{0}_{2\times 5}}_{R} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{\varepsilon_t} \tag{7b}$$

State : $X_t = AX_{t-1} + C\varepsilon_t,$

$$\underbrace{\begin{bmatrix} y_t^* \\ y_{t-1}^* \\ g_t \\ r_t^* \\ r_{t-1}^* \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ r_{t-1}^* \\ r_{t-2}^* \\ \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \\ \varepsilon_{4t-1} \\ \varepsilon_{5t-1} \end{bmatrix}}_{X_{t-1}} + \underbrace{\begin{bmatrix} 0 & 0 & \sigma_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_4 & 0 \\ 0 & 0 & 0 & 4c\sigma_4 & \sigma_5 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}}_{\varepsilon_t} \tag{7c}$$

The Kalman Filter and Smoother based steady-state $\text{diag}(P)$ are, respectively:

$$\text{diag}(P^*_{t|t}) = \begin{bmatrix} 0.9995 \\ 0.2006 \\ 0.1608 \end{bmatrix} \quad \text{and} \quad \text{diag}(P^*_{t|T}) = \begin{bmatrix} 0.9439 \\ 0.0561 \\ 0.0561 \end{bmatrix}. \tag{8}$$