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The solutions of this mock are to be discussed in **ISI CMI 2025 forum**

Dedicated to Palak

UGAB

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1 P1

For $n \geq 1$, let S_n denote the set of all permutations of $\{1, 2, \ldots, n\}$. Let p_n denote the probability of the event that a randomly chosen permutation does not fix any integer in its original position. Find

$$\lim_{n\to\infty} p_n.$$

- (A) e
- (B) e^2
- (C) $\frac{1}{e}$
- (D) 1

2 P2

Let a, b, c be positive real numbers satisfying the system of equations

$$\sqrt{2a - ab} + \sqrt{2b - ab} = 1$$

$$\sqrt{2b - bc} + \sqrt{2c - bc} = \sqrt{2}$$

$$\sqrt{2c - ca} + \sqrt{2a - ca} = \sqrt{3}$$

Then $[(1-a)(1-b)(1-c)]^2$ can be written as $\frac{m}{n}$.Then find $\lceil m+n \rceil$. where,

$$\lceil x \rceil = \min\{k \in \mathbb{Z} \mid k \geq x\}.$$

- (A) 34
- (B) 36
- (C) 39
- (D) 33

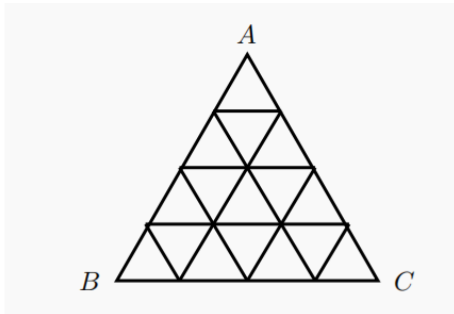


Figure 1: image for P3

3 P3

3. You live in $\triangle$-City, which is a large equilateral triangle with each side of length 2km. The corners are named A, B, and C. The city is partitioned into triangular blocks with sides of 500m each, and there are roads at the boundaries of the blocks.

You live in corner A and your office is at corner B.

You have decided to walk from home to the office every day, and you are willing to change your route, as long as the total distance is at most 2.5kms. How many such routes are possible from A to B?

- (A) 5
- (B) 21
- (C) 45
- (D) 11

4 P4

The number of ordered pairs (a, b) satisfying the equation

$$\frac{a}{b} + \frac{b+1}{a} = 4.$$

are

- (A) 00
- (B) 02
- (C) finitely many
- (D) infinitely many

5 P5

Suppose that the sequence $(a_n)_{n\geq 1}$ of real numbers is such that $a_{n+1} + a_n \to 0$. Then the set of limits of convergent subsequences of this sequence (select the incorrect option)

- (A)is a finite set
- (B) is an infinite set
- (C)can have at most two elements
- (D) cannot be an empty set

6 P6

Let $f : [0, 1] \to \mathbb{R}$ be a monotonic function such that the number of its discontinuity points is finite. For $t \geq 0$ let

$$m_t = \int_0^1 x^t f(x) dx.$$

Then which of the following statements is true?

- (A) There exists a $t \geq 0$ for which m_t is not defined.
- (B) $m_t < \infty$ for all $t \geq 0$.
- (C) m_t is defined and is equal to ∞ for all $t \geq 0$.
- (D) m_t is defined for all $t \geq 0$ and there exists $t_1, t_2 \geq 0$ such that m_{t_1} is equal to ∞ and $m_{t_2} < \infty$.

7 P7

Consider the circular disc on the xy-plane with centre $(0, 1)$ and radius 1 unit. Let $A = (0, 2)$. The disc is rolled on the x-axis so that its centre is now at $(1, 1)$. The point A has now moved to a point, say, B. Then the coordinates of B are

- (A) $\left(1 + \sin \frac{1}{2\pi}, 1 + \cos \frac{1}{2\pi}\right)$
- (B) $(1 + \sin 1, 1 + \cos 1)$
- (C) $\left(1 + \sin \frac{1}{2\pi}, 1 - \cos \frac{1}{2\pi}\right)$
- (D) $(1, 2)$

8 P8

Define $f : \mathbb{N} \to \mathbb{N}$ such that $\forall n \in \mathbb{N}$

$$f(f(f(n))) + f(f(n)) + f(n) = 3n,$$

$$A = \sum_{i=1}^{2023} \left(i^2 \prod_{k=1}^{i+1} f(k) \right),$$

the remainder when A is divided by 2025 is

- (A) 01
- (B) 02
- (C) 03
- (D) 04

9 P9

Let α, β, γ be the roots of the equation

$$x^3 - 13x^2 + 31x - 11 = 0$$

The value of

$$\frac{\alpha^4}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^4}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^4}{(\gamma-\alpha)(\gamma-\beta)}$$

is

- (A)
- (B)
- (C)
- (D)

10 P10

$$\left\{ \frac{5^{99}}{13} \right\}$$

can be written as $\frac{\alpha}{\zeta}$ then $\alpha + \zeta$ is [where $\{\cdot\}$ represents the fractional part function.]

- (A) 24

- (B) 18
- (C) 16
- (D) 21

11 P11

A curved wedge is cut from a cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at an angle 45° at the center of the cylinder. the volume of the wedge is

- (A)09
- (B)24
- (C)18
- (D)27

12 P12

Which of the following is/are true for a series of real numbers $\sum a_n$?

- p. If $\sum a_n$ converges then $\sum a_n^2$ converges
- q.If $\sum a_n^2$ converges then $\sum a_n$ converges
- r. If $\sum a_n^2$ converges then $\sum \frac{1}{n} a_n$ converges
- s. If $\sum |a_n|$ converges then $\sum \frac{1}{n} a_n$ converges
- (A) only r and s are true
- (B) p and q both are false
- (C) p and r are true
- (D) p,r and s are true

13 P13

How many permutations of the word 17295364 do not contain a three-term decreasing subsequence, (where a subsequence refers to a three-consecutive-digit string)

- (A) 1460
- (B) 1336
- (C) 1221
- (D) 1430

14 P14

Number of integers z such that all the roots of the following polynomial are also integers:

$$f(x) = x^3 - (z-3)x^2 - 11x + (9z - 28).$$

- (A) 0
- (B) 4
- (C) 1
- (D) 2

15 P15

7. Consider a continuous function $f : [-1, 1] \to \mathbb{R}$ which is differentiable everywhere in the interval $(-1, 1)$. Further, suppose that $f(-1) = -\frac{1}{2}$ and $f'(x) \leq 1$ for all $x \in (-1, 1)$. Which of the following statements is **false**?

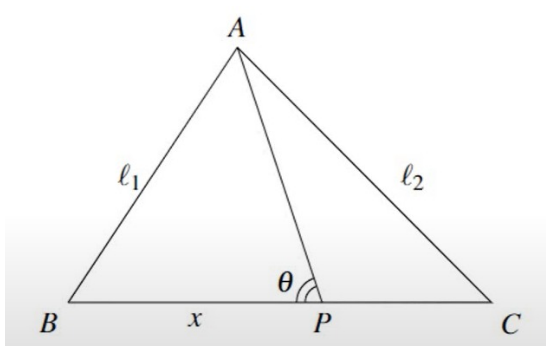
- (A) $f(1)$ can be greater than 1 but not greater than 2.
- (B) $f(1)$ can be greater than 2.
- (C) $f(1)$ can be less than -2.
- (D) $f(1)$ can be less than -1 but not less than -2.

16 P16

A triangle ABC, as shown below, is given. Assume $\ell_1 = \overline{AB}$, $\ell_2 = \overline{AC}$, $x = \overline{BP}$, and $\ell = \overline{BC}$, Given that $\ell_1 = 12$ and $\ell_2 = 16$, evaluate the integral:

$$\int_0^\ell \cos(\theta(x))\, dx$$

where $\theta(x) = \angle CPA$.



- (A)24
- (B)12
- (C)04
- (D)16

17 P17

The minimum of the function $f : (0, \infty)^3 \to \mathbb{R}$ is,

$$f(x, y, z) = x^z + y^z - (xy)^{z/4}.$$

- (A) -0.125
- (B) 1
- (C) -0.25
- (D) 0

18 P18

Let $(a_n)_{n\geq 1}$ be a non-constant arithmetic progression of positive numbers. then the limit

$$\lim_{n\to\infty} \frac{n\left(a_1 \cdots a_n\right)^{1/n}}{a_1 + \cdots + a_n}.$$

is

- (A)0
- (B)$\frac{1}{e}$
- (C) $\frac{2}{e}$
- (D) 1

19 P19

Let $f \in C^1(a, b)$, $\lim\limits_{x\to a^+} f(x) = \infty$, $\lim\limits_{x\to b^-} f(x) = -\infty$ and $f'(x) + f^2(x) \geq -1$ for $x \in (a, b)$. Then the minimum value of $b - a$ is

- (A)2π
- (B) 4π
- (C) 0
- (D)$\pi/2$

20 P20

Consider the following mappings from $\mathbb{C} \to \mathbb{C}$: $f(z) = \frac{1}{z}$, $g(z) = z^2$, $h(z) = A + Bz$ where $A, B \in \mathbb{C}$ with $|B| \neq 1$. Which of the following is false?

- (A) f maps circles into circles
- (B) g maps semi-circles into semi-circles
- (C) h maps squares into squares
- (D) None of the above

21 P21

The number of such functions $f : \{1, 2, \ldots, 10\} \to \{1, 2, \ldots, 2024\}$ that satisfy for every $1 \leq i \leq 9$, $f(i+1) - f(i) \geq 20$, equals

- (A) $10! \times \binom{1853}{10}$
- (B) $11! \times \binom{1854}{11}$
- (C) $\binom{1853}{10}$
- (D) $\binom{1854}{11}$

22 P22

Two line segments AB and CD are restricted to move along the X and Y axes, respectively, so that the points A, B, C, D are concyclic. If $AB = a$ and $CD = b$, then the locus of the centre of the circle passing through A, B, C, D in polar coordinates is

- (A) $r^2 = \frac{a^2+b^2}{4}$
- (B)$r^2 \cos 2\theta = \frac{a^2-b^2}{4}$
- (C) $r^2 = 4(a^2 + b^2$
- (D)$r^2 \cos 2\theta = 4(a^2 - b^2)$

23 P23

$$S = \sum_{r=1}^{\infty} \frac{1}{(3r-1)(3r-2)}.$$

If $S = \frac{m}{n}$ in its simplest form, find:

$$\lceil m + n \rceil .$$

- (A)07
- (B)05
- (C)03
- (D)08

24 P24

The sequence $(a_n)_{n\geq 0}$ satisfies

$$a_{m+n} + a_{m-n} = \frac{1}{2}(a_{2m} + a_{2n})$$

for all integers $m, n \geq 0$ with $m \geq n$.

If $a_1 = 1$, find largest prime factor of $a_{45} - 1$.

- (A)23
- (B)05
- (C)337
- (D)17

25 P25

Let $f_0(x) = \max(|x|, \cos(x))$, and $f_{k+1}(x) = \max(|x| - f_k(x), f_k(x) - \cos(x))$. Compute

$$\lim_{n\to\infty} \int_{-\pi/2}^{\pi/2} (f_{n-1}(x) + f_n(x))\, dx.$$

- (A)$\pi^2/4$
- (B)$\pi^2/2$
- (C)$\pi^4/16$
- (D)π

26 P26

If $\alpha + \beta + \theta = \pi$, and

$$\tan\left(\frac{\alpha+\beta-\theta}{4}\right) \times \tan\left(\frac{\theta+\alpha-\beta}{4}\right) \times \tan\left(\frac{\beta+\theta-\alpha}{4}\right) = 1,$$

the value of $1 + \cos\alpha + \cos\beta + \cos\theta$ equals

- (A)2
- (B)-1
- (C)0
- (D)1

27 P27

Let $f : \mathbb{N} \to \mathbb{N}$ be a bijective function then

$$\lim_{n\to\infty} \sum_{n=1}^{n} \frac{f(k)}{k^2}$$

is

- (A) 0
- (B) not 0 but possibly some finite number
- (C)infinite
- (D) the limit varies for different f

28 P28

Let ABC be a triangle in which $C = 90°$. Let m_a, m_b, m_c be its medians from the vertices A, B, C on to BC, CA, AB respectively. The maximum value of $\left(\frac{m_a+m_b}{m_c}\right)^2$ is

- (A) 25
- (B) 12
- (C) 10
- (D) 16

29 P29

Let S be a square of side length 1. Two points are chosen independently at random on the sides of S. The probability that the straight-line distance between the points is at least $\frac{1}{2}$ is

$$\frac{a - b\pi}{c}$$

where a, b, and c are positive integers and $\gcd(a, b, c) = 1$. What is $a+b+c$?

- (A) 59
- (B) 60
- (C) 61
- (D)69

30 P30

Let $f(x)$ and $g(x)$ be two polynomials in $\mathbb{Q}[X]$ such that

$$\lim_{|x|\to\infty}(\sqrt{f(x)} - g(x)) = 0.$$

which of the following assertion is false?

- (A)$f(x)$ must be of even degree
- (B)for such $g(x)$ to exis $f(x)$ must be a monic polynomial
- (C)if the leading coefficient of $f(x)$ is a^2 then leading coefficent of $g(x)$ must be $|a|$
- (D)all the roots of $g(x)$ must be roots of $f(x)$

An image to motivate myself