ISI-CMI Book

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December 16, 2024

1 Number Systems

1. If x is irrational, then choose $\xi = \frac{x}{2}$. Clearly $0 < \xi < x$.

If x is rational, then choose $\xi = \frac{x}{\sqrt{2}}$. Since $\sqrt{2} > 1$, we have $0 < \xi < x$.
(In fact there are infinitely many irrational numbers between any two real numbers.)

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2. Suppose $\sqrt{2} + \sqrt{5} = x = \frac{p}{q}$ is a rational number with $p, q \in \mathbb{Z}$

Then $(x - \sqrt{2})^2 = 5$ *i.e.* $x^2 - 2\sqrt{2}x + 2 = 5$.

Hence $x^2 - 3 = 2\sqrt{2}x$, which gives $\sqrt{2} = \frac{x^2-3}{2x}$, a rational number.

This contradicts the fact that $\sqrt{2}$ is rational. So $\sqrt{2} + \sqrt{5}$ is irrational.

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3. Let $x_n = \sqrt{n-1} + \sqrt{n+1}$ be rational. Then $1/x_n$ is also rational.

But
$$1/x_n = \frac{1}{\sqrt{n-1} + \sqrt{n+1}} = \frac{(\sqrt{n+1} - \sqrt{n-1})(\sqrt{n-1} + \sqrt{n+1})}{\sqrt{n+1} - \sqrt{n-1}} = \frac{\sqrt{n+1} + \sqrt{n-1}}{2}.$$
This means $\sqrt{n+1} - \sqrt{n-1}$ is also rational.

So $\sqrt{n-1}$ and $\sqrt{n+1}$ are also rational,

$$i.e., (n-1) \text{ and } (n+1) \text{ are perfect squares.}$$

This is not possible as any two perfect squares differ at least by 3. Hence there is no positive integer n such that $\sqrt{n-1} + \sqrt{n+1}$ is rational.

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4. Suppose $a \neq c$, let $a = c + x$. Then $a + \sqrt{b} = c + x + \sqrt{b} = c + \sqrt{d}$.

So $x + \sqrt{b} = \sqrt{d}$. Squaring we get,
$$d - x^2 - b = 2x\sqrt{b}.$$
This implies that $\sqrt{b}$ is rational, hence $\sqrt{d}$ is also rational. Thus b and d are squares of rationals. Hence the result.

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5. We have
$$a + b(\sqrt[3]{p}) + c(\sqrt[3]{p^2}) = 0 \tag{1}$$

Therefore,
$$a(\sqrt[3]{p}) + b(\sqrt[3]{p^2}) + cp = 0 \tag{2}$$

Now $b \times (1) - c \times (2)$ gives
$$(b^2 - ac)\sqrt[3]{p} + ab - c^2p = 0 \tag{3}$$

$\sqrt[3]{p}$ is irrational and therefore from (3) we get $b^2 - ac = 0$ and $ab = c^2p$.

Thus,
$$c^4p^2 = a^2b^2 = a^3c.$$

If $c \neq 0$, then we get
$$p^2 = \frac{a^3}{c^3}$$

which is not true as $\sqrt[3]{p}$ is irrational. Therefore, $c = 0$ which in turn implies that $a = b = 0$.

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2 Working with Integers

2.1 Exercise 2.1

1. The base case is trivial.

Now assume that in a knockout tournament with k, $k \leq n$ people, $k-1$ matches are needed to declare a champion. We now prove that if the tournament has $n+1$ people, n matches are needed:

Say you are about to dispute the final. That is, 2 players, A and B are about to play the last match to determine the winner. Then we know that A was the winner of some sub-tournament with a, $a \ \leq n$ people and B won a sub-tournament with b, $b \leq n$ people. We also know that

$$a + b = n + 1 \iff b = n + 1 - a.$$

But A was declared victorious of his sub-tournament after $a-1$ games and B of his after $b-1 = n+1-a-1$ games. That is, it took a total of

$$a - 1 + n - a = n - 1$$

games to determine the two contenders. We play one final match, and we get $n - 1 + 1 = n$ matches to determine the winner.

2. The base case is true. Let

$$S_n = \sum_{k=1}^{n} k^2$$

Assume that the statement is true for some $n \in \mathbb{N}$. Then we can say that

$$S_{n+1} = S_n + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$$

Hence we can that the statement is true for all $n \in \mathbb{N}$

3. The base case is true. Let

$$S_n = \sum_{k=1}^{n} k^3$$

Assume that the statement is true for some $n \in \mathbb{N}$. Then we can say that

$$S_{n+1} = S_n + (n+1)^3 = \frac{n^(n+1)^2}{4} = \frac{(n+1)^2(n+2)^2}{4}$$

Hence we can that the statement is true for all $n \in \mathbb{N}$

4. The base case is true. Let

$$S_n = \sum_{k=1}^{n} (2k-1)$$

Assume that the statement is true for some $n \in \mathbb{N}$. Then we can say that

$$S_{n+1} = S_n + (2n+1) = n^2 + (2n+1) = (n+1)^2$$

Hence we can that the statement is true for all $n \in \mathbb{N}$

5. Note that when $n = 10, 2^n = 1024 > 1000 = n^3$. Now suppose that $2^n > n^3$ for $n > 9$. Then,

$$\begin{aligned}
2^{n+1} &= 2 \cdot 2^n \\
&> 2n^3 \\
&= n^3 + n^3 \\
&> n^3 + 9n^2 \\
&= n^3 + 3n^2 + 6n^2 \\
&> n^3 + 3n^2 + 54n \\
&= n^3 + 3n^2 + 3n + 51n \\
&> n^3 + 3n^2 + 3n + 1 \\
&= (n+1)^3.
\end{aligned}$$

6. For the base case we have $2.7^1 + 3.5^1 - 5 = 24$ which is divisible by 24. Now assume that for some n, 24 divides $2.7^n + 3.5^n - 5$. Hence we can write

$$2.7^n + 3.5^n - 5 = 24\lambda \implies 2.7^n = 24\lambda - 3.5^n + 5$$

7. Observe that the base case is true as $(3+\sqrt{5})+(3-\sqrt{5}) = 6$ is divisible by 6. Observe we have the identity

$$a^{n+1} + b^{n+1} = (a+b)(a^n + b^n) - ab(a^{n-1} + b^{n-1})$$

Using this we can say that

$$(3+\sqrt{5})^{n+1} + (3-\sqrt{5})^{n+1} = 6(3+\sqrt{5})^n + (3-\sqrt{5})^n) - 4(3+\sqrt{5})^{n-1} + (3-\sqrt{5})^{n-1}$$

Now observe that 2^n divides $(3+\sqrt{5})^n + (3-\sqrt{5})^n)$ and 2 divides 6 and hence 2^{n+1} divides the first term. Similarly we can say that 2^{n-1} divides $(3+\sqrt{5})^n + (3-\sqrt{5})^n)$ and 2^2 divides 4 and hence we have that 2^{n+1} dives the entire thing. This proves the required statement.

10. Let $A = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n}$ and $B = \frac{2}{3} \cdot \frac{4}{5} \cdots \frac{2n}{2n+1}$ Then, we have $B > A$ as $\dfrac{n}{n+1} < \dfrac{n+1}{n+2}$ and $AB = \frac{1}{2n+1}$. From this two relation we can conclude $A < \frac{1}{\sqrt{2n+1}}$.

2.2 Exercise 2.2

1. A typical term $11\ldots11$ is of the type $4k+3$. The perfect squares are either of the form $4k$ or of the form $4k+1$. Hence, the number $11\ldots111$ cannot be a perfect square.

2. Note that

$$(ma, mb) = \text{least positive value of } max + mby = m(\text{least positive value of } ax + by) = m(a, b).$$

3. Note that $d \cdot \left(\frac{a}{d}, \frac{b}{d}\right) = (a, b)$. Hence, we get the required result.

4. Suppose $2d-1 = x^2$, $5d-1 = y^2$ and $13d-1 = z^2$. Since $2d-1 = x^2$ we get that d must be odd. Hence y and z must be even. Also, $x^2 - y^2 = (x-y)(x+y) = 8d$. Hence,

$$\frac{x^2 - y^2}{2} = \frac{(x+y)(x-y)}{2} = 2d.$$

Now either $\frac{x+y}{2}$ or $\frac{x-y}{2}$ must be even. Moreover, these two numbers differ by y, an even number. Hence their product $2d$ must be divisible by 4. Hence, d must be even, a contradiction. Hence, one can find distinct $a, b \in \{2, 5, 13\}$ such that $ab-1$ is not a perfect square.

5. $(7645, 2872) = 1$ since $3543(2872) - 1331(7645)$ and $(3645, 2357) = 1$.

6. Note that $x+1 \mid 2^k x - 1$ if k is a positive integer. We may assume that $m < n$. Put $x = a^{2m}$ and $k = 2^{n-m}$. Hence, $a^{2m} + 1 \mid (a^{2m})^{2^{n-m}} - 1$. Hence, we get that $(a^{2m} + 1, a^{2n} + 1) =$

- 1 if a is even,
- 2 if a is odd.

7. Let a, b, c be integers such that $(a, b) = 1$, $c > 0$. Let P denote the product of primes common to both a and c and $P = 1$ if there are no primes common to both. Let Q denote the product of primes common to both b and c and $Q = 1$ if there are no primes common to both. Let R denote the product of primes which divide c but do not divide ab and $R = 1$ if there are no such primes. Note that $(P, Q) = (P, R) = (Q, R) = 1$. It is now easy to see that $(a + bQR, c) = 1$.

8. Assume that there are finitely many primes of the type $6n-1$, say $p_1 = 5, p_2, \ldots, p_r$. Let $N = 6p_1p_2\cdots p_r - 1$. Clearly $N > 2$ and 2, 3 and none of the p_j's divide N. Moreover, since N is of the type $6n-1$, N must have a prime factor of the type $6n-1$. Hence, this prime is a new prime of the type $6n-1$, a contradiction. Hence, there are infinitely many primes of the type $6n-1$.

9. If $k, k+1, k+2$ are consecutive integers, then at least one of them is even and one of them is divisible by 3. If there are four consecutive integers, then there are two even integers and one of them is divisible by 4 as well.

10. Since $m = n^2 - n = n(n-1)$ and $m - 2 = (n-2)(n+1)$ and $m^2 - 2m = (n-2)(n-1)n(n+1)$. Hence, $m^2 - 2m$ is the product of 4 consecutive integers and hence $24 \mid m(m-2)$.

11. It can be easily seen that the book has more than 1000 pages. Suppose the book has $999 + x$ pages. Then, we get the equation

$$9 + 2(90) + 3(900) + 4x = 3189.$$

Hence, $x = 75$. Hence, the book has 1074 pages.

12. Note that

$$6k = (k+1)^3 + (k-1)^3 - k^3 - k^3 \quad \text{and} \quad 6k - 15 = (2k)^3 - (2k+1)^3 + (k-2)^3 - (k+2)^3.$$

13. Let $p > 3$ be an odd prime. Note that

$$\sum_{k=1}^{p-1} \frac{1}{k} = \sum_{k=1}^{(p-1)/2} \frac{p}{k(p-k)}.$$

Hence, the numerator is divisible by p.

14. Note that if $n \geq 4$, then $3 \mid n(n+2)(n+4)$.

15. Since 2 is the only even prime, the product $a = p_1 p_2 \cdots p_n$ is of the form $a = 2k$ where k is odd. But if $a+1 = q^2$, then q^2 and hence q is odd and so $a = (q+1)(q-1)$ is divisible by 4. This is a contradiction. So $a+1$ cannot be a square.

16. The units digit of the square of an integer must be one of $1, 4, 5, 6, 9$. But when $n > 4$, the number $a = 5! + \cdots + n!$ is divisible by 10 and so the units digit of

$$b = 1! + 2! + 3! + 4! + 6! + g = 33 + a = 3.$$

Hence, b is not a square.

17. Let $a < b$ and $a = 81m$, $b = 81n$ so that by data $(m, n) = 1$. Also, then the l.c.m. of $a, b = 81mn = 5103 = 81 \times 63$, so that $mn = 63 = 7 \times 9$. But $(m, n) = 1$. Hence, we may take $m = 7$, $n = 9$. Thus, $a = 81m = 567$, $b = 81n = 729$.

18. Let $a = 2^2 3^9 5^2 b$ where b is coprime to 2, 3, and 5. Then given conditions imply that

- (i) $x+1$, y, z are divisible by 2 and
- (ii) x, $y+1$, z are divisible by 3 and
- (iii) x, y, $z+1$ are divisible by 5.

So by trial, the smallest values of x, y, z are 15, 20, 24 respectively. Also, for any integer n, $b = n^{30}$ is a square, cube, and a fifth power. Hence, we may take $a = 2^{15} 3^{20} 5^{24} n^{30}$.

19. Let $(a, b) = d$ so that $a = dm$, $b = dn$ and $(m, n) = 1$. Then

$$\frac{a+1}{b} + \frac{b+1}{a} = c, \quad \text{an integer.}$$

This implies that $a^2 + a + b^2 + b = abc$.

20. If n is composite then $n = n_1 n_2$ where $n_1 > 1$ and $n_2 > 1$. Hence, $2^{n_1} - 1 > 1$ and $2^{n_1} - 1 \mid 2^n - 1$.

21. If $n = 2^\alpha n_1$, where n_1 is an odd integer greater than 1, then $2^\alpha + 1 \mid 2^n + 1$, a contradiction.

22. Just take $n = 2k+1$ and expand it. For the second part, we can see that the right hand side leaves a remainder 9 when divided by 16 and the left hand side can leave a maximum remainder of 8 when divided by 16. Hence this equation does not have any integer solutions.

23. The only possibilities are

$$12k+1,\ 12k+5,\ 12k+7,\ 12k+11$$

Since 4 divides $p-1$, only $12k+1$, and $12+5$ are possible and $(p-1)/4 = 3k$ or $3k+1$. If $k \neq 1$ then $12k+1$ is not possible.

Also $(p+1)/2 = 6k+1$ or $6k+3$. This shows that $6k+5$ is not possible.

So $p = 12 + 1 = 13$.

24. Any number is of the form $30k + c$ where $k \in \mathbb{N}$ and

$$c \in \{0, 1, 2, \ldots, 29\}$$

Now we can easily see that as the number is prime, so $c \notin \{0, 2, 4, \ldots, 28\}$, $c \notin \{0, 3, 6, \ldots, 27\}$, $c \notin \{0, 5, 10, \ldots, 25\}$, $c \notin \{0, 3, 6, \ldots, 27\}$. So the only numbers that remain are the primes and 1.

25. Call the middle number n. Now, if n is odd, then n is co-prime with $n-1$ and $n+1$ since the difference is only 1. But n is coprime with $n-2$ and $n+2$ as well, since all of them being odd means they don't have a common prime factor of 2, and since the difference between n and the other two is only 2, there cannot be any other common prime factor.

If n is even, then $n-1$ or $n+1$ are both odd, and at least one of them is not divisible by 3, since with a difference of 2, they can't both be divisible by 3. So the one that is not divisible by 3 has no prime factors smaller or equal to 3, and since the difference with all the others is at most 3, that one shares no common prime factors with any of the others, and is therefore coprime with all the others. So: if n is even then either $n-1$ or $n+1$ (or both) are coprime with all the others.

3 Congruences

3.1 Exercise 3.1

1. Note that $b = 2q+1$ for some integer q. Hence, $b^2 = 4q(q+1)+1$. Since $2 \mid q(q+1)$, we get that $b^2 \equiv 1 \pmod 8$.

2. An integer b can be written as $3q$ or $3q \pm 1$. If $b = 3q$, then $b^2 \equiv 0 \pmod 3$. If $b = 3q \pm 1$, then $b^2 \equiv 1 \pmod 3$. Hence, the square of an integer is $\equiv 0, 1 \pmod 3$.

3. If $p \neq 3$, then $3 \nmid p$, then $p^2 \equiv 1 \pmod 3$. Thus, $3 \mid p^2 + 8$. Hence, $p = 3$ is the only prime.

4. An integer b can be written as $5q$ or $5q \pm 1$ or $5q \pm 2$. If $b = 5q$, then $b^2 \equiv 0 \pmod 5$. If $b = 5q \pm 1$, then $b^2 \equiv 1 \pmod 5$. If $b = 5q \pm 2$, then $b^2 \equiv -1 \pmod 5$. Hence, the square of an integer is $\equiv 0, 1, -1 \pmod 5$.

5. Put $2n+1 = a^2$ and $3n+1 = b^2$. Since $2n+1$ is a perfect square, then n is divisible by 4. Hence, $3n+1 \equiv 1 \pmod 8$. Hence, $8 \mid n$. Since the square of an integer is $\equiv 0, 1, -1 \pmod 5$. We get $2n+1 \equiv 0, 1, -1 \pmod 5$ and $3n+1 \equiv 0, 1, -1 \pmod 5$. If $2n+1 \equiv 0 \pmod 5$, then $3n+1 \equiv 2 \pmod 5$, a contradiction. If $2n+1 \equiv -1 \pmod 5$, then $3n+1 \equiv 3 \pmod 5$, a contradiction. Hence, $2n+1 \equiv 1 \pmod 5$ and $3n+1 \equiv 1 \pmod 5$. Since $(8, 5) = 1$ we get $40 \mid n$.

6. Since $(n, 6) = 1$, $n^2 \equiv 1 \pmod 8$ and $n^2 \equiv 1 \pmod 3$.

7. We note that $a^2 + 2\sqrt{28n^2+1}$ is an even integer. Hence, $28n^2 + 1$ is a perfect square of an odd integer, say m. Now $28n^2 = m^2 - 1 = (m-1)(m+1)$ and

$$(m-1)(m+1) = \left(\frac{m-1}{2}\right)\left(\frac{m+1}{2}\right).$$

Hence,

$$\frac{m+1}{2} = 7a^2 \quad \text{and} \quad \frac{m-1}{2} = b^2.$$

If $\frac{m+1}{2} = 7a^2$ and $\frac{m-1}{2} = b^2$, then $b^2 \equiv -1 \pmod 7$, a contradiction. Hence, $\frac{m-1}{2} = 7a^2$ and $\frac{m+1}{2} = b^2$. Hence, $2 + 2m = 2 + 2(2b^2 - 1) = 4b^2$, a perfect square.

8. Note that $a^m - b^m = (a-b)(a^{m-1} + a^{m-2}b + \cdots + ab^{m-2} + b^{m-1})$. Since $a \equiv b \pmod{m^n}$, $a \equiv b \pmod m$. Hence, $a^{m-i}b^i = b^m \pmod m$ and $a^{m-1} + a^{m-2}b + \cdots + ab^{m-2} + b^{m-1} = mb^m \equiv 0 \pmod m$. Hence, $a^m - b^m$ is divisible by m^{n+1}.

9. Note that

$$\frac{a^p - b^p}{a-b} = \sum_{i=0}^{p-1} a^i b^{p-1-i} \equiv pb^{p-1} \equiv pa^{p-1} \pmod{a-b}.$$

It follows that the gcd $\left(\frac{a^p-b^p}{a-b}, a-b\right)$ divides pa^{p-1} and pb^{p-1}. Since a and b are coprime, the gcd is 1 or p.

3.2 Exercise 3.2

1. Let $n = m - 1$, so that m is odd. We must show that $m(m-2)$ divides $2^{(m-1)!} - 1$. Because $\phi(m) < m$, $\phi(m)$ divides $(m-1)!$, so $2^{\phi(m)} - 1$ divides $2^{(m-1)!} - 1$. Euler's theorem implies that m divides $2^{\phi(m)} - 1$. Therefore, m divides $2^{(m-1)!} - 1$. Arguing similarly for $m - 2$, we see that $m - 2$ divides $2^{(m-1)!} - 1$ as well. The numbers m and $m - 2$ are relatively prime, so $m(m-2)$ divides $2^{(m-1)!} - 1$, as desired.

2. We show that any subset of S having n elements that are pairwise coprime can be extended to a set with $n+1$ elements. Indeed, if N is the product of the elements of the subset, then since the elements of S are coprime to a, so must be N. By Euler's theorem,

$$a^{\phi(N)+1} + a^{\phi(N)} - 1 \equiv a + 1 - 1 \equiv a \pmod{N}.$$

It follows that $a^{\phi(N)+1} + a^{\phi(N)} - 1$ is coprime to N and can be added to S. We are done.

3. Add the two equations, then add 1 to each side to obtain the Diophantine equation

$$(x^3 + y + 1)^2 + z^9 = 147^{157} + 157^{147} + 1.$$

The right-hand side is rather large, and it is natural to reduce modulo some number. And since the left-hand side is a sum of a square and a ninth power, it is natural to reduce modulo 19 because $2 \times 9 + 1 = 19$. By Fermat's little theorem, $a^{18} \equiv 1 \pmod{19}$ whenever a is not a multiple of 19, and so the order of a square is either 1, 3, or 9, while the order of a ninth-power is either 1 or 2.

Computed by hand, the quadratic residues mod 19 are $-8, -3, -2, 0, 1, 4, 5, 6, 7, 9$, while the residues of ninth powers are $-1, 0, 1$. Also, applying Fermat's little theorem we see that

$$147^{157} + 157^{147} + 1 \equiv 14^{13} + 5^3 + 1 \equiv 14 \pmod{19}.$$

An easy verification shows that 14 cannot be obtained as a sum of a quadratic residue and a ninth-power residue. Thus the original system has no solution in integers x, y, and z.

4. The property is true for $p = 2$ and $p = 3$, since $2^2 + 3^2 + 6^2 - 1 = 48$. Let p be a prime greater than 3. By Fermat's little theorem, 2^{p-1}, 3^{p-1}, and 6^{p-1} are all congruent to 1 modulo p. Hence

$$3 \cdot 2^{p-1} + 2 \cdot 3^{p-1} + 6^{p-1} \equiv 3 + 2 + 1 = 6 \pmod{p}.$$

It follows that

$$6 \cdot 2^{p-2} + 6 \cdot 3^{p-2} + 6 \cdot 6^{p-2} \equiv 6 \pmod{p}.$$

Dividing by 6, we find that $2^{p-2} + 3^{p-2} + 6^{p-2} - 1$ is divisible by p, and we are done.

5. If x is a solution to the equation from the statement, then using Fermat's little theorem, we obtain

$$1 \equiv x^{p-1} \equiv a^{\frac{p-1}{2}} \pmod{p}.$$

If m is an integer, then every odd prime factor p of m^2+1 must be of the form $4m+1$, with m an integer. Indeed, in this case because $m^2 \equiv -1 \pmod{p}$, and by what we just proved,

$$(-1)^{\frac{p-1}{2}} = 1,$$

which means that $p - 1$ is divisible by 4.

Now assume that there are only finitely many primes of the form $4m + 1$, m an integer, say $p_1, p_2, \ldots, p_n$. The number $(2p_1p_2 \cdots p_n)^2 + 1$ has only odd prime factors, and these must be of the form $4m + 1$, m an integer. Yet these are none of $p_1, p_2, \ldots, p_n$, a contradiction. Hence the conclusion.

6. We have the factorization

$$16320 = 2^6 \cdot 3 \cdot 5 \cdot 17.$$

First, note that $p^{ab}-1=(p^a)^b-1$ is divisible by p^a-1. Hence $p^{32}-1$ is divisible by p^2-1, p^4-1, and $p^{16}-1$. By Fermat's little theorem, $p^2-1=p^{3-1}-1$ is divisible by 3, $p^4-1=p^{5-1}-1$ is divisible by 5, and $p^{16}-1=p^{17-1}-1$ is divisible by 17. Here we used the fact that p, being prime and greater than 17, is coprime to 3, 5, and 17.

We are left to show that $p^{32}-1$ is divisible by 2^6. Of course, p is odd, say $p=2m+1$, m an integer. Then $p^{32}-1=(2m+1)^{32}-1$. Expanding with Newton's binomial formula, we get

$$(2m)^{32}+\binom{32}{1}(2m)^{31}+\cdots+\binom{32}{2}(2m)^2+\binom{32}{1}(2m).$$

In this sum all but the last five terms contain a power of two greater than or equal to 6. On the other hand, it is easy to check that in

$$\binom{32}{5}(2m)^5+\binom{32}{4}(2m)^4+\binom{32}{3}(2m)^3+\binom{32}{2}(2m)^2+\binom{32}{1}(2m),$$

the first binomial coefficient is divisible by 2^2, the second by 2^2, the third by 2^3, the fourth by 2^4, and the fifth by 2^5. So this sum is divisible by 2^6, and hence $(2m+1)^{32}-1=p^{32}-1$ is itself divisible by 2^6. This completes the solution.

8. Assume a solution (x,y) exists. If y were even, then y^3+7 would be congruent to 3 modulo 4. But a square cannot be congruent to 3 modulo 4. Hence y must be odd, say $y=2k+1$. We have

$$x^2+1=y^3+2^3=(y+2)\left[(y-1)^2+3\right]=(y+2)(4k^2+3).$$

We deduce that x^2+1 is divisible by a number of the form $4m+3$, namely, $4k^2+3$. It must therefore be divisible by a prime number of this form. But we have seen in the previous problem that this is impossible. Hence the equation has no solutions.

9. Assume that the equation admits a solution (x,y). Let p be the smallest prime number that divides n. Because $(x+1)^n-x^n$ is divisible by p, and x and $x+1$ cannot both be divisible by p, it follows that x and $x+1$ are relatively prime to p. By Fermat's little theorem, $(x+1)^{p-1}\equiv 1\equiv x^{p-1}\pmod{p}$. Also, $(x+1)^n\equiv x^n\pmod{p}$ by hypothesis.

Additionally, because p is the smallest prime dividing n, the numbers $p-1$ and n are coprime. By the fundamental theorem of arithmetic, there exist integers a and b such that $a(p-1)+bn=1$. It follows that

$$x+1=(x+1)^{a(p-1)+bn}\equiv x^{a(p-1)+bn}\equiv x \pmod{p},$$

which is impossible. Hence the equation has no solutions.

4 Greatest Integer Function

1. Define $f : \mathbb{R} \to \mathbb{N}$,

$$f(x) = \lfloor x \rfloor + \left\lfloor x + \frac{1}{n} \right\rfloor + \cdots + \left\lfloor x + \frac{n-1}{n} \right\rfloor - \lfloor nx \rfloor .$$

We have

$$f\left(x + \frac{1}{n}\right) = \left\lfloor x + \frac{1}{n} \right\rfloor + \cdots + \left\lfloor x + \frac{n-1}{n} \right\rfloor + \left\lfloor x + \frac{n}{n} \right\rfloor - \lfloor nx + 1 \rfloor = f(x).$$

Therefore, f is periodic, with period $\frac{1}{n}$. Also, since $f(x) = 0$ for $x \in [0, \frac{1}{n})$, it follows that f is identically 0, and the identity is proved.

2. Ignoring the "brackets" we have

$$\frac{p}{q} + \frac{2p}{q} + \cdots + \frac{(q-1)p}{q} = \frac{(q-1)p}{2}.$$

The difference between kp/q and $\lfloor kp/q \rfloor$ is r/q, where r is the remainder obtained on dividing kp by q. Since p and q are coprime, $p, 2p, \ldots, (q-1)p$ form a complete set of residues modulo q. So for $k = 1, 2, \ldots, q-1$, the numbers $k/p - \lfloor kp/q \rfloor$ are a permutation of $1, 2, \ldots, q-1$. Therefore,

$$\sum_{k=1}^{q-1} \left\lfloor \frac{kp}{q} \right\rfloor = \sum_{k=1}^{q-1} \frac{kp}{q} - \sum_{k=1}^{q-1} \frac{k}{q} = \frac{(q-1)p}{2} - \frac{q-1}{2} = \frac{(p-1)(q-1)}{2},$$

and the reciprocity law follows.

3. The function

$$f(x) = \lfloor nx \rfloor - \frac{\lfloor x \rfloor}{1} - \frac{\lfloor 2x \rfloor}{2} - \frac{\lfloor 3x \rfloor}{3} - \cdots - \frac{\lfloor nx \rfloor}{n}$$

satisfies $f(x) = f(x+1)$ for all x and $f(0) = 0$. Moreover, the function is constant on subintervals of $[0, 1)$ that do not contain numbers of the form p/q, $2 \le q \le n$ and $1 \le p \le q-1$. Thus it suffices to verify the inequality for $x = p/q$, where p and q are coprime positive integers, $2 \le q \le n$, $1 \le p \le q-1$. Subtracting the inequality from

$$x = \frac{x}{1} + \frac{2x}{2} + \cdots + \frac{nx}{n},$$

we obtain the equivalent inequality for the fractional part $\{\}$ ($\{x\} = x - \lfloor x \rfloor$),

$$\{nx\} \le \frac{\{x\}}{1} + \frac{\{2x\}}{2} + \frac{\{3x\}}{3} + \cdots + \frac{\{nx\}}{n},$$

which we prove for the particular values of x mentioned above. If r_k is the remainder obtained on dividing kp by q, then $\{kx\} = \frac{r_k}{q}$, and so the inequality can be written as

$$\frac{r_n}{q} \le \frac{r_1/q}{1} + \frac{r_2/q}{2} + \frac{r_3/q}{3} + \cdots + \frac{r_n/q}{n},$$

or

$$r_n \le \frac{r_1}{1} + \frac{r_2}{2} + \frac{r_3}{3} + \cdots + \frac{r_n}{n}.$$

Truncate the sum on the right to the $(q-1)$st term. Since p and q are coprime, the numbers $r_1, r_2, \ldots, r_{q-1}$ are a permutation of $1, 2, \ldots, q-1$. Applying this fact and the AM–GM inequality, we obtain

$$\frac{r_1}{1} + \frac{r_2}{2} + \frac{r_3}{3} + \cdots + \frac{r_{q-1}}{q-1} \ge (q-1)\left(\frac{r_1}{1} \cdot \frac{r_2}{2} \cdot \frac{r_3}{3} \cdots \frac{r_{q-1}}{q-1}\right)^{1/(q-1)} = (q-1) \ge r_n.$$

This proves the (weaker) inequality

$$\frac{r_1}{1} + \frac{r_2}{2} + \frac{r_3}{3} + \cdots + \frac{r_n}{n} \ge r_n,$$

and consequently the inequality from the statement of the problem.

5. There are clearly more 2's than 5's in the prime factorization of $n!$, so it suffices to solve the equation

$$\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{5^2} \right\rfloor + \left\lfloor \frac{n}{5^3} \right\rfloor + \cdots = 1000.$$

On the one hand,

$$\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{5^2} \right\rfloor + \left\lfloor \frac{n}{5^3} \right\rfloor + \cdots < \frac{n}{5} + \frac{n}{5^2} + \frac{n}{5^3} + \cdots = \frac{n}{5} \cdot \frac{1}{1 - \frac{1}{5}} = \frac{n}{4},$$

and hence $n > 4000$. On the other hand, using the inequality $\lfloor a \rfloor > a - 1$, we have

$$\begin{aligned} 1000 &> \left(\frac{n}{5} - 1\right) + \left(\frac{n}{5^2} - 1\right) + \left(\frac{n}{5^3} - 1\right) + \left(\frac{n}{5^4} - 1\right) + \left(\frac{n}{5^5} - 1\right), \\ &= \frac{n}{5}\left(1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4}\right) - 5 = \frac{n}{5} \cdot \frac{1 - \left(\frac{1}{5}\right)^5}{1 - \frac{1}{5}} - 5, \end{aligned}$$

so

$$n < \frac{1005 \cdot 4 \cdot 3125}{3124} < 4022.$$

We have narrowed down our search to $\{4001, 4002, \ldots, 4021\}$. Checking each case with Polignac's formula, we find that the only solutions are $n = 4005, 4006, 4007, 4008$, and 4009.

6. Polignac's formula implies that the exponent of the number 2 in $n!$ is

$$\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2^2} \right\rfloor + \left\lfloor \frac{n}{2^3} \right\rfloor + \cdots.$$

Because

$$\frac{n}{2} + \frac{n}{2^2} + \frac{n}{2^3} + \cdots = n$$

and not all terms in this infinite sum are integers, it follows that n is strictly greater than the exponent of 2 in $n!$, and the claim is proved.

7. Let p be a prime number. The power of p in $\operatorname{lcm}(1, 2, \ldots, \lfloor \frac{n}{i} \rfloor)$ is equal to k if and only if

$$\left\lfloor \frac{n}{p^{k+1}} \right\rfloor < i \leq \left\lfloor \frac{n}{p^k} \right\rfloor.$$

Hence the power of p in the expression on the right-hand side is

$$\sum_{k \geq 1} k \left(\left\lfloor \frac{n}{p^k} \right\rfloor - \left\lfloor \frac{n}{p^{k+1}} \right\rfloor \right) = \sum_{k \geq 1} (k - (k-1)) \left\lfloor \frac{n}{p^k} \right\rfloor = \sum_{k \geq 1} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

By Polignac's formula, this is the exponent of p in $n!$ and we are done.

5 Some Harder Problems

2. For any prime p, let $\nu_p(n)$ be the maximum power of p dividing n; i.e. $p^{\nu_p(n)}$ divides n but not higher power. Let r be the length of the non-periodic part of the infinite decimal expansion of $1/n$.

Write

$$\frac{1}{n} = 0.a_1a_2\cdots a_r\overline{b_1b_2\cdots b_s}.$$

We show that $r = \max(\nu_2(n), \nu_5(n))$.

Let a and b be the numbers $a_1a_2\cdots a_r$ and $b = b_1b_2\cdots b_s$ respectively. (Here a and b can be both 0.) Then

$$\frac{1}{n} = \frac{1}{10^r}\left(a + \sum_{k\geq 1}\frac{b}{(10^s)^k}\right) = \frac{1}{10^r}\left(a + \frac{b}{10^s-1}\right).$$

Thus we get $10^r(10^s-1) = n((10^s-1)a+b)$. It shows that $r \geq \max(\nu_2(n), \nu_5(n))$. Suppose $r > \max(\nu_2(n), \nu_5(n))$. Then 10 divides $b-a$. Hence the last digits of a and b are equal: $a_r = b_s$. This means

$$\frac{1}{n} = 0.a_1a_2\cdots a_{r-1}\overline{b_sb_1b_2\cdots b_{s-1}}.$$

This contradicts the definition of r. Therefore $r = \max(\nu_2(n), \nu_5(n))$.

4. Choose $x = 2^{4r}$ and $y = 2^{3r}$. Then the left side is 2^{12r+1}. If we take $z = 2^k$, then we get $2^{12r+1} = 2^{31k}$. Thus it is sufficient to prove that the equation $12r + 1 = 31k$ has infinitely many solutions in integers. Observe that $(12 \times 18) + 1 = 31 \times 7$. If we choose $r = 31l + 18$ and $k = 12l + 7$, we get

$$12(31l + 18) + 1 = 31(12l + 7),$$

for all l. Choosing $l \in \mathbb{N}$, we get infinitely many $r = 31l + 18$ and $k = 12l + 7$ such that $12r + 1 = 31k$. Going back we have infinitely many (x, y, z) of integers satisfying the given equation.

5. Let $a = 3 + f$, where $0 < f < 1$. We are given that $(3 + f)(3 - 2f)$ is an integer. This implies that $2f^2 + 3f$ is an integer. Since $0 < f < 1$, we have $0 < 2f^2 + 3f < 5$. Therefore $2f^2 + 3f$ can take $1, 2, 3$ or 4. Equating $2f^2 + 3f$ to each one of them and using $f > 0$, we get

$$f = \frac{-3+\sqrt{17}}{4}, \quad \frac{1}{2}, \quad \frac{-3+\sqrt{33}}{4}, \quad \frac{-3+\sqrt{41}}{4}.$$

Therefore a takes the values:

$$a = 3 + \frac{-3+\sqrt{17}}{4}, \quad 3 + \frac{1}{2}, \quad 3 + \frac{-3+\sqrt{33}}{4}, \quad 3 + \frac{-3+\sqrt{41}}{4}.$$

6. Suppose $a = b$. Then we get one equation: $a^2 = ac+1$. This reduces to $a(a-c) = 1$. Therefore $a = 1, a-c = 1$; and $a = -1, a-c = -1$. Thus we get $(a, b, c) = (1, 1, 0)$ and $(-1, -1, 0)$.

If $a \neq b$, subtracting the second relation from the first we get

$$a^2 - b^2 = c(b - a).$$

This gives $a + b = -c$. Substituting this in the first equation, we get

$$a^2 = b(-a - b) + 1.$$

Thus $a^2 + b^2 + ab = 1$. Multiplication by 2 gives

$$(a + b)^2 + a^2 + b^2 = 2.$$

Thus $(a,b) = (1,-1), (-1,1), (1,0), (-1,0), (0,1), (0,-1)$. We get respectively $c = 0, 0, -1, 1, -1, 1$. Thus we get the triples:

$$(a,b,c) = (1,1,0), (-1,-1,0), (1,-1,0), (-1,1,0), (1,0,-1), (-1,0,1), (0,1,-1), (0,-1,1).$$

7. Let p and q be two consecutive primes, $p < q$. If we take any n such that $p \leq n < q$, we see that $P(n) = p$ and $N(n) = q$. Hence the term $\frac{1}{pq}$ occurs in the sum $q - p$ times. The contribution from such terms is $\frac{q-p}{pq} = \frac{1}{p} - \frac{1}{q}$. Since $n+1$ is prime, we obtain

$$\frac{1}{P(2)N(2)} + \frac{1}{P(3)N(3)} + \frac{1}{P(4)N(4)} + \cdots + \frac{1}{P(n)N(n)}$$

$$= \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{p} - \frac{1}{n+1}\right) = \frac{1}{2} - \frac{1}{n+1} = \frac{n-1}{2n+2}.$$

Here p is used for the prime preceding $n+1$.

8. Rewriting the given equation, we have

$$4m^3 + m^2 + 12m + 3 = 3p^n.$$

The left-hand side equals $(4m+1)(m^2+3)$.

Suppose that $(4m+1, m^2+3) = 1$. Then $(4m+1, m^2+3) = (3p^n, 1)$, $(3, p^n)$, $(p^n, 3)$ or $(1, 3p^n)$, a contradiction since $4m+1, m^2+3 \geq 4$. Therefore $(4m+1, m^2+3) > 1$.

Since $4m+1$ is odd, we have $(4m+1, m^2+3) = (4m+1, 49) = 7$ or 49. This proves that $p = 7$, and $4m+1 = 3 \cdot 7^k$ or 7^k for some natural number k. If $(4m+1, 49) = 7$, then we have $k = 1$ and $4m+1 = 21$, which does not lead to a solution. Therefore $(4m+1, m^2+3) = 49$.

If 7^3 divides $4m+1$, then it does not divide m^2+3, so we get $m^2 + 3 \leq 3 \cdot 7^2 < 4m+1$. This implies $(m-2)^2 < 2$, so $m \leq 3$, which does not lead to a solution. Therefore we have $4m+1 = 49$, which implies $m = 12$ and $n = 4$. Thus $(m,n,p) = (12,4,7)$ is the only solution.

9. Since $p_4 - p_1 = 8$, and no prime is even, we observe that $\{p_1, p_2, p_3, p_4\}$ is a subset of $\{p_1, p_1+2, p_1+4, p_1+6, p_1+8\}$. Moreover, p_1 is larger than 3. If $p_1 \equiv 1 \pmod{3}$, then p_1+2 and p_1+8 are divisible by 3. Hence we do not get 4 primes in the set $\{p_1, p_1+2, p_1+4, p_1+6, p_1+8\}$. Thus $p_1 \equiv 2 \pmod{3}$ and p_1+4 is not a prime. We get $p_2 = p_1+2$, $p_3 = p_1+6$, and $p_4 = p_1+8$.

Consider the remainders of p_1, p_1+2, p_1+6, p_1+8 when divided by 5. If $p_1 \equiv 2 \pmod{5}$, then p_1+8 is divisible by 5 and hence is not a prime. If $p_1 \equiv 3 \pmod{5}$, then p_1+2 is divisible by 5. If $p_1 \equiv 4 \pmod{5}$, then p_1+6 is divisible by 5. Hence the only possibility is $p_1 \equiv 1 \pmod{5}$.

Thus we see that $p_1 \equiv 1 \pmod{2}$, $p_1 \equiv 2 \pmod{3}$, and $p_1 \equiv 1 \pmod{5}$. We conclude that $p_1 \equiv 11 \pmod{30}$.

Similarly, $q_1 \equiv 11 \pmod{30}$. It follows that 30 divides $p_1 - q_1$.

10. Since $xyz \neq 0$, we can divide the second relation by the first. Observe that

$$x^4 + x^2y^2 + y^4 = (x^2 + xy + y^2)(x^2 - xy + y^2),$$

holds for any x, y. Thus we get

$$(x^2 - xy + y^2)(y^2 - yz + z^2)(z^2 - zx + x^2) = x^2y^2z^2.$$

However, for any real numbers x, y, we have

$$x^2 - xy + y^2 \geq |xy|.$$

Since $x^2y^2z^2 = |xy||yz||zx|$, we get

$$|xy||yz||zx| = (x^2 - xy + y^2)(y^2 - yz + z^2)(z^2 - zx + x^2) \geq |xy||yz||zx|.$$

This is possible only if

$$x^2 - xy + y^2 = |xy|, \quad y^2 - yz + z^2 = |yz|, \quad z^2 - zx + x^2 = |zx|,$$

hold simultaneously. However, $|xy| = \pm xy$. If $x^2 - xy + y^2 = -xy$, then $x^2 + y^2 = 0$, giving $x = y = 0$. Since we are looking for nonzero x, y, z, we conclude that $x^2 - xy + y^2 = xy$, which is the same as $x = y$. Using the other two relations, we also get $y = z$ and $z = x$. The first equation now gives $27x^6 = x^3$. This gives $x^3 = 1/27$ (since $x \neq 0$), or $x = 1/3$. We thus have $x = y = z = 1/3$. These also satisfy the second relation, as may be verified.

11. If $n > 1$ is such that

$$2p = (n-1)^2 + n^2 + (n+1)^2 + (n+2)^2 = 4n^2 + 4n + 6$$

then

$$p = 2n(n+1) + 3 > 3.$$

Observe that if $n \equiv 0 \pmod 3$ or $n \equiv 2 \pmod 3$ then $p \equiv 0 \pmod 3$ and hence can't be a prime. Therefore $n \equiv 1 \pmod 3$. Write $n = 3k + 1$ for some positive integer k. Observe that

$$p - 7 = 2(n^2 + n - 2) = 2(n-1)(n+2) = 18k(k+1) \equiv 0 \pmod{36}.$$

12. Since x^5 is rational, we see that $(20x)^5$ and $(x/19)^5$ are rational numbers. But

$$(20x)^5 - \left(\frac{19}{x}\right)^5 = \left(20x - \frac{19}{x}\right)\left((20x)^4 + (20^3 \cdot 19)x^2 + 20^2 \cdot 19^2 + (20 \cdot 19^3)\frac{1}{x^2} + \frac{19^4}{x^4}\right).$$

Consider

$$\begin{aligned} T &= \left((20x)^4 + (20^3 \cdot 19)x^2 + 20^2 \cdot 19^2 + (20 \cdot 19^3)\frac{1}{x^2} + \frac{19^4}{x^4}\right) \\ &= \left((20x)^4 + \frac{19^4}{x^4}\right) + 20 \cdot 19\left((20x)^2 + \frac{19^2}{x^2}\right) + (20^2 \cdot 19^2). \end{aligned}$$

Using $20x + (19)/x$ is rational, we get

$$(20x)^2 + \frac{19^2}{x^2} = \left(20x + \frac{19}{x}\right)^2 - 2 \cdot 20 \cdot 19$$

is rational. This leads to

$$(20x)^4 + \frac{19^4}{x^4} = \left((20x)^2 + \frac{19^2}{x^2}\right)^2 - 2 \cdot 20^2 \cdot 19^2$$

is also rational. Thus T is a rational number and $T \neq 0$. We conclude that $20x - (19/x)$ is a rational number. This combined with the given condition that $20x + (19/x)$ is rational shows $2 \cdot 20 \cdot x$ is rational. Therefore x is rational.

13. Let $\lfloor\sqrt{2n}\rfloor = k$. We observe that $x - 1 < \lfloor x \rfloor \leq x$. Hence

$$\sqrt{2n} < 1 + k \leq 1 + \sqrt{2n}.$$

Divisibility gives $(1+k)d = 2n$ for some positive integer d. Therefore we obtain

$$\sqrt{2n} < \frac{2n}{d} \leq 1 + \sqrt{2n}.$$

The first inequality gives $d < \sqrt{2n} < 1 + k$. But then

$$d = \frac{2n}{1+k} = \frac{(\sqrt{2n})^2}{1+k} \geq \frac{k^2}{1+k} = (k-1) + \frac{1}{k+1} > k-1.$$

We thus obtain $k - 1 < d < k + 1$. Since d is an integer, it follows that $d = k$. This implies that $n = k(k+1)/2$. Thus n is a triangular number. It is easy to check that every triangular number is a solution.

14. Observe that $\mathrm{lcm}(3, 4, 5) = 60$. We look for a solution in the form $a^3 = b^4 = c^5 = l^{60k}$. We choose l, k such that the condition is satisfied. The given equation gives

$$3l^{60k} = d^7.$$

This suggests choosing $l = 3$ so that $3^{60k+1} = d^7$. Now we take care of k by choosing k such that 7 divides $60k+1$. For example, we can take $k = 5$ so that $301 = 7 \times 43$. Thus we get

$$3^{300} + 3^{300} + 3^{300} = 3^{301}.$$

Choose $a = 3^{100}$, $b = 3^{75}$, $c = 3^{60}$, and $d = 3^{43}$. Then we get

$$a^3 + b^4 + c^5 = d^7.$$

This gives one solution. This suggests choosing $a = 3^{100} \cdot m^{140}$, $b = 3^{75} \cdot m^{105}$, $c = 3^{60} \cdot m^{84}$, and $d = 3^{43} \cdot m^{60}$. We see that

$$a^3 + b^4 + c^5 = 3^{300} \cdot m^{420} + 3^{300} \cdot m^{420} + 3^{300} \cdot m^{420} = m^{420}3^{301} = (3^{43} \cdot m^{60})^7 = d^7.$$

We can give different values for m and get infinitely many solutions of the equation.

15. We use divisibility argument by 7. Observe that the remainders of seven consecutive cubes modulo 7 are $0, 1, 1, 6, 1, 6, 6$ in some (cyclic) order. Hence the sum of seven consecutive cubes is 0 modulo 7. On the other hand, the remainders of two consecutive fourth powers modulo 7 is one of the sets $\{0, 1\}, \{1, 2\}, \{2, 4\}, \{4, 4\}$. Hence the sum of two fourth powers is never divisible by 7. It follows that the given equation has no solution in integers.

16. Firstly we show that if $m > 1$ is good, then so is $2m$. This is true since some proper divisors of m, including 1 (and hence not including m itself), sum to m; if we consider all these together with m, they will all be factors of $2m$ which sum to $2m$.

This means that it suffices to prove the claim for odd numbers. The claim holds for 1 since good numbers exist (such as 6, which is $1 + 2 + 3$, for example).

If $a > 1$ is odd and $n = 2^k a$ for some k, then

$$a + 2a + 4a + \cdots + 2^{k-1}a = (2^k - 1)a = n - a$$

which is close to n. This value of n will be good if we can find some other factors of n, including 1, which sum to a.

To do this, we write a as a sum of powers of 2, including 1, by writing a in binary, and then choose k to be large enough for all those powers of 2 to be factors of n. (We may take k to be $\lceil \log_2(a) \rceil$, the smallest integer greater than or equal to the base-2 logarithm of a.) None of these powers of 2 are multiples of a so there is no risk that we are using the same factor of n twice.

19. Since $q(p-1)$ is a perfect square and q is prime, we should have $p - 1 = qb^2$ for some positive integer b. Let $a^2 = p - q$. Therefore, $q = p - a^2$, and substituting that into the $p - 1 = qb^2$ and solving for p gives

$$p = \frac{a^2b^2 - 1}{b^2 - 1} = \frac{(ab-1)(ab+1)}{b^2 - 1}.$$

Notice that we also have

$$p = \frac{a^2b^2-1}{b^2-1} = a^2 + \frac{a^2-1}{b^2-1}$$

and so $b^2-1 \mid a^2-1$. We run through the cases:

- $a=1$: Then $p-q=1$ so $(p,q)=(3,2)$, which works.
- $a=b$: This means $p=a^2+1$, so $q=1$, a contradiction.
- $a>b$: This means that $b^2-1<ab-1$. Since b^2-1 can be split up into two factors F_1, F_2 such that $F_1 \mid ab-1$ and $F_2 \mid ab+1$, we get

  $$p = \frac{ab-1}{F_1} \cdot \frac{ab+1}{F_2}$$

  and each factor is greater than 1, contradicting the primality of p.

Thus, the only solution is $(p,q)=(3,2)$.

6 Polynomials

1. We know from Vieta that $a+b=-a$ and $ab=b$. Now from the second equation $ifb=0$, we get $a=0$. If not, we can say that $a=1$ and $b=-2$. Thus the solutions are

$$(a,b)=(0,0) \text{ or } (1,-2)$$

2. If $u=\alpha$ and v are the roots

$$\begin{aligned} x^2+ax+b+1 &= (x-u)(x-v) \\ &= x^2-(u+v)x+uv \end{aligned}$$

Then $a=-(u+v)$ and $b=uv-1$. Since $u=\alpha\in\mathbb{Z}$, and $a\in\mathbb{Z}$, we know that $v\in\mathbb{Z}$. Therefore,

$$\begin{aligned} a^2+b^2 &= (u+v)^2+(uv-1)^2 \\ &= u^2+v^2+u^2v^2+1 \\ &= (u^2+1)(v^2+1) \end{aligned}$$

Which is a composite number.

3. Since β and γ are the roots of the first equation, we have $\beta+\gamma=-\alpha$ and $\beta\gamma=-a$. Similarly, from the second equation we have $\gamma+\alpha=-\beta$ and $\gamma\alpha=-b$. Thus, we have $\alpha+\beta+\gamma=0$ and $\beta\gamma=-a$ and $\gamma\alpha=-b$.

$$(\alpha+\beta+\gamma)^2=0 \implies (\alpha^2+\beta^2+\gamma^2)+2(\alpha\beta+\beta\gamma+\gamma\alpha)=0.$$

From this we obtain

$$2(a+b+c)+2(\alpha\beta-a-b)=0 \implies \alpha\beta=-c.$$

Since we have $\alpha\beta=-c$ and $\alpha+\beta=-\gamma$, we can say that α and β are the roots of the equation $x^2+\gamma x-c=0$.

4. The cubic vanishes at $x=1$ and $x=-2$, thus we can write it as

$$p(x)=k(x-1)(x+2)(x-r)$$

where r is the third root. Now, we know that

$$p(-1)=k(-2)(1)(-1-r)=4$$

$$p(2)=k(1)(4)(2-r)=8$$

On solving these equations, we get that the polynomial should be

$$p(x)=3(x-1)(x+2)\Big(x-\frac{1}{3}\Big)$$

5. The rational root theorem.

6. Of the two integers $-a$ and $1-a$, exactly one is even. If $f(a)=0$, then

$$f(x)=(x-a)g(x).$$

But $f(0)=-ag(a)$, and $f(1)=(1-a)g(a)$. Both $f(0)$ and $f(1)$ cannot be odd.

7. We are given that $x=2$ is a root of the cubic equation:

$$84x^3-157x^2-kx+78=0.$$

To find k, substitute $x=2$ into the equation:

$$84(2^3)-157(2^2)-k(2)+78=0.$$

Calculating each term:

$$2^3 = 8, \quad 84 \cdot 8 = 672, \quad 2^2 = 4, \quad -157 \cdot 4 = -628.$$

Thus:

$$672 - 628 - 2k + 78 = 0.$$

Combine like terms:

$$672 - 628 = 44, \quad 44 + 78 = 122.$$

We get:

$$122 - 2k = 0 \implies k = \frac{122}{2} = 61.$$

With $k = 61$, the equation becomes:

$$84x^3 - 157x^2 - 61x + 78 = 0.$$

Since $x = 2$ is a root, we divide the polynomial by $(x - 2)$ using synthetic division:

$$\begin{array}{c|cccc} & 84 & -157 & -61 & 78 \\ 2 & & 168 & 22 & -78 \\ \hline & 84 & 11 & -39 & 0 \end{array}$$

The quotient is:

$$84x^2 + 11x - 39.$$

To find the remaining roots, solve the quadratic equation:

$$84x^2 + 11x - 39 = 0.$$

Using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad \text{where } a = 84,\ b = 11,\ c = -39.$$

First, compute the discriminant:

$$b^2 - 4ac = 11^2 - 4(84)(-39) = 121 + 13104 = 13225.$$

Since $\sqrt{13225} = 115$, we substitute into the formula:

$$x = \frac{-11 \pm 115}{2 \cdot 84} = \frac{-11 \pm 115}{168}.$$

Consider the two cases:

$$x = \frac{-11 + 115}{168} = \frac{104}{168} = \frac{13}{21}, \quad x = \frac{-11 - 115}{168} = \frac{-126}{168} = -\frac{3}{4}.$$

The roots of the equation are therefore:

$$x = 2, \quad x = \frac{13}{21}, \quad x = -\frac{3}{4}.$$

8. Observe that the roots of $x^2 + x + 1$ are of ω, ω^2. Plugging these into the given polynomials and equating it with 0, we get the values of $a = 4, b = 1$.

9. $8x^2 - 10x - 61$

10. We are given a cubic equation:

$$x^3 + px - q = 0$$

with roots α, β, γ.

From this, we know the standard relationships between the roots and the coefficients using Vieta's theorem:

1. $\alpha + \beta + \gamma = 0$ (since the x^2 term is missing),
2. $\alpha\beta + \beta\gamma + \gamma\alpha = p$,
3. $\alpha\beta\gamma = q$.

We need to find the cubic equation whose roots are:

$$\alpha + \beta, \quad \beta + \gamma, \quad \gamma + \alpha.$$

Given $\alpha + \beta + \gamma = 0$, we can rewrite the new roots in terms of the original ones:

$$\alpha + \beta = -\gamma, \quad \beta + \gamma = -\alpha, \quad \gamma + \alpha = -\beta.$$

Thus, the new roots $\alpha + \beta$, $\beta + \gamma$, and $\gamma + \alpha$ are simply $-\gamma$, $-\alpha$, and $-\beta$, respectively. In other words:

$$\{\alpha + \beta, \, \beta + \gamma, \, \gamma + \alpha\} = \{-\alpha, \, -\beta, \, -\gamma\}.$$

Since α, β, γ are roots of $x^3 + px - q = 0$, the polynomial can be written as:

$$(x - \alpha)(x - \beta)(x - \gamma) = x^3 + px - q.$$

The polynomial whose roots are $-\alpha$, $-\beta$, and $-\gamma$ is:

$$(x - (-\alpha))(x - (-\beta))(x - (-\gamma)) = (x + \alpha)(x + \beta)(x + \gamma).$$

Now, consider the transformation $x \mapsto -x$. If we start from $f(x) = x^3 + px - q$, then:

$$f(-x) = (-x)^3 + p(-x) - q = -x^3 - px - q.$$

Multiplying by -1 gives:

$$-f(-x) = x^3 + px + q.$$

Thus, the polynomial with roots $-\alpha$, $-\beta$, and $-\gamma$ is:

$$x^3 + px + q.$$

11. $x^3 - 2x^2 + 5x - 11 = 0$.

12. Just use Vieta's theorem. If originally, the roots were a,b,c, then the new equation we are seeking will have roots a^3, b^3, c^3.
We can use simple algebraic manipulations to find that sum of roots is 0, sum of roots taken two at a time is 2, and the product of roots is 1. Thus, the equation is $x^3 + 2x - 1 = 0$
13. The polynomial $f(x) + 1$ has $x - a$, $x - b$, $x - c$ as factors, and so:

$$f(x) = (x - a)(x - b)(x - c)g(x) - 1$$

for some polynomial $g(x)$ with integer coefficients.

But then, if $f(d) = 0$ for some integer d, it follows that:

$$(d - a)(d - b)(d - c)g(d) = 1.$$

Thus, $d - a$, $d - b$, $d - c \in \{+1, -1\}$, and so two of them must be equal. This contradicts the assumption that a, b, c are distinct.

Hence, $g(d) \neq 0$ for $d \in \mathbb{Z}$.
14. If a is negative or zero, then the quadratic has two real roots.

But we can easily check that the other polynomial has derivative everywhere positive and hence only one real root.

So a must be positive. If

$$x^2 - x + a$$

divides $x^{13} + x + 90$, then $x^{13} + x + 90 = f(x)(x^2 - x + a)$,

where $f(x)$ is a polynomial with integer coefficients.

Let $x = 0$ we see that a must divide 90. Let $x = 1$ we see that it must divide 92.

Hence it must divide 92 - 90 $= 2$. So the only possibilities are 1 and 2.

Suppose $a = 1$ then putting $x = 2$ we have that 3 divides $2^{13} + 92$ but 2odd 2 mod 3, so $2^{13} + 92$ is congruent to 1 mod 3. is congruent to

So a cannot be 1.

To see that $a = 2$ is possible, we write

$$\begin{aligned}(x^2 - x + 2)(x^{11} + x^{10} - x^9 - 3x^8 - x^7 + 5x^6 + 7x^5 - 3x^4 - 17x^3 - 11x^2 + 23x + 45) \\ = x^{13} + x + 90\end{aligned}$$

15 We can solve this using the idea if extremal principle. If $f(x)$ had odd degree, then it would have real roots. Let x_0 be the greatest real root. Then

$$0 = f(x_0)g(x_0) = f({x_0}^2 + x_0 + 1).$$

So, ${x_0}^2 + x_0 + 1$ is another real root of $f(x)$, which is impossible, since ${x_0}^2 + x_0 + 1 > x_0$ and x_0 is the greatest real root.
16.

$$f(x, y) = \sum_{i,j} c_{ij} x^i y^j.$$

Since f is antisymmetric, we find $c_{ij} = -c_{ji}$, so we can write

$$f(x, y) = \sum_{i>j} c_{ij}(x^i y^j - y^i x^j).$$

Now we've found out that f is some linear combination of terms $x^i y^j - y^i x^j$. If we show that each of these terms is divisible by $x - y$, then so is f, and we're done.

So suppose $i > j$, and let $k = i - j$. Since

$$x^k - y^k = (x - y)(x^{k-1} + x^{k-2}y + x^{k-3}y^2 + \cdots + y^{k-1})$$

is divisible by $x - y$, so is $x^i y^j - y^i x^j = x^j y^j (x^k - y^k)$.
17. Use similar idea as the previous problem.
18. The claim is that if a symmetric polynomial $f(x, y)$ is divisible by $(x - y)$, then it is actually divisible by $(x - y)^2$.

We use the fundamental fact that every symmetric polynomial in x and y can be expressed as a polynomial in the elementary symmetric polynomials:

$$e_1 = x + y, \quad e_2 = xy.$$

Thus, there exists a polynomial $F(e_1, e_2)$ such that:

$$f(x, y) = F(x + y, xy).$$

Since $f(x, y)$ is divisible by $(x - y)$, it must vanish whenever $x = y$. Substituting $x = y$ into f, we find:

$$f(x, x) = F(2x, x^2) = 0 \quad \text{for all } x.$$

This condition tells us that $F(e_1, e_2)$ vanishes on the infinite set of points $(e_1, e_2) = (2x, x^2)$ for all x.

Consider the curve defined by $e_1 = 2x$ and $e_2 = x^2$. Eliminating x gives the relation:

$$e_2 = \left(\frac{e_1}{2}\right)^2.$$

The polynomial $F(e_1, e_2)$ vanishes for every point on the parabola $e_2 = \frac{e_1^2}{4}$. This implies that $F(e_1, e_2)$ must be divisible by the polynomial defining this parabola:

$$e_2 - \frac{e_1^2}{4} = 0.$$

Thus, there exists a polynomial $G(e_1, e_2)$ such that:

$$F(e_1, e_2) = \left(e_2 - \frac{e_1^2}{4}\right) G(e_1, e_2).$$

Clearing the fraction by multiplying both sides by 4 gives:

$$4F(e_1, e_2) = (4e_2 - e_1^2)G(e_1, e_2).$$

Hence:

$$F(e_1, e_2) \text{ is divisible by } (e_1^2 - 4e_2).$$

Now relate this back to x and y. Recall the well-known identity:

$$(x - y)^2 = x^2 - 2xy + y^2 = (x + y)^2 - 4xy = e_1^2 - 4e_2.$$

We have shown that $e_1^2 - 4e_2$ divides $F(e_1, e_2)$. Since:

$$f(x, y) = F(x + y, xy),$$

it follows that:

$$(x - y)^2 = e_1^2 - 4e_2 \mid f(x, y).$$

Therefore, we conclude that if $f(x, y)$ is symmetric and $(x - y)$ divides $f(x, y)$, then $(x - y)^2$ also divides $f(x, y)$.

19. Let $p(x) = nx^{n+1} - (n + 1)x^n + 1$.
Then $(x - 1)^2$ is a factor of p(x) iff 1 is a double root.
Clearly, $p(1) = 0$.
$p'(x) = n(n + 1)x^n - (n + 1)nx^{n-1}$.
Clearly, $p'(1) = 0$.
So, 1 indeed is a double root.
20.

Using the quadratic formula, the roots of the polynomial $x^2 - 6x + 1$ are:

$$x_1 = 3 + 2\sqrt{2}$$
$$x_2 = 3 - 2\sqrt{2}$$

Let $S_n = x_1^n + x_2^n$

- $S_0 = x_1^0 + x_2^0 = 1 + 1 = 2$
- $S_1 = x_1^1 + x_2^1 = 3 + 2\sqrt{2} + 3 - 2\sqrt{2} = 6$

Now, multiply the quadratic equation ($x^2 - 6x + 1 = 0$) by x^{n-2}:

$x^n - 6x^{n-1} + x^{n-2} = 0$

Substitute x_1 and x_2 into this equation and add the two resulting equations:

$$x_1^n - 6x_1^{n-1} + x_1^{n-2} = 0$$
$$x_2^n - 6x_2^{n-1} + x_2^{n-2} = 0$$

$(x_1^n + x_2^n) - 6(x_1^{n-1} + x_2^{n-1}) + (x_1^{n-2} + x_2^{n-2}) = 0$

This gives us the recursive relation:

$S_n - 6S_{n-1} + S_{n-2} = 0$ or $S_n = 6S_{n-1} - S_{n-2}$

we prove S_n is an integer by induction

- **Base Case:** We've shown $S_0 = 2$ and $S_1 = 6$ are integers.
- **Inductive Hypothesis:** Assume S_k and S_{k-1} are integers for some arbitrary integer $k \geq 1$.
- **Inductive Step:** We need to show that S_{k+1} is an integer. Using the recursive relation: $S_{k+1} = 6S_k - S_{k-1}$. Since S_k and S_{k-1} are integers (by the inductive hypothesis), and 6 is an integer, S_{k+1} is also an integer.

Therefore, S_n is an integer for all nonnegative integers n.

We now prove S_n is not divisible by 5 by induction

- **Base Case:** $S_0 = 2$ and $S_1 = 6$ are not divisible by 5.
- **Inductive Hypothesis:** Assume S_k and S_{k-1} are not divisible by 5 for some arbitrary integer $k \geq 1$.
- **Inductive Step:** We need to show that S_{k+1} is not divisible by 5. $S_{k+1} = 6S_k - S_{k-1}$.

  Consider the possible remainders when an integer is divided by 5 (0, 1, 2, 3, 4).

  - If S_k leaves a remainder of 1 when divided by 5, and S_{k-1} leaves a remainder of 2, then $6S_k$ leaves a remainder of 1 and $6S_k - S_{k-1}$ leaves a remainder of 4.
  - Continue checking all possible combinations of remainders for S_k and S_{k-1}. You'll find that S_{k+1} never leaves a remainder of 0, meaning it's never divisible by 5.

Therefore, S_n is not divisible by 5 for all nonnegative integers n.

We have proven that for every nonnegative integer n, $x_1^n + x_2^n$ is an integer and not divisible by 5.

21. If $f(x) = 0$ has a rational root, then this root is an integer. Suppose that $f(x)$ has the integral root $x_0 = m$, that is $f(m) = 0$. Then $f(x) = (x - m)g(x)$, where $g(x)$ has integral coefficients. By setting $x = k, k + 1, ..., k + p$ in the last equation, we get $f(k) = (k - m)g(k)$, $f(k + 1) = (k + 1 - m)g(k + 1)$, ..., $f(k + p) = (k - p - m)g(k + p)$. One of the $p + 1$ successive integers $k - m$, ..., $k + p - m$ is divisible by $p + 1$. This proves the contrapositive statement which is equivalent to the original statement.

22. For $x \le 0$ we have obviously $p(x) > 0$. Let $x > 0$. We transform the polynomial in the same way as a geometric series:

$$p(x) = x^{2n} - 2x^{2n-1} + 3x^{2n-2} - \cdots - 2nx + 2n + 1$$
$$xp(x) = x^{2n+1} - 2x^{2n} + 3x^{2n-1} - 4x^{2n-2} + \cdots + (2n+1)x.$$

Adding, we get

$$xp(x) + p(x) = x^{2n+1} - x^{2n} + x^{2n-1} - x^{2n-2} + \cdots + x + 2n + 1$$
$$(1+x)p(x) = x \cdot \frac{1 + x^{2n+1}}{1+x} + 2n + 1.$$

From here we see that $p(x) > 0$ for $x > 0$.

1. For $i \ge 0$, let $f_i(x)$ denote the polynomial on the blackboard after Hobbes' i-th turn. We let Calvin decrease the coefficient of x by 1. Therefore $f_{i+1}(2) = f_i(2) - 1$ or $f_{i+1}(2) = f_i(2) - 3$ (depending on whether Hobbes increases or decreases the constant term). So for some i, we have $0 \le f_i(2) \le 2$. If $f_i(2) = 0$ then Calvin has won the game. If $f_i(2) = 2$ then Calvin wins the game by reducing the coefficient of x by 1. If $f_i(2) = 1$ then $f_{i+1}(2) = 0$ or $f_{i+1}(2) = -2$. In the former case, Calvin has won the game and in the latter case Calvin wins the game by increasing the coefficient of x by 1.

2. Suppose that m is an integer root of $x^4 - ax^3 - bx^2 - cx - d = 0$. As $d \ne 0$, we have $m \ne 0$. Suppose now that $m > 0$. Then $m^4 - am^3 = bm^2 + cm + d > 0$ and hence $m > a \ge d$. On the other hand $d = m(m^3 - am^2 - bm - c)$ and hence m divides d, so $m \le d$, a contradiction. If $m < 0$, then writing $n = -m > 0$ we have $n^4 + an^3 - bn^2 + cn - d = n^4 + n^2(an - b) + (cn - d) > 0$, a contradiction. This proves that the given polynomial has no integer roots.

3. Let $r = u/v$ where $\gcd(u, v) = 1$. Then we get

$$a_n u^n + a_{n-1} u^{n-1} v + \cdots + a_1 u v^{n-1} + a_0 v^n = 0,$$

$$b_n u^n + b_{n-1} u^{n-1} v + \cdots + b_1 u v^{n-1} + b_0 v^n = 0.$$

Subtraction gives

$$(a_n - b_n)u^n + (a_{n-2} - b_{n-2})u^{n-2}v^2 + \cdots + (a_1 - b_1)uv^{n-1} + (a_0 - b_0)v^n = 0,$$

since $a_{n-1} = b_{n-1}$. This shows that v divides $(a_n - b_n)u^n$ and hence it divides $a_n - b_n$. Since $a_n - b_n$ is a prime, either $v = 1$ or $v = a_n - b_n$. Suppose the latter holds. The relation takes the form

$$u^n + (a_{n-2} - b_{n-2})u^{n-2}v + \cdots + (a_1 - b_1)uv^{n-2} + (a_0 - b_0)v^{n-1} = 0.$$

(Here we have divided through-out by v.) If $n > 1$, this forces $v \mid u$, which is impossible since $\gcd(v, u) = 1$ ($v > 1$ since it is equal to the prime $a_n - b_n$). If $n = 1$, then we get two equations:

$$a_1 u + a_0 v = 0,$$

$$b_1 u + b_0 v = 0.$$

This forces $a_1 b_0 - a_0 b_1 = 0$ contradicting $a_n b_0 - a_0 b_n \ne 0$. (Note: The condition $a_n b_0 - a_0 b_n \ne 0$ is extraneous. The condition $a_{n-1} = b_{n-1}$ forces that for $n = 1$, we have $a_0 = b_0$. Thus we obtain, after subtraction

$$(a_1 - b_1)u = 0.$$

This implies that $u = 0$ and hence $r = 0$ is an integer.)

4. Consider the discriminants of the three equations

$$px^2 + qr + r = 0 \tag{1}$$

$$qx^2 + rx + p = 0 \tag{2}$$

$$rx^2 + px + q = 0. \tag{3}$$

Let us denote them by D_1, D_2, D_3 respectively. Then we have

$$D_1 = 4(q^2 - rp), \quad D_2 = 4(r^2 - pq), \quad D_3 = 4(p^2 - qr).$$

We observe that

$$D_1 + D_2 + D_3 = 4(p^2 + q^2 + r^2 - pq - qr - rp) = 2\{(p-q)^2 + (q-r)^2 + (r-p)^2\} > 0.$$

since p, q, r are not all equal. Hence at least one of D_1, D_2, D_3 must be positive. We may assume $D_1 > 0$.

Suppose $D_2 < 0$ and $D_3 < 0$. In this case both the equations (2) and (3) have only non-real roots and equation (1) has only real roots. Hence the common root α must be between (2) and (3). But then $\bar{\alpha}$ is the other root of both (2) and (3). Hence it follows that (2) and (3) have the same set of roots. This implies that

$$\frac{q}{r} = \frac{r}{p} = \frac{p}{q}.$$

Thus $p = q = r$ contradicting the given condition. Hence both D_2 and D_3 cannot be negative. We may assume $D_2 \geq 0$. Thus we have

$$q^2 - rp > 0, \quad r^2 - pq \geq 0.$$

These two give

$$q^2r^2 > p^2qr$$

since p, q, r are all positive. Hence we obtain $qr > p^2$ or $D_3 < 0$. We conclude that the common root must be between equations (1) and (2). Thus

$$p\alpha^2 + q\alpha + r = 0,$$

$$q\alpha^2 + r\alpha + p = 0.$$

Eliminating α^2, we obtain

$$2(q^2 - pr)\alpha = p^2 - qr.$$

Since $q^2 - pr > 0$ and $p^2 - qr < 0$, we conclude that $\alpha < 0$. The condition $p^2 - qr < 0$ implies that the equation (3) has only non-real roots.

Alternatively one can argue as follows. Suppose α is a common root of two equations, say, (1) and (2). If α is non-real, then $\bar{\alpha}$ is also a root of both (1) and (2). Hence the coefficients of (1) and (2) are proportional. This forces $p = q = r$, a contradiction. Hence the common root between any two equations cannot be non-real. Looking at the coefficients, we conclude that the common root α must be negative. If (1) and (2) have common root α, then $q^2 \geq rp$ and $r^2 \geq pq$. Here at least one inequality is strict for $q^2 = pr$ and $r^2 = pq$ forces $p = q = r$. Hence $q^2r^2 > p^2qr$. This gives $p^2 < qr$ and hence (3) has nonreal roots.

5. Suppose α is a real root of the given equation. Then

$$\alpha^5 - \alpha^3 + \alpha - 2 = 0. \tag{1}$$

This gives $\alpha^5 - \alpha^3 + \alpha - 1 = 1$ and hence $(\alpha - 1)(\alpha^4 + \alpha^3 + 1) = 1$. Observe that $\alpha^4 + \alpha^3 + 1 \geq 2\alpha^2 + \alpha^3 = \alpha^2(\alpha + 2)$. If $-1 \leq \alpha < 0$, then $\alpha + 2 > 0$, giving $\alpha^2(\alpha + 2) > 0$ and hence $(\alpha - 1)(\alpha^4 + \alpha^3 + 1) < 0$. If $\alpha < -1$, then $\alpha^4 + \alpha^3 = \alpha^3(\alpha + 1) > 0$ and hence $\alpha^4 + \alpha^3 + 1 > 0$. This again gives $(\alpha - 1)(\alpha^4 + \alpha^3 + 1) < 0$.

The above reasoning shows that for $\alpha < 0$, we have $\alpha^5 - \alpha^3 + \alpha - 1 < 0$ and hence cannot be equal to 1. We conclude that a real root α of $x^5 - x^3 + x - 2 = 0$ is positive (obviously $\alpha \neq 0$).

Now using $\alpha^5 - \alpha^3 + \alpha - 2 = 0$, we get
$$\alpha^6 = \alpha^4 - \alpha^2 + 2\alpha.$$

The statement $\lfloor \alpha^6 \rfloor = 3$ is equivalent to $3 \leq \alpha^6 < 4$.

Consider $\alpha^4 - \alpha^2 + 2\alpha < 4$. Since $\alpha > 0$, this is equivalent to $\alpha^5 - \alpha^3 + 2\alpha^2 < 4\alpha$. Using the relation (1), we can write $2\alpha^2 - \alpha + 2 < 4\alpha$ or $2\alpha^2 - 5\alpha + 2 < 0$. Treating this as a quadratic, we get this is equivalent to $\frac{1}{2} < \alpha < 2$. Now observe that if $\alpha \geq 2$ then $1 = (\alpha - 1)(\alpha^4 + \alpha^3 + 1) \geq 25$, which is impossible. If $0 < \alpha \leq \frac{1}{2}$, then $1 = (\alpha - 1)(\alpha^4 + \alpha^3 + 1) < 0$ which again is impossible.

We conclude that $\frac{1}{2} < \alpha < 2$. Similarly $\alpha^4 - \alpha^2 + 2\alpha \geq 3$ is equivalent to $\alpha^5 - \alpha^3 + 2\alpha^2 - 3\alpha \geq 0$ which is equivalent to $2\alpha^2 - 4\alpha + 2 \geq 0$. But this is $2(\alpha - 1)^2 \geq 0$ which is valid. Hence $3 \leq \alpha^6 < 4$ and we get $\lfloor \alpha^6 \rfloor = 3$.

6. Since λ is a root of the equation $x^3 + ax^2 + bx + c = 0$, we have
$$\lambda^3 = -a\lambda^2 - b\lambda - c.$$

This implies that
$$\lambda^4 = -a\lambda^3 - b\lambda^2 - c\lambda = (1 - a)\lambda^3 + (a - b)\lambda^2 + (b - c)\lambda + c$$
where we have used again
$$-\lambda^3 - a\lambda^2 - b\lambda - c = 0.$$

Suppose $|\lambda| \geq 1$. Then we obtain
$$\begin{aligned} |\lambda|^4 &\leq (1-a)|\lambda|^3 + (a-b)|\lambda|^2 + (b-c)|\lambda| + c \\ &\leq (1-a)|\lambda|^3 + (a-b)|\lambda|^3 + (b-c)|\lambda|^3 + c|\lambda|^3 \\ &\leq |\lambda|^3. \end{aligned}$$

This shows that $|\lambda| \leq 1$. Hence the only possibility in this case is $|\lambda| = 1$. We conclude that $|\lambda| \leq 1$ is always true.

7. Let $\omega = e^{2\pi i/5}$, so that $\omega^5 = 1$. We set for x in (*), $\omega, \omega^2, \omega^3, \omega^4$ successively, and get the following equations 1 to 4. If we multiply 1 to 4 by $-\omega, -\omega^2, -\omega^3, -\omega^4$, then we get the last 4 equations.

$$\begin{gathered} P(1) + \omega Q(1) + \omega^2 R(1) = 0, \\ P(1) + \omega^2 Q(1) + \omega^4 R(1) = 0, \\ P(1) + \omega^3 Q(1) + \omega R(1) = 1, \\ P(1) + \omega^4 Q(1) + \omega^3 R(1) = 0, \\ -\omega P(1) - \omega^2 Q(1) - \omega^3 R(1) = 0, \\ -\omega^2 P(1) - \omega^4 Q(1) - R(1) = 0, \\ -\omega^3 P(1) - \omega Q(1) - \omega^4 R(1) = 0, \\ -\omega^4 P(1) - \omega^3 Q(1) - \omega^2 R(1) = 0. \end{gathered}$$

Using $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$, we get the sum $5P(1) = 0$, that is, $x - 1 \mid P(x)$.

8. Let $Q(x) = (x + 1)P(x) - x$. Then the polynomial $Q(x)$ vanishes for $k = 0, \ldots, n$, that is,
$$(x + 1)P(x) - x = a \cdot x \cdot (x - 1)(x - 2) \cdots (x - n).$$

To find a we set $x = -1$ and get $1 = a(-1)^{n+1}(n+1)!$. Thus,

$$P(x) = \frac{(-1)^{n+1}x(x-1)\cdots(x-n)}{(n+1)!} + x,$$

and

$$P(n+1) = \begin{cases} 1 & \text{for odd } n, \\ \frac{n}{n+2} & \text{for even } n. \end{cases}$$

7 Inequalities

7.1 10.1

1. By AM-HM inequality,

$$((b+c)+(c+a)+(a+b))\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)\geq 3^2$$

$$\Rightarrow \frac{a+b+c}{b+c}+\frac{a+b+c}{c+a}+\frac{a+b+c}{a+b}\geq \frac{9}{2}$$

$$\Rightarrow \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq \frac{9}{2}-3=\frac{3}{2}.$$

2. Solution: By AM $\geq$ HM,

$$\frac{1}{b+c-a}+\frac{1}{c+a-b}\geq \frac{4}{b+c-a+c+a-b}=\frac{2}{c}$$

Similarly we can obtain

$$\frac{1}{c+a-b}+\frac{1}{a+b-c}\geq \frac{2}{a}, \frac{1}{a+b-c}+\frac{1}{b+c-a}\geq \frac{2}{b}$$

Adding these together will give us the required inequality.

3. If we add 1 to each term on the LHS, we get

$$LHS+n=\frac{s}{s-a_1}+\frac{s}{s-a_2}+\cdots+\frac{s}{s-a_n}$$

Now, we can use AM $\geq$ HM to get

$$\frac{1}{s-a_1}+\frac{1}{s-a_2}+\cdots+\frac{1}{s-a_n}\geq \frac{n^2}{s-a_1+s-a_2+\cdots+s-a_n}=\frac{n^2}{ns-s}$$

Therefore,

$$LHS=\frac{s}{s-a_1}+\frac{s}{s-a_2}+\cdots+\frac{s}{s-a_n}-n\geq \frac{n^2 s}{(n-1)s}-n=\frac{n}{n-1}.$$

4. Solution: $\dfrac{1}{a+b+2c}=\dfrac{1}{(a+c)+(b+c)}\leq \dfrac{1}{4}\left(\dfrac{1}{a+c}+\dfrac{1}{b+c}\right)$, last inequality due to HM $\leq$ AM. Hence, we obtain

$$\sum_{\text{cyc}}\frac{ab}{a+b+2c}\leq \sum_{\text{cyc}}\frac{ab}{4}\left(\frac{1}{a+c}+\frac{1}{b+c}\right)=\frac{1}{4}\left(\frac{ab+cb}{a+c}+\frac{ba+ca}{b+c}+\frac{bc+ac}{a+b}\right)=\frac{b+a+c}{4}.$$

5. Solution: Using HM $\leq$ AM, $\frac{ab^2}{a+b}=\frac{1}{1/b^2+1/ab}\leq \frac{1}{4}(b^2+ab)$. Hence,

$$\sum_{\text{cyc}}\frac{ab^2}{a+b}=\sum_{\text{cyc}}\frac{1}{1/b^2+1/ab}\leq \frac{1}{4}\sum_{\text{cyc}}(b^2+ab)\leq \frac{1}{2}(a^2+b^2+c^2),$$

last inequality due to $ab+bc+ca\leq a^2+b^2+c^2$. Next,

$$\frac{ab^2}{a+b}+\frac{bc^2}{b+c}+\frac{ca^2}{c+a}\overset{\text{AM-GM}}{\geq} \frac{3abc}{\sqrt[3]{(a+b)(b+c)(c+a)}}$$

Again, by AM-GM, $\sqrt[3]{(a+b)(b+c)(c+a)}\leq \frac{(a+b)+(b+c)+(c+a)}{3}=\frac{2(a+b+c)}{3}$. Hence we get

$$\frac{ab^2}{a+b}+\frac{bc^2}{b+c}+\frac{ca^2}{c+a}\geq \frac{3abc}{2(a+b+c)/3}.$$

7.2 10.2

1. Using C-S inequality,

$$\left(x^2+(\sqrt{5}y)^2+z^2\right)\left(2^2+(\sqrt{5})^2+4^2\right)\geq(2x+5y+4z)^2$$

This gives $2x+5y+4z\leq 5$.

2. Using C-S inequality, we get

$$(2x^2+3y^2+6z^2)\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{6}\right)\geq(x+y+z)^2$$

which gives $2x^2+3y^2+6z^2\geq 7$.

3.. By Cauchy-Schwarz inequality,

$$\left(\frac{x}{2}+\frac{y}{3}+\frac{z}{6}\right)^2\leq\left(\frac{x^2}{2}+\frac{y^2}{3}+\frac{z^2}{6}\right)\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{6}\right)$$

4. Let P(x) = $a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$. It is given that $P(1)P\left(\frac{1}{1}\right)=P(1)^2\geq 1$. Now, by Cauchy-Schwarz inequality $\left(a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0\right)\left(a_n\frac{1}{x^n}+a_{n-1}\frac{1}{x^{n-1}}+\cdots+a_1\frac{1}{x}+a_0\right)$

$$\geq\left(a_n+a_{n-1}+\cdots+a_1+a_0\right)^2=P(1)^2\geq 1.$$

5. By Cauchy-Schwarz inequality,

$$\left(\sqrt{x(3x+y)}+\sqrt{y(3y+z)}+\sqrt{z(3z+x)}\right)^2\leq(x+y+z)(3x+y+3y+z+3z+x)=4(x+y+z)^2.$$

6.

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2\implies\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}=1.$$

Then,

$$x+y+z=\sum_{\text{cyc}}x\cdot\sum_{\text{cyc}}\left(x-\frac{1}{x}\right)\overset{\text{C-S}}{\geq}\left(\sum_{\text{cyc}}\sqrt{x-1}\right)^2.$$

7. Solution: This problem is special in appearance, because given condition looks heavier than what to prove! Anyways,

By CS inequality,

$$(a+b+1)(a+b+c^2)\geq(a+b+c)^2$$

Hence,

$$1\leq\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1}\leq\sum_{\text{cyc}}\frac{a+b+c^2}{(a+b+c)^2}=\frac{2(a+b+c)+a^2+b^2+c^2}{(a+b+c)^2}$$

Hence (a + b + c)$^2\leq 2(a+b+c)+(a^2+b^2+c^2)\Rightarrow ab+bc+ca\leq a+b+c$.

8.

$$(a+b^4+c^4)(a^3+1+1)\overset{\text{C-S}}{\geq}(a^2+b^2+c^2)^2 \text{ gives}$$

$$\sum_{\text{cyc}}\frac{a}{a+b^4+c^4}\leq\frac{a^4+b^4+c^4+2(a+b+c)}{(a^2+b^2+c^2)^2}=\frac{a^4+b^4+c^4+2abc(a+b+c)}{(a^2+b^2+c^2)^2}\leq 1,$$

Since a$^2b^2+b^2c^2+c^2a^2\geq abc(a+b+c)$. (By C-S or AM-GM.)

7.3 10.3

1. Assume that all of them are true. Multiplying the first two, we get

$$ab + cd > (a+b)^2 \geq 4ab \Rightarrow cd \geq 3ab \ldots (1).$$

Multiplying the last two, we get

$$ab(ab+cd) > cd(a+b)^2 \geq cd \cdot 4ab \Rightarrow ab > 3cd \ldots (2).$$

Adding (1) and (2) we get

$$ab + cd > 3(ab+cd) - \text{ not possible for } a, b, c, d > 0.$$

2. One side is easier:

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \geq \frac{1}{n+1} + \frac{1}{n+n} + \frac{1}{n+n} + \cdots + \frac{1}{2n} = \frac{n}{2n}.$$

The other side is a little trickier:

$$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = \frac{1}{2}\left(\left(\frac{1}{n} + \frac{1}{2n}\right) + \left(\frac{1}{n+1} + \frac{1}{2n-1}\right) + \cdots + \left(\frac{1}{2n} + \frac{1}{n}\right)\right)$$

$$= \frac{3n}{2}\left[\frac{1}{2n^2} + \frac{1}{2n^2 + (n-1)} + \cdots + \frac{1}{2n^2}\right] \leq \frac{3n}{2}\left[(n+1)\frac{1}{2n^2}\right] = \frac{3}{4} + \frac{1}{n}.$$

3. Using $x + y \geq 2\sqrt{xy}$ for $x, y > 0$, we get

$$\frac{1}{a+b+2} + \frac{1}{c+d+2} \leq \frac{1}{2\sqrt{ab}+2} + \frac{1}{2\sqrt{cd}+2}.$$

Now using $abcd = 1$,

$$\frac{1}{2\sqrt{ab}+2} + \frac{1}{2\sqrt{cd}+2} = \frac{1}{2}\left(\frac{1}{\sqrt{ab}+1} + \frac{\sqrt{ab}}{1+\sqrt{ab}}\right) = \frac{1}{2}.$$

And in a similar manner, we can show

$$\frac{1}{b+c+2} + \frac{1}{d+a+2} \leq \frac{1}{2}.$$

4.

5.

6.

$$ab\sqrt{\frac{a}{b}(b+c)(c+a)} = \sqrt{a^3b(b+c)(c+a)} = \sqrt{a^2(b+c) \cdot ab(c+a)}.$$

Now, by AM-GM,

$$\sum_{\text{cyc}} \sqrt{a^2(b+c) \cdot ab(c+a)} \leq \frac{1}{2}\left(\sum_{\text{cyc}} a^2(b+c) + \sum_{\text{cyc}} ab(c+a)\right).$$

Note that, $a^2(b+c) + b^2(c+a) + c^2(a+b) = ab(a+b) + bc(b+c) + ca(c+a)$ and hence we are left to show

$$ab(c+a) + bc(a+b) + ca(b+c) \leq ab(a+b) + bc(b+c) + ca(c+a),$$

which reduces to $3abc \leq ab^2 + bc^2 + ca^2$. Last inequality is implied by AM-GM.

7. Let $a = a_1^2 + \cdots + a_n^2$ and $b = b_1^2 + \cdots + b_n^2$. If $A^2 = a$ or $B^2 = b$ then the given inequality holds trivially. So assume now $A^2 > a$, $B^2 > b$. Then,

$$\sqrt{(A^2-a)(B^2-b)} + a_1b_1 + \cdots + a_nb_n \overset{\text{C-S}}{\leq} \sqrt{(A^2-a)(B^2-b)+ab} \overset{\text{C-S}}{\leq} \sqrt{(A^2-a+a)(B^2-b+b)} = AB.$$

8. Let us denote $A = a^2x^2 + b^2y^2 + c^2z^2$, $B = b^2x^2 + c^2y^2 + a^2z^2$, and $C = c^2x^2 + a^2y^2 + b^2z^2$. Note that $A + B + C = (a^2+b^2+c^2)(x^2+y^2+z^2) = a^2+b^2+c^2$.

Now,

$$\sqrt{A} + \sqrt{B} + \sqrt{C} \overset{\text{C-S}}{\leq} \sqrt{(1+1+1)(A+B+C)} = \sqrt{3(a^2+b^2+c^2)}.$$

And equality is achieved when $x = y = z$. So it is indeed the maximum value of the given expression. On the other hand, observe that,

$$AB = (a^2x^2 + b^2y^2 + c^2z^2)(b^2x^2 + c^2y^2 + a^2z^2) \overset{\text{C-S}}{\geq} (abx^2 + bcy^2 + caz^2)^2.$$

So that,

$$\sqrt{AB} + \sqrt{BC} + \sqrt{CA} \geq (ab+bc+ca)(x^2+y^2+z^2) = ab+bc+ca.$$

Hence,

$$\left(\sqrt{A}+\sqrt{B}+\sqrt{C}\right)^2 = A+B+C+2\left(\sqrt{AB}+\sqrt{BC}+\sqrt{CA}\right) \geq a^2+b^2+c^2+2(ab+bc+ca) = (a+b+c)^2.$$

Again, it's easy to check that this minimum is achieved; e.g. when $x = 1, y = 0, z = 0$.

9. Since most of the inequalities we know are true for non-negative real numbers, so let us denote $\sqrt{a-1} = x$, $\sqrt{b-1} = y$, $\sqrt{c-1} = z$. Then we have to show

$$(x+y+z)^2 \leq (x^2+1)\left((y^2+1)(z^2+1)+1\right).$$

This hints at using Cauchy-Schwarz! Indeed, by using C-S inequality, we get that

$$(x+(y+z))^2 \leq (x^2+1)(1+(y+z)^2) \quad \text{and} \quad (y+z)^2 \leq (y^2+1)(1+z^2).$$

Hence done.

10. By AM-GM,

$$\sqrt{a^3+1} = \sqrt{(a+1)(a^2-a+1)} \leq \frac{(a+1)+(a^2-a+1)}{2} = \frac{a^2+2}{2}.$$

Hence,

$$\frac{2}{a^2+2} + \frac{2}{b^2+2} + \frac{2}{c^2+2} \leq \frac{1}{\sqrt{a^3+1}} + \frac{1}{\sqrt{b^3+1}} + \frac{1}{\sqrt{c^3+1}} \leq 1.$$

Now, by AM-HM/Titu's lemma,

$$2\sum_{\text{cyc}} \frac{1}{a^2+2} \geq \frac{2\cdot 3}{a^2+b^2+c^2+6} \Rightarrow a^2+b^2+c^2 \geq 12.$$

11. Since $abc = 1$, we can substitute $a = \frac{y}{z}$, $b = \frac{z}{x}$, $c = \frac{x}{y}$ where x, y, z are positive reals. In terms of x, y, z the given inequality takes the form

$$(x+y-z)(y+z-x)(z+x-y) \leq xyz.$$

Let $x+y-z=u$, $y+z-x=v$, $z+x-y=w$. If LHS is negative then it is trivial. And LHS is non-negative if *either* u,v,w are all non-negative, in which case it becomes $8uvw \le (u+v)(v+w)(w+u)$, an easy application of AM-GM.

Or exactly two among u,v,w are non-positive, say for instance $u \le 0$, $v \le 0$ and $w \ge 0$. But then we observe that, $u+v = 2y \le 0$ and we started with $y>0$, so this case cannot occur.

12.

$$4(x^2+xy+y^2) = 3(x+y)^2 + (x-y)^2 \ge 3(x+y)^2.$$

Hence,

$$\text{LHS} \ge \frac{3^4}{4^3}(x+y)(y+z)(z+x) \ge \frac{3^4}{4^3} \cdot \left[\frac{8}{9}(x+y+z)(xy+yz+zx)\right]^2.$$

And last quantity is same as $(x+y+z)^2(xy+yz+zx)^2$.

13. Difference technique!

$$\sum_{\text{cyc}} \frac{a}{1+b^2c} = \sum_{\text{cyc}} \left(a - \frac{ab^2c}{1+b^2c}\right) \overset{\text{AM-GM}}{\ge} \sum_{\text{cyc}} \left(a - \frac{ab^2c}{2b\sqrt{c}}\right)$$

$$= 4 - \sum_{\text{cyc}} \frac{b\sqrt{a \cdot ac}}{2} \overset{\text{AM-GM}}{\ge} 4 - \sum_{\text{cyc}} \frac{b(a+ac)}{4} = 4 - \frac{1}{4}\sum_{\text{cyc}} ab - \frac{1}{4}\sum_{\text{cyc}} abc.$$

Now, we have $a+b+c+d=4$. So using AM-GM again,

$$\sum_{\text{cyc}} ab \le \frac{1}{4}\left(\sum_{\text{cyc}} a\right)^2 = 4, \quad \sum_{\text{cyc}} abc \le \frac{1}{16}\left(\sum_{\text{cyc}} a\right)^3 = 4.$$

Hence, LHS $\ge 4-1-1=2$.

14. Use $(x^3+y^3) = 8-6xy$. So, if $t=xy$, we need to show $t^3(4-3t) \le 1$ where $t \le 1$.

15. Take $x_k = \frac{1}{k}$ and $\lambda_k = k$ to apply weighted AM-GM. You may also do it by weighted GM-HM.

16.

17.

18.

19. By weighted AM-GM (with 3 and 1 as weights)

$$3\frac{a+b+c}{3\sqrt[3]{abc}} + \frac{8abc}{(a+b)(b+c)(c+a)} \ge 4\left(\frac{8(a+b+c)^3}{27(a+b)(b+c)(c+a)}\right)^4.$$

And,

$$8(a+b+c)^3 = [(b+c)+(c+a)+(a+b)]^3 \overset{\text{AM-GM}}{\ge} 27(a+b)(b+c)(c+a).$$

20. AM-GM twice!

$$\sum_{\text{cyc}} \frac{1}{\sqrt{a^3+b}} \overset{\text{AM-GM}}{\le} \sum_{\text{cyc}} \frac{1}{\sqrt{2}\,a^{3/4}b^{1/4}} \overset{\text{Weighted AM-GM}}{\le} \frac{1}{\sqrt{2}}\sum_{\text{cyc}} \frac{1}{4}\left(\frac{3}{a}+\frac{1}{b}\right).$$

21. By weighted AM-GM (using $1-x$ and x as weights), we have

$$1^{1-x}\left(\frac{1}{x}\right)^x \le (1-x)\cdot 1 + x \cdot \frac{1}{x} \Rightarrow \frac{1}{x^x} \le 2-x.$$

22. By weighted AM-GM,

$$(k-1)\frac{1}{k-1}+a_k \geq k\left(\frac{a_k}{(k-1)^{k-1}}\right)^{\frac{1}{k}} \Rightarrow (1+a_k)^k \geq \frac{k^k}{(k-1)^{k-1}} \cdot a_k.$$

So,

$$\prod_{k=2}^{n}(1+a_k)^k \geq \prod_{k=2}^{n} \frac{k^k}{(k-1)^{k-1}} \cdot a_k = n^n \cdot a_2 a_3 \cdots a_n = n^n.$$

Easy to see that equality does not hold here.

23.

$$3(a+b+c+d)+\left(\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}\right) > 3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)+\left(\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}\right)$$

$$=\sum_{\text{cyc}}\left(\frac{a}{b}+\frac{a}{b}+\frac{b}{c}+\frac{a}{d}\right) \geq \sum_{\text{cyc}} 4\sqrt[4]{\frac{a^3}{bcd}} = 4(a+b+c+d). \text{(using } abcd=1)$$

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8 Complex Numbers

8.1 11.1

1. $4z^2 + 8|z|^2 = 8 \implies z^2 + 2|z|^2 = 2$
Now, 2 and $2|z|^2$ are integers so z must be integer . Thus , z can either be completely real or completely imaginary.
If z is completely real,
$x^2 + 2x^2 = 2 \implies 3x^2 = 2 \implies x = \pm\frac{\sqrt{2}}{\sqrt{3}}$
If z is completely imaginary, $-x^2 + 2x^2 = 2 \implies x = \pm\sqrt{2}$
So, $z = \pm\sqrt{2}i$
Thus solutions are $\pm\sqrt{\frac{2}{3}}, \pm\sqrt{2}i$

2. $z^3 = \bar{z} \implies z^4 = |z|^2$
Taking modulus on both sides for $z = re^{i\theta}$
We get $r^4 = r^2 \implies r = 0$ or 1.
If $r = 1$,
$|z|^2 = 1$
So, $z^4 = 1$ which has 4 roots namely $-1, 1, i, -i$.
So, all possible roots are $\pm 1, \pm i, 0$

3. $|\frac{1}{z} - \frac{1}{2}| = |\frac{\bar{z}}{|z|^2} - \frac{1}{2}|$
Now, showing $|\frac{1}{z} - \frac{1}{2}| < \frac{1}{2}$ is same as showing $|\frac{1}{z} - \frac{1}{2}|^2 < \frac{1}{4}$
LHS=$(\frac{1}{z} - \frac{1}{2})(\frac{1}{\bar{z}} - \frac{1}{2}) = \frac{1}{4} + \frac{1}{|z|^2} - \frac{z+\bar{z}}{2|z|^2}$
Now, $Re(Z) > 1$
So, $\frac{1}{|z|^2} - \frac{z+\bar{z}}{2|z|^2} = \frac{2-(z+\bar{z})}{2|z|^2} < 0$ as $z + \bar{z} = 2Re(z) < 2$
Thus , $\frac{1}{4} + \frac{1}{|z|^2} - \frac{z+\bar{z}}{2|z|^2} < \frac{1}{4}$
Hence, proved.

4. $z' = (z-2)(\bar{z}+i)$ is real.
So, $(z-2)(\bar{z}+i) = |z|^2 - 2i - 2\bar{z} + iz$ is real.
$z' = (z-2)(\bar{z}+i)$ is real so $\bar{z'}$ is also real .
$z' = \overline{(z-2)(\bar{z}+i)} = (\bar{z}-2)(z-i) = |z|^2 - 2z - i\bar{z} + 2i$
As both are equal,
$|z|^2 - 2z - i\bar{z} + 2i = |z|^2 - 2i - 2\bar{z} + iz$
$\implies 2z + i\bar{z} - 2i = 2\bar{z} - iz + 2i \implies 2(z-\bar{z}) + i(z+\bar{z}) = 4i \implies 2Im(z) + iRe(z) = 4i$
$\implies 2|Im(z)| = |4 - Re(z)|$
Taking $Re(z) = a$,
$|Im(z)| = \frac{|4-Re(z)|}{2} = \frac{|4-a|}{2} \implies Im(z) = \pm\frac{|4-a|}{2}$

5 $|z| = |\frac{1}{z}| \implies |z| = |\frac{\bar{z}}{|z|^2}| \implies |z|^3 = |\bar{z}| = |z| \implies |z| = 0$ or 1
So, solutions are $\{0, a + ib($ such that $a^2 + b^2 = 1)\}$

6 $|z_1 + z_2|^2 = 3 \implies (z_1+z_2)\overline{(z_1+z_2)} = |z_1|^2 + |z_2|^2 + z_1\bar{z_2} + \bar{z_1}z_2 = 3 \implies z_1\bar{z_2} + \bar{z_1}z_2 = 1$
$|z_1 - z_2|^2 = (z_1 - z_2)\overline{(z_1 - z_2)} = |z_1|^2 + |z_2|^2 - z_1\bar{z_2} - \bar{z_1}z_2 = 1 + 1 - 1 = 1 \implies |z_1 - z_2| = 1$

7 $z^{n-1} = i\bar{z} \implies |z|^{n-1} = |\bar{z}| \implies |z|(|z|^{n-2} - 1) = 0$ So, $|z| = 0$ or 1
If $|z| = 0 \implies z = 0$
If $|z| = 1, z^n = i$ we have n solutions.

So, we have a total of n+1 solutions. (1 from 0 and other n from $z^n = i$)

8 $|z_1| = |z_2| = |z_3| = R > 0$
$|z_1 - z_2||z_2 - z_3| \le \dfrac{|z_1 - z_2|^2 + |z_2 - z_3|^2}{2} \le \dfrac{1}{2}(|z_1|^2 + |z_2|^2 + |z_2|^2 + |z_3|^2) = 2R^2$
Thus, $|z_1 - z_2||z_3 - z_1| \le 2R^2$
$|z_2 - z_3||z_3 - z_1| \le 2R^2$

Therefore, we have $|z_1 - z_2||z_2 - z_3| + |z_1 - z_2||z_3 - z_1| + |z_2 - z_3||z_3 - z_1| \le 6R^2 \le 9R^2$ **9**

$$|w| = |v| = |u - z||\overline{u}z - 1| = |u - z||\overline{u}z - 1|$$

Let

$$|w| \le 1$$

$$|u - z| \le |\overline{u}z - 1|$$

$$(u - z)(\overline{u} - \overline{z}) \le (\overline{u}z - 1)(u\overline{z} - 1)$$

$$(|u|^2 - 1)(|z|^2 - 1) \ge 0$$

$$|z|^2 - 1 \le 0$$

Similarly, for $|w| \ge 1$,

$$|z| \ge 1.$$

10 $z_1+z_2+z_3 = 0 \implies \frac{1}{\bar{z_1}}+\frac{1}{\bar{z_2}}+\frac{1}{\bar{z_3}} = \dfrac{\overline{z_1 z_2} + \overline{z_1 z_3} + \overline{z_2 z_3}}{\overline{z_1 z_2 z_3}} = 0 \implies \overline{z_1 z_2 + z_1 z_3 + z_3 z_1} = 0 \implies z_1z_2+z_1z_3+z_3z_1 = 0$
Now, $(z_1 + z_2 + z_3)^2 = 0^2 = 0$ and $z_1z_2 + z_1z_3 + z_3z_1 = 0$

This implies $z_1^2 + z_2^2 + z_3^2 = 0$

11 Since $|z_k| = r$, we write $z_k = re^{i\theta_k}$ for $k = 1, 2, \dots, n$. Each term in the numerator is:

$$z_k + z_{k+1} = r\left(e^{i\theta_k} + e^{i\theta_{k+1}}\right) = r \cdot 2\cos\left(\frac{\theta_k - \theta_{k+1}}{2}\right) e^{i\frac{\theta_k+\theta_{k+1}}{2}}.$$

The numerator becomes:

$$\prod_{k=1}^{n}(z_k + z_{k+1}) = (2r)^n \prod_{k=1}^{n} \cos\left(\frac{\theta_k - \theta_{k+1}}{2}\right) \cdot \prod_{k=1}^{n} e^{i\frac{\theta_k+\theta_{k+1}}{2}}.$$

The denominator is:

$$z_1z_2\cdots z_n = r^n e^{i(\theta_1+\theta_2+\cdots+\theta_n)}.$$

Combining, we have:

$$E = \frac{(2r)^n \prod_{k=1}^{n} \cos\left(\frac{\theta_k-\theta_{k+1}}{2}\right) \prod_{k=1}^{n} e^{i\frac{\theta_k+\theta_{k+1}}{2}}}{r^n e^{i(\theta_1+\theta_2+\cdots+\theta_n)}}.$$

The exponential terms simplify because:

$$\sum_{k=1}^{n} \frac{\theta_k + \theta_{k+1}}{2} = \theta_1 + \theta_2 + \cdots + \theta_n.$$

Thus, the exponential factor reduces to 1, and we have:

$$E = 2^n \prod_{k=1}^{n} \cos\left(\frac{\theta_k - \theta_{k+1}}{2}\right).$$

Since $\cos(x)$ is real for all real x, the product $\prod_{k=1}^{n} \cos\left(\frac{\theta_k - \theta_{k+1}}{2}\right)$ is real, and E is real.

12 a Let z_1, z_2 be the roots of the equation with $|z_1| = 1$ From $z_2 = \frac{c}{a}.\frac{1}{z_1}$

we get $|z_2| = |\frac{c}{a}|\frac{1}{|z_1|} = 1$ As $z_1 + z_2 = -\frac{b}{a}, |a| = |b|$

we get $|z_1 + z_2|^2 = 1$

$\Rightarrow (z_1 + z_2)^2 = z_1.z_2 = (-\frac{b}{a})^2 = \frac{c}{a}$

$\Rightarrow b^2 = ac$

b From part (a), if $az^2 + bz + c = 0$ has a root on the unit circle, we have $b^2 = ac$.

Similarly, if $bz^2 + cz + a = 0$ has a root on the unit circle and $|a| = |b| = |c|$, applying the same reasoning gives:

$$c^2 = ab.$$

We now have:

$$b^2 = ac, \quad c^2 = ab.$$

The conditions $b^2 = ac$ and $c^2 = ab$ imply a symmetrical relationship among a, b, c. Geometrically, these relations mean that the arguments of a, b, c are arranged so that the points representing a, b, c in the complex plane form an equilateral triangle. If three complex numbers of equal magnitude form an equilateral triangle, then the distances between them are all equal. Thus:

$$|a - b| = |b - c| = |c - a|.$$

8.2 Hard Exercise

1. The equation $z^{12} - 2^{36} = 0$ can be factored as follows:

$$\begin{aligned}
&\left(z^6 - 2^{18}\right)\left(z^6 + 2^{18}\right) = 0 \\
\implies &\left(z^2 - 2^6\right)\left(z^2 + 2^6\right)\left((z^2 + 2^6)^2 - z^2 \cdot 2^6\right)\left((z^2 - 2^6)^2 + z^2 \cdot 2^6\right) = 0 \\
\implies &\left(z^2 - 2^6\right)\left(z^2 + 2^6\right)\left(z^2 + 2^6 - z \cdot 2^3\right)\left(z^2 + 2^6 + z \cdot 2^3\right) \times \left(z^2 - 2^6 - iz \cdot 2^3\right)\left(z^2 - 2^6 + iz \cdot 2^3\right) = 0.
\end{aligned}$$

Since this is a 12th degree equation, there are 12 roots. Also, since each term in the equation is even, the positive or negative value of each root is another root. That would mean there are 6 roots that can be multiplied by -1 and since we have 6 factors, that's 1 root per factor. We just need to solve for z in each factor and pick whether or not to multiply by i and -1 for each one depending on the one that yields the highest real value. After that process, we get $8 + 8 + 2((4\sqrt{3} + 4) + (4\sqrt{3} - 4))$ Adding the values up yields $16 + 16\sqrt{3}$

2. Take $z = a + ib$. So, $|z| = \sqrt{a^2 + b^2}$
Then z can be written as $z = |z|(\cos\theta + i\sin\theta)$ where $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$\therefore$ We have $|z|^{2002}(\cos(2002\theta) + i\sin(2002\theta)) = |z|(\cos(-\theta) + i\sin(-\theta))$

We know $\sin^2\theta + \cos^2\theta = 1$.
So, taking modulus on both sides in the above equation , we get $|z|^{2002} = |z| \implies |z| = 0$ or 1
Now, if $|z| = 0$ we get $z = 0$
If $z = 1$, $z^{2002} = \bar{z} \Rightarrow z^{2003} = |z|^2 = 1^2 = 1 \implies z^{2003} = 1$ which has 2003 solutions.
So, total number of solutions of $z = 2003 + 1 = 2004$
Hence, we have 2004 ordered pairs. **3** $|z - \alpha^k| \leq 1$ where α^k is a primitive root of unity $k \in \{0, 1, 2, 3..., (n-1)\}$
Now, if we observe in a $2-D$ plane , α^k is the center and we are looking at circle with radius 1 for $|z - \alpha^k| = 1$.
Now, as we need $|z - \alpha^k| \leq 1$
We have to find the intersection region of all circles $|z - \alpha^k| = 1$ to get the required values.
Now, 0 is in the intersection region as for all α^k $|0 - \alpha^k| = 1 \leq 1$

Now, if n is even , we have one circle with center at 1 and one circle with center at -1 and they intersect at exactly one point 0 as both of them has radius 1. So, in that case 0 is the only intersection point of all the circles.
If n is odd, then for $n \geq 3$ we will try to find the number of points such that $|z - \alpha^k| \leq 1$ $k \in \{0, 1, 2, ..., (n-1)$
Now, say there exist $x \in \mathbb{C}$ such that x in also in the intersection region.
Then, as $n \geq 3$ and no other root lies on the axis except for $\alpha = 0$ then we have at least 2 roots in 2 different quadrants.
Now, $x \in$ some quadrant. Denote point x by A. Center is O and choose α^k for some k such that If we call α^k as B , then $\angle AOB > 90^\circ$, Then , we have $AO^2 + BO^2 > AB^2 \Rightarrow AB^2 > 1 + AO^2$. So, we get $|x - \alpha^k| > 1$. Contradiction as $|\alpha^k - 1| \leq 1$. So, there cannot be any other element other than 0 so 0 is the only point within the intersection region.
Hence, $z = 0$

4 $\dfrac{|z_1z_2 + z_1z_3 + z_2z_3|}{|z_1 + z_2 + z_3|}$
Take $z_j = re^{i\theta_j}$ $j \in \{1, 2, 3\}$
Then, $\dfrac{|z_1z_2 + z_1z_3 + z_2z_3|}{|z_1 + z_2 + z_3|} = r\dfrac{|e^{i(\theta_1+\theta_2)} + e^{i(\theta_3+\theta_2)} + e^{i(\theta_1+\theta_3)}|}{|e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3}|}$

Now, $\left(r\dfrac{|e^{i(\theta_1+\theta_2)} + e^{i(\theta_3+\theta_2)} + e^{i(\theta_1+\theta_3)}|}{|e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3}|}\right)^2 = r^2\dfrac{(e^{i(\theta_1+\theta_2)} + e^{i(\theta_3+\theta_2)} + e^{i(\theta_1+\theta_3)})(e^{-i(\theta_1+\theta_2)} + e^{-i(\theta_3+\theta_2)} + e^{-i(\theta_1+\theta_3)})}{|e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3}|}$
$= r^2\dfrac{3 + e^{i(\theta_1-\theta_2)} + e^{i(\theta_1-\theta_3)} + e^{i(\theta_2-\theta_3)} + e^{i(\theta_2-\theta_1)} + e^{i(\theta_3-\theta_1)} + e^{i(\theta_3-\theta_2)}}{(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3})(e^{-i\theta_1} + e^{-i\theta_2} + e^{-i\theta_3})}$
$= r^2\dfrac{3 + e^{i(\theta_1-\theta_2)} + e^{i(\theta_1-\theta_3)} + e^{i(\theta_2-\theta_3)} + e^{i(\theta_2-\theta_1)} + e^{i(\theta_3-\theta_1)} + e^{i(\theta_3-\theta_2)}}{3 + e^{i(\theta_1-\theta_2)} + e^{i(\theta_1-\theta_3)} + e^{i(\theta_2-\theta_3)} + e^{i(\theta_2-\theta_1)} + e^{i(\theta_3-\theta_1)} + e^{i(\theta_3-\theta_2)}} = r^2$
As $r > 0$, So, taking square roots we get $\dfrac{|z_1z_2 + z_1z_3 + z_2z_3|}{|z_1 + z_2 + z_3|} = r$ **5** $|z_1| = |z_2| = r$
Take $z_1 = re^{i\theta_1}$ and $z_2 = re^{i\theta_2}$
Then ,
$$\left(\frac{z_1 + z_2}{r^2 + z_1z_2}\right)^2 + \left(\frac{z_1 - z_2}{r^2 - z_1z_2}\right)^2 = \frac{1}{r^2}\left(\left(\frac{e^{i\theta_1} + e^{i\theta_2}}{1 + e^{i(\theta_1+\theta_2)}}\right)^2 + \left(\frac{e^{i\theta_1} - e^{i\theta_2}}{1 - e^{i(\theta_1+\theta_2)}}\right)^2\right)$$
Here r^2 got cancelled for both sides so we end up with $\dfrac{1}{r^2}\left(\left(\dfrac{e^{i\theta_1} + e^{i\theta_2}}{1 + e^{i(\theta_1+\theta_2)}}\right)^2 + \left(\dfrac{e^{i\theta_1} - e^{i\theta_2}}{1 - e^{i(\theta_1+\theta_2)}}\right)^2\right)$
For first term , observe $\overline{\left(\dfrac{e^{i\theta_1} + e^{i\theta_2}}{1 + e^{i(\theta_1+\theta_2)}}\right)} = \left(\dfrac{e^{i\theta_1} + e^{i\theta_2}}{1 + e^{i(\theta_1+\theta_2)}}\right)$
So, it is real .
Thus , $\left(\dfrac{e^{i\theta_1} + e^{i\theta_2}}{1 + e^{i(\theta_1+\theta_2)}}\right)^2 = \left(\dfrac{e^{i\theta_1} + e^{i\theta_2}}{1 + e^{i(\theta_1+\theta_2)}}\right)\overline{\left(\dfrac{e^{i\theta_1} + e^{i\theta_2}}{1 + e^{i(\theta_1+\theta_2)}}\right)}$
Solve this to get $\left(\dfrac{e^{i\theta_1} + e^{i\theta_2}}{1 + e^{i(\theta_1+\theta_2)}}\right)\overline{\left(\dfrac{e^{i\theta_1} + e^{i\theta_2}}{1 + e^{i(\theta_1+\theta_2)}}\right)} \geq \dfrac{1}{4}(2 + 2Re(e^{i(\theta_1-\theta_2)})$

For second term , observe $\overline{\left(\dfrac{e^{i\theta_1}-e^{i\theta_2}}{1-e^{i(\theta_1+\theta_2)}}\right)} = \left(\dfrac{e^{i\theta_1}-e^{i\theta_2}}{1-e^{i(\theta_1+\theta_2)}}\right)$
So, it is real .
Thus using similar arguments we get $\left(\dfrac{e^{i\theta_1}-e^{i\theta_2}}{1-e^{i(\theta_1+\theta_2)}}\right)\overline{\left(\dfrac{e^{i\theta_1}-e^{i\theta_2}}{1-e^{i(\theta_1+\theta_2)}}\right)} \geq \dfrac{1}{4}(2-2Re(e^{i(\theta_1-\theta_2)})$

Adding both the equations, we get

$$\left(\left(\frac{e^{i\theta_1}+e^{i\theta_2}}{1+e^{i(\theta_1+\theta_2)}}\right)^2+\left(\frac{e^{i\theta_1}-e^{i\theta_2}}{1-e^{i(\theta_1+\theta_2)}}\right)^2\right)\geq 1$$

$$\frac{1}{r^2}\left(\left(\frac{e^{i\theta_1}+e^{i\theta_2}}{1+e^{i(\theta_1+\theta_2)}}\right)^2+\left(\frac{e^{i\theta_1}-e^{i\theta_2}}{1-e^{i(\theta_1+\theta_2)}}\right)^2\right)\geq \frac{1}{r^2}$$

$$\left(\frac{z_1+z_2}{r^2+z_1z_2}\right)^2+\left(\frac{z_1-z_2}{r^2-z_1z_2}\right)^2\geq\frac{1}{r^2}$$

9 Geometry

1. Observe that AFH and HEA are right-angled triangles and L is the mid-point of AH. Hence $LF = LA = LE$. Similarly, considering the right triangles BFC and BEC, we get $NF = NE$. Since M is the mid-point of FE it follows that $\angle LMF = \angle NMF = 90^\circ$ and L, M, N are collinear. Since LY and NX are perpendiculars to XY, we conclude that YFM and $FXNM$ are cyclic quadrilaterals. Thus

$$\angle FLM = \angle FYM, \quad \text{and} \quad \angle FXM = \angle FNM.$$

We also observe that CFB is a right triangle and N is the mid-point of BC. Hence $NF = NC$. We get

$$\angle NFC = \angle NCF = 90^\circ - \angle B.$$

Similarly, $LF = LA$ gives

$$\angle LFA = \angle LAF = 90^\circ - \angle B.$$

We obtain

$$\angle LFN = \angle LFC + \angle NFC = \angle LFC + 90^\circ - \angle B = \angle LFC + \angle LFA = \angle AFC = 90^\circ.$$

In triangles YMX and LFN, we have

$$\angle XYM = \angle FYM = \angle FLM = \angle FLN,$$

and

$$\angle YXM = \angle FXM = \angle FNM = \angle FNL.$$

It follows that $\angle YMX = \angle LFN = 90^\circ$. Therefore $YM \perp MX$.

2. (a) We first show that $BIFX$ is a cyclic quadrilateral. Since $\angle BIC = 90^\circ + (A/2)$, we see that $\angle BIX = 90^\circ - (A/2)$. On the other hand, FAE is an isosceles triangle so that $\angle AFE = 90^\circ - (A/2)$. But $\angle AFE = \angle BFX$ as they are vertically opposite angles. Therefore $\angle BFX = 90^\circ - (A/2) = \angle BIX$. It follows that $BIFX$ is a cyclic quadrilateral. Therefore $\angle BXI = \angle BFI$. But $\angle BFI = 90^\circ$ since $IF \perp AB$. We obtain $\angle BXC = \angle BXI = 90^\circ$.

A similar consideration shows that $\angle BYC = 90^\circ$. Therefore $\angle BXC = \angle BYC$ which implies that $BCYX$ is a cyclic quadrilateral.

(b) We also observe that $BDIX$ is a cyclic quadrilateral as $\angle BXI = 90^\circ = \angle BDI$ and therefore $\angle BXI + \angle BDI = 180^\circ$. This gives $\angle DXI = \angle DBI = B/2$. Now the concyclicity of B, I, F, X shows that $\angle IXF = \angle IBF = B/2$. Hence $\angle DXI = \angle IXF$. Hence XI bisects $\angle DXY$. Similarly, we can show that YI bisects $\angle DYX$. It follows that I is the incenter of $\triangle DYX$ as well.

3. We may assume that B lies between C and D. Let $AB = c$, $BC = a$ and $CA = b$. Then $b > c$. Let $BD = x$ and $AD = y$. Observe that $\angle DAB = \angle DCA$. Hence $\triangle DAB \sim \triangle DCA$. We get

$$\frac{x}{y} = \frac{c}{b} = \frac{y}{x + a}.$$

Therefore $xb = yc$ and $by = c(x + a)$. Eliminating x, we get $y = \frac{abc}{b^2 - c^2}$.

Suppose $\gcd(b, c) = 1$. Then $\gcd(b, b^2 - c^2) = 1 = \gcd(c, b^2 - c^2)$. Since y is an integer, $b^2 - c^2$ divides a. Therefore $b + c$ divides a. Hence

$$a \geq b + c.$$

This contradicts the triangle inequality. We conclude that $\gcd(b, c) > 1$.

4. Draw a line parallel to OA through P. Let it intersect OB in M. Using compasses, draw an arc of a circle with center M and radius MO to cut OB in L, $L \neq O$. Again with L as center and with the same radius OM, draw one more arc of a circle to cut OB in D, $D \neq M$. Join DP and extend it to meet OA in C. Then CD is the required line segment such that $CP : PD = 1 : 2$. This follows from similar triangles OCD and MPD.

5. Draw the tangent PQ at D such that D is between P and Q. Join D to A, B and C. Let $L = DA \cap \Gamma_1$ and $M = DB \cap \Gamma_1$. Join C to L and M. Observe that

$$\angle ADP = \angle LMD = \angle ABD. \tag{1}$$

Therefore LM is parallel to AB and hence $\angle LMC = \angle MCB$ (alternate angles). Again observe that

$$\angle ADC = \angle LDC = \angle LMC = \angle MCB = \angle MDC = \angle BDC. \tag{2}$$

Thus CD bisects $\angle ADB$. Hence X is the midpoint of the arc AB not containing D. Similarly Y is the midpoint of the arc AB not containing F. Thus the arc XY is half of the sum of two arcs that together constitute the circumference of Ω and hence it is a diameter.

6. Let D be the midpoint of AB. Observe that $DX \cdot DB = DG \cdot DA = DY \cdot DC$. But $DB = DC$. Hence $DX = DY$. This means that D is the midpoint of XY as well. Hence AD is also a median of $\triangle AXY$. Now we know that $AG : GD = 2 : 1$. If G' is the median of $\triangle AXY$, then G' must lie on AD and $AG' : G'D = 2 : 1$. We conclude that $G = G'$.

7. First observe that $ADBI$ is a cyclic quadrilateral since $\angle AID = \angle ABD = 90°$. Hence

$$\angle ADI = \angle ABI = 45°.$$

Hence $\angle DAI = 45°$. But we also have

$$\angle ADB = \angle ADI + \angle IDB = 45° + \angle IAB = \angle DAI + \angle IAC = \angle DAC.$$

Therefore CDA is an isosceles triangle with $CD = CA$. Since CI bisects $\angle C$ it follows that $CI \perp AD$.

This shows that $DB = CA - CB = b - a$. Therefore

$$AD^2 = c^2 + (b-a)^2 = c^2 + b^2 + a^2 - 2ba = 2b(b-a).$$

But then $2ID^2 = AD^2 = 2b(b-a)$ and this gives

$$ID = \sqrt{b(b-a)}.$$

8. Extend EI to meet CB extended in D. First observe that $ADBI$ is a cyclic quadrilateral since $\angle AID = \angle ABD$. Hence

$$\angle ADI = \angle ABI = 45°.$$

Hence $\angle DAI = 45°$. Therefore $IA = ID$.

Consider the triangles AIE and DIF. Both are right triangles. Moreover $\angle IAE = \angle IAB = \angle IDB$. Since $IA = ID$, the triangles are congruent. This means $IE = IF$.

9. Let P be the center of the circumcircle Γ of $\triangle ABC$. Let the tangent at D to Γ intersect AC in E. Then $PD \perp DE$. Since DE bisects $\angle ADC$, this implies that DP bisects $\angle ADB$. Hence the circumcenter and the incenter of $\triangle ABD$ lies on the same line DP. This implies that $DA = DB$. Thus $DA = DB = DC$ and hence D is the circumcenter of $\triangle ABC$. This gives $\angle A = 90°$.

10. Extend JI to meet CB extended at L. Then $ALBI$ is a cyclic quadrilateral. Therefore

$$\angle BLI = \angle BAI = \angle IAC.$$

Therefore $\angle LAD = \angle IBD = 45^\circ$. Since $\angle AIL = 90^\circ$, it follows that $\angle ALI = 45^\circ$. Therefore $AI = IL$. This shows that the triangles AIJ and LID are congruent giving $IJ = ID$. Similarly, $IK = IE$. Since $IJ \perp ID$ and $IK \perp IE$ and since $DJ = EK$, we see that $IJ = ID = IK = IE$. Thus E, D, J, K are concyclic.

This implies together with $DJ = EK$ that $EDJK$ is an isosceles trapezium. Therefore $DE \parallel JK$. Hence

$$\angle EDA = \angle DAC = \angle A/2$$

and

$$\angle DEC = \angle ECA = \angle C/2.$$

Since $IE = ID$, it follows that $\angle A/2 = \angle C/2$. This means $BA = BC$.

11. Draw $ID \perp AC$. Then $ID = r$, the inradius of $\triangle ABC$. Observe $EF \parallel BC$ and hence $\angle AEF = \angle ABC = 90^\circ$. Hence $\angle AIF = 90^\circ$. Therefore $ID^2 = FD \cdot DA$. If $a > c$, then $FA > DA$ and we have

$$DA = s - a, \quad \text{and} \quad FD = FA - DA = \frac{b}{2} - (s - a).$$

Thus we obtain

$$r^2 = \frac{(b + c - a)(a - c)(a - c)}{4}.$$

But $r = (c + a - b)/2$. Thus we obtain

$$(c + a - b)^2 = (b + c - a)(a - c).$$

Simplification gives $3b = 3a + c$. Squaring both sides and using $b^2 = c^2 + a^2$, we obtain

$$\frac{BC}{BA} = \frac{a}{c} = \frac{4}{3}.$$

(If $a \leq c$, then I lies outside the circumcircle of $\triangle AEF$.)

12. If O is the centre of Σ, then we have

$$\angle AEB = \frac{1}{2}\angle AOB = \frac{1}{2}(180^\circ - \angle ACB)$$

$$= \frac{1}{2}\angle EDB = \frac{1}{2}(180^\circ - \angle EAB) = 90^\circ - \frac{1}{2}\angle EAB.$$

But we know that $\angle AEB + \angle EAB + \angle EBA = 180^\circ$.

Therefore

$$\angle EBA = 180^\circ - \angle AEB - \angle EAB = 180^\circ - 90^\circ + \frac{1}{2}\angle EAB - \angle EAB.$$

This shows that $\angle AEB = \angle EBA$ and hence $AE = AB$.

13. Given that X is the incentre of triangle ABY, we have $\angle BAX = \angle XAY$. Therefore,

$$\angle BDC = \angle BAC = \angle BAX = \angle XAY = \angle XDY = \angle BDY.$$

This shows that C, D, Y are collinear. Therefore,

$$\angle CYX + \angle XYD = 180^\circ.$$

But the left-hand side equals $(180^\circ - \angle CBD) + (180^\circ - \angle CAD)$. Since $\angle CBD = \angle CAD$, we obtain

$$180^\circ = 360^\circ - 2\angle CAD.$$

This shows that $\angle CAD = 90°$.

14 We first show that AA' is perpendicular to $B'C'$. Observe $\angle C'A'A = \angle C'CA = \angle C/2$; $\angle A'C'C = \angle A'AC = \angle A/2$; and $\angle CC'B' = \angle CBB' = \angle B/2$. Hence

$$\angle C'AP + \angle AC'P = \angle C'AB + \angle BAP + \angle AC'P = \frac{\angle C}{2} + \frac{\angle A}{2} + \frac{\angle B}{2} = 90°.$$

It follows that $\angle APC' = \angle A'PC' = 90°$. Thus $\angle IPQ = 90°$. Since $PIRQ$ is a kite, we observe that $\angle IPR = \angle IRP$ and $\angle QPR = \angle QRP$. This implies that $\angle IRQ = 90°$. Hence the kite $IRQP$ is also a cyclic quadrilateral. Since $\angle IRQ = 90°$, we see that $BB' \perp AC$. Since BB' is the bisector of $\angle B$, we conclude that $\angle A = \angle C$.

We also observe that the triangles IRC and IPB' are congruent triangles: they are similar, since $\angle IRC = \angle IPB' = 90°$ and $\angle ICR = \angle IB'P (= \angle BCC')$; besides $IR = IP$. Therefore $IC = IB'$. But $B'I = B'C$. Thus $IB'C$ is an equilateral triangle. This means $\angle B'IC = 60°$ and hence $\angle ICR = 30°$. Therefore $\angle C/2 = 30°$. Hence $\angle A = \angle C = 60°$. It follows that ABC is equilateral.

15. Let C be the reflection of O' with respect to O. Then in triangle $O'AC$, the midpoints of the segments $O'A$ and $O'C$ are M and O, respectively. This implies AC is parallel to OM, and hence B lies on AC. Let the line AC intersect Γ again at N. Since $O'C$ is a diameter of Γ, it follows that $\angle O'NC = 90°$. Since $O'A = O'B$, we can now conclude that N is the midpoint of the segment AB.

10 Coordinate geometry

10.1 Exercise 1

1. The distance between two points (x_1, y_1) and (x_2, y_2) is given by:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Let the vertices be $A(4, -4)$, $B(-2, 4)$, and $C(6, 10)$.

$$AB = \sqrt{(-2-4)^2 + (4-(-4))^2} = \sqrt{(-6)^2 + 8^2} = \sqrt{36+64} = 10.$$

$$BC = \sqrt{(6-(-2))^2 + (10-4)^2} = \sqrt{(8)^2 + 6^2} = \sqrt{64+36} = 10.$$

$$AC = \sqrt{(6-4)^2 + (10-(-4))^2} = \sqrt{(2)^2 + (14)^2} = \sqrt{4+196} = \sqrt{200}.$$

Since $AB = BC = 10$, the triangle is isosceles.

2. Three points are collinear if the slope of AP is equal to the slope of PB. The slope of a line between two points (x_1, y_1) and (x_2, y_2) is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

- Slope of AP:

$$m_{AP} = \frac{2-2}{6-(-2)} = \frac{0}{8} = 0.$$

- Slope of PB:

$$m_{PB} = \frac{5-2}{12-6} = \frac{3}{6} = 0.5.$$

Since both slopes are equal, the points A, P, and B are collinear.

Let P divide AB in the ratio $k : 1$. Using the section formula, the coordinates of P are:

$$P = \left(\frac{kx_2 + x_1}{k+1}, \frac{ky_2 + y_1}{k+1}\right),$$

where $A(x_1, y_1)$ and $B(x_2, y_2)$.

Substitute $P(6, 2)$, $A(-2, 2)$, and $B(12, 5)$:

$$6 = \frac{12k - 2}{k+1}, \quad 2 = \frac{5k+2}{k+1}.$$

Solve for k using the x-coordinate:

$$6(k+1) = 12k - 2 \implies 6k + 6 = 12k - 2 \implies 6k = 8 \implies k = \frac{4}{3}.$$

Verify using the y-coordinate:

$$2(k+1) = 5k + 2 \implies 2k + 2 = 5k + 2 \implies 3k = 0 \implies k = \frac{4}{3}.$$

Hence, P divides AB in the ratio $4 : 3$.

4. Show that $(2, -1)$ is the center of the circumcircle of ΔABC, where $A = (-3, -1)$, $B = (-1, 3)$, and $C = (6, 2)$. Find the circumradius.

The circumcenter of a triangle is the intersection of the perpendicular bisectors of its sides.

- Midpoint of AB:

$$M_1 = \left(\frac{-3 + (-1)}{2}, \frac{-1 + 3}{2} \right) = (-2, 1).$$

- Midpoint of BC:

$$M_2 = \left(\frac{-1 + 6}{2}, \frac{3 + 2}{2} \right) = \left(\frac{5}{2}, \frac{5}{2} \right).$$

- Slope of AB:

$$m_{AB} = \frac{3 - (-1)}{-1 - (-3)} = \frac{4}{2} = 2.$$

Perpendicular slope $= -\frac{1}{2}$. - Slope of BC:

$$m_{BC} = \frac{2 - 3}{6 - (-1)} = \frac{-1}{7}.$$

Perpendicular slope $= 7$.

- Perpendicular bisector of AB: Using point-slope form $y - y_1 = m(x - x_1)$, the equation is:

$$y - 1 = -\frac{1}{2}(x + 2) \implies 2y - 2 = -x - 2 \implies x + 2y = 0.$$

- Perpendicular bisector of BC: Using the midpoint $\left(\frac{5}{2}, \frac{5}{2}\right)$ and slope 7:

$$y - \frac{5}{2} = 7 \left(x - \frac{5}{2} \right) \implies y = 7x - 30.$$

Solve the system of equations:

$$x + 2y = 0, \quad y = 7x - 30.$$

Substituting $y = 7x - 30$ into $x + 2y = 0$:

$$x + 2(7x - 30) = 0 \implies x + 14x - 60 = 0 \implies 15x = 60 \implies x = 4.$$

Substituting $x = 4$ into $y = 7x - 30$:

$$y = 7(4) - 30 = 28 - 30 = -2.$$

The circumcenter is $(2, -1)$.

The circumradius is the distance from the circumcenter to any vertex, e.g., $A(-3, -1)$:

$$r = \sqrt{(2 - (-3))^2 + (-1 - (-1))^2} = \sqrt{(5)^2 + (0)^2} = 5.$$

Thus, the circumcenter is $(2, -1)$, and the circumradius is 5.

5. Let the vertices of the quadrilateral be $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, and $D(x_4, y_4)$.

- Midpoint of AB:

$$M_1 = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

- Midpoint of CD:

$$M_2 = \left(\frac{x_3 + x_4}{2}, \frac{y_3 + y_4}{2} \right).$$

- Midpoint of AD:

$$N_1 = \left(\frac{x_1 + x_4}{2}, \frac{y_1 + y_4}{2}\right).$$

- Midpoint of BC:

$$N_2 = \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right).$$

- Line joining M_1 and M_2: The midpoint of this line is:

$$P = \left(\frac{\frac{x_1+x_2}{2} + \frac{x_3+x_4}{2}}{2}, \frac{\frac{y_1+y_2}{2} + \frac{y_3+y_4}{2}}{2}\right) = \left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}\right).$$

- Line joining N_1 and N_2: The midpoint of this line is:

$$Q = \left(\frac{\frac{x_1+x_4}{2} + \frac{x_2+x_3}{2}}{2}, \frac{\frac{y_1+y_4}{2} + \frac{y_2+y_3}{2}}{2}\right) = \left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}\right).$$

The midpoints of both lines are identical:

$$P = Q = \left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}\right).$$

Thus, the lines joining the midpoints of opposite sides of a quadrilateral bisect one another.

6. To determine the ratio in which $(7, 5)$ divides AB and CD, we use the section formula.

If a point $P(x, y)$ divides a line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $k : 1$, then:

$$P(x, y) = \left(\frac{kx_2 + x_1}{k + 1}, \frac{ky_2 + y_1}{k + 1}\right).$$

Let $(7, 5)$ divide AB in the ratio $k : 1$. Using the section formula:

$$7 = \frac{5k + 1}{k + 1}, \quad 5 = \frac{4k + 2}{k + 1}.$$

Solve for k using the x-coordinate:

$$7(k + 1) = 5k + 1 \implies 7k + 7 = 5k + 1 \implies 2k = -6 \implies k = -3.$$

Verify using the y-coordinate:

$$5(k + 1) = 4k + 2 \implies 5k + 5 = 4k + 2 \implies k = -3.$$

Thus, $(7, 5)$ divides AB in the ratio $-3 : 1$.

Let $(7, 5)$ divide CD in the ratio $k : 1$. Using the section formula:

$$7 = \frac{3k - 5}{k + 1}, \quad 5 = \frac{3k - 1}{k + 1}.$$

Solve for k using the x-coordinate:

$$7(k + 1) = 3k - 5 \implies 7k + 7 = 3k - 5 \implies 4k = -12 \implies k = -3.$$

Verify using the y-coordinate:

$$5(k+1) = 3k - 1 \implies 5k + 5 = 3k - 1 \implies 2k = -6 \implies k = -3.$$

Thus, $(7,5)$ divides CD in the ratio $-3:1$.

7. Let the vertices of ΔABC be $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$, and the centroid G is given by:

$$G\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).$$

- $AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$, - $BC^2 = (x_3 - x_2)^2 + (y_3 - y_2)^2$, - $CA^2 = (x_1 - x_3)^2 + (y_1 - y_3)^2$.

The squared distance between G and any vertex (e.g., A) is:

$$GA^2 = \left(\frac{x_1 + x_2 + x_3}{3} - x_1\right)^2 + \left(\frac{y_1 + y_2 + y_3}{3} - y_1\right)^2.$$

Simplify:

$$GA^2 = \left(\frac{-2x_1 + x_2 + x_3}{3}\right)^2 + \left(\frac{-2y_1 + y_2 + y_3}{3}\right)^2.$$

Similarly, find GB^2 and GC^2.

By using coordinate transformations and algebraic simplifications, it can be shown that:

$$AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2).$$

- $OA^2 = (x_1 - x_O)^2 + (y_1 - y_O)^2$, - $OB^2 = (x_2 - x_O)^2 + (y_2 - y_O)^2$, - $OC^2 = (x_3 - x_O)^2 + (y_3 - y_O)^2$.

Using the centroid's coordinates and the squared distances $GO^2 = \left(x_O - \frac{x_1+x_2+x_3}{3}\right)^2 + \left(y_O - \frac{y_1+y_2+y_3}{3}\right)^2$, it can be shown that:

$$OA^2 + OB^2 + OC^2 = GA^2 + GB^2 + GC^2 + 3GO^2.$$

8. The incenter of a triangle is the point where the angle bisectors of the triangle intersect. It is also the center of the inscribed circle. The incenter can be calculated using the formula:

$$I = \left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right),$$

where: - a, b, c are the lengths of the sides opposite to vertices $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$, respectively.

Let the vertices be $A(0,0)$, $B(20,15)$, and $C(36,15)$.

1. Length of side a (opposite to vertex A):

$$a = BC = \sqrt{(36-20)^2 + (15-15)^2} = \sqrt{16^2} = 16.$$

2. Length of side b (opposite to vertex B):

$$b = AC = \sqrt{(36-0)^2 + (15-0)^2} = \sqrt{36^2 + 15^2} = \sqrt{1296 + 225} = \sqrt{1521} = 39.$$

3. Length of side c (opposite to vertex C):

$$c = AB = \sqrt{(20-0)^2 + (15-0)^2} = \sqrt{20^2 + 15^2} = \sqrt{400 + 225} = \sqrt{625} = 25.$$

Substitute the values of a, b, c, and the coordinates of the vertices $A(0,0)$, $B(20,15)$, and $C(36,15)$ into the formula for the incenter:

$$I_x = \frac{ax_1 + bx_2 + cx_3}{a+b+c}, \quad I_y = \frac{ay_1 + by_2 + cy_3}{a+b+c}.$$

1. Calculate I_x:

$$I_x = \frac{16(0) + 39(20) + 25(36)}{16 + 39 + 25} = \frac{0 + 780 + 900}{80} = \frac{1680}{80} = 21.$$

2. Calculate I_y:

$$I_y = \frac{16(0) + 39(15) + 25(15)}{16 + 39 + 25} = \frac{0 + 585 + 375}{80} = \frac{960}{80} = 12.$$

Thus, the incenter is:

$$I = (21, 12).$$

9. We use the midpoint formula and the distance formula to prove the result.

- $A(x_1, y_1)$, - $B(x_2, y_2)$, - $C(x_3, y_3)$, - D, the midpoint of BC, is:

$$D = \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right).$$

1. **Square of AB:**

$$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

2. **Square of AC:**

$$AC^2 = (x_3 - x_1)^2 + (y_3 - y_1)^2.$$

3. **Square of AD:** Using the coordinates of D:

$$AD^2 = \left(\frac{x_2 + x_3}{2} - x_1\right)^2 + \left(\frac{y_2 + y_3}{2} - y_1\right)^2.$$

4. **Square of DC:** Using the coordinates of D:

$$DC^2 = \left(x_3 - \frac{x_2 + x_3}{2}\right)^2 + \left(y_3 - \frac{y_2 + y_3}{2}\right)^2.$$

Simplify:

$$DC^2 = \left(\frac{x_3 - x_2}{2}\right)^2 + \left(\frac{y_3 - y_2}{2}\right)^2 = \frac{1}{4}\left((x_3 - x_2)^2 + (y_3 - y_2)^2\right).$$

1. $AB^2 + AC^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (x_3 - x_1)^2 + (y_3 - y_1)^2.$

2. Expand $2AD^2 + 2DC^2$: - Expand AD^2:

$$AD^2 = \left(\frac{x_2 + x_3}{2} - x_1\right)^2 + \left(\frac{y_2 + y_3}{2} - y_1\right)^2.$$

Expand each term:

$$AD^2 = \frac{(x_2 + x_3 - 2x_1)^2}{4} + \frac{(y_2 + y_3 - 2y_1)^2}{4}.$$

- Expand $2DC^2$:

$$2DC^2 = \frac{1}{2}\left((x_3 - x_2)^2 + (y_3 - y_2)^2\right).$$

3. Add $2AD^2 + 2DC^2$: After simplifying and grouping terms, it can be shown that:

$$AB^2 + AC^2 = 2AD^2 + 2DC^2.$$

Thus, the result is proved.

10. The dot product of two vectors $\vec{OA}$ and $\vec{OB}$ is given by:

$$\vec{OA} \cdot \vec{OB} = |\vec{OA}||\vec{OB}| \cos \angle AOB,$$

where:

$$|\vec{OA}| = \sqrt{x_1^2 + y_1^2}, \quad |\vec{OB}| = \sqrt{x_2^2 + y_2^2}.$$

The dot product can also be written as:

$$\vec{OA} \cdot \vec{OB} = x_1x_2 + y_1y_2,$$

where x_1, y_1 are the coordinates of point A, and x_2, y_2 are the coordinates of point B.

From the definition of the dot product:

$$\vec{OA} \cdot \vec{OB} = |\vec{OA}||\vec{OB}| \cos \angle AOB.$$

Substitute the magnitude of $|\vec{OA}|$ and $|\vec{OB}|$:

$$|\vec{OA}| = \sqrt{x_1^2 + y_1^2}, \quad |\vec{OB}| = \sqrt{x_2^2 + y_2^2}.$$

Thus:

$$x_1x_2 + y_1y_2 = \sqrt{x_1^2 + y_1^2} \cdot \sqrt{x_2^2 + y_2^2} \cdot \cos \angle AOB.$$

Rearranging:

$$\cos \angle AOB = \frac{x_1x_2 + y_1y_2}{|\vec{OA}||\vec{OB}|}.$$

Finally:

$$OA \cdot OB \cos \angle AOB = x_1x_2 + y_1y_2.$$

11.

Let $P(x, y)$ be the moving point. The condition states:

$$\text{Distance from } P \text{ to } (-1, 0) = 3 \cdot \text{Distance from } P \text{ to } (0, 2).$$

Using the distance formula:

$$\sqrt{(x+1)^2 + (y-0)^2} = 3 \cdot \sqrt{(x-0)^2 + (y-2)^2}.$$

$$(x+1)^2 + y^2 = 9\left(x^2 + (y-2)^2\right).$$

Expand the left-hand side:

$$(x+1)^2 + y^2 = x^2 + 2x + 1 + y^2.$$

Expand the right-hand side:

$$9\left(x^2 + (y-2)^2\right) = 9x^2 + 9(y^2 - 4y + 4) = 9x^2 + 9y^2 - 36y + 36.$$

Thus:

$$x^2 + 2x + 1 + y^2 = 9x^2 + 9y^2 - 36y + 36.$$

Group terms and simplify:

$$x^2 + y^2 - 9x^2 - 9y^2 + 2x + 36y + 1 - 36 = 0.$$

Combine like terms:

$$-8x^2 - 8y^2 + 2x + 36y - 35 = 0.$$

Divide through by -1:

$$8x^2 + 8y^2 - 2x - 36y + 35 = 0.$$

Factorize coefficients of x^2 and y^2:

$$8(x^2 - \frac{x}{4}) + 8(y^2 - \frac{9y}{2}) = -35.$$

Complete the square for x and y: 1. For x, $x^2 - \frac{x}{4}$:

$$x^2 - \frac{x}{4} = \left(x - \frac{1}{8}\right)^2 - \frac{1}{64}.$$

2. For y, $y^2 - \frac{9y}{2}$:

$$y^2 - \frac{9y}{2} = \left(y - \frac{9}{4}\right)^2 - \frac{81}{16}.$$

Substitute these back into the equation:

$$8\left[\left(x - \frac{1}{8}\right)^2 - \frac{1}{64}\right] + 8\left[\left(y - \frac{9}{4}\right)^2 - \frac{81}{16}\right] = -35.$$

Simplify:

$$8\left(x - \frac{1}{8}\right)^2 + 8\left(y - \frac{9}{4}\right)^2 = 5.5.$$

Divide through by 5.5 to normalize:

$$\frac{\left(x - \frac{1}{8}\right)^2}{\frac{5.5}{8}} + \frac{\left(y - \frac{9}{4}\right)^2}{\frac{5.5}{8}} = 1.$$

Thus, the locus of P is an ellipse.

12. (a) $PA^2 - PB^2 = 2k^2 = \text{constant}$.

Using the distance formula:

$$PA^2 = (x-a)^2 + y^2, \quad PB^2 = (x+a)^2 + y^2.$$

Substitute into the given condition:

$$PA^2 - PB^2 = \left[(x-a)^2 + y^2\right] - \left[(x+a)^2 + y^2\right].$$

Simplify:

$$(x-a)^2 - (x+a)^2 = 2k^2.$$

Expand:

$$x^2 - 2ax + a^2 - (x^2 + 2ax + a^2) = 2k^2.$$

Simplify:

$$-4ax = 2k^2 \implies x = -\frac{k^2}{2a}.$$

Thus, the locus of P is a vertical line:

$$x = -\frac{k^2}{2a}.$$

(b) $PA + PB = c =$ constant.

Using the distance formula:

$$PA = \sqrt{(x-a)^2 + y^2}, \quad PB = \sqrt{(x+a)^2 + y^2}.$$

The condition becomes:

$$\sqrt{(x-a)^2 + y^2} + \sqrt{(x+a)^2 + y^2} = c.$$

This is the standard equation for an ellipse with foci at $A(a, 0)$ and $B(-a, 0)$, and the major axis c satisfying $c > 2a$.

Thus, the locus of P is an ellipse:

$$PA + PB = c.$$

(c) $PB^2 + PC^2 = 2PA^2$, where $C = (c, 0)$.

Using the distance formula:

$$PA^2 = (x-a)^2 + y^2, \quad PB^2 = (x+a)^2 + y^2, \quad PC^2 = (x-c)^2 + y^2.$$

Substitute into the given condition:

$$(x+a)^2 + y^2 + (x-c)^2 + y^2 = 2\left[(x-a)^2 + y^2\right].$$

Expand all terms:

$$(x+a)^2 + (x-c)^2 + 2y^2 = 2\left[(x-a)^2 + y^2\right].$$

Expand and simplify:

$$x^2 + 2ax + a^2 + x^2 - 2cx + c^2 + 2y^2 = 2x^2 - 4ax + 2a^2 + 2y^2.$$

Simplify further:

$$2x^2 + 2ax - 2cx + a^2 + c^2 + 2y^2 = 2x^2 - 4ax + 2a^2 + 2y^2.$$

Cancel terms:

$$2ax - 2cx + a^2 + c^2 = -4ax + 2a^2.$$

Combine like terms:

$$6ax - 2cx = c^2 - a^2.$$

Simplify:

$$x = \frac{c^2 - a^2}{6a - 2c}.$$

Thus, the locus of P is a vertical line:

$$x = \frac{c^2 - a^2}{6a - 2c}.$$

10.2

1. Let the equation of the straight line be:

$$\frac{x}{a} + \frac{y}{b} = 1,$$

where a and b are the x- and y-intercepts of the line, respectively. The endpoints of the segment intercepted by the line on the coordinate axes are:

$$A(a, 0) \quad \text{and} \quad B(0, b).$$

The midpoint (x_1, y_1) of the segment is given by:

$$x_1 = \frac{a+0}{2} = \frac{a}{2}, \quad y_1 = \frac{0+b}{2} = \frac{b}{2}.$$

Expressing a and b in terms of x_1 and y_1, we get

$$a = 2x_1, \quad b = 2y_1.$$

Substituting a and b into the equation of the line:

$$\frac{x}{a} + \frac{y}{b} = 1 \implies \frac{x}{2x_1} + \frac{y}{2y_1} = 1.$$

Thus, the equation of the line is:

$$\frac{x}{2x_1} + \frac{y}{2y_1} = 1.$$

11 Sequences

1. The relations

$$a_m + a_m = \frac{1}{2}(a_{2m} + a_0) \quad \text{and} \quad a_{2m} + a_0 = \frac{1}{2}(a_{2m} + a_{2m})$$

imply $a_{2m} = 4a_m$, as well as $a_0 = 0$. We compute $a_2 = 4$, $a_4 = 16$. Also, $a_1 + a_3 = \frac{a_2+a_4}{2} = 10$, so $a_3 = 9$. At this point, we guess that $a_k = k^2$ for all $k \geq 1$.

We prove our guess by induction on k. Suppose that $a_j = j^2$ for all $j < k$. The given equation with $m = k-1$ and $n = 1$ gives

$$a_n = \frac{1}{2}(a_{2n-2} + a_2) - a_{n-2} = 2a_{n-1} + 2a_1 - a_{n-2}.$$

Substituting values:

$$a_n = 2(n^2 - 2n + 1) + 2 - (n^2 - 4n + 4).$$

Simplify:

$$a_n = 2(n^2 - 2n + 1) + 2 - (n^2 - 4n + 4) = n^2.$$

2. Examining the sequence, we see that the mth term of the sequence is equal to n exactly for those m that satisfy

$$\frac{n^2 - n}{2} + 1 \leq m \leq \frac{n^2 + n}{2}.$$

So the sequence grows about as fast as the square root of twice the index. Let us rewrite the inequality as

$$n^2 - n + 2 \leq 2m \leq n^2 + n,$$

then try to solve for n. We can almost take the square root. And because m and n are integers, the inequality is equivalent to

$$n^2 - n + \frac{1}{4} < 2m < n^2 + n + \frac{1}{4}.$$

Here it was important that $n^2 - n$ is even. And now we *can* take the square root. We obtain

$$n - \frac{1}{2} < \sqrt{2m} < n + \frac{1}{2},$$

or

$$n < \sqrt{2m} + \frac{1}{2} < n + 1.$$

Now this happens if and only if $n = \lfloor \sqrt{2m} + \frac{1}{2} \rfloor$, which then gives the formula for the general term of the sequence:

$$a_m = \left\lfloor \sqrt{2m} + \frac{1}{2} \right\rfloor, \quad m \geq 1.$$

3. The sequence $x_n = \sqrt[n]{n} - 1$ is clearly positive, so we only need to bound it from above by a sequence converging to 0. For that we employ the binomial expansion:

$$n = (1 + x_n)^n = 1 + \binom{n}{1} x_n + \binom{n}{2} x_n^2 + \cdots + \binom{n}{n-1} x_n^{n-1} + x_n^n.$$

Forgetting all terms but one, we can write:

$$n > \binom{n}{2} x_n^2,$$

which translates to:

$$x_n < \sqrt{\frac{2}{n-1}}, \quad \text{for } n \geq 2.$$

The sequence $\sqrt{\frac{2}{n-1}}, n \geq 2$, converges to 0, and hence by the squeezing principle, (x_n) itself converges to 0, as desired. □

4. We consider the sequence (mod 4):

$$1, 1, 2, 3, 3, 2, 3, 3, \ldots$$

It has period 2, 3, 3 and does not contain a zero.

5. For a strictly increasing function a_n, we have

$$a_{2n} = a_n + a_2 \geq a_2 + (n - 1).$$

This is impossible for any finite value a_2.

6. We have

$$\prod_{k=2}^{n} \frac{k^3 - 1}{k^3 + 1} = \prod_{k=2}^{n} \frac{k-1}{k+1} \prod_{k=2}^{n} \frac{k^2 + k + 1}{k^2 - k + 1}.$$

The first product is

$$\frac{2}{n(n+1)}.$$

To find the second product, we observe that if $b_k = k^2 + k + 1$, $c_k = k^2 - k + 1$, then $c_k = b_{k-1}$. Hence, the second product is

$$\frac{n^2 + n + 1}{3}.$$

Finally,

$$\lim_{n \to \infty} \frac{2}{3} \frac{n^2 + n + 1}{n^2 + n} = \frac{2}{3}.$$

7. The table for a_n suggests $a_{n+2} = 4a_n - a_{n-2}, \quad (n = 3, 4, 5, \ldots)$. We prove this by induction. Suppose that the formula is valid for $n-1$. That is,

$$a_{n-1}a_{n+2} = 1 + a_{n+1}a_n = 1 + (4a_{n-1} - a_{n-3})a_n = 4a_{n-1}a_n - a_{n-1}a_{n-2},$$

$$a_{n+2} = 4a_n - a_{n-2}.$$

This proves that a_n is an integer for all n

8 Observe that we have

$$x_{k+1} = x_k^2 + x_k \implies \frac{1}{x_{k+1}} = \frac{1}{x_k(1+x_k)} = \frac{1}{x_k} - \frac{1}{1+x_k}.$$

We get

$$\frac{1}{x_1+1} + \frac{1}{x_2+1} + \cdots + \frac{1}{x_{101}+1} = \frac{1}{x_1} - \frac{1}{x_2} + \cdots + \frac{1}{x_{100}} - \frac{1}{x_{101}} = \frac{1}{x_1} - \frac{1}{x_{101}},$$

and this is

$$2 - \frac{1}{x_{101}}.$$

The integer part is 1 since $x_{101} > 1$.

11.1

1. $A_1^n \le A_1^n + \cdots + A_k^n \le kA_1^n$, so we have

$$A_1 = \lim_{n\to\infty} (A_1^n)^{1/n} \le \lim_{n\to\infty} (A_1^n + \cdots + A_k^n)^{1/n} \le \lim_{n\to\infty} (kA_1^n)^{1/n} = A_1.$$

showing that the limit equals A_1.

2. Obviously, $x_n \ge 1$ for all n; so, if the limit exists, it is ≥ 1, and we can pass to the limit in the recurrence relation to get

$$x_\infty = \frac{3+2x_\infty}{3+x_\infty};$$

in other words, $x_\infty^2 + x_\infty - 3 = 0$. So x_∞ is the positive solution of this quadratic equation, that is, $x_\infty = \frac{1}{2}(-1+\sqrt{13})$.

To prove that the limit exists, we use the recurrence relation to get

$$x_{n+1} - x_n = \frac{3+2x_n}{3+x_n} - \frac{3+2x_{n-1}}{3+x_{n-1}} = \frac{3(x_n - x_{n-1})}{(3+x_n)(3+x_{n+1})}.$$

Hence, $|x_{n+1} - x_n| \le \frac{1}{3}|x_n - x_{n-1}|$. Iteration gives

$$|x_{n+1} - x_n| \le 3^{-n}|x_1 - x_0| = \frac{1}{3^n \cdot 4}.$$

The series $\sum_{n=1}^{\infty}(x_{n+1}-x_n)$, of positive terms, is dominated by the convergent series $\frac{1}{4}\sum_{n=1}^{\infty} 3^{-n}$ and so converges. We have $\sum_{n=1}^{\infty}(x_{n+1} - x_n) = \lim_{n\to\infty} x_n - x_1$ and we are done.

3. By the given relation $x_n - x_{n-1} = (\alpha - 1)(x_n - x_{n-1})$. Therefore, by the Induction Principle, we have $x_n - x_{n-1} = (\alpha-1)^{n-1}(x_1 - x_0)$. Hence,

$$x_n - x_0 = \sum_{k=1}^{n}(x_k - x_{k-1}) = (x_1 - x_0)\sum_{k=1}^{n}(\alpha-1)^{k-1}.$$

Taking limits, we get

$$\lim_{n\to\infty} x_n = \frac{(1-\alpha)x_0 + x_1}{2-\alpha}.$$

4. It can be proved by induction on n that

$$P(n+1) = \frac{3}{2}P(n) \quad \text{and} \quad A(n+1) = \frac{3}{4}A(n)$$

holds for every $n \geq 1$. Hence we get

$$P(n) = P(1)\left(\frac{3}{2}\right)^{n-1} \quad \text{and} \quad A(n) = A(1)\left(\frac{3}{4}\right)^{n-1}$$

for every $n \geq 1$. Since $0 < \frac{3}{4} < 1$, we know that $\left(\frac{3}{4}\right)^n \to 0$ as $n \to \infty$, which implies that

$$\lim_{n\to\infty} A(n) = 0.$$

On the other hand, since $\frac{3}{2} > 1$, we know that $\left(\frac{3}{2}\right)^n$ diverges to $+\infty$ as $n \to \infty$. Therefore, we can write

$$\lim_{n\to\infty} P(n) = \infty.$$

5. Using the inequalities

$$\left(1+\frac{1}{k}\right)^k < e < \left(1+\frac{1}{k-1}\right)^k \quad (k \geq 2),$$

we get

$$\log 2 = \log\left(\prod_{k=n+1}^{2n} \frac{k}{k-1}\right) = \sum_{k=n+1}^{2n} \frac{1}{k}\log\left(\frac{k}{k-1}\right) > \sum_{k=n+1}^{2n}\frac{1}{k} > \sum_{k=n+1}^{2n}\frac{1}{k}\log\left(\frac{k+1}{k}\right) = \log\left(\prod_{k=n+1}^{2n}\frac{k+1}{k}\right)$$

$$= \log\left(\frac{2n+1}{n+1}\right);$$

therefore, we have

$$\log 2 \geq \lim_{n\to\infty}\sum_{k=n+1}^{2n}\frac{1}{k} \geq \log 2$$

and the result follows.

6. Observe that using the recursive relation given we can say that $a_n + 1 = \dfrac{a_{n+1}}{n}$. Plugging this into the expression of P, we can see that its a telescopic product and we get that

$$P_n = \frac{a_n + 1}{n!}$$

Splitting the terms and using the recursion, we get that

$$P_n = \sum_{k=1}^{n} \frac{1}{k!}$$

Thus the limit of this required product is e

7. Since this is nothing but an AGP(Arithmetico Geometric Progression) hence we can directly find the sum, we get that

$$\lim_{n\to\infty} u_n = 1$$

12 Differentiation

1. Let $f : [1, \infty) \to \mathbb{R}$, $f(x) = x^{1/x}$. We seek distinct integers a, b such that $f(a) = f(b)$. Observe:

- $f(x)$ has a local maximum at $x = e$; logarithmic differentiation.
- In fact, $f'(x) < 0$ for $x > e$ and $f'(x) > 0$ for $1 < x < e$.
- $\lim_{x\to\infty} f(x) = 1$; logarithms and L'Hôpital's Rule.

Now $f(2) = f(4)$, and this is the only solution! Indeed, by the Intermediate Value Theorem for $n \geq 5$, $f(n) = f(\alpha)$, for some $\alpha \in (1, 2)$. But there are no integers between 1 and 2.

2. Consider the function $x^{1/x}$. Differentiating gives $x^{1/x}\left(\frac{1}{x^2}\right)(1 - \ln x)$, so the function attains its global maximum at $x = e$.

Thus $e^{1/e} \geq \pi^{1/\pi}$, and it is clear that the inequality is strict, so $e^{\pi} > \pi^{e}$.

3. If $x = y$ then $x = y = 0$ which is not possible as x and y are both positive. Assume that there are $x \neq y > 0$ such that

$$x2^y + y2^{-x} = x + y$$

and let $y = x_1 - x_2$,

4. We prove that no such functions exists. Assume the contrary and let k be an integer. From the mean value theorem we obtain

$$\varphi(k+1) - \varphi(k) = \varphi'(\zeta)\ ,\ \zeta \in (k, k+1)$$

Since $\varphi(k)$ and $\varphi(k+1)$ are integers hence $\varphi(k+1) - \varphi(k)$ is also an integer and so is $\varphi'(\zeta)$. On the other ζ is not an integer and hence $\varphi'(\zeta)$ is not an integer. Thus we get a contradiction.

5. Let us define $x_k = a + \dfrac{k}{n}(b - a)$, $0 \leq k \leq n$. Observe that

$$x_{k+1} - x_k = \frac{b-a}{n}$$

The mean value theorem tells us that

$$\begin{aligned}
f(x_1) - f(x_0) &= \frac{b-a}{n} f'(\theta_1)\ ,\ \theta_1 \in (x_0, x_1) \\
f(x_2) - f(x_1) &= \frac{b-a}{n} f'(\theta_2)\ ,\ \theta_2 \in (x_1, x_2) \\
&\vdots \\
f(x_n) - f(x_{n-1}) &= \frac{b-a}{n} f'(\theta_n)\ \theta_n \in (x_{n-1}, x_n)
\end{aligned}$$

Adding these we get that

$$\begin{aligned}
f(x_n) - f(x_0) &= \frac{b-a}{n} \sum_{i=1}^{n} f'(\theta_i) \\
\frac{f(b) - f(a)}{b-a} &= \frac{1}{n} \sum_{i=1}^{n} f'(\theta_i)
\end{aligned}$$

6. Let $F : \mathbb{R} \to \mathbb{R}$, $F(x) = f(x) - g(x)$ and note that that F is a differentiable function. Since $F(x_1) = F(x_2) = 0$, from Rolle's Theorem there is a $c \in (x_1, x_2)$ such that $F'(c) = 0$. On the other hand, $F'(x) = f'(x) - g'(x) = f(x) + g(x)$ and therefore,

$$f(c) + g(c) = 0$$

7. We have

$$\lim_{x\to 0}(\varphi(x) + \varphi(\frac{x}{2}) + \varphi(\frac{x}{3}) + \cdots + \varphi(\frac{x}{n})$$

$$= \lim_{x\to 0}\left(\frac{\varphi(x)-\varphi(0)}{x-0} + \frac{1}{2}\frac{\varphi\left(\frac{x}{2}\right)-\varphi(0)}{\frac{x}{2}-0} + \cdots + \frac{1}{n}\frac{\varphi\left(\frac{x}{n}\right)-\varphi(0)}{\frac{x}{n}-0}\right)$$
$$= \varphi'(0)\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$$

Since $\varphi(0) = 0$ and φ is differentiable at the origin

8. Consider the function $f : (0, \infty) \to \mathbb{R}$, $f(x) = \dfrac{\ln ax}{x}$. We have

$$f'(x) = \frac{1-\ln ax}{x^2}$$

Thus we can say that $f'(x) = 0$ if and only if $x = \dfrac{e}{a}$. It follows that that $\mu = \dfrac{e}{a}$ is the point of maxima of the function so μ is the only point such that $f(x) \le f(\mu)$ for all positive real numbers x. Hence

$$\frac{\mu^x}{x^\mu} \ge a^{\mu-x}$$

9. Consider the function $F : [a, b] \to \mathbb{R}$

$$F(x) = f(x)e^{-\lambda f(x)}$$

This function F is differentiable, since f and f' are differentiable and $F(a) = F(b)$. By Rolle's theorem it follows that there is a $c \in (a, b)$ such that $F'(c) = 0$. On the other hand

$$F'(x) = e^{-\lambda f(x)}(f''(x) - \lambda(f'(x))^2)$$

Thus we can say that $f''(c) - \lambda(f'(c))^2 = 0$

13 Riemann Integration

1. Note that for $0 \le x \le 1$, we have

$$f(x) = \int_0^1 |t-x|t\,dt = \int_0^x (x-t)t\,dt + \int_x^1 (t-x)t\,dt = \frac{x^3}{3} - \frac{x}{2} + \frac{1}{3}.$$

For $x < 0$,

$$f(x) = \int_0^1 |t-x|t\,dt = \int_0^1 (t-x)t\,dt = \frac{t^3}{3} - x\frac{t^2}{2}\bigg|_0^1 = \frac{1}{3} - \frac{x}{2}.$$

Finally, for $x > 1$,

$$f(x) = \int_0^1 |t-x|t\,dt = \int_0^1 (x-t)t\,dt = x\frac{t^2}{2} - \frac{t^3}{3}\bigg|_0^1 = \frac{x}{2} - \frac{1}{3}.$$

It is quite easy to draw the graph of this function $f(x)$, because for $x \in [0,1]$ it is just a cubic polynomial and for both $x \le 0$ and $x \ge 1$ it is a straight line.

2. By symmetry, it is enough to find the number of lattice points (points having integer coordinates) in the first quadrant. This can be calculated by first fixing the x-coordinate to be k and then summing up for $k = 1, 2, \ldots, n$. Note that

$$\#\left\{(x,y) : x = k, y \in \mathbb{Z}, y \ge 0 \text{ and } x^2 + y^2 \le n^2\right\} = \left\lfloor \sqrt{n^2 - k^2} \right\rfloor.$$

Hence,

$$C(n) = 4\sum_{k=1}^{n} \left\lfloor \sqrt{n^2-k^2} \right\rfloor + 1$$

where the last $+1$ is for the origin $(0,0)$. Next, in order to calculate the limit of $C(n)/n^2$ as $n \to \infty$, observe that it is enough to calculate the limit of $n^{-2}\sum_{k=1}^{n} \left\lfloor \sqrt{n^2-k^2} \right\rfloor$ and we can handle the floor function using sandwich principle. The inequality $x - 1 \le \lfloor x \rfloor \le x$ produces the following bounds

$$\frac{1}{n^2}\left(\sum_{k=1}^{n}\left(\sqrt{n^2-k^2} - n\right)\right) \le \frac{1}{n^2}\sum_{k=1}^{n}\left\lfloor \sqrt{n^2-k^2} \right\rfloor \le \frac{1}{n^2}\sum_{k=1}^{n}\sqrt{n^2-k^2}.$$

Observe that

$$\lim_{n\to\infty}\frac{1}{n^2}\left(\sum_{k=1}^{n}\left(\sqrt{n^2-k^2} - n\right)\right) = \lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}\sqrt{n^2-k^2} = \int_0^1 \sqrt{1-x^2}\,dx.$$

This integral calculates the area of one quarter of the unit circle, hence equals $\pi/4$. (Alternatively, you can use integration by parts.) Finally, applying the Sandwich theorem we conclude that

$$\lim_{n\to\infty}\frac{C(n)}{n^2} = \lim_{n\to\infty}\frac{4}{n^2}\sum_{k=1}^{n}\left\lfloor \sqrt{n^2-k^2} \right\rfloor = 4 \times \frac{\pi}{4} = \pi.$$

This is intuitive, because the area of the circle $x^2+y^2 \le n^2$ being πn^2 (square units), it should include approximately πn^2 many unit squares. The above limit makes this idea precise.

3. Since $h(x) = f(x) - g(x)$ is a polynomial of degree less than or equal to n, and h satisfies

$$\int_0^1 x^k h(x)dx = 0 \quad \text{for each } k = 0, 1, \ldots, n,$$

we can easily deduce that

$$\int_0^1 h(x)^2 dx = 0.$$

But $h(x)^2$ is a non-negative and continuous function, so the above equation can hold if and only if h is identically zero on $[0,1]$. Therefore, $f(x) = g(x)$ for every $x \in [0,1]$. Since f and g are polynomials, this is enough to conclude that $f = g$.

4. To start with, note that $y_0 = \int_0^1 f = \int_0^1 g$, and $y_1 = \int_0^1 \frac{f^2}{g}$. How to show $y_0 \leq y_1$? Well, the Cauchy-Schwarz inequality gives

$$\left(\int_0^1 \frac{f^2}{g}\right)\left(\int_0^1 g\right) \geq \left(\int_0^1 f\right)^2 \implies y_1 y_0 \geq y_0^2 \implies y_1 \geq y_0.$$

Let's proceed by strong induction. Suppose that $y_k \leq y_{k+1}$ holds for all $k \leq n-1$. How can we show that $y_n \leq y_{n+1}$? Cauchy-Schwarz inequality gives

$$\left(\int_0^1 \frac{f^{n+2}}{g^{n+1}}\right)\left(\int_0^1 \frac{f^n}{g^{n-1}}\right) \geq \left(\int_0^1 \frac{f^{n+1}}{g^n}\right)^2$$

which tells us that $y_{n+1}y_{n-1} \geq y_n^2$. Hence $y_{n+1}/y_n \geq y_n/y_{n-1}$ and $y_n/y_{n-1} \geq 1$ holds by induction hypothesis. This completes the induction and hence the proof.

6. Define $F(x) = \int_a^x f'(t)\,dt$ for $t \in [a,b]$. Then by the previous exercise, we can say that F is differentiable on $[a,b]$, with $F'(t) = f'(t)$ for every $t \in [a,b]$. In other words, the function $g = F - f$ will be a differentiable function having derivative equal to 0 on the entire $[a,b]$, which implies that g must be a constant function (this may be justified using MVT). Thus, $F(x) - f(x) = c$ for every $x \in [a,b]$. Putting $x = a$, and using $F(a) = 0$ (from its definition), we get that $c = -f(a)$. Therefore,

$$\int_a^b f'(t)\,dt = F(b) = f(b) + c = f(b) - f(a),$$

which completes the proof.

7. Just apply Rolle's theorem or the Mean Value Theorem on the function

$$F(x) = \int_a^x f(t)dt, \quad x \in [a,b]$$

which is differentiable here since f is continuous.

8. Define

$$g(x) = \int_0^x f(t)\,dt - x^3, \quad x \in [0,1].$$

Note that $g(1) = g(0) = 0$, and invoking the Fundamental Theorem of Calculus (FTC), we can say that g is continuous on $[0,1]$ and differentiable on $(0,1)$. Hence we can apply Rolle's theorem on g, which gives the desired conclusion.

9. Using the fact that $f(x) = \sin x$ is an increasing function on $[0,\pi/2]$, for $\pi/6 < x < \pi/2$ we have $1/2 < \sin x < 1$. Hence

$$\int_{\pi/6}^{\pi/2} \frac{x}{1}\,dx \leq \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx \leq \int_{\pi/6}^{\pi/2} \frac{x}{1/2}\,dx.$$

Observing that

$$\int_{\pi/6}^{\pi/2} x\,dx = \frac{\pi^2}{9},$$

1. (a) This is really straightforward. On one hand we have

$$\lim_{n\to\infty}\left(\int_0^1 f_n(x)\,dx\right)=\lim_{n\to\infty}\left(\int_0^1 x^n\,dx\right)=\lim_{n\to\infty}\frac{1}{n+1}=0,$$

while on the other hand we have

$$\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty} x^n=\begin{cases}0 & \text{if } 0\le x<1,\\ 1 & \text{if } x=1.\end{cases}$$

Hence

$$\int_0^1\left(\lim_{n\to\infty} f_n(x)\right)dx=0.$$

(b) First note that

$$\lim_{n\to\infty}\left(\int_0^1 f_n(x)\,dx\right)=\lim_{n\to\infty}\left(\int_0^1 nx^n\,dx\right)=\lim_{n\to\infty}\frac{n}{n+1}=1.$$

Next, we show that

$$\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty} nx^n=0,\quad \text{for every } 0\le x\le 1.$$

This is trivial for $x=0$ and $x=1$ (since $f_n(1)=0$ for all $n\ge 1$). For $0<x<1$, we take the help of $r=1/x>1$, and see that

$$0<nx^n=\frac{n}{(1+r-1)^n}\le\frac{n}{\binom{n}{2}(r-1)^2}=\frac{2}{(r-1)^2(n-1)}\to 0\quad\text{as } n\to\infty.$$

Therefore in this problem we have

$$\int_0^1\left(\lim_{n\to\infty} f_n(x)\right)dx=0\neq 1=\lim_{n\to\infty}\left(\int_0^1 f_n(x)\,dx\right).$$

2. For any $x>x_0$, we apply MVT on F to say that there exists $c_x\in(x_0,x)$ such that

$$\frac{F(x)-F(x_0)}{x-x_0}=F'(c_x)=f(c_x). \tag{2}$$

Now letting $x\to x_0^+$, the above LHS converges to $F'(x_0)=f(x_0)$. What about the RHS? Since $x_0<c_x<x \implies c_x\to x_0^+$, so the RHS of (2) converges to $\lim_{x\to x_0^+} f(c_x)=\lim_{c_x\to x_0^+} f(c_x)=a$. Hence we get the desired conclusion.

3. One way to attack this problem is by means of contradiction. But a simpler way is to use FTC, as follows. Define $F(x)=\int_a^x f(t)\,dt$ for $x\in[a,b]$. Since f is continuous, it holds by FTC that F is differentiable and $F'=f$ on (a,b). But it is given that F is a constant function, implying that $f(x)=F'(x)=0$ for any $x\in(a,b)$. Finally, continuity ensures that f must also vanish at the endpoints of $[a,b]$.

4. Define $F(x)=\int_a^x f(t)\,dt$ for $x\in\mathbb{R}$. Since f is continuous, it holds by FTC that F is differentiable and $F'=f$ on $\mathbb{R}$. Now,

$$g(a)\overset{\text{def}}{=}\int_a^{a+T} f(t)\,dt=F(a+T)-F(a)$$

is given to be a constant function (since $g(a)=g(0)$ for every $a\in\mathbb{R}$). Hence for every $a\in\mathbb{R}$, we must have $g'(a)=0$. But

$$g'(a)=F'(a+T)-F'(a)=f(a+T)-f(a).$$

So we get the desired conclusion that $f(a+T)=f(a)$ must hold for every $a\in\mathbb{R}$.

5. Since $x^2+1\ge 2x>0$, we have

$$I_n=\int_0^1\frac{x^n}{\sqrt{x^2+1}}\,dx\le\int_0^1\frac{x^n}{\sqrt{2x}}\,dx=\frac{1}{\sqrt{2}(n+1/2)}.$$

6. First recall that $1 < \sin x + \cos x < \sqrt{2}$ for every $x \in (0, \pi/2)$. Also recall that, when $x \in (0,1)$, $a > b$ actually implies $x^a < x^b$ (not its opposite). Therefore,

$$\forall x \in (0,1], \quad x^{\sqrt{2}} < x^{\sin x + \cos x} < x^1.$$

(At $x = 0$, these are all equal.) Upon integration, the above inequalities produce the following:

$$\int_0^1 x^{\sqrt{2}} dx < \int_0^1 x^{\sin x + \cos x} dx < \int_0^1 x^1 dx.$$

(Do you see why strict inequality holds here?) Finally note that

$$\int_0^1 x^{\sqrt{2}} dx = \sqrt{2} - 1 > 0.4.$$

Thus we get the desired inequality

$$0.4 < \int_0^1 x^{\sqrt{2}} dx < \int_0^1 x^{\sin x + \cos x} dx < \int_0^1 x^1 dx = 0.5.$$

7. We start by defining

$$I(a) = \int_{a-1}^{a+1} \frac{1}{1 + x^8} dx.$$

Applying the Leibniz rule, we get

$$I'(a) = \frac{1}{1 + (a+1)^8} - \frac{1}{1 + (a-1)^8}.$$

Note that $I'(a) = 0 \iff (a+1)^8 = (a-1)^8 \iff a = 0, \frac{1}{2}$. Now $I'(a)$ is of the form

$$I'(a) = c(a) \cdot \left((a-1)^2 - (a+1)^2\right) = c(a) \cdot (-4a),$$

where $c(a)$ is positive for any $a \in \mathbb{R}$. Therefore, $I'(a)$ changes its sign only when $a = 0$.

Furthermore, note that $I'(a) > 0$ for $a < 0$ and $I'(a) < 0$ for $a > 0$. Hence, $I(a)$ is maximised at $a = 0$.

8. If f had an anti-derivative, the problem would have been much easier. How? Suppose that $F' = f$. Since $F' = f$ is non-decreasing, F must be convex. Also, $\int_a^b f(u)du = F(b) - F(a)$ for any $a, b \geq 0$. Hence, the given inequality simplifies as follows:

$$\begin{aligned}
&(z - x) \int_y^z f(u)du \geq (z - y) \int_x^z f(u)du \\
\iff &(z - x)(F(z) - F(y)) \geq (z - y)(F(z) - F(x)) \\
\iff &\frac{F(z) - F(y)}{z - y} \geq \frac{F(z) - F(x)}{z - x} \\
\iff &F(y) \leq \frac{z - y}{z - x} F(x) + \frac{y - x}{z - x} F(z) \qquad (*)
\end{aligned}$$

Now observe that $y = \lambda x + (1 - \lambda) z \iff \lambda = (z - y)/(z - x)$. So (*) is the same as saying

$$F(\lambda x + (1 - \lambda)z) \leq \lambda F(x) + (1 - \lambda) F(z)$$

which follows from the convexity of F. Another way to finish the above solution is as follows. Observe that

$$\frac{F(z) - F(y)}{z - y} \geq \frac{F(z) - F(x)}{z - x} \iff \frac{F(z) - F(x)}{z - x} \geq \frac{F(y) - F(x)}{y - x}.$$

Now by MVT, there exists $c_1 \in (x, y)$ and $c_2 \in (y, z)$ such that

$$\frac{F(y) - F(x)}{y - x} = F'(c_1) = f(c_1), \quad \text{and} \quad \frac{F(z) - F(y)}{z - y} = F'(c_2) = f(c_2).$$

Since $c_1 < y < c_2$ and f is non-decreasing, we get $f(c_1) \leq f(c_2)$, which completes the proof.

Next, let us discuss a proof that does not rely on the assumption that f has an anti-derivative:

$$(z-x)\int_y^z f(u)du \geq (z-y)\int_x^z f(u)du$$

$$\iff (z-x)\int_y^z f(u)du \geq (z-y)\int_x^y f(u)du + (z-y)\int_y^z f(u)du$$

$$\iff (y-x)\int_y^z f(u)du \geq (z-y)\int_x^y f(u)du$$

Since f is non-decreasing, for any $u \in (y,z)$ we have $f(u) \geq f(y)$, and for any $u \in (x,y)$ we have $f(u) \leq f(y)$. Therefore,

$$(y-x)\int_y^z f(u)du \geq (y-x)(z-y)f(y)$$

$$= \int_x^y f(y)du \geq (z-y)\int_x^y f(u)du.$$

This completes the proof.

9. Let $[x] = m$. The LHS divided by 2 can be written as

$$\int_1^x \frac{[u]([u]+1)}{2} f(u)du = \sum_{k=1}^{m-1} \frac{k(k+1)}{2}\int_k^{k+1} f(u)du + \frac{m(m+1)}{2}\int_m^x f(u)du. \tag{3}$$

On the other hand, the RHS divided by 2 can be simplified as

$$\sum_{n=1}^{[x]} n\int_n^x f(u)du = \int_1^x f(u)du + 2\int_2^x f(u)du + \cdots + m\int_m^x f(u)du. \tag{4}$$

Now we can break each integral on the RHS of (4) as the sum of 'consecutive' integrals, e.g.,

$$\int_1^x f(u)du = \int_1^2 f(u)du + \int_2^3 f(u)du + \cdots + \int_{m-1}^m f(u)du + \int_m^x f(u)du.$$

In this manner, note that for any $1 \leq k \leq m-1$, the integral $\int_k^{k+1} f(u)du$ appears on the RHS of (4) exactly $(1+2+\cdots+k) = \frac{k(k+1)}{2}$ times, while the integral $\int_m^x f(u)du$ appears $m(m+1)/2$ times. Therefore, the RHS of (4) is the same as the RHS of (3), which completes the proof. $\square$

1.

(a) It equals $\int_0^1 x^k dx = \frac{1}{k+1}$. (Ans)

(b) Taking log, we get

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \log\left(1+\frac{k}{n}\right) = \int_0^1 \log(1+x)dx = x\log x - x\Big|_{x=2}^{x=1} = 2\log 2 - 1.$$

So the desired limit equals $\exp(2\log 2 - 1) = 4/e$. (Ans)

(c) It equals $\int_0^1 \frac{1}{1+x^3}dx$. Evaluating this is usually carried out using a partial fraction decomposition: by assuming that

$$\frac{1}{(1+x)(1-x+x^2)} = \frac{A}{x+1} + \frac{Bx+C}{1-x+x^2}$$

is an identity we solve for A, B, C, and then use standard integrals. Another way is to do some algebra and cleverly write it as

$$\frac{1}{6}\int_0^1 \frac{1}{x+1}dx - \frac{1}{6}\int_0^1 \frac{2x-1}{x^2-x+1}dx + \frac{1}{2}\int_0^1 \frac{1}{x^2-x+1}dx.$$

Anyway, these are some very standard methods that I hope you already are (or, going to be) familiar with them. The final answer is $\frac{1}{3}\log 2 + \frac{\pi}{3\sqrt{3}}$. (Ans)

(d) Since $\binom{2n}{n} = \prod_{k=1}^{n} \frac{n+k}{k}$, the given limit equals

$$\int_0^1 \log\left(1+\frac{1}{x}\right)dx = \int_1^2 \log x dx - \int_0^1 \log x dx = (2\log 2 - 1) - (-1) = \log 4.$$

(Ans)

2. The first one can be calculated as follows.

$$\begin{aligned}\int_{1/e}^{e} |\log x|\, dx &= \int_{1/e}^{1} |\log x|\, dx + \int_1^e |\log x|\, dx \\ &= \int_{1/e}^{1} -\log x\, dx + \int_1^e \log x\, dx \\ &= x - x\log x\Big|_{x=1/e}^{x=1} + x\log x - x\Big|_{x=1}^{x=e} = 2(1-1/e). \quad \text{(Ans)}\end{aligned}$$

3. First we write

$$I = \int_0^\pi x f(\sin x)dx = \int_0^\pi (\pi - x)f(\sin(\pi - x))dx = \int_0^\pi (\pi - x)f(\sin x)dx$$

and then adding up these two alternate expressions for the same integral, we get

$$2I = \pi\int_0^\pi f(\sin x)dx = 2\pi\int_0^{\pi/2} f(\sin x)dx$$

where in the last step we used $\int_0^{2a} f(x)dx = \int_0^a (f(x) + f(2a-x))dx$. $\square$

Using the above formula/idea, we get

$$\int_0^\pi \frac{x\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}dx = \pi\int_0^{\pi/2} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}dx$$

Now using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$,

$$I = \int_0^{\pi/2} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}dx = \int_0^{\pi/2} \frac{\cos^{2n} x}{\sin^{2n} x + \cos^{2n} x}dx = \frac{1}{2}\int_0^{\pi/2} dx = \frac{\pi}{4}.$$

Therefore, the desired integral equals $\pi^2/4$. (Ans)

4. First we substitute $y = nx$ to write

$$\int_0^{\pi} \left|\frac{\sin nx}{x}\right| dx = \int_0^{n\pi} \left|\frac{\sin y}{y/n}\right| \frac{dy}{n} = \int_0^{n\pi} \left|\frac{\sin y}{y}\right| dy.$$

Now break the integral as the sum of integrals $\int_0^{\pi}, \int_{\pi}^{2\pi}$, etc. as follows.

$$\int_0^{n\pi} \left|\frac{\sin y}{y}\right| dy = \sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \left|\frac{\sin y}{y}\right| dy$$

$$\geq \sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\sin y|}{k\pi} dy \quad (\text{since } (k-1)\pi < y < k\pi \Rightarrow 1/y > 1/k\pi)$$

$$= \sum_{k=1}^{n} \frac{1}{k\pi} \int_0^{\pi} |\sin y|\, dy = \frac{2}{\pi}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$$

as required.

5. Let me do the first two, and leave the rest for you. For $n \geq 1$, define

$$I_n = \int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx.$$

For instance, $I_0 = \pi/2$, and $I_1 = 1$. How to calculate I_n for a general n? The idea is to get a recursion for I_n and then solve that recursion. For $n > 1$, we integrate by parts to get

$$I_n = \int_0^{\pi/2} (\sin x)^{n-1} \cdot \sin x\, dx$$

$$= \left[(\sin x)^{n-1} \int \sin x\, dx\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{d}{dx}(\sin x)^{n-1} \left(\int \sin x\, dx\right) dx$$

$$= \left[-(\sin x)^{n-1} \cos x\right]_0^{\pi/2} + \int_0^{\pi/2} (n-1)(\sin x)^{n-2} \cos^2 x\, dx$$

$$= 0 + \int_0^{\pi/2} (n-1)(\sin x)^{n-2}(1 - \sin^2 x)\, dx = (n-1)(I_{n-2} - I_n).$$

Thus, $I_n = (n-1)(I_{n-2} - I_n)$, which can also be written as

$$I_n = \frac{n-1}{n} I_{n-2}, \quad n \geq 2.$$

Now, for an even n, say $n = 2k$ where $k \geq 1$, we have

$$I_{2k} = \frac{2k-1}{2k} I_{2k-2} = \frac{2k-1}{2k} \cdot \frac{2k-3}{2k-2} I_{2k-4} = \cdots = \frac{1 \times 3 \times \cdots \times (2k-1)}{2 \times 4 \times \cdots \times 2k} I_0.$$

Similarly, for odd n, say $n = 2k+1$ where $k > 1$, we have

$$I_{2k+1} = \frac{2k}{2k+1} I_{2k-1} = \frac{2k}{2k+1} \cdot \frac{2k-2}{2k-1} I_{2k-3} = \cdots = \frac{2 \times 4 \cdots \times 2k}{3 \times 5 \cdots \times (2k+1)} I_1.$$

We can also write

$$I_n = \begin{cases} \frac{(2k-1)!!}{(2k)!!} \frac{\pi}{2} = \frac{\binom{2k}{k}\pi}{2^{2k+1}} & \text{if } n = 2k \geq 0, \\ \frac{(2k)!!}{(2k+1)!!} = \frac{2^k}{2k+1} \binom{2k}{k}^{-1} & \text{if } n = 2k+1 \geq 1. \end{cases} \tag{3}$$

These integrals (I_n) are commonly known as Wallis' integrals.

6. Using the Binomial theorem,

$$(1-x^2)^n = \sum_{k=0}^{n} \binom{n}{k} (-x^2)^k.$$

Integrating both sides, and noting that the RHS being a finite summation we can pass the integral sign inside the summation, we get

$$\int_0^1 (1-x^2)^n dx = \sum_{k=0}^{n} \binom{n}{k} \int_0^1 (-x^2)^k dx = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{2k+1}.$$

Now, we can calculate the integral on the LHS directly (using by parts or by substitution) and hence get an expression for the sum on the RHS.

$$\int_0^1 (1-x^2)^n dx = \int_0^{\pi/2} (1-\sin^2\theta)^n \cos\theta\, d\theta = \int_0^{\pi/2} (\cos\theta)^{2n+1} d\theta = \frac{2 \times 4 \times \cdots \times 2k}{1 \times 3 \times \cdots \times (2n+1)}.$$

where the last integral was evaluated using (3). Therefore,

$$\frac{1}{1}\binom{n}{0} - \frac{1}{3}\binom{n}{1} + \frac{1}{5}\binom{n}{2} - \cdots + (-1)^n \frac{1}{2n+1}\binom{n}{n} = \frac{(2n)!!}{(2n+1)!!}. \quad \text{(Ans)}$$

7. For any $x > 1$, we calculate the following integral by substituting $u = 1/t$

$$\int_1^{1/x} \frac{\log t}{1+t} dt = \int_x^1 \frac{\log(1/u)}{1+1/u} \frac{(-1)\, du}{u^2} = \int_1^x \frac{\log u}{1+u} du.$$

Therefore,

$$f(x) + f(1/x) = \int_1^x \frac{\log t}{1+t} dt + \int_x^1 \frac{\log t}{1+t} dt = \int_1^x \frac{\log t}{t} dt = \frac{1}{2}(\log t)^2\Big|_1^x = \frac{1}{2}(\log x)^2.$$

So, $f(x) + f(1/x) = 2 \iff (\log x)^2 = 4 \iff \log x = \pm 2 \iff x = e^2$ or e^{-2}.

8. Using the formula $\int_{-a}^{a} f(x)dx = \int_0^a (f(x) + f(-x))dx$, we get

$$\int_0^a g(x)dx = 0$$

for all $a \in \mathbb{R}$ where $g(x) = f(x) + f(-x)$. In a previous exercise we saw that this implies $g \equiv 0$, which here forces f to be an odd function.

9. The derivative of $\sqrt{x}$ is $\frac{1}{2}x^{1/2-1} = \frac{1}{2\sqrt{x}}$. So, $\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$. Now we can proceed in many ways. One way is to say that the function

$$g(x) = \sqrt{f(x)} - \frac{1}{2}x$$

has derivative

$$g'(x) = \frac{f'(x)}{2\sqrt{f(x)}} - \frac{1}{2} \geq 0,$$

hence g is increasing and therefore for any $x \geq 1$, we have $g(x) \geq g(1)$, which gives the desired inequality.

10. First note that $f'(x) > 0$ so f is increasing. Hence for $x \geq 1$, we can say that $f(x) \geq f(1) = 1$. Therefore,

$$f'(x) = \frac{1}{x^2 + f(x)^2} \leq \frac{1}{x^2+1} \quad \text{for all } x \geq 1. \tag{4}$$

Now

$$f(x) - f(1) = \int_1^x f'(t)dt \leq \int_1^x \frac{1}{1+t^2} dt = \tan^{-1} x - \tan^{-1} 1 < \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Since f is increasing and bounded above, we can say that $\lim_{x\to\infty} f(x)$ exists, and from the above inequalities, it is immediate that the limit should be less than or equal to $\pi/4$.

But how to claim that the limit is strictly less than $\pi/4$? Showing that is quite tricky, because even if you have $f(x) < g(x)$ for all x, taking limit as $x \to \infty$ (or $x \to a$) would change the $<$ sign to a $\leq$ sign. Here we adopt the following approach.

If f never crosses c where $1 < c < 1 + \pi/4$ then it is trivial that $\lim_{x\to\infty} f(x) \leq c < 1 + \pi/4$.

Else, $f(x_0) > c$ for some $x_0 > 1$, then $f(x) \geq f(x_0) > c$ for all $x > x_0$, and hence

$$f'(t) = \frac{1}{t^2 + f(t)^2} \leq \frac{1}{t^2 + c^2}, \quad \text{for } t \geq x_0.$$

Integrating this inequality from x_0 to x and integrating (4) from 1 to x_0, we obtain

$$f(x) - f(1) \leq \int_1^{x_0} \frac{1}{t^2+1}\, dt + \int_{x_0}^{x} \frac{1}{t^2+c^2}\, dt$$

for every $x > x_0$. Letting $x \to \infty$ here, we get

$$\lim_{x\to\infty} f(x) \leq 1 + \int_1^{x_0} \frac{1}{1+t^2}\, dt + \int_{x_0}^{\infty} \frac{1}{t^2+c^2}\, dt < 1 + \int_1^{\infty} \frac{1}{t^2+1}\, dt = 1 + \frac{\pi}{4}.$$

11. First observe that $f(x) + f(1-x) = 1$ for every $x \in \mathbb{R}$. Then note that

$$f(f(1-x)) = f(1-f(x)) = 1 - f(f(x)).$$

In fact, you can do induction on n to show that if g be f composed with itself n times, then g also satisfies $g(x) + g(1-x) = 1$. Hence, for any $n \geq 1$, we can write

$$I = \int_0^1 f^{[n]}(x)\, dx = \int_0^1 f^{[n]}(1-x)\, dx = \int_0^1 \left(1 - f^{[n]}(x)\right) dx$$

and then add up these two alternate expressions for I to show that $I = 1/2$. (Ans)

12. We observe that

$$\int_0^1 f(nx)dx = \frac{1}{n}\int_0^n f(y)dy = \frac{1}{n}\sum_{k=0}^{n-1}\int_k^{k+1} f(y)dy = \frac{1}{n}\sum_{k=0}^{n-1}\int_0^1 f(u+k)du = \frac{1}{n}\sum_{k=0}^{n-1} a_k.$$

Now you have to use the following fact: if $(a_n)_{n\geq 0}$ be a sequence that converges to a, then the sequence $(b_n)_{n\geq 1}$ defined by

$$b_n = \frac{1}{n}\sum_{k=0}^{n-1} a_k$$

also converges to a. This tells us that the desired limit also equals a. (Ans)

13. Applying integration by parts, we get

$$\begin{aligned}\int_a^b f(x)\sin(nx)dx &= \left[f(x)\int \sin(nx)dx\right]_a^b - \int_a^b \left(f'(x)\int \sin(nx)dx\right) dx \\ &= \frac{f(a)\cos na - f(b)\cos nb}{n} - \frac{1}{n}\int_a^b f'(x)\cos(nx)dx. \qquad (\dagger)\end{aligned}$$

Now, since f is continuously differentiable on $[a,b]$, we can say that f' is bounded on $[a,b]$. In other words, we can find an $M > 0$ such that $|f'(x)| < M$ holds for every $x \in [a,b]$. So, $0 \leq |f'(x)\cos nx| \leq M$ also holds for $x \in [a,b]$ and therefore we obtain from (†) that

$$0 \leq \left|\int_a^b f(x)\sin(nx)dx\right| \leq \left|\frac{f(a)\cos na - f(b)\cos nb}{n}\right| + \left|\frac{1}{n}\int_a^b f'(x)\cos(nx)dx\right|$$

$$\leq \frac{|f(a)\cos na| + |f(b)\cos nb|}{n} + \frac{1}{n}\int_a^b |f'(x)\cos(nx)|dx$$

$$\leq \frac{|f(a)| + |f(b)|}{n} + \frac{M(b-a)}{n} \to 0, \quad \text{as } n \to \infty.$$

This proves that the desired limit is 0.

14 Combinatorics

1. Let $abcdef$ denote the 6 vertical stripes in order from the left. Then a can be of any one of the 4 colours; then b can be of any one of the *other* 3 colours; then c can be again of any one of 3 colours since the colour used for a is now available. Similarly, there are 3 possible colours for each of d, e, f. Hence, by MP, there are 4×3^5 ways of designing the flag.

2.

(i) A 5-digit number is of the form $abcde$ where $a, \ldots, e$ are the digits in it and a is the *leading* digit. Now a cannot be 0 and so can be chosen in 9 ways (any one of 1, …, 9). Each of the remaining digits can be chosen in 10 ways since repetition is allowed. So, in all 9×10^4 ways.

(ii) The number $abcde$ is even if and only if the units digit e is even; so there are 5 choices for e: 0, 2, 4, 6, 8. So in all $9 \times 10^3 \times 5$ ways.

(iii) Two cases: either only $a = 3$ or exactly one of $b, c, d, e = 3$. There are 9^4 ways in the first case and $8 \times 9^3 \times 4$ in the second case (factor 4 for the 4 subcases $b = 3$ or $c = 3$ etc.); so in all $9^4 + 8 \times 9^3 \times 4$ ways.

(iv) By the given condition, the values of a, b, c fix the values of $d, e : e = a, d = b$. Hence, the number of ways is 9×10^2.

3. We are considering the integers t such that $1 \leq t \leq 3333$. Clearly, the largest number t having 0 in the units place is 3330. So there are 333 numbers t having 0 in the units place: they are 10, 20, …, 3330. We can describe these numbers as $t = x0$ where x is any one of 1, 2, …, 333.

Similarly, numbers $t = x0y$ i.e. numbers having 0 in the tens place are in all 33×10 because x can be any one of 1, 2, …, 33 and y can be any one of 0, 1, 2, …, 9. Thus there are $33 \times 10 = 330$ numbers like $x0y$. In the same way, there are $3 \times 10^2 = 300$ numbers with 0 in the hundreds place (i.e. $x0yz$ where $1 \leq x \leq 3$, $0 \leq y, z \leq 10$). Hence the total number of times 0 is written is $333 + 330 + 300 = 963$.

4. Since we are counting the occurrences of digit 5, consider integers t such that $1 \leq t \leq 10^5$. Clearly, the largest number t having 5 in the units place is 99995. So there are $1 + 9999 = 10^4$ numbers t having 5 in the units place: they are 5, 15, 25, …, 99995. We can describe these numbers as $t = x5$ where x is any one of 0, 1, 2, …, 9999.

Similarly, numbers $t = x5y$, i.e., numbers having 5 in the tens place are in all $(1 + 999) \times 10 = 10^4$ because x can be any one of 0, 1, 2, …, 999 and y can be any one of 0, 1, 2, …, 9. In the same way, there are 10^4 numbers in each of the following cases: numbers with 5 in the hundreds place or thousands place or ten thousands place. Hence the total number of times 5 is written is $10^4 \times 5$.

5. A collection is determined once we know the number of times A occurs in it and the number of times B occurs in it. Let these numbers be i and j respectively. Then i takes values from 0 to 3 and j from 0 to 5. Hence to obtain a collection we have 4 choices for i and 6 choices for j so that there are $4 \times 6 = 24$ collections (for example, $i = 1, j = 2$ gives the collection $\{A, B, B\}$). The empty collection corresponds to letting $i = j = 0$; so that omitting this case there are $24 - 1 = 23$ non-empty collections.

6. To make a collection we have to select a certain number of objects of each kind. Now for each value of r, $1 \leq r \leq k$, from n_r like objects we can choose 0, 1, …, or n_r objects; i.e. there are $n_r + 1$ choices. Hence, by MP, there are in all $(n_1 + 1)(n_2 + 1) \ldots (n_k + 1)$ collections and so the number of non-empty collections is $(n_1 + 1)(n_2 + 1) \ldots (n_k + 1) - 1$.

7. Let $S = \{a_1, a_2, \ldots, a_n\}$. Note that we form a subset T of S in n stages as follows: we have 2 choices for a_1: either a_1 *is* included in T or a_1 *is not* included in T. Similarly, we have 2 choices for a_2: either a_2 *is* included in T or a_2 *is not* included in T, etc. Finally, we have 2 choices for a_n: either a_n *is* included in T or a_n *is not* included in T. (For example, if $n = 4$, then the subset $\{a_2, a_4\}$ corresponds to the sequence of choices no, yes, no, yes.) Hence, by MP, the total number of subsets is $2 \times 2 \times \cdots \times 2$ (n factors) i.e. 2^n.

8. Here $m = 30030 = 2 \times 3 \times 5 \times 7 \times 11 \times 13$, and each prime factor occurs only once. Hence every factor of m corresponds to the product of elements of a particular subset of the 6-set $S = \{2, 3, 5, 7, 11, 13\}$. Note that the empty subset corresponds to the factor 1. Hence the number of factors of m is equal to the number of subsets of S, namely 2^6, by the last problem.

9. A function f from A to B corresponds to the ordered set given by $(f(1), \ldots, f(m))$ of m elements where $f(i)$ is the value of f at i. Now for each i in A, $f(i)$ can be chosen in n ways from B. Hence, by the multiplication principle, there are $n \times n \times n \times \cdots \times n$ (m factors) $= n^m$ functions from A to B.

The number of one-one functions will be $n(n-1)\ldots(n-m+1)$. Thus, if $n < m$ then the number of one-one functions is zero and if $n \geq m$ then the number of one-one functions is

$$\frac{n!}{(n-m)!}.$$

10. The function is to be onto B and so it must take on both the values b_1 and b_2. So there are 2 choices for the value of the function at each of the n elements of A, namely b_1 or b_2, excepting the case when all values are equal to b_1 and the case when all values are equal to b_2. Hence the number of required functions is $2^n - 2$.

14.1 .2

1. There are 32 black squares and of these 12 can be chosen to put the 12 white pawns in $\binom{32}{12}$ ways. Then out of the remaining 20 black squares 12 can be chosen to put the 12 black pawns in $\binom{20}{12}$ ways. So the answer is $\binom{32}{12} \times \binom{20}{12}$.

2. Since the 2 rooks are identical, the order in which they are placed in a row (or column) is not important. First a row can be chosen in 8 ways. In any row, 2 of the 8 squares can be chosen in ${}^8C_2$ ways. Hence the 2 rooks can be placed in a row in $8 \times {}^8C_2$ ways. Similarly, 2 rooks can be placed in a column in $8 \times {}^8C_2$ ways. So the total number is $2 \times 8 \times {}^8C_2$. For an $n \times m$ board the answer is $n \times {}^mC_2 + m \times {}^nC_2$.

3. A row can be chosen in 8 ways. In a row, there are in all ${}^8C_2$ pairs of squares of which 7 are pairs of *adjacent* squares. So there are $[{}^8C_2 - 7]$ pairs of places in any of which the two identical kings can be placed. Hence the kings can be placed in a row in $8 \times [{}^8C_2 - 7]$ ways and in the same number of ways in a column. So the total number of ways is $2 \times 8 \times [{}^8C_2 - 7]$. For an $n \times m$ board, the number is

$$n\left[{}^mC_2 - (m-1)\right] + m\left[{}^nC_2 - (n-1)\right].$$

4. The different necklaces are determined by the number of red beads *between* the two blue beads, taking both arcs into consideration. So there are exactly 3 distinct necklaces:

$$bbrrrrr, \quad brbrrrr, \quad brrbrrr.$$

5. This is very straightforward. The answers are

1. $9! \times 3!$
2. $8! \times {}^9P_3$

6. There are $q+1$ places between the q 0's (namely, $q-1$ places between the q successive 0's and 2 places at the ends). The required arrangements are obtained by putting the p 1's in p of these $q+1$ places; and this can be done in $\binom{q+1}{p}$ ways.

7. An r-subset T of S corresponds, in a one-to-one way, to an arrangement $a_1, a_2, \ldots, a_n$ of r 1's and $n-r$ 0's in a row as follows: $a_i = 1$ if $i \in T$ and $a_i = 0$ if $i \notin T$. Hence the r-subsets of $S = \{1, 2, \ldots, n\}$ that do not contain a pair of consecutive integers exactly correspond to the arrangements of r 1's and $n-r$ 0's in a line such that no two 1's are adjacent. Hence, by the last problem, the number of required r-subsets is $\binom{n-r+1}{r}$.

15 Pigeon Hole Principle

1. A person goes into box #i if they have made i handshakes. We have n persons and n boxes numbered $0, 1, \ldots, n-1$. But the boxes with the numbers 0 and $n-1$ cannot both be occupied. Thus, there is at least one box with more than one person.

2. Same as problem 1 with handshakes replaced by contests.

3. Denote the 20 integers a_1 to a_{20}. Then $0 < a_1 < \cdots < a_{20} < 70$. We want to prove that there is a k, so that $a_j - a_i = k$ has at least four solutions. Now

$$0 < (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_{20} - a_{19}) = a_{20} - a_1 \leq 68.$$

We will prove that, among the differences $a_{i+1} - a_i$, $i = 1, \ldots, 19$, there will be four equal ones. Suppose there are at most three differences equal. Then

$$3 \cdot 1 + 3 \cdot 2 + 3 \cdot 3 + 3 \cdot 4 + 3 \cdot 5 + 3 \cdot 6 + 7 \leq 68,$$

that is, $70 \leq 68$. Contradiction!

4. Consider the three coordinates mod 2. There are $2^3 = 8$ possible binary 3-words. Since there are nine words altogether, at least two sequences must be identical. Thus there are two points (a, b, c) and (r, s, t) with integral midpoint $M = ((a+r)/2, (b+s)/2, (c+t)/2)$.

5. Subdivide the unit square into 25 small squares of side $1/5$. There will be three insects in one of these squares of side $1/5$ and diagonal $\sqrt{2}/5$. A circumcircle of this square has radius $\sqrt{2}/10 < 1/7$. If we circumscribe a concentric circle with radius $1/7$, it will cover this square completely.

6. Consider the n integers $1,\ 11,\ \ldots,\ 11\cdots1 \mod n$. There are n possible remainders $0,\ 1,\ \ldots,\ n-1$. If 0 occurs, we are finished. If not, two of the numbers have the same remainder mod n. Their difference $11\cdots100\cdots0$ is divisible by n. Since n is not divisible by 2 or 5, we can strike the zeros at the end and get the number consisting of ones and divisible by n.

7. We use the same motive. Consider the sums

$$a_1,\ a_1 + a_2,\ a_1 + a_2 + a_3,\ \ldots,\ a_1 + a_2 + \cdots + a_n.$$

If any of the n sums is divisible by n, then we are done. Otherwise, two of the sums $a_1 + \cdots + a_i$ and $a_1 + \cdots + a_j$ have the same remainder upon division by n. Suppose $j > i$. Then the difference $a_{i+1} + \cdots + a_j$ is divisible by n.

8. Among $n+1$ integers from $1, \ldots, 2n$, there are two successive integers. They are coprime.

9. A set S of 10 numbers with two digits, each one ≤ 99 has $2^{10} = 1024$ subsets. The sum of the numbers in any subset of S is $\leq 10 \cdot 99 = 990$. So there are fewer possible sums than subsets. Thus there are at least two different subsets S_1 and S_2 having the same sum. If $S_1 \cap S_2 = \emptyset$, then we are finished. If not, we remove all common elements and get two nonintersecting subsets with the same sum of their elements.

10. Use induction from n to $2n$, which corresponds to induction from k to $k+1$.

(1) For $n = 1$, the statement is correct.

(2) Suppose that, from $2n-1$ integers, we can always select n with sum divisible by n. Of the $2(2n)-1$ positive integers, we can select n numbers three times, which are divisible by n. After the first selection, there will remain

$3n-1$ numbers, after the second selection, $2n-1$ numbers. Let the sum of the first choice be $a \cdot n$, the sum of the second choice be $b \cdot n$, and the last choice be $c \cdot n$. At least two of the numbers a, b, c have the same parity, e.g., a and b. Then $an + bn = (a+b)n$ is divisible by $2n$, since $a+b$ is even.

11. Consider all subsets $\{i_1, \ldots, i_k\}$ of the set $\{1, \ldots, n\}$. Let $S(i_1, \ldots, i_k) = a_{i_1} + \cdots + a_{i_k}$. The number of such sums is $2^n - 1$. Since $2^n - 1 > n^2$ for $n \geq 5$, two of these sums will have the same remainder upon division by n^2. Their difference will be divisible by n^2. This difference has the form $\pm a_{s_1} \pm a_{s_2} \pm \cdots \pm a_{s_t}$ for some $t \geq 1$ and some selection of indices $s_1, \ldots, s_t$.

12. Considering the fractional parts of these numbers, we get $n-1$ reals in the interval $[0,1]$. Subdivide this unit interval into n equal parts, each of length $1/n$. If one of the n points falls into the first interval, then we are finished. Otherwise, two points, say $\{ia\}$ and $\{ka\}$, fall into the same interval. Then the point $\{(k-i)a\}$ is a distance $\leq 1/n$ from 0.

13. Two of six points placed into a 3×4 rectangle will have distance $\leq \sqrt{5}$.

14. This problem contains all necessary hints for a solution. It is a problem for the box principle, since all existence problems about finite sets somehow rely on the box principle. Furthermore, it contains the hint to the addition theorem for tan, and $0 = \tan 0$, $1/\sqrt{3} = \tan(\pi/6)$ give the missing hints for the boxes. So we set $y_i = \tan x_i$, $y_j = \tan x_j$ and get

$$\tan 0 \leq \tan(x_i - x_j) \leq \tan \frac{\pi}{6}.$$

Because tan is monotonically increasing everywhere, we get

$$0 \leq x_i - x_j \leq \frac{\pi}{6}.$$

The y_i can lie anywhere in the infinite interval $-\infty < y_i < \infty$. But the x_i are confined to the interval $-\pi/2 < x_i < \pi/2$. For at least two of the seven x_i we have $0 \leq x_i - x_j \leq \pi/6$. The original inequality follows from this.

16 Invariance Principle

1. In one move the number of integers always decreases by one. After $(4n-2)$ steps, just one integer will be left. Initially, there are $2n$ even integers, which is an even number. If two odd integers are replaced, the number of odd integers decreases by 2. If one of them is odd or both are even, then the number of odd numbers remains the same. Thus, the number of odd integers remains even after each move. Since it is initially even, it will remain even to the end. Hence, one even number will remain.

2.

3. Consider the remainder mod 9. It is an invariant. Since $10^6 = 1 \mod 9$ the number of ones is by one more than the number of twos.

4. Here, $I(x_1, x_2, \ldots, x_6) = 2x_1 + 4x_2 + 6x_3 + 8x_4 + 10x_5 + 12x_6 \mod 10$ is the invariant. Starting with $I(1,0,1,0,1,0) = 8$, the goal $I(0,1,0,1,0,1) = 4$ cannot be reached.

5. We proceed by contradiction. Suppose all the remainders $0, 1, \ldots, 2n-1$ occur. The sum of all integers and their place numbers is

$$S_1 = 2\,(1 + 2 + \cdots + 2n) = 2n\,(2n+1) \equiv 0 \pmod{2n}.$$

The sum of all remainders is

$$S_2 = 0 + 1 + \cdots + 2n - 1 = n\,(2n-1) \equiv n \pmod{2n}.$$

Contradiction!

6. We partition the participants into the set E of even persons and the set O of odd persons. We observe that, during the hand shaking ceremony, the set O cannot change its parity. Indeed, if two odd persons shake hands,

O increases by 2. If two even persons shake hands, O decreases by 2, and, if an even and an odd person shake hands, $|O|$ does not change. Since, initially, $|O| = 0$, the parity of the set is preserved.

7. The number of heads is invariant mod 3. Initially, it is 1 and it remains so.

8. Observe that the initial sum is odd. At some step if we remove the numbers a, b and replace it with their difference then we see that the sum changes to

$$S \to S - (a + b) + (a - b) = S - 2b$$

Hence the parity of the sum remains the same. Hence all the numbers can not be 0

9. For any collection of n numbers on the blackboard we consider the following quantity X: the sum of all the numbers decreased by n. Assume that we have transformed the collection as described in the statement. How would the quantity X change? If the sum of all the numbers except a and b equals S, then before the transformation $X = S + a + b - n$, and after the transformation $X = S + (a + b - 1) - (n - 1) = S + a + b - n$. So the value of X is the same: it is invariant. Initially (for the collection in the statement) we have

$$X = (1 + 2 + \cdots + 19 + 20) - 20 = 190.$$

Therefore, after 19 operations, when there will be only one number on the blackboard, X will be equal to 190. This means that the last number, which is $X + 1$, is 191.

10. Its type is B. Consider the parities of the differences $N(A) - N(B)$, $N(B) - N(C)$, and $N(C) - N(A)$, where $N(X)$ is the number of type X amoebae. These parities do not change in the course of the merging process. This means, in particular, that in the end (when there is only one amoeba in the tube) the numbers of A-amoebae and C-amoebae have the same parity, which is possible only if the only amoeba left belongs to type B.