A Legen-*wait for it*-dary mock

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§1 P1

Find all integers $n \geq 3$ so that , if we list the divisors of $n!$, including 1 and itself, in increasing order then we have

$$d_2 - d_1 \leq d_3 - d_2 \leq ... \leq d_k - d_{k-1}$$

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§2 P2

(a) Determine the number of permutations $a_1, a_2, ...a_{2025}$ of the numbers $\{1, 2, ...2025\}$ for which $|a_i - i|$ is constant for all positive integer i.
(b) Determine the number of permutations $a_1, a_2, ...a_{2024}$ of the numbers $\{1, 2, ...2024\}$ for which $|a_i - i|$ is constant for all positive integer i.
(Okay , you should be able to see that you guys are appearing on a lucky year)

§3 P3

Say all the set of natural numbers is partitioned into n(disjoint) AP-s with common differences $r_1, r_2, ...r_n$. Prove that

$$\frac{1}{r_1} + \frac{1}{r_2} + ... + \frac{1}{r_n} = 1$$

§4 P4

Let $f : [0, 1] \to (0, \infty)$ be continuous. Prove the following.
(a) for $n \in \mathbb{N}$,$n \geq 2$ there exists a unique positive real number $a_n \in (0, 1)$ such that

$$\int_0^{a_n} f(t)\,dt = \frac{1}{n} \int_0^1 f(t)\,dt$$

(b) Determine whether $\lim_{n\to\infty} na_n$ exists, and if it does find the limit.

§5 P5

Arnab, being bored(you may like to dm him why), is playing a game. He starts with $\{0, 10\}$. At each step he takes a polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0$ with $a_i \in S$, $\forall\, i$ and if P has an integer root then he includes that root in S . Show that irrespective of how he plays he cannot expand $|S|$ over a certain value. Find the S at the final case when it cannot be expanded further.

§6 P6

Suppose you have a not necessarily unbiased coin with the probability of showing head p. Define a "run" to be consecutive same outcomes e.g. this sequence of 10 tosses, $HHHHTTHTTH$ has 5 runs with "length"s 4,2,1,2,1 respectively. Prove that the expected length of first run is at least that of the second run.

§7 P7

Let k be a natural number. A sequence $a_1, a_2, ..., a_k$ is said to be $k-$depressed if for all $n \in \mathbb{N}$, a_n = number of distinct elements in the set$\{a_1, a_2, ..., a_{n+k}\}$. Determine, in terms of k , how many $k-$depressed sequence are there?

§8 P8

Let $x_1 = 2$ and $x_{n+1} = x_n^2 - x_n + 1$ for $n \geq 1$ Then prove that

$$1 - \frac{1}{2^{2^{n-1}}} < \frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n} < 1 - \frac{1}{2^{2^n}}$$

In life, you will be challenged in each and every step. This mock is nothing in comparison. So, better suit up! For boosting your confidence here is a legendary specimen of what your attitude should be :