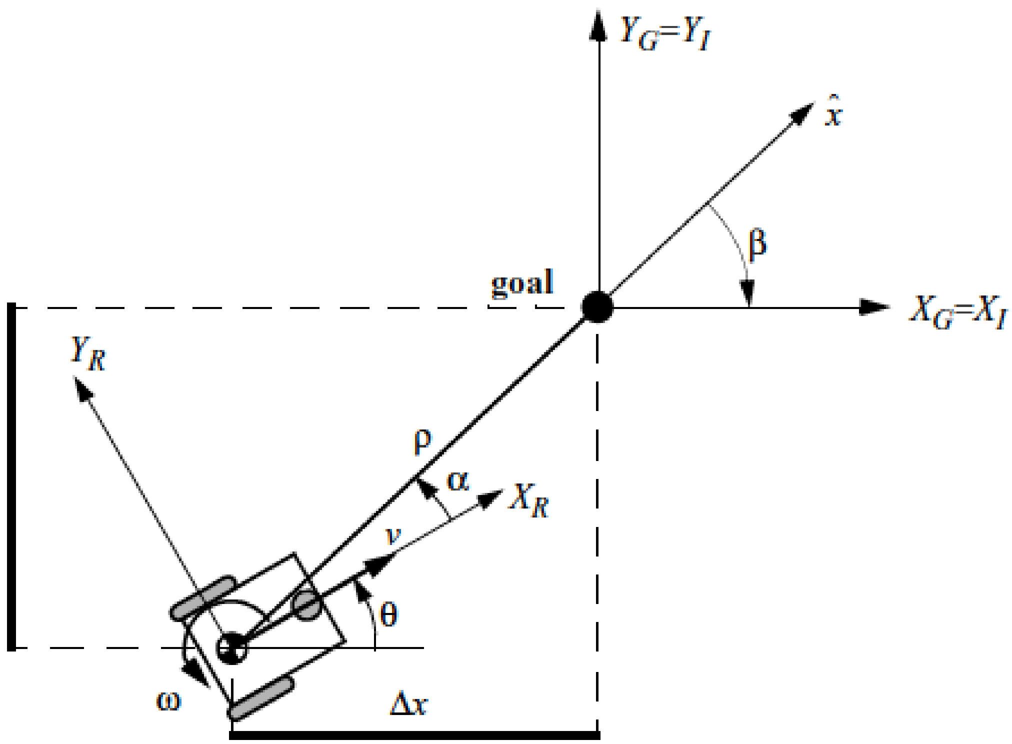
**Model:**




**Definitions:** We define all terms to be explicit

- $\mathbf{p}$ is the current error vector $\mathbf{p} = (\Delta x, \Delta y)^T$; $\rho$ and $\beta$ are its polar coordinates.
- $\rho$ is the magnitude of the error vector $\mathbf{p}$, $\rho = \|\mathbf{p}\|$
- $\beta$ is the negative of the angle between error vector $\mathbf{p}$ and the x-axis, which can be computed as $-\mathrm{atan2}(\Delta y, \Delta x)$
- $\mathbf{v}$ is the current velocity vector $\mathbf{v} = \dot{\mathbf{p}} = (\Delta \dot{x}, \Delta \dot{y})^T$
- $\theta$ is the angle between the x-axis and velocity vector $\mathbf{v}$
- $v$ is the magnitude of $\mathbf{v}$, $v = \|\mathbf{v}\|$
- $\omega$ is the angular velocity of the robot about the z-axis (assumed there is none about the x- and y-axes), $\omega = \dot{\theta}$. The vector form of this is denoted $\Omega$
- $\alpha$ is the angle between our velocity $\mathbf{v}$ and the error vector $\mathbf{p}$

From the slides we have the following relationships:

$$\begin{aligned}
\rho &= \sqrt{\Delta x^2 + \Delta y^2} \\
\Delta x &= \rho\cos(\alpha+\theta) \\
\Delta y &= \rho\sin(\alpha+\theta) \\
\beta &= -\alpha - \theta = -\text{atan}\left(\frac{\Delta y}{\Delta x}\right)
\end{aligned}$$

**Want to show:**

$$\begin{pmatrix}\dot\rho \\ \dot\alpha \\ \dot\beta\end{pmatrix} = \begin{pmatrix}\cos\alpha & 0 \\ \frac{\sin\alpha}{\rho} & -1 \\ -\frac{\sin\alpha}{\rho} & 0\end{pmatrix}\begin{pmatrix}v \\ \omega\end{pmatrix} = \begin{pmatrix}(\cos\alpha)v \\ \left(\frac{\sin\alpha}{\rho}\right)v - \omega \\ -\left(\frac{\sin\alpha}{\rho}\right)v\end{pmatrix}$$

**Proof:**

- For $\rho$, see that the change of $\rho$ is the component of the velocity $\mathbf{v}$ along the direction $\mathbf{p}$. This is just the projection of $\mathbf{v}$ along the direction of $p$, which is $\|\mathbf{v}\|\cos(\alpha) = v\cos(\alpha)$ (since $\alpha$ is the angle between $\mathbf{v}$ and $\mathbf{p}$). Now we show a calculus based proof. We can derive it directly from the definition:

  $$\begin{aligned}
  \rho =& \sqrt{\Delta x^2 + \Delta y^2} \\
  \implies \dot\rho =& \frac{(\Delta x)(\Delta\dot x) + (\Delta y)(\Delta\dot y)}{\sqrt{\Delta x^2 + \Delta y^2}} \\
  =& \frac{\mathbf{p}^T\mathbf{v}}{\|\mathbf{p}\|} \\
  =& \left(\frac{\mathbf{p}}{\|\mathbf{p}\|}\right)^T\left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right)v
  \end{aligned}$$

  The inner product between two unit vectors is defined to be the cosine angle between them. We defined $\alpha$ as the angle between $\mathbf{p}$ and $\mathbf{v}$. Thus:

  $$= (\cos(\alpha))v$$

  This is what we wanted to prove, so we are done with $\rho$.

- We will prove the formula for $\beta$ before $\alpha$. For $\beta$, we see that the angle is changing with the perpendicular component of the velocity, which has length $v\sin(\alpha)$. However the same speed gives different changes in $\beta$ inversly proportional the distance $\rho$. The sign change comes from the fact that $\beta$ is a clockwise angle. We formalize using the atan2 definition of $\beta$.

  $$\begin{aligned}
  & \tan(-\beta) = \frac{\Delta y}{\Delta x} \\
  \implies & -\sec^2(-\beta)\dot\beta = \frac{\Delta\dot y\Delta x - \Delta y\Delta\dot x}{(\Delta x)^2} && \text{(Using } \dot{(\tan(x))} = \sec^2(x)\text{)} \\
  \implies & -\left(1+\tan^2(x)\right)\dot\beta = \frac{\Delta\dot y\Delta x - \Delta y\Delta\dot x}{(\Delta x)^2} && \text{(Using } 1+\tan^2(x) = \sec^2(x)\text{)} \\
  \implies & -\left(1+\frac{(\Delta y)^2}{(\Delta x)^2}\right)\dot\beta = \frac{\Delta\dot y\Delta x - \Delta y\Delta\dot x}{(\Delta x)^2} && \text{(Using definition of } \beta\text{)} \\
  \implies & -\left((\Delta x)^2 + (\Delta y)^2\right)\dot\beta = \Delta\dot y\Delta x - \Delta y\Delta\dot x && \text{(Rearranging terms)} \\
  \implies & -\rho^2\dot\beta = \begin{pmatrix}\Delta\dot x \\ \Delta\dot y\end{pmatrix}^T\begin{pmatrix}\Delta y \\ -\Delta x\end{pmatrix} && \text{(Using definition of } \rho \text{ and dot product)}
  \end{aligned}$$

The vector on the left of the product is just $\mathbf{v}$, the one on the right is just $\mathbf{p}$ rotated $\pi/2$ radians (90 degrees) couterclockwise. We denote this vector $\tilde{\mathbf{p}}$. Due to this rotation, the angle between $\mathbf{v}$ and $\tilde{\mathbf{p}}$ is $\pi/2 - \alpha$ (so it still has norm $\rho$). This formalizes our notion of the perpendicular component of $\mathbf{v}$ in the $\mathbf{p}$ direction. Note that we could have just as easily picked the vector coordinates so that $\mathbf{v}$ was rotated, so it doesn't matter which we chose to rotate in principle.

$$\begin{aligned}
\implies && -\rho^2\dot{\beta} &= \mathbf{v}^T\tilde{\mathbf{p}} && \text{(Names from above)} \\
\implies && -\rho\dot{\beta} &= \left(\frac{\mathbf{v}}{v}\right)^T \left(\frac{\tilde{\mathbf{p}}}{\rho}\right) v && \text{(Rearranging terms)} \\
\implies && \dot{\beta} &= -\frac{\cos\left(\frac{\pi}{2} - \alpha\right)}{\rho} v && \text{(Definitions from above and rearranging terms)} \\
\implies && \dot{\beta} &= -\frac{\sin(\alpha)}{\rho} v && \text{(Trigonometric identities)}
\end{aligned}$$

Which is our desired formula

- For $\alpha$, the intuition is that it is the negative of $\beta$ in the rotating reference frame of the robot, so the additive offset of $-\omega$ must be added. The calculus proof for this is straightforward. We use the alternate definition of $\beta$

$$\begin{aligned}
&& \beta &= -\alpha - \theta && \\
\implies && \alpha &= -\beta - \theta && \\
\implies && \dot{\alpha} &= -\dot{\beta} - \dot{\theta} && \\
\implies && \dot{\alpha} &= -\left(-\frac{\sin(\alpha)}{\rho} v\right) - \omega && \text{(Definition of } \dot{\omega} \text{ and } \dot{\beta}\text{)} \\
\implies && \dot{\alpha} &= \frac{\sin(\alpha)}{\rho} v - \omega &&
\end{aligned}$$

Which is the definition we wanted.