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## Week 1-2: Base 2, 8 and 16

- Place value for digits:
  - $110110_2 = 1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 = 32 + 16 + 4 + 2 = 54_{10}$
  - $111111_2 = 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 32 + 16 + 8 + 4 + 2 + 1 = 63_{10}$
  - $111111_2 = 1000000_2 - 1_{10}$
- Decimals to binary:  $193_{10} = 1100001_2$ 
  - $193/2 = 96 + 1$
  - $96/2 = 48 + 0$
  - $48/2 = 24 + 0$
  - $24/2 = 12 + 0$
  - $12/2 = 6 + 0$
  - $6/2 = 3 + 0$
  - $3/2 = 2 + 1$
  - $1/2 = 0 + 1$

	1024	512	256	128	64	32	16	8	4	2	1
199	0	0	0	1	1	0	0	0	1	1	1
313	0	0	1	0	0	1	1	1	0	0	1
488	0	0	1	1	1	1	0	1	0	0	0
1025	1	0	0	0	0	0	0	0	0	0	1

- Place value for fractional numbers
  - . = fractional point

Binary	8	4	2	1	1/2	1/4	1/8	Decimal
1101.101	1	1	0	1.	1	0	1	13 5/8

Decimal	256	128	64	32	16	8	4	2	1	1/2	1/4	1/8	Binary
271.25	1	0	0	0	0	1	1	1	1.	0	1		10001111.01
0.75									0.	1	1		0.11

- $0.25_{10}$  in binary: 0.01
  - $0.25 \times 2 = 0.5$
  - $0.5 \times 2 = 1.0$  <- stop
- $0.75_{10}$  in binary: 0.11
  - $0.75 \times 2 = 1.5$
  - $0.5 \times 2 = 1.0$  <- stop
- Rational and irrational numbers
  - $0.117 \rightarrow 117 / 1000$
- Rational decimals to binary
  - $1/6 < 1$  0
  - $1/6 \times 2 = 1/3 < 1$  0
  - $1/3 \times 2 = 2/3 < 1$  0 repeat
  - $2/3 \times 2 = 4/3 > 1$  1 repeat
  - $1/3 \times 2 = 2/3 < 1$  0
  - $1/6 = 0.0\bar{0}1$

Total conversion:

296 (10->2)	296 (10->8)	296 (10->16)
296/2 = 148+ 0	296/8 = 37+ 0	296/16 = 18+ 8
148/2 = 74+ 0	37/8 = 4+ 5	18/16 = 1+ 2
74/2 = 37+ 0	4/8 = 0+ 4	1/16 = 0+ 1
37/2 = 18+ 1		
18/2 = 9+ 0	100 101 000	1 0010 1000
9/2 = 4+ 1	100 4	1 1
4/2 = 2+ 0	101 5	0010 2
2/2 = 1+ 0	000 0	1000 8
1/2 = 0+ 1		
1 0010 1000	450	128

Arithmetic in binary

▪ Addition

Digit 1	(27)		1	1	0	1	1
Digit 2	(11)			1	0	1	1
Carry over			1		1	1	
Result	(38)	1	0	0	1	1	0

Digit 1	(22)		1	0	1	1	0
Digit 2	(15)			1	1	1	1
Carry over			1	1	1		
Result	(37)	1	0	0	1	0	1

▪ Multiplication

Digit 1	(7)				1	1	1
Digit 2	(5)				1	0	1
					1	1	1
	+			0	0	0	
	+		1	1	1		
Carry over			1	1			
Result	(35)	1	0	0	0	1	1

## Week 3: Modulo

- $a|b$ 
  - $a$  divides  $b$  if there is an integer  $c$  that  $b=ac$  ( $a \neq 0$ )
  - then  $a$  is a divisor of  $b$  and  $b$  is a multiple of  $a$
  - if  $0 < a < b$ ,  $a$  is a **proper divisor** of  $b$
- **trivial divisor** of  $n$  is a divisor that equals  $n$  or  $1$ ; else nontrivial divisor
- integer values
  - if  $a|b$  and  $a|c$ , then  $a|(b+c)$ ;                       $3|6$  and  $3|18$ , then  $a|(18+6)$
  - if  $a|b$ , then  $a|bc$  for any integer  $c$
  - if  $a|b$  and  $b|c$ , then  $a|c$ ;                       $3|6$  and  $6|12$ , then  $3|12$
- Prime
  - A positive integer  $n$  greater than  $1$  is called **prime**, if its only divisors are  $n$  and  $1$ .
  - If  $n$  is an integer  $\geq 1$ , then there is a prime  $p$  such that  $n < p \leq n!+1$
  - Given any real number  $X \geq 1$ , there exists a prime between  $x$  and  $2x$
  - If  $n$  is an integer  $\geq 2$ , then there are no primes between  $n!+2$  and  $n!+n$
- Composites
  - A positive integer  $n$  that is greater than  $1$  and is not prime is called **composite**
  - If  $n$  is a composite, then  $n$  has a prime divisor  $p$  such that  $p \leq \sqrt{n}$

## Theory of Congruences

- Modulo
  - Let  $a$  be an integer and  $n$  a positive integer greater than  $1$ ;  $r$  is the remainder of " $a \bmod n$ ";                      e.g.  $35 \bmod 12 = 11$
  - " $r$  is equal to  $a$  reduced modulo  $n$ "
  - " $a$  is congruent to  $b$  modulo  $n$ " ( $a \equiv b \pmod{n}$ ) if  $n$  is a divisor of  $a-b$  or if  $n|(a-b)$

Modular addition:  $(A + B) \bmod X = A \bmod X + B \bmod X$

Additive identity:  $A + B \equiv A \bmod C$

- $1 + X \equiv 0 \bmod 5$

Additive inverse:  $A + B \equiv 0 \bmod C$

- $-23 + B \equiv 0 \bmod 5$ ;  $B = 2$

Modular multiplication:  $(A * B) \bmod X = A \bmod X * B \bmod X$

- $(159 * 943) \bmod 5 = 159 \bmod 5 * 943 \bmod 5 = 4 \bmod 5 * 3 \bmod 5 = 12 \bmod 5 \equiv 2$
- $(-569 * -662) \bmod 10 = 376 \bmod 10 * 662 \bmod 10 = 6 \bmod 10 * 2 \bmod 10 \equiv 8$
- $206^9 \pmod{13} \equiv ???$   
 $206 \pmod{13} \equiv 11$   
 $206^2 \pmod{13} = 11^2 \pmod{13} = 121 \pmod{13} \equiv 4$   
 $206^4 \pmod{13} = 4^2 \pmod{13} = 16 \pmod{13} = 3$   
 $206^9 \pmod{13} = 206^4 * 206^4 * 206 \pmod{13} = 3 * 3 * 11 \pmod{13} = 99 \pmod{13} \equiv 8$

Multiplicative inverse:  $A * B \equiv 1 \pmod{X}$

- $3 * 5 \equiv 1 \bmod 7$
- $4 * 2 \equiv 1 \bmod 7$
- Must not be co-prime, e.g.  $2 * B \equiv \bmod 8$  ( $2$  and  $8$  ( $2*2*2*2$ ) are co-prime)

## Week 4: Sequences

### Arithmetic progressions:

-5	-3	-1	1	...	3	5	7	# 2n - 7
100	107	114	121	...	128	135	142	# 7n + 93

### Geometric progressions

2	4	8	16	...	32	64	128	# 2^n
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## Week 5: Series

### Arithmetic Series starting with i=0

Series with  $i^0$

$$\sum_{i=0}^n 3 = 3 * n \qquad \sum_{i=0}^{10} 3 = 3 * 10 = 30$$

Series with  $i^1$

$$\sum_{i=0}^n 3i = 3 * \frac{n(n+1)}{2} \qquad \sum_{i=0}^{10} 3i = 3 * \frac{10(10+1)}{2} = \frac{3}{2} * 110 = 165$$

Series with  $i^2$

$$\sum_{i=0}^n 3i^2 = 3 * \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=0}^{10} 3i^2 = 3 * \frac{10(10+1)(2*10+1)}{6} = \frac{1}{2} * 110 * 21 = 1155$$

Series with compounds

$$\sum_{i=0}^n 3i^2 + 5i + 7 = \sum_{i=0}^n 3i^2 + \sum_{i=0}^n 5i + \sum_{i=0}^n 7$$
$$\sum_{i=0}^n 3i^2 + 5i + 7 = 3 * \frac{n(n+1)(2n+1)}{6} + 5 * \frac{n(n+1)}{2} + 7 * n$$

### Arithmetic Series not starting with i=0

$$\sum_{i=10}^{n=20} 4i = \sum_{i=0}^{n=20} 4i - \sum_{i=0}^{n=9} 4i = 4 * \frac{20(20+1)}{2} - 4 * \frac{9(9+1)}{2} = 840 - 180 = 660$$

### Geometric series

$$\sum_{i=1}^n a * b^{i-1} = a * \sum_{i=1}^n b^{i-1} = a * \frac{b^i - 1}{b - 1}$$
$$\sum_{i=1}^5 0.1 * -\frac{1}{2}^i = 0.1 * \sum_{i=1}^5 -\frac{1}{2}^i = 0.1 * \frac{-\frac{1}{2}^5 - 1}{-\frac{1}{2} - 1} = 0.06875$$

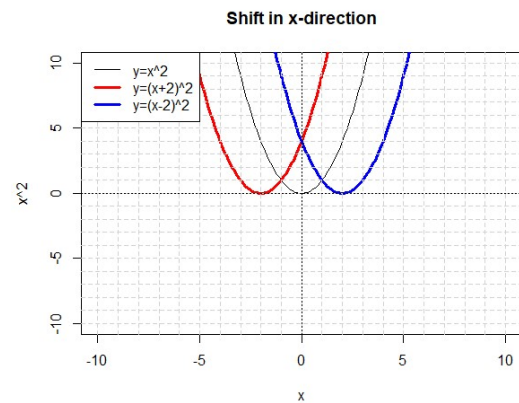
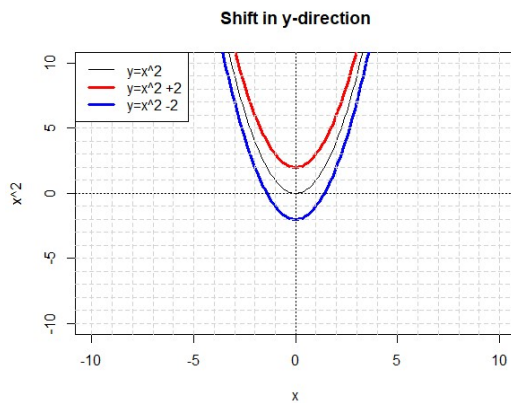
## Week 6: Transformations

Intervals:

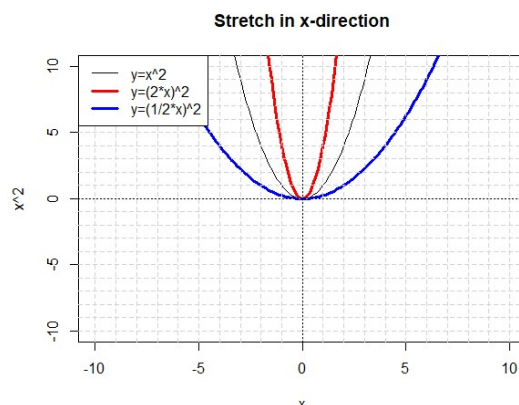
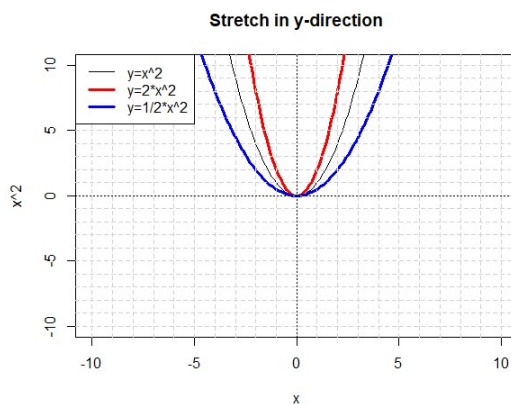
- Exclusive:  $(5, 8)$ :  $\{6, 7\}$
- Inclusive:  $[5, 8]$ :  $\{5, 6, 7, 8\}$
- Mixed:  $[5, 8)$ :  $\{5, 6, 7\}$

Translations:

- |                              |         |    |             |
|------------------------------|---------|----|-------------|
| ▪ Shift in y-direction (+2): | $y=x^2$ | -> | $y=x^2 + 2$ |
| ▪ Shift in y-direction (-2): | $y=x^2$ | -> | $y=x^2 - 2$ |
| ▪ Shift in x-direction (+2): | $y=x^2$ | -> | $y=(x-2)^2$ |
| ▪ Shift in x-direction (-2): | $y=x^2$ | -> | $y=(x+2)^2$ |



- |                                  |         |    |             |
|----------------------------------|---------|----|-------------|
| ▪ Stretch in y-direction (*1/2): | $y=x^2$ | -> | $y=(x^2)/2$ |
| ▪ Stretch in y-direction (*2):   | $y=x^2$ | -> | $y=2(x^2)$  |
| ▪ Stretch in x-direction (*1/2): | $y=x^2$ | -> | $y=(2x)^2$  |
| ▪ Stretch in x-direction (*2):   | $y=x^2$ | -> | $y=(x/2)^2$ |



<https://www.mathsisfun.com/sets/function-transformations.html>

## Week 7: Triangles

### Angles

- Acute: biggest angle  $< 90^\circ$
- Obtuse: biggest angle  $> 90^\circ$
- Right: biggest angle  $= 90^\circ$

### Degree to radians

- $1^\circ = \frac{1^\circ}{180^\circ} \pi \text{ rad} = 0.0174 \text{ rad}$
- $18^\circ = \frac{18^\circ}{180^\circ} \pi \text{ rad} = \frac{1}{10} \pi \text{ rad} = 0.314 \text{ rad}$
- $180^\circ = \frac{180^\circ}{180^\circ} \pi \text{ rad} = \pi \text{ rad} = 3.141 \text{ rad}$
- $360^\circ = \frac{360^\circ}{180^\circ} \pi \text{ rad} = 2\pi \text{ rad} = 6.283 \text{ rad}$

### Degree to radians

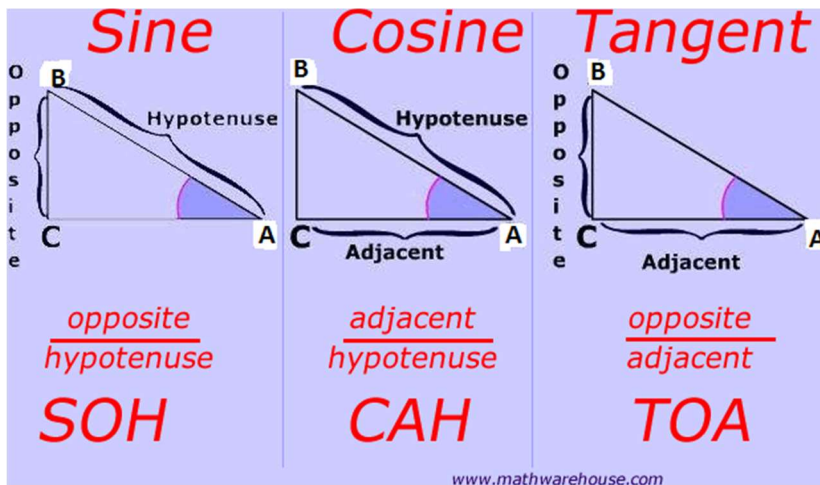
- $1 \text{ rad} = 1 * \frac{180^\circ}{\pi} = 57.296^\circ$
- $\pi \text{ rad} = \pi \frac{180^\circ}{\pi} = 180^\circ$
- $2\pi \text{ rad} = 2\pi \frac{180^\circ}{\pi} = 360^\circ$
- $\frac{5}{6} \pi \text{ rad} = \frac{5}{6} \pi * \frac{180^\circ}{\pi} = \frac{5}{1} * \frac{30^\circ}{1} = 150^\circ$
- $\frac{20}{9} \pi \text{ rad} = \frac{20}{9} \pi * \frac{180^\circ}{\pi} = \frac{20}{1} * \frac{20^\circ}{1} = 400^\circ$

### Squareroots / surds

- $\sqrt{9} = 3$
- $\sqrt{n * m} = \sqrt{n} * \sqrt{m}$ :  $\sqrt{20} = \sqrt{2 * 2 * 5} = \sqrt{2} * \sqrt{2} * \sqrt{5} = 2\sqrt{5} =$
- $\frac{\sqrt{n}}{\sqrt{m}} = \sqrt{\frac{n}{m}}$ :  $\frac{\sqrt{54}}{\sqrt{6}} = \sqrt{\frac{54}{6}} = \sqrt{9} = 3$  OR  $\frac{\sqrt{54}}{\sqrt{6}} = \frac{\sqrt{6*9}}{\sqrt{6}} = \frac{\sqrt{6}*\sqrt{9}}{\sqrt{6}} = \sqrt{9} = 3$
- $n\sqrt{x} + m\sqrt{x} = (n + m)\sqrt{x}$ :  $4\sqrt{5} + 7\sqrt{5} = 11\sqrt{5}$
- $\sqrt{18} - \sqrt{2} = 3\sqrt{2} - \sqrt{2} = 2\sqrt{2}$
- $3(\sqrt{4} + \sqrt{5}) = 3\sqrt{4} + 3\sqrt{5} = \sqrt{9*4} + \sqrt{9*5} = \sqrt{36} + \sqrt{45} = 6 + 3\sqrt{5}$
- $2\sqrt{2} * 5\sqrt{10} = \sqrt{4*2} * \sqrt{25*10} = \sqrt{8} * \sqrt{250} = \sqrt{2000} = 10\sqrt{20} = 20\sqrt{5}$

## Week 8: sin, cos, tan

Right triangles:



Any triangle:

- Sine rule:  $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
- Cosine rule:  $c^2 = a^2 + b^2 - 2 * a * b * \cos(C)$

Amplitude: half distance between maximum and minimum values

- $y = a * \sin(bx + c)$ ; amplitude =  $|a|$

Period: Horizontal spread of a full cycle

- $y = a * \sin(bx + c)$ ; period =  $|2\pi/b|$
- $y = a * \sin(bx + c)$ ; amplitude =  $|a|$

Phase: Horizontal shift

- $y = a * \sin(bx + c)$ ; period =  $c$



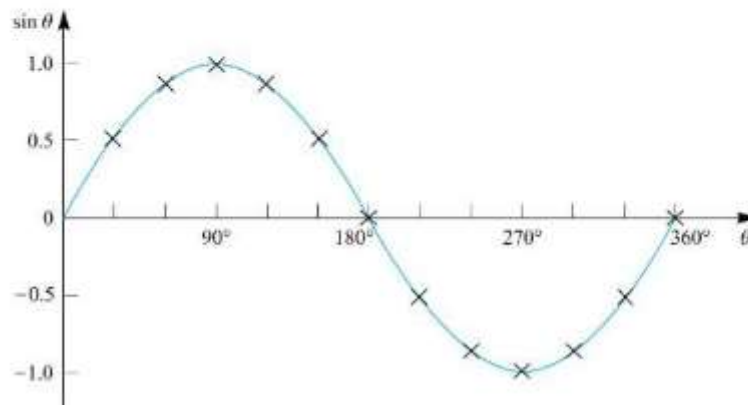
## Week 9: Trigonometric Functions

### Sinus

$\theta$	$0^\circ$	$30^\circ$	$60^\circ$	$90^\circ$	$120^\circ$	$150^\circ$	$180^\circ$
$\sin \theta$	0	0.5	0.8660	1	0.8660	0.5	0

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$\theta$	$210^\circ$	$240^\circ$	$270^\circ$	$300^\circ$	$330^\circ$	$360^\circ$
$\sin \theta$	-0.5	-0.8660	-1	-0.8660	-0.5	0

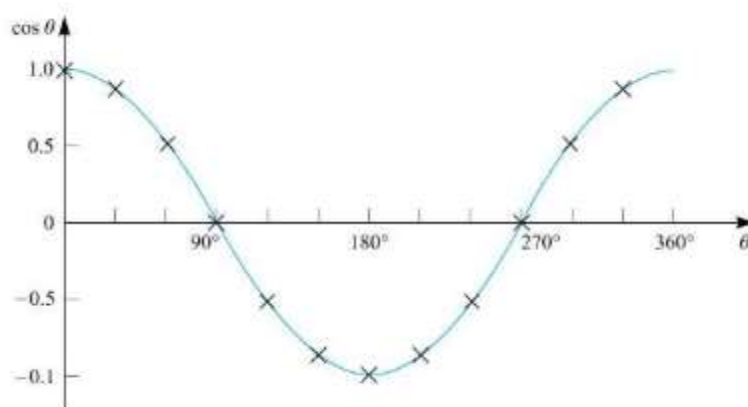


### Cosinus

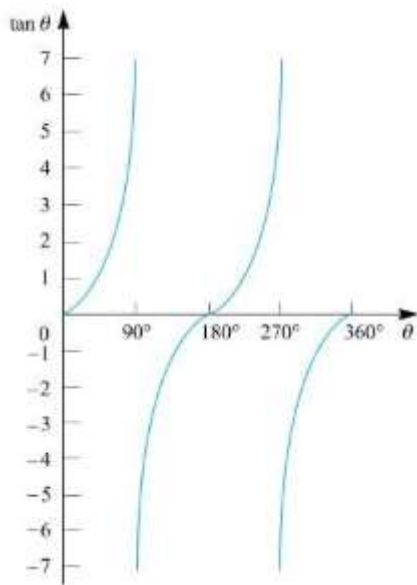
$\theta$	$0^\circ$	$30^\circ$	$60^\circ$	$90^\circ$	$120^\circ$	$150^\circ$	$180^\circ$
$\cos \theta$	1	0.8660	0.5	0	-0.5	-0.8660	-1

---

$\theta$	$210^\circ$	$240^\circ$	$270^\circ$	$300^\circ$	$330^\circ$	$360^\circ$
$\cos \theta$	-0.8660	-0.5	0	0.5	0.8660	1



## Tangens



Trigonometrical identities: The following equations are true for all angles:

- $\frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$
- $\sin^2 A + \cos^2 A = 1$
- $\theta = \frac{s}{r}$

Angle associated with coordinates

- $(-3,2)$ : Quadrant II,  $180^\circ - \tan(2/3) = 146.31^\circ$
- $(-3,-2)$ : Quadrant III,  $180^\circ + \tan(2/3) = 183.69^\circ$

Unit circle

- $x=4/5 \rightarrow y= \pm (3/5)$
- angle:  $-240^\circ \rightarrow 240^\circ - 180^\circ = 60^\circ$ ,  $x=\cos 60^\circ=0.5$ ,  $y=\sin 60^\circ=0.87$

## Week 10: Coordinate systems

### Polar to Cartesian

- Quadrant I:  $P(\text{radius} \cdot \cos(\text{angle}), \text{radius} \cdot \sin(\text{angle}))$
- Quadrant II:  $P(-\text{radius} \cdot \cos(\text{angle}), \text{radius} \cdot \sin(\text{angle}))$
- Quadrant III:  $P(-\text{radius} \cdot \cos(\text{angle}), -\text{radius} \cdot \sin(\text{angle}))$
- Quadrant IV:  $P(\text{radius} \cdot \cos(\text{angle}), -\text{radius} \cdot \sin(\text{angle}))$
- $P(11, 36.4^\circ)$ :  $x=11 \cdot \cos 36.4^\circ=8.85$ ,  $y=11 \cdot \sin 36.4^\circ=6.53$

### Cartesian to Polar

- $P(\text{radius}, \text{angle})$
- $\text{angle} = \tan(y/x)$
- $\text{radius} = (x^2 + y^2)^{0.5}$
- $P(3,8)$ :  $\text{theta}=\tan(8/3)=69.44^\circ$ ,  $r=(3^2 + 8^2)^{0.5}=8.54$ ,  $P(8.54, 69.44^\circ)$
- $P(-4,7)$ :  $\text{II}$ ,  $\text{theta} = 180-\tan(7/4)=119.75^\circ$ ,  $r=(4^2 + 7^2)^{0.5}=8.06$ ,  $P(8.06, 119.75^\circ)$

■

## Week 11: Exponential functions

- $a^x \dots$  a: base, x: power (or index)
- $n^2 * n^4 = n^{2+4} = n^6$
- $\frac{n^5}{n^3} = n^{5-3} = n^2$
- $(n^2)^3 = n^{2*3} = n^6$

## Week 12: Logarithms

- $y = a^x$  equivalent to:  $\log_a y = x$
- $\log A + \log B = \log AB$
- $\log A - \log B = \log A/B$
- $\log 1 = 0$
- $n \log A = \log A^n$
- $\log_a x = -\log_{y_a} x$
- Calculator:  $\log_n m = \log n / \log m$ :  $\log_{16} 2 = \log 16 / \log 2 = 4$

## Week 13-14: Limits and differentiation

### Limits

- $\lim_{n \rightarrow \infty} n = \infty$
- $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$
- $\lim_{n \rightarrow \infty} \frac{n+1}{n} = \frac{n}{n} = 1$
- $\lim_{n \rightarrow \infty} \frac{2^n}{10^{n-1}} = 0$

### Differentiation

Power rule:  $f(x) = x^n$   $f'(x) = \frac{1}{n} x^{n-1}$

- $f(x) = 2x^3$ ,  $f'(x) = \frac{2}{3} x^{3-1} = \frac{2}{3} x^2$
- $f(x) = \sqrt{x} = x^{\frac{1}{2}}$   $f'(x) = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$
- $f(x) = \frac{1}{x^2} = x^{-2}$   $f'(x) = -\frac{1}{2} x^{-2-1} = -\frac{1}{2x^3}$

Multiplication rule:  $f(x) = u * v$   $f'(x) = u'v + uv'$

- $f(x) = (x^2 - 3x)(4x - 1) = uv$   $f'(x) = (2x - 3)(4x - 1) + 4(x^2 - 3x)$

Division rule:  $f(x) = \frac{u}{v}$   $f'(x) = \frac{u'v - uv'}{v^2}$

- $f(x) = \frac{x^2 - 3x}{4x - 1} = \frac{u}{v}$   $f'(x) = \frac{(2x-1)(4x-1) - 4(x^2-3x)}{(4x-1)^2}$

Chain rule:  $f(x) = g(h(x))$   $f'(x) = g'(h(x)) * h'(x)$

- $f(x) = g(h(x)) = (x^3 + 12x^2)^2$   $g(x) = x^2, h(x) = x^3 + 12x^2$
- $f'(x) = g'(h(x)) * h'(x) = 2(x^3 + 12x^2) * (3x^2 + 24x)$

### Trigonometric functions:

- $f(x) = \sin(x)$   $f'(x) = \cos(x)$
- $f(x) = \cos(x)$   $f'(x) = -\sin(x)$
- $f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$   $f'(x) = \frac{1}{\cos^2(x)}$
- $f(x) = \sin(x^2)$   $f'(x) = 2x \cos(x^2)$
- $f(x) = x^2 \sin(x)$   $f'(x) = 2x \sin(x) + x^2 \cos(x)$
- $f(x) = \sin(x) \cos(x)$   $f'(x) = \cos(x)^2 - \sin(x)^2$

### Exponential functions:

- $f(x) = \ln(x)$   $f'(x) = \frac{1}{x}$
- $f(x) = \ln(x^2 - 2x) = \ln(x) + \ln(x - 2)$   $f'(x) = \frac{1}{x} + \frac{1}{x-2} = \frac{2(x-1)}{x(x-2)}$
- $f(x) = \log_3(x)$   $f'(x) = \frac{1}{x \ln 3}$
- $f(x) = e^x$   $f'(x) = e^x$
- $f(x) = e^{x^2 - 4x + 1}$   $f'(x) = (2x - 4)e^{x^2 - 4x + 1}$
- $f(x) = 0.1^x = (e^{\ln(0.1)})^x$   $f'(x) = \ln(0.1) 0.1^x$
- $f(x) = \frac{2^x}{3^{x-2}} = 9 \left(\frac{2}{3}\right)^x$   $f'(x) = 9 \ln\left(\frac{2}{3}\right) \left(\frac{2}{3}\right)^x$

## Extreme points

- Maxima:  $f'(x) = 0$   $f''(x) < 0$
- Minima:  $f'(x) = 0$   $f''(x) > 0$

## Week 15: Vectors and matrices

### Vectors

- $\vec{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$
- $\vec{AB} = B - A = \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
- $\vec{AB} + A = B - A + A = B$
- Magnitude  $|\vec{a}|$  of  $\vec{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ :  $|\vec{a}| = \sqrt{3^2 + 2^2} = \sqrt{13}$

Dot product:  $u * v = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} * \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = u_1 * v_1 + u_2 * v_2$

- $\begin{pmatrix} 1 \\ 2 \end{pmatrix} * \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 1 * 3 + 2 * 4 = 3 + 8 = 11$

Angle between two vectors:  $\cos(\theta) = \frac{u*v}{|u|*|v|}$

- $u = \begin{pmatrix} 6 \\ 3 \end{pmatrix}, v = \begin{pmatrix} 5 \\ 13 \end{pmatrix}$
- $\cos(\theta) = \frac{6*5+3*13}{\sqrt{6^2+3^2}*\sqrt{5^2+13^2}} = \frac{(30+39)}{\sqrt{36+9}*\sqrt{25+1}} = \frac{69}{\sqrt{45}*\sqrt{194}}, \theta = 42^\circ$

### Matrices

- 2 by 3:  $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$
- Transposed matrix:  $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$
- Identity matrices:  $Id_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, Id_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- Addition:  $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 3 & 5 \end{pmatrix}$
- Multiplication:
  - $X_1$  by  $X_2$ ,  $Y_1$  by  $Y_2$ , multiplication possible if  $X_1 = Y_2$
  - $(u_1 \ u_2 \ u_3 \ u_4) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = (u_1 * v_1 + u_2 * v_2 + u_3 * v_3 + u_4 * v_4)$
  - $(3 \ -3 \ -1 \ 4) \begin{pmatrix} 1 \\ 2 \\ -1 \\ 5 \end{pmatrix} = (3 * 1 + (-3) * 2 + (-1) * (-1) + 4 * 5) = (18)$
  - $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix} = \begin{pmatrix} aA + bD + cG & aB + bE + cH & aC + bF + cI \\ dA + eD + fG & dB + eE + fH & dC + eF + fI \\ gA + hD + iG & gB + hE + iH & gC + hF + iI \end{pmatrix}$
  - $\begin{pmatrix} -3 & 3 & 6 \\ 2 & -1 & 5 \\ -5 & -2 & 4 \end{pmatrix} \begin{pmatrix} -1 & 3 & -2 \\ 1 & 4 & -5 \\ 6 & 3 & 6 \end{pmatrix} = \begin{pmatrix} 42 & 21 & 27 \\ 27 & 17 & 31 \\ 27 & -11 & 44 \end{pmatrix}$

## Determinants 2x2

- $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$
- $\det \begin{pmatrix} -1 & 3 \\ 4 & 5 \end{pmatrix} = -1 * 5 - 3 * 4 = -5 - 12 = -17$
- The determinant of a 2x2 matrix is zero when one row is a multiple of the other row
- $\det \begin{pmatrix} -2 & 4 \\ 3 & -6 \end{pmatrix} = -2 * (-6) - 3 * 4 = 12 - 12 = 0$

## Determinants 3x3

- $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$
- $M = \begin{pmatrix} -1 & 3 & 3 \\ 5 & 3 & -6 \\ 3 & 4 & -3 \end{pmatrix}$
- $\det(M) = (+1) * (-1) \det \begin{pmatrix} 3 & -6 \\ 4 & -3 \end{pmatrix} + (-1) * 3 \det \begin{pmatrix} 5 & -6 \\ 3 & -3 \end{pmatrix} + (+1) * 3 \det \begin{pmatrix} 5 & 3 \\ 3 & 4 \end{pmatrix}$

## Week 16: Inverting matrices, cross product of vectors

### Inversion of matrices

1. Check dimensions: must be the same
  2. Calculate determinant (if 0, it is not invertible)
  3. Calculate determinants of smaller matrices obtained by removing one row and one column
  4. Transpose co-factor matrix
  5. Divide adjoint matrix by determinant
- <https://arndt-bruenner.de/mathe/scripts/inversematrix.htm>

1.  $M = \begin{pmatrix} 4 & -3 \\ 3 & 5 \end{pmatrix} : 2 \times 2$
2.  $\det(M) = 4 * 5 - (-3) * 3 = 20 + 9 = 29$
3.  $\frac{1}{29} \begin{pmatrix} 5 & 3 \\ -3 & 4 \end{pmatrix} :$

### Cross product of vectors

- $x = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, y = \begin{pmatrix} d \\ e \\ f \end{pmatrix}, x \times y = \begin{pmatrix} bg - cf \\ ce - ag \\ af - be \end{pmatrix}$   $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} h & i \\ f & g \end{vmatrix} - b \begin{vmatrix} d & i \\ g & e \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$
- $a = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}, b = \begin{pmatrix} -6 \\ -6 \\ 2 \end{pmatrix}, a \times b = \begin{pmatrix} 3 * 2 - (-1 * (-6)) \\ -1 * (-6) - 3 * 2 \\ 3 * (-6) - 3 * (-6) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

### Multiplication of matrices and vectors

- $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1a + 2b + 3c \\ 1d + 2e + 3f \\ 1g + 2h + 3i \end{pmatrix}$
- $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 * 2 + 0 * 3 + 0 * 4 \\ -1 * 2 + 1 * 3 + 0 * 4 \\ 0 * 2 + 1 * 3 + 2 * 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 11 \end{pmatrix}$

## Week 17: Linear transformation and matrices

### Scaling

- No scaling:  $Id_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- 2x in x (dilation):  $Id_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$
- 2x in y (dilation):  $Id_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$
- $\frac{1}{2}$ x in x (dilation):  $Id_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$

### Reflection

- No reflection:  $Id_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- Reflection on y-axis:  $M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
- Reflection on x-axis:  $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- Reflection on x-axis:  $M = \begin{pmatrix} \cos 60^\circ & \sin 60^\circ \\ \sin 60^\circ & -\cos 60^\circ \end{pmatrix}$

### Rotation

- No rotation:  $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- Anticlockwise by  $90^\circ$ :  $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{pmatrix}$
- Anticlockwise by  $30^\circ$ :  $M = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{pmatrix}$

### Combinations

- 2x dilation in x, 3x dilation in y, rotation on y-axis:  $M = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$
- 2x dilation in x, 3x dilation in y, rotation by  $180^\circ$ :  $M = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}$

### Homogeneous coordinates

- $M = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Rightarrow M = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$
- $M = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \Rightarrow M = \begin{pmatrix} \frac{2}{4} \\ \frac{3}{4} \\ \frac{4}{4} \end{pmatrix} \Rightarrow M = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{4} \\ 1 \end{pmatrix}$
- $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow M = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

### Translations

- Translation 2 in x and 3 in y direction:  $M = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

## Week 18: Affine transformation in homogeneous coordinates

### Homogeneous coordinates

- $M = \begin{pmatrix} 4 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \Rightarrow M = \begin{pmatrix} 4 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ , dilation with scaling 4x x and 1/2x y direction.
- $M^{-1} = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- $P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow M = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ , rotation 90° anticlockwise
- $P^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ , rotation 90° clockwise

### Combining translations and linear transformations

- $M = \begin{pmatrix} 2 & 0 & -1 \\ 0 & \frac{1}{3} & 5 \\ 0 & 0 & 1 \end{pmatrix}$ , translation by  $\begin{pmatrix} -1 \\ 5 \end{pmatrix}$ , dilation x: 2, y: 1/3
- $M^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 1 \\ 0 & 3 & -5 \\ 0 & 0 & 1 \end{pmatrix}$
- $P = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ , rotation 90° anticlockwise, translation by  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

### Solving equations using matrices

- $\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$
- $\begin{pmatrix} x - y = 1 \\ 2x + y = 3 \end{pmatrix}$



## Week 19: Combinatorics and probability

### Permutations (ordered combinations)

- Permutation of  $n$  objects is  $n!$ 
  - X, Y, Z: XYZ, XZY, YXZ, YZX, ZXY, ZYX ( $3! = 6$ )
- Select  $m$  of  $n$  objects:  $\frac{n!}{(n-m)!}$ 
  - 4 of 6:  $\frac{6!}{(6-4)!} = \frac{6!}{2!} = 6 * 5 * 4 * 3 = 360$
  - Distinct combinations of TOTTENHAM:
    - 9 letters: 3\*T, 1\*O, 1\*E, 1\*N, 1\*H, 1\*A, 1\*M
    - $\frac{9!}{3!1!1!1!1!1!} = \frac{9!}{3!} = 60480$
  - Distinct combinations of LONDON:
    - 6 letters: 1\*L, 2\*O, 2\*N, 1\*D
    - $\frac{6!}{2!2!} = 180$

### Combinations: Sets of objects (order does not matter)

- ${}^nC_m = \frac{n!}{m!(n-m)!}$
- 3 cards out of 52:  ${}^{52}C_3 = \frac{52!}{3!(52-3)!} = \frac{52!}{3!(49)!} = \frac{52*51*50*49!}{1*2*3*49!} = 25 * 17 * 52 = 22\ 100$
- 2 of 5 and 3 of 10:  ${}^{15}C_2 * {}^{10}C_3 = \frac{15!}{2!13!} * \frac{10!}{3!7!} = \frac{14*15}{2!} * \frac{(8*9*10)}{3!} = 105 * 120$

### Independent events:

- Having 1 Heads from coin toss:  $\frac{1}{2}$
- Having 2 Heads from two coin tosses:  $\frac{1}{2^2} = \frac{1}{4}$

### Probability of mutually exclusive events

- Two mutually exclusive events:  $P(A \cup B) = P(A) + P(B)$  ...or:  $\cup$
- Two not mutually exclusive events:  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$  ... and:  $\cap$ 
  - Draw King (A) or Spades (B):  $\frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{4}{13}$

### The complement of an event: $P(A') = 1 - P(A)$

- 15 C cakes, 10 V cakes:
  - Chance of not getting a C cake:  $P(A') = 1 - P(A) = 1 - \frac{15}{25} = \frac{10}{25}$

## Week 20: Conditional probability

### Conditional probability

- $P(A \cap B) = P(A|B) * P(B)$
- $P(A|B) = \frac{P(A \cap B)}{P(B)}$

### Mean and standard deviations

- Mean:
  - sum of values divided by number of values
  - 2.4, 3.4, 5.1, 6.7, 3.0, 2.2, 3.3, 3.5 ... mean = 3.7
- Median:
  - central value (or mean of two central values)
  - 2.2, 2.4, 3.0, 3.3, 3.4, 3.5, 5.1, 6.7 ... median = 3.35
- Mode: most common value
- Standard deviation:
  - Mean squared distance:  $s = \sqrt{\frac{\sum_{i=1}^N (x_i - \bar{x})^2}{N-1}}$
  - 2.2, 2.4, 3.0, 3.3, 3.4, 3.5, 5.1, 6.7 ... median  $s = \sqrt{\frac{\sum_{i=1}^N (x_i - 3.7)^2}{N-1}} = \sqrt{\frac{15.68}{7}} = 1.497$