CM1015: Numerical Mathematics

Summary

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1 Number bases and modular arithmetic

Calculator Hint: Use $\boxed{\text{base n}}$ to convert a number between bases.

1.1 Decimal to Any Base

- The way to convert decimal numbers to any base is to divide by the base, e.g. 2 (or 8, or 16) repeatedly and note the remainder.
- To obtain the converted number we write out the remainder, reading from the bottom one to the top one.

Example Convert 32 from decimal to binary.

$$\begin{aligned}
32 &\equiv 0 \pmod 2\\
16 &\equiv 0 \pmod 2\\
8 &\equiv 0 \pmod 2\\
4 &\equiv 0 \pmod 2\\
2 &\equiv 0 \pmod 2\\
1 &\equiv 1 \pmod 2
\end{aligned}$$

1.2 Other Bases to Decimal

- Evaluate the place values of the non-decimal number one by one.

1.3 Converting Fractional Parts

- Decimal numbers are converted using normal place value evaluation. The fractional digits are negative powers of 2 in binary ($2^{-1}, 2^{-2}$ and so on) or another base
- Other base numbers are separated into a fractional part and an integer part. The integer part is converted using the division algorithm above. The fractional part is multiplied by 2 over and over again, and the integer part of the results are noted as the digits, until the result of the multiplication has no factional part anymore.

Example Convert the decimal fraction 0.15 to binary.

$$
\begin{aligned}
0.15 \cdot 2 &= 0.3 \\
0.3 \cdot 2 &= 0.6 \\
0.6 \cdot 2 &= 1.2 \\
0.2 \cdot 2 &= 0.4 \\
0.4 \cdot 2 &= 0.8 \\
0.8 \cdot 2 &= 1.6 \\
0.6 \cdot 2 &= 1.2
\end{aligned}
$$

The fractional part is *repeating* as $0.00\overline{1001}$.

1.4 Adding and Subctracting in Other Bases

- Addition is straightforward, resulting in carry over digits
- Subtraction requires *borrowing digits* from higher place values and striking out the higher place values

1.5 Multiplication in Other Bases

- Multiply the entire top number with the first digit (from the right) of the bottom number
- Do this with each digit of the bottom number, but shift over the result by one place value to the left each time

1.6 Conversion from binary to other bases

- Group the binary digits in 3-digit octets or 4-digit doublets and convert the groups to their respective bases. **Important:** Remove any leading or trailing zeroes. This will shorten octets or doublets but the number will be valid.

1.7 Modular Arithmetic

Encryption & Decryption Given a message M, we encrypt is to C using the public encryption keys e and p.

$$
\begin{aligned}
C &\equiv M^e \pmod{p} \\
M &\equiv C^d \pmod{p}
\end{aligned}
$$

Fermat's Little Theorem Given a prime number p, for any integer a:

$$a^p \equiv a \pmod{p}$$
$$a^{p-1} \equiv 1 \pmod{p}$$
$$a^{p-2} \equiv a^{-1} \pmod{p}$$

Finding new encryption keys While encryption and decryption is done in one modulo p, e.g. (mod 23), finding new keys happens in a modulo $p-1$, e.g. (mod 22). We need to find the multiplicative inverse of $a \pmod{p-1}$. If $p-1$ is a prime number, Fermat's little theorem applies.

Otherwise, using the Euclidian table, we can look up the *special power* ***sp***. Otherwise the special power is $p-1$.

Therefore, we calculate the inverse as follows:

$$e^{sp} \equiv 1 \pmod{p-1}$$
$$e^{sp-1} \cdot e \equiv 1 \pmod{p-1}$$
$$e^{-1} \equiv e^{sp-1} \pmod{p-1}$$

2 Sequences and Series

Calculator Hint: Use table to generate a list of n terms of a sequence. Notations using Σ can be done with math sum

1. An **infinite sequence** is a function whose domain is the set of positive integers $a_1, a_2, a_3 \ldots$. A finite sequence is a sequence whose domain is restricted to only the first n positive integers.

2. If the sequence is alternating in sign, this can be expressed by multiplying the sequence by $(-1)^{n+1}$ if the first term is positive, and $(-1)^n$ if the first term is negative.

3. Note that sequences often use factorial notation $n!$

4. 0! is defined as 1.

5. When simplifying a fraction with factorial expressions, simply start expanding the factorial to identify common terms and divide out.

2.1 Interval Notation

1. A **closed interval** includes the endpoints and is noted with square brackets, e.g. [01]

2. An **open interval** excludes the endpoints and is noted with round brackets, e.g. (01)

3. An **unbounded interval** has ∞ as one of the endpoints, e.g. $[0\infty)$

2.2 Arithmetic Sequences

A sequence is arithmetic when the **difference** between consecutive terms is the same. The nth term of an arithmetic sequence is given by

$$a + (n-1)d \tag{1}$$

where a is the first term and d is the *common difference.*

2.3 Geometric Sequences

A sequence is geometric when the **ratio** between consecutive terms is the same. The nth term of a geometric sequence is given by

$$a_n = a_0 r^n \tag{2}$$

where a_0 is the first term and r is the *common ratio.*

2.4 Sums

The following summation formulas describe the most common types of sums and their algebraic expansions.

$$\sum_{i=1}^{n} c = cn \tag{3}$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \tag{4}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \tag{5}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \tag{6}$$

$$\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i \tag{7}$$

$$\sum_{i=1}^{n} k a_i = k \sum_{i=1}^{n} a_i, k \text{ is a constant.} \tag{8}$$

2.5 Finite Series of Arithmetic Sequences

A *series* is a sum of a sequence.

$$S_n = \frac{n}{2}(2a + (n-1)d) \tag{9}$$

$$S_n = \frac{n}{2}(a_1 + a_n) \tag{10}$$

2.6 Finite Series of Geometric Sequences

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r} \tag{11}$$

In order to apply this formula the sum needs to be of the form $\sum_{i=1}^{n} ar^{n-1}$. **Note:** When evaluating a sum Σ, be sure to count the terms before applying this formula. For example, $\sum_{n=0}^{20}$ has 21 terms.

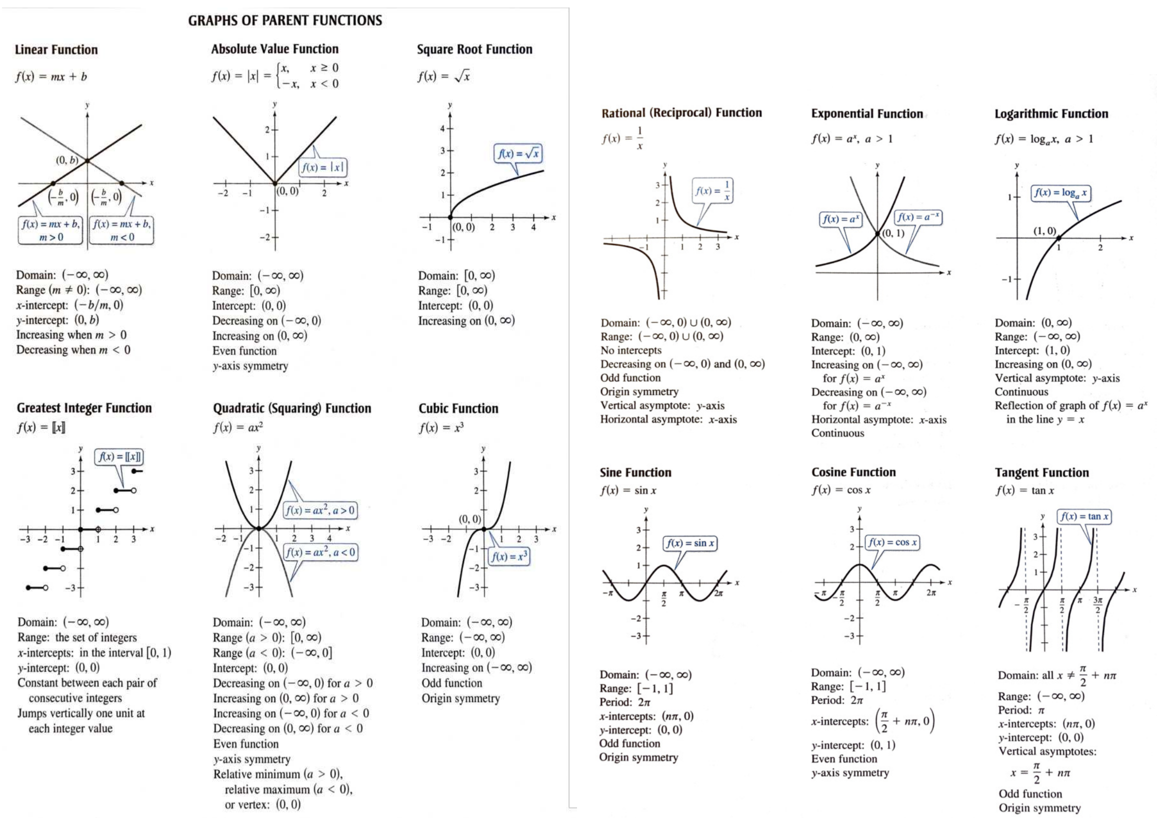
2.7 Ininite Series of Geometric Sequences

$$S_\infty = \frac{a}{1 - r} \quad \text{provided } \; -1 < r < 1 \tag{12}$$

Note that in order to apply this formula the sum needs to be of the form $\sum_{i=1}^{\infty} a_1 r^n$. If $|r| > 0$, the sequence is **divergent** and therefore the series is undefined. This may sometimes be referred to as having an *infinite sum*.

3 Graphs & Kinematics

3.1 Parent Graphs



3.2 Transformations

Let c be a positive real number. Vertical and horizontal shifts in the graph of $y = f(x)$ are represented as follows.

1. Vertical shift c units *up*: $h(x) = f(x) + c$

2. Vertical shift c units *down*: $h(x) = f(x) - c$

3. Horizontal shift c units to the right: $h(x) = f(x - c)$

4. Horizontal shift c units to the left: $h(x) = f(x + c)$

5. Reflection on x-axis: $h(x) = -f(x)$

6. Reflection on y-axis: $h(x) = f(-x)$,

7. Vertical Dilation: $h(x) = cf(x), |c| > 1 = \text{stretch}, |c| < 1 = \text{shrink}$

8. Horizontal Dilation: $h(x) = f(cx), |c| > 1 = \text{shrink}, |c| < 1 = \text{stretch}$

Note: Order matters, therfore note e.g. if reflections need to be applied to the entire function after or before other transformations.

3.3 Inverse Functions

1. Rewrite $f(x)$ using y in place of x

2. Solve for y. The new function is the inverse function $f^{-1}(x)$ of $f(x)$

3.4 Kinematics & SUVAT

The kinematic quantities are:

Variable	Quantity
s	displacement
u	initial velocity
v	final velocity
a	acceleration
t	time taken for the change in velocity

The four *SUVAT* equations are:

$$v = u + at \tag{13}$$

$$v^2 = u^2 + 2as \tag{14}$$

$$s = ut + \frac{1}{2}at^2 \tag{15}$$

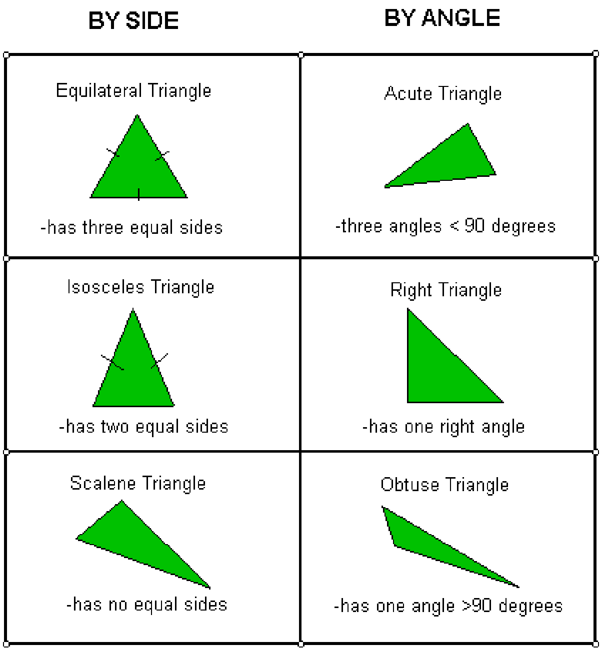
$$s = v - \frac{1}{2}at^2 \tag{16}$$

$$s = \frac{1}{2}(u + v)t \tag{17}$$

4 Angles & Triangles

4.1 Types of Triangles

The following lists the common types of triangles.



4.2 Triangle Invariants

Angle Sum The sum of all angles in a triangle is always 180° or π rad.

$$\alpha + \beta + \gamma = 180° = \pi\text{rad}$$

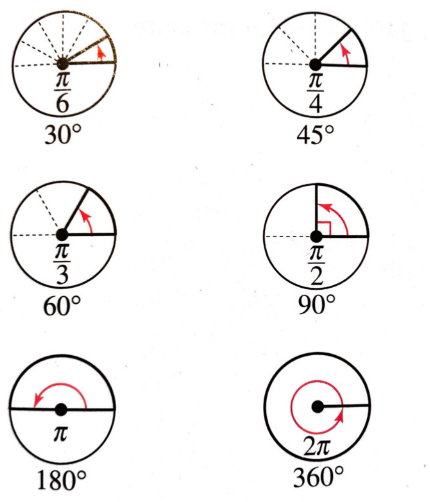
Triangle Inequality The sum of two sides is strictly larger than the third side.

$$a + b > c$$

4.3 Radians and Degrees

Calculator Hint: Use [math] [DMS] to convert between radians and degrees.

1. To convert degrees to radians, multiply degrees by $\frac{\pi\text{rad}}{180°}$
2. To convert radians to degrees, multiply radians by $\frac{180°}{\pi\text{rad}}$



4.4 Working with Surds

Working with roots (surds, radicals), these properties should be kept in mind.

$$
\begin{aligned}
\sqrt[n]{a^m} &= (\sqrt[n]{a})^m \\
\sqrt[n]{a} \cdot \sqrt[n]{b} &= \sqrt[n]{ab} \\
\frac{\sqrt[n]{a}}{\sqrt[n]{b}} &= \sqrt[n]{\frac{a}{b}}, \quad b \neq 0 \\
\sqrt[m]{\sqrt[n]{a}} &= \sqrt[mn]{a} \\
\sqrt[n]{a}^n &= a \\
\sqrt[n]{a^n} &= |a|, n \\
\sqrt[n]{a^n} &= a \\
a^{1/n} &= \sqrt[n]{a}
\end{aligned}
$$

5 Trigonometric Functions

5.1 Trigonometric Ratios in Right Triangles

The main trigonometric rations are as follows.

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \tag{18}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \tag{19}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \tag{20}$$

Further, the following identities are important.

$$\sin^2 A + \cos^2 A = 1 \tag{21}$$
$$\frac{\sin A}{\cos A} = \tan A \tag{22}$$
$$\sin\theta = -\sin(\theta - 180^\circ) \tag{23}$$
$$\cos\theta = -\cos(\theta - 180^\circ) \tag{24}$$
$$\tan\theta = \tan(\theta - 180^\circ) \tag{25}$$
$$\sin\theta = -\sin(360^\circ - \theta) \tag{26}$$
$$\cos\theta = \cos(360^\circ - \theta) \tag{27}$$
$$\tan\theta = -\tan(360^\circ - \theta) \tag{28}$$
$$\tag{29}$$

5.2 Trigonometric Laws

5.2.1 Law of Sines

The sine rule states

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \tag{30}$$

This rule can *only* be used when given two angles and one side, or two sides and a non-included angle.

5.2.2 Law of Cosines

The cosine rule states

$$a^2 = b^2 + c^2 - 2bc\cos A \tag{31}$$
$$b^2 = a^2 + c^2 - 2ac\cos B \tag{32}$$
$$c^2 = a^2 + b^2 - 2ab\cos C \tag{33}$$
$$\tag{34}$$

This rule is used when given three sides or two sides and the included angle.

5.2.3 Application of the Laws

When dealing with the case SSA with enclosed angle, the law of sines can produce two angles as possible answers. In this case:

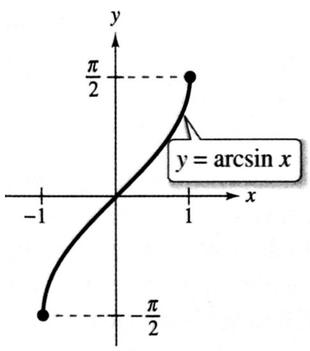
1. See if you are given two sides and the angle not in between (SSA). This is the situation that may have 2 possible answers.

Angles / Sides	1 Side	2 Sides	3 Sides
1 Angle	not solvable	Encl.: Sines, Unencl.: Cosines	Law of Cosines
2 Angles	Law of Sines	Law of Cosines	Law of Cosines
3 Angles	Law of Sines	Law of Sines	Solved

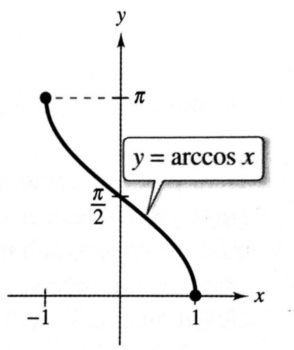
2. Find the value of the unknown angle.
3. Once you find the value of your angle, subtract it from 180° to find the possible second angle.
4. Add the new angle to the original angle. If their sum is less than 180°, you have two valid answers. If the sum is over 180°, then the second angle is not valid.

5.3 Inverses of Trigonometric Functions

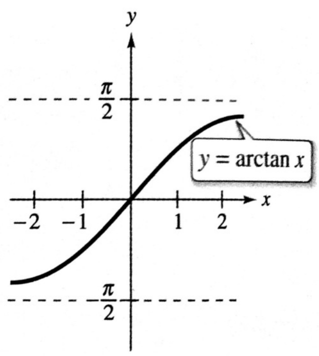
To find an inverse, we need to find a part of the function that is one-to-one. Hence, we restrict the domain of the three trigonmetric functions. Domain and range are flipped.



Domain: $[-1, 1]$
Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
Intercept: $(0, 0)$
Symmetry: origin
Odd function



Domain: $[-1, 1]$
Range: $[0, \pi]$
y-intercept: $\left(0, \frac{\pi}{2}\right)$



Domain: $(-\infty, \infty)$
Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
Horizontal asymptotes: $y = \pm\frac{\pi}{2}$
Intercept: $(0, 0)$
Symmetry: origin
Odd function

5.4 Transformations of Trigonometric Functions

Let c be a positive real number. Vertical and horizontal shifts in the graph of $y = f(x)$ are represented as follows.

1. Vertical shift c units *up*: $h(x) = \sin(x) + c$
2. Vertical shift c units *down*: $h(x) = \sin(x) - c$
3. Horizontal shift c units to the right: $h(x) = \sin(x - c)$
4. Horizontal shift c units to the left: $h(x) = \sin(x + c)$
5. Reflection on x-axis: $h(x) = -\sin(x)$
6. Reflection on y-axis: $h(x) = \sin(-x)$
7. Vertical Dilation, c is the *amplitude and range*: $h(x) = c\sin(x)$
8. Horizontal Dilation, c adjusts the *period*: $h(x) = f(cx)$

Remember that sin and cos are horizontal translations of each other by $\frac{\pi}{2}$.

5.5 Solving Trigonometric Equations

1. Check if the question is asking for values of x within $[0, 360°]$ or $[-360°, 360°]$
2. After solving the equation, ensure covering all possible angles. To do this, find the *corresponding angle* by finding the other angle with the same sign.
3. For $\sin(\theta)$, the corresponding angle is $180° - \theta$
4. For $\cos(\theta)$, the corresponding angle is $360° - \theta$
5. For $\tan(\theta)$, the corresponding angle is $\theta + 180°$
6. Finally, calculate two further angles by subtracting $360k, k = -1$ from them (if these are in range).

5.6 Polar Coordinates

Calculator Hint: Use math R P to convert between polar and cartesian. Note that you can only receive output of one coordinate at a time (x, y, r, θ).

Cartesian to Polar

$$r = \sqrt{x^2 + y^2} \quad (35)$$

$$\theta = \tan^{-1}(\frac{y}{x}) \quad (36)$$

Polar to Cartesian

$$x = r\cos(\theta) \tag{37}$$

$$y = r\sin(\theta) \tag{38}$$

6 Exponential Functions & Logarithms

6.1 Exponential Functions

e is the exponential constant 2.71828... Exponential expressions can be simplified using the rules below.

$$\mathrm{e}^a\mathrm{e}^b = \mathrm{e}^{a+b} \tag{39}$$

$$\frac{\mathrm{e}^a}{\mathrm{e}^b} = \mathrm{e}^{a-b} \tag{40}$$

$$\mathrm{e}^0 = 1 \tag{41}$$

$$\left(e^a\right)^b = \mathrm{e}^{ab} \tag{42}$$

The exponential function is

$$y = e^x \tag{43}$$

- The exponential function is never negative.
- When $x = 0$, the function value is 1.
- As x increases, then e^x increases. This is known as *exponential growth*.

6.2 Logarithmic Functions

Logarithm $y = a^x$ and $\log_a y = x$ are equivalent. The notation $\log_5 125 = 3$ is read as "The logarithm to the base 5 of 125 is 3." The following identities are of importance.

$$\log_a X = \frac{\log_{10} X}{\log_{10} a} \tag{44}$$

$$\log_a X = \frac{\ln X}{\ln a} \tag{45}$$

$$\log_a a = 1 \tag{46}$$

$$\log_a 1 = 0 \tag{47}$$

6.2.1 Laws of Logarithms

Product Property $\log A + \log B = \log AB$

Quotient Property $\log A - \log B = \log \frac{A}{B}$

Power Property $n \log A = \log A^n$

Some useful identities to solve equations are listed below. Note that even if an equation has solutions, these may not be valid given the original equation (e.g. negative values in a logfunction). Use the **inverse** and **one-to-one** properties to solve these equations.

$$\ln(e^x) = x \tag{48}$$

$$e^{\ln(x)} = x \tag{49}$$

7 Limits & Differentiation

7.1 Limits

If $f(x)$ become arbitrarily close to a unique number L as x approaches c from either side, then the **limit** of $f(x)$ as x approaches c is L.

$$\lim_{x \to c} f(x) = L \tag{50}$$

7.1.1 Estimating Limits

1. Check what the variable is supposed to approach
2. Build a table of values that allows you to investigate the approached x-value
3. Investigate the limit. It may be that the function is not defined at the approached x-value. The limit may still exist.
4. Check if the limit can exist (see below).

7.1.2 Conditions Under Which Limits Do Not Exist

The limit of $f(x)$ as $x \to c$ does not exist when any of the conditions listed below are true.

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side of c.
2. $f(x)$ increases or decreases without bound as x approaches c.
3. $f(x)$ oscillates between two fixed values as x approaches c.

7.1.3 Basic Limits

Let b and c be real numbers and let n be a positive integer. Using the properties below, you can use **direct substitution** to evaluate the limit.

$$\lim_{x \to c} f(x) = f(c)$$

$$\lim_{x \to c} b = b \qquad \text{Limit of a constant function} \tag{51}$$

$$\lim_{x \to c} x = c \qquad \text{Limit of the identity function} \tag{52}$$

$$\lim_{x \to c} x^n = c^n \qquad \text{Limit of a power function} \tag{53}$$

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c} \qquad \text{Limit of a radical function} \tag{54}$$

7.1.4 Properties of Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the limits $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = K$

$$\lim_{x \to c} [bf(x)] = bL \tag{55}$$

$$\lim_{x \to c} [f(x) \pm g(x)] = L \pm K \tag{56}$$

$$\lim_{x \to c} [f(x)g(x)] = LK \tag{57}$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K} \quad , K \neq 0 \tag{58}$$

$$\lim_{x \to c} [f(x)]^n = L^n \tag{59}$$

7.1.5 Dividing out & Rationalising

1. If direct subsitution fails, e.g. it produces a quotient such as $\frac{0}{0}$, try factorising or rationalising (multiply by the conjugate) the denominator or numerator
2. Divide out any common factors

3. Use direct substitution on the newly formed function

7.1.6 Limits at Infinity

If r is a real positive number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

Given a function in rational form $\frac{f(x)}{g(x)}$, if the order of both numerator and denominator are the same, the limit is composed of the leading coefficients of the highest order.

Otherwise, use the following technique:

1. Divide both numerator and denominator by the highest power in the denominator
2. Evaluate the limits of individual terms; fractions with x in the denominator evaluate to 0.

7.2 Differentiation

A gradient function $f'(x)$describes the slope of the tangent line at any point of a function $f(x)$. It is also called the **first derivative**.

7.2.1 Common Functions

Table 1 lists common functions and their first derivatives.

7.2.2 Differentiation Rules

Table 2 lists the most important rules to find the derivative of a function.

7.2.3 Finding Minima & Maxima

Stationary points are found by setting the gradient function to 0, i.e. $f' = 0$.

Using the second derivative $f''(x)$, the point can be tested further:

1. If f'' is **positive** at a stationary point, then the point is a **minimum**.
2. If f'' is **negative** at a stationary point, then the point is a **maximum**.
3. If f'' is 0, this test does not tell us anything and the points to the left and right of $f'(x)$ should be examined. This might indicate a point of inflection.

7.2.4 Finding Asymptotes

Vertical Asymptotes These are found by finding points where a function $f(x)$ is undefined. In rational functions, these can be found by finding points where the denominator is 0.

Horizontal Asymptotes These are found by finding the limit of a function $f(x)$ as x approaches ∞. A function can at most have two horizontal asymptotes, one in each direction.

Table 1. *Common functions & their derivatives*

Function	$f(x)$	$f'(x)$
Constant	c	0
Line	x	1
Square	x^2	$2x$
Square Root	$\sqrt{x}$	$\frac{1}{2}x^{-\frac{1}{2}}$
Exponential	e^x	e^x
	a^x	$\ln(a)a^x$
Logarithms	$\ln(x)$	$\frac{1}{x}$
	$\log_a(x)$	$\frac{1}{\ln(a)x}$
Trigonometry	$\sin(x)$	$\cos(x)$
	$\cos(x)$	$-\sin(x)$
	$\tan(x)$	$\frac{1}{\cos^2(x)} = \sec^2$

8 Vectors & Matrices

8.1 Vectors

A vector has both magnitude and direction. A **unit vector** has magnitude 1. When a vector is multplied by a scalar, it's magnitude is manipulated and the direction remains.

The unit vector $\hat{a}$ in direction of $\vec{a}$:

$$\hat{a} = \frac{1}{|a|}\vec{a} \tag{60}$$

The magnitude of a vector:

Table 2. *Differentiation Rules*

Rule	$f(x)$	$f'(x)$
Multiplication by a constant	cf	$cf'(x)$
Power Rule	x^n	nx^{n-1}
Sum Rule	$f+g$	$f'+g'$
Difference Rule	$f-g$	$f'-g'$
Product Rule	fg	$f'g+fg'$
Quotient Rule	$\frac{f}{g}$	$\frac{f'g-fg'}{g^2}$
Reciprocal Rule	$\frac{1}{f}$	$-\frac{f'}{f^2}$
Chain Rule	$f(g(x))$	$f'(g(x))g'(x)$

$$|\mathbf{a}| = \sqrt{x^2+y^2} \tag{61}$$

8.1.1 Dot Product

The dot product of two vectors is a scalar quantity and represents the length of the projection of one vector onto another.

Given two vectors

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

the dot product is defined as follows.

$$\vec{a} \bullet \vec{b} = |a||b|\cos(\theta) \qquad \text{Geometric Definition} \tag{62}$$

$$\vec{a} \bullet \vec{b} = a_1b_1 + a_2b_2 + a_3b_3 \qquad \text{Algebraic Definition} \tag{63}$$

8.1.2 Cross Product

The cross product of a vector only exists for 2 vectors in a 3-dimensional space. Therefore it can be derived using the Laplace expansion for the 3×3 determinant. So, given two vectors $\vec{a} \times \vec{b}$, we can note a matrix such as

$$\begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \tag{64}$$

Then the cross product's resultant can be calculated using the minor determinants as follows.

$$\vec{a} \times \vec{b} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} i - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} j + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} k \tag{65}$$

This is a similar approach as the **Laplace expansion**.

A geometric definition of the **magnitude** of the cross product is given as

$$\|\vec{a} \times \vec{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta \tag{66}$$

8.2 Matrices

8.2.1 Systems of Linear Equations

A system of linear equations

$$\text{System:} \begin{cases} x - 4y + 3z = 5 \\ -x + 3y - z = -3 \\ 2x - 4z = 6 \end{cases} \tag{67}$$

can be written as an augmented matrix

$$\text{Augmented:} \left[\begin{array}{ccc:c} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{array}\right] \tag{68}$$

This form can be used to solve the system by applying elementary row operations to achieve a **row-echelon form**, which means the matrix has a 1 along the diagonal and zeroes in all positions below the diagonal.

8.2.2 Elementary Row Operations

The elementary row operations are:

$$\begin{array}{ll} \text{Interchange two rows.} & R_a \leftrightarrow R_b \\ \text{Multiply a row by a nonzero constant.} & cR_a \quad (c \neq 0) \\ \text{Add a multiple of a row to another row.} & cR_a + R_b \end{array} \tag{69}$$

8.2.3 Matrix Addition & Scalar Multiplication

Two matrices of the same dimension $m \times n$ can be added by adding the individual components of the matrix.

$$A + B = [a_{ij} + b_{ij}] \tag{70}$$

A matrix can be multiplied with a scalar by multiplying each component with the scalar.

$$cA = [ca_{ij}] \tag{71}$$

8.2.4 Matrix Multiplication

Two matrices can multiplied if and only if the columns of the first matrix matches the number of rows of the second matrix. A matrix with dimensions $m \times n$ multiplied with a matrix of dimensions $n \times q$ will result in a matrix of dimensions $m \times q$.

To calculate the component c_{ij} of a matrix AB, multiply each ith row-component of A with the jth column-component of B and add them together.

8.2.5 Identity Matrix

The identity matrix $\mathbf{I}$ is the matrix with leading ones across the diagonal and zeroes everywhere else. It is always **square**.

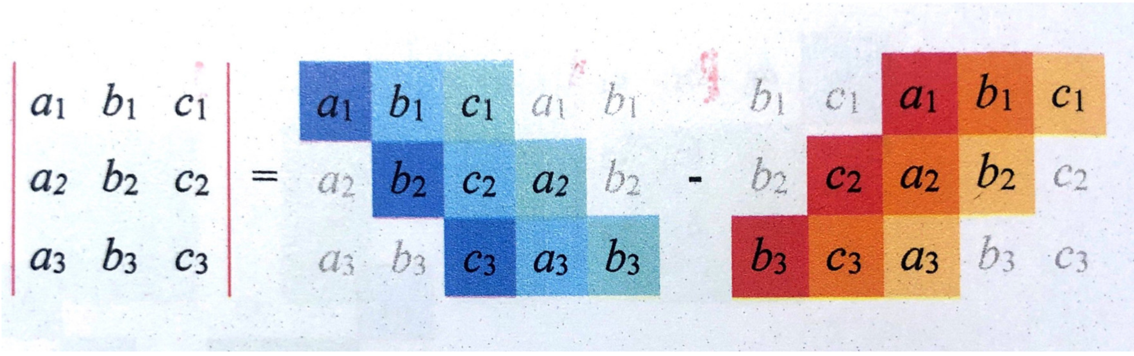
8.2.6 Determinant of a 2×2 matrix

The determinant of a 2×2 matrix is calculated as follows

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

8.2.7 Determinant of a $m \times n$ matrix

The determinant of a 3 $\times$3 matrix can be calculated using Sarrus' rule:



$$\det(C) = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_1c_2b_3 - b_1a_2c_3 - c_1b_2a_3 \tag{72}$$

For a higher-order matrix, the determinant can be calculated using the **Laplace expansion**. This multiplies the minor determinants with the elements of the first row and adds them together.

$$\begin{vmatrix} a_{11}a_{12}a_{13} \\ a_{21}a_{22}a_{23} \\ a_{31}a_{32}a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22}a_{23} \\ a_{32}a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \tag{73}$$

8.2.8 Inverse of a 2×2 matrix

The inverse **A** of a matrix has the property

$$\mathbf{AA^{-1}} = \mathbf{I} \tag{74}$$

The inverse of a 2×2 matrix is given by

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{75}$$

8.2.9 Inverse of a $m \times n$ matrix

The inverse of a $m \times n$ matrix is calculated by building a matrix of cofactors first. This matrix is derived by calculating the **minor determinants** of the original matrix.

The signs of the cofactor matrix are then flipped to match the following pattern

$$\begin{bmatrix} + - + \\ - + - \\ + - + \end{bmatrix} \tag{76}$$

The inverse is then given by

$$A^{-1} = \frac{(\text{ cofactor matrix of } \mathbf{A})^{\mathrm{T}}}{\det\ \mathbf{A}} \tag{77}$$

8.3 Transformations

Transformations on a vector can be expressed as a matrix using homogenous coordinates. These are only needed in the case of translations, otherwise a reduced 2×2 transformation matrix may be used.

Translation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{78}$$

Scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{79}$$

Reflection Reflection along the y-axis is given as

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{80}$$

while reflection along the x-axis is given as

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{81}$$

Rotation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{82}$$

9 Combinatorics & Probability

9.1 Counting Problems

9.1.1 Fundamental Counting Principle

Let E_1 and E_2 be two events. The first event E_1 can occur in m_1 different ways: After E_1 has occurred, E_2 can occur in m_2 different ways. The number of ways the two events can occur is $m_1 \cdot m_2$.

9.1.2 Permutations

A permutation of n different elements is an ordering of the elements such tha one element is first, one is second, one is third, and so on.

Calculator Hint: Use nPr to calculate permutations taken r at a time.

The number of permutations of n elements is

$$n \cdot (n-1) \cdot \cdot 4 \cdot 3 \cdot 2 \cdot 1 = n!$$

In other words, there are $n!$ different ways of ordering n elements.

The number of permutations of n elements taken r at a time is

$$_nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-2)\cdots(n-r+1)$$

Consider a set of n objects that has n_1 of one kind of object, n_2 of a second kind, and so on. The number of **distinguishable permutations** of the n objects is

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot \cdots \cdot n_k!} \tag{83}$$

9.1.3 Combinations

Combinations consider only the possible sets of objects *regardless* of the order in which the members of the set are arranged.

Calculator Hint: Use nCr to calculate combinations taken r at a time.

The number of possible combinations of n elements taken r at a time is

$$_nC_r = \frac{n!}{(n-r)!r!} = \frac{_nP_r}{r!} \tag{84}$$

9.2 Probability

For an event E with $n(E)$ outcomes that meet the restriction and that are equally likely, the probability $P(E)$ of an event within a sample space S is given as

$$P(E) = \frac{n(E)}{n(S)} \tag{85}$$

The sum of all probabilities of an event must equal 1.

Addition Rule The probability of two events A **or** B ocurring is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{86}$$

If the events are mutually exclusive, then $P(A \cap B) = 0$

Multiplication Rule The probability of two events A **and** B ocurring is given by

$$P(A \cap B) = P(A) \cdot P(B) \tag{87}$$

This applies when A and B are independent from each other.

Conditional Probability The probability of event A ocurring given event B has already ocurred is given by

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{88}$$

The multiplication rule given that A is conditional to B is given by

$$P(A \cap B) = P(A) \cdot P(B|A) \tag{89}$$

9.3 Statistics

9.3.1 Mean

Mean of a Population

$$\begin{aligned} \mu = \frac{\sum x_i}{N} \text{ where } \mu = &\text{ population mean} \\ x_i = &\text{ the } i \text{ th data value in the population} \\ \sum = &\text{ the sum of} \\ N = &\text{ number of data values in the population} \end{aligned} \tag{90}$$

Mean of a Sample

$$\begin{aligned} \overline{x} = \frac{\sum x_i}{n} \text{ where } \mu = &\text{ sample mean} \\ x_i = &\text{ the } i \text{ th data value in the sample} \\ \sum = &\text{ the sum of} \\ n = &\text{ number of data values in the sample} \end{aligned} \tag{91}$$

9.3.2 Variance & Standard Deviation

Variance of a Population

$$\begin{aligned} \sigma^2 = \frac{\sum (x_i - \mu)^2}{N} \text{ where } \sigma^2 = &\text{ population variance} \\ x = &\text{ sample mean} \\ x_i = &\text{ the } i \text{ th data value} \\ N = &\text{ number of data values in the sample} \end{aligned} \tag{92}$$

Variance of a Sample

$$\begin{aligned} s^2 = \frac{\sum (x_i - \overline{x})^2}{n-1} \text{ where } s^2 &= \text{ sample variance} \\ x &= \text{ sample mean} \\ x_i &= \text{ the } i \text{ th data value} \\ n &= \text{ number of data values in the sample} \end{aligned} \tag{93}$$

Standard Deviation

$$\begin{array}{cc} \text{For a Population} & \text{For a Sample} \\ \sigma = \sqrt{\sigma^2} & s = \sqrt{s^2} \end{array} \tag{94}$$

9.3.3 Normal Distribution

The probability of an event in a normal distribution is given as

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{(2\sigma^2)}} \tag{95}$$