Week 17 Algebra, Vectors, and Matrices Reading Note 2

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Cornell Notes	**Topic:** **Algebra, Vectors, and Matrices**	Course: BSc Computer Science Class: Computational Mathematics[Reading] Date: July 27, 2020

Essential Question:

What are vectors and matrices?

Questions/Cues:

- Are transformations commutative?
- What are the different notations in which transformations can be defined?
- What are the rules for matrix multiplication?
- What is the determinant of a matrix?
- What are homogeneous coordinates?
- What is the notation for a 2D translation and 2D scaling?
- What is a 2D reflection?
- What is a 2D shearing?
- What is 2D rotation?
- How is 2D scaling performed from a matrix perspective?
- How is 2D reflection performed from a matrix perspective?
- How is 2D rotation performed for an arbitrary point?

Notes

7.2 Matrices

Matrix notation was investigated by the British mathematician Arthur Cayley around 1858. Caley formalized matrix algebra, along with the American mathematicians Benjamin and Charles Pierce. Also, by the start of the 19th century Carl Gauss (1777–1855) had proved that transformations were not commutative, i.e. $T_1 \times T_2 \neq T_2 \times T_1$, and Caley's matrix notation would clarify such observations. For example, consider the transformation T_1:

$$\mathbf{T}_1 \quad \begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \qquad (7.4)$$

and another transformation T_2 that transforms T_1:

$$\mathbf{T}_2 \times \mathbf{T}_1 \begin{array}{l} x'' = Ax' + By' \\ y'' = Cx' + Dy' \end{array} \tag{7.5}$$

If we substitute the full definition of T_1 we get

$$\mathbf{T}_2 \times \mathbf{T}_1 \begin{array}{l} x'' = A(ax + by) + B(cx + dy) \\ y'' = C(ax + by) + D(cx + dy) \end{array} \tag{7.6}$$

which simplifies to

$$\mathbf{T}_2 \times \mathbf{T}_1 \begin{array}{l} x'' = (Aa + Bc)x + (Ab + Bd)y \\ y'' = (Ca + Dc)x + (Cb + Dd)y \end{array} \tag{7.7}$$

Caley proposed separating the constants from the variables, as follows:

$$\mathbf{T}_1 \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \tag{7.8}$$

where the square matrix of constants in the middle determines the transformation. The algebraic form is recreated by taking the top variable x', introducing the $=$ sign, and multiplying the top row of constants $[a\ b]$ individually by the last column vector containing x and y. We then examine the second variable y', introduce the $=$ sign, and multiply the bottom row of constants $[c\ d]$ individually by the last *column* vector containing x and y, to create

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \tag{7.9}$$

Using Caley's notation, the product $\mathrm{T}_2 \times \mathrm{T}_1$ is

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} x' \\ y' \end{bmatrix} \tag{7.10}$$

But the notation also intimated that

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \tag{7.11}$$

and when we multiply the two *inner matrices* together they must produce

$$\begin{aligned} x'' &= (Aa + Bc)x + (Ab + Bd)y \\ y'' &= (Ca + Dc)x + (Cb + Dd)y \end{aligned} \tag{7.12}$$

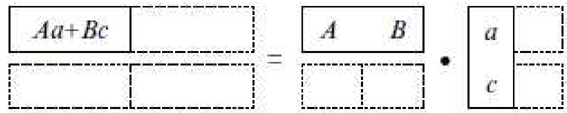
or in matrix form

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} Aa + Bc & Ab + Bd \\ Ca + Dc & Cb + Dd \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \tag{7.13}$$

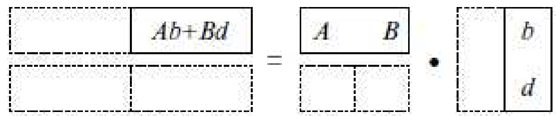
otherwise the two systems of notation will be inconsistent. This implies that

$$\begin{bmatrix} Aa + Bc & Ab + Bd \\ Ca + Dc & Cb + Dd \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{7.14}$$

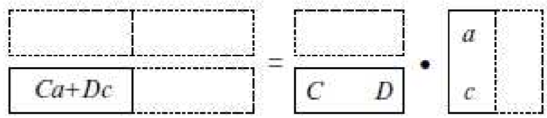
which demonstrates how matrices must be multiplied. Here are the rules for matrix multiplication:



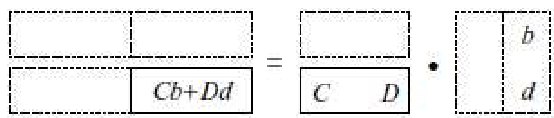
1 The top left-hand corner element $Aa + Bc$ is the product of the top row of the first matrix by the left column of the second matrix.



2 The top right-hand element $Ab + Bd$ is the product of the top row of the first matrix by the right column of the second matrix.



3 The bottom left-hand element $Ca + Dc$ is the product of the bottom row of the first matrix by the left column of the second matrix.



4 The bottom right-hand element $Cb + Dd$ is the product of the bottom row of the first matrix by the right column of the second matrix.

It is now a trivial exercise to confirm Gauss's observation that $\mathrm{T}_1 \times \mathrm{T}_2 \neq \mathrm{T}_2 \times \mathrm{T}_1$, because if we reverse the transforms $\mathrm{T}_2 \times \mathrm{T}_1$ to $\mathrm{T}_1 \times \mathrm{T}_2$ we get

$$\begin{bmatrix} Aa + Bc & Ab + Bd \\ Ca + Dc & Cb + Dd \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{7.15}$$

which shows conclusively that the product of two transforms is not commutative.

7.2.2 The Determinant of a Matrix

The *determinant* of a 2×2 matrix is a scalar quantity computed. Given a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

its determinant is $ad - cb$ and is represented by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \tag{7.18}$$

For example, the determinant of $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ is $3 \times 2 - 1 \times 2 = 4$

Basically, homogeneous coordinates define a point in a plane using three coordinates instead of two. Initially, Plücker located a homogeneous point relative to the sides of a triangle, but later revised his notation to the one employed in contemporary mathematics and computer graphics. This states that for a point P with coordinates (x, y) there exists a homogeneous point (x, y, t) such that $X = x/t$ and $Y = y/t$. For example, the point (3, 4) has homogeneous coordinates (6, 8, 2), because $3 = 6/2$ and $4 = 8/2$. But the homogeneous point (6, 8, 2) is not unique to (3, 4); (12, 16, 4), (15, 20, 5) and (300, 400, 100) are all possible homogeneous coordinates for (3, 4).

The reason why this coordinate system is called 'homogeneous' is because it is possible to transform functions such as $f(x, y)$ into the form $f(x/t, y/t)$ without disturbing the degree of the curve. To the non-mathematician this may not seem anything to get excited about, but in the field of projective geometry it is a very powerful concept.

For our purposes, we can imagine that a collection of homogeneous points of the form (x, y, t) exist on an xy-plane where t is the z-coordinate, as illustrated in Figure 7.4. The figure shows a triangle on the $t = 1$ plane, and a similar triangle, much larger, on a more distant plane. Thus instead of working in two dimensions, we can work on an arbitrary xy-plane in three dimensions. The t- or z-coordinate of the plane is immaterial because the x- and y-coordinates are eventually scaled by t. However, to keep things simple it seems a good idea to choose $t = 1$. This means that the point (x, y) has homogeneous coordinates $(x, y, 1)$, making scaling unnecessary.

If we substitute 3D homogeneous coordinates for traditional 2D Cartesian coordinates, we must attach a 1 to every (x, y) pair. When a point $(x, y, 1)$ is transformed, it will emerge as $(x', y', 1)$, and we discard the 1. This may seem a futile exercise, but it resolves the problem of creating a translation transformation.

Consider the following transformation on the homogeneous point $(x, y, 1)$:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.19}$$

This expands to

$$\begin{aligned} x' &= ax + by + c \\ y' &= dx + ey + f \\ 1 &= 1 \end{aligned} \tag{7.20}$$

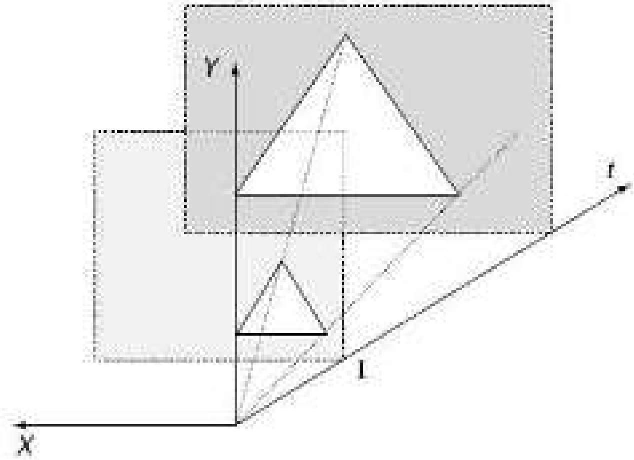


Fig. 7.4. 2D homogeneous coordinates can be visualized as a plane in 3D space, generally where $t = 1$, for convenience.

which solves the above problem of adding a constant.

7.3.1 2D Translation

The algebraic and matrix notation for 2D translation is

$$\begin{aligned} x' &= x + t_x \\ y' &= y + t_y \end{aligned} \tag{7.21}$$

or, using matrices,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.22}$$

7.3.2 2D Scaling

The algebraic and matrix notation for 2D scaling is

$$\begin{aligned} x' &= s_x x \\ y' &= s_y y \end{aligned} \tag{7.23}$$

or, using matrices,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.24}$$

The scaling action is relative to the origin, i.e. the point (0,0) remains (0,0) All other points move away from the origin. To scale relative to another point (p_x, p_y) we first subtract (p_x, p_y) from (x, y) respectively. This effectively translates the reference point (p_x, p_y) back to the origin. Second, we perform the scaling operation, and third, add (p_x, p_y) back to (x, y) respectively, to compensate for the original subtraction. Algebraically this is

$$\begin{aligned} x' &= s_x(x - p_x) + p_x \\ y' &= s_y(y - p_y) + p_y \end{aligned} \tag{7.25}$$

which simplifies to

$$\begin{aligned} x' &= s_x x + p_x(1 - s_x) \\ y' &= s_y y + p_y(1 - s_y) \end{aligned} \tag{7.26}$$

or in a homogeneous matrix form

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & p_x(1 - s_x) \\ 0 & s_y & p_y(1 - s_y) \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.27}$$

For example, to scale a shape by 2 relative to the point (1, 1) the matrix is

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

7.3.3 2D Reflections

The matrix notation for reflecting about the y-axis is:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.28}$$

or about the x-axis

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.29}$$

However, to make a reflection about an arbitrary vertical or horizontal axis we need to introduce some more algebraic deception. For example, to make a reflection about the vertical axis $x = 1$, we first subtract 1 from the x-coordinate. This effectively makes the $x = 1$ axis coincident with the major y-axis. Next we perform the reflection by reversing the sign of the modified

x-coordinate. And finally, we add 1 to the reflected coordinate to compensate for the original subtraction. Algebraically, the three steps are

$$x_1 = x - 1$$
$$x_2 = -(x - 1)$$
$$x' = -(x - 1) + 1$$

which simplifies to

$$\begin{aligned} x' &= -x + 2 \\ y' &= y \end{aligned} \tag{7.30}$$

or in matrix form,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.31}$$

Figure 7.5 illustrates this process.

In general, to reflect a shape about an arbitrary y-axis, $y = a_x$, the following transform is required:

$$\begin{aligned} x' &= -(x - a_x) + a_x \quad = -x + 2a_x \\ y' &= y \end{aligned} \tag{7.32}$$

or, in matrix form,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2a_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.33}$$

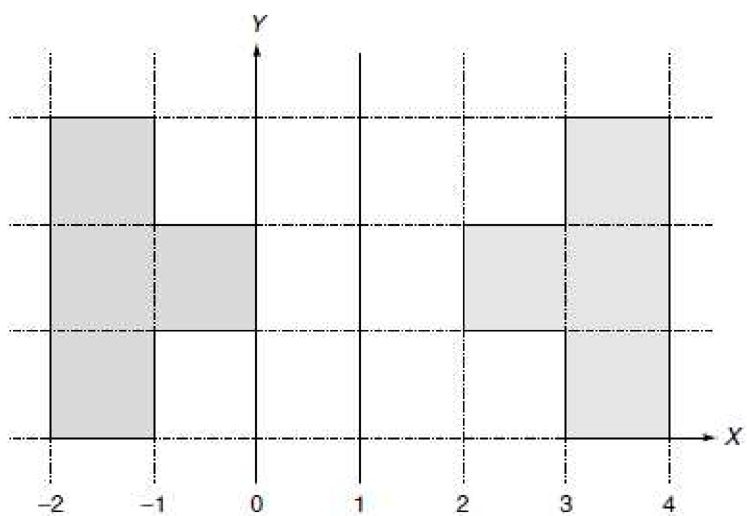


Fig. 7.5. The shape on the right is reflected about the $x = 1$ axis.

Similarly, this transform is used for reflections about an arbitrary x-axis, $y = a_y$:

$$\begin{aligned} x' &= x \\ y' &= -(y - a_y) + a_y = -y + 2a_y \end{aligned} \tag{7.34}$$

or, in matrix form,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2a_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.35}$$

7.3.4 2D Shearing

A shape is sheared by leaning it over at an angle β. Figure 7.6 illustrates the geometry, and we see that the y-coordinate remains unchanged but the x-coordinate is a function of y and $\tan(\beta)$.

$$\begin{aligned} x' &= x + y\tan(\beta) \\ y' &= y \end{aligned} \tag{7.36}$$

or, in matrix form,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \tan(\beta) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.37}$$

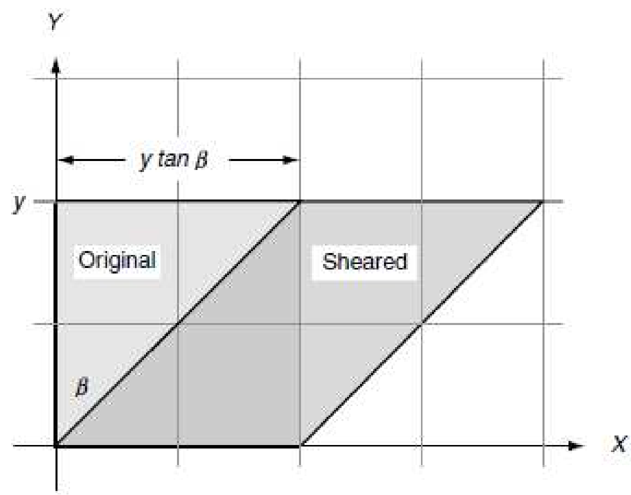


Fig. 7.6. The original square shape is sheared to the right by an angle β, and the horizontal shift is proportional to $ytan(\beta)$.

7.3.5 2D Rotation

Figure 7.7 shows a point $P(x, y)$ which is to be rotated by an angle β about the origin to $P'(x', y')$. It can be seen that

$$\begin{aligned} x' &= R\cos(\theta + \beta) \\ y' &= R\sin(\theta + \beta) \end{aligned} \tag{7.38}$$

therefore

$$\begin{aligned} x' &= R(\cos(\theta)\cos(\beta) - \sin(\theta)\sin(\beta)) \\ y' &= R(\sin(\theta)\cos(\beta) + \cos(\theta)\sin(\beta)) \\ x' &= R\left(\frac{x}{R}\cos(\beta) - \frac{y}{R}\sin(\beta)\right) \\ y' &= R\left(\frac{y}{R}\cos(\beta) + \frac{x}{R}\sin(\beta)\right) \\ x' &= x\cos(\beta) - y\sin(\beta) \\ y' &= x\sin(\beta) + y\cos(\beta) \end{aligned} \tag{7.39}$$

or, in matrix form,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\beta) & -\sin(\beta) & 0 \\ \sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.40}$$

For example, to rotate a point by 90° the matrix becomes

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

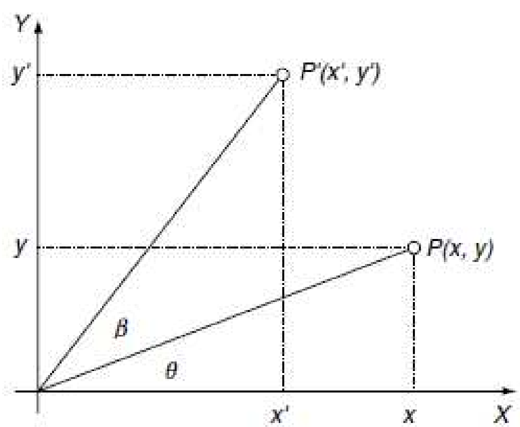


Fig. 7.7. The point $P(x, y)$ is rotated through an angle β to $P'(x', y')$.

Thus the point (1, 0) becomes (0, 1). If we rotate by 360° the matrix becomes

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Such a matrix has a null effect and is called an *identity matrix*.

To rotate a point (x, y) about an arbitrary point (p_x, p_y) we first subtract (p_x, p_y) from the coordinates (x, y) respectively. This enables us to perform the rotation about the origin. Second, we perform the rotation, and third, we add (p_x, p_y) to compensate for the original subtraction. Here are the steps:

1 Subtract (p_x, p_y):

$$\begin{aligned} x_1 &= (x - p_x) \\ y_1 &= (y - p_y) \end{aligned}$$

2 Rotate β about the origin:

$$\begin{aligned} x_2 &= (x - p_x)\cos(\beta) - (y - p_y)\sin(\beta) \\ y_2 &= (x - p_x)\sin(\beta) + (y - p_y)\cos(\beta) \end{aligned}$$

3 Add (p_x, p_y):

$$\begin{aligned} x' &= (x - p_x)\cos(\beta) - (y - p_y)\sin(\beta) + p_x \\ y' &= (x - p_x)\sin(\beta) + (y - p_y)\cos(\beta) + p_y \end{aligned}$$

Simplifying,

$$\begin{aligned} x' &= x\cos(\beta) - y\sin(\beta) + p_x(1 - \cos(\beta)) + p_y\sin(\beta) \\ y' &= x\sin(\beta) + y\cos(\beta) + p_y(1 - \cos(\beta)) - p_x\sin(\beta) \end{aligned} \tag{7.41}$$

and, in matrix form,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\beta) & -\sin(\beta) & p_x(1 - \cos(\beta)) + p_y\sin(\beta) \\ \sin(\beta) & \cos(\beta) & p_y(1 - \cos(\beta)) - p_x\sin(\beta) \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.42}$$

If we now consider rotating a point 90° about the point (1, 1) the matrix operation becomes

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

A simple test is to substitute the point (2, 1) for (x, y): it is transformed correctly to (1, 2).

7.3.6 2D Scaling

The strategy we used to scale a point (x, y) relative to some arbitrary point (p_x, p_y) was to first, translate $(-p_x, -p_y)$; second, perform the scaling; and third, translate (p_x, p_y). These three transforms can be represented in matrix form as follows:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = [\text{translate}(p_x, p_y)] \cdot [\text{scale}(s_x, s_y)] \cdot [\text{translate}(-p_x, -p_y)] \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

which expands to

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & p_x \\ 0 & 1 & p_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -p_x \\ 0 & 1 & -p_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.43}$$

Note the sequence of the transforms, as this often causes confusion. The first transform acting on the point $(x, y, 1)$ is translate $(-p_x, -p_y)$, followed by scale (s_x, s_y), followed by translate (p_x, p_y). If they are placed in any other sequence, you will discover, like Gauss, that transforms are not commutative!

We can now concatenate these matrices into a single matrix by multiplying them together. This can be done in any sequence, so long as we preserve the original order. Let's start with scale (s_x, s_y) and translate $(-p_x, -p_y)$. This produces

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & p_x \\ 0 & 1 & p_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & -s_x p_x \\ 0 & s_y & -s_y p_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

and finally

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & p_x(1-s_x) \\ 0 & s_y & p_y(1-s_y) \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.44}$$

which is the same as the previous transform (7.27).

7.3.7 2D Reflections

A reflection about the y-axis is given by

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.45}$$

Therefore, using matrices, we can reason that a reflection transform about an arbitrary axis $x = a_x$, parallel with the y-axis, is given by

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = [\text{translate}(a_x, 0)] \cdot [\text{reflection}] \cdot [\text{translate}(-a_x, 0)] \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

which expands to

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -a_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

We can now concatenate these matrices into a single matrix by multiplying them together. Let's begin by multiplying the reflection and the translate $(-a_x, 0)$ matrices together. This produces

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & a_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

and finally

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2a_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.46}$$

which is the same as the previous transform (7.33).

7.3.8 2D Rotation about an Arbitrary Point

A rotation about the origin is given by

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\beta) & -\sin(\beta) & 0 \\ \sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{7.47}$$

Therefore, using matrices, we can develop a rotation about an arbitrary point (p_x, p_y) as follows:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = [\text{translate}(p_x,\ p_y)] \cdot [\text{rotate } \beta] \cdot [\text{translate}(-p_x, -p_y)] \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

which expands to

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & p_x \\ 0 & 1 & p_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos(\beta) & -\sin(\beta) & 0 \\ \sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -p_x \\ 0 & 1 & -p_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

We can now concatenate these matrices into a single matrix by multiplying them together. Let's begin by multiplying the rotate β and the translate $(-p_x, -p_y)$ matrices together. This produces

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & p_x \\ 0 & 1 & p_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos(\beta) & -\sin(\beta) & -p_x\cos(\beta) + p_y\sin(\beta) \\ \sin(\beta) & \cos(\beta) & -p_x\sin(\beta) - p_y\cos(\beta) \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

and finally

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\beta) & -\sin(\beta) & p_x(1-\cos(\beta)) + p_y\sin(\beta) \\ \sin(\beta) & \cos(\beta) & p_y(1-\cos(\beta)) - p_x\sin(\beta) \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad (7.48)$$

which is the same as the previous transform (7.42).

I hope it is now is obvious to the reader that one can derive all sorts of transforms either algebraically, or by using matrices – it is just a question of convenience.

Summary

In this week, we learned about the rules of matrix multiplication, what linear transformations (rotation, shears, scaling, reflection) are in 2 dimensions, what a matrix is, what a translation operation is in 2 dimensions,and finally what homogeneous coordinates are.