FCS Week 1 Reading Note

Notebook:	Fundamentals of Computer Science		
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Cornell Notes	**Topic:** Logic: Part 1	Course: BSc Computer Science Class: CM1025 Fundamentals of Computer Science[Reading] Date: April 13, 2021

Essential Question:

What is propositional logic?

Questions/Cues:

- What is a proposition?
- What is a truth value?
- What is a paradox?
- What are compound propositions?
- What is logical conjunction?
- What is logical disjunction?
- What is logical negation?
- What is logical Implication?
- What is the Converse, Inverse, and Contrapositive?
- What is the logical bi-conditional?
- What is the order of precedence for logical operators?
- What is a tautology, contradiction and/or contingency?
- What does it mean when two propositions are said to be logically equivalent?
- What are the laws of logic?

Notes

A declarative sentence that is either true or false, but *not* both, is a **proposition** (or a **statement**), which we will denote by the lowercase letter p, q, r, s, or t. The variables p, q, r, s, or t are **boolean variables** (or **logic variables**).

EXAMPLE 1.1 The following sentences are propositions:

(1) Socrates was a Greek philosopher.
(2) $3 + 4 = 5$.
(3) $1 + 1 = 0$ and the moon is made of green cheese.
(4) If $1 = 2$, then roses are red.

The following sentences are *not* propositions:

- Let me go! (exclamation)
- $x + 3 = 5$ (x is an unknown.)
- Close the door! (command)
- Kennedy was a great president of the United States. (opinion)
- What is my line? (interrogation)

The truthfulness or falsity of a proposition is called its **truth value**, denoted by T(true) and F(false), respectively. (These values are often denoted by 1 and 0 by computer scientists.) For example, the truth value of statement (1) in Example 1.1 is T and that of statement (2) is F.

Consider the sentence, *This sentence is false*. It is certainly a valid declarative sentence, but is it a proposition? To answer this, assume the sentence is true. But the sentence says it is false. This contradicts our assumption. On the other hand, suppose the sentence is false. This implies the sentence is true, which again contradicts our assumption. Thus, if we assume that the sentence is true, it is false; and if we assume that it is false, it is true. It is a meaningless and self-contradictory sentence, so it is *not* a proposition, but a **paradox**.

The truth value of a proposition may not be known for some reason, but that does not prevent it from being a proposition.

Propositions (1) and (2) in Example 1.1 are **simple propositions**. A **compound proposition** is formed by combining two or more simple propositions called **components**. For instance, propositions (3) and (4) in Example 1.1 are compound. The components of proposition (4) are *1 = 2* and *Roses are red*. The truth value of a compound proposition depends on the truth values of its components.

Conjunction

The **conjunction** of two arbitrary propositions p and q, denoted by $p \wedge q$, is the proposition p *and* q. It is formed by combining the propositions using the word *and*, called a **connective**.

EXAMPLE 1.2 Consider the statements

p: *Socrates was a Greek philosopher*
and q: *Euclid was a Chinese musician.*

Their conjunction is given by

$p \wedge q$: *Socrates was a Greek philosopher and Euclid was a Chinese musician.* ■

To define the truth value of $p \wedge q$, where p and q are arbitrary propositions, we need to consider four possible cases:

- p is true, q is true.
- p is true, q is false.
- p is false, q is true.
- p is false, q is false.

(See the **tree diagram** in Figure 1.1 and Table 1.1.) If both p and q are true, then $p \wedge q$ is true; if p is true and q is false, then $p \wedge q$ is false; if p is false and q is true, then $p \wedge q$ is false; and if both p and q are false, then $p \wedge q$ is also false.

Figure 1.1

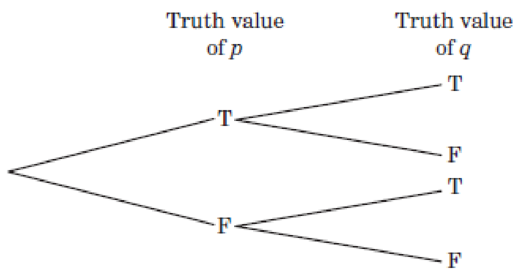

Table 1.1

p	q	$p \wedge q$
T	T	
T	F	
F	T	
F	F	

This information can be summarized in a table. In the third column of Table 1.1, enter the truth value of $p \wedge q$ corresponding to each pair of truth values of p and q. The resulting table, Table 1.2, is the **truth table** for $p \wedge q$.

Table 1.2

Truth table for $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

A second way of combining two propositions p and q is by using the connective *or*. The resulting proposition p *or* q is the **disjunction** of p and q and is denoted by $p \vee q$.

EXAMPLE 1.4 Consider the statements

p: *Harry likes pepperoni pizza for lunch*

and

q: *Harry likes mushroom pizza for lunch.*

Their disjunction is given by

$p \vee q$: *Harry likes pepperoni pizza for lunch or Harry likes mushroom pizza for lunch.*

This sentence, however, is often written as

$p \vee q$: *Harry likes pepperoni or mushroom pizza for lunch.* ■

An interesting observation: In this example, Harry could like pepperoni pizza or mushroom pizza, or both, for lunch. In other words, the connective *or* is used in the inclusive sense *and/or* to mean *at least one, maybe both*. Such a disjunction is an **inclusive disjunction**.

Table 1.3 gives the truth table for an inclusive disjunction.

Table 1.3

Truth table for $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The disjunction of two propositions is true if at least one component is true; it is false only if both components are false.

EXAMPLE 1.5 Consider the statements

r: *Bernie will play basketball at 3 P.M. today*

and

s: *Bernie will go to a matinee at 3 P.M. today.*

Then $r \vee s$: *Bernie will play basketball or go to a matinee at 3 P.M. today.* ■

In this example, Bernie cannot play basketball and go to a matinee at the *same* time, so the word *or* is used in the exclusive sense to mean at least one, but *not* both. Such a disjunction is an **exclusive disjunction**. (See Exercise 31.) Throughout our discussion, we will be concerned with only inclusive disjunction, so the word "disjunction" will mean "inclusive disjunction."

Negation

The **negation** of a proposition p is *It is not the case that* p, denoted by $\sim p$. You may read $\sim p$ as *the negation of* p or simply *not* p.

EXAMPLE 1.6 Let p: Paris is the capital of France
and q: Apollo is a Hindu god.
The negation of p is

$\sim p$: It is not the case that Paris is the capital of France.

This sentence, however, is often written as

$\sim p$: Paris is not the capital of France.

Likewise, the negation of q is

$\sim q$: Apollo is not a Hindu god. ■

If a proposition p is true, then $\sim p$ is false; if p is false, then $\sim p$ is true. This definition is summarized in Table 1.4.

Table 1.4

Truth table for $\sim p$

p	$\sim p$
T	F
F	T

EXAMPLE 1.7 Evaluate each boolean expression, where $a = 3, b = 5$, and $c = 6$.
(1) $[\sim(a > b)] \wedge (b < c)$ (2) $\sim[(a \leq b) \vee (b > c)]$

SOLUTION:

(1) Since $a > b$ is false, $\sim(a > b)$ is true. Also, $b < c$ is true. Therefore,

$$[\sim (a > b)] \wedge (b < c) \text{ is true. (See row 1 of Table 1.2.)}$$

(2) $a \leq b$ is true; but $b > c$ is false. So $(a \leq b) \vee (b > c)$ is true. (See row 2 of Table 1.3.) Therefore, $\sim[(a \leq b) \vee (b > c)]$ is false. ■

Implication

Two propositions p and q can be combined to form statements of the form: *If p, then q*. Such a statement is an **implication**, denoted by $p \to q$. Since it involves a condition, it is also called a **conditional statement**. The component p is the **hypothesis** (or **premise**) of the implication and q the **conclusion**.

EXAMPLE 1.8 Let p: $\triangle$ABC is equilateral

and q: $\triangle$ABC is isosceles.

Then

$p \to q$: *If $\triangle ABC$ is equilateral, then it is isosceles.*

Likewise,

$$q \to p\text{: } \textit{If } \triangle ABC \textit{ is isosceles, then it is equilateral.}$$

(*Note*: In the implication $q \to p$, q is the hypothesis and p is the conclusion.) ■

Implications occur in a variety of ways. The following are some commonly known occurrences:

- If p, then q.
- If p, q.
- p implies q.
- p only if q.
- q if p.
- p is sufficient for q.
- q is necessary for p.

Accordingly, the implication $p \to q$ can be read in one of these ways. For instance, consider the proposition

$$p \to q\text{: If } \triangle\text{ABC is equilateral, then it is isosceles.}$$

It means exactly the same as any of the following propositions:

- If △ABC is equilateral, it is isosceles.
- △ABC is equilateral implies it is isosceles.
- △ABC is equilateral only if it is isosceles.
- △ABC is isosceles if it is equilateral.
- That △ABC is equilateral, is a sufficient condition for it to be isosceles.
- That △ABC is isosceles, is a necessary condition for it to be equilateral.

Warning: The statement p *only if* q is often misunderstood as having the same meaning as the statement p *if* q. Remember, p *if* q means *If* q, *then* p. So be careful. Think of *only if* as one phrase; do not split it.

To construct the truth table for an implication *If* p, *then* q, we shall think of it as a conditional promise. If you do p, then I promise to do q. If the promise is kept, we consider the implication true; if the promise is not kept, we consider it false. We can use this analogy to construct the truth table, as shown below.

Consider the following implication:

$$p \to q\text{: If you wax my car, then I will pay you \$25.}$$

If you wax my car (p true) and if I pay you \$25 (q true), then the implication is true. If you wax my car (p true) and if I do not pay you \$25 (q false), then the promise is violated; hence the implication is false. What if you do not wax my car (p false)? Then I may give you \$25 (being generous!) or not. (So q may be true or false). In either case, my promise has not been tested and hence has not been violated. Consequently, the implication has not been proved false. If it is not false, it must be true. In other words, if p is false, the implication $p \to q$ is true by default. (If p is false, the implication is said to be **vacuously true**.)

This discussion is summarized in Table 1.5.

Table 1.5

Truth table for $p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

In the ordinary use of implications in the English language, there is a relationship between hypothesis and conclusion, as in the car waxing example. This relation, however, does not necessarily hold for formal implications. For instance, in the implication, *If the power is on, then $3+5=8$*, the conclusion $3+5=8$ is not even related to the hypothesis; however, from a mathematical point of view, the implication is true. This is so because the conclusion is true regardless of whether or not the power is on.

Converse, Inverse, and Contrapositive

The **converse** of the implication $p \to q$ is $q \to p$ (switch the premise and the conclusion in $p \to q$). The **inverse** of $p \to q$ is $\sim p \to \sim q$ (negate the premise and the conclusion). The **contrapositive** of $p \to q$ is $\sim q \to \sim p$ (negate the premise and the conclusion, and then switch them).

EXAMPLE 1.9 Let $p \to q$: If $\triangle$ABC is equilateral, then it is isosceles.

Its converse, inverse, and contrapositive are given by:

Converse	$q \to p$:	If $\triangle$ABC is isosceles, then it is equilateral.
Inverse	$\sim p \to \sim q$:	If $\triangle$ABC is not equilateral, then it is not isosceles.
Contrapositive	$\sim q \to \sim p$:	If $\triangle$ABC is not isosceles, then it is not equilateral. ■

A word of caution: Many people mistakenly think that an implication and its converse mean the same thing; they usually say one to mean the other. In fact, they need not have the same truth value. You will, however, learn in Example 1.18 that an implication and its contrapositive have the same truth value, and so do the converse and the inverse.

Biconditional Statement

Two propositions p and q can be combined using the connective *if and only if*. The resulting proposition, p *if and only if* q, is the conjunction of two implications: (1) p only if q, and (2) p if q, that is, $p \to q$ and $q \to p$. Accordingly, it is called a **biconditional statement**, symbolized by $p \leftrightarrow q$.

EXAMPLE 1.13 Let p: $\triangle ABC$ is equilateral

and q: $\triangle ABC$ is equiangular.

Then the biconditional statement is given by

$p \leftrightarrow q$: $\triangle ABC$ is equilateral if and only if it is equiangular. ■

Since the biconditional $p \leftrightarrow q$ means exactly the same as the statement $(p \to q) \wedge (q \to p)$, they have the same truth value in every case. We can use this fact, and columns 1, 2, and 5 of Table 1.6 to construct the truth table for $p \leftrightarrow q$, as in Table 1.9.

> Notice that the statement $p \leftrightarrow q$ is true if both p and q have the same truth value; conversely, if $p \leftrightarrow q$ is true, then p and q have the same truth value.

Table 1.9

Truth table for $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Order of Precedence

> To evaluate complex logical expressions, you must know the order of precedence of the logical operators. The order of precedence from the highest to the lowest is: (1) $\sim$ (2) $\wedge$ (3) $\vee$ (4) $\to$ (5) $\leftrightarrow$. Note that parenthesized subexpressions are always evaluated first; if two operators have equal precedence, the corresponding expression is evaluated from left to right. For example, the expression $(p \to q) \wedge {\sim q} \to {\sim p}$ is evaluated as $[(p \to q) \wedge (\sim q)] \to (\sim p)$, and $p \to q \leftrightarrow {\sim q} \to {\sim p}$ is evaluated as $(p \to q) \leftrightarrow [(\sim q) \to (\sim p)]$.

EXAMPLE 1.15 Construct a truth table for $(p \to q) \leftrightarrow (\sim p \vee q)$.

SOLUTION:
We need columns for p, q, $p \to q$, $\sim p$, $\sim p \vee q$, and $(p \to q) \leftrightarrow (\sim p \vee q)$. First, fill in the first two columns with the four pairs of truth values for p and q. Then use the truth tables for implication, negation, disjunction, and biconditional to complete the remaining columns. Table 1.10 shows the resulting table.

Table 1.10

p	q	$p \to q$	$\sim p$	$\sim p \vee q$	$(p \to q) \leftrightarrow (\sim p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

always true!

Tautology, Contradiction, and Contingency

An interesting observation: It is clear from Table 1.10 that the compound statement $(p \to q) \leftrightarrow (\sim p \vee q)$ is always true, regardless of the truth values of its components. Such a compound proposition is a **tautology**; it is an eternal truth. For example, $p \vee \sim p$ is a tautology. (Verify this.)

On the other hand, a compound statement that is always false is a **contradiction**. For instance, $p \wedge \sim p$ is a contradiction (Verify this). A compound proposition that is neither a tautology nor a contradiction is a **contingency**. For example, $p \vee q$ is a contingency.

Two compound propositions p and q, although they may look different, can have identical truth values for all possible pairs of truth values of their components. Such statements are **logically equivalent**, symbolized by $p \equiv q$; otherwise, we write $p \not\equiv q$. If $p \equiv q$, the columns headed by them in a truth table are identical.

EXAMPLE 1.17 Verify that $p \to q \equiv \sim p \vee q$.

SOLUTION:

Construct a truth table containing columns headed by $p \to q$ and $\sim p \vee q$, as in Table 1.11. Use the truth tables for implication, negation, and disjunction to fill in the last three columns. Since the columns headed by $p \to q$ and $\sim p \vee q$ are identical, the two propositions have identical truth values. In other words, $p \to q \equiv \sim p \vee q$.

Table 1.11

p	q	$p \to q$	$\sim p$	$\sim p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

identical columns

(*Note*: This example shows that an implication can be expressed in terms of negation and disjunction.)

EXAMPLE 1.18 Show that $p \to q \equiv \sim q \to \sim p$; that is, an implication is logically equivalent to its contrapositive.

SOLUTION:

Once again, construct a truth table, with columns headed by $p, q, p \to q$, $\sim q, \sim p$, and $\sim q \to \sim p$. Use the truth tables for implication and negation to complete the last four columns. The resulting table (see Table 1.12) shows that the columns headed by $p \to q$ and $\sim q \to \sim p$ are identical; therefore, $p \to q \equiv \sim q \to \sim p$.

Table 1.12

p	q	$p \to q$	$\sim q$	$\sim p$	$\sim q \to \sim p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

identical columns

■

This is an extremely useful, powerful result that plays an important role in proving theorems, as we will see in Section 1.5.

It follows by this example that $q \to p \equiv \sim p \to \sim q$ (Why?); that is, the converse of an implication and its inverse have identical truth values.

Laws of Logic

Let p, q, and r be any three propositions. Let t denote a tautology and f a contradiction. Then:

Idempotent laws

1. $p \wedge p \equiv p$
2. $p \vee p \equiv p$

Identity laws

3. $p \wedge t \equiv p$
4. $p \vee f \equiv p$

Inverse laws

5. $p \wedge (\sim p) \equiv f$
6. $p \vee (\sim p) \equiv t$

Domination laws

7. $p \vee t \equiv t$
8. $p \wedge f \equiv f$

Commutative laws

9. $p \wedge q \equiv q \wedge p$
10. $p \vee q \equiv q \vee p$

Double negation

11. $\sim (\sim p) \equiv p$

Associative laws

12. $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
13. $p \vee (q \vee r) \equiv (p \vee q) \vee r$

Distributive laws

14. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
15. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

De Morgan's laws

16. $\sim (p \wedge q) \equiv \sim p \vee \sim q$
17. $\sim (p \vee q) \equiv \sim p \wedge \sim q$

Implication conversion law

18. $p \to q \equiv \sim p \vee q$

Contrapositive law

19. $p \to q \equiv \sim q \to \sim p$

Reductio ad absurdum law

20. $p \to q \equiv (p \wedge \sim q) \to f$

We can make a few observations about some of the laws. The commutative laws imply that the order in which we take the conjunction (or disjunction) of two propositions does not affect their truth values. The associative laws say that the way we group the components in a conjunction (or disjunction) of three or more propositions does not alter the truth value of the resulting proposition; accordingly, parentheses are *not* needed to indicate the grouping. In other words, the expressions $p \wedge q \wedge r$ and $p \vee q \vee r$ are no longer ambiguous, but do make sense.

Nonetheless, parentheses are essential to indicate the groupings in the distributive laws. For instance, if we delete the parentheses in law 14, then its left-hand side (LHS) becomes $p \wedge q \vee r \equiv (p \wedge q) \vee r$, since $\wedge$ has higher priority than $\vee$. But $(p \wedge q) \vee r \not\equiv p \wedge (q \vee r)$. (You may verify this.)

EXAMPLE 1.19 Verify that $\sim(p \wedge q) \equiv \sim p \vee \sim q$.

SOLUTION:

Construct a truth table with columns headed by $p, q, p \wedge q$, $\sim(p \wedge q)$, $\sim p$, $\sim q$, and $\sim p \vee \sim q$. Since columns 4 and 7 in Table 1.14 are identical, it follows that $\sim(p \wedge q) \equiv \sim p \vee \sim q$. ■

Table 1.14

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	T	F	F	F	T
T	F	F	T	F	T	F
F	T	F	T	T	F	T
F	F	F	T	T	T	T

identical columns

De Morgan's laws, although important, can be confusing, so be careful when you negate a conjunction or a disjunction. The negation of a conjunction (or disjunction) of two statements is the disjunction (or conjunction) of their negations.

EXAMPLE 1.20 Let

p: Peter likes plain yogurt

and

q: Peter likes flavored yogurt.

Then

$p \wedge q$: Peter likes plain yogurt and flavored yogurt.

$p \vee q$: Peter likes plain yogurt or flavored yogurt.

By De Morgan's laws,

$\sim(p \wedge q) \equiv \sim p \vee \sim q$: Peter does not like plain yogurt or does not like flavored yogurt

and

$\sim(p \vee q) \equiv \sim p \wedge \sim q$: Peter likes neither plain yogurt nor flavored yogurt. ■

De Morgan's laws can be used in reverse order also; that is, $\sim p \vee \sim q \equiv \sim(p \wedge q)$ and $\sim p \wedge \sim q \equiv \sim(p \vee q)$. For instance, the sentence, *Claire does not like a sandwich or does not like pizza for lunch* can be rewritten as *It is false that Claire likes a sandwich and pizza for lunch*. Likewise, the sentence, *The earth is not flat and not round* can be restated as *It is false that the earth is flat or round*.

EXAMPLE 1.21 Determine whether or not the statement $x \leftarrow x + 1$ will be executed in the following sequence of statements:

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a ← 7; b ← 4
if ~[(a < b) ∨ (b ≥ 5)] then
  x ← x + 1
```

SOLUTION:
The statement $x \leftarrow x + 1$ will be executed if the value of the boolean expression $\sim[(a < b) \vee (b \geq 5)]$ is true. By De Morgan's law,

$$\begin{aligned} \sim[(a < b) \vee (b \geq 5)] &\equiv \sim(a < b) \wedge \sim(b \geq 5) \\ &\equiv (a \geq b) \wedge (b < 5) \end{aligned}$$

Since $a = 7$ and $b = 4$, both $a \geq b$ and $b < 5$ are true; so, $(a \geq b) \wedge (b < 5)$ is true. Therefore, the assignment statement will be executed.

EXAMPLE 1.22 Using the laws of logic simplify the boolean expression $(p \wedge \sim q) \vee q \vee (\sim p \wedge q)$.

SOLUTION:
[The justification for every step is given on its right-hand-side (RHS).]

$$\begin{aligned} (p \wedge \sim q) \vee q \vee (\sim p \wedge q) &\equiv [(p \wedge \sim q) \vee q] \vee (\sim p \wedge q) && \text{assoc. law} \\ &\equiv [q \vee (p \wedge \sim q)] \vee (\sim p \wedge q) && \text{comm. law} \end{aligned}$$

$$\begin{aligned} &\equiv [(q \vee p) \wedge (q \vee \sim q)] \vee (\sim p \wedge q) && \text{dist. law} \\ &\equiv [(q \vee p) \wedge t] \vee (\sim p \wedge q) && q \vee \sim q \equiv t \\ &\equiv (q \vee p) \vee (\sim p \wedge q) && r \wedge t \equiv r \\ &\equiv (\sim p \wedge q) \vee (p \vee q) && \text{comm. law} \\ &\equiv [\sim p \vee (p \vee q)] \wedge [q \vee (p \vee q)] && \text{dist. law} \\ &\equiv [(\sim p \vee p) \vee q] \wedge [q \vee (p \vee q)] && \text{assoc. law} \\ &\equiv (t \vee q) \wedge [q \vee (p \vee q)] && \sim p \vee p \equiv t \\ &\equiv t \wedge [q \vee (p \vee q)] && t \vee q \equiv t \\ &\equiv q \vee (p \vee q) && t \wedge r \equiv r \\ &\equiv q \vee (q \vee p) && \text{comm. law} \\ &\equiv (q \vee q) \vee p && \text{assoc. law} \\ &\equiv q \vee p && \text{idem. law} \\ &\equiv p \vee q && \text{comm. law} \end{aligned}$$

For any propositions p, q, and r, it can be shown that $p \rightarrow (q \vee r) \equiv (p \wedge \sim q) \rightarrow r$ (see Exercise 12). We shall employ this result in Section 1.5.

Summary

In this week, we learned about the basic building blocks of propositional logic, their order and their use. In addition to this, we learned about logical equivalence and the laws of logic.