

10.2 Further techniques

Notebook: Discrete Mathematics [CM1020]

Created: 2019-10-07 2:31 PM

Updated: 2020-01-20 6:44 PM

Author: SUKHJIT MANN

Cornell Notes

Topic:
10.2 Further techniques

Course: BSc Computer Science

Class: Discrete Mathematics-
Lecture

Date: January 20, 2020

Essential Question:

What are the rules/strategies used when counting objects when they are sampled with or without replacement?

Questions/Cues:

- What is a binomial expression?
- What is the Binomial Theorem?
- What is Pascal's Identity?
- What is Pascal's Triangle?
- What are permutations with repetition?
- What are permutations without repetition?
- What are combinations with repetition?
- What are combinations without repetition?
- How do we choose which formula to use when selecting k-objects from a set with n-elements?
- How do we distribute objects into boxes?
- What is meant by distinguishable/indistinguishable?
- What is meant by with/without exclusion?

Notes

Binomial expression

An expression consisting of two terms, connected by a + or – sign is called a binomial expression.

Examples of binomial expressions:

$$x + a; 2x - y; x^2 - y^2; 2x - 3y, \dots$$

Binomial theorem

$$\begin{aligned}(x+y)^1 &= x+y \\ (x+y)^2 &= x^2 + 2xy + y^2 \\ (x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\ &\vdots \\ (x+y)^{30}\end{aligned}$$

Let x and y be variables, and n a non-negative integer.
The expansion of $(x+y)^n$ can be formalised as:

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

Example

What is the coefficient of x^8y^7 in the expansion of $(3x-y)^{15}$?

Solution:

- We can view the expression as $(3x+(-y))^{15}$
- By the binomial theorem:

$$(3x+(-y))^{15} = \sum_{k=0}^{15} \binom{15}{k} (3x)^k (-y)^{15-k}$$

- Consequently, the coefficient of x^8y^7 in the expansion is obtained when $k=8$:
 - $\binom{15}{8} (3)^8 (-1)^7 = -3^8 \frac{15!}{8!7!}$

Application of the binomial theorem

Let's prove the identity $2^n = \sum_{k=0}^n \binom{n}{k}$

Using binomial theorem:

- With $x=1$ and $y=1$, from the binomial theorem we see that the identity is verified.

Using Sets:

- Consider the subsets of a set with n elements
- There are subsets with zero elements, with one element, with two elements and so on ... with n elements
- Therefore the total number of subsets is: $\sum_{k=0}^n \binom{n}{k}$
- Also, since we know that a set with n elements has 2^n subsets, we can conclude that: $2^n = \sum_{k=0}^n \binom{n}{k}$

Pascal's identity

If n and k are integers with $n \geq k \geq 1$, then:

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

Proof:

- Let T be a set where $|T| = n + 1$, $a \in T$, and $S = T - \{a\}$
- There are $\binom{n+1}{k}$ subsets of T containing k elements. Each of these subsets either:
 - contains a with $k - 1$ other elements, or
 - contains k elements of S and not a
- There are:
 - $\binom{n}{k-1}$ subsets of k elements that contain a
 - $\binom{n}{k}$ subsets of k elements of T that don't contain a
- Hence $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

Pascal's triangle

Pascal's triangle is a number triangle with numbers arranged in staggered rows such that $a_{n,r}$ is the binomial coefficient $\binom{n}{r}$:



$$\binom{4}{3} = \binom{3}{2} + \binom{3}{3} = 4 + 1 = 5$$

Using Pascal's identity, we can show that the result of adding two **adjacent** coefficients in this triangle is **equal** to the binomial coefficient in the **next row between these two coefficients**.

Permutations with repetition

The number of **r-permutations** of a set of n objects with repetition allowed is n^r .

Proof:

- Since we have n choices each time, there are n possibilities for the 1st choice, n possibilities for the 2nd choice, ..., and n possibilities when choosing the last number
- By the product rule, multiplying each time:

$$\underbrace{\boxed{n} \times \boxed{n} \times \boxed{n} \times \dots \times \boxed{n}}_{r \text{ times}} = n^r$$

Example

How many strings of length r can be formed if we are using only uppercase letters in the English alphabet?

Solution:

The number of such strings is 26^r , which is the number of r -permutations with repetition of a set with 26 elements.

Permutations without repetition

In the case of permutations without repetition, we reduce the number of available choices each time by 1. The number of **r -permutations** of a set with n objects without repetition is:

$$\boxed{n} \times \boxed{n-1} \times \boxed{n-2} \times \dots \times \boxed{n-r+1}$$

$\xrightarrow{\text{r times}}$

$$P(n, r) = P_n^r = n(n-1)(n-2) \dots (n-r+1) = \frac{n!}{(n-r)!}$$

Example

During a running competition how many different ways can the first and the second place be awarded if 10 runners are taking part in the race?

Solution:

$$P(10, 2) = P_{10}^2 = \frac{10!}{(10-2)!} = \frac{10!}{8!} = 90$$

Combination with repetition

The number of ways in which k objects can be selected from n categories of objects, with repetition permitted, can be calculated as: $\binom{k+n-1}{k} = \frac{(k+n-1)!}{k!(n-1)!}$

- It is also the total **number** of ways to put **k identical balls** into n distinct boxes.
- It is also the total **number of functions** from a set of k identical elements to a set of n distinct elements.

Example

Let's find all multisets of size 3 from the set $\{1, 2, 3, 4\}$.

Solution:

- Using bars and crosses, think of the values 1, 2, 3, 4 as four categories
- We will denote each multiset of size 3 by placing three crosses in the various categories
- For instance, the multiset $\{1, 1, 3\}$ is represented by $\times \times || \times |$
- This counting problem can be modelled as distributing the 3 crosses among the $3+4-1$ positions, the remaining positions being occupied by bars
- Thus the number of multisets of size 3 is: $C(6,3) = \frac{6!}{3!3!} = 20$.

Combination without repetition

- The number of ways in which r objects can be selected from n categories of objects with repetition not permitted can be calculated as: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- This counting problem is the same as the **number** of ways of **putting k identical balls** into n distinct boxes, where each box receives at most one ball
- It is also the **number of one-to-one functions** from a set of k identical elements into a set of n distinct elements
- It is also the number of **k -element subsets** of an n -element set.

Choice of formulas

- We have discussed four different ways of selecting k objects from a set with n elements:
 - the order in which the choices are made may or may not matter,
 - repetition may or may not be allowed.
- The following table summarises the formula in each case:

	Order matters	Order does not matter
<i>Repetition is not permitted</i>	$\frac{n!}{(n-k)!}$	$\frac{n!}{k!(n-k)!}$
<i>Repetition is permitted</i>	n^k	$\frac{(k+n-1)!}{k!(n-1)!}$

Example

John is the chair of a committee. In how many ways can a committee of 3 be chosen from 10 people, given that John must be one of the people selected?

Solution:

- Since John is already chosen, we need to choose another 2 out of 9 people.
- In choosing a committee, the order doesn't matter, so we need to apply the combination without repetition formula: $C(9,2) = \frac{9!}{2!(9-2)!} = 36$ ways.

Distributing objects into boxes

Counting problems can be phrased in terms of distributing k **objects** into n **boxes** under various conditions:

- The **objects** can be either distinguishable or indistinguishable
- The **boxes** can be either distinguishable or indistinguishable
- The distribution can be done either with exclusion or without exclusion.

Distinguishable = refers to objects or boxes that are marked in some way that makes each one distinguishable from the other

Indistinguishable = refers to objects or boxes that are identical, so that there is no way to tell them apart

****Note**** When placing indistinguishable objects into distinguishable boxes, it makes no difference which object is placed into which box

With exclusion = means that no box can contain more than one object

Without exclusion = means that a box may contain more than one object

Distinguishable objects and distinguishable boxes with exclusion

In this case, we want to **distribute k balls, numbered from 1 to k , into n boxes, numbered from 1 to n , in such a way that no box receives more than one ball.**

This is equivalent to making an **ordered selection of k boxes from n boxes**, where the balls do the selecting for us:

- the ball labelled 1 chooses the first box
- the ball labelled 2 chooses the second box
- and so on ...

Distinguishable objects and distinguishable boxes with exclusion

Theorem:

Distributing **k distinguishable balls** into n distinguishable **boxes**, with exclusion, is equivalent to forming a permutation of size k from a set of size n .

Therefore, the number of ways of placing k distinguishable balls into n distinguishable boxes is as follows:

$$P(n, k) = n(n-1)(n-2) \dots (n-k+1) = \frac{n!}{(n-k)!}$$

Distinguishable objects and distinguishable boxes without exclusion

In this case, we want to **distribute k balls, numbered from 1 to k , into n boxes, numbered from 1 through n , without restrictions on the number of balls in each box.**

This is equivalent to making an **ordered selection of k boxes from n , with repetition**, where the balls do the selecting for us:

- the ball labelled 1 chooses the first box
- the ball labelled 2 chooses the second box
- and so on ...

Distinguishable objects and distinguishable boxes without exclusion

Theorem:

Distributing k distinguishable balls into n distinguishable boxes, without exclusion, is equivalent to forming a permutation of size k from a set of size n , with repetition.

Therefore, there are:
 n^k different ways.

Indistinguishable objects and distinguishable boxes with exclusion

In this case, we want to **distribute k balls**, into n **boxes, numbered** from 1 through n , in such a way that **no box** receives more **than one ball**.

Theorem:

Distributing k indistinguishable balls into n distinguishable boxes, with exclusion, is equivalent to forming a combination of size k from a set of size n .

Therefore, there are
 $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ different ways

Indistinguishable objects and distinguishable boxes without exclusion

In this case, we want to **distribute k balls**, into n **boxes, numbered** from 1 through n , **without restrictions** on the number of balls in each box.

Theorem :

Distributing k indistinguishable balls into n distinguishable boxes, without exclusion, is equivalent to forming a combination of size k from a set of size n , with repetition.

Therefore, there are
 $\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}$ different ways

Example

How many ways are there of placing 8 indistinguishable balls into 6 distinguishable boxes?

$$\binom{8+6-1}{8} = \binom{13}{8} = \frac{13!}{8!5!} = 1,287$$

Summary

In this week, we learned about distinguishable & indistinguishable objects/boxes and the different permutation/combination formulas to apply in various such scenarios.