Week 19 Combinatorics and Probabilities Reading Note 2

Notebook:	Computational Mathematics
Created:	2020-04-21 2:48 PM
Updated:	2020-07-30 8:00 PM
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URL:	https://www.coursera.org/learn/uol-cm1015-computational-mathematics/home/week/13

Cornell Notes	**Topic:** **Combinatorics and Probabilities**	Course: BSc Computer Science Class: Computational Mathematics[Reading] Date: July 30, 2020

Essential Question:

What is the probability of an event and how can we use the principles of counting to evaluate such probabilities?

Questions/Cues:

- What is the fundamental counting principle?
- What is a permutation?
    - What is the number of permutations of n elements?
    - What is the permutations of n elements taken r at a time?
- What are distinguishable permutations?
- What are combinations?
- What is an experiment, its outcomes, the sample space of the experiment and the related event?
- What is the probability of an event?
- What are mutually exclusive events?
- What are independent events?
- What is the complement of an event?

Notes

EXAMPLE 1 Selecting Pairs of Numbers at Random

You place eight pieces of paper, numbered from 1 to 8, in a box. You draw one piece of paper at random from the box, record its number, and *replace* the paper in the box. Then, you draw a second piece of paper at random from the box and record its number. Finally, you add the two numbers. How many different ways can you obtain a sum of 12?

Solution To solve this problem, count the different ways to obtain a sum of 12 using two numbers from 1 to 8.

First number	4	5	6	7	8
Second number	8	7	6	5	4

So, a sum of 12 can occur in five different ways.

EXAMPLE 2 Selecting Pairs of Numbers at Random

You place eight pieces of paper, numbered from 1 to 8, in a box. You draw one piece of paper at random from the box, record its number, and *do not* replace the paper in the box. Then, you draw a second piece of paper at random from the box and record its number. Finally, you add the two numbers. How many different ways can you obtain a sum of 12?

Solution To solve this problem, count the different ways to obtain a sum of 12 using two *different* numbers from 1 to 8.

First number	4	5	7	8
Second number	8	7	5	4

So, a sum of 12 can occur in four different ways.

Fundamental Counting Principle

Let E_1 and E_2 be two events. The first event E_1 can occur in m_1 different ways. After E_1 has occurred, E_2 can occur in m_2 different ways. The number of ways the two events can occur is $m_1 \cdot m_2$.

The Fundamental Counting Principle can be extended to three or more events. For example, the number of ways that three events E_1, E_2, and E_3 can occur is

$$m_1 \cdot m_2 \cdot m_3.$$

EXAMPLE 3 Using the Fundamental Counting Principle

How many different pairs of letters from the English alphabet are possible?

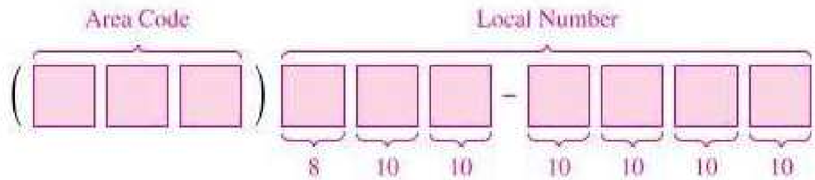
Solution There are two events in this situation. The first event is the choice of the first letter, and the second event is the choice of the second letter. The English alphabet contains 26 letters, so it follows that the number of two-letter pairs is

$$26 \cdot 26 = 676.$$

EXAMPLE 4 **Using the Fundamental Counting Principle**

Telephone numbers in the United States have 10 digits. The first three digits are the *area code* and the next seven digits are the *local telephone number*. How many different telephone numbers are possible within each area code? (Note that a local telephone number cannot begin with 0 or 1.)

Solution The first digit of a local telephone number cannot be 0 or 1, so there are only eight choices for the first digit. For each of the other six digits, there are 10 choices.



So, the number of telephone numbers that are possible within each area code is

$$8 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 8{,}000{,}000.$$

Permutations

One important application of the Fundamental Counting Principle is in determining the number of ways that n elements can be arranged (in order). An ordering of n elements is called a **permutation** of the elements.

> **Definition of a Permutation**
>
> A **permutation** of n different elements is an ordering of the elements such that one element is first, one is second, one is third, and so on.

EXAMPLE 5 **Finding the Number of Permutations**

How many permutations of the letters

A, B, C, D, E, and F

are possible?

Solution Consider the reasoning below.

First position: Any of the *six* letters

Second position: Any of the remaining *five* letters

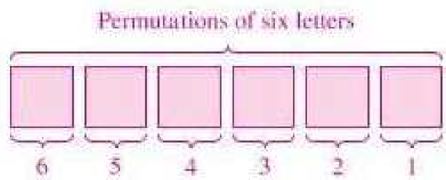
Third position: Any of the remaining *four* letters

Fourth position: Any of the remaining *three* letters

Fifth position: Either of the remaining *two* letters

Sixth position: The *one* remaining letter

So, the numbers of choices for the six positions are as shown in the figure.



The total number of permutations of the six letters is

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720.$$

Generalizing the result in Example 5, the number of permutations of n different elements is $n!$.

Number of Permutations of n Elements

The number of permutations of n elements is

$$n \cdot (n-1) \cdots 4 \cdot 3 \cdot 2 \cdot 1 = n!.$$

In other words, there are $n!$ different ways of ordering n elements.

It is useful, on occasion, to order a *subset* of a collection of elements rather than the entire collection. For example, you may want to order r elements out of a collection of n elements. Such an ordering is called a **permutation of n elements taken r at a time.** The next example demonstrates this ordering.

EXAMPLE 6 Counting Horse Race Finishes

Eight horses are running in a race. In how many different ways can these horses come in first, second, and third? (Assume that there are no ties.)

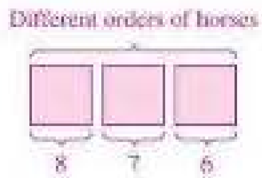
Solution Here are the different possibilities.

Win (first position): *Eight* choices

Place (second position): *Seven* choices

Show (third position): *Six* choices

The numbers of choices for the three positions are as shown in the figure.



So, using the Fundamental Counting Principle, there are

$$8 \cdot 7 \cdot 6 = 336$$

different ways in which the eight horses can come in first, second, and third.

Generalizing the result in Example 6 gives the formula below.

Permutations of n Elements Taken r at a Time

The number of permutations of n elements taken r at a time is

$$_nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-2) \cdot \cdot \cdot (n-r+1).$$

Using this formula, rework Example 6 to find that the number of permutations of eight horses taken three at a time is

$$\begin{aligned} {}_8P_3 &= \frac{8!}{(8-3)!} \\ &= \frac{8!}{5!} \\ &= \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}} \\ &= 336 \end{aligned}$$

which is the same answer obtained in the example.

Remember that for permutations, order is important. For example, to find the possible permutations of the letters A, B, C, and D taken three at a time, count A, B, D and B, A, D as different because the *order* of the elements is different.

Consider, however, the possible permutations of the letters A, A, B, and C. The total number of permutations of the four letters is ${}_4P_4 = 4!$. However, not all of these arrangements are *distinguishable* because there are two A's in the list. To find the number of distinguishable permutations, use the formula below.

Distinguishable Permutations

Consider a set of n objects that has n_1 of one kind of object, n_2 of a second kind, n_3 of a third kind, and so on, with

$$n = n_1 + n_2 + n_3 + \cdots + n_k.$$

The number of **distinguishable permutations** of the n objects is

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot \cdots \cdot n_k!}.$$

EXAMPLE 7 Distinguishable Permutations

See LarsonPrecalculus.com for an interactive version of this type of example.

In how many distinguishable ways can the letters in BANANA be written?

Solution This word has six letters, of which three are A's, two are N's, and one is a B. So, the number of distinguishable ways the letters can be written is

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3!} = \frac{6!}{3! \cdot 2! \cdot 1!}$$

$$= \frac{6 \cdot 5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!} \cdot 2!}$$

$$= 60.$$

The 60 different distinguishable permutations are as listed below.

AAABNN	AAANBN	AAANNB	AABANN
AABNAN	AABNNA	AANABN	AANANB
AANBAN	AANBNA	AANNAB	AANNBA
ABAANN	ABANAN	ABANNA	ABNAAN
ABNANA	ABNNAA	ANAABN	ANAANB
ANABAN	ANABNA	ANANAB	ANANBA
ANBAAN	ANBANA	ANBNAA	ANNAAB
ANNABA	ANNBAA	BAAANN	BAANAN
BAANNA	BANAAN	BANANA	BANNAA
BNAAAN	BNAANA	BNANAA	BNNAAA
NAAABN	NAAANB	NAABAN	NAABNA
NAANAB	NAANBA	NABAAN	NABANA
NABNAA	NANAAB	NANABA	NANBAA
NBAAAN	NBAANA	NBANAA	NBNAAA
NNAAAB	NNAABA	NNABAA	NNBAAA

Combinations

When you count the number of possible permutations of a set of elements, order is important. As a final topic in this section, you will look at a method of selecting subsets of a larger set in which order is *not* important. Such subsets are called **combinations of n elements taken r at a time.** For example, the combinations

$$\{A, B, C\} \qquad \text{and} \qquad \{B, A, C\}$$

are equivalent because both sets contain the same three elements, and the order in which the elements are listed is not important. So, you would count only one of the two sets. Another example of how a combination occurs is in a card game in which players are free to reorder the cards after they have been dealt.

EXAMPLE 8 Combinations of n Elements Taken r at a Time

In how many different ways can three letters be chosen from the letters

A, B, C, D, and E?

(The order of the three letters is not important.)

Solution The subsets listed below represent the different combinations of three letters that can be chosen from the five letters.

$\{A, B, C\}$ $\{A, B, D\}$
$\{A, B, E\}$ $\{A, C, D\}$
$\{A, C, E\}$ $\{A, D, E\}$
$\{B, C, D\}$ $\{B, C, E\}$
$\{B, D, E\}$ $\{C, D, E\}$

So, when order is not important, there are 10 different ways that three letters can be chosen from five letters.

Combinations of n Elements Taken r at a Time

The number of combinations of n elements taken r at a time is

$${}_nC_r = \frac{n!}{(n-r)!r!}$$

which is equivalent to ${}_nC_r = \dfrac{{}_nP_r}{r!}$.

To see how to use this formula, rework the counting problem in Example 8. In that problem, you want to find the number of combinations of five elements taken three at a time. So, $n = 5$, $r = 3$, and the number of combinations is

$${}_5C_3 = \frac{5!}{2!3!} = \frac{5 \cdot \overset{2}{\cancel{4}} \cdot \cancel{3!}}{\cancel{2} \cdot 1 \cdot \cancel{3!}} = 10$$

which is the same answer obtained in the example.

EXAMPLE 9 **Counting Card Hands**

A standard poker hand consists of five cards dealt from a deck of 52 (see Figure 9.2). How many different poker hands are possible? (Order is not important.)

Solution To determine the number of different poker hands, find the number of combinations of 52 elements taken five at a time.

$$
\begin{aligned}
{}_{52}C_5 &= \frac{52!}{(52-5)!5!} \\
&= \frac{52!}{47!5!} \\
&= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47!}{47! \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\
&= 2{,}598{,}960
\end{aligned}
$$

A ♥	A ♦	A ♣	A ♠
2 ♥	2 ♦	2 ♣	2 ♠
3 ♥	3 ♦	3 ♣	3 ♠
4 ♥	4 ♦	4 ♣	4 ♠
5 ♥	5 ♦	5 ♣	5 ♠
6 ♥	6 ♦	6 ♣	6 ♠
7 ♥	7 ♦	7 ♣	7 ♠
8 ♥	8 ♦	8 ♣	8 ♠
9 ♥	9 ♦	9 ♣	9 ♠
10 ♥	10 ♦	10 ♣	10 ♠
J ♥	J ♦	J ♣	J ♠
Q ♥	Q ♦	Q ♣	Q ♠
K ♥	K ♦	K ♣	K ♠

Ranks and suits in a standard deck of playing cards

Figure 9.2

EXAMPLE 10 **Forming a Team**

You are forming a 12-member swim team from 10 girls and 15 boys. The team must consist of five girls and seven boys. How many different 12-member teams are possible?

Solution There are ${}_{10}C_5$ ways of choosing five girls. There are ${}_{15}C_7$ ways of choosing seven boys. By the Fundamental Counting Principle, there are ${}_{10}C_5 \cdot {}_{15}C_7$ ways of choosing five girls and seven boys.

$$
{}_{10}C_5 \cdot {}_{15}C_7 = \frac{10!}{5! \cdot 5!} \cdot \frac{15!}{8! \cdot 7!} = 252 \cdot 6435 = 1{,}621{,}620
$$

So, there are 1,621,620 12-member swim teams possible.

The Probability of an Event

Any happening for which the result is uncertain is an **experiment.** The possible results of the experiment are **outcomes,** the set of all possible outcomes of the experiment is the **sample space** of the experiment, and any subcollection of a sample space is an **event.**

For example, when you toss a six-sided die, the numbers 1 through 6 can represent the sample space. For this experiment, each of the outcomes is *equally likely.*

To describe sample spaces in such a way that each outcome is equally likely, you must sometimes distinguish between or among various outcomes in ways that appear artificial. Example 1 illustrates such a situation.

EXAMPLE 1 Finding a Sample Space

Find the sample space for each experiment.

a. You toss one coin.

b. You toss two coins.

c. You toss three coins.

Solution

a. The coin will land either heads up (denoted by H) or tails up (denoted by T), so the sample space is

$$S = \{H, T\}.$$

b. Either coin can land heads up or tails up, so the possible outcomes are as follows.

HH = heads up on both coins

HT = heads up on the first coin and tails up on the second coin

TH = tails up on the first coin and heads up on the second coin

TT = tails up on both coins

So, the sample space is

$$S = \{HH, HT, TH, TT\}.$$

Note that this list distinguishes between the two cases HT and TH, even though these two outcomes appear to be similar.

c. Using notation similar to that used in part (b), the sample space is

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Note that this list distinguishes among the cases HHT, HTH, and THH, and among the cases HTT, THT, and TTH.

To find the probability of an event, count the number of outcomes in the event and in the sample space. The *number of equally likely outcomes* in event E is denoted by $n(E)$, and the number of equally likely outcomes in the sample space S is denoted by $n(S)$. The probability that event E will occur is given by $n(E)/n(S)$.

The Probability of an Event

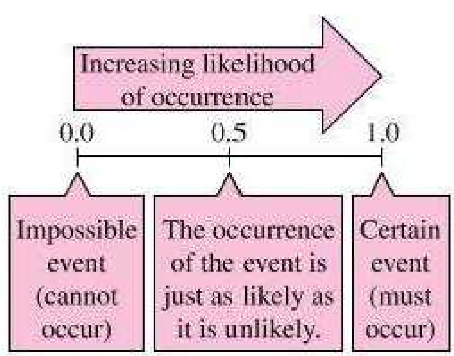
If an event E has $n(E)$ equally likely outcomes and its sample space S has $n(S)$ equally likely outcomes, then the **probability** of event E is

$$P(E) = \frac{n(E)}{n(S)}.$$

The number of outcomes in an event must be less than or equal to the number of outcomes in the sample space, so the probability of an event must be a number between 0 and 1, inclusive. That is,

$$0 \le P(E) \le 1$$

as shown in the figure. If $P(E) = 0$, then event E *cannot occur*, and E is an **impossible event.** If $P(E) = 1$, then event E *must occur*, and E is a **certain event.**



EXAMPLE 2 Finding the Probability of an Event

See LarsonPrecalculus.com for an interactive version of this type of example.

a. You toss two coins. What is the probability that both land heads up?

b. You draw one card at random from a standard deck of 52 playing cards. What is the probability that it is an ace?

Solution

a. Using the results of Example 1(b), let

$$E = \{HH\} \quad \text{and} \quad S = \{HH, HT, TH, TT\}.$$

The probability of getting two heads is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{4}.$$

b. The deck has four aces (one in each suit), so the probability of drawing an ace is

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{52} = \frac{1}{13}.$$

EXAMPLE 3 Finding the Probability of an Event

You toss two six-sided dice. What is the probability that the total of the two dice is 7?

Solution There are six possible outcomes on each die, so by the Fundamental Counting Principle, there are $6 \cdot 6$ or 36 different outcomes when you toss two dice. To find the probability of rolling a total of 7, you must first count the number of ways in which this can occur.

First Die	Second Die
1	6
2	5
3	4
4	3
5	2
6	1

So, a total of 7 can be rolled in six ways, which means that the probability of rolling a total of 7 is

$$P(E) = \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}.$$

EXAMPLE 4 Finding the Probability of an Event

Twelve-sided dice, as shown in Figure 9.3, can be constructed (in the shape of regular dodecahedrons) such that each of the numbers from 1 to 6 occurs twice on each die. Show that these dice can be used in any game requiring ordinary six-sided dice without changing the probabilities of the various events.

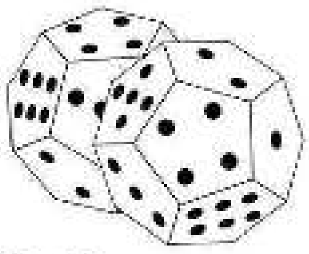


Figure 9.3

Solution For an ordinary six-sided die, each of the numbers

1, 2, 3, 4, 5, and 6

occurs once, so the probability of rolling any one of these numbers is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{6}.$$

For one of the 12-sided dice, each number occurs twice, so the probability of rolling each number is

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{12} = \frac{1}{6}.$$

EXAMPLE 5 Random Selection

The figure shows the numbers of degree-granting postsecondary institutions in various regions of the United States in 2015. What is the probability that an institution selected at random is in one of the three southern regions? *(Source: National Center for Education Statistics)*

Solution From the figure, the total number of institutions is 4622. There are $858 + 302 + 451 = 1611$ institutions in the three southern regions, so the probability that the institution is in one of these regions is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1611}{4622} \approx 0.349.$$

EXAMPLE 6 Finding the Probability of Winning a Lottery

In Arizona's The Pick game, a player chooses six different numbers from 1 to 44. If these six numbers match the six numbers drawn (in any order), the player wins (or shares) the top prize. What is the probability of winning the top prize when the player buys one ticket?

Solution To find the number of outcomes in the sample space, use the formula for the number of combinations of 44 numbers taken six at a time.

$$n(S) = {}_{44}C_6$$

$$= \frac{44 \cdot 43 \cdot 42 \cdot 41 \cdot 40 \cdot 39}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$= 7{,}059{,}052$$

When a player buys one ticket, the probability of winning is

$$P(E) = \frac{1}{7{,}059{,}052}.$$

Mutually Exclusive Events

Two events A and B (from the same sample space) are **mutually exclusive** when A and B have no outcomes in common. In the terminology of sets, the intersection of A and B is the empty set, which implies that

$$P(A \cap B) = 0.$$

For example, when you toss two dice, the event A of rolling a total of 6 and the event B of rolling a total of 9 are mutually exclusive. To find the probability that one or the other of two mutually exclusive events will occur, *add* their individual probabilities.

Probability of the Union of Two Events

If A and B are events in the same sample space, then the probability of A or B occurring is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

EXAMPLE 7 Probability of a Union of Events

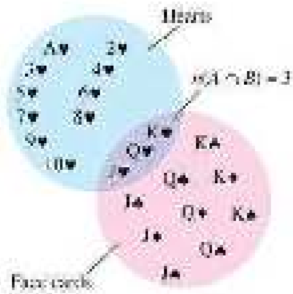


Figure 9.4

You draw one card at random from a standard deck of 52 playing cards. What is the probability that the card is either a heart or a face card?

Solution The deck has 13 hearts, so the probability of drawing a heart (event A) is

$$P(A) = \frac{13}{52}.$$

Similarly, the deck has 12 face cards, so the probability of drawing a face card (event B) is

$$P(B) = \frac{12}{52}.$$

Three of the cards are hearts and face cards (see Figure 9.4), so it follows that

$$P(A \cap B) = \frac{3}{52}.$$

Finally, applying the formula for the probability of the union of two events, the probability of drawing either a heart or a face card is

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} \\ &= \frac{22}{52} \\ &\approx 0.423. \end{aligned}$$

EXAMPLE 8 Probability of Mutually Exclusive Events

The human resources department of a company has compiled data showing the number of years of service for each employee. The table shows the results.

DATA

Years of Service	Number of Employees
0–4	157
5–9	89
10–14	74
15–19	63
20–24	42
25–29	38
30–34	35
35–39	21
40–44	8
45 or more	2

Spreadsheet at LarsonPrecalculus.com

a. What is the probability that an employee chosen at random has 4 or fewer years of service?

b. What is the probability that an employee chosen at random has 9 or fewer years of service?

Solution

a. To begin, add the number of employees to find that the total is 529. Next, let event A represent choosing an employee with 0 to 4 years of service. Then the probability of choosing an employee who has 4 or fewer years of service is

$$\begin{aligned} P(A) &= \frac{157}{529} \\ &\approx 0.297. \end{aligned}$$

b. Let event B represent choosing an employee with 5 to 9 years of service. Then

$$P(B) = \frac{89}{529}.$$

Event A from part (a) and event B have no outcomes in common, so these two events are mutually exclusive and

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) \\ &= \frac{157}{529} + \frac{89}{529} \\ &= \frac{246}{529} \\ &\approx 0.465. \end{aligned}$$

So, the probability of choosing an employee who has 9 or fewer years of service is about 0.465.

Independent Events

Two events are **independent** when the occurrence of one has no effect on the occurrence of the other. For example, rolling a total of 12 with two six-sided dice has no effect on the outcome of future rolls of the dice. To find the probability that two independent events will occur, *multiply* the probabilities of each.

> **Probability of Independent Events**
>
> If A and B are independent events, then the probability that both A and B will occur is
>
> $$P(A \text{ and } B) = P(A) \cdot P(B).$$
>
> This rule can be extended to any number of independent events.

EXAMPLE 9 Probability of Independent Events

A random number generator selects three integers from 1 to 20. What is the probability that all three numbers are less than or equal to 5?

Solution Let event A represent selecting a number from 1 to 5. Then the probability of selecting a number from 1 to 5 is

$$P(A) = \frac{5}{20} = \frac{1}{4}.$$

So, the probability that all three numbers are less than or equal to 5 is

$$\begin{aligned} P(A) \cdot P(A) \cdot P(A) &= \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) \\ &= \frac{1}{64}. \end{aligned}$$

EXAMPLE 10 **Probability of Independent Events**

In 2015, approximately 65% of Americans expected much of the workforce to be automated within 50 years. In a survey, researchers selected 10 people at random from the population. What is the probability that all 10 people expected much of the workforce to be automated within 50 years? *(Source: Pew Research Center)*

Solution Let event A represent selecting a person who expected much of the workforce to be automated within 50 years. The probability of event A is 0.65. Each of the 10 occurrences of event A is an independent event, so the probability that all 10 people expected much of the workforce to be automated within 50 years is

$$[P(A)]^{10} = (0.65)^{10} \approx 0.013.$$

The Complement of an Event

The **complement of an event** A is the collection of all outcomes in the sample space that are *not* in A. The complement of event A is denoted by A'. Because $P(A \text{ or } A') = 1$ and A and A' are mutually exclusive, it follows that $P(A) + P(A') = 1$. So, the probability of A' is

$$P(A') = 1 - P(A).$$

Probability of a Complement

Let A be an event and let A' be its complement. If the probability of A is $P(A)$, then the probability of the complement is

$$P(A') = 1 - P(A).$$

For example, if the probability of *winning* a game is $P(A) = \frac{1}{4}$, then the probability of *losing* the game is $P(A') = 1 - \frac{1}{4} = \frac{3}{4}$.

EXAMPLE 11 **Probability of a Complement**

A manufacturer has determined that a machine averages one faulty unit for every 1000 it produces. What is the probability that an order of 200 units will have one or more faulty units?

Solution To solve this problem as stated, you would need to find the probabilities of having exactly one faulty unit, exactly two faulty units, exactly three faulty units, and so on. However, using complements, it is much less tedious to find the probability that all units are perfect and then subtract this value from 1. The probability that any given unit is perfect is 999/1000, so the probability that all 200 units are perfect is

$$P(A) = \left(\frac{999}{1000}\right)^{200} \approx 0.819$$

and the probability that at least one unit is faulty is

$$P(A') = 1 - P(A) \approx 1 - 0.819 = 0.181.$$

Summary

In this week, we learned about what the fundamental counting principle is, what a permutation is, what distinguishable permutations are, what combinations are, what an experiment, its sample space and outcomes are, what the probability of an event is, what mutually exclusive events are, what independent events are, and what the complement of an event is.