Week 19 Combinatorics and Probabilities Reading Note 1

Notebook:	Computational Mathematics		
Created:	2020-04-21 2:48 PM	**Updated:**	2020-07-30 6:42 PM
Author:	SUKHJIT MANN		
URL:	https://www.coursera.org/learn/uol-cm1015-computational-mathematics/home/week/13		

Cornell Notes	**Topic:** **Combinatorics and Probabilities**	Course: BSc Computer Science Class: Computational Mathematics[Reading] Date: July 30, 2020

Essential Question:

What is the probability of an event and how can we use the principles of counting to evaluate such probabilities?

Questions/Cues:

- What is an average?
- What is the arithmetic mean?
- What is the median?
- What is the mode?
- What is the variance?
- What is standard deviation?
- What is the probability scale and complementary events?
- What is meant by unbiased?
- What is a theoretical probability?
- What is an experimental probability?
- What are independent events?

Notes

- Average = To represent a whole set of data by a single number that in some way represents the set

The arithmetic mean

The **arithmetic mean**, or simply the **mean**, of a set of values is found by adding up all the values and dividing the result by the total number of values in the set. It is given by the following formula:

Key point

$$\text{mean} = \frac{\text{sum of the values}}{\text{total number of values}}$$

30.1 Eight students sit a mathematics test and their marks out of 10 are 4, 6, 6, 7, 7, 7, 8 and 10. Find the mean mark.

Solution The sum of the marks is $4+6+6+7+7+7+8+10=55$. The total number of values is 8. Therefore,

$$\text{mean} = \frac{\text{sum of the values}}{\text{total number of values}} = \frac{55}{8} = 6.875$$

The examiner can quote 6.875 out of 10 as the 'average mark' of the group of students.

30.2 In a hospital a patient's body temperature is recorded every hour for six hours. Find the mean temperature over the six-hour period if the six temperatures, in °C, were

36.5 36.8 36.9 36.9 36.9 37.0

Solution To find the mean temperature the sum of all six values is found and the result is divided by 6. That is,

$$\text{mean} = \frac{36.5+36.8+36.9+36.9+36.9+37.0}{6}$$

$$= \frac{221.0}{6} = 36.83\ ^{\circ}\text{C}$$

In more advanced work we make use of a formula for the mean that requires knowledge of some special notation. Suppose we have n values and we call these x_1, x_2, x_3 and so on up to x_n. The mean of these values is given the symbol $\bar{x}$, pronounced 'x bar'. To calculate the mean we must add up these values and divide by n, that is

$$\text{mean} = \bar{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$

A notation is often used to shorten this formula. In mathematics, the Greek letter sigma, written $\sum$, stands for a sum. The sum $x_1 + x_2$ is written

$$\sum_{i=1}^{2} x_i$$

and the sum $x_1 + x_2 + x_3 + x_4$ is written

$$\sum_{i=1}^{4} x_i$$

Note that i runs through all integer values from 1 to n.

where the values below and above the sigma sign give the first and last values in the sum. Similarly $x_1 + x_2 + x_3 + \cdots + x_n$ is written $\sum_{i=1}^{n} x_i$. Using this notation, the formula for the mean can be written in the following way:

Key point

$$\text{mean} = \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

WORKED EXAMPLES

30.3 Express the following in sigma notation:

(a) $x_1 + x_2 + x_3 + \cdots + x_8 + x_9$ (b) $x_{10} + x_{11} + \cdots + x_{100}$

Solution (a) $x_1 + x_2 + x_3 + \cdots + x_8 + x_9 = \sum_{i=1}^{9} x_i$

(b) $x_{10} + x_{11} + \cdots + x_{100} = \sum_{i=10}^{100} x_i$

30.4 Find the mean of the values $x_1 = 5$, $x_2 = 7$, $x_3 = 13$, $x_4 = 21$ and $x_5 = 29$.

Solution The number of values equals 5, so we let $n = 5$. The sum of the values is

$$\sum_{i=1}^{5} x_i = x_1 + x_2 + x_3 + x_4 + x_5 = 5 + 7 + 13 + 21 + 29 = 75$$

The mean is

$$\text{mean} = \bar{x} = \frac{\sum_{i=1}^{5} x_i}{n} = \frac{75}{5} = 15$$

When the data is presented in the form of a frequency distribution the mean is found by first multiplying each data value by its frequency. The results are added. This is equivalent to adding up all the data values. The mean is found by dividing this sum by the sum of all the frequencies. Note that the sum of the frequencies is equal to the total number of values. Consider the following example.

WORKED EXAMPLE

30.5 Thirty-eight students sit a mathematics test and their marks out of 10 are shown in Table 30.1. Find the mean mark.

Solution Each data value, in this case the mark, is multiplied by its frequency, and the results are added. This is equivalent to adding up all the 38 individual marks. This is shown in Table 30.2.

Table 30.1
Marks of 38 students in a test

Mark, m	Frequency, f
0	0
1	0
2	1
3	0
4	1
5	7
6	16
7	8
8	3
9	1
10	1
Total	38

Table 30.2
Marks of 38 students multiplied by frequency

Mark, m	Frequency, f	$m \times f$
0	0	0
1	0	0
2	1	2
3	0	0
4	1	4
5	7	35
6	16	96
7	8	56
8	3	24
9	1	9
10	1	10
Totals	38	236

Note that the sum of all the frequencies is equal to the number of students taking the test. The number 236 is equal to the sum of all the individual marks. The mean is found by dividing this sum by the sum of all the frequencies:

$$\text{mean} = \frac{236}{38} = 6.21$$

The mean mark is 6.21 out of 10.

Using the sigma notation the formula for the mean mark of a frequency distribution with N classes, where the frequency of value x_i is f_i, becomes

Key point

$$\text{mean} = \bar{x} = \frac{\sum_{i=1}^{N} f_i \times x_i}{\sum_{i=1}^{N} f_i}$$

Note that $\sum_{i=1}^{N} f_i = n$; that is, the sum of all the frequencies equals the total number of values.

When the data is in the form of a grouped distribution the class midpoint is used to calculate the mean. Consider the following example.

WORKED EXAMPLE

30.6 The heights, to the nearest centimetre, of 100 students are given in Table 30.3. Find the mean height.

Solution Because the actual heights of students in each class are not known we use the midpoint of the class as an estimate. The midpoint of the class 164–165 is 164.5. Other midpoints and the calculation of the mean are shown in Table 30.4.

Then

$$\text{mean} = \bar{x} = \frac{\sum_{i=1}^{N} f_i \times x_i}{\sum_{i=1}^{N} f_i} = \frac{17150}{100} = 171.5$$

The mean height is 171.5 cm.

Table 30.3
Heights of 100 students

Height (cm)	Frequency
164–165	4
166–167	8
168–169	10
170–171	27
172–173	30
174–175	10
176–177	6
178–179	5
Total	100

Table 30.4
Heights of 100 students with midpoints multiplied by frequency

Height (cm)	Frequency f_i	Midpoint x_i	$f_i \times x_i$
164–165	4	164.5	658.0
166–167	8	166.5	1332.0
168–169	10	168.5	1685.0
170–171	27	170.5	4603.5
172–173	30	172.5	5175.0
174–175	10	174.5	1745.0
176–177	6	176.5	1059.0
178–179	5	178.5	892.5
Total	100		17150.0

The median

A second average that also typifies a set of data is the **median**.

Key point

The **median** of a set of numbers is found by listing the numbers in ascending order and then selecting the value that lies halfway along the list.

WORKED EXAMPLE

30.7 Find the median of the numbers

1 2 6 7 9 11 11 11 14

Solution The set of numbers is already given in order. The number halfway along the list is 9, because there are four numbers before it and four numbers after it in the list. Hence the median is 9.

When there is an even number of values, the median is found by taking the mean of the two middle values.

WORKED EXAMPLE

30.8 Find the median of the following salaries: £24,000, £12,000, £16,000, £22,000, £10,000 and £25,000.

Solution The numbers are first arranged in order as £10,000, £12,000, £16,000, £22,000, £24,000 and £25,000. Because there is an even number of values there are two middle figures: £16,000 and £22,000. The mean of these is

$$\frac{16{,}000 + 22{,}000}{2} = 19{,}000$$

The median salary is therefore £19,000.

The mode

A third average is the **mode**.

Key point The **mode** of a set of values is that value that occurs most often.

WORKED EXAMPLE

30.9 Find the mode of the set of numbers

1 1 4 4 5 6 8 8 8 9

Solution The number that occurs most often is 8, which occurs three times. Therefore 8 is the mode. Usually a mode is quoted when we want to represent the most popular value in a set.

Sometimes a set of data may have more than one mode.

WORKED EXAMPLE

30.10 Find the mode of the set of numbers

20 20 21 21 21 48 48 49 49 49

Solution In this example there is no single value that occurs most frequently. The number 21 occurs three times, but so does the number 49. This set has two modes. The data is said to be **bimodal**.

Suppose we have n values x_1, x_2, x_3 up to x_n. Their mean is $\bar{x}$ given by

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

The variance is found from the following formula:

Key point

$$\text{variance} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}$$

If you study this carefully you will see that to calculate the variance we must:

- calculate the mean value $\bar{x}$
- subtract the mean from each value in turn, that is find $x_i - \bar{x}$
- square each answer to get $(x_i - \bar{x})^2$
- add up all these squared quantities to get $\sum_{i=1}^{n} (x_i - \bar{x})^2$
- divide the result by n to get

  $$\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}$$

  which is the variance

The standard deviation is found by taking the square root of the variance:

Key point

$$\text{standard deviation} = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}}$$

WORKED EXAMPLE

30.11 Find the variance and standard deviation of 10, 15.8, 19.2 and 8.7.

Solution First the mean is found:

$$\bar{x} = \frac{10 + 15.8 + 19.2 + 8.7}{4} = \frac{53.7}{4} = 13.425$$

The calculation to find the variance is given in Table 30.7.

$$\text{variance} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n} = \frac{73.049}{4} = 18.262$$

The standard deviation is the square root of the variance:

$$\sqrt{18.262} = 4.273$$

Table 30.7

x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
10	$10 - 13.425 = -3.425$	$(-3.425)^2 = 11.731$
15.8	$15.8 - 13.425 = 2.375$	$2.375^2 = 5.641$
19.2	$19.2 - 13.425 = 5.775$	$5.775^2 = 33.351$
8.7	$8.7 - 13.425 = -4.725$	$(-4.725)^2 = 22.326$
Total		73.049

When dealing with a grouped frequency distribution with N classes the formula for the variance becomes

Key point

$$\text{variance} = \frac{\sum_{i=1}^{N} f_i (x_i - \bar{x})^2}{\sum_{i=1}^{N} f_i}$$

As before, the standard deviation is the square root of the variance.

WORKED EXAMPLE

30.12 In a period of 30 consecutive days in July the temperature in °C was recorded as follows:

18	19	20	23	24	24	21	18	17	16
16	17	17	17	18	19	20	20	22	23
24	24	25	23	21	21	20	19	19	18

(a) Produce a grouped frequency distribution showing data grouped from 16 to 17 °C, 18 to 19 °C and so on.
(b) Find the mean temperature of the grouped data.
(c) Find the standard deviation of the grouped data.

Solution (a) The data is grouped using a tally chart as in Table 30.8.

Temperature range (°C)	Tally	Frequency
16–17	~~||||~~ |	6
18–19	~~||||~~ |||	8
20–21	~~||||~~ ||	7
22–23	||||	4
24–25	~~||||~~	5
Total		30

(b) When calculating the mean temperature from the grouped data we do not know the actual temperatures. We do know the frequency of each class. The best we can do is use the midpoint of each class as an estimate of the values in that class. The class midpoints and the calculation to determine the mean are shown in Table 30.9. The mean is then

$$\bar{x} = \frac{\sum_{i=1}^{N} f_i \times x_i}{\sum_{i=1}^{N} f_i} = \frac{603}{30} = 20.1\ ^\circ\mathrm{C}$$

Temperature range (°C)	Frequency f_i	Class midpoint x_i	$f_i \times x_i$
16–17	6	16.5	99.0
18–19	8	18.5	148.0
20–21	7	20.5	143.5
22–23	4	22.5	90.0
24–25	5	24.5	122.5
Total	30		603

(c) To find the variance and hence the standard deviation we must subtract the mean, 20.1, from each value, square and then add the results. Finally this sum is divided by 30. The complete calculation is shown in Table 30.10.

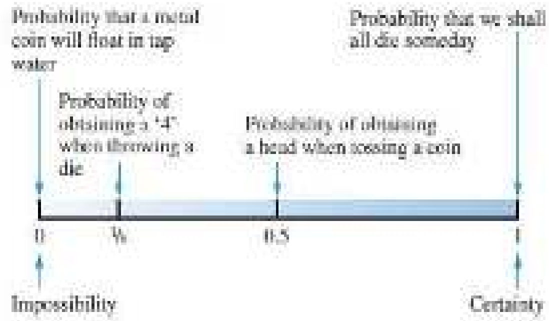
Finally the variance is given by

Temperature range (°C)	Frequency f_i	Class midpoint x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
16–17	6	16.5	−3.60	12.96	77.76
18–19	8	18.5	−1.60	2.56	20.48
20–21	7	20.5	0.40	0.16	1.12
22–23	4	22.5	2.40	5.76	23.04
24–25	5	24.5	4.40	19.36	96.80
Total	30				219.20

$$\text{variance} = \frac{\sum_{i=1}^{N} f_i(x_i - \bar{x})^2}{\sum_{i=1}^{N} f_i} = \frac{219.20}{30} = 7.307$$

The standard deviation is the square root of the variance, that is $\sqrt{7.307} = 2.703$.

Figure 31.1
The probability scale



probability scale in Figure 31.1. It is important to note that no probability can lie outside the range 0 to 1.

Key point

All probabilities lie in the range [0, 1].

Complementary events

Consider the following situation. A light bulb is tested. Clearly it either works or it does not work. Here we have two events: the first is that the light bulb works and the second is that the light bulb does not work. When the bulb is tested, one or other of these events must occur. Furthermore, each event excludes the other. In such a situation we say the two events are **complementary**. In general, two events are complementary if one of them must happen and, when it does, the other event cannot. The sum of the probabilities of the two complementary events must always equal 1. This is known as **total probability**. We shall see how this result can be used to calculate probabilities shortly.

- Unbiased = the outcome or probability has the same chance of occurrence as any other outcome or probability
- Theoretical Probability = A probability that is quite unlikely, although not impossible, when all events are equally likely it is calculated from the following formula:

Key point

When all events are equally likely

$$P\begin{pmatrix}\text{obtaining our}\\ \text{chosen event}\end{pmatrix} = \frac{\text{number of ways the chosen event can occur}}{\text{total number of possibilities}}$$

WORKED EXAMPLES

31.1 A fair die is thrown. What is the probability of obtaining an even score?

Solution The chosen event is throwing an even score, that is a '2', '4' or '6'. There are therefore three ways that this chosen event can occur out of a total of six, equally likely, ways. So

$$P(\text{even score}) = \frac{3}{6} = \frac{1}{2}$$

31.2 A fair coin is tossed. What is the probability that it will land with its head uppermost?

Solution There are two equally likely ways the coin can land: head uppermost or tail uppermost. The chosen event, that the coin lands with its head uppermost, is just one of these ways. The chance of getting a head is the same as the chance of getting a tail. Therefore,

$$P(\text{head}) = \frac{1}{2}$$

Note from the previous example that $P(\text{tail}) = \frac{1}{2}$ and also that

$P(\text{tail}) + P(\text{head}) = 1$

Note that the two events, getting a head and getting a tail, are complementary because one of them must happen and either one excludes the other. Therefore the sum of the two probabilities, the total probability, equals 1.

WORKED EXAMPLE

31.3 Two coins are tossed.

(a) Write down all the possible outcomes.

(b) What is the probability of obtaining two tails?

Solution (a) Letting H stand for head and T for tail, the possible outcomes are

$H, H \qquad H, T \qquad T, H \qquad T, T$

There are four possible outcomes, each one equally likely to occur.

(b) Obtaining two tails is just one of the four possible outcomes. Therefore, the probability of obtaining two tails is $\frac{1}{4}$.

In some circumstances we do not have sufficient information to calculate a theoretical probability. We know that if a coin is unbiased the probability of obtaining a head is $\frac{1}{2}$. But suppose the coin is biased so that it is more likely to land with its tail uppermost. We can experiment by tossing the coin a large number of times and counting the number of tails obtained. Suppose we toss the coin 100 times and obtain 65 tails. We can then estimate the probability of obtaining a tail as $\frac{65}{100}$ or 0.65. Such a probability is known as an **experimental probability** and is accurate only if a very large number of experiments are performed. Generally, we can calculate an experimental probability from the following formula:

Key point

$$P\left(\begin{array}{c}\text{chosen event}\\ \text{occurs}\end{array}\right) = \frac{\text{number of ways the chosen event occurs}}{\text{total number of times the experiment is repeated}}$$

31.4 A biased die is thrown 1000 times and a score of '6' is obtained on 200 occasions. If the die is now thrown again what is the probability of obtaining a score of '6'?

Solution Using the formula for the experimental probability we find

$$P(\text{throwing a '6'}) = \frac{200}{1000} = 0.2$$

If the die were unbiased the theoretical probability of throwing a '6' would be $\frac{1}{6} = 0.167$, so the die has been biased in favour of throwing a '6'.

31.5 A manufacturer produces microwave ovens. It is known from experience that the probability that a microwave oven is of an acceptable standard is 0.92. Find the probability that an oven selected at random is not of an acceptable standard.

Solution When an oven is tested either it is of an acceptable standard or it is not. An oven cannot be both acceptable and unacceptable. The two events, that the oven is acceptable or that the oven is unacceptable, are therefore complementary. Recall that the sum of the probabilities of complementary events is 1 and so

$$P(\text{oven is not acceptable}) = 1 - 0.92 = 0.08$$

Two events are **independent** if the occurrence of either one in no way affects the occurrence of the other. For example, if an unbiased die is thrown twice the score on the second throw is in no way affected by the score on the first. The two scores are independent. The **multiplication law** for independent events states the following:

Key point

If events A and B are independent, then the probability of obtaining A and B is given by

$$P(A \text{ and } B) = P(A) \times P(B)$$

WORKED EXAMPLE

31.6 A die is thrown and a coin is tossed. What is the probability of obtaining a '6' and a head?

Solution These events are independent since the score on the die in no way affects the result of tossing the coin, and vice versa. Therefore

$$\begin{aligned} P(\text{throwing a '6' and tossing a head}) &= P(\text{throwing a '6'}) \\ &\quad \times P(\text{tossing a head}) \\ &= \frac{1}{6} \times \frac{1}{2} \\ &= \frac{1}{12} \end{aligned}$$

When several events are independent of each other the multiplication law becomes

$$P(A \text{ and } B \text{ and } C \text{ and } D \ldots) = P(A) \times P(B) \times P(C) \times P(D) \ldots$$

WORKED EXAMPLE

31.7 A coin is tossed three times. What is the probability of obtaining three heads?

Solution The three tosses are all independent events since the result of any one has no effect on the others. Therefore

$$P(3 \text{ heads}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

Summary

In this week, we learned about what an average is, what mean, median, and mode are, what variance and standard deviation is, what complementary and independent events are, and what a theoretical and experimental probability is.