Base Conversion

Decimal to base$_n$ (quotient):

- Take the quotient of the decimal and divide by the base
- Note the remainder
- Repeat from step 1 until the quotient is zero
- Return the list of remainders, the first is the least significant digit of the new quotient. The last is the most significant.
- Eg: 273_{10} to base$_2$
  - 273 / 2 = 136 REM 1
  - 136 / 2 = 68 REM 0
  - 68 / 2 = 34 REM 0
  - 34 / 2 = 17 REM 0
  - 17 / 2 = 8 REM 1
  - 8 / 2 = 4 REM 0
  - 4 / 2 = 2 REM 0
  - 2 / 2 = 1 REM 0
  - 1 / 2 = 0 REM 1
  - $273_{10} = 100010001_2$

Decimal to base$_n$ (fractional)

- Take the fractional of the decimal and multiply by the base
- Take the new quotient and note it separately
- Repeat from step 1 until the result is 0
- Return the list of quotients as the new fractional. The first is the most significant digit, the last is the least.
- Eg: 0.625_{10} to base$_2$
  - 0.625 * 2 = 1.25 QUOT 1
  - 0.25 * 2 = 0.5 QUOT 0
  - 0.5 * 2 = 1.0 QUOT 1
  - 0.0 * 2 = 0.0
  - $0.625_{10} = 0.101_2$

Base$_n$ to decimal (quotient)

- Multiply each digit by the base raised to the power of that digit’s number.
- Add the results
- Eg: 101101_2
  - $1 * 2^0 = 1$
  - $0 * 2^1 = 0$
  - $1 * 2^2 = 4$
  - $1 * 2^3 = 8$
  - $0 * 2^4 = 0$
  - $1 * 2^5 = 32$
  - Sum = 45

Base$_n$ to decimal (fractional)

- Multiply each digit by the base raised to the inverse power of that digit’s number.

- Add the results
- Convert fraction to decimal
- Eg: 0.1101_2
  - $1 * 2^{-1} = 1/2$
  - $0 * 2^{-2} = 0$
  - $1 * 2^{-3} = 1/8$
  - $1 * 2^{-4} = 1/16$
  - Sum = 11/16
  - 11/16 = 0.6875

Base Arithmetic

As normal, but carry on the base number, not always 10.

Modular Arithmetic

Basics

Modular math = clock arithmetic

It is concerned with the remainders of division by a particular number

Remainders are always between 0 and N – 1. So working in (mod 5) we would expect remainders between 0 and 4.

N modulo K:

- If N < K then Mod = N.
  - Eg.: 5 (mod 7) ≡ 5 (mod 7)
- If N > K then Mod = N - (quotient(N/K) * K). In other words, find the remainder of N/K, it’s not rocket science.
  - Eg.: 29 (mod 7) ≡ 29 / 7 = 4 REM 1 ≡ 1 (mod 7)
- Negation: N (mod K) can also be expressed as -(K – N) (mod K)
  - Eg.: 255 (mod 257) ≡ -2 (mod 257)

Special cases:

- (mod 10) Just take the least significant digit
- (mod 5) Least significant digit, if greater than five subtract five.
- (mod 9) Casting of 9s, probably a product of working in $base_{10}$.. Add together all of the digits. Then add together the digits of the result. Continue until you have a single number. This is the remainder.
- (mod 3) Cast of 9s. Take result and divide by 3. Remainder is mod.
- (mod 11) Sum (even order digits) – Sum (odd order digits) = mod.

Arithmetic summary:

- (X + Y) (mod K) ≡ ((X (mod K)) + (Y (mod K)) (mod K))
- (X - Y) (mod K) ≡ ((X (mod K)) - (Y (mod K)) (mod K))
- (X * Y) (mod K) ≡ ((X (mod K)) * (Y (mod K)) (mod K))

Additive identity:

The additive identity of any (mod K) is 0, as when added to anything else it causes not change.

Additive inverse:

Any pair of mod results which add up to K are “additive inverses” as they result in mod 0.

Eg: 2 + 3 = 0 (mod 5)

This can also be stated as 3 being the -2 of (mod 5), as when added to 2 it produces 0.

May be expressed as:

-23 (mod 5)

Ans:

-23 (mod 5) ≡
-3 ≡
2

Multiplication:
As in arithmetic summary, calculate A(mod K) and B(mod K), multiply the results then calculate Result(mod K)

Eg:
11 * 13 (mod 9) ≡
2 * 4 (mod 9) ≡
8

Multiplicative identity:
The multiplicative identity of any (mod K) is 1, as when multiplied by anything else it doesn’t change it.

Multiplicative inverse:
With linear numbers the inverse is the number which, when multiplied by the original, produces 1.

Eg. The inverse of 2 (or 2/1 or 2^1) is 1/2 (or 2^{-1})

The inverse of modular numbers is the same: The numbers which when multiplied will produce 1. This is arrived at by multiplying them normally then finding the modulo of the product.

Eg. The inverse of 2 (mod 5) is 3. As 2 * 3 = 6 and 6 (mod 5) is 1
This can also be written as 2^{-1} (mod 5) ≡ 3

Naive calculation of X^{-1} (mod K):

- Multiply X by every N from 0 to K-1
- Find the (mod K) of each of those numbers
- X^{-1} (mod K) = whichever one results in 1

Calculation of X^{-1} (mod K) when K is prime:

- Fermat’s little theorem: X^{K-1} ≡ 1 (mod K)
- If we multiply both sides by X^{-1} we find that:
- X^{-1} ≡ X^{K-2} (mod K)
- So, we raise X to the power of K-2
- Find the result
- Then find the modulo (mod K)
- Which leaves us the inverse

Calculation of X^{-1} (mod K) when K is not prime:

- Count how many numbers from 1 -> K1 that are coprime with K
- Coprime: Has no factors in common other than 1
- Call this number N
- $X^{-1} \equiv X^{n-1}$ (mod K)
- Eg:
  - 9^{-1} (mod 22)
  - factors of 22 are: 2,11
  - [1->21]= [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21]
  - Remove numbers coprime with 22 (ie. Divisible by 2 or 11)
  - [1,~~2~~,3,~~4~~,5,~~6~~,7,~~8~~,9,~~10~~,~~11~~,~~12~~,13,~~14~~,15,~~16~~,17,~~18~~,19,~~20~~,21]
  - n = len[1,3,5,7,9,13,15,17,19,21] = 10
  - $9^{-1} \equiv 9^9 \equiv 5$ (mod 22)
- Check your results: $(X * X^{-1})\%K$ should equal 1
  - (9*5)%22 = 45%22 = 1

Exponentiation:

- Simplify the base to the lowest congruant integer.
  - Eg. $277^8 \equiv 2^8$ (mod 5)
- Gradually raise this integer to the required power. This can be achieved by working out basic powers and multiplying them together to “add” the exponent.
  - Eg. $2^8 = 2^5 * 2^3 = 32 * 8 = 256$
- We can also multiply the exponent using powers
  - Eg. $2^{64} = (2^8)^8$
- Eg in (mod 5):
  - $2^3 \equiv 8 \equiv 3$ (mod 5)
  - $2^5 \equiv 32 \equiv 2$ (mod 5)
  - $2^8 \equiv 2^5 * 2^3 \equiv 3 * 2 \equiv 6 \equiv 1$ (mod 5)
- NB: X^{K-1} (mod K) = 1 PROVIDED X/K is not a whole number
- This means that if K is prime, X will not evenly divide by K
- So all prime numbers are easier to work with in modular arithmetic, as we only ever have to calculate K-1 powers to calculate any X^Y (mod K)

Encryption using Modular Arithmetic:

RSA encryption works using a public/private key system (you already know this)

- For a given (mod K), take any base (the plaintext character) and raise it to the power E.
- Resolve this power (find it’s modulo) and transmit it, it is now encrypted.
- At the other and, raise it to the power D.
- When this new power is resolved, the resulting modulo is the original plaintext character.
- The public key is E combined with K
- The private key is D combined with K

Finding key pairs in mod K (where K is prime):

- E is the multiplicative inverse of D in (mod K-1)
- So E * D ≡ 1 (mod K-1)
- So by fermat's little theorem:
- $D \equiv E^{K-3}$ (Mod K-1)

Sequences

Sequences:

- Sequences are any list of numbers
- Some sequencences have a logical progression, some do not
- If a sequence is infinite this is noted by placing a ... at the end
  - Eg.: 1,2,3,4,5,6...
- If there is a gap in the middle, we can also use ...
  - Eg.: 1,2,3...4,5,6
- Sequences can also be defined with formulae
  - Eg.: $X_n = n$

Arithmetic Sequences:

- An arithmetic sequence exists where each term is found by adding a constant number to the previous term.
  - Eg.: 1,5,9,13,17 (n+4)
- One expression is the “recurrance relation”: $a_{n+1} = a_n + d$
  - This uses the current term to define the next one
  - D is the “common difference”
  - N is the current iteration
  - In order to use this definition the first term must be provided separately
- One can also write the “general term”: $A_n = A_1 + d*(n-1)$
- In order to obtain term n using the recurrance relation, then we must calculate all the terms before n. In order to obtain n with the general term we simply slot our number into the formula.

Geometric Sequences:

- A geometric sequence exists where each term is found by multiplying the previous term by a constant number.
  - Eg.: 1,3,9,27,81,243
- The “recurrance relation” of this sequence is: $A_n = R*A_{n-1}$
  - R = The common ratio, or the factor used to create the new term
- The “general term” of this sequence is: $A_n = A_1 * R^{n-1}$
- As a rule: $A_n/A_{n+1} = R$
- This can be used on multiple terms to determine if the sequence is truly geometric
- Checking if a X is a term in a geometric sequence
  - Write out the general term
  - Resolve so that the formula is a single power
  - If X is a power of the base then yes. If not then no
- Negative ratios will result in alternating positive and negative terms
- Fractional ratios: Result in decreasing and convergant sequences

Series

- A series is the sum of the terms of a sequence
- The notation is:

$$\sum_{n=0}^{63} 2^n$$

  - Sum of terms from A_0 to A_{63}
  - Where each term is 2^n
- Triangular numbers: An arithmetic series where D = 1 produces triangular numbers
  - Eg.: 1+2+3+4+5 = 15
  - They are called this as that number of objects can be arranged in a regular triangle, like bowling balls:
  - Eg.: 10
  - 0
  - 00
  - 000
  - 0000
- If a general term consists of the sum of two components, it can be treated as the summation of two separate summations.
- Eg.:
  - $(4)\Sigma(n=1)\ (n + 2^n) = (4)\Sigma(n=1)(n) + (4)\Sigma(n=1)(2^n)$
- The same is not true of two components multiplied together.
- If a series is multiplied by a number, or has another operation applied to it, it is the same as applying that operation to the general term.
- Eg.:
  - 5*(4)Σ(n=1)(n) = (4)Σ(n=1)(5n)

Summing arithmetic sequences:
- Add the first and last terms
- Multiply this number by half the number of terms
- Eg.:
  - 1+4+7+10+13+16+19+22+25+28 =
  - (1+28)*5 =
  - 145
- If the number of terms is odd, this formula still works, you just multiply by the half.
- Eg.:
  - 1+4+7+10+13 =
  - (1+13)*2.5 =
  - 35

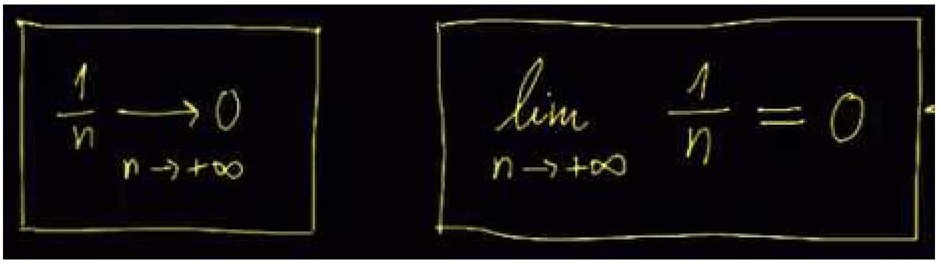
Summing geometric sequences:

- Where S is the sum of the terms of the sequencene, N is the number of terms and R is the common ratio.
- $S = A_1*((1-R^n)/(1-R))$
- Eg.
  - Sequence = 1,2,4,8,16
  - $A_n = 2^{n-1}$
  - R = 2
  - N = 5
  - $S = (2^5-1)/(2-1)$
  - = (32-1)/(1)
  - = 31/1
  - = 31
- This can be proven:
  - S = 1+2+4+8+16
  - 2S = 2+4+8+16+32
  - If we subtract S from 2S we get S, or:
  - S =-1 +32
  - S = 32-1
  - S = 31
  - So we multiplied S by the common ratio (SR), then subtracted S by the common ratio – 1 (S*(R-1))
  - The result was $A_{n+1} – A_1*(R-1)$

Other summation formulae:

- Sum of numbers from 1->N
  - N(N+1)/2
  - Ie. 1+2+3+4+5+6...+N
  - Based on the formula for the sum of arithmetic series
- Sum of squares from 1->N
  - N(N+1)(2N+1)/6
  - Ie. $1^2+2^2+3^2+4^2+5^2+6^2...+N^2$
- Sum of cubes from 1->N
  - $N^2(N+1)^2/4$
  - Ie. $1^3+2^3+3^3+4^3+5^3+6^3...+N^3$

Convergence, Divergence & Limits

- Limits are a number which is approached but not reached by a sequence.
- Eg.:
  - 1,1/2,1/3,1/4,1/5... -> 0
  - May be noted as:
  - 

- In order to find a limit, one can separate the series into parts.
  - Eg, Σ (1/n + 5) = Σ (1/n) + Σ(5)
  - Σ (1/n) -> 0 Therefore the limit is 5

- All arithmetic series are divergent and approach +inf or -inf
- Geometric series are divergent if R > 1, and approach either +inf or -inf
- If R<-1, then they are divergent with no limit
- If R<1 && R>-1 they are convergent on 0

- If you're stuck, you can always sum a finite series to see where it goes.
- Harmonic series =
  - 1/1, 1/2, 1/3, 1/4... 1/inf
  - ^ Not convergent

- If we add two convergent sequences, the result is also convergent and the sum of that series is equal to the sum of the original series added together.
  - Take two convergent sequences S_1 and S_2
  - S_1 + S_2 is also convergent
  - $\Sigma(S_1 + S_2) = \Sigma(S_1) + \Sigma(S_2)$

- If we multiply a convergent sequence by a constant, the result is convergent. Also, the limit of the new sequence is the same as the limit of the old sequence multiplied by the constant.
  - Where A_n is a convergent sequence and C is a constant:
  - $Lim(A_n)*C = Lim(A_n*C)$
  - And both are convergent

- When deciding the convergence of a sequence, one approach can be to consider similar sequences

- If $\Sigma A_n < \Sigma B_n < \Sigma C_n$ and both A_n and C_n are convergent, then B_n will be convergent also.
- As a bonus: $\mathrm{Lim}(a_n) < \mathrm{Lim}(b_n) < \mathrm{Lim}(C_n)$

- Hypothesis: If you divide the terms of a series by N, the sum is = Σ/N

Functions and Graphing

2D Cartesian coordinate system:
I don’t think I need to cover this

Function notation and definitions:

- Two common styles of function notation:
  - $f(x):x^2$
  - $y = x^2$
- Independent variable:
  - Normally plotted on the x-axis
  - So called because we can input any value into it
- Dependant variable:
  - Normally plotted on y-axis
  - So called because it’s value is dependant on the value of the independant variable
- Domain: The series of values inputted as the independant variable
- Range: The series of values solved for the dependant variable

Interval notation:

- Expresses a set of values, eg: 0,1,2,3,4,5 = [0,5]
- Square brackets “[” include the value they enclose
- Round brackets “(” exclude the value they enclose
- So [0,5) means the set {0,1,2,3,4}
- If there is a gap in a set it may be expressed as two sets using venn notation
- Eg all positive and negative numbers except zero:
  - (-inf,0)U(0,inf) using interval notation. The “U” represents the “union” of the two sets.
  - This can also be written as “R \ {0}”
- Another example, all positive numbers and zero:
  - [0,inf)
  - R^+_0

Asymptote: Graphical equivalent of a limit. A line the graph approaches but will never touch.

Intercept: The point (or points) at which a graph cuts an axis.
Vertical intercept: Where it cuts the y axis
Horizontal intercept: Where it cuts the x axis
Vertex: A graph’s point of greatest extent along a particular axis.

Types of graph:

- Line: $y = mx + c$
  - m is the slope
  - c is the vertical intercept

- Quadratic:
  - An arch type shape (parabola)
  - Caused by three terms summed together:
    - $y=ax^2+bx+c$
    - a is the coefficient of x^2 and affects the width of the arch
    - If a is positive, the parabola will be “u” shaped, if it is negative the arch will be drawn “n” shaped.
    - b is the coefficient of x and affects the position of the arch
    - c is the independent term (as it is not affected by the value of x) and defines the vertical intercept. Altering this value will move the parabola up or down the y axis.
  - We can also solve for x:
    - $x = (-b+-sqrt(b^2-4ac))/2a$
  - Vertex may be found by finding two values for x at the same y, then finding their center. Then solve for y with this x value.
  - Alternately, if the function can be restated as
    - y=(x+a)2 + b
    - Then the vertex will be (a,b)
- Cubic:
  - Draws a compound curve which travels from one quarter to the opposite quarter (eg. Top-left to bottom-right)
  - As it approaches the origin it transitions through an S shaped bend
  - Caused by three terms summed together:
    - $y=ax^3+bx+c$
    - a is the coefficient of x3 and affects the slope of the curves. Higher = narrower, lower = broader
      - If it is positive, it produces an ascending curve (bottom-left to top-right)
      - If it is negative it produces a descending curve (top-left to bottom-right)
    - b is the coefficient of x and affects the eccentricity of the s bend
      - A higher value will produce a more pronounced bend if the sign is opposite “a”
      - A higher value will produce a straighter bend if the sign is the same as “a”
    - c (the independent term) is the vertical intercept
      - A more negative value moves the curve down the y axis
      - A more positive value moves it up
- Higher order polynomials
  - Involving powers higher than x^3
  - If a term can be reduced to zero, then the graph will cross the x-axis at that point
    - Eg: y = (x-4)(x+2)(x-2)(x+5)
    - The first term will resolve as zero when x is 4, therefore there is a point at (4,0)
- Reciprocal function
  - X is the denominator of a fraction, 1 is the numerator

        - Eg.: y = 1/(x-5)
    - Forms a rectangular hyperbola
    - Always has a value for X which will result in division by zero and cannot be computed. In this case x = 5
    - This is a vertical asymptote
    - So the domain is real numbers except 5:
        - R\{5}
        - (-inf,5)U(5,inf)
- Rational function
    - X is the numerator, X^2 minus a constant is the denominator
        - Eg.: $x/(x^2-3)$
    - Has two vertical asymptotes (in this case, sqrt(3) and negative sqrt(3)
    - Also has a horizontal asymptote
- Piecewise function
    - Function broken down into separate conditional statements
        - Eg: if(y<=5){y=x}, if(y>5){y=2x-2}
        - The sub function can be any other type of function

Transformations:

- Translation
    - Horizontal translation: Add/subtract value to/from every X before any other operation is performed
        - Eg: $y = x^2 + 3x$
        - Shift right by three units:
        - $y = (x - 3)^2 + 3(x - 3)$
    - Vertical translation: Add/subtract value after all other operations are performed.
        - Eg: $y = x^2 + 3x$
        - Shift up by two units:
        - $y = x^2 + 3x + 2$
- Rotation
- Scaling
    - Vertical scaling: Multiply all terms by the scalar after every other operation
        - Eg: $y = x^3 + x^2 + 5$
        - Scaled on the y axis by a factor of 2 becomes:
            - $y = 2x^3 + 2x^2 + 20$
            - or
            - $y = 2(x^3 + x^2 + 10)$
    - Horizontal scaling: Multiply every x by the inverse of the scalar prior to any other operation
        - Eg: $y = x^3 + x^2 + 5$
        - Scaled on the x axis by a factor of 2 becomes:
            - $y = (x*1/2)^3 + (x*1/2)^2 + 5$
- Reflection
    - Change the sign
    - Eg.: (5,2)
        - Reflected on the X axis becomes (5,-2)
        - Reflected on the Y axis becomes (-5,2)
    - Eg.: $y = x^2 + 2x +3$
        - Y axis reflections: Invert the sign of the final result

        - $y = -(x^2 + 2x + 3)$
        - X axis reflections: Invert the sign of each X
        - $y = (-x)^2\ 2(-x) + 3$

Kinematics:

- Displacement: Net distance moved from beginning to end. On a round trip displacement is zero
- Speed: Distance moved per unit of time
- Velocity:
- Acceleration: Rate of change of velocity. “Delta V”
- SUVAT Equations:
    - S: Position in time
    - U: Initial velocity
    - V: Final velocity
    - A: Acceleration
    - T: Time elapsed
    - $s=ut+1/2at^2$
    - $s=1/2(u+v)t$
    - $v2=u^2+2as$
    - $v=u+at$

Trigonometry

Angles:
- Acute: < 90, > 0
- Right angle: = 90
- Obtuse: > 90, < 180
- Reflex: > 180, < 360

Triangles:
- Types:
  - Equilateral: Three equal sides
  - Isoceles: Two equal sides
  - Scalene: No equal sides
  - Right angle: One angle = 90 deg
- Rules:
  - All angles add up to 180 deg
  - Triangular inequality: The length of one side is less than the length of the other two.
- Notation:
  - ABC: Angles
  - abc: Sides opposite those angles
    - Eg:. If C = 90 deg then c = hypotenuse

Radians:
- 1 rad = the angle corresponding to a chord with length of r
- 360 degrees = 2pi rad
- 360/(Angle in degrees) = 2pi rad/(Angle in radians)
- Conversions:
  - Angle in Radians = (Degrees * 2pi)/360
  - Angle in Radians = Degrees * (pi/180)
  - Angle in Degrees = (Radians * 360)/2pi
  - Angle in Degrees = Rads * (180/pi)
- Rule of thumb: Calculate rad angles to 3 decimal places where you'd calculate degrees to 1 decimal place.

Radicals/Surds:
- A surd is an irrational square root
  - Eg sqrt(2)
- A root * a root = the contents multiplies
  - Eg.: sqrt(2) * sqrt(3) = sqrt(6)
- This can be used to simplify radicals of the same base
  - Eg.: sqrt(12) = sqrt(4*3) = sqrt(4)*sqrt(3) = 2*sqrt(3)
- Xsqrt(Y)* Asqrt(B) = XAsqrt(YB)
- When factorising try for prime numbers
- A root / a root = the contents divided
  - Eg.: sqrt(30)/sqrt(10) = sqrt(30/10) = sqrt(3)

Pythagoras' Theorem:
- $a^2 + b^2 = c^2$

- Where abc is a right angled triangle
- And c is the hypotenuse
- Can be used to find a missing side
- Also to test if a triangle is right angled

Trigonometric Ratios:

- Unit triangle
  - Right angled triangle
  - Hypotenuse = 1
- To find the length of sides:
  - Sine = Opposite/Hypotenuse
  - Cosine = Adjacent/Hypotenuse
  - Tangent = Opposite/Adjacent
- To find the value of angles:
  - Arc-sine = Theta, given a certain O/H
  - Arc-cosine = Theta, given a certain A/H
  - Arc-tan = Theta, given a certain O/A
  - Also may be noted as: $\sin^{-1}$ $\cos^{-1}$ and $\tan^{-1}$

Trigonometric Rules:

- Sine rule:
  - a/sin(A) = b/sin(B) = c/sin(C)
  - We can use this to solve triangles if:
    - We have two lengths and an angle opposite one of them
    - We have two angles and a length opposite one of them
- Cosine rule:
  - Length forms:
    - $a^2 = b^2 + c^2 - 2bc \cos(A)$
    - $b^2 = a^2 + c^2 - 2ac \cos(B)$
    - $c^2 = a^2 + b^2 - 2ab \cos(C)$
  - Angle forms:
    - $\cos(A)=(b^2 + c^2 - a^2)/2bc$
    - $\cos(B)=(a^2 + c^2 - b^2)/2ac$
    - $\cos(C)=(a^2 + b^2 - c^2)/2ab$
  - We can use this to solve triangles if:
    - We have two lengths and one of the opposite angles
    - We have two lengths and the angle between them
    - We have all three lengths
- Other rules:
  - $\sin^2(A)+\cos^2(A)=1$
  - Cos(A)=Sin(A+90deg)
  - Sin(A)/Cos(A)=Tan(A)

Trigonometric Functions

Angles, Quadrants and Coordinates:

- Quadrants run anticlockwise
  - 1st quadrant: Between 3:00 and 12:00
  - 2nd quadrant: Between 12:00 and 9:00
  - 3rd quadrant: Between 9:00 and 6:00
  - 4th quadrant: Between 6:00 and 3:00
- Measure angles going anticlockwise starting with the positive subaxis of X as the starting point.
  - Eg. (0,5) would have an angle of 90deg (1/2pi rad)

Unit circle:

- Circle with origin (0,0) and radius 1
- Used to define trigonometric ratios
- Trigonometric ratios only cover angles between 0 and 90 degrees. In order to use larger angles:
  - 90-180: Theta = 180 – angle
  - 180-270: Theta = angle - 180
  - 270-360: Theta = 360 – angle
- The sign of cos and sin will also need to be adjusted to account for projections into -x and -y space.
  - Quadrant 1: +x(cos) +y(sin) +y/x(tan)
  - Quadrant 2: -x(cos) +y(sin) -y/x(tan)
  - Quadrant 3: -x(cos) -y(sin) +y/x(tan)
  - Quadrant 4: +x(cos) -y(sin) -y/x(tan)
- P is a point in the first quadrant
  - Theta is the angle between the positive X semi-axis and the radius pointing to P
  - The compliment of P (with angle = 180-theta) in the second quadrant may be noted as P^{I}
  - Similarly the compliment of P in the third and fourth quadrant may be noted as P^{II} and P^{III} respectively
- Converting from Sine to Cosine
  - To convert Sin(x) to Cos(x), translate left by pi/2
  - So Cos(x) = Sin(x + pi/2)
  - Equally, to convert Cos(x) to Sin(x), translate right by pi/2
  - So Sin(x) = Cos(x - pi/2)

Common Angles:

- 0 deg:
  - Sin: 0
  - Cos: 1
  - Tan: 0
- 30 deg:
  - Sin: 0.5
  - Cos: sqrt(3)/2
  - Tan: 1/sqrt(3)
- 45 deg:

    - Sin: 1/sqrt(2)
    - Cos: 1/sqrt(2)
    - Tan: 1
- 60 deg:
    - Sin: sqrt(3)/2
    - Cos: 0.5   pi/3
    - Tan: sqrt(3)
- 90 deg:
    - Sin: 1
    - Cos: 0
    - Tan: NaN**olar to Cartesian:**
- Convert angle to first quadrant
    - Eg: 320 = 40
- X = Cos(A) * R
- Y = Sin(A) * R
    - Adjust sign appropriately
- 180 deg:
    - Sin: 0
    - Cos: -1
    - Tan: 0
- 270 deg:
    - Sin: 1
    - Cos: 0
    - Tan: NaN

Graphs of trigonometric functions:
- Sine is (unsurprisingly) a sine wave with a period of 2pi and an origin at (0,0)
- Cosine is an offset sine wave with the same period and an origin at -pi/2
- Tan is an s shaped curve which ranges from -infinity to infinity with an asymptote whenever sine = 0. It's origin is (0,0) and it's period is pi.
- Solving graphically:
    - If y = Sin(x)
    - And Sin(x) = 1/2
    - This means we are solving the function for y=1/2
    - Depending on the domain, this will produce a number of points.

Inverses of trigonometric functions:
- Inverses may be graphed as the reflection of a section of their compliment across the diagonal line y=x.
- When they are inverted, the domain and the range also invert
- ArcSin:
    - Domain: [-1,1]
    - Range: [-1/2pi,1/2pi]
- ArcCos:
    - Domain: [-1,1]
    - Range: [0,pi]
- ArcTan:
    - Domain: R

    - Range: (-1,1)

Translations of trigonometric functions:

- The same as regular functions...
- Translation:
    - Vertical: add/subtract value after all other operations
    - Horizontal: add/subtract value to Xs before all other operations. Sign is inverse, so negative goes right, positive goes left.
- Scaling:
    - Vertical: Multiply by scalar after all other operations
    - Horizontal: Multiply all Xs by inverse of scalar before all other operations. Eg. If the scalar is 2, multiply by 1/2.
- Reflection:
    - Vertical: Change the sign of the solution after all other operations
    - Horizontal: Change the sign of all Xs before any other operation

Solving Trigonometric Equations:

- Be aware that equations may have more than one valid solution.
- Such equations are often restricted to a specific range, eg. -180 > x > 180
- x must be solved for every potential value within this range:
    - Eg: Sin(x – 30) = Sin (80)
    - x = 110
    - Is also equivalent to:
    - Sin(x – 30) = Sin (100)
    - x = 130
- If the ratio is negative, restate it with a positive angle in the negative quadrant.
    - Eg. -tan(30) = tan(150)
- When finding equivalents of tan(x), tan has a period of pi or 180. So simply find the first value and add/subtract 180 from it.
- When considering possible values, search for the equivalent angles that share the same sign.
    - For example, Sin(100) is the equivalent of Sin(80) as 80 is in the first quadrant where the sign of sines are positive, and 100 is in the second quadrant where they are also positive.
    - For negative values we take all of the solutions and append 360k
    - x = 110 + 360k
    - x = 130 + 360k
    - Next we substitute values for k and see if they are congruant with our range.
    - In this case k=1 will produce values that exceed our range. K = 0 will work, k = -1 will work, k=-2 will exceed our range.
    - This gives us:

- k = 0
- x = 110
- x = 130
- k = -1
- x = 110 - 360
- x = 130 - 360
- We then solve for x
- x = 110
- x = 130
- x = -230
- x = -250

- When noting results, use set notation:
  - X element{-250,-230,110,130}
- Or longer hand notation:
  - x=-250 or x=-230 or x=110 or x=130

Polar coordinates:

- Cartesian coordinates: Your distance from the x and y axes *(x deflection, y deflection)*
- Polar coordinates: Your distance from the origin and the angle from the positive X semi-axis *(radius, angle)*
- Eg : Cartesian(1,1) = Polar(sqrt(2), 45)

Converting from Cartesian to Polar:

- Eg, (2,1) to (r,a)
- The distance from the origin can be found using pythagoras' theorem
  - r = sqrt(2^2 + 1^2) = sqrt(5) = 2.23
  - r = 2.23
- The angle from the pos x semi-axis may be found via tan and cotan
  - Tan(a) = 1/2
  - a = Cotan(1/2) = 26.56
  - a = 26.56
- Cartesian (2,1) = Polar (2.23,26.56)
- Take into account the signs of the coordinates and convert to the relevant quadrant

Converting from Polar to Cartesian:

- Convert angle to first quadrant
  - Eg: 320 = 40
- X = Cos(A) * R
- Y = Sin(A) * R
- Adjust sign appropriately

Exponential and Logarithmic Functions

Exponentiation:

- Whole integers: 4^3
  - Equivalent to 4 * 4 * 4
  - Raising to a power of 0 = 1
  - Eg, $3^0 = 1$
  - Raising to a power of 1 = itself
  - Eg $6^1 = 6$
- Negative powers: 4^{-3}
  - Invert the base
  - So: $(1/4)^3$
  - Equivalent to 1/4 * 1/4 * 1/4
- Fractional powers: $4^{2/3}$
  - Becomes: $\sqrt[3]{4^2}$
  - Or: $(\sqrt[3]{4})^2$
  - Denominator of the fraction is the root
  - Numerator of the fraction becomes the new power
  - Base remains the base
- Irrational powers $4^{\sqrt{3}}$
  - $4^{1.73}$
  - $\sqrt[100]{2^{173}}$
  - Approximate to n decimal places

Exponential Arithmetic:

- When multiplying two powers with the same base:
  - Add the powers
  - Eg: $2^3 * 2^4 = 2^7$
- When dividing two powers with the same base:
  - Subtract the powers
  - Eg: $2^4 / 2^2 = 2^2$
- When multiplying two powers with the same exponent:
  - Multiply the bases
  - Eg: $2^2 * 5^2 = 10^2$
- When dividing two powers with the same exponent:
  - Divide the bases
  - Eg: $4^5 / 2^5 = 2^5$
- To find the inverse of an exponential function:
  - Invert the power and apply it to the result
  - Eg: $4^2 = 8$
  - So: $8^{1/2} = 4$
- Change the base of a power
  - Eg: $8^2 = 2^x$
  - Rephrase 8 as a power of 2
  - So: $8^2 = (2^3)^2 = 2^6$

Euler's Number:

- E = the limit of $(1+x/n)^n$

- Originally used when analysing compound interest
- Called the “natural base” as it is the exponential function equal to it’s own derivative
- I think this means that the limit of it’s own exponential function is itself

Solving exponential equations:
- Use exponential arithmetic techniques to find a common base
- Then use regular arithmetic with the exponents

Transforming exponential graphs:
- X axis reflection: Change sign of entire equation
  - Eg: $y = e^x$
  - Becomes: $y = -(e^x)$
- Y axis reflection: Change sign of exponent
  - Eg: $y = e^x$
  - Becomes: $y = e^{-x}$
- Horizontal translation: Add/subtract from x
  - Move left: $y = e^{x+1}$
  - Move right: $y = e^{x-1}$
- Vertical translation: Add/subtract from result of equation
  - Move up: $y = (e^x) + 1$
  - Move down: $y = (e^x) – 1$
- Horizontal scaling: Multiply/divide x by inverse of scalar
  - $y = e^{2x}$
- Vertical scaling: Multiply/divide result of equation by inverse of scalar
  - $y = 2(e^x)$

Logarithms:
- Inverse of exponentiation
  - Where $X = b^Y$
  - $Y = \mathrm{Log}_b(X)$
- Logarithms in base e are “natural logarithms”
  - Eg. $e^x = 24$
  - Or: $\mathrm{Log}_e(24) = x$
  - Or: x = Ln24
  - Ln means “natural logarithm”
- On a calculator:
  - Ln = Log_e
  - Log = $\log_{10}$
- Cannot find the log of a negative number: These are imaginary
- Also cannot find the log of 0

Logarithmic Arithmetic:
- Adding logs of same base:
  - Multiply the logs
  - $\mathrm{Log}_2(4) + \mathrm{Log}_2(3) = \mathrm{Log}_2(4 * 3) = \mathrm{Log}_2(12)$
- Subtracting logs of same base:
  - Divide the logs
  - $\mathrm{Log}_2(9) – \mathrm{Log}_2(3) = \mathrm{Log}_2(9/3) = \mathrm{Log}_2(3)$
- Logarithm of a power:

    - Exponent multiplied by Log of base
    - $Log_2(9^3) = 3 * Log_2(9)$
- For inverse bases with the same log
    - Change the sign of the result
    - $Log_2 16 = 4$
    - $Log_{1/2} 16 = -4$
    - This is just another way of saying that negative powers produce fractions
- Changing base:
    - Changing from base b to d
    - $log_b(x) = log_d(x)/log_d(b)$
    - So, $log_3(5) = log_{10}(5)/log_{10}(3) = 1.46$

Logarithmic functions:
- Inverse of exponential functions
    - Reflection of an exponential curve across the 45 degree line y = x
    - Take the table of values for exponential functions, and flip the axes
    - So if a point is (1, 2) it will now be (2, 1)
- When the base is greater than zero
    - The result is a relatively flat curve, it's x value grows extremely quickly as y increases
    - It has a vertical asymptote of 0 (if not translated)
    - When x = 1, y is 0
    - When x < 1, y is negative
    - When x > 1, y is positive
- When the base is between one and zero:
    - The graph is reflected on the x axis
    - This is presumably the inverse graph of the whole version of that number
- For all bases:
    - Where X = 1, Y = 0
    - Where X = (the base), Y = 1

Logarithmic Equations:
- Graphical solution:
    - Literally just plot it on a graph and find the result
- Calculator:
    - In some cases it may be possible to simply solve the equation directly on your calculator
    - Most calculators only have ln and log(base 10) functions so we may need to change the base
- Algebraically:
    - Refer to arithmatic section
    - Not always necessary to provide a numeric answer. Rephrasing it as a power can be acceptable
    - Remember that we can't find the log of negative numbers

Transformations of Logarithmic Graphs:
- X axis reflection: Change sign of entire equation
    - Eg: y = Log(x)

    - Becomes: y = -Log(x)
- Y axis reflection: Change sign of log
    - Eg: Log(x)
    - Becomes: Log(x)
- Horizontal translation: Add/subtract from x
    - Move left: $y = e^{x+1}$
    - Move right: $y = e^{x-1}$
- Vertical translation: Add/subtract from result of equation
    - Move up: $y = (e^{x}) + 1$
    - Move down: $y = (e^{x}) – 1$
- Horizontal scaling: Multiply/divide x by inverse of scalar
    - $y = e^{2x}$
- Vertical scaling: Multiply/divide result of equation by inverse of scalar
    - $y = 2(e^{x})$

Inverting a Function:
- Eg: $y = 5e^{x-1}$
- First step, swap the variables
    - $x = 5e^{y-1}$
- Solve for y
    - $x/5 = e^{y-1}$
    - Ln(x/5) = y - 1
    - y = Ln(x/5) + 1

Calculus

Blah

Vectors

Vector basics:
- Vectors are a direction and magnitude
- As opposed to scalars which are just magnitude
- They have a tail (origin) and head (destination)
- Notation:
  - Vector = $\vec{v}(\frac{1}{2})$
    - Where 1 is the x displacement and 2 is the y displacement
  - Magnitude = $|\vec{v}| = \sqrt{1^2+2^2} = \sqrt{5}$
- V(displacement) = C(head point) – A(tail point)

Arithmetic:
- In order to check if vectors are parallel, find out if one is a multiple of the other
- The complement of a vector:
  - The same magnitude in the opposite direction
  - Switch the sign on each coordinate
  - $\vec{v} = (1,2)$
  - $-\vec{v} = (-1,-2)$
- A point plus a vector is a point
  - The tail is point A, the vector extends from it, the head is point B
- A vector plus a vector is a vector
  - The head of Vector A is the tail of Vector B. Vector C extends from tail A to head B
- A vector minus a vector is a vector
  - Same as adding, but add the complement
- A point (head) minus a point (tail) is a vector
  - This is essentially reversing a point plus a vector. We start at the head and work backwards to find the tail. This reverse operation defines the vector.
- A vector multiplied by a value (scalar) is a vector
  - Just multiply each coordinate by the value
- A vector divided by a value (scalar) is a vector
  - Just divide each coordinate by the value
- Dot Product:
  - Notation: $\vec{v} \cdot \vec{w}$
  - Operation: $(v(x) * w(x)) + (v(y) * w(y))$
  - Where θ is the angle between $\vec{v}$ and $\vec{w}$
  - The dot product is $\cos\theta * |\vec{v}| * |\vec{w}|$
  - This can be used to find:
    - $\vec{v}$ projected on to $\vec{w}$ = $\vec{v} \cdot \vec{w} / |\vec{w}|$
    - $\vec{w}$ projected on to $\vec{v}$ = $\vec{v} \cdot \vec{w} / |\vec{v}|$
    - $\theta = \arccos((\vec{v} \cdot \vec{w})/(|\vec{w}|*|\vec{v}|))$
- A unit vector has magnitude of 1
  - Notation:
    - Vector: $\vec{v}$
    - Unit Vector: $\vec{w}$

    - $\vec{w} = \hat{v}$
    - Circumflex indicated vector converted to unit vector
  - To convert from vector $\vec{v}$ to unit vector $\vec{w}$:
    - Find $|\vec{v}|$ (magnitude of $\vec{v}$)
    - Divide $\vec{v}$ by $|\vec{v}|$
    - Ie. Multiply $\vec{v}$ by $1/|\vec{v}|$
    - This gives $\vec{w}$ or $\hat{v}$

Matrices

Blah

Combinatorics and Probability

Definitions:

- Combination: Set of objects from a superset. Order is unimportant
- Permutation: Set of objects from a superset. Order is important

Permutations:

- Order matters
- Number of permutations may be calculated with factorials
  - Eg: A,B,C,D
  - Arranged as all possible permutations of four letters
  - 4! = 4*3*2*1 = 24
- If the permutation set is smaller than the superset, then divide by the difference in number of places
  - Eg: A,B,C,D,E,F
  - Superset length is 6
  - Arranged as all possible permutations of 4 letters
  - Leaves a remainder of 2
  - (4!)/(2!) = (4*3*2*1)/(2*1) = 24/2 = 12
- If some of the elements of the superset are indistinguishable
  - We may wish to restrict the subsets to "distinguishable permutations"
  - This means that permutations containing indistinguishable elements in different orders will not be counted
  - Formula is:
    - $(n!)/(n_1!n_2!...n_k!)$
    - Where n! Is the total number of permutations
    - n_x is the total number of that indistinguishable element
  - Eg: M,I,S,S,I,S,S,I,P,P,I
    - n_1 (M) = 1
    - n_2 (I) = 4
    - n_3 (S) = 4
    - n_4 (P) = 2
    - k = 4
    - n = 11
    - $n!/n_1!*n_2!*n_3!*n_4!$
    - (11!)/(1!)(4!)(4!)(2!)
    - 34650

Combinations:

- Order unimportant
  - Eg: C,O,M,P,U,T,E,R
  - C,O,M = M,C,O
- Find the number of permutations, then divide by the number of permutations of each subset
  - Eg: C,O,M,P,U,T,E,R
  - Find the number of possible combinations of 3 letters
  - First find the number of permutations:

        - 8 letters
        - 8!/5! = 336
    - Next divide by the number of permutations of each subset
        - 336/3! = 336/6 = 56
    - Also can use the notation ^nC_r
        - N = number of elements in the superset
        - R = number of elements in each subset
        - ^nC_r = n!/((n-r)!*r!)

Probability:

- Notation: Probability of x occurring = P(x)
- Sample space: The set of all possible outcomes
- Event: The set of outcome(s) whose probability is being assessed
- Probability = # of outcomes in the event / # of outcomes in sample space
    - Eg, flipping a coin:
    - Sample space: TT, TH, HH, HT
    - Event: HH
    - Probability = 1/4
- Maximum probability under normal conditions is 1. The only way to get a higher probability is to have an event space larger than the sample space.

Probability of event A *OR* event B:

- If there is no overlap in the event space (events are mutually exclusive), add them
    - P(A∪B) = P(A) + P(B)
- If there is an overlap (not mutuall exclusive), add them, and subtract the overlap
    - P(A∪B) = P(A) + P(B) - P(A∩B)
- Eg: Probability of rolling at least one “2” when rolling two dice
    - Events 1: The first dice has a 2 {21,22,23,24,25,26}
    - Events 2: The second dice has a 2 {12,22,32,42,52,62}
    - Overlap: Both dice have a 2 {22}
    - Probability 1 = 6/36 = 1/6
    - Probability 2 = 6/36 = 1/6
    - Overlap = 1/36
    - Total probability = 6/36 + 6/36 – 1/36 = 11/36

Probability of event A *AND* event B

- If they are independent (the outcome of one does not affect the outcome of the other) then multiply their probabilities
    - P(A∩B) = P(A) * P(B)
    - Eg, rolling a dice twice. What is the chance of getting a 2 and then a 3.
        - Probability of rolling a 2 = 1/6
        - Probability of rolling a 3 = 1/6
        - Probability of rolling 2 followed by 3 = 1/36

Complement of an event:

- Sample space – event space = complement set
- Notation: Complement A = A' or A^C or $\bar{A}$
- Sara prefers: A'
- P(A') = 1 – P(A)

Conditional probability: Probability of B given that A has happened

- Our sample space is the event space of A
- $P(A|B) = P(A \cap B) / P(A)$

Statistics

Arithmetic mean:
- Tries to find an average number to represent the dataset
- Notation: Arithmetic mean of x = $\bar{x}$ (sample mean) or μ (actual mean)
- Formula: (Sum of data points)/(number of data points)

Variance:
- Tries to give you an idea of how close or far the dataset is to the arithmetic mean
- Long method:
    - Find the difference between each datapoint and the mean
    - Square the results
    - Find the mean of the results
- Notation: σ^2 (actual variance) s^2 (sample variance)
- Actual Formula: (Sum of datapoints - $\mu)^2$/(number of data points)
- Sample Formula: (Sum of datapoints - $\mu)^2$/(number of data points - 1)

Standard Deviation:
- Square root of variance
- More accurate measure of deviance from the mean
- Notation: σ (actual SD) s (sample SD)

Normal Distribution:
- Approximation of the natural curve most natural distributions of properties follow
- Inputs: μ, σ^2
- Formula: $1/(\sqrt{\sigma^2}2\pi) * e^{-(x-\mu)\text{^}2/2\sigma\text{^}2}$

Bayes Theorem:
- Given that: $P(A|B) = P(A\cap B) / P(B)$
    - $P(A\cap B) = P(A|B) * P(B)$
    - $P(B|A) = (P(A|B) * P(B))/ P(A)$
- Terminology:
    - Posterior = $P(B|A)$
    - Prior = $P(A|B)$
    - Likelyhood = P(B)
    - Evidence = P(A)