FCS Week 6 Reading Note

Notebook:	Fundamentals of Computer Science		
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Cornell Notes	**Topic:** Basic Combinatorial Principles: Part 2	Course: BSc Computer Science Class: CM1025 Fundamentals of Computer Science[Reading] Date: April 23, 2021

Essential Question:

What is counting?

Questions/Cues:

- What is the addition principle?
- What is the inclusion-exclusion principle?
- What is the multiplication principle?
- What is a permutation?
- What is a cyclic permutation?
- What is a combination?
- What are permutations and combinations with repetitions?

Notes

EXAMPLE 6.1 Find the number of ways of drawing a red queen or a black king from a standard deck of playing cards.

SOLUTION:
Let A denote the set of red queens and B the set of black kings. Clearly, $|A| = 2 = |B|$. Since A and B are disjoint sets, by the addition principle, $|A \cup B| = |A| + |B| = 2 + 2 = 4$. Thus there are four different ways of drawing a red queen or a black king. ■

In this example, drawing a red queen can be considered a **task**, say, task A. Likewise, drawing a black king can be considered task B. Since the two tasks cannot occur simultaneously, they are **mutually exclusive**. So finding the number of ways of drawing a red queen or a black king is equivalent to finding the number of ways task A or task B can be performed.

Accordingly, the addition principle can be restated in terms of tasks as follows.

THEOREM 6.1 **(Addition Principle)** Let A and B be two mutually exclusive tasks. Suppose task A can be done in m ways and task B in n ways. Then task A or task B can take place in $m + n$ ways. ■

The addition principle can be extended to any finite number of pairwise mutually exclusive tasks, using induction. For instance, let $T_1, \ldots, T_n$ be n pairwise mutually exclusive tasks. Suppose task T_i can be done in m_i ways, where $1 \le i \le n$. Then task $T_1, T_2, \ldots,$ or T_n can be done in $m_1 + m_2 + \cdots + m_n$ ways, as the next example illustrates.

EXAMPLE 6.2 A freshman has selected four courses and needs one more course for the next term. There are 15 courses in English, 10 in French, and 6 in German she is eligible to take. In how many ways can she choose the fifth course?

SOLUTION:
Let E be the task of selecting a course in English, F the task of selecting a course in French, and G that of selecting a course in German. These tasks can be done in 15, 10, and 6 ways, respectively, and are mutually exclusive, so, by the addition principle, the fifth course can be selected in $|\mathrm{E}| + |\mathrm{F}| + |\mathrm{G}| = 15 + 10 + 6 = 31$ ways. ■

Like the addition principle, the inclusion–exclusion principle can be restated in terms of tasks in an obvious way. It can also be extended to a finite number of tasks (see Theorem 6.19).

THEOREM 6.2 **(Inclusion–Exclusion Principle)** Suppose a task A can be done in m ways, task B in n ways, and both can be accomplished in k different ways. Then task A or B can be done in $m + n - k$ ways. ■

EXAMPLE 6.3 In how many ways can you deal a king or a black card from a standard deck of cards?

SOLUTION:
A king can be selected in four different ways and a black card in 26 different ways. These two tasks can be done simultaneously in two ways, namely, by selecting a black king; so, by Theorem 6.2, a king or a black card can be selected in $4 + 26 - 2 = 28$ ways. ■

Before stating the next counting principle, let us return to Example 2.23 in Chapter 2. The task of selecting a mode of transportation for the trip from Boston to London via New York consists of two subtasks A and B: A is selecting a mode of transportation from Boston to New York — car, plane, or ship — and B is selecting a mode of transportation from New York to London — plane or ship. They can be done in $|A| = 3$ and $|B| = 2$ ways. Recall that the trip can be made in $6 = |A| \cdot |B|$ ways; that is, $|A \times B| = 6 = |A| \cdot |B|$.

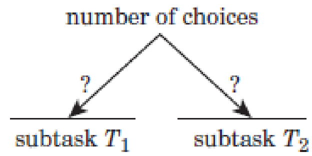
More generally, we have the following result.

THEOREM 6.3 **(Multiplication Principle)** Suppose a task T is made up of two subtasks, subtask T_1 followed by subtask T_2. If subtask T_1 can be done in m_1 ways and subtask T_2 in m_2 different ways for each way subtask T_1 can be done, then task T can be done in $m_1 m_2$ ways. ■

EXAMPLE 6.4 Find the number of two-letter words that begin with a vowel—a, e, i, o, or u.

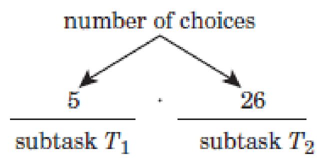
SOLUTION:
The task of forming a two-letter word consists of two subtasks T_1 and T_2: T_1 consists of selecting the first letter and T_2 selecting the second letter, as Figure 6.1 shows.

Figure 6.1



Since each word must begin with a vowel, T_1 can be accomplished in five ways. There are no restrictions on the choice of the second letter, so T_2 can be done in 26 ways (see Figure 6.2). Therefore, by the multiplication principle, the task can be performed in $5 \cdot 26 = 130$ different ways. In other words, 130 two-letter words begin with a vowel.

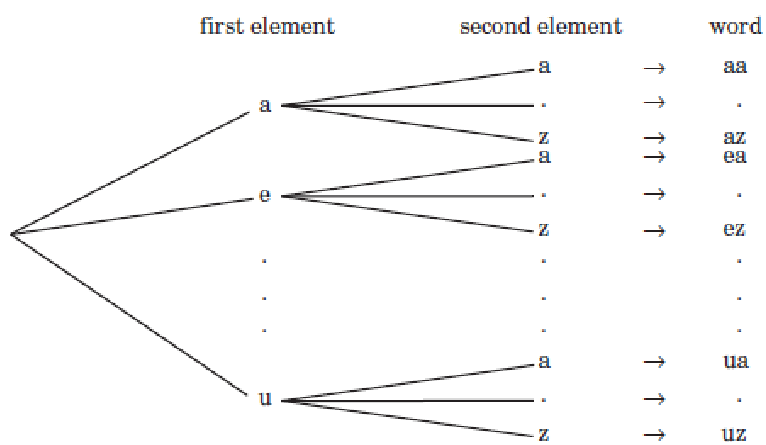
Figure 6.2



■

The various two-letter words in this example can be enumerated systematically by constructing a tree diagram, as in Figure 6.3. All desired words can be obtained by traversing the various branches of the tree, as indicated.

Figure 6.3



The multiplication principle can also be extended to any finite number of subtasks. Suppose a task T can be done by n successive subtasks, T_1, $T_2, \ldots, T_n$. If subtask T_i can be done in m_i different ways after T_{i-1} has been completed, where $1 \le i \le n$, then task T can be done in $m_1 m_2 \cdots m_n$ ways.

The multiplication principle can be applied to prove that a set with size n has 2^n subsets, as shown below.

EXAMPLE 6.5 Show that a set S with n elements has 2^n subsets.

SOLUTION:
Every subset of S can be uniquely identified by an n-bit word (see Figure 6.4). The task of forming an n-bit word can be broken down to n subtasks: selecting a bit for each of the n positions. Each position in the word

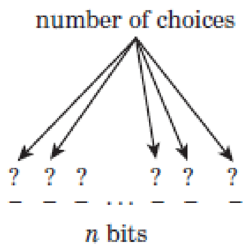


Figure 6.4

has two choices, 0 or 1; so, by the multiplication principle, the total number of n-bit words that can be formed is $\underbrace{2 \cdot 2 \cdot \cdots \cdot 2}_{n \text{ times}} = 2^n$ (see Figure 6.5). In other words, S has 2^n subsets.

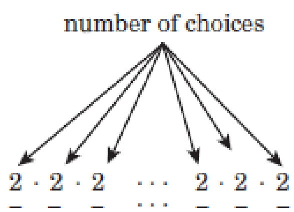


Figure 6.5

■

We now turn to solving the problem posed at the beginning of the chapter.

EXAMPLE 6.6 One type of automobile license plate number in Massachusetts consists of one letter and five digits. Compute the number of such license plate numbers possible.

SOLUTION:
For convenience, we decompose the task into three subtasks:

- Choosing a letter. It can be done in 26 ways.
- Choosing the position of the letter. It has six possible slots.

- Choosing the five digits. They can be selected in $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 100{,}000$ ways.

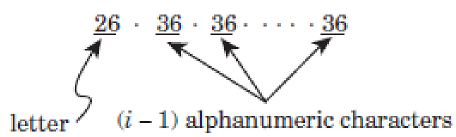
Now we are ready to find the final answer. By the multiplication principle, the total number of license plates is $26 \cdot 6 \cdot 100000 = 15{,}600{,}000$. ■

The next example depends on the multiplication and addition principles.

EXAMPLE 6.7 (optional) An identifier in a programming language consists of a letter followed by alphanumeric characters.* Find the number of legal identifiers of length at most 10.

SOLUTION:
Let S_i denote the set of identifiers of length i, where $1 \le i \le 10$. Then $|S_i| = 26 \cdot 36^{i-1}$ (see Figure 6.6). Since the subtasks $S_1, \ldots, S_{10}$ are mutually

Figure 6.6



exclusive, by the addition principle, the total number of identifiers of length ≤ 10 is given by

$$\sum_{i=1}^{10} |S_i| = \sum_{i=1}^{10} 26 \cdot 36^{i-1} = 26\left(\sum_{i=0}^{9} 36^i\right)$$
$$= 26 \cdot \frac{(36^{10}-1)}{36-1} = \frac{26(36^{10}-1)}{35}$$
$$= 2{,}716{,}003{,}412{,}618{,}210$$
$$\approx 2.7 \text{ quadrillion!}$$ ■

The final example in this opening section employs the multiplication and the inclusion–exclusion principles.

EXAMPLE 6.8 An eight-bit word is called a **byte**. Find the number of bytes with their second bit 0 or the third bit 1.

SOLUTION:

Number of bytes with second bit $0 = 2 \cdot 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7$

Number of bytes with third bit $1 = 2 \cdot 2 \cdot 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7$

Since these two subtasks are *not* mutually exclusive, we cannot add these two partial answers and claim that the answer is $2^7 + 2^7 = 128 + 128 = 256$.

So, we must find the number of bytes that have both properties. The number of bytes with second bit 0 and third bit 1 equals $2 \cdot 1 \cdot 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6 = 64$, so, by the inclusion–exclusion principle, the number of bytes with the given properties is $2^7 + 2^7 - 2^6 = 128 + 128 - 64 = 192$. ■

Permutation

A **permutation** of a set of n (distinct) elements taken r ($0 \leq r \leq n$) at a time is an arrangement of r elements of the set. For convenience, it is called an ***r*-permutation**. If $r = n$, then the r-permutation is called simply a **permutation**. The number of r-permutations of a set of size n is denoted by $P(n, r)$.

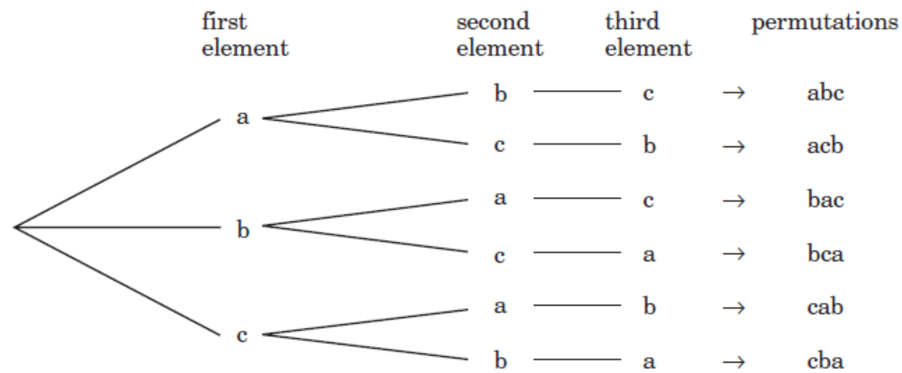
We begin our discussion with a simple example.

EXAMPLE 6.9 Find the number of permutations; that is, 3-permutations of the elements of the set $\{a, b, c\}$.

SOLUTION:
By the multiplication principle, the number of 3-permutations of three elements is $3 \cdot 2 \cdot 1 = 6$. Thus $P(3, 3) = 6$. ■

The various permutations in Example 6.9 can be obtained systematically using a tree diagram, as Figure 6.7 shows. They are abc, acb, bac, bca, cab, and cba.

Figure 6.7



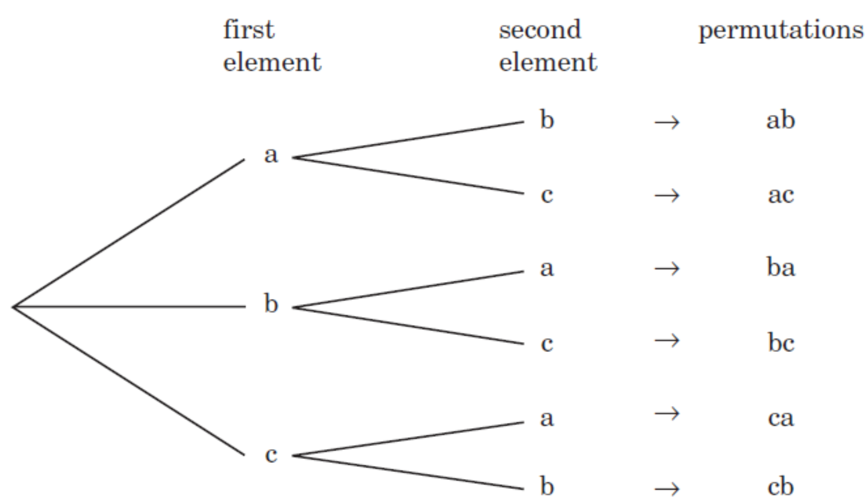
EXAMPLE 6.10 Find the number of 2-permutations of the elements of the set $\{a, b, c\}$.

SOLUTION:
Again, by the multiplication principle, the number of 2-permutations is $3 \cdot 2 = 6$; that is, $P(3, 2) = 6$. ■

The various 2-permutations in Example 6.10 are ab, ac, ba, bc, ca, and cb. They can be obtained using the tree diagram in Figure 6.8.

Examples 6.9 and 6.10 can be interpreted as follows: Suppose you have three books in your hands and would like to arrange them in a bookcase. If there is enough room for three books, they can be arranged in $3 \cdot 2 \cdot 1 = 6$ different ways. On the other hand, if there is room for only two books, they can be arranged in $3 \cdot 2 = 6$ different ways.

Figure 6.8



THEOREM 6.4 The number of r-permutations of a set of n (distinct) elements is given by $P(n,r) = \dfrac{n!}{(n-r)!}$.

PROOF:
Since there are n elements, the first element can be chosen in n ways. Now $n-1$ elements are left; so the second element can be chosen in $n-1$ ways. Continue like this until the rth element is ready to be chosen. At this point there are $n-r+1$ elements left. Consequently, the rth element can be chosen in $n-r+1$ ways. Thus, by the multiplication principle,

$$\begin{aligned} P(n,r) &= n(n-1)(n-2)\cdots(n-r+1) \\ &= \frac{n(n-1)\cdots(n-r+1)(n-r)\cdots 2\cdot 1}{(n-r)\cdots 2\cdot 1} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

■

Although it is easy to remember the value of $P(n,r)$ using this formula, $P(n,r)$ is often computed using the formula $P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$. The values $n!$ and $(n-r)!$ may be too large even for a calculator to compute. Then the value $n!/(n-r)!$ may not be exact. You will find the fact that $n! = n(n-1)!$ useful in computing $P(n,r)$. For example,

$$\begin{aligned} P(25,5) &= \frac{25!}{(25-5)!} = \frac{25!}{20!} = \frac{25\cdot 24\cdot 23\cdot 22\cdot 21\cdot 20!}{20!} \\ &= 25\cdot 24\cdot 23\cdot 22\cdot 21 = 6{,}375{,}600 \end{aligned}$$

Suppose we let $r = n$ in Theorem 6.4. Then

$$P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$$

Accordingly, we have the following result.

THEOREM 6.5 The number of permutations of a set of size n is given by $P(n,n) = n!$. That is, n elements can be arranged in $n!$ ways. ■

EXAMPLE 6.11 A photographer would like to arrange 10 cats for a television commercial. How many ways can she arrange them in a row?

SOLUTION:
Since all the cats have to be in the commercial at the same time, $r = n = 10$. Therefore, the number of possible arrangements is $P(10, 10) = 10! = 3{,}628{,}800$. ■

EXAMPLE 6.12 Find the number of words that can be formed by scrambling the letters of the word SCRAMBLE. (Remember, a word is just an arrangement of symbols; it need not make sense.)

SOLUTION:
The word SCRAMBLE contains eight distinct letters. Therefore, the number of words that can be formed equals the number of arrangements of the letters in the word, namely, $P(8, 8) = 8! = 40{,}320$. ■

The next example uses the multiplication principle, as well as Theorem 6.5.

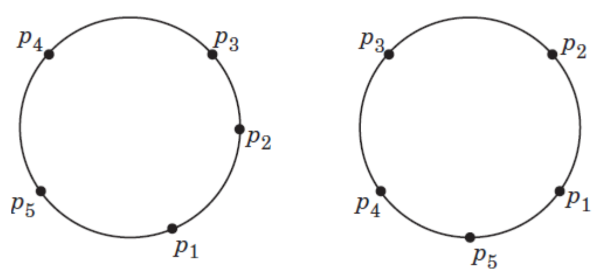
EXAMPLE 6.13 A salesperson at a computer store would like to display six models of personal computers, five models of computer monitors, and four models of keyboards. In how many different ways can he arrange them in a row if items of the same family are to be next to each other?

SOLUTION:
There are three types of items: personal computers, monitors, and keyboards. Think of the items in each family as *tied together* into one unit. These families can be arranged in $P(3, 3) = 3!$ ways. Now the items within each family can be rearranged. The six models of personal computers can be arranged in $P(6, 6) = 6!$ ways, the monitors in $P(5, 5) = 5!$ ways, and the keyboards in $P(4, 4) = 4!$ different ways. Thus, by the multiplication principle, the total number of possible arrangements is $3!\,6!\,5!\,4! = 12{,}441{,}600$. ■

Cyclic Permutation

In how many different ways can you place five beads on a necklace? The answer is *not* $5! = 120$, but far less, since it contains a lot of duplicate arrangements. For instance, the two circular arrangements shown in Figure 6.9 are identical. (Look at the relative positions of the beads p_1 through p_5.) Each circular arrangement is a **cyclic permutation**.

Figure 6.9

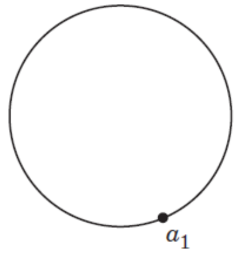


Before we find the number of cyclic permutations of the five beads in Example 6.14, the following general result will be useful to prove.

THEOREM 6.6 The number of cyclic permutations of n (distinct) items is $(n - 1)!$.

PROOF:
To avoid duplicates, let us assign a fixed position to the first item a_1 around the circle (see Figure 6.10). Now $n-1$ positions are left. So the second item a_2 can be placed in any one of the $n-1$ positions. Now $n-2$ positions

Figure 6.10



are left. Therefore, the third item a_3 can be placed in any of the $n-2$ positions. Continue like this until all items have been placed. Thus, by the multiplication principle, the number of cyclic permutations is $1 \cdot (n-1)(n-2)\cdots 2 \cdot 1 = (n-1)!$ ■

EXAMPLE 6.14 Find the number of different ways five zinnias can be planted in a circle.

SOLUTION:

$$\begin{aligned}\begin{pmatrix}\text{Number of ways of planting}\\ \text{five zinnias in a circle}\end{pmatrix} &= \begin{pmatrix}\text{Number of cyclic permutations}\\ \text{of five items}\end{pmatrix}\\ &= (5-1)! = 24\end{aligned}$$ ■

THEOREM 6.7 The number of r-permutations of n distinct elements satisfies the recurrence relation $P(n,r) = P(n-1,r) + rP(n-1,r-1)$, where $0 < r < n$.

PROOF:
Let X be a set with n elements and x an arbitrary element in it. The set of r-permutations of X can be partitioned into two subsets: A, the set of permutations *not* containing x, and B, the set of permutations containing x.

- *To find the number of elements in A*: Since no permutations in A contain x, every element in A is an r-permutation of $n-1$ elements. The number of such permutations is $P(n-1,r)$.
- *To find the number of elements in B*: Since every permutation in B contains x, $n-1$ candidates are left in X for the remaining $r-1$ positions. They can be arranged in $P(n-1,r-1)$ ways. Now the position of x in a permutation has r choices. Therefore, by the multiplication principle, $rP(n-1,r-1)$ permutations contain x.

Since A and B are disjoint sets, by the addition principle,

$$\begin{aligned}P(n,r) &= |A| + |B|\\ &= P(n-1,r) + rP(n-1,r-1)\end{aligned}$$ ■

Note that it's much easier to compute $P(n,r)$ using the explicit formula in Theorem 6.4 rather than by using the recursive. Try $P(5,3)$ both ways to see the difference.

Combination

An ***r*-combination** of a set of n elements, where $0 \leq r \leq n$, is a subset containing r elements.

The number of r-combinations of a set with n elements is denoted by $C(n,r)$ or $\binom{n}{r}$.* Both notations frequently appear in combinatorics. The number of combinations is also called the **binomial coefficient**

EXAMPLE 6.20 Find the number of r-combinations of the set $\{a, b, c\}$, when $r = 0$, 1, 2, or 3.

SOLUTION:

- Exactly one subset contains zero elements: the null set.
  Number of 0-combinations $= C(3,0) = 1$.
- Three subsets contain one element each: $\{a\}$, $\{b\}$, and $\{c\}$.
  Number of 1-combinations $= C(3,1) = 3$.
- Three subsets contain two elements each: $\{a,b\}$, $\{b,c\}$, and $\{c,a\}$.
  Number of 2-combinations $= C(3,2) = 3$.
- Finally, exactly one subset contains three elements: the set itself.
  Number of 3-combinations $= C(3,3) = 1$. ■

THEOREM 6.10 The number of r-combinations of a set of n elements is given by $C(n,r) = \dfrac{n!}{r!(n-r)!}, \quad 0 \leq r \leq n$.†

PROOF:

By definition, there are $C(n,r)$ r-combinations of a set of n elements. Each combination contains r elements and contributes $P(r,r) = r!$ r-permutations, so the total number of r-permutations is $r!C(n,r)$. But, by definition, there are $P(n,r) = \dfrac{n!}{(n-r)!}$ r-permutations. Therefore,

$$r!C(n,r) = \frac{n!}{(n-r)!}$$

That is,

$$C(n,r) = \frac{n!}{r!(n-r)!}$$ ■

In particular, $C(n,0) = \dfrac{n!}{0!(n-0)!} = \dfrac{n!}{0!\,n!} = 1$. That is, the number of 0-combinations of a set with n elements is one (see Example 6.20). Also, $C(n,n) = \dfrac{n!}{n!(n-n)!} = \dfrac{n!}{n!\,0!} = 1$. That is, the number of n-combinations of a set with n elements is also one (see Example 6.20).

A word of caution: To compute $C(n,r)$ when n is fairly large, do *not* compute $n!$ and $r!(n-r)!$, and then divide. The value of $n!$ may be very large for your calculator to hold without approximating it, so you will find the following fact useful:

$$C(n,r) = \frac{n(n-1)\cdots(n-r+1)}{r!}$$

EXAMPLE 6.21 Compute the number of subcommittees of three members each that can be formed from a committee of 25 members.

SOLUTION:

$$\begin{aligned}\begin{pmatrix}\text{Number of subcommittees}\\ \text{of three people each}\end{pmatrix} &= \begin{pmatrix}\text{number of 3-combinations}\\ \text{of a set of 25 people}\end{pmatrix}\\ &= C(25,3)\\ &= \frac{25\cdot 24\cdot 23}{3!}\\ &= 2300\end{aligned}$$ ■

EXAMPLE 6.22 **(The Pizza problem)** Let f_n denote the maximum number of places into which a pizza can be divided with n cuts. Find a formula for f_n.

SOLUTION:
Clearly, the maximum number of regions can be realized when every two chords, that is, cuts, intersect and no three chords are concurrent.

It follows by Example 5.5 in Chapter 5 that f_n can be defined recursively as

$$f_0 = 1$$
$$f_n = f_{n-1} + n, \quad n \geq 1$$

Solving this recurrence relation (see Exercise 3 in Section 5.2) yields

$$f_n = 1 + \frac{n(n+1)}{2} \quad \text{(Verify this.)}$$

This formula can be rewritten as

$$\begin{aligned} f_n &= 1 + n + \frac{n(n-1)}{2} \quad \text{(Verify this.)} \\ &= C(n,0) + C(n,1) + C(n,2), \quad n \geq 0 \end{aligned}$$

EXAMPLE 6.23 How many committees of three blondes and four brunettes can be formed from a group of five blondes and six brunettes?

SOLUTION:
Three blondes can be selected from five blondes in $C(5,3) = 10$ different ways and four brunettes from six brunettes in $C(6,4) = 15$ different ways. Therefore, by the multiplication principle, the number of committees with three blondes and five brunettes is $10 \cdot 15 = 150$. ■

EXAMPLE 6.24 Let A be a 10-element subset of the set $\{1, 2, \ldots, 15\}$. Let A_s be a subset of A containing three elements, where the subscript s denotes the sum of the elements in A_s. For example, the subset $\{2, 4, 5\}$ is denoted by A_{11}. Determine if each subset of A can be identified by a unique name A_s. In other words, does every sum $i + j + k$ have a unique value s, where $1 \leq i < j < k \leq 15$?

SOLUTION:

We let the pigeonhole principle do the job for us. The least value of s is $1 + 2 + 3 = 6$ and the largest value of s is $13 + 14 + 15 = 42$. Thus $6 \leq s \leq 42$; there are at most 37 possible values of s.

There are $C(10, 3) = 120$ three-element subsets (pigeons) of A and only 37 possible sums (pigeonholes), so, by the pigeonhole principle, at least two subsets must yield the same sum; that is, *not* every three-element subset of A can have a unique name.

For example, let $A = \{1, 2, \ldots, 10\}$. Since subsets, $\{1, 2, 5\}$ and $\{1, 3, 4\}$, yield the same sum, 8, they have the same name, A_8. ■

The next theorem will in many cases reduce your workload with combinations, as seen in Example 6.25.

THEOREM 6.11 $C(n, r) = C(n, n - r)$, where $0 \leq r \leq n$.

PROOF:

$$C(n, n-r) = \frac{n!}{(n-r)![n-(n-r)]!}$$

$$= \frac{n!}{(n-r)!r!} = C(n, r)$$ ■

> According to Theorem 6.11, the number of r-combinations of a set with n elements equals that of the $(n - r)$-combinations of the set; for example, $C(5, 2) = C(5, 3) = 10$. This result can be used to cut down the amount of work needed to compute the number of combinations in several applications.

EXAMPLE 6.25 Find the number of groups that can be formed from a group of seven marbles if each group must contain at least three marbles.

SOLUTION:

Since each group must contain at least three marbles, it can contain three, four, five, six, or seven marbles.

Number of groups containing three marbles $= C(7, 3) = 35$

Number of groups containing four marbles $= C(7, 4) = C(7, 3) = 35$

Number of groups containing five marbles $= C(7, 5) = 21$

Number of groups containing six marbles $= C(7, 6) = C(7, 1) = 7$

Number of groups containing seven marbles $= C(7, 7) = 1$

Total number of groups $= 35 + 35 + 21 + 7 + 1 = 99$ ■

This problem can be done in a clever, shorter way as follows:

$$\begin{aligned}\begin{pmatrix}\text{number of groups containing}\\ \text{at most two marbles}\end{pmatrix} &= \begin{pmatrix}\text{number of groups containing}\\ \text{0, 1, or 2 marbles}\end{pmatrix}\\ &= C(7,0)+C(7,1)+C(7,2)\\ &= 1+7+21=29\end{aligned}$$

So

$$\begin{aligned}\begin{pmatrix}\text{number of groups}\\ \text{containing at least}\\ \text{three marbles}\end{pmatrix} &= \begin{pmatrix}\text{total number}\\ \text{of possible}\\ \text{groups}\end{pmatrix} - \begin{pmatrix}\text{number of}\\ \text{groups}\\ \text{containing}\\ \text{at most two}\\ \text{marbles}\end{pmatrix}\\ &= 2^7-29=99\end{aligned}$$

THEOREM 6.12 $$C(n,r)=C(n-1,r-1)+C(n-1,r),\ \text{where } 0<r<n. \tag{6.4}$$

Permutations with Repetitions

Consider the word REFERENCE. If we swap the second E with the fourth E in the word, we do *not* get a new word. How can we compute the number of permutations in such cases?

EXAMPLE 6.27 Find the number of different arrangements of the letters of the word REFERENCE.

SOLUTION:
The word REFERENCE contains nine letters. If they were all distinct, the answer would be $9! = 362{,}880$. But, since duplicate letters exist, the answer is indeed much less.

Let N denote the number of different words. We shall find the value of N in an indirect way.

The word REFERENCE contains two R's and four E's; the remaining letters are distinct. Think of the two R's as two distinct letters, R_1 and R_2, and the four E's as four distinct letters, E_1 through E_4. The letters R_1 and R_2 can be arranged in 2! ways and the four E's in 4! ways. Therefore, if all the letters were distinct, there would be a total of $2!\,4!\,N$ different words. Thus $2!\,4!\,N = 9!$; so

$$\begin{aligned}N &= \frac{9!}{2!\,4!} \qquad (6.5)\\ &= 7560\end{aligned}$$

■

An interesting observation: The number 9 in the numerator of Equation (6.5) indicates the number of letters in the word. Each number in the denominator indicates the frequency of each repeating letter.

THEOREM 6.13 The number of permutations of n items of which n_1 items are of one type, n_2 are of a second type, ... , and n_k are of a kth type, is $n!/(n_1!\,n_2!\cdots n_k!)$.

PROOF:

Let N denote the total number of permutations. As in Example 6.27, we shall find the value of N indirectly.

Let $A_1,\ldots,A_{n_1}$ denote the items of the first type; $B_1,\ldots,B_{n_2}$ items of the second type; ...; and $Z_1,\ldots,Z_{n_k}$ items of the kth type. If all items were distinct, the total would be $n!$ permutations.

If items $A_1,\ldots,A_{n_1}$ are distinct, they can be arranged in $n_1!$ ways. Items $B_1,\ldots,B_{n_2}$, if distinct, can be arranged in $n_2!$ ways, and so on. Items $Z_1,\ldots,Z_{n_k}$, if distinct, can be arranged in $n_k!$ ways. Thus, by the multiplication principle, if all items are distinct, there would be $(n_1!\,n_2!\cdots n_k!)N$ permutations. So $n! = (n_1!\,n_2!\cdots n_k!)N$. Thus

$$N = \frac{n!}{n_1!\,n_2!\cdots n_k!} \qquad \blacksquare$$

EXAMPLE 6.28 Find the number of bytes containing exactly three 0's.

SOLUTION:

$$\begin{aligned}
\begin{pmatrix}\text{number of bytes containing}\\ \text{exactly three 0's}\end{pmatrix} &= \begin{pmatrix}\text{number of bytes containing}\\ \text{three 0's and five 1's}\end{pmatrix}\\
&= \begin{pmatrix}\text{number of permutations of}\\ \text{eight symbols of which}\\ \text{three are alike (0's) and}\\ \text{five are alike (1's)}\end{pmatrix}\\
&= \frac{8!}{3!\,5!}, \quad \text{by Theorem 6.13}\\
&= 56 \qquad \blacksquare
\end{aligned}$$

Combinations with Repetitions

Just as permutations can deal with repeated elements, so can combinations (called **selections**). For example, suppose five friends go to a local restaurant for beverages: iced tea, hot tea, or coffee. The waitress puts all five requests on the same order. How many different orders for the table are possible? The order in which the beverages are selected is immaterial and the same beverage can be selected by more than one person. Also, not every beverage need be selected.

EXAMPLE 6.30 Find the number of 3-combinations of the set $S = \{a, b\}$.

SOLUTION:

S contains $n=2$ elements. Since each combination must contain three elements, $r=3$. Since $r>n$, the elements of each combination must be repeated. Consequently, a combination may contain three a's, two a's and one b, one a and two b's, or three b's. Using the set notation, the 3-combinations are $\{a,a,a\}$, $\{a,a,b\}$, $\{a,b,b\}$ and $\{b,b,b\}$. So there are four 3-combinations of a set of two elements. ■

EXAMPLE 6.31 Find the number of 3-combinations of the set $\{a,b,c\}$, where the elements may be repeated.

SOLUTION:

Again, using the set notation, the various 3-combinations are:

$$\{a,a,a\} \quad \{a,a,b\} \quad \{a,a,c\} \quad \{a,b,b\} \quad \{a,b,c\} \quad \{a,c,c\}$$
$$\{b,b,b\} \quad \{b,b,c\} \quad \{b,c,c\}$$
$$\{c,c,c\}$$

Thus the set $\{a,b,c\}$ has 10 3-combinations. ■

EXAMPLE 6.32 Five friends would like to order beverages with their dinner at a local restaurant that serves iced tea, hot tea, or coffee. Find the number of beverage orders possible.

SOLUTION:

Denote each type of beverage by a dash and separate them using two slashes, as shown below:

$$\underset{\text{iced tea}}{\underline{\qquad\qquad}} \;/\; \underset{\text{hot tea}}{\underline{\qquad\qquad}} \;/\; \underset{\text{coffee}}{\underline{\qquad\qquad}}$$

Mark each person's selection by an X in the appropriate area.

For instance, the distribution XX / X / XX indicates that two people selected iced tea, one selected hot tea, and two selected coffee; the distribution XXX / / XX means, three people selected iced tea, none ordered hot tea, and two selected coffee.

Thus the number of possible beverage orders equals the number of permutations of seven items (five X's and two /'s) of which five are alike (X's) and the other two are alike (/'s):

$$\frac{7!}{5!\,2!} = 21$$ ■

THEOREM 6.14 The number of r-combinations with repetitions from a set of n elements is $C(n+r-1,\, r)$.

PROOF:

Each r-combination with repeated elements from a set of n elements can be considered a string of r X's and $(n-1)$ slashes, as in Example 6.32. Each string contains $r+n-1 = n+r-1$ symbols, of which r are alike (X's) and $n-1$ are alike (slashes). Therefore, by Theorem 6.13, the number of such strings, that is, r-combinations, equals

$$\frac{(n+r-1)!}{r!(n-1)!} = C(n+r-1, r)$$ ■

EXAMPLE 6.33 There are five types of soft drinks at a fast food restaurant: Coke Classic, Diet Coke, root beer, Pepsi, and Sprite. Find the number of beverage orders 11 guests can make.

SOLUTION:
Since there are five types of soft drinks, $n = 5$. Each beverage order is a selection containing 11 items, that is, an 11-combination with repeating elements. Therefore, by Theorem 6.14, the number of possible beverage orders equals

$$\begin{aligned} C(n+r-1, r) &= C(5+11-1, 11) \\ &= C(15, 11) \\ &= \frac{15!}{11!\,4!} = 1365 \end{aligned}$$

■

This problem has a nice interpretation. Let x_i denote the number of guests ordering soft drink i, where $1 \le i \le 5$. Then $x_1+x_2+x_3+x_4+x_5 = 11$, where $x_i \ge 0$. The number of nonnegative integer solutions of this equation is the same as the number of possible beverage orders, so the number of integer solutions of this equation is $C(5+11-1, 11) = C(15, 11) = 1365$.

THEOREM 6.15 Let $x_1, x_2, \ldots, x_n$ be n nonnegative integer variables and r a nonnegative integer. The equation $x_1 + x_2 + \cdots + x_n = r$ has $C(n+r-1, r)$ integer solutions. ■

EXAMPLE 6.34 Find the number of solutions of the equation

$$x_1 + x_2 + x_3 = 5 \tag{6.6}$$

where x_1, x_2, and x_3 are nonnegative integer variables.

SOLUTION:
Here $r = 5$ and $n = 3$. By Theorem 6.15, the number of solutions is

$$\begin{aligned} C(n+r-1, r) &= C(3+5-1, 5) \\ &= C(7, 5) = 21 \end{aligned}$$

(Can you list all the solutions? See Example 6.35 also.) ■

Taking this example a step further, suppose you would like to find all solutions of Equation (6.6), where $x_1, x_2, x_3 \ge 1$. Make the substitution $y_i = x_i - 1, 1 \le i \le 3$. Clearly, $y_i \ge 0$. Equation (6.6) becomes

$$y_1 + y_2 + y_3 = 2$$

By Theorem 6.15, this equation has $C(n+r-1, r) = C(3+2-1, 2) = C(4, 2) = 6$ solutions: $(0,1,1), (1,0,1), (1,1,0), (2,0,0), (0,2,0)$, and $(0,0,2)$. Consequently, Equation (6.6) with $x_i \ge 1$ has six solutions: $(1,2,2)$, $(2,1,2), (2,2,1), (3,1,1), (1,3,1)$, and $(1,1,3)$.

Summary

In this week, we learned about the principle rules of counting, permutations, combinations, permutations with repetition and combinations with repetition.