10.1 The Basics

Notebook:	Discrete Mathematics [CM1020]		
Created:	2019-10-07 2:31 PM	**Updated:**	2020-01-18 6:03 PM
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Cornell Notes	**Topic:** 10.1 The Basics	Course: BSc Computer Science Class: Discrete Mathematics-Lecture Date: January 18, 2020

Essential Question:

What are the rules/strategies used when counting objects when they are sampled with or without replacement?

Questions/Cues:

- What is the Product Rule?
- What is the Product Rule in terms of Sets?
- What is the Addition Rule?
- What is the Addition/Sum Rule in terms of Sets?
- What is the Subtraction Rule?
- What is the Division Rule?
- What is the Pigeonhole principle?
- What is the generalized pigeonhole principle?
- What is a permutation on a set?
- What is the number of permutations possible on a set?
- What is a combination on a set?
- What is the number of combinations possible on a set?

Notes

Product rule

To determine the **number** of different **possible outcomes** in a complex process, we can break the problem into a sequence of two independent tasks:

- if there are **n** ways of doing the first task
- for each of these ways of doing the first task, there are **m** ways of doing the second task
- then there are **n·m** different ways of doing the whole process.

Example

Let's consider a restaurant offering a **combination meal** where a person can order one from each of the following categories: 2 salads, 3 main dishes, 4 side dishes and 3 desserts.

How many different combination meals are possible?

Solution

Solution
The problem can be **broken** down into **4 independent events**:
- selecting a salad, selecting a main dish, selecting a side dish and selecting a dessert.

For each event, the **number** of available options is:
- 2 for the first event
- 3 for the second event
- 4 for the third event
- 3 for the fourth event

Thus, there are $2 \cdot 3 \cdot 4 \cdot 3 = \mathbf{72}$ possible combination meals.

Product rule in terms of sets

Let **A** be the set of ways to do the first task and **B** the set of ways to do second task. If A and B are disjoint, then:

The number ways to do both task 1 and 2 can be represented as $|A \times B| = |A| \cdot |B|$

In other words, the number of elements in the Cartesian product of these sets is the product of the number of elements in each set.

Addition rule

- Suppose a **task 1** can be done **n** ways and a **task 2** can be done in **m** ways
- Assume that both tasks are independent, that is, performing task 1 doesn't mean performing task 2 and vice versa
- In this case, the number of ways of executing task 1 or task 2 is equal to n + m.

Example

- The computing department must choose either a student or a member of academic staff as a representative for a university committee
- How many ways of choosing this representative are there if there are 10 academic staff and 77 mathematics students, and no one is both a member of academic staff and a student?

Solution:

- By the addition rule, there are 10 + 77 ways of choosing this representative.

The sum rule in terms of sets

Let A be the set of ways to do **task 1** and B the set of ways to do **task 2**, where A and B are disjoint sets

- The sum rule can be phrased in terms of sets
- $|A \cup B|=|A|+|B|$ as long as A and B are disjoint sets.

Combining the sum and product rules

Combining the sum and product rules allows us to solve more complex problems.

Example:

- Suppose a label in a programming language can be either a single letter or a letter followed by two digits. What is the number of possible labels?

Solution:

- The number of labels with one letter only is 26
- Using the product rule the number of labels with a letter followed by 2 digits is 26 x 10 x 10
- Using the sum rule the total number of labels is $26 + 26.10.10 = 2{,}626$.

Subtraction rule

- Suppose a task can be done either in one of n_1 ways or in one of n_2 ways.
- Then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways common to the two different ways.
- This is also known as the **principle of inclusion-exclusion**

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example

How many binary bit strings of length eight either start with a 1 bit or end with the two bits 00?

Solution:

- Number of bit strings of length eight that start with a 1 bit: 2.2.2.2.2.2.2 = 2^7 = 128
- Number of bit strings of length eight that end with the two bits 00: 2^6 = 64
- Number of bit strings of length eight that start with a 1 bit and end with bits 00 is 2^5 = 32
- Using the subtraction rule:
  - the number of bit strings either starting with a 1 or ending with 00 is 128 + 64 – 32 = 160.

Division rule

- Suppose a task can be done using a procedure that can be carried out in **n** ways, and for every way **w**, exactly **d** of the **n** ways correspond to **w**. Then this task can done in **n/d** ways
- **In terms of sets**: if the finite set **A** is the union of **n** pair-wise disjoint subsets each with **d** elements, then **n = |A|/d**
- **In terms of functions**: if f is a function from A to B, where both are finite sets, and for every value $y \in B$ there are exactly d values $x \in A$ such that f(x) = y, then |B| = |A|/d

Example

In how many ways can we **seat 4 people** around a table, where two seating arrangements are considered the same when each person has the same left and right neighbour?

Solution:

Let's first number the seats around the table from 1 to 4 proceeding clockwise:

- There are **four ways** to select the person for seat 1, three for seat 2, two for seat 3, and one for seat 4
- Thus there are 4.3.2.1 = 24 ways to order the four people
- Since two seating arrangements are the same when each person has the same left and right neighbour, for every choice for seat 1, we get the same seating
- Therefore, **by the division rule**, there are 24/4 = 6 different seating arrangements.

Pigeonhole principle

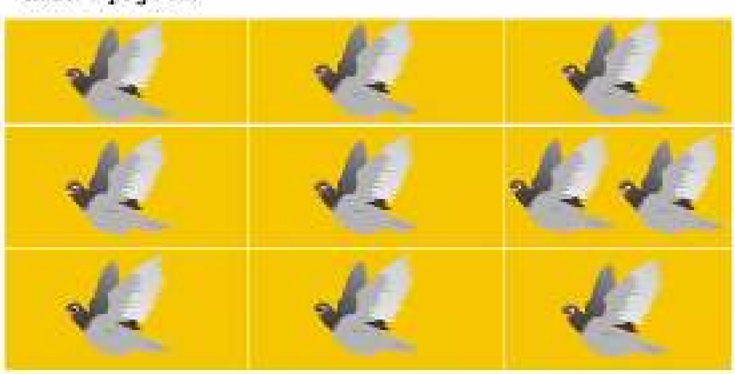
If k is a positive integer and k + 1 objects are placed into k boxes, then at least one box contains two or more objects.

Proof by contrapositive:

- Let's suppose none of the k boxes has more than one object
- Then the total number of objects would be at most k
- Which contradicts the statement that we have k + 1 objects.

Example

If a flock of 10 pigeons roosts in a set of 9 pigeonholes, one of the pigeonholes must have more than 1 pigeon.



Exercise

Prove that a function f from a set with k + 1 elements to a set with k elements is not one-to-one.

Solution: We can prove this using the pigeonhole principle as follows:

- Create a box_y for each element y in the co-domain of f
- Put all of the elements x from the domain in the box for y such that f(x) = y
- Because there are k + 1 elements and only k boxes, at least one box has two or more elements
- Hence, f can't be one-to-one.

The generalised pigeonhole principle

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects, where $\lceil x \rceil$ is called the ceiling function, which represents the round-up value of x.

Let's prove it by contrapositive:

- Suppose that none of the boxes contains more than $\lceil N/k \rceil - 1$ objects
- Then the total number of objects is at most
  $$k(\lceil \frac{N}{k} \rceil - 1) < k((\frac{N}{k} + 1) - 1) = N$$
- This is a contradiction because there is a total of N objects.

Example

How many cards must be selected from a standard deck of **52 cards** to guarantee that **at least four cards of the same suit** are chosen?

Solution:

- We assume four boxes, one for each suit
- Using the generalised pigeonhole principle, at least one box contains at least $\lceil \frac{N}{4} \rceil$ cards, where N is the number of cards selected
- At least four cards of one suit are selected if $\lceil \frac{N}{4} \rceil \geq 4$
- The smallest integer N such that $\lceil \frac{N}{4} \rceil \geq 4$ is equal to 13.

Definition of a permutation

- A permutation of a set of distinct objects is **an ordered arrangement** of these objects
- An ordered arrangement of r elements of a set is called an **r-permutation**
- The number of r-permutations of a set with n elements is denoted by **P(n,r)**.

Example

Let S = {1,2,3}

- The ordered arrangement **3,1,2 is a 3-permutation** of S
- The ordered arrangement **3,2 is a 2-permutation** of S

- The 2-permutations of S = {1,2,3} are 1,2; 1,3; 2,1; 2,3; 3,1; and 3,2
- Hence, P(3,2) = 6.

Number of permutations

If n is a positive integer and r is an integer with $r \leq n$, then there are $P(n,r) = n(n-1)(n-2) \cdots (n-(r-1))$ r-permutations of a set with n distinct elements.

$$P(n,r) = \frac{n!}{(n-r)!}$$

Proof:

- By the product rule:
  - there are n different ways for choosing the 1^{st} element
  - $n-1$ ways for choosing the 2^{nd} element
  - $n-2$ ways for choosing the 3^{rd} element , and so on
  - there are $(n-(r-1))$ ways to choose the last element
  - hence, $P(n,r) = n(n-1)(n-2) \cdots (n-(r-1))$
  - $P(n,0) = 1$, since there is only one way to order zero.

Example

How many possible ways are there of selecting a **first** prize winner, a **second** prize winner and a **third-prize** winner from 50 different people?

Solution:

$P(50,3) = 50 \cdot 49 \cdot 48 = 117{,}600$

Definition of combinations

- An **r-combination** of elements of a set is an **unordered** selection of r elements from the set
- An r-combination is a **subset** of the set with **r** elements
- The number of r-combinations of a set with n distinct elements is denoted by $C(n,r) = \binom{n}{r}$
- The notation used is also called a **binomial coefficient**.

Number of combinations

- The number of r-combinations of a set with n distinct elements can be formulated as:
  $$C(n,r) = \frac{n!}{(n-r)!r!} = \frac{P(n,r)}{r!}$$
- $C(n,r)$ can be referred to as n choose r
- It follows that $C(n,r) = C(n,n-r)$.

Example

How many ways are there of selecting **six players** from a **20**-member tennis team to make a trip to an international competition?

Solution:

$$C(20,6) = \frac{20!}{6!\,14!} = \frac{20.19.18.17.16.15}{6.5.4.3.2} = 38{,}760$$

Summary

In this week, we learned about the Product, Addition, Subtraction & Division rules pertaining to counting. Alongside this, we explored the pigeonhole principle, combinations & permutations.