Week 3 Sequences & Series Reading Note

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Cornell Notes	**Topic:** Sequences and Series	Course: BSc Computer Science Class: Computational Mathematics[Reading] Date: April 28, 2020

Essential Question:

What are sequences and series and how we define each?

Questions/Cues:

- What is a Sequence?
- What is an Arithmetic Sequence/Progression?
- What is the formula for calculating the nth term of an Arithmetic Sequence/Progression?
- What is a Geometric Sequence/Progression?
- What is the formula for calculating the nth term of a Geometric Sequence/Progression?
- What is a Infinite Sequence?
- What is a Series?
- What is Sigma Notation?
- What is an Arithmetic Series?
- What is a Geometric Series?
- What is a Infinite Geometric Series?

Notes

- Sequence = set of numbers written down in a specific order
    - Each number in the sequence is called a term
    - Sometimes we use the symbol "..." to indicate that the sequence continues
        - Sequences with a finite number of terms are called finite sequences. e.g. 1, 2, 3, ... , 20
        - Sequences that go on forever and are indicate with "..." on the end are called infinite sequences
        - We use a subscript notation to refer to different terms in a sequence. For example, if a sequence is denoted by x then the first term is labeled x_1 if you start counting from 1 or x_0 if you start counting from zero

WORKED EXAMPLES

12.1 The terms of a sequence x are given by $x_k = 2k + 3$. Write down the terms x_1, x_2 and x_3.

Solution If $x_k = 2k + 3$ then replacing k by 1 gives $x_1 = 2 \times 1 + 3 = 5$. Similarly $x_2 = 7$ and $x_3 = 9$.

12.2 The terms of a sequence x are given by $x_k = 4k^2$. Write down the terms x_0, x_1, x_2 and x_3.

Solution If $x_k = 4k^2$ then replacing k by 0 gives $x_0 = 4 \times 0^2 = 0$. Similarly $x_1 = 4$, $x_2 = 16$ and $x_3 = 36$.

- Arithmetic sequence/progression = a sequence formed by adding a fixed amount to the previous term
  - The fixed amount added each time is a constant called the common difference

WORKED EXAMPLES

12.3 Write down four terms of the arithmetic progression that has first term 10 and common difference 3.

Solution The second term is found by adding the common difference 3 to the first term 10. This gives a second term of 13. Similarly to find the third term we add on another 3, and so on. This gives the sequence 10, 13, 16, 19 ....

12.4 Write down six terms of the arithmetic progression that has first term 5 and common difference -2.

Solution Note that in this example the common difference is negative. To find each new term we must add -2. In other words we must subtract 2. This results in

$$5, 3, 1, -1, -3, -5$$

An arithmetic progression can be written

$$a, a + d, a + 2d, a + 3d \ldots$$

a is the **first term**, d is the **common difference**.

The nth term of an arithmetic progression is given by $a + (n - 1)d$.

WORKED EXAMPLE

12.5 Use the formula for the nth term to find the 10th term of an arithmetic progression with first term 3 and common difference 5.

Solution Using the formula nth term $= a + (n - 1)d$ with $a = 3$, $d = 5$ and $n = 10$ we find

$$10\text{th term} = 3 + (10 - 1) \times 5 = 3 + 9 \times 5 = 48$$

The 10th term is therefore 48. Using the formula allows us to state the 10th term without having to calculate all the previous terms.

- Geometric Sequence/Progression = a sequence formed by multiplying the previous term by a fixed amount
  - The fixed by which each term is multiplied is called the common ratio
    - If you're given the first term of a sequence & and the common ratio it is easy to generate subsequent terms.

WORKED EXAMPLES

12.6 Find the first six terms of the geometric sequence with first term 3 and common ratio 2.

Solution The first term is 3. The second term is found by multiplying the first by the common ratio 2. So, the second term is 6. The third term is found by multiplying the second by 2, to give 12. We continue in this fashion to generate the sequence 3, 6, 12, 24, 48, 96.

12.7 Write down the first five terms of the geometric sequence with first term 5 and common ratio $\frac{2}{3}$.

Solution The second term is found by multiplying the first by $\frac{2}{3}$. So the second term is $5 \times \frac{2}{3} = \frac{10}{3}$. The third term is found by multiplying this by $\frac{2}{3}$. Continuing in this way we generate the sequence

$$5, \frac{10}{3}, \frac{20}{9}, \frac{40}{27}, \frac{80}{81}$$

12.8 A geometric progression is given by $1, \frac{1}{2}, \frac{1}{4}, \ldots$. What is its common ratio?

Solution The first term is 1. We must ask by what number must the first term be multiplied to give us the second. The answer is clearly $\frac{1}{2}$. The common ratio is $\frac{1}{2}$. This can be checked by verifying that the third term is indeed $\frac{1}{4}$. More generally, the common ratio of a geometric sequence can be found by dividing any term by the previous term. Observe this in the earlier examples.

12.9 Write down the first six terms of the geometric sequence with first term 4 and common ratio -1.

Solution Note that in this example the common ratio is negative. This will have an interesting effect, as you will see. Starting with the first term and repeatedly multiplying by -1 gives the sequence $4, -4, 4, -4, 4, -4$. Notice that because the common ratio is negative, the terms of the sequence alternate in sign.

A geometric progression can be written

$$a, ar, ar^2, ar^3 \ldots$$

a is the **first term**, r is the **common ratio**.

Where:

the first term is a
the second term is ar
the third term is ar^2
the fourth term is ar^3

The nth term of a geometric progression is given by ar^{n-1}.

WORKED EXAMPLE

12.10 Write down the seventh term of the geometric progression that has first term 2 and common ratio 3.

Solution The nth term is ar^{n-1}. Here $a = 2$ and $r = 3$. The seventh term is $(2)(3)^{7-1} = 2 \times 3^6 = 2 \times 729 = 1458$.

- Infinite Sequence = a sequence that continues indefinitely and we use the … to indicate this.
  - It can sometimes be the case that as we move along the sequence, the terms get closer and closer to a fixed value. For example, $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots$, terms are getting smaller and smaller if we continue forever these terms approach the

value 0, sequence can also be written as $x_k = \frac{1}{k}$, for $k = 1,2,3....$ As k gets larger and larger and approaches infinity, terms in the sequence get closer and closer to 0; we say that "$\frac{1}{k}$ tends to 0 as k tends to infinity" or "as k tends to infinity, the limit of the sequence is 0"

- Written as $\lim_{k \to \infty} \frac{1}{k} = 0$, where 'lim' stands for limit
    - $\lim_k \to \infty$ means we must examine the behaviour of the sequence as k gets larger and larger
    - $*$ Very important point to note here, when a sequence possesses a limit it is said to **converge**. However, not all sequences possess a limit or a value they approach, thus are said to **diverge**. For example, the sequence defined by $x_k = 3k - 2$, which is $1, 4, 7, 10...$ is such an example. In this example, as k gets larger and larger, so too do the terms of the sequence; tending to no such value L as in the previous examples above.

WORKED EXAMPLE

12.11 (a) Write down the first four terms of the sequence $x_k = 3 + \frac{1}{k^2}$, $k = 1, 2, 3 ....$

(b) Find, if possible, the limit of this sequence as k tends to infinity.

Solution (a) The first four terms are given by

$$x_1 = 3 + \frac{1}{1} = 4, \quad x_2 = 3 + \frac{1}{4} = 3\tfrac{1}{4}, \quad x_3 = 3 + \frac{1}{9} = 3\tfrac{1}{9},$$
$$x_4 = 3 + \frac{1}{16} = 3\tfrac{1}{16}$$

(b) As more terms are included we see that x_k approaches 3 because the quantity $\frac{1}{k^2}$ becomes smaller and smaller. We can write

$$\lim_{k \to \infty} \left(3 + \frac{1}{k^2}\right) = 3$$

This sequence converges to the limit 3.

- Series = when the terms of a sequence are added together, the result is known as a series. Eg. 1 + 2 + 3 + 4 + 5
    - A series is a sum. If the series contains a finite number of terms we can add them all up and obtain the sum of the series
        - If the series contains an infinite number of terms it is more complicated because an infinite series may have a finite sum in which case it is said to converge or alternatively it may not have a finite sum in which case it said to diverge.
- Sigma notation($\sum$) = a concise and convenient way of writing long sums.
    - For example, $1 + 2 + 3 + 4 + 5 + ... + 10 + 11 + 12$ can be written concisely using sigma notation as: $\sum_{k=1}^{k=12} k$, which stands for a sum of all the values of k as k ranges through all whole numbers from 1 to 12.
        - The lowermost and uppermost values of k are written at the bottom and top of the sigma sign respectively
            - The lowermost value of k is commonly $k = 1$ or $k = 0$ but values are possible. When working with sigma notation, always check the lowermost value of k
            - Sometimes the sigma notation is itself is used in shorthand, meaning the '$k =$' part written at the bottom can be omitted.

For example, our previous example can be written as : $\sum_{k=1}^{12} k$ or $\sum_{1}^{12} k$

WORKED EXAMPLES

12.12 Write out explicitly what is meant by

$$\sum_{k=1}^{k=5} k^3$$

Solution We must let k range from 1 to 5, cube each value of k, and add the results:

$$\sum_{k=1}^{k=5} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3$$

12.13 Express $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ concisely using sigma notation.

Solution Each term takes the form $\frac{1}{k}$, where k varies from 1 to 4. In sigma notation we could write this as

$$\sum_{k=1}^{k=4} \frac{1}{k}$$

12.14 Write the sum

$$x_1 + x_2 + x_3 + x_4 + \cdots + x_{19} + x_{20}$$

using sigma notation.

Solution The sum may be written as

$$\sum_{k=1}^{k=20} x_k$$

- ∗ In general, $(-1)^k$ takes on values of 1 when k is even, and - 1 when k is odd. A factor of $(-1)^k$ in the terms of a series means that the terms alternate in sign

WORKED EXAMPLES

12.15 Write out fully what is meant by $\sum_{k=1}^{4}(-1)^k 2^k$.

Solution When $k = 1$, $(-1)^k 2^k = (-1)^1 2^1 = -2$.

When $k = 2$, $(-1)^k 2^k = (-1)^2 2^2 = 4$.

When $k = 3$, $(-1)^k 2^k = (-1)^3 2^3 = -8$.

When $k = 4$, $(-1)^k 2^k = (-1)^4 2^4 = 16$.

So

$$\sum_{k=1}^{4} (-1)^k 2^k = -2 + 4 - 8 + 16$$

Note that the signs are alternating owing to the presence of $(-1)^k$ in the sum.

12.16 Write out fully what is meant by

$$\sum_{i=0}^{5} \frac{(-1)^{i+1}}{2i+1}$$

Solution Note that the lowermost value of i is $i = 0$.

When $i = 0$, $\frac{(-1)^{i+1}}{2i+1} = \frac{(-1)^1}{1} = -1$

When $i = 1$, $\frac{(-1)^{i+1}}{2i+1} = \frac{(-1)^2}{3} = \frac{1}{3}$

Substituting in values for $i = 2$, $i = 3$, $i = 4$ and $i = 5$ produces the sum

$$\sum_{i=0}^{5} \frac{(-1)^{i+1}}{2i+1} = -1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11}$$

- Arithmetic Series = When the terms of an arithmetic sequence are added, the result is known as an arithmetic series
    - If the arithmetic series has a large number of terms then finding its sum by directly adding all the terms is ill-advised. Fortunately, the following formula makes it easier to find the sum of an arithmetic series:

The sum of the first n terms of an arithmetic series with first term a and common difference d is denoted by S_n and given by

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

WORKED EXAMPLE

12.17 Find the sum of the first 10 terms of the arithmetic series with first term 3 and common difference 4.

Solution Use the formula $S_n = \frac{n}{2}(2a + (n-1)d)$ with $n = 10$, $a = 3$ and $d = 4$:

$$S_{10} = \frac{10}{2}(2 \times 3 + (10 - 1) \times 4) = 5(6 + 36) = 210$$

- Geometric Series = When the terms of a geometric sequence are added, the result is known as a geometric series. The following is the formula that allows us to find the sum of a geometric series:

The sum of the first n terms of a geometric series with first term a and common ratio r is denoted by S_n and given by

$$S_n = \frac{a(1 - r^n)}{1 - r} \qquad \text{provided } r \text{ is not equal to } 1$$

        - The formula excludes the case where $r = 1$ because the denominator becomes zero and division by zero is never allowed

WORKED EXAMPLE

12.18 Find the sum of the first five terms of the geometric series with first term 2 and common ratio 3.

Solution Use the formula $S_n = \frac{a(1-r^n)}{1-r}$ with $n = 5$, $a = 2$ and $r = 3$:

$$S_5 = \frac{2(1-3^5)}{1-3} = \frac{2(1-243)}{-2} = 242$$

- Infinite Geometric Series = When terms of an infinite geometric sequence are added, the result is an infinite geometric series, in the case that sum of the terms is finite and can be found.
    - Consider the special of an infinite geometric series in which the common ratio r lies between -1 and 1. In this case the sum always exists and its value can be found using the following formula:

The sum of an infinite number of terms of a geometric series is denoted by S_∞ and is given by

$$S_\infty = \frac{a}{1-r} \qquad \text{provided } -1 < r < 1$$

- Note in this formula if the common ratio is larger than 1 or less than -1, $r > 1 \text{or } r < -1$, then the series doesn't converge and the sum of an infinite geometric series can't be found

WORKED EXAMPLE

12.19 Find the sum of the infinite geometric series with first term 2 and common ratio $\frac{1}{3}$.

Solution Using the formula $S_\infty = \frac{a}{1-r}$ with $a = 2$ and $r = \frac{1}{3}$:

$$S_\infty = \frac{2}{1-\frac{1}{3}} = \frac{2}{\frac{2}{3}} = 3$$

Notice that we can only make use of the formula because the value of r lies between -1 and 1.

Summary

In this week, we learned about what a sequence and what is series is. Alongside this, we explored the difference between an arithmetic and geometric sequence/series. Secondly, we looked at how arithmetic/geometric series from their respective sequences. Finally, we touched on the notion of the infinite and infinity with infinite sequences and a infinite geometric series.