Week 17 Algebra, Vectors, and Matrices Lecture Note

Notebook:	Computational Mathematics		
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Author:	SUKHJIT MANN		
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Cornell Notes	**Topic:** **Algebra, Vectors, and Matrices**	Course: BSc Computer Science Class: Computational Mathematics[Lecture] Date: July 25, 2020

Essential Question:

What are vectors and matrices?

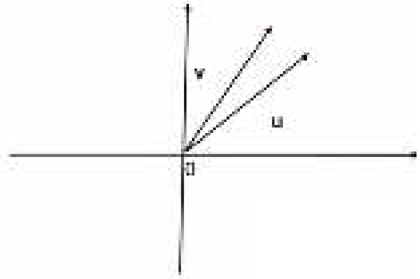
Questions/Cues:

- What is a vector space and field?
- What is scalar multiplication?
- What is the sum/difference of two vectors?
- How do we use the unit vectors to represent a general vector?
- What is the scalar product of two vectors?
- What is the cross product of two vectors?
- How do we perform the rotation of a vector?
- What is meant by a linear operation?
- What is a matrix?
- What are some operations which be performed on matrices?
- What is a translation?
- What are homogeneous coordinates?

Notes

Vector Space and Vectors

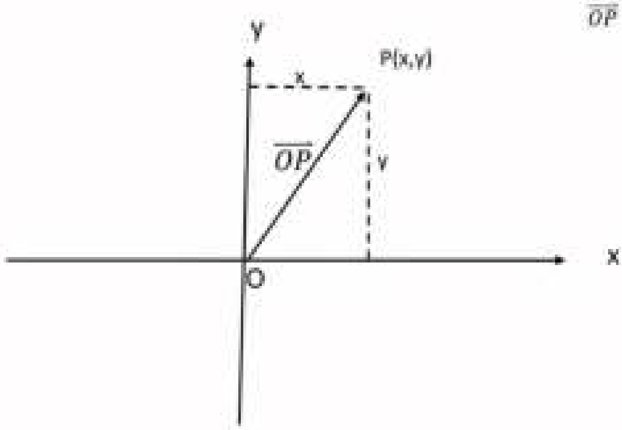
Given a 'Field' usually **R** the set of real numbers
(but can be also **C**, complex numbers)
a Vector Space on **R** is an abstract Space of
objects v called vectors for which you can
define a sum and a product with a
'scalar' (a number in **R**),
that fulfill the following properties:



Associativity	$u + (v + w) = (u + v) + w$
Commutativity	$u + v = v + u$
Identity or null vector	$v + 0 = v$
Inversa	$v + (-v) = 0$
Compatibility	$a(bv) = (ab)v$
Distributivity	$a(u + v) = au + av$
Distributivity	$(a + b)v = av + bv$

Vectors: examples

'Euclidean vectors'
Position or displacement vectors
$\overline{OP}$

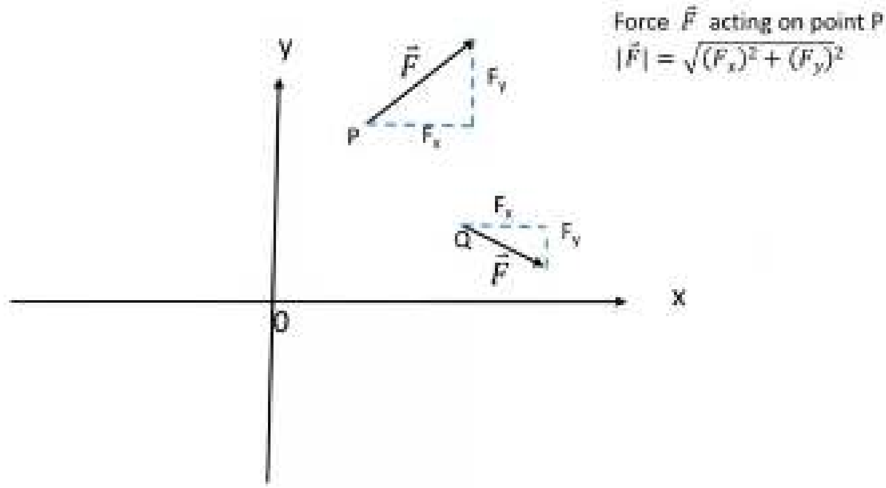


$|\overline{OP}| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$

$x_0 = y_0 = 0$

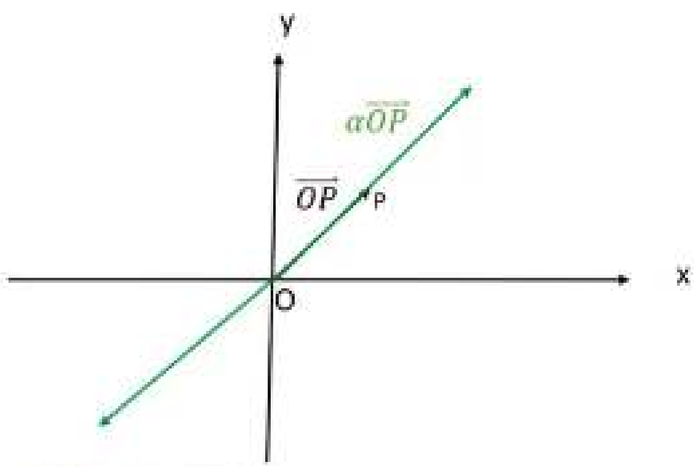
$\overline{OP} = \begin{pmatrix} x \\ y \end{pmatrix} = (x,y) = [x,y]$

Vector Spaces: examples



Other examples:
Rgb vectors (1,227,1) for determining
colour of pixel

Multiplication by a scalar

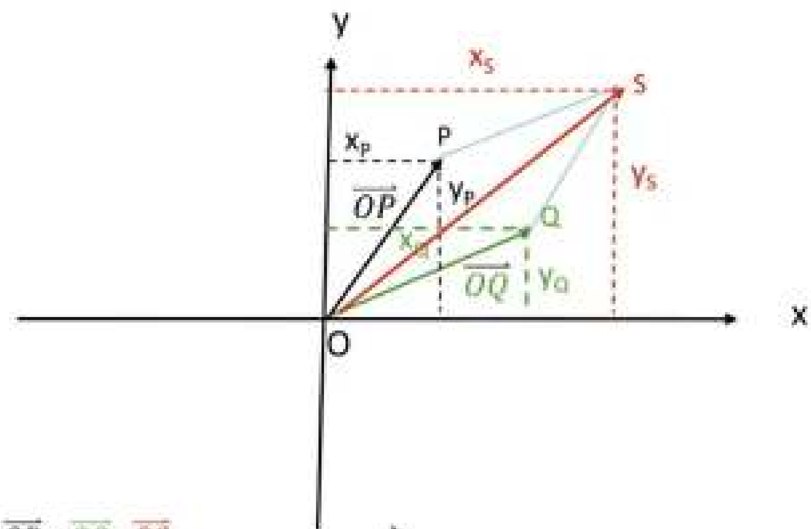


$|\alpha\,\overrightarrow{OP}| = |\alpha|\,|\overrightarrow{OP}|$

$$\begin{pmatrix} x_P \\ y_P \end{pmatrix} \rightarrow \begin{pmatrix} \alpha x_P \\ \alpha y_P \end{pmatrix}$$

- Multiplying by a negative constant changes the direction of the vector
- Scalar multiplication increases the magnitude of the vector by a factor by the absolute value of the constant

Sum of vectors



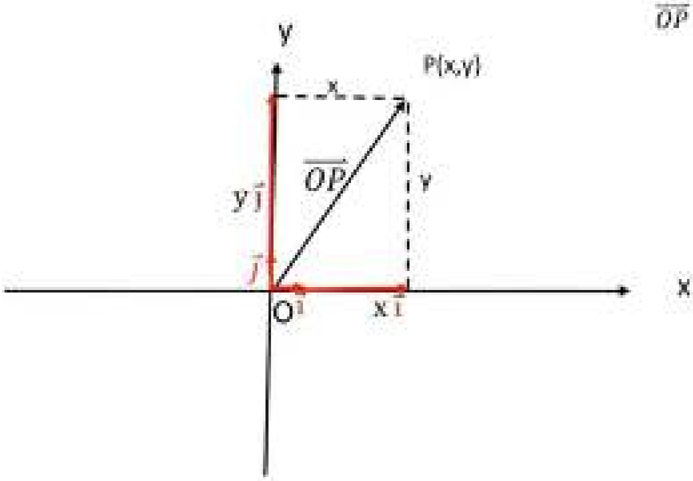
$\overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OS}$

$$\begin{pmatrix} x_P \\ y_P \end{pmatrix} + \begin{pmatrix} x_Q \\ y_Q \end{pmatrix} = \begin{pmatrix} x_S = x_P + x_Q \\ y_S = y_P + y_Q \end{pmatrix}$$

Parallelogram rule

- The difference of two vector can be found by summing together **OP** and **OQ**, but with the x and y coordinates of **OQ** having a negative in front

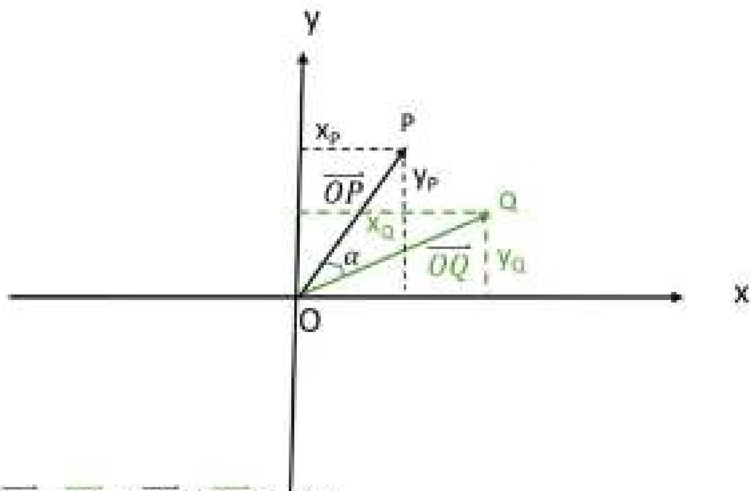
'Euclidean vectors'
Position or displacement vectors
$\overrightarrow{OP}$



$\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix} = (x,y) = [x,y] = x\vec{i} + y\vec{j}$

- i is the unit vector (vector of magnitude 1) in the x-direction a vector of length unit you are using on the x-axis. j is the unit vector in y-direction

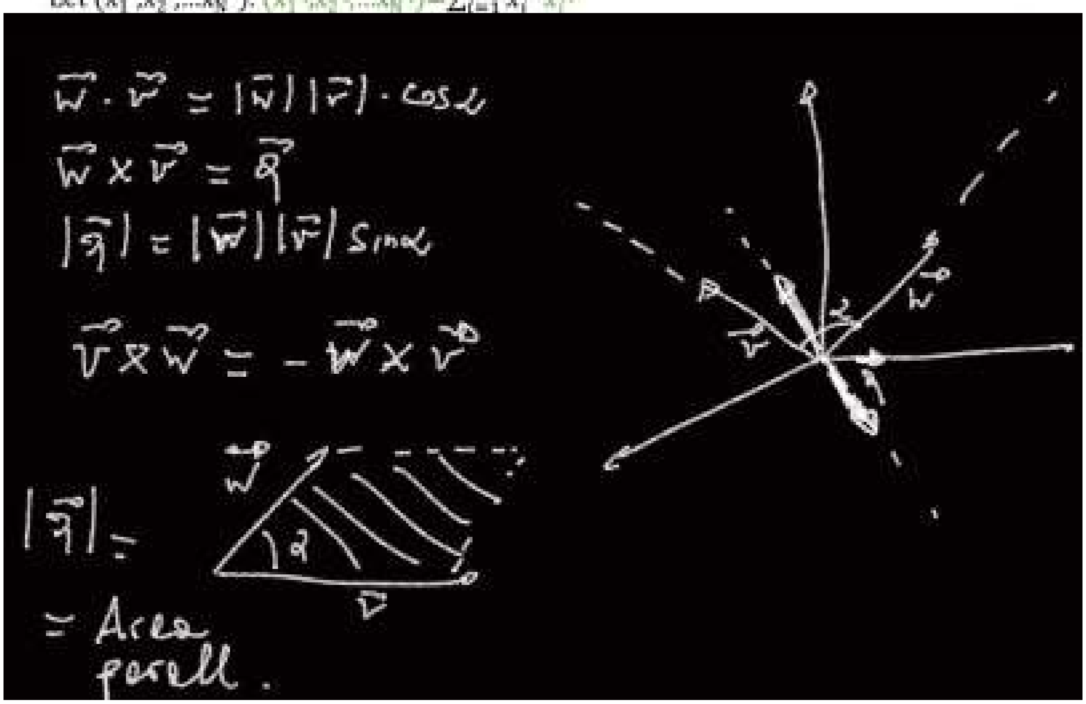
Scalar product of vectors



$\overrightarrow{OP} \cdot \overrightarrow{OQ} = |\overrightarrow{OP}||\overrightarrow{OQ}|\text{Cos}(\alpha)$

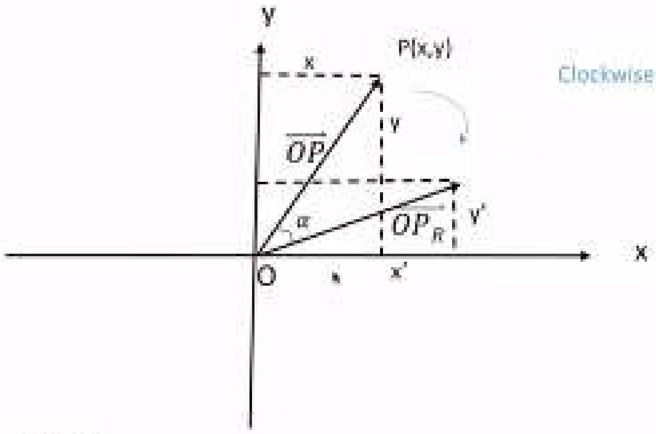
$\begin{pmatrix} x_P \\ y_P \end{pmatrix} \cdot \begin{pmatrix} x_Q \\ y_Q \end{pmatrix} = x_P x_Q + y_P y_Q$

Let $(x_1^P, x_2^P, \dots x_N^P) \cdot (x_1^Q, x_2^Q, \dots x_N^Q) = \sum_{i=1}^{N} x_i^P x_i^Q$



- Area of a parallelogram = b X h

Linear Transformations: Rotations of a vector



$\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\overrightarrow{OP_R} = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x\,\mathrm{Cos}(\alpha) + y\,\mathrm{Sin}(\alpha) \\ -x\,\mathrm{Sin}(\alpha) + y\,\mathrm{Cos}(\alpha) \end{pmatrix}$

Prove:

$\sqrt{(x)^2 + (y)^2} = \sqrt{(x')^2 + (y')^2}$

Linear Transformations: Rotations of a vector

$\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\overrightarrow{OP_R} = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x\mathrm{Cos}(\alpha) + y\mathrm{Sin}(\alpha) \\ -x\mathrm{Sin}(\alpha) + y\mathrm{Cos}(\alpha) \end{pmatrix}$

Linear Operator $\boldsymbol{M}$ s.t. $\overrightarrow{OP_R} = \boldsymbol{M}\,\overrightarrow{OP}$

Linear means if $\overrightarrow{OP} = a\,\overrightarrow{OQ} + b\,\overrightarrow{OR}$

$\rightarrow \boldsymbol{M}\,\overrightarrow{OP} = \boldsymbol{M}(a\,\overrightarrow{OQ} + b\,\overrightarrow{OR}) = a\,\boldsymbol{M}\,\overrightarrow{OQ} + b\,\boldsymbol{M}\,\overrightarrow{OR}$

$Q(x) \rightarrow (x^2)$ <u>not linear:</u>

if $x = (ax_1 + bx_2)$

$\rightarrow Q(x) \rightarrow (a\,x_1 + b\,x_2)^2 \neq a\,x^2_1 + b\,x^2_2 = aQ(x_1) + bQ(x_2)$

- A linear operation applied to a vector means that the components of the vector are recombined, which means they are either added or subtracted together or multiplied by a scalar (constant)

Linear Transformations: Matrices

As a vector can be represented as $\begin{pmatrix} x \\ y \end{pmatrix}$

a Linear Operator can be represented as a matrix

What is a matrix?

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & \cdots & n \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ \vdots \\ m \end{matrix} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

Each element has two indeces a_{ij}

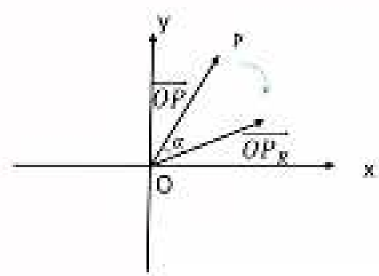
2x2 Matrix: $\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$

2 rows and 2 columns

Applied to a column vector (row times column rule)

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} m_{11}x + m_{12}y \\ m_{21}x + m_{22}y \end{pmatrix}$$

Linear Transformation Example: Rotation of a vector

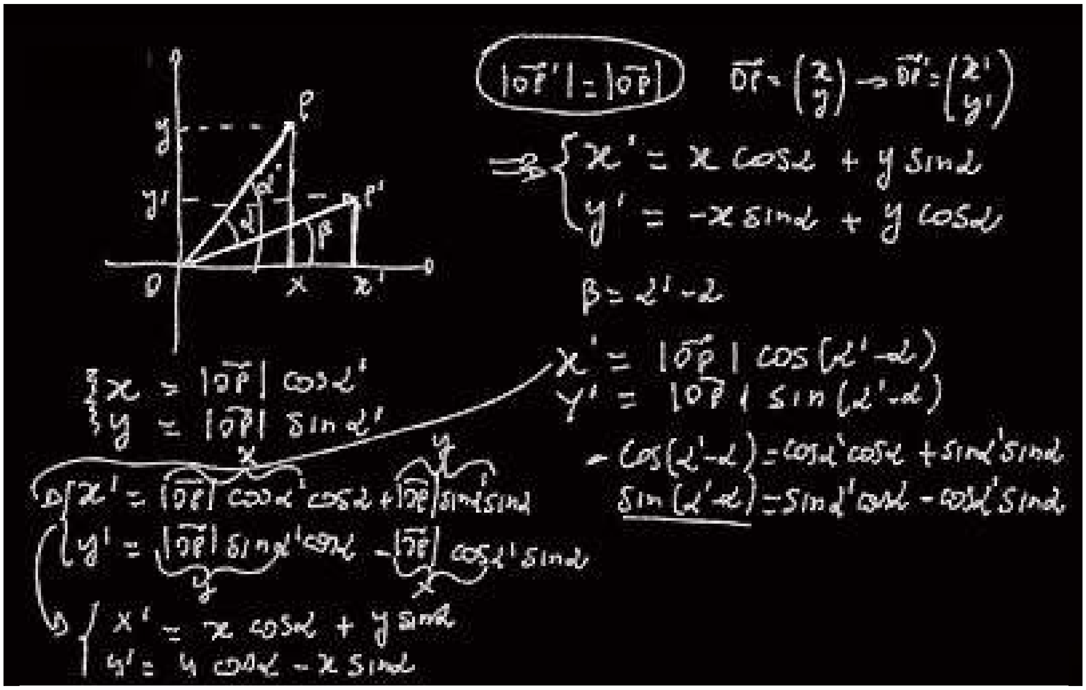


$\overline{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\overline{OP_R} = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x\mathrm{Cos}(\alpha) + y\mathrm{Sin}(\alpha) \\ -x\mathrm{Sin}(\alpha) + y\mathrm{Cos}(\alpha) \end{pmatrix}$

Matrix $M = \begin{pmatrix} \mathrm{Cos}(\alpha) & \mathrm{Sin}(\alpha) \\ -\mathrm{Sin}(\alpha) & \mathrm{Cos}(\alpha) \end{pmatrix}$ applied to a vector:

$M\overline{OP} = \begin{pmatrix} \mathrm{Cos}(\alpha) & \mathrm{Sin}(\alpha) \\ -\mathrm{Sin}(\alpha) & \mathrm{Cos}(\alpha) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\mathrm{Cos}(\alpha) - y\mathrm{Sin}(\alpha) \\ -x\mathrm{Sin}(\alpha) + y\mathrm{Cos}(\alpha) \end{pmatrix} = \overline{OP_R}$



Operations with matrices

Moltiplication by a scalar

$$a\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}=\begin{pmatrix} a\,m_{11} & a\,m_{12} \\ a\,m_{21} & a\,m_{22} \end{pmatrix}$$

Sum of two matrices

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}+\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}=\begin{pmatrix} m_{11}+a_{11} & m_{12}+a_{12} \\ m_{21}+a_{21} & m_{22}+a_{22} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 5 & 0 \end{pmatrix}+\begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}=\begin{pmatrix} 1 & 4 \\ 8 & 0 \end{pmatrix}$$

Operations with matrices

Multiplication of two matrices

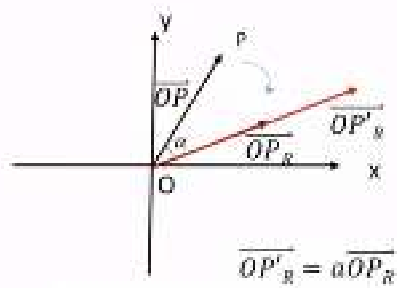
$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 5 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 1\times0 + 2\times3 & 1\times2 + 2\times0 \\ 5\times0 + 0\times3 & 5\times2 + 0\times0 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 2 \\ 0 & 10 \end{pmatrix}$$

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} m_{11}a_{11}+m_{12}a_{21} & m_{21}a_{11}+m_{22}a_{21} \\ m_{11}a_{12}+m_{12}a_{22} & m_{21}a_{12}+m_{22}a_{22} \end{pmatrix}$$

Examples: Multiplication by a scalar



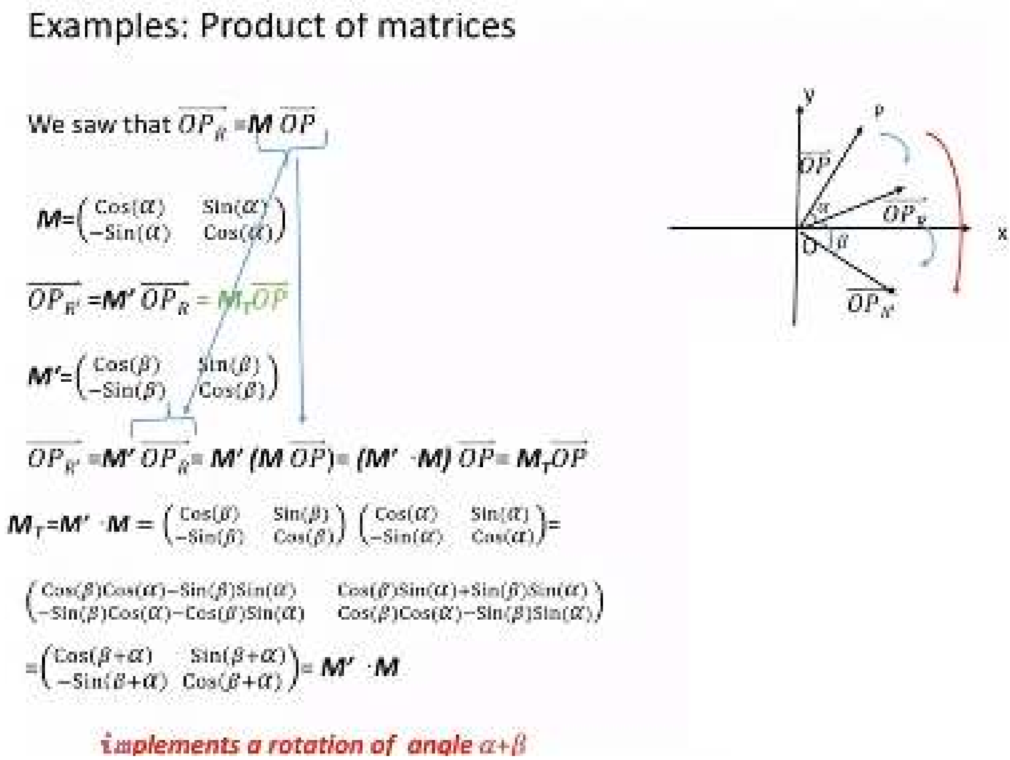
$\overrightarrow{OP'_R} = a\overrightarrow{OP_R}$

We saw that $\overrightarrow{OP_R}$ =$\boldsymbol{M}\,\overrightarrow{OP}$

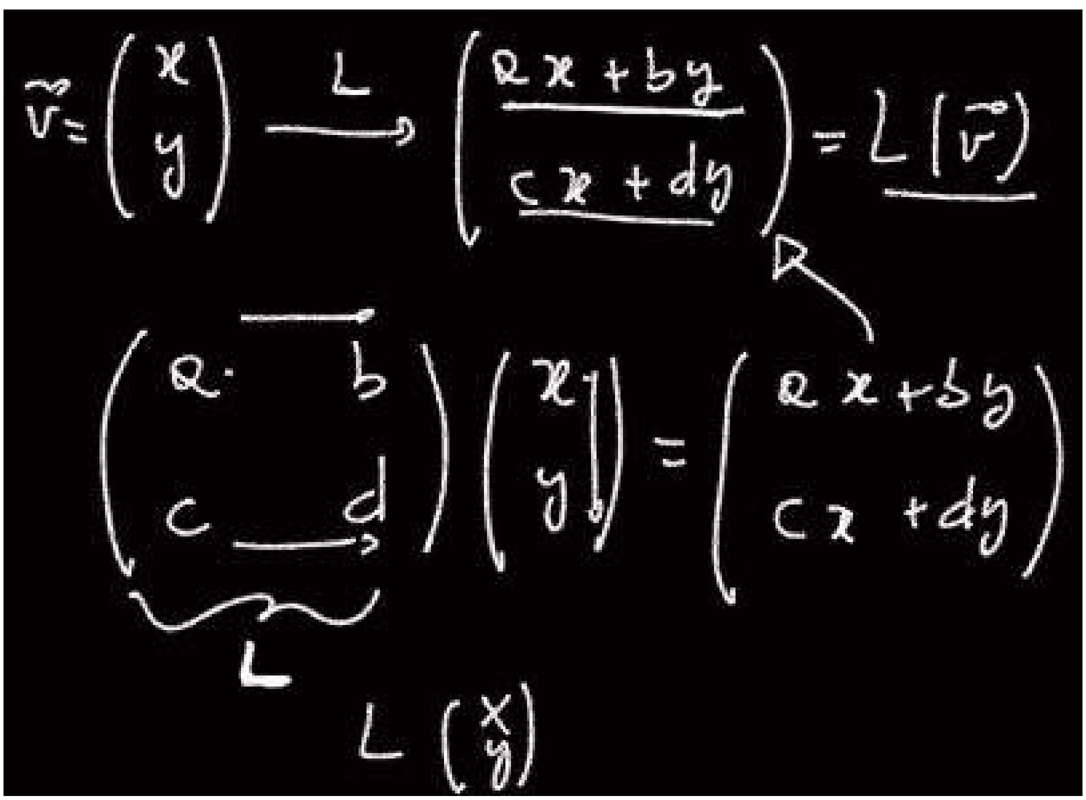
$$\boldsymbol{M} = \begin{pmatrix} \mathrm{Cos}(\alpha) & \mathrm{Sin}(\alpha) \\ -\mathrm{Sin}(\alpha) & \mathrm{Cos}(\alpha) \end{pmatrix}$$

$$\boldsymbol{M'} = a\,\boldsymbol{M} = \begin{pmatrix} a\,\mathrm{Cos}(\alpha) & a\,\mathrm{Sin}(\alpha) \\ -a\,\mathrm{Sin}(\alpha) & a\,\mathrm{Cos}(\alpha) \end{pmatrix}$$

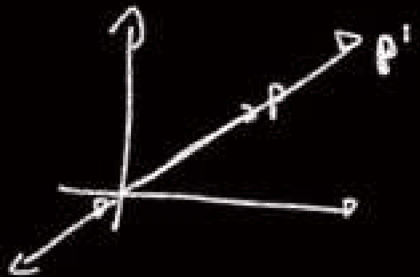
$$\overrightarrow{OP'_R} = \boldsymbol{M'}\,\overrightarrow{OP} = a\boldsymbol{M}\,\overrightarrow{OP} = a\overrightarrow{OP_R}$$



- The composition of two linear transformations is equal to the product of the matrices of the single two transformations. If you do the product of the matrices associated with two single transformations, the product of those two matrices implements the composition of those two transformations.



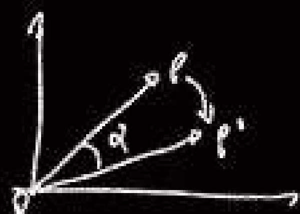
1) $\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix} = \alpha \begin{pmatrix} x \\ y \end{pmatrix}$



$|\vec{OP'}| = |\alpha|\,|\vec{OP}|$

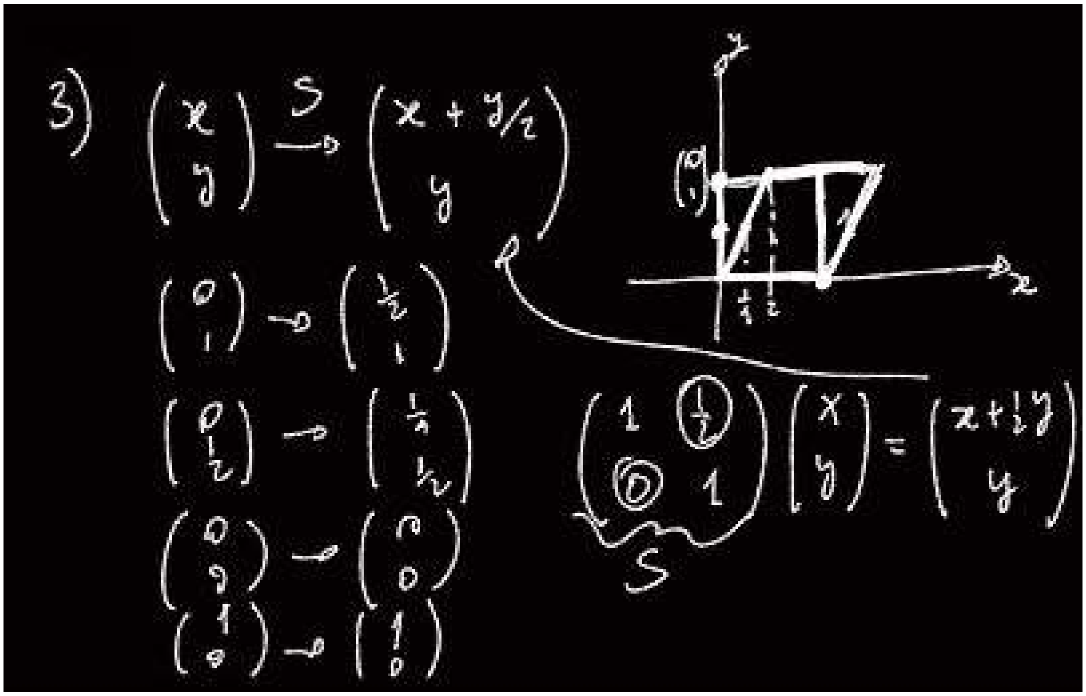
$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix}$$

2) $\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{R} \begin{pmatrix} x\cos\alpha + y\sin\alpha \\ -x\sin\alpha + y\cdot\cos\alpha \end{pmatrix}$

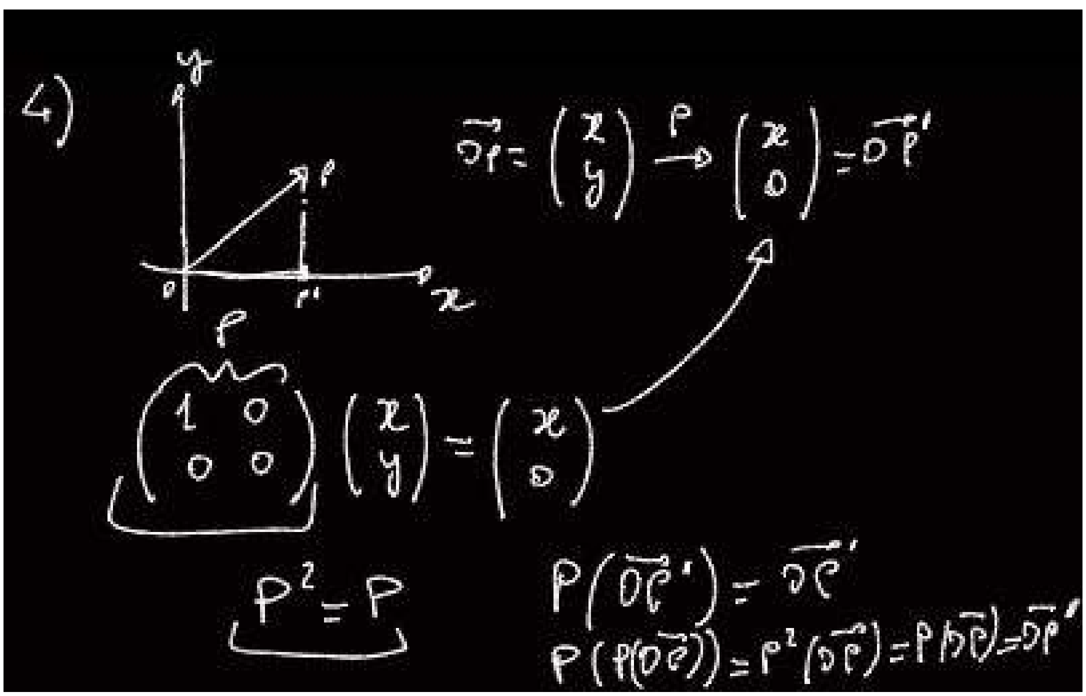


$|\vec{OP'}| = |\vec{OP}|$

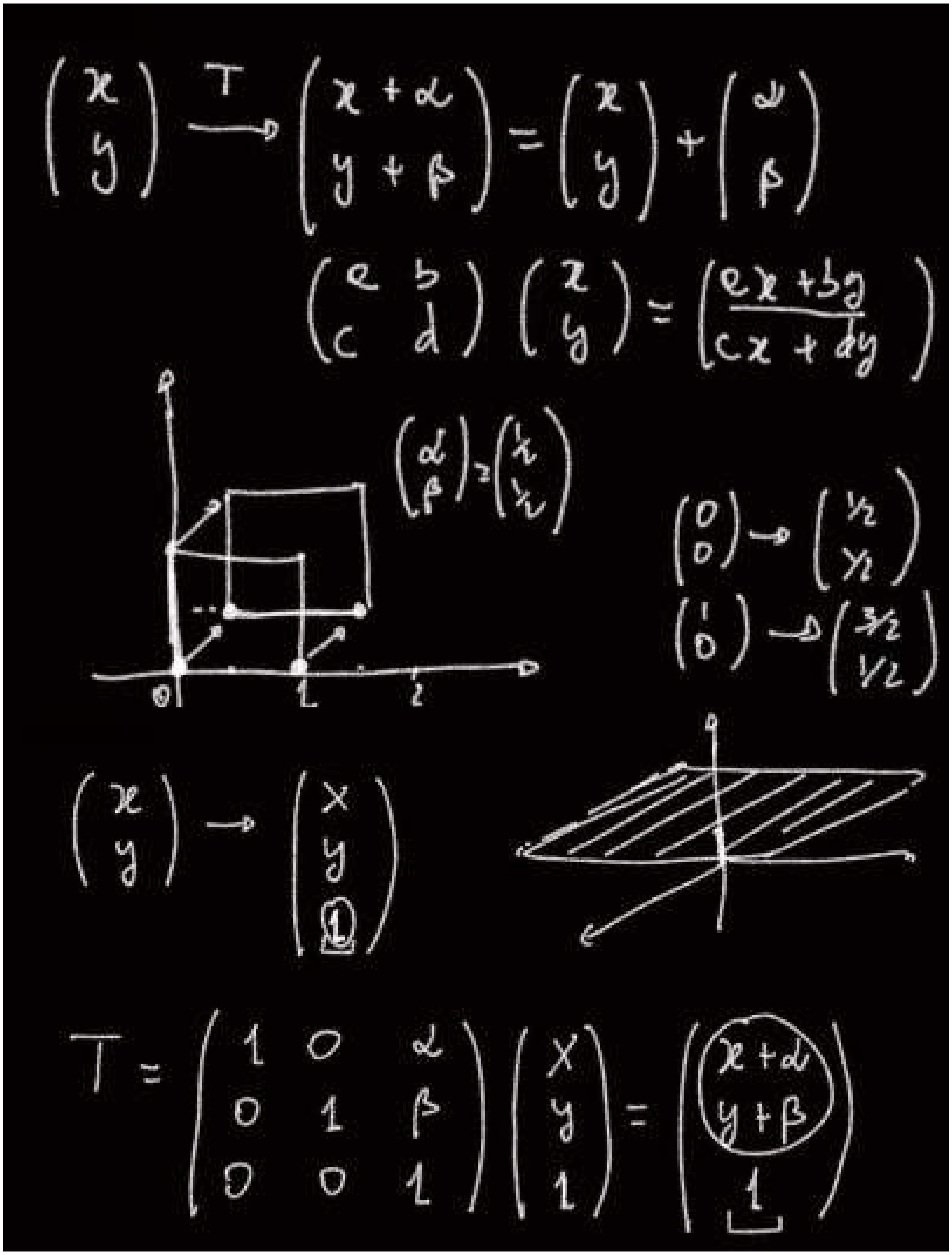
$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\alpha + y\sin\alpha \\ -x\sin\alpha + y\cos\alpha \end{pmatrix}$$

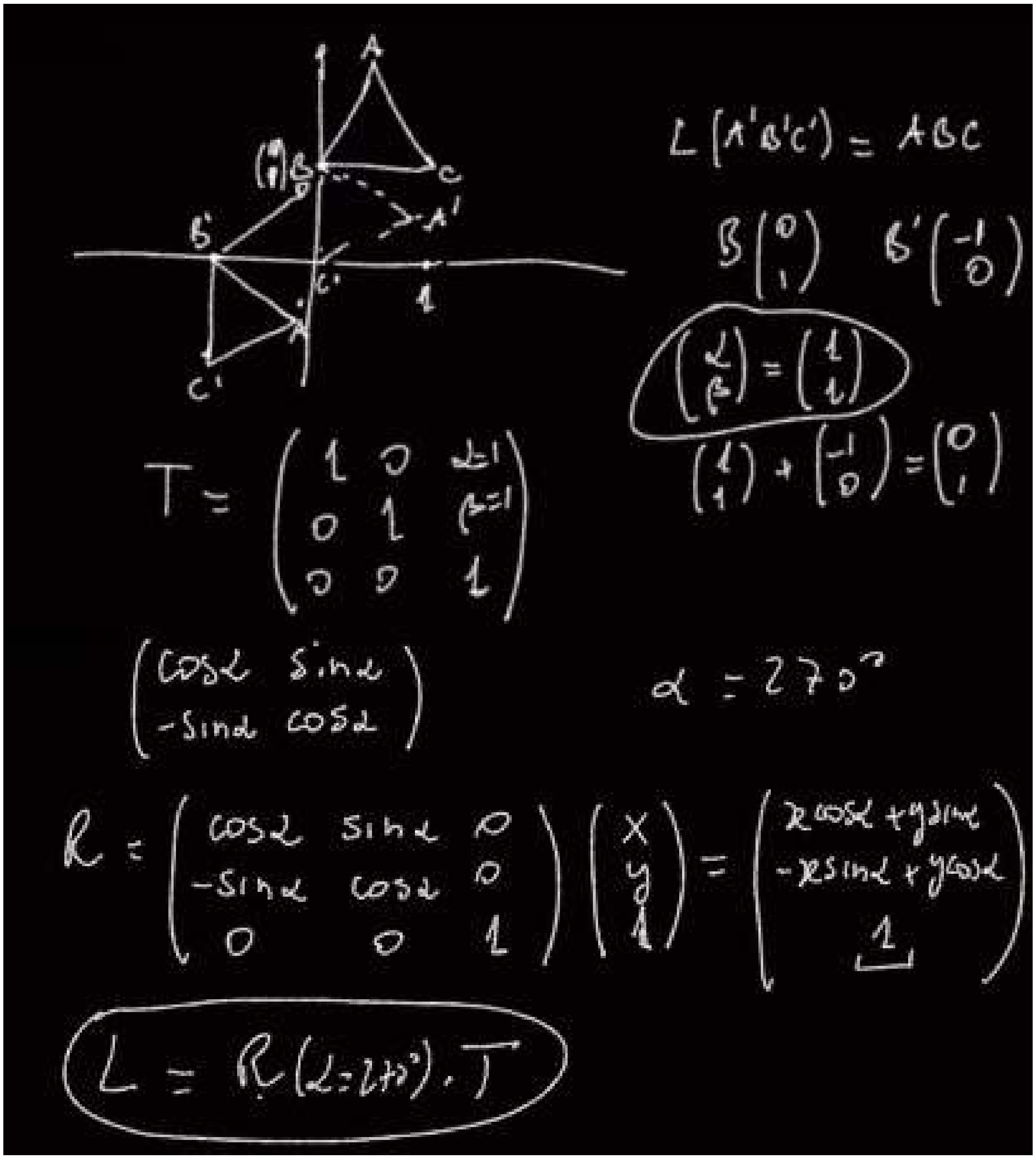


- A shear transformation fixes a direction, which in two-dimensions means just a straight line with direction fixed. It displaces all points of the vector along the direction by an amount proportional to the distance of that point from that direction, so from that straight line.
- x component is shifted by an amount proportional to y, which is the distance from the x-axis



- For a projection you have choose a direction of reference, so in this case a straight line. It takes a vector and transforms it into another vector, which has a component, the same component of the initial vector along the direction of reference and has 0 as a component along the direction perpendicular to the direction reference
- Translation= a transformation that takes a vector of component xy into a vector in which the components of the original vector where a constant is added


(x, y) →T (x+α, y+β) = (x, y) + (α, β)
(a b; c d)(x, y) = (ax+by, cx+dy)
(α, β) = (1/2, 1/2)
(0, 0) → (1/2, 1/2)
(1, 0) → (3/2, 1/2)
0 1 2
(x, y) → (x, y, 1)
T = (1 0 α; 0 1 β; 0 0 1)(x, y, 1) = (x+α, y+β, 1)



Summary

In this week, we learned about what a vector space and field are, what scalar multiplication is, how to calculate the sum/difference of two vectors, how to use unit vector notation to represent a general vector, what the scalar/cross of two vectors is, linear transformations (rotation, shears), what a matrix is, matrix operations, what a translation operation is and finally what homogeneous coordinates are.