Mathematical Economics: Application of Integration in Economic Theory

28.1 Suppose a firm is faced with a marginal revenue function $MR = 40 - 0.6q$.

a) What will be the total revenue of the firm if quantity supplied is 10 units?

b) What will be the change in total revenue when the quantity changes by one unit? Interpret.

c) Find the area under the curve when quantity changes from q=0 to q=10

Solution:

a) By definition **(Module 27, Section 27.6),** Total revenue function is given by

$$TR = \int (40 - 0.6q)dq$$

$$or\ \ TR = \int 40dq + c_1 - \int 0.6qdq\ + c_2$$

$$or\ \ TR = 40q - 0.6\frac{q^2}{2} + c \qquad (c = c_1 + c_2)$$

Now, if $q = 0, TR = 0, hence\ c = 0$

$$\therefore\ TR = 40q - 0.6\frac{q^2}{2}$$

$$or\ \ TR = 40q - 0.3q^2$$

When $q = 10, TR = 40(10) - 0.3(10)^2 = 370$

b) By definition, the change in total revenue for a unit change in quantity is given by marginal revenue, that is, $MR_q = \frac{\Delta TR}{\Delta q}$

Using the MR equation and putting the value of $q = 10$, $MR = 40 - 0.6(10) = 34$

This means that when quantity changes by one unit, the total revenue changes by 34

Therefore,

If $q = 11, TR = 370 + 34 = 404$ and if $q = 9, TR = 370 - 34 = 336$

This can also be attained by putting the value of q in the TR function.

Thus, for $q = 11,\ TR = 40(11) - 0.3(11)^2 = 440 - 36.3 = 403.7 \approx 404$

And for $q = 9,\ TR = 40(9) - 0.3(9)^2 = 360 - 24.3 = 335.7 \approx 336$

Conversely, $if\ TR_2 = 370\ and\ TR_1 = 336,\ MR = TR_2 - TR_1 = 34$

Note:

For simplicity we say one unit. Sugar may be weighed in milligrams or in tons while cars are in large units.

By using the concept of definite integral:

In order to find the area under the MR curve between two points ($q = 9 \; and \; q = 10$), we may write

$$Area\ under\ the\ curve = \int_{9}^{10} (40 - 0.6q)dq$$

$$or, \qquad Area\ under\ the\ curve = \left[40q - \frac{0.6q^2}{2} + c\right]_9^{10}$$

$$or, \qquad Area\ under\ the\ curve = \left[40(10) - \frac{0.6(10)^2}{2} + c\right] - \left[40(9) - \frac{0.6(9)^2}{2} + c\right]$$

$$Area\ under\ the\ curve = 370 - 336 = 34$$

c) Graphical representation:

In order to find the area under the MR curve, let us draw the graph of the marginal revenue function $MR = 40 - 0.6q$

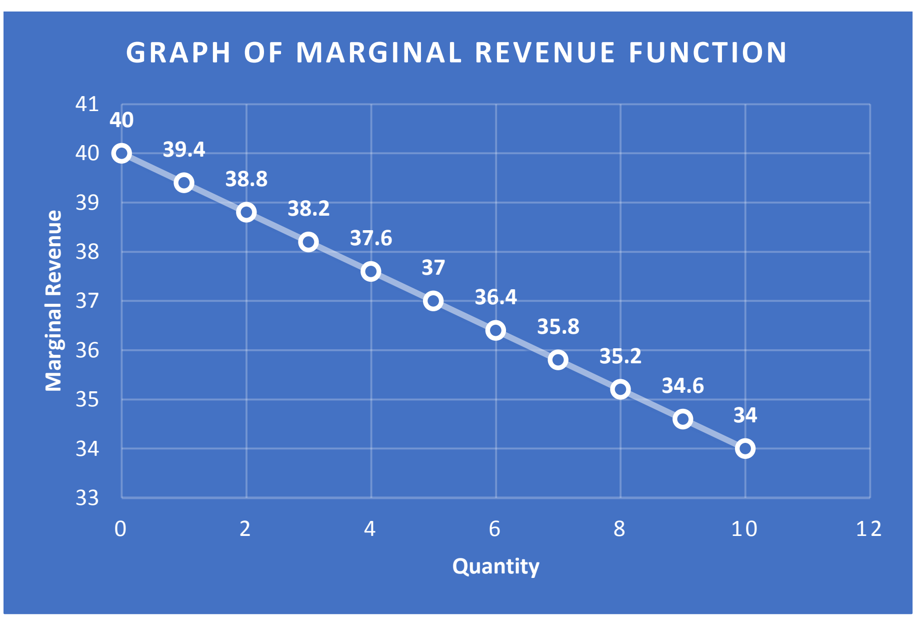

Fig. 28.1: Graph of Marginal Revenue Function

In order to find the area under the curve when $q\ changes\ from\ 0\ to\ 10$, we find the area of the triangle and the rectangle shown in Fig. 28.2.

Fig. 28.2: Area under Marginal Revenue Curve between q=0 and q=10

Now,

$$Area\ of\ the\ triangle = \frac{1}{2} \times base \times altitude$$

$$or, Area\ of\ the\ triangle = \frac{1}{2} \times 10 \times (40 - 34) = 30$$

And
$$Area\ of\ the\ rectangle = length \times breadth$$

$$or, Area\ of\ the\ rectangle = 10 \times 34 = 340$$

Therefore,

$$Area\ under\ the\ curve = Area\ of\ the\ triangle + area\ of\ the\ rectangle = 30 + 340 = 370$$

The area under the curve is noting but the total revenue.

This area may be calculated by using the concept of definite integration from $q = 0\ to\ q = 10$

Therefore,

$$Area\ under\ the\ curve = \int MR\ dq$$

$$or, \qquad Area\ under\ the\ curve = \int (40 - 0.6q)dq$$

$$or, \qquad Area\ under\ the\ curve = \left[40q - \frac{0.6q^2}{2} + c\right]_0^{10}$$

$$or, \qquad Area\ under\ the\ curve = \left[40(10) - \frac{0.6(10)^2}{2} + c\right] - \left[40(0) - \frac{0.6(0)^2}{2} + c\right] = 370$$

28.2 Suppose the marginal cost function of a firm is given as $MC = 4q^2 - 16q + 25$,

a) Determine the change in total cost when quantity is changed from 5 units to 10 units

b) Find the area under the MC curve when quantity changes from q=5 to q=10

Solution:

a) Given the marginal cost function, the total cost function may be found by taking the integration of the MC function.

Therefore,

$$TC = \int_5^{10} MC\ dq$$

$$or, \qquad TC = \int_5^{10} (4q^2 - 16q + 25)\ dq$$

$$or, \ \ TC = \left[4\frac{q^3}{3} - 16\frac{q^2}{2} + 25q + c\right]_5^{10}$$

$$or, \qquad TC = \left[4\frac{(10)^3}{3} - 8(10)^2 + 25(10) + c\right] - \left[4\frac{(5)^3}{3} - 8(5)^2 + 25(5) + c\right]$$

$$or, \ \ TC = 783.33 - 91.66 = 691.67$$

Therefore, when quantity changes from 5 units to 10 units, the total cost changes by 691.67

b) In order to find the area under the curve between $q = 5\ qnd\ q = 10$, let us draw the graph of the MC curve, $MC = 4q^2 - 16q + 25$

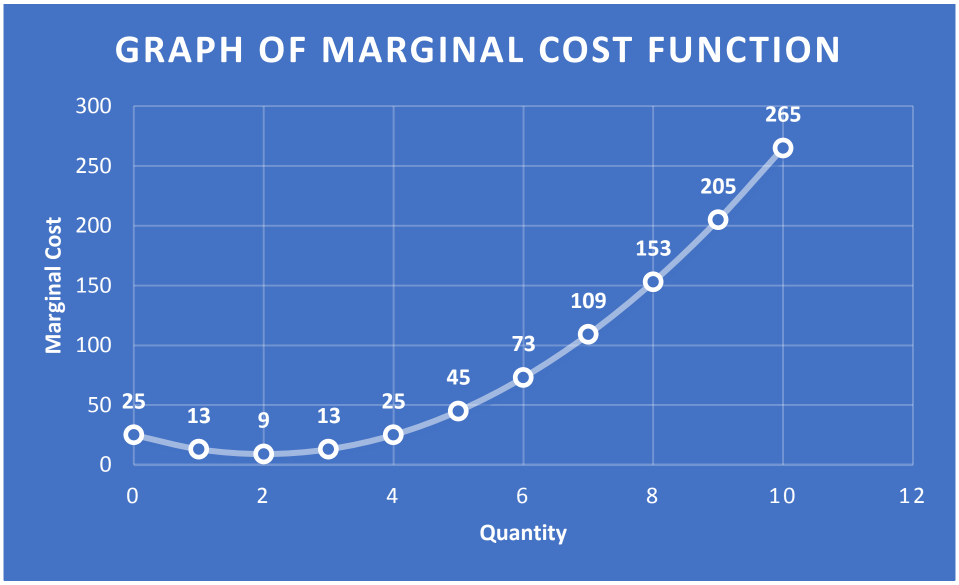

Fig. 28.3: Graph of Marginal Cost Function

Since the MC curve is a non-linear equation, the area under the curve is given by the shaded area in yellow as shown in Fig. 28.4

Fig. 28.4: Area under Marginal Cost curve between q=5 and q=10

If we add the area of the rectangle ABCD and the area of the triangle ADE, we get the Area of the figure coloured in yellow and orange, which is greater than the area under the MC curve.

Thus,

$Area\ of\ rectangle\ ABCD = 45 \times 5 = 225$ and

$$Area\ of\ traingle\ ADE = \frac{1}{2} \times 5 \times (265 - 45) = 550$$

$$Now, area\ of\ rectangle\ +\ area\ of\ triangle\ =\ 225 + 550 = 775$$

The concept of integration helps to find the area under the curve. This is obtained in section (a) of question 28.2 as 691.67.

Therefore, the orange area excluded = $775 - 691.67 = 83.33$

28.3. Consumer's Surplus:

When consumers demand for a particular good or commodity, they are willing to pay any price to get that good or commodity. However, the price that is fixed in the market is the price at which demand is equal to the supply. Generally, the willingness to pay of the consumer is greater than the market price and the difference between the willingness to pay and the market price is termed as consumer's surplus.

This concept may be explained with the graphical method and calculated using the concept of integration.

Suppose a consumer is faced with a demand function for a particular good as $Q = 50 - 2P$ and the market price of the good is ₹ 20. Calculate the consumer's surplus.

Solution:

Given the demand function, $Q = 50 - 2P$, when $P = 20,\ Q = 10$. The consumer's surplus may be graphically shown as in Fig. 28.5

For one unit of the good, the market price is 20, but the consumer is willing to pay somewhere close to 25. Since the market price is less than the consumer's willingness to pay, there exist a surplus and this surplus is termed as consumer's surplus.

The total consumer's surplus till the equilibrium quantity is therefore given by the yellow shaded region.

Fig. 28.5: Consumer's Surplus for P=20

If the market price is 10, equilibrium quantity is 30 and the consumer's surplus will increase as shown by the orange region in Fig. 28.6.

Fig. 28.6: Consumer's Surplus for P =10

This consumer's surplus may be calculated by using the concept of definite integral.

Now,

$$Q = 50 - 2P$$

$$or, \qquad P = 25 - 0.5Q$$

When market price $P = 20$, for equilibrium quantity $Q = 10$, the consumer will pay

Expenditure= $20 \times 10 = 200$

Therefore, $Consumer's\ Surplus = Willingness\ to\ pay - Expenditure$

$$Or, \qquad Consumer'ssurplus = \int_0^{10} (25 - 0.5Q)dQ - 200$$

$$or, Consumer's\ Surplus = \left[25Q - \frac{0.5q^2}{2} + c\right]_0^{10} - 200$$

$$or, Consumer's\ Surplus = [25(10) - 0.25(10)^2 + c] - [25(0) - 0.25(0)^2 + c] - 200$$

$$or, \qquad \boldsymbol{Consumer's\ Surplus = 225 - 200 = 25}$$

Since the graph is linear, the consumer's surplus can also be found out by finding the area of the yellow shaded region.

Thus,

$$Area\ under\ Demand\ curve\ and\ above\ the\ market\ price\ (P = 20) = \frac{1}{2} \times 10 \times 5 = 25$$

Similarly, the consumer's surplus when $P = 10$ may be calculated. (Try it!)

28.4 Producer's Surplus:

When suppliers supply a good or commodity, they receive more than what they are willing to accept. This is excess amount is termed as producer's surplus.

Suppose that the supply function is given by $Q = -3 + 2P$. If the market price is ₹6, what is the producer's surplus?

Solution:

Given the supply function, $Q = -3 + 2P$, when $P = 6$, $Q = 9$. The producer's surplus may be calculated by using the concept of definite integral.

Now,

$$Q = -3 + 2P$$

$$or, \ P = 0.5Q + 1.5$$

At equilibrium quantity, $Total\ revenue\ = 6 \times 9 = 54$

Therefore,

$$Producer's\ Surplus = Actual\ revenue - Willingness\ to\ accept$$

$$or, \qquad Producer's\ Surplus = 54 - \int_0^9 (0.5Q + 1.5)\ \mathrm{dQ}$$

$$or,, \qquad Producer's\ Surplus = 54 - \left[\frac{0.5Q^2}{2} + 1.5Q + c\right]_0^9$$

$$or, Producer's\ Surplus = 54 - [0.25(9)^2 + 1.5(9) + c] - [0.25(0) - 1.5(0)^2 + c]$$

$$or, \qquad \boldsymbol{Producer's\ Surplus = 54 - 33.75 = 20.25}$$

The producer's surplus can be graphically shown as in Fig. 28.7

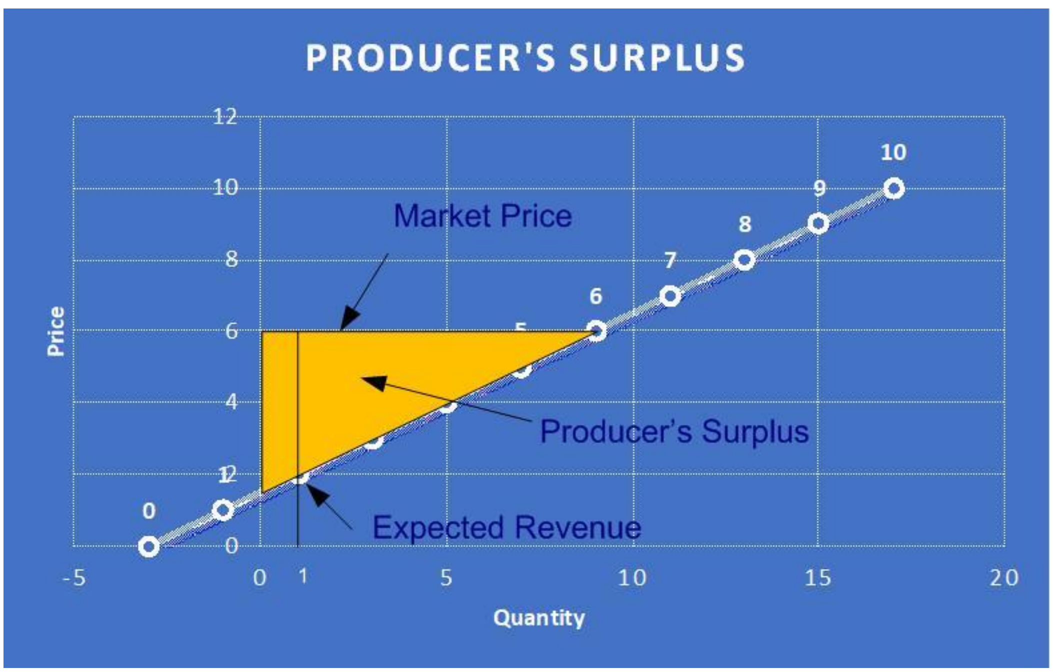

Fig. 28.7: Producer's Surplus

In order to determine the producer's surplus by the graphical method, we find the area of the orange shaded region.

Thus,

$$Area\ above\ the\ supply\ curve\ and\ below\ the\ market\ price\ = \frac{1}{2} \times base \times altitude$$

$$or, \qquad \boldsymbol{Producer's\ surplus = \frac{1}{2} \times 9 \times 4.5 = 20.25}$$

