Solved Problems: Matrix Algebra

Introduction:

To solve problems using matrix algebra, there are certain rules that must be followed. The basic principles underlying the mathematical operations of addition, subtracting, multiplication and division in matrix algebra are discussed through solved problems.

A. Order of a matrix:

The order of a matrix is the product of the rows and columns in a matrix represented by $r\ x\ c$, where r is the number of rows and c is the number of columns. The rule is that the number of rows must be written first. Therefore, a matrix with 3 rows and 2 columns will be written as 3×2.

Example:

Write the order of the following matrices:

a) $\begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 2 \end{bmatrix}$ c) $\begin{bmatrix} 5 \\ 7 \end{bmatrix}$ d) $\begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & 5 \end{bmatrix}$ e) $\begin{bmatrix} 4 & 3 \\ 6 & 1 \\ 7 & 3 \end{bmatrix}$

Solution:

a) There are two rows and two columns. Therefore, the order of the matrix is 2×2
b) There is one row and two columns. The order of the matrix may be written as 1×2. This may also be termed as a row vector.
c) There are two rows and one column. The order of the matrix may be written as 2×1. This may also be termed as a column vector.
d) There are 2 rows and three columns. The order of the matrix is 2×3
e) There are 3 rows and 2 columns. The order of the matrix is 3×2

B. Addition and subtraction of matrices:

Two or more matrices may be added or subtracted. The rule to be followed is that the order of the matrices must be the same. Therefore, a matrix of order 2×3 cannot add or subtract from another matrix of order 3×2 because the number of rows in the first matrix is not equal to the number of rows in the second matrix.

The resultant matrix (matrix obtained after using the mathematical operation) will have the same order as the matrices added or subtracted. Thus, if two matrices of order 3×2 is added, the order of the resultant matrix will also be 3×2

Example 1:

Check if the following matrix can be added and find the resultant matrix.

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix}$$

Solution: To add two matrices, the condition required is that the number of rows and columns in the first matrix must be equal to the number of rows and columns in the second matrix.

Given

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix}$$

Matrix A is of order 2×2 and Matrix B is of order 2×2. Since the order of matrix A and B is the same, they can be added.

The resultant matrix is

$$A + B = \begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2+2 & 1+1 \\ 2+2 & 4+4 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 4 & 8 \end{bmatrix}$$

Example 2:

A manufacturer based in Dehradun produces two products X and Y and sells in two different markets, one situated in Rajput Road and the other in Patel Nagar. The weekly sales for each market are given below:

Market in Rajpur Road				
	Week 1	**Week 2**	**Week 3**	**Week 4**
Product X	10	5	12	8
Product Y	15	10	8	7
Market in Patel Nagar				
Product X	5	8	3	6
Product Y	4	7	5	3

Find the weekly total sales of the two products using matrix algebra.

Solution:

Let the market in Rajpur Road be called M and market in Patel Nagar be called N. Then, the given information may be arranged in matrix form as follows:

A = $\begin{bmatrix} 10 & 5 & 12 & 8 \\ 15 & 10 & 8 & 7 \end{bmatrix}$ and B = $\begin{bmatrix} 5 & 8 & 3 & 6 \\ 4 & 7 & 5 & 3 \end{bmatrix}$

The rows represent the two products and the columns represent the weeks of a particular month.

Now, matrix addition gives

A + B = $\begin{bmatrix} 10+5 & 5+8 & 12+3 & 8+6 \\ 15+4 & 10+7 & 8+5 & 7+3 \end{bmatrix} = \begin{bmatrix} 15 & 13 & 15 & 14 \\ 19 & 17 & 13 & 10 \end{bmatrix}$

The resultant matrix gives the total weekly sales on Product X and Y.

C. Multiplication of matrices:

In order to multiply two matrices, the rule that is followed is that the number of columns in the first matrix must be equal to the number of rows in the second matrix. The order of the resultant matrix is determined by the product of the number of rows in the first matrix and the number of columns in the second matrix.

Note:

If A and B are two matrices, it is not necessary that the product $AB \neq BA$

Example:

Check if the following matrices can be multiplied. How is the order of the resultant matrix determined? Find the resultant matrix and write the order of the resultant matrix.

a) $A = \begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 4 & 1 \end{bmatrix}$

c) $A = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 4 & 2 \end{bmatrix}$

d) $A = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

Solution: To multiply two matrices, the number of columns in the first matrix must be equal to the number of rows in the second matrix. The order of the resultant matrix is determined by the product of the number of rows in the first matrix and the number of columns in the second matrix.

a) Given $A = \begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

Matrix A is of order 2×2 and Matrix B is of order 2×2.

In the product AB, since the number of columns in the first matrix is equal to the number of rows in the second matric, the product AB can be calculated. The product BA can also be calculated.
The resultant matrix is given by:

$$AB = \begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 \times 2 + 1 \times 1 & 2 \times 1 + 1 \times 3 \\ 2 \times 2 + 4 \times 1 & 2 \times 1 + 4 \times 3 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 8 & 14 \end{bmatrix}$$

The order of AB is given as 2×2

Similarly,

$$BA = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 \times 2 + 1 \times 2 & 2 \times 1 + 1 \times 4 \\ 1 \times 2 + 3 \times 2 & 1 \times 1 + 3 \times 4 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 8 & 13 \end{bmatrix}$$

Note: $\boldsymbol{AB \neq BA}$

b) Given $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 1 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 4 & 1 \end{bmatrix}$

Matrix A is of order 3×2 and matrix B is of order 2×3.

The product AB can be calculated because the number of columns in A, that is 2; is equal to the number of rows in B, that is, 2.

Thus,

$$AB = \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 3 \\ 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 \times 2 + 3 \times 2 & 1 \times 1 + 3 \times 4 & 1 \times 3 + 3 \times 1 \\ 3 \times 2 + 2 \times 2 & 3 \times 1 + 2 \times 4 & 3 \times 3 + 2 \times 1 \\ 1 \times 2 + 2 \times 2 & 1 \times 1 + 2 \times 4 & 1 \times 3 + 2 \times 1 \end{bmatrix}$$

$$\therefore AB = \begin{bmatrix} 8 & 13 & 6 \\ 10 & 11 & 11 \\ 6 & 9 & 5 \end{bmatrix}$$

The order of the resultant matrix is 3×3

The product BA can also be calculated by the same reasoning.

Thus,

$$BA = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 \times 1 + 1 \times 3 + 3 \times 1 & 2 \times 3 + 1 \times 2 + 3 \times 2 \\ 2 \times 1 + 4 \times 3 + 1 \times 1 & 2 \times 3 + 4 \times 2 + 1 \times 2 \end{bmatrix}$$

$$\therefore BA = \begin{bmatrix} 8 & 14 \\ 15 & 16 \end{bmatrix}$$

The order of the resultant matrix is 2×2

Therefore, $AB \neq BA$

c) Given $A = [2 \quad 3 \quad 5]$ $B = [3 \quad 4 \quad 2]$

The order of A is 1×3 and the order of B is 1×3.
The product AB cannot be calculated because the number of columns in A, that is 3; is not equal to the rows in B, that is 1.
The product BA also cannot be calculated because the number of columns in B, that is, 3; is not equal to the number of rows in A, that is, 1

d) Given $A = [2 \quad 3 \quad 5]$ $B = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

The order of A is 1×3 and the order of B is 3×1

The product AB can be calculated because the number of columns in A, that is 3 is equal to the number of rows in B, that is, 3.

Thus,

$$AB = [2 \quad 3 \quad 5] \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = 2 \times 3 + 3 \times 4 + 5 \times 5 = 43$$

The product is a scalar, order is 1×1

The product BA can be calculated because the number of columns in B that is 1 is equal to the number of rows in A that is 1.

Thus,

$$BA = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \begin{bmatrix} 2 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 3 \times 2 & 3 \times 3 & 3 \times 5 \\ 4 \times 2 & 4 \times 3 & 4 \times 5 \\ 5 \times 2 & 5 \times 3 & 5 \times 5 \end{bmatrix} = \begin{bmatrix} 6 & 9 & 15 \\ 8 & 12 & 20 \\ 10 & 15 & 25 \end{bmatrix}$$

The order of the resultant matrix is 3×3

e) If a matrix is multiplied by a scalar value, then every element in the matrix will be multiplied by the scalar.

Example:

i) $2\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 \times 1 & 2 \times 3 \\ 2 \times 2 & 2 \times 4 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 4 & 8 \end{bmatrix}$

ii) $\frac{1}{2}\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \times 2 & \frac{1}{2} \times 4 \\ \frac{1}{2} \times 6 & \frac{1}{2} \times 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

13.1. A firm uses two inputs, Labour (L) and Capital (K) to produce two products X and Y. To produce one unit of X, 4 units of labour and 1 unit of capital is used. To produce one unit of Y, 6 units of labour and 2 units of capital is used.

a) Represent the given data in matrix form.
b) Find the total units of labour used in producing both X and Y using matrix addition.
c) If the price of labour is ₹ 50 per unit and price of capital is ₹ 100 per unit, find the total cost of producing X and the total cost of producing Y.

Solution:

a) The given data may be represented in matrix form in two different ways as shown in Table 13.1 and Table 13.2:

Table 13.1:

	Product X (1 unit)	**Product Y (1 unit)**
Labour	4	6
Capital	1	2

In Matrix notation:

$$A = \begin{bmatrix} 4 & 6 \\ 1 & 2 \end{bmatrix}$$

Table 13.2:

	Labour	**Capital**
Product X (1 unit)	4	1
Product Y (1 unit)	6	2

In Matrix notation:

$$B = \begin{bmatrix} 4 & 1 \\ 6 & 2 \end{bmatrix}$$

Note: Matrix B is the transpose of matrix A

b) To find the total units of labour used in producing both X and Y, we may use two vectors, say L and K. The rows represent Labour and Capital and the column in L represent Product X and that in K represent Product Y

Let $L = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $K = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$,

then $L + K = \begin{bmatrix} 4+6 \\ 1+2 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$

Hence, total units of labour used in producing X and Y is 10 and total units of capital used in producing X and Y is 3

c) The price of labour and capital may be written as a column vector or a row vector.

Thus,

$P_1 = \begin{bmatrix} 50 \\ 100 \end{bmatrix}$ or $P_2 = \begin{bmatrix} 50 & 100 \end{bmatrix}$

P_1 is a column vector of order 2×1 and P_2 is a row vector of order 1×2

Or P_2 may be said to be the transpose of P_1

To find the total cost of producing X and Y, let us first see what are the meaningful results that may be obtained using the matrices constructed in section a and b.

1. Using Matrix A and vector P_1, and applying the product of a matrix, we have

   $AP_1 = \begin{bmatrix} 4 & 6 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 50 \\ 100 \end{bmatrix} = \begin{bmatrix} 4 \times 50 + 6 \times 100 \\ 1 \times 50 + 2 \times 100 \end{bmatrix} = \begin{bmatrix} 800 \\ 250 \end{bmatrix}$, which is not meaningful, because

   6 is the units of labour used in producing Y and it is multiplied with the price of capital, i.e ₹ 100.

2. P_1A cannot be calculated as it does not satisfy the condition for the product of two matrices, that is, the number of columns in the first matrix is not equal to the number of rows in the second matrix.

3. AP_2 cannot be calculated as it does not satisfy the condition for the product of two matrices, that is, the number of columns in the first matrix is not equal to the number of rows in the second matrix.

4. $P_2A = [50 \quad 100]\begin{bmatrix}4 & 6\\1 & 2\end{bmatrix} = [50 \times 4 + 100 \times 1 \qquad 50 \times 6 + 100 \times 2] = [300 \quad 500]$

This product gives the total cost of producing Product X and product Y. The total cost of producing X is ₹ 300 and the total cost of producing Y is ₹ 500.

Therefore, P_2A gives the answer to the question in section c.

What happens if we multiply Matrix B with the price vector? (Try it!)

13.2. A producer sells two products X and Y in two different markets M_1 and M_2. In market M_1, he sells 10 units of product X and 20 units of product Y in the month of January. In February, he sells 20 units of product X and 40 units of product Y. In market M_2, he sells 5 units of Product X and 10 units of product Y in January and in February; he sells 15 units of product X and 15 units of product Y.

a) Arrange the available information in rows and columns.
b) Calculate the total units of product X sold in January in both the markets.
c) Calculate the total units of product Y sold in January in both the markets.
d) Calculate the total units of X and Y sold in January in Market 1.
e) Calculate total units of product X sold in both the markets in each month.
f) If the price of product X is ₹ 5 and price of product Y is ₹10, calculate the total revenue earned.

Solution:

a) The available information are:
   i) In January, units of product X sold in Market 1 = 10
      units of product Y sold in Market 1 = 20
   ii) In February, units of product X sold in Market 1 = 20
      units of product Y sold in Market 1 = 40
   iii) In January, units of product X sold in Market 2 = 5
      units of product Y sold in Market 2 = 10
   iv) In February, units of product X sold in Market 2 = 15
      units of product Y sold in Market 2 = 15

The above information may be arranged in rows and columns as follows in two ways:

Matrix 1

Product	**Market 1**		**Market 2**		**Market 1+Market 2**
	January	**February**	**January**	**February**	
Product X	10	20	5	15	50
Product Y	20	40	10	15	85
X + Y	30	60	15	30	135

Product X and Y may be arranged in Rows and Market 1 and Market 2 may be arranged in Columns.

Alternative arrangement:

The given information may also be arranged as follows:

Matrix 2

Product	**Product X**		**Product Y**		**X+Y**
	January	**February**	**January**	**February**	
Market 1	10	20	20	40	90
Market 2	5	15	10	15	45
Market 1+ Market 2	15	35	30	55	135

Product X and Y may be arranged in Columns and Market 1 and Market 2 may be arranged in Rows.

b) To find the total units of product X sold in January in both the markets, we may want to refer to Matrix 2. This matrix will easily give us the result. Taking the column sum for January, gives total units of product X sold in Market 1 and 2 combined and that is 15.

Note: Matrix 1 will not give this result. Therefore, it is important to construct the matrix in such a way that the desired results can be calculated easily.

c) The total units of product Y sold in January in both the markets may also be obtained from Matrix 2. The result is 30. This result again will be difficult to obtain directly from Matrix 1.

d) The total units of product X and Y sold in January in Market 1 may be obtained from Matrix 1. The result is 30. This result will be difficult to obtain from Matrix 2.

e) To calculate the total units of product X sold in both the market in each month, we may refer to Matrix 1.

Let $A = \begin{bmatrix} J & F \\ 10 & 20 \\ 20 & 40 \end{bmatrix}$ and $B = \begin{bmatrix} J & F \\ 5 & 15 \\ 10 & 15 \end{bmatrix}$

The first row in A represents product X and second row represents product Y.

The first column in A represents month of January in Market 1 and second row represents month of February in Market 1.

Similarly,

The first row in B represents product X and second row represents product Y.

The first column in B represents month of January in Market 2 and second row represents month of February in Market 2.

Now, $A + B = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix} + \begin{bmatrix} 5 & 15 \\ 10 & 15 \end{bmatrix} = \begin{bmatrix} 15 & 35 \\ 30 & 55 \end{bmatrix}$

Let $C = \begin{bmatrix} 15 & 35 \\ 30 & 55 \end{bmatrix}$

The element 15 gives the total units of X sold from both Market 1 and 2 in January

The element 35 gives the total units of X sold from both Market 1 and 2 in February

The element 30 gives the total units of Y sold from both Market 1 and 2 in January

The element 55 gives the total units of Y sold from both Market 1 and 2 in February

f) If the price of product X is ₹ 5 and price of product Y is ₹10, the total revenue earned may be obtained from Matrix 1.

Let $R = \begin{bmatrix} 50 \\ 85 \end{bmatrix}$ and $P = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$

R is the matrix of total units of product X and Y sold in both markets

P is the matrix of prices of product X and Y.

To calculate the total revenue from Product X and Y, we may use matrix multiplication. But before using matrix multiplication matrix R must be transposed. (to fulfill the condition for matrix multiplication)

Thus $R' = [50 \quad 85]$

Therefore, $R'P = [50 \quad 85]\begin{bmatrix} 5 \\ 10 \end{bmatrix} = 50 \times 5 + 85 \times 10 = 250 + 850 = 1100$

