Mathematical Economics

Solution to Linear Programming - The Graphical Method

24.1. Introduction:

Before solving any problem, it is important to understanding the gist of the problem so that accurate solutions may be obtained. In module 23, we learned that linear programming technique is used to solve many practical problems in different fields such as agriculture, manufacturing, transportation, energy, health care, military operations and economics. What is the general structure of a linear programming problem? How can it be solved?

This module will try to answer these questions.

A. **Objectives**

The objectives of the module are:

1. *Interpret* the structure of a linear programming problem.
2. *Demonstrate* the mathematical formulation of a linear program.
3. *Determine* the optimum solution by the graphical method.

Terminology

1. Choice variables: variables whose values are to be determined and optimized
2. Objective function: a function that states the goal (objective) of optimization
3. Performance variable: the dependent variable in an objective function
4. Constraints: the restrictions of resources or requirements of an optimization problem
5. Non-negativity constraint: a constraint that is included in an optimization problem so as to have positive optimum solution
6. Feasible region: set of co-ordinates that satisfy all the constraints of the problem
7. Basic feasible solution: the corner points of a feasible region, intersection of two constraints
8. Optimum solution: the solution that satisfies all the constraints and optimizes the objective function, selected from among the basic feasible solutions

24.2. Mathematical formulation:

The main purpose of mathematical economics is to express economic theories in mathematical form to explain an economic situation more accurately and for efficient decision-making. While solving problems related to Linear Programming, the first step is to express the problem in mathematical form. Mathematical formulation of a linear programming problem is an art of converting words, sentences, phrases into mathematical expressions using numbers, symbols, equations, inequalities that represent the relationship among different choice variables subject to some constraints.

The first model of a linear programming problem was developed by George B. Dantzig, an American mathematical scientist, to solve military logistics problems while he was working with the US Air Force during the Second World War. The concept of linear programming was used to plan the movement and maintenance of military forces.

Image 24.1: Allied cargo ship convoy crossing the Atlantic Ocean, 1944

[Source: https://en.wikipedia.org/wiki/Military_logistics#/media/File:Convoy001.jpg]

24.3. Components of a Linear Programming problem

Constructing a linear programming model is an art. Though there are software available for mathematical programming, it is necessary to have an idea about the basic components and structure of a linear programming problem before proceeding to solve it.

A linear programming model consists of the following components:

a) **Choice variables:** These are the unknowns whose values are to be determined and optimized. They are also termed as decision variables or activities. The choice variables are generally denoted by $x_1, x_2, \dots \dots x_n$ or $y_1, y, \dots \dots y_n$.

b) **Objective function:** A linear program is a tool to solve optimization problems. The main aim or objective is to maximize or minimize a function, which is termed as the 'objective function'. The dependent variable in the objective function is known as the 'performance variable'. For example, the objective may be maximizing the profit, maximizing the revenue, minimizing the cost and so on. Therefore, profit, revenue, cost and so on are the 'performance variables', written on the LHS of the function. The RHS of the objective function consists of the choice variables that contributes to the 'performance variable'.

c) Constraints: An economy has unlimited wants but limited or scarce resources to fulfill these wants. Therefore, the objective of maximizing or minimizing a function has to be fulfilled subject to certain constraints (restrictions). Generally, for a maximization problem, the constraints are ≤ type and for minimization problem, the constraints are ≥ type. However, in more complex situations, one may have both ≤ and ≥ constraints as well as constraints that are equations. The optimum solution attained must satisfy all the constraints in the LP problem. The presence of various alternatives in inequality form is the essence of the formulation of a Linear programming model.

d) Non-negativity constraint: This constraint tells that the choice variables whose values are to be determined and optimized are non-negative. That is, $x_1 \geq 0, x_2 \geq 0, \dots \dots x_n \geq 0$ or $y \geq 0, y_2 \geq 0, \dots \dots y_n \geq 0$

24.4. Structure of a Linear Programming problem

The components of a linear programming problem are systematically arranged to arrive at the final structure. The structure is based on certain assumptions to simplify the complex real-world problems.

A. Assumptions: The major assumptions for formulation of a linear programming problem are given below:

1. **Linear:** The choice variables are linearly related, which means that the contribution made by each of the choice variables to the 'performance variable' are proportional and is a constant.
2. **Continuous and divisible:** The quantities of the decision variables obtained must be continuous and divisible. For instances where quantities are not divisible, linear programming problem cannot be applied.
3. **Additive:** The value of the objective function and the constraints used in the LP model is the sum of the individual contributions of the choice variables.
4. **Certainty:** An important assumption is that the parameters used in the model, such as availability of resources, profit, cost, time, per unit contribution are known.

B. Basic Structure

Based on the assumptions and the components, the structure of a Linear Programming problem is given as follows:

$$
\begin{array}{ll}
\multicolumn{2}{l}{Maximize\ /Minimize} \\
 & Objective\ function \\
subject\ to & \\
 & Constraint\ 1 \\
 & Constraint\ 2 \\
 & \qquad \ldots\ldots. \\
 & Constraint\ n \\
and & Non-negativity\ constraint
\end{array}
$$

Note: Linear programming problem has no probabilistic elements

24.5. General formulation

If Y is a 'performance variable' and $x_1, x_2, \ldots \ldots x_n$ are $'n'$ different choice variables that contributes to the 'performance variable' with $'n'$ different constraints, then the linear programming model may be written as follows:

$$
\left.\begin{array}{ll}
\textit{Maximize} & \\
& Y = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \\
\textit{subject to} & \\
& a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq c_1 \\
& a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \leq c_2 \\
& \text{---} \\
& \text{---} \\
& a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n \leq c_n \\
\quad\quad and & x_1 \geq 0, x_2 \geq 0, \dots \dots x_n \geq 0
\end{array}\right\} \quad (24.1)
$$

OR

$$
\left.\begin{array}{ll}
\textit{Minimize} & \\
& Y = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \\
\textit{subject to} & \\
& a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \geq c_1 \\
& a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \geq c_2 \\
& \text{---} \\
& \text{---} \\
& a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n \geq c_n \\
\quad\quad and & x_1 \geq 0, x_2 \geq 0, \dots \dots x_n \geq 0
\end{array}\right\} \quad (24.2)
$$

Equation (24.1) and (24.2) are Linear programming models whose objective is to maximize Y or minimize Y subject to given constraints. The solution of the model seeks to find the optimum values of the choice variables $x_1, x_2, \dots \dots x_n$ that will satisfy all the constraints.

Note:

a) It is not necessary that each of the constraints contain all the decision variables, as shown in the example.

b) Constraints may take any form ($\leq, = or \geq$) depending upon the statement of the problem

24.6. Solving the LP problem

A linear programming problem may be solved by two methods:

i) **Graphical Method:** Graphical method is used to solve simple linear programming problems with two choice variables.

ii) **Simplex Method:** The word 'simplex' is topology word for polyhedron. This method was invented by George Dantzig. Simplex method is used to solve more complex linear programming problems with more than two choice variables. Since the 3 dimensional or higher dimensional graphs are difficult to draw and understand, for more than 3 choice variables, simplex method helps to solve it.

Note: This course will concentrate on the graphical method.

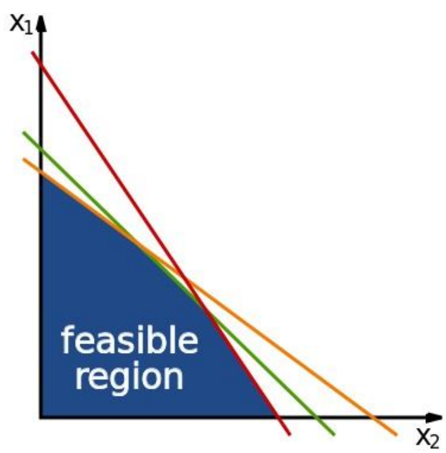

Image 24.2: LP with two choice variables

[Source: https://en.wikipedia.org/wiki/File:Linear_Programming_Feasible_Region.svg]

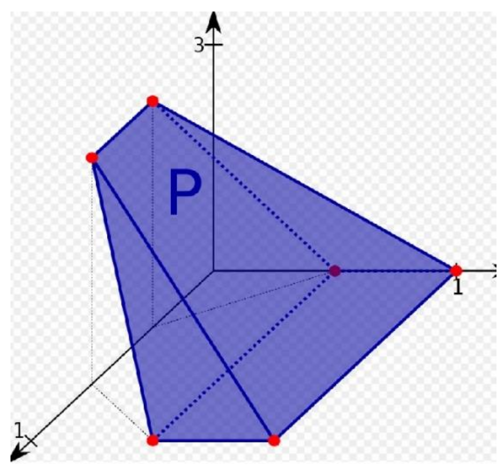

Image 24.3: LP with three choice variables

[Source: https://en.wikipedia.org/wiki/Linear_programming#/media/File:3dpoly.svg]

Image 24.2 is a case of two choice variables X_1 and X_2. The red, green and yellow lines are linear constraints. The shaded blue region is the feasible region from which the optimum solution is to be determined. Image 24.3 is a case of three choice variables. The linear programming problem is to find a point on the polyhedron that is on the plane (not shown) with the highest possible value.

24.7. The Graphical Method

The graphical method is generally used for solving linear programming problems with two choice variables. It is important to have some idea about few basic concepts and terms and before proceeding to solve the LP problem by the graphical method.

A. Basic concepts:

1) **Feasible region and feasible solution:** The feasible region is a set of all the possible points (co-ordinates) that satisfy all the constraints in the linear programming problem. Co-ordinates that satisfy one constraint but fail to satisfy other constraints in the linear program are not considered feasible solutions. A feasible region may be bounded or unbounded. All the points (ordered pairs) in the feasible region is a feasible solution.

A. Bounded **B. Unbounded**

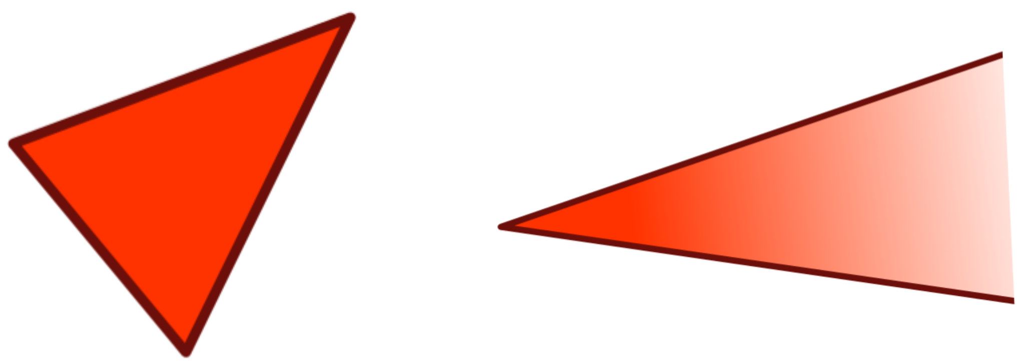

Image 24.4: Bounded and Unbounded feasible region

[Source: https://en.wikipedia.org/wiki/File:Bounded_unbounded.svg]

A bounded feasible region is a closed region, that is, the feasible solutions are limited by the constraints. On the other hand, an unbounded feasible region is an open region, that is the values of the feasible region are not limited by the constraints. The figure in B is open towards the right which means that it is not bounded.

Generally, a maximization problem with constraints of $\leq$ type is bounded. And a minimization problem with constraints of $\geq$ type is unbounded.

2) **Basic feasible solution:** Graphically, the corner points of the feasible region are the basic feasible solutions. The intersection of the graphs of two constraints gives a corner point. The co-ordinates of the corner point may be found out by solving the two equations that intersect. The intersection of two lines outside the feasible region are not considered as basic feasible solutions.

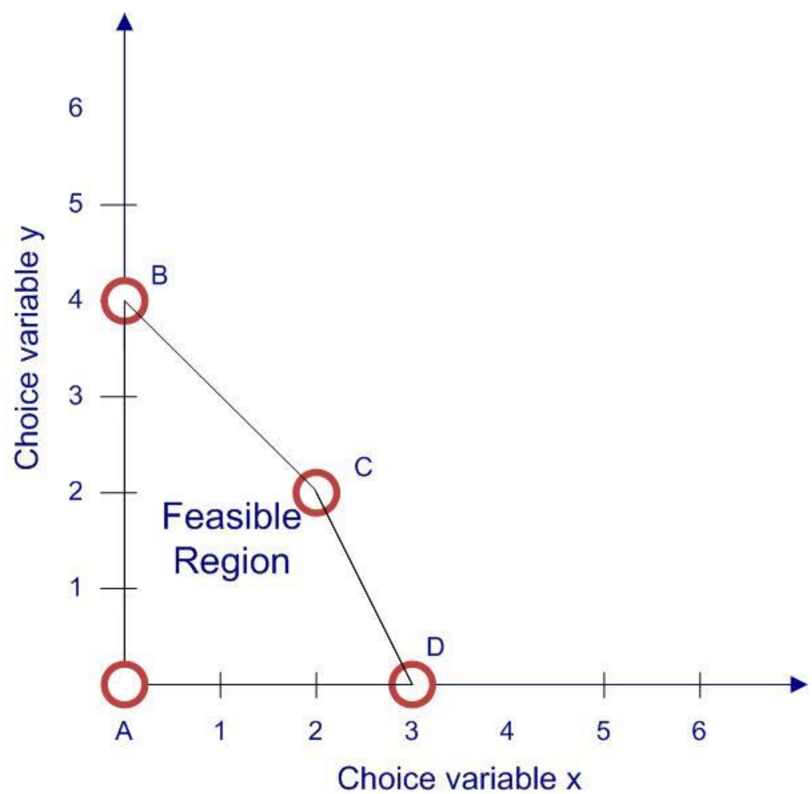

Fig. 24.1(a): Corner points in a maximization problem denoted by A, B, C, and D

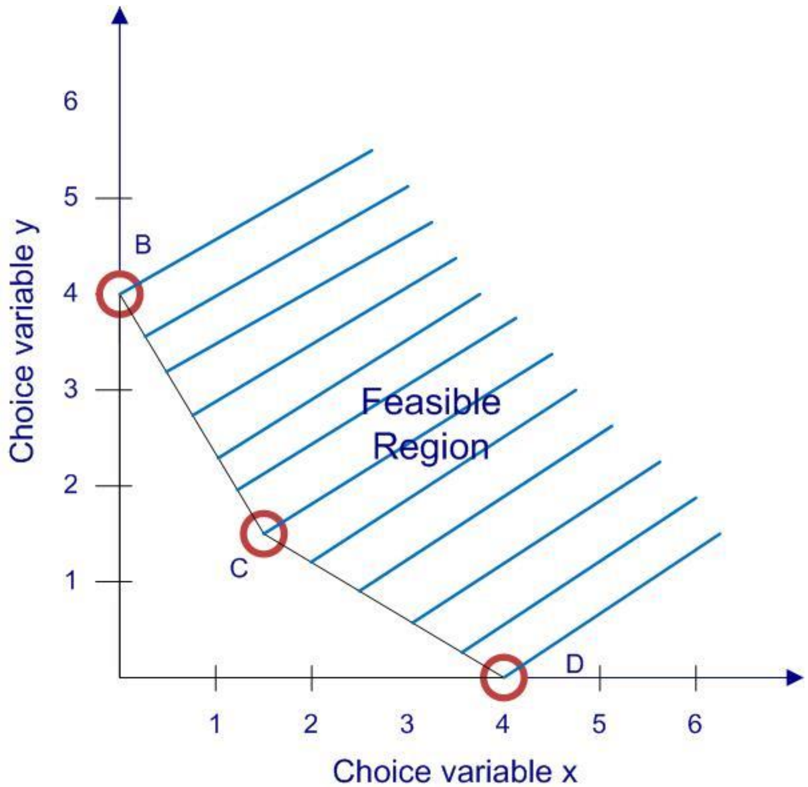

Fig. 24.1(b): Corner points in a minimization problem denoted by B, C and D

3) **Optimum solution:** The solution that satisfies all the constraints and optimizes the objective function is the optimum solution. The optimum solution is selected from among the basic feasible solutions. Graphically, the point where the graph of the objective function is tangent to a corner point gives the optimum solution.

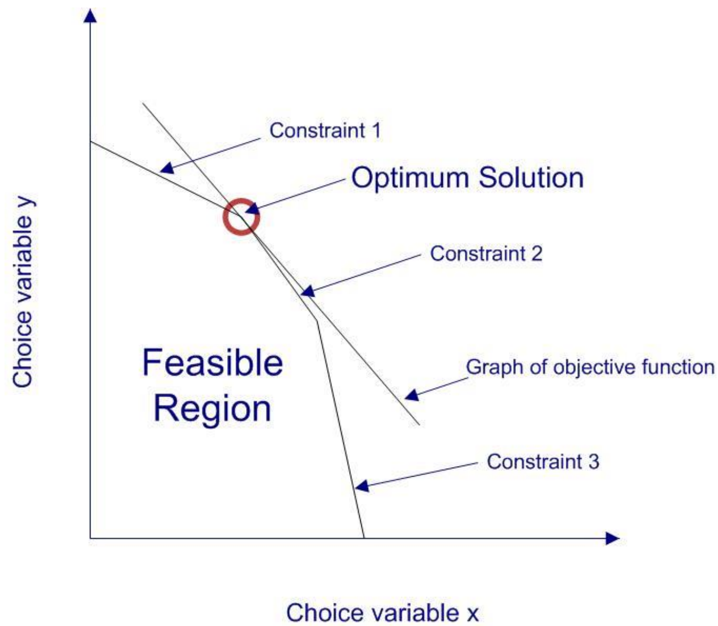

Fig. 24.2(a): Optimum solution for maximization problem

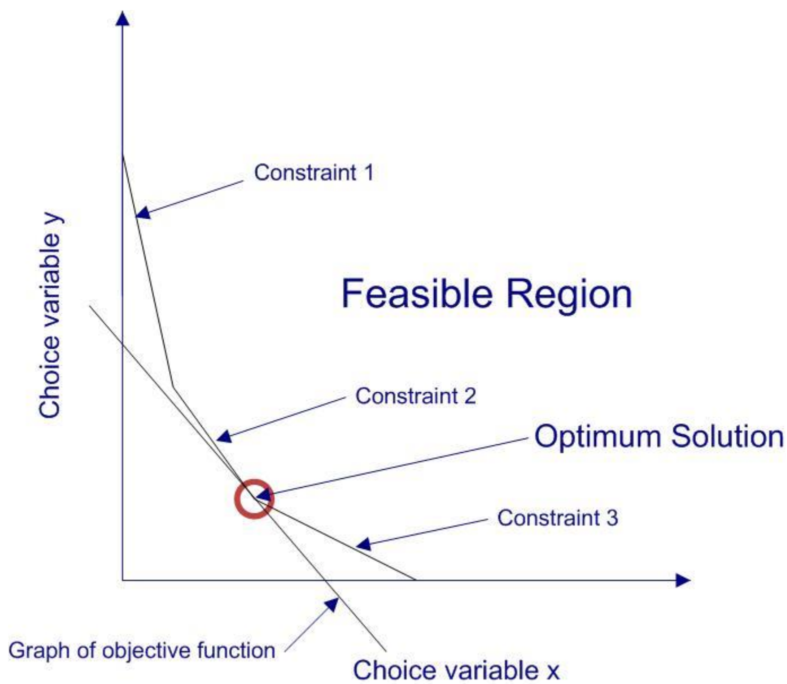

Fig. 24.2(b): Optimum solution for minimization problem

B. Procedure for solving the LP problem

The graphical method of Linear Programming problem involves finding out the optimum values of the decision variables from among a set of feasible solutions. As the name suggests, the problem is represented on a graph paper. The graphical method is generally used for problems with two choice variables. If the number of choice variables is more than two, expressing the problem in 3-

D (Image 3) or more becomes difficult and such problems may be solved by using the Simplex method.

Consider a simple structure of a linear programming problem with two choice variables $x \text{ and } y$ and two constraints.

$$\text{Maximize}$$
$$\pi = 4x + 3y$$
$$\text{subject to}$$
$$x + y \leq 4$$
$$2x + y \leq 6$$
$$\text{and} \qquad x \geq 0, y \geq 0$$

π is the performance variable

$x \text{ and } y$ are the choice variables

The steps for solving the linear programming problem are explained as follows:

Step 1: Rewrite the inequalities in the constraints as strict equations

Therefore,

$$x + y = 4 - - - - - (24.3)\ (Constraint\ 1)$$
$$2x + y = 6 - - - - - (24.4)\ (Constraint\ 2)$$

Step 2: Trace out the graph of both the constraints in equation (24.3) and (24.4)

Hint: To obtain the graph of an equation, find two extreme points and then join the points. To obtain the extreme points, put $x = 0$ in the equation and find the value of y then put $y = 0$ and find the value of x. Example: In equation (24.3), putting $x = 0$, we get $y = 4$ and putting $y = 0$, gives $x = 4$.

Therefore,

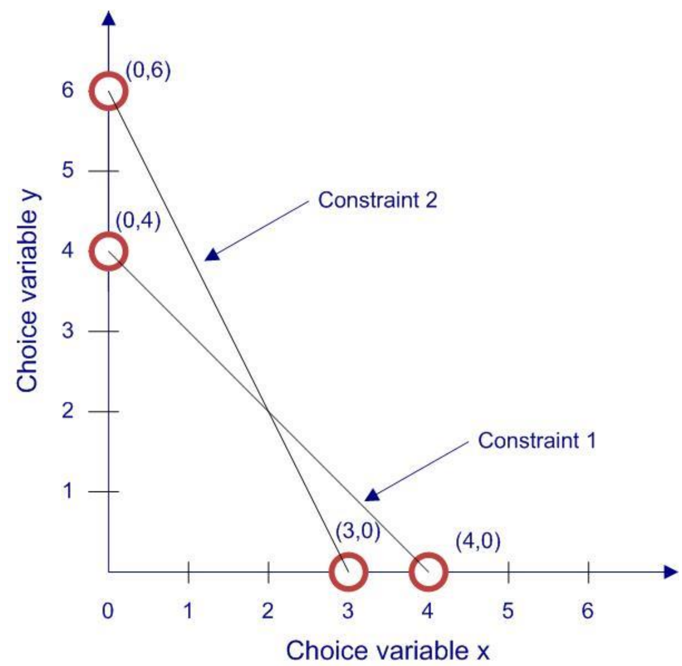

Fig. 24.3: Graph of constraints

Step 3: Identify the feasible region. The feasible region for the above problem is given by the common area below the graph of the constraint equations.

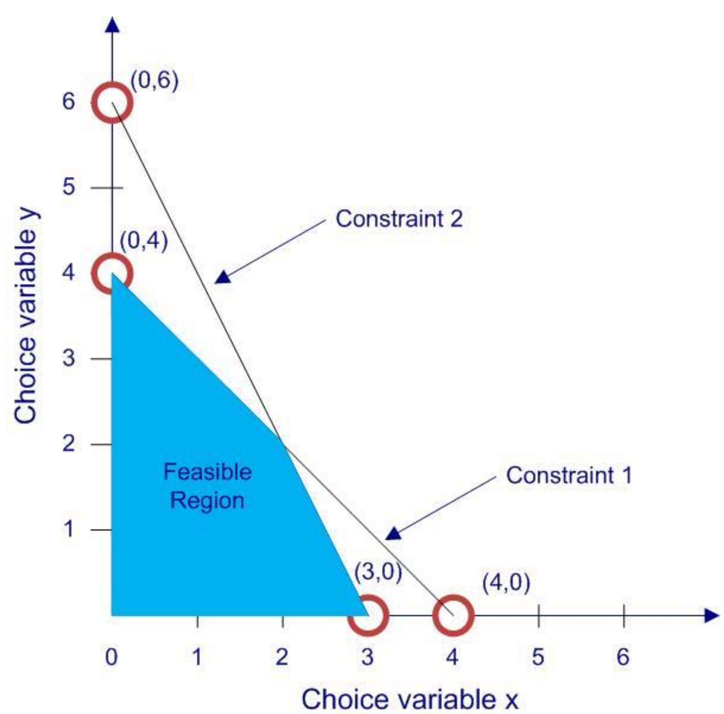

Fig. 24.4(a): Feasible Region

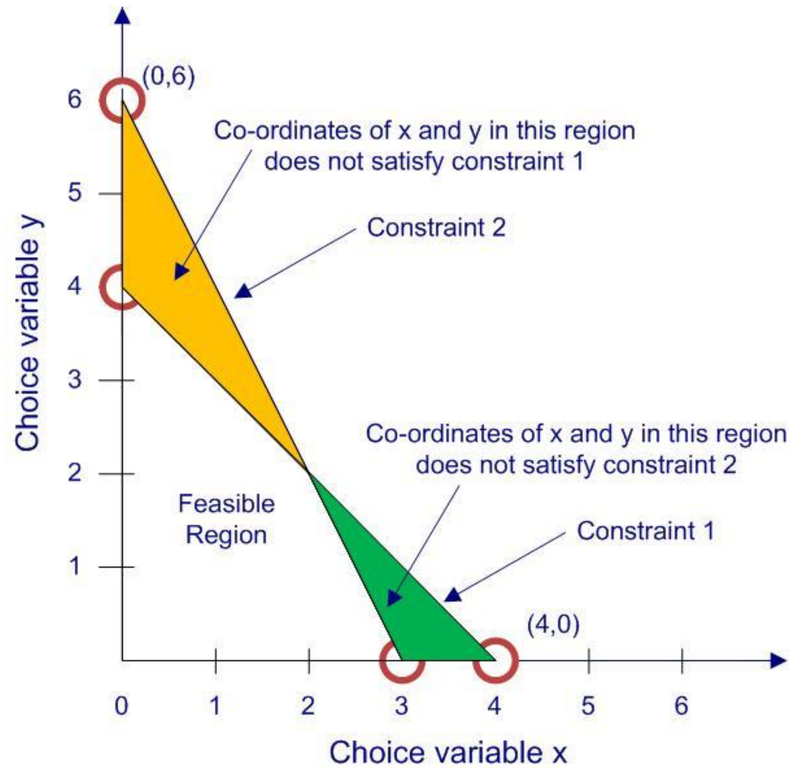

Fig. 24.4(b): Region which is not feasible

The upper part (from the point of intersection of the two graphs) of the graph of constraint 2 lie above the border line of constraint 1. Therefore, the co-ordinates of x and y above the line of constraint 1 will not satisfy constraint 1. Similarly, the lower part of the graph of constraint 1 lie above the border line of constraint 2. Therefore, the co-ordinates of x and y above the line of constraint 2 will not satisfy constraint 2. Hence, co-ordinates lying in both these regions shaded orange and green are not feasible solutions as they satisfy just one constraint.

Step 4: Identify the basic feasible solutions (corner points)

The corner points of the feasible region give the basic feasible solutions. In the graph obtained for the given maximization problem, there are four corner points; three of which are on the x-axis and y axis.

The co-ordinates of the corner point in the co-ordinate plane may be obtained by solving the equations that intersect.

Solving

$$x + y = 4$$

$$2x + y = 6$$

We get $x = 2 \; and \; y = 2$

Therefore, the co-ordinates of the corner points are R (0,0), S (0,4), T (2,2) and U (3,0)

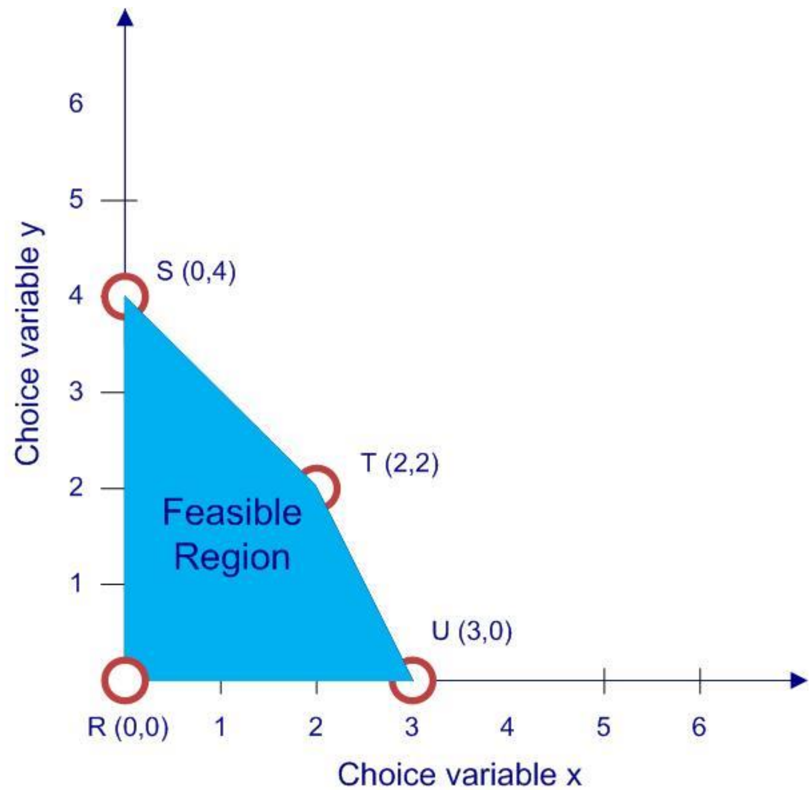

Fig. 24.5: Basic Feasible solutions

Note:

If the graph of constraints does not intersect, we do not get a feasible region and there does not exist any solution.

Step 5: Re-write the objective function in the form of the equation of a straight line $y = mx + c$ and obtain the slope co-efficient $'m'$.

Therefore,

$$\pi = 4x + 3y$$

=> $3y = \pi - 4x$ (considering y as dependent variable)

=> $y = \frac{\pi}{3} - \frac{4}{3}x$--------------- (24.5)

From equation (24.5), slope = $-\frac{4}{3}$

Step 6: Trace the objective function with slope $\frac{4}{3}$. Since the slope is negative, the objective function will be a downward sloping straight line.

Therefore,

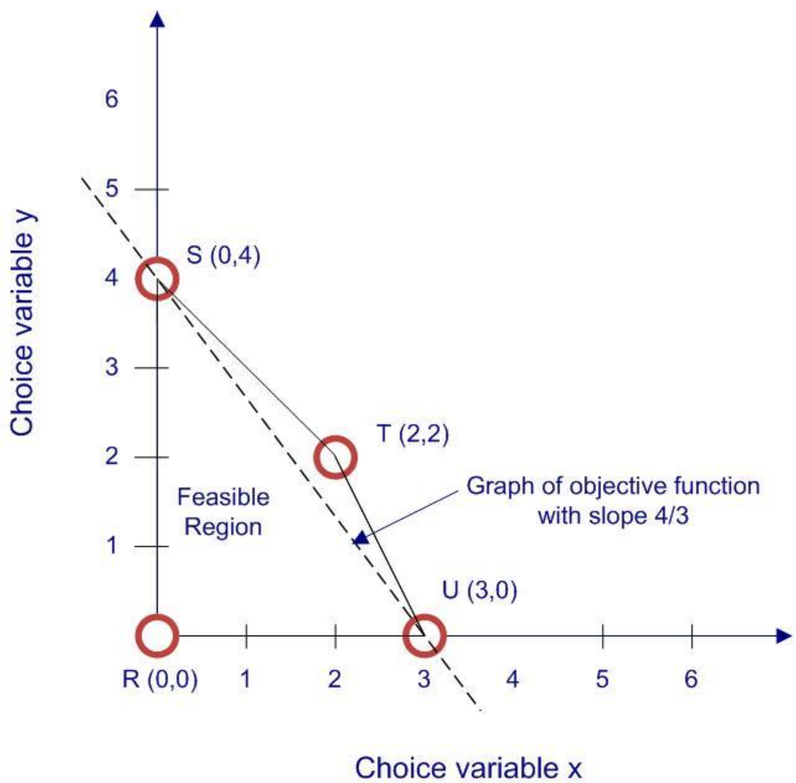

Fig. 24.6: Graph of the objective function

The dotted line is the graph of the objective function.

Hint for tracing the graph of objective function: $Slope = \frac{\Delta y}{\Delta x} = \frac{Change\ in\ the\ Perpendicular}{Change\ in\ the\ Base}$

Therefore, from the origin (0,0), $\Delta y = 4$ and $\Delta x = 3$

Step 7: Trace out similar dotted straight lines parallel to the objective function.

For a maximization: Move towards the border of the feasible region.

For minimization: Move towards the origin.

The corner point in the feasible region that touches (not cross) the graph of the objective function gives the optimum solution. The x co-ordinate and y co-ordinate of the optimum solution satisfies all the constraints and maximizes the 'performance variable' π.

Therefore,

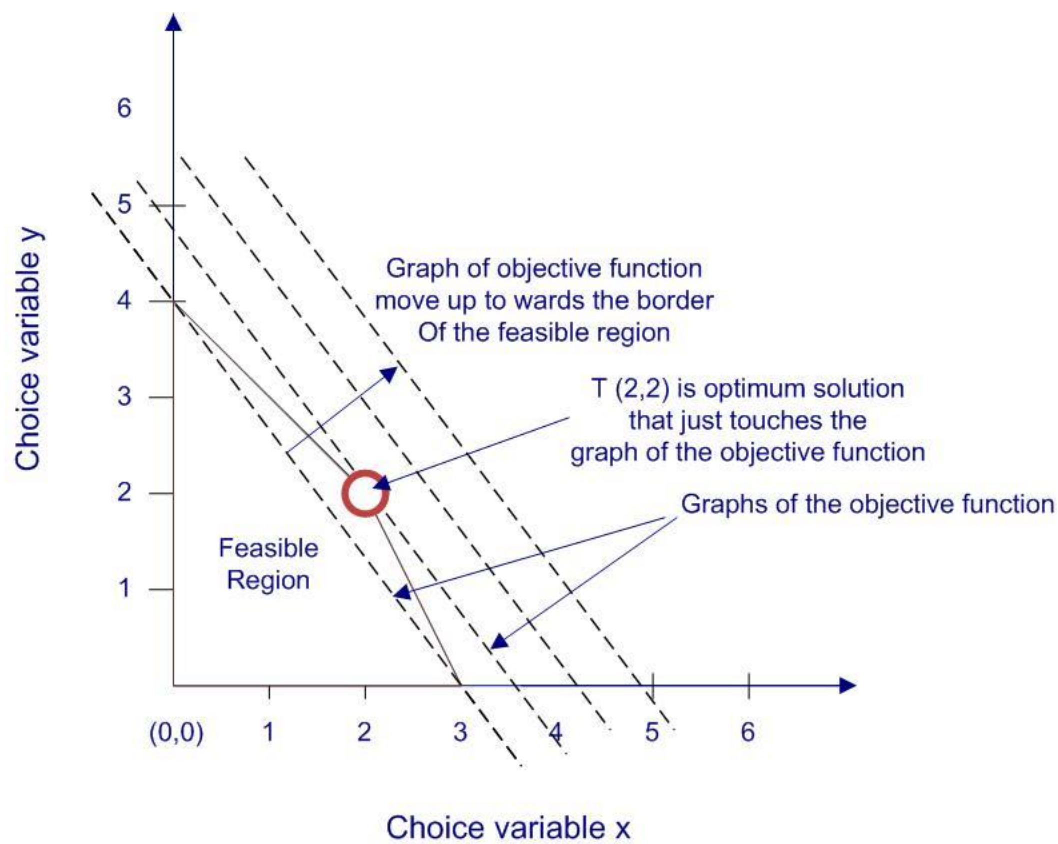

Fig. 24.7: Optimum Solution

The optimum solution out of the basic feasible solutions, R (0,0), S (0,4), T (2,2) and U (3,0) is T (2,2). Therefore, x= 2 and y=2 maximizes the objective function and also satisfies both the constraints.

Note:

For a solution to exist, the number of constraints must at least be equal to the number of choice variables.

24.8. Alternative method (Algebraical method):

There is an alternative method for finding the optimum solution of a linear programming problem. As mentioned in Section 24.7, Step 1 to Step 4 remains the same. In Step 5, instead of tracing out the graph of the objective function, the basic feasible solutions are used in the equation of the objective function to find out the optimum solution. Since the optimum solution is found out algebraically, this method may be called the "algebraical method'.

This method is explained as follows:

In Step 4, the basic feasible solutions that were identified are R (0,0), S (0,4), T (2,2) and U (3,0).

Using each basic feasible solution, we find the value of the 'performance variable' $\pi = 4x + 3y$

Therefore,

For R (0,0), $\pi = 4(0) + 3(0) = 0$

For S (0,4), $\pi = 4(0) + 3(4) = 12$

For T (2,2), $\pi = 4(2) + 3(2) = 14$

For U (3,0), $\pi = 4(3) + 3(0) = 12$

It is seen that the value of π is maximum for T (2,2) and hence it can be concluded that $x = 2\ and\ y = 2$ is the optimum solution that maximizes π.