Mathematical Economics: Integration- An Introduction

27.1. Introduction:

In Economics, marginal means additional. The marginal functions such as marginal cost, marginal revenue function, marginal propensity to consume function or marginal propensity to save function, all are obtained from the total functions respectively by the technique of differentiation. Is there any possible way to get back the total function from a given marginal function?

This module will try to explore the answer to this question.

Objectives

The objectives of the module are:

1. *Define and explain* the concept of integration and integral
2. *Determine* the area under a curve

Terminology

1. Integrate: mix, combine, put together two or more groups or elements
2. Integration: a process of integrating
3. Integral: values to a function that describes area, volume, displacement from a combination of infinitesimal data.
4. Antiderivative: reverse of a derivative
5. Indefinite integral: an integral expressed without limits and that has an arbitrary constant
6. Definite integral: an integral with numerical values that represent the area under a curve and above the x axis.
7. Infinitesimal: infinitely small

27.2. Meaning and History of Integration

Integration is a process of integrating. To 'integrate' means to mix, combine, join, put together two or more groups of people, things or elements. In Mathematics, 'integration' carries a similar meaning. It is a mathematical process studied under the branch of Calculus called Integral Calculus.

Cavalieri's principle is seen as the early step towards Integral Calculus and integration. This principle was originally called the method of indivisibles. Cavalieri principle is named after an Italian mathematician Bonaventura Cavalieri.

Image 27.1: Bonaventura Cavalieri

[Source: https://en.wikipedia.org/wiki/File:Bonaventura_Cavalieri.jpeg]

The Integration Symbol

It was Newton and Leibniz who separately invented calculus, though it is still a controversy as to who invented calculus first. The contribution of both Newton and Leibniz is reflected in the notation used for integration. Today, the notation that is used for integration is $\int$ and is read as 'integral of'

27.3. Definition

In Mathematics, Integration is a technique of finding out a function, the derivative of which is the original function.

If $y = f(x)$ is a function, then $\frac{dy}{dx}$ $or\ f'(x)$ is the differentiation of $y\ with\ respect\ to\ x$.

By definition of integration,

$$\int f'(x)dx = f(x) + c$$

$f(x)$ is the original function found by integration

c is an arbitrary constant of integration.

Note:

The arbitrary constant c is important because the same derivative may be obtained from the differentiation of different functions.

Example:

If there are two functions $y = x^2 + 10$ and $y = x^2 + 20$

Differentiation of both the functions give the same derivative $\frac{dy}{dx} = 2x$ **(see Module 20, Section 20.3)** But the functions have different constant values.

Hence, to distinguish between the two functions the arbitrary constant c is used in integration.

By rule, ***(see section 27.4)*** integrating $2x$ gives the original function. Symbolically,

$$\int 2x\, dx = x^2 + c$$

In this example c may be 10 or 20 which gives the original functions $x^2 + 10$ or $x^2 + 20$

Integration is the reverse process of differentiation.

27.4. Basic rules of integration:

The basic rules of integration and the operation rules are given in Table 27.1.

Table 27.1: Basic rules and operation rules of integration

A. Basic Rules	B. Operation Rules
***Power Rule*:** $\int x^n\, dx = \frac{1}{n+1} x^{n+1} + c$ $n \neq -1$	**Addition Rule:** $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$
Exponential Rule: $\int e^x\, dx = e^x + c$ $\int f'(x)e^{f(x)}\, dx = e^{f(x)} + c$	**Product Rule:** $\int kf(x)dx = k\int f(x)$ k **is a constant**
Logarithmic Rule: $\int \frac{1}{x} dx = ln\, x + c \quad for \quad x > 0$ $\int \frac{f'(x)}{f(x)} dx = lnf(x)\ for\ f(x) > 0$ $or\ \ ln\lvert f(x)\rvert + c\ \ for\ \ f(x) \neq 0$	**Substitution Rule:** $\int f(u)\frac{du}{dx} dx = \int f(u)du\ \ u\ is\ function\ of\ x$ ***Integration by Parts*:** $\int vdu = uv - \int udv\ \ u, v\ are\ functions\ of\ x$

27.5. Integral:

Integral means essential, fundamental or important part of a whole. In mathematics, an integral assigns values to a function in such a way that describes area, volume, displacement from a combination of infinitesimal data. Infinitesimal data are infinitely small quantities that cannot be measured.

An Integral may be indefinite or definite.

Note: This module is confined to area

A. Indefinite Integral or Antiderivative:

Indefinite Integral or Antiderivative is an integration of the general form where there are no upper and lower limits. Therefore, an Antiderivative or indefinite integral of a function f is a differentiable function F whose derivative, that is F' is equal to the original function. In other words, a function F is an antiderivative of a function f if the derivative of $F\ is\ f$

Symbolically, $\int f dx = F(x) + c$, where $F' = f$

The expression $\int f dx$ is known as the indefinite integral and the result is the antiderivative.

Example with Geometric interpretation

Suppose there are three functions of the form $y = f(x)$

$$f(x) = x^2$$

$$f(x) = x^2 + 1$$

$$and\ f(x) = x^2 - 1$$

Then, derivative of $f(x)\ or\ f'(x) = 2x$ for all three functions.

$$(\because derivative\ of\ a\ constant\ is\ zero)$$

Recalling that a derivative is the rate of change of a function or the slope of a curve, the rate at which y changes with respect to x or the slope of the three functions is $2x$.

Antiderivative is the opposite of derivative. That is, what is the function whose slope is $2x$?

Clearly, there may be a group of functions such as $x^2, x^2 + 1\ or\ x^2 - 1$ or in general $x^2 + c$. It may also be said that there are a group of curves with the same slope.

The process of obtaining the function from a given slope is known as integration.

Thus,

$$\int 2x\, dx = x^2 + c \text{ -----------(27.1)}$$

The expression $\int 2x\, dx$ is known as the indefinite integral and $x^2 + c$ is the antiderivative. A can be seen, the RHS (antiderivative) is equal to the original function.

Graphically, the antiderivative may be obtained by putting the values $0{,}1\ and\ -1\ for\ c$ in equation (1)

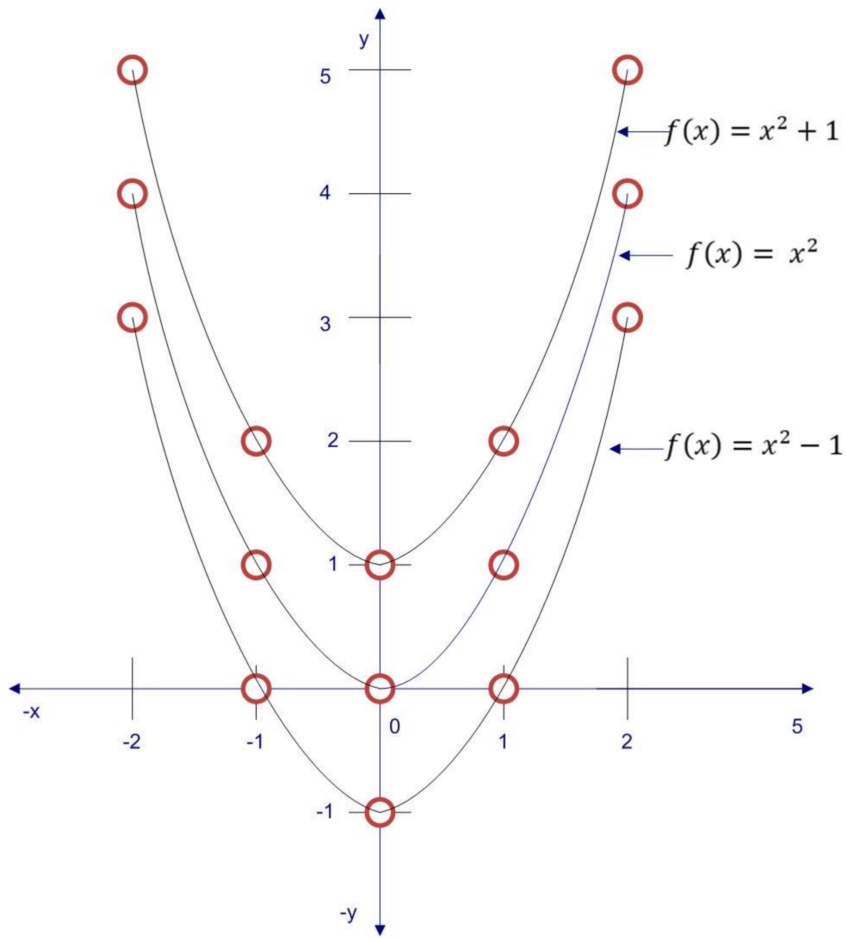

Fig. 27.1: Graph of antiderivative

In Fig. 27.1, it is seen that the antiderivatives or indefinite integrals looks similar. In this example it is a set of similar parabolas passing through different points on the y axis. In general, it may be said that the antiderivative or indefinite integral is a set of similar curves with the same slope.

B. Definite Integral:

The concept of integration is used to find the area under a curve using definite integral. Definite integrals are integrals with numerical values that represent the area under a curve and above the x axis. There is a lower limit and an upper limit.

Symbolically, a definite integral of a function with interval [a, b], where a is the lower limit and b is the upper limit, is written as

$$\int_a^b f(x)\,dx$$

Graphically, it may be represented as follows:

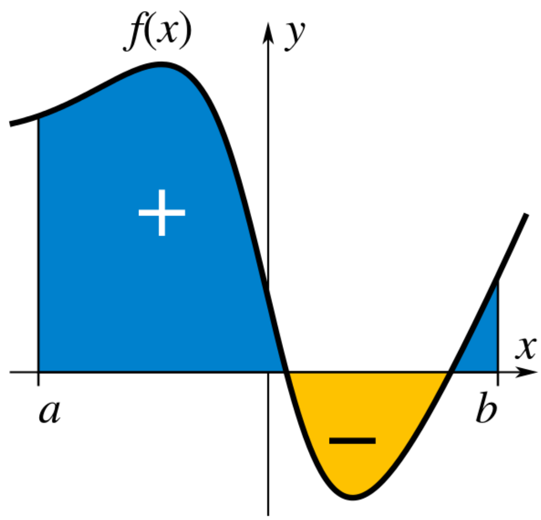

Image 27.2: Definite Integral

[Source: https://en.wikipedia.org/wiki/File:Integral_example.svg]

A definite integral of a function within the range [a, b] is the shaded area under the curve. The shaded area above the *x*-axis adds to the total and that below the *x*-axis subtracts from the total.

Antiderivatives are related to definite integrals in such a way that definite integral is the difference between the values of an antiderivative at the upper limit and the lower limit.

Thus, symbolically

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

where $F(x)$ is the antiderivative.

Example:

Consider the function $y = x^2$, graph of which is given in Fig 27.2.

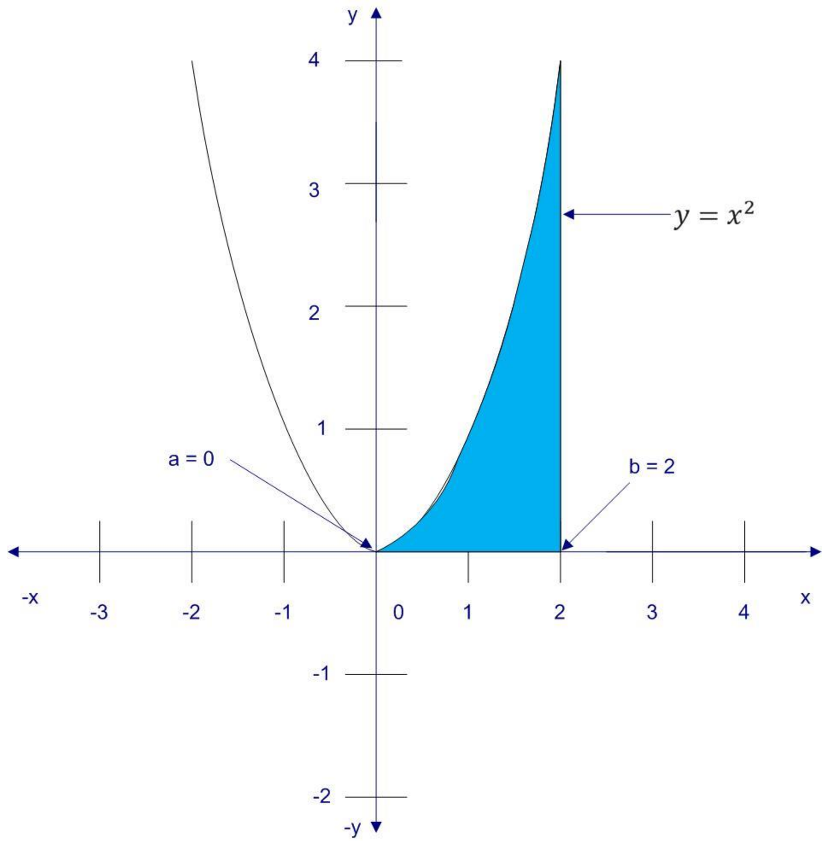

Fig.27.2: Area under curve for function $y = x^2$

In order to find the area under the curve between lower limit a = 0 and upper limit b = 2,

The concept of definite integral gives

$$\int_0^2 x^2\, dx = [F(x)]_0^2 = F(2) - F(0)$$

By power rule of integration, $F(x) = \frac{x^3}{3} + c$, therefore,

$$\int_0^2 x^2\, dx = \left[\frac{x^3}{3} + c\right]_0^2 = \left[\frac{2^3}{3} + c\right] - \left[\frac{0^3}{3} + c\right] = \frac{8}{3}$$

27.6. Geometric Interpretation of Definite Integral:

By definition, definite integral represents the area under a curve, above the x axis and between two defined values.

For a given function $y = f(x)$, suppose we have the graph as shown in Fig. 27.3.

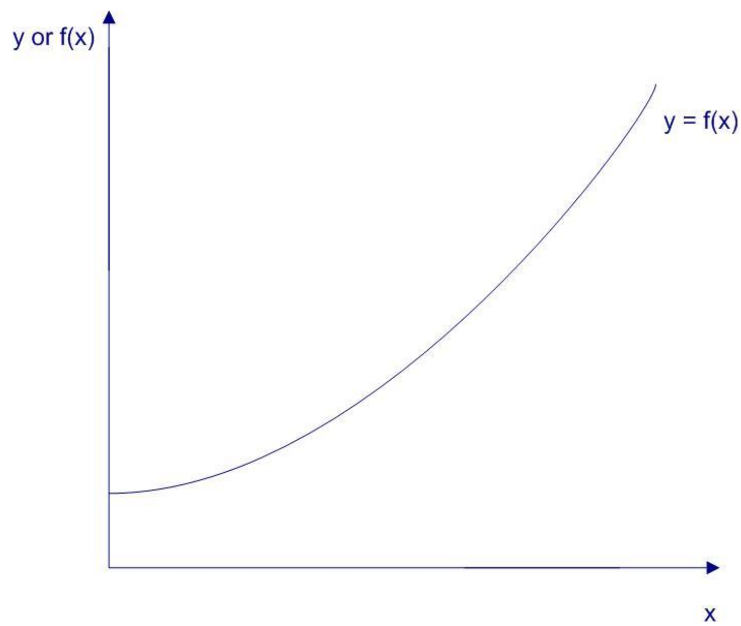

Fig. 27.3: Graph of function y =f(x)

Case 1:

In order to find the area below the curve and above the x axis between two points $x = a \; and \; x = b$ (Fig 27.4), a rectangle PQRS is constructed. The change of x from a to b may be represented as Δx. At point $x = b$, the value of $y \; or \; f(x)$ will be $f(b)$.

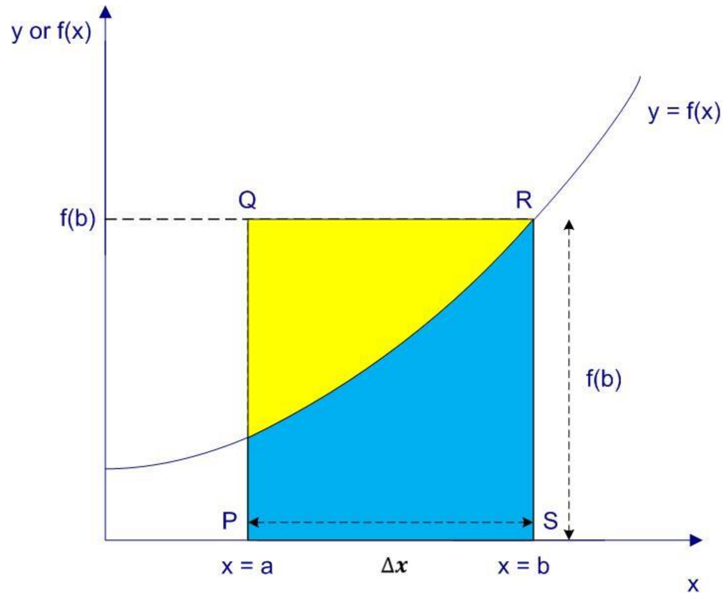

Fig. 27.4: Area under the curve

Therefore,

The area of rectangle PQRS = $f(b) \times \Delta x$ $\quad \because area\ of\ a\ rectangle = \text{(length x breadth)}$

Clearly, the area of the rectangle is greater than the area under the curve between the two points $x = a\ and\ x = b$

$The\ area\ of\ the\ rectangle\ - yellow\ region = area\ under\ the\ curve\ (blue\ region)$

Case 2:

Now, dividing the area under the curve into two parts, we get a case as shown in Fig. 27.5. In this case, the area under the curve may be shown as the sum of the area of two rectangles.

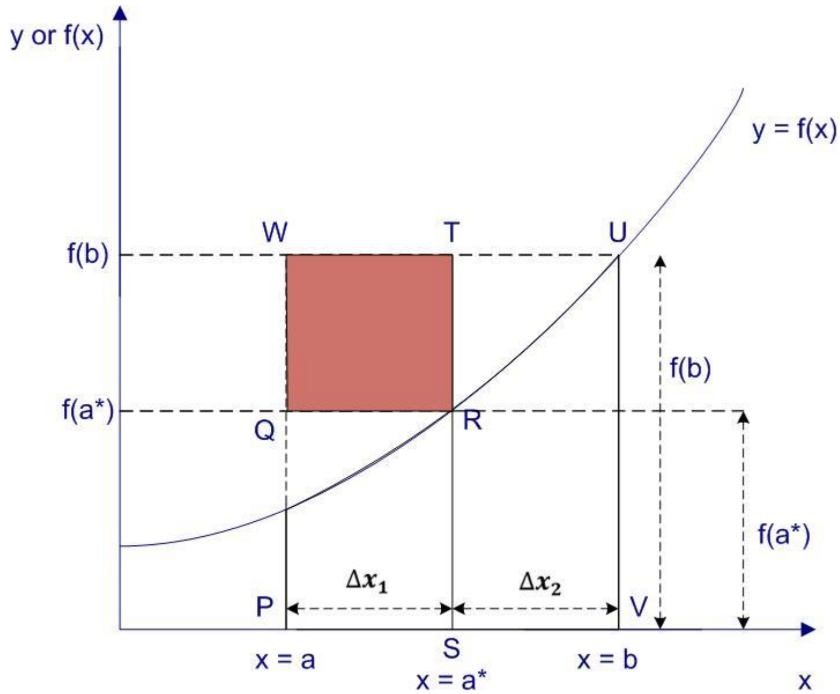

Fig. 27.25: Area using two rectangles

The area of rectangle STUV + area of rectangle PQRS = $\{f(b) \times \Delta x_2\} + \{f(a)^* \times \Delta x_1\}$

Clearly,

The sum of the area of the two rectangles is still greater than the area under the curve. However, the yellow region in Case 1 is reduced by the area of the region QWTR that is shaded brown. Therefore, we have moved a step closer to the area under the curve.

Case 3:

If the number of rectangles is increased to four, the sum of the area of the rectangles moves even more closer to the area under the curve, as the yellow region is reduced further. In this case, the yellow region reduces by the shaded regions A+B+C.

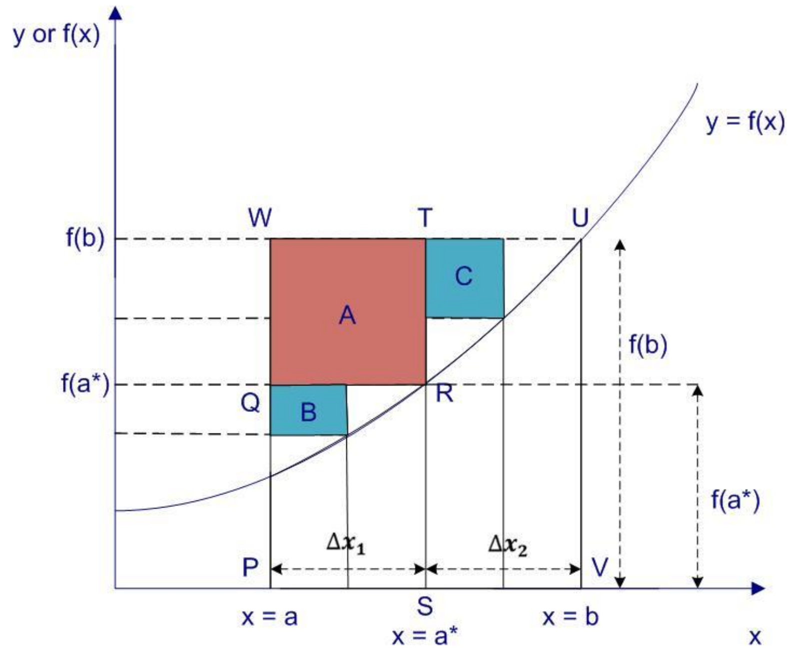

Fig. 27.6: Area using four rectangles

Similarly, by drawing smaller and smaller rectangles, the yellow region in Fig. 27.4 will be erased completely, and the sum of all the rectangles will approximate to the area under the curve.

Note that, by drawing infinite number of rectangles, $\Delta x \to 0$ and the sum of the area of infinite rectangles will tend to the area under the curve.

Mathematically, this situation may be expressed as

$$Area\ under\ the\ curve\ =\ \lim_{n\to\infty} \sum f(x_i)\Delta x_i$$

$$i = 1,2,3, \dots . n$$

Therefore, definite integral

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a) = Area\ under\ the\ curve$$

Note:

a) For infinitesimal changes, Δ changes to d, That is, Δx changes to dx. In other words, area shrinks to line (height) when Δx shrinks to point. Hence X^2 becomes X. That is derivative.

b) Summation is used for discrete values. Integration is used for continuous values.

