Test of Mathematics at 10+2 Level

TOMATO OBJECTIVE SOLUTIONS

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1. A worker suffers a 20% cut in wages. He regains his original pay by obtaining a rise of
   (a) 20%
   (b) 22.5%
   (c) 25%
   (d) 27.5%

Solution :

Let his wage is x.

After 20% cut his wage is x – 20x/100 = 80x/100

In 80x/100 he needs to regain 20x/100

In 1 he needs to regain (20x/100)/(80x/100) = ¼

In100 he needs to regain 100*(1/4) = 25

Therefore, he needs a rise of 25% to regain his wage.

Option (c) is correct.

2. If m men can do a job in d days, then the number of days in which m + r men can do the job is
   (a) d + r
   (b) (d/m)(m + r)
   (c) d/(m + r)
   (d) md/(m + r)

Solution :

m men can do a job in d days

1 man can do the job in md days

m + r men can do the job in md/(m + r) days.

Option (d) is correct.

3. A boy walks from his home to school at 6 km per hour (kmph). He walks back at 2 kmph. His average speed, in kmph, is

(a) 3
(b) 4
(c) 5
(d) $\sqrt{12}$

Solution :

Let the distance from his home to school is x km.

Therefore, total distance covered = x + x = 2x.

Time taken to go to school = x/6 hours.

Time taken to come home from school = x/2 hours.

Total time = x/6 + x/2

So, average speed = total distance/total time = 2x/(x/6 + x/2) = 2/(1/6 + ½) = 3 kmph

Option (a) is correct.

4. A car travels from P to Q at 30 kilometres per hour (kmph) and returns from Q to P at 40 kmph by the same route. Its average speed, in kmph, is nearest to
(a) 33
(b) 34
(c) 35
(d) 36

Solution :

Let the distance between P and Q is x km.

Total distance covered = x + x = 2x

Time taken to go from P to Q = x/30 hours.

Time taken to go from Q to P = x/40 hours.

Total time = x/30 + x/40

Average speed = total distance/total time = 2x/(x/40 + x/30) = 2/(1/40 + 1/30) = 2*40*30/70 = 240/7 = 34.285 (approx.)

Option (b) is correct.

5. A man invests Rs. 10000 for a year. Of this Rs. 4000 is invested at the interest rate of 5% per year, Rs. 3500 at 4% per year and the rest at α% per year. His total interest for the year is Rs. 500. Then α equals
(a) 6.2
(b) 6.3
(c) 6.4
(d) 6.5

Solution :

Interest from Rs. 4000 = 5*4000/100 = Rs. 200

Interest from Rs. 3500 = 4*3500/100 = Rs. 140

Rest money = 10000 – (4000 + 3500) = Rs. 2500

Interest from Rs. 2500 = α*2500/100 = 25α

As per the question, 200 + 140 + 25α = 500

⇨ α = 160/25 = 6.4

Option (c) is correct.

6. Let $x_1, x_2, \ldots, x_{100}$ be positive integers such that $x_i + x_{i+1} = k$ for all i, where k is constant. If $x_{10} = 1$, then the value of x_1 is
(a) k
(b) k – 1
(c) k + 1
(d) 1

Solution :

Clearly, $x_9 = k - 1$, $x_8 = 1$, $x_7 = k - 1$, ….., $x_1 = k - 1$

Option (b) is correct.

7. If $a_0 = 1$, $a_1 = 1$ and $a_n = a_{n-1}a_{n-2} + 1$ for n > 1, then

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(a) a_{465} is odd and a_{466} is even
(b) a_{465} is odd and a_{466} is odd
(c) a_{465} is even and a_{466} is even
(d) a_{465} is even and a_{466} is odd.

Solution :

As, a_0 and a_1 both odd so, a_2 = 2 = even.

As a_2 is even, both a_3 and a_4 will be odd because between a_{n-1} and a_{n-2} one is even and hence added to 1 becomes odd.

Then a_5 will be even as a_3 and a_4 are both odd.

So, the sequence will go in the way, a_0, a_1 odd, a_2 even, a_3, a_4 odd, a_5 even, a_6, a_7 odd, a_8 even and so on..

So, the numbers which are congruent to 2 modulus 3 are even and rest are odd.

Now, $465 \equiv 0 \pmod 3$ and $466 \equiv 1 \pmod 3$

⇨ a_{465} and a_{466} are both odd.

Option (b) is correct.

8. Two trains of equal length L, travelling at speeds V_1 and V_2 miles per hour in opposite directions, take T seconds to cross each other. Then L in feet (1 mile 5280 feet) is
   (a) $11T/15(V_1 + V_2)$
   (b) $15T/11(V_1 + V_2)$
   (c) $11(V_1 + V_2)T/15$
   (d) $11(V_1 + V_2)/15T$

Solution :

Speed = V_1 miles per hour = V_1*5280/3600 feet/second = $22V_1/15$ feet/second

Relative velocity = $22V_1/15 + 22V_2/15 = 22(V_1 + V_2)/15$ feet/second

Total distance covered = sum of train lengths = L + L = 2L

Therefore, $2L = \{22(V_1 + V_2)/15\}*T$

⇨ $L = 11(V_1 + V_2)T/15$

Option (c) is correct.

9. A salesman sold two pipes at Rs. 12 each. His profit on one was 20% and the loss on the other was 20%. Then on the whole, he
   (a) Lost Re. 1
   (b) Gained Re. 1
   (c) Neither gained nor lost
   (d) Lost Rs. 2

Solution :

Let the cost price of the pipe on which he made profit = x.

⇨ x + 20x/100 = 12
⇨ 120x/100 = 12
⇨ x = 12*100/120
⇨ x = 10

Let the cost price of the pipe on which he lost = y.

⇨ y – 20y/100 = 12
⇨ 80y/100 = 12
⇨ y = 12*100/80
⇨ y = 15

Therefore, total cost price = 10 + 15 = 25

Total selling price = 2*12 = 24

So, he lost (25 – 24) = Re. 1

Option (a) is correct.

10. The value of $(256)^{0.16}(16)^{0.18}$ is
   (a) 4
   (b) 16
   (c) 64
   (d) 256.25

Solution :

$(256)^{0.16}(16)^{0.18} = 2^{8*0.16}*2^{4*0.18} = 2^{8*0.16 + 4*0.18} = 2^{4(2*0.16 + 0.18)} = 2^{4*0.5} = 2^2 = 4$

Option (a) is correct.

11. The digit in the unit place of the integer 1! + 2! + 3! + ….. + 99! Is
(a) 3
(b) 0
(c) 1
(d) 7

Solution :

Now, after 5! = 120 all the terms end with 0 i.e. unit place digit is 0.

So, the unit place digit of the given integer is the unit place digit of the integer 1! + 2! + 3! + 4! = 1 + 2 + 6 + 24 = 33 i.e. 3

Option (a) is correct.

12. July 3, 1977 was a SUNDAY. Then July 3, 1970 was a
(a) Wednesday
(b) Friday
(c) Sunday
(d) Tuesday

Solution :

In 1970 after July 3 there are = 28 + 31 + 30 + 31 + 30 + 31 = 181 days.

In 1971 there are 365 days.

In 1972 there are 366 days.

In 1973, 1974, 1975 there are 3*365 = 1095 days.

In 1976 there are 366 days.

In 1977 up to July 3 there are 31 + 28 + 31 + 30 + 31 + 30 + 3 = 184 days.

Therefore total number of days up to July 3, 1970 from July 3, 1977 is 181 + 365 + 366 + 1095 + 366 + 184 = 2557

2557 ≡ 2 (mod 7)

- ⇨ July 3, 1970 was a Friday. (Sunday – 2)

Option (b) is correct.

13. June 10, 1979 was a SUNDAY. Then May 10, 1972, was a
  - (a) Wednesday
  - (b) Thursday
  - (c) Tuesday
  - (d) Friday

Solution :

After May 10, 1972 there are 21 + 30 + 31 + 31 + 30 + 31 + 30 + 31 = 235 days.

In 1973, 1974, 1975 there are 3*365 = 1095 days.

In 1976 there are 366 days.

In 1977, 1978 there are 365*2 = 730 days.

In 1979 up to June 10, there are 31 + 28 + 31 + 30 + 31 + 10 = 161 days.

Therefore, total number of days from May 10, 1972 to June 10, 1979 is 235 + 1095 + 366 + 730 + 161 = 2587 days.

Now, 2587 ≡ 4 (mod 7)

- ⇨ May 10, 1972 was a Wednesday. (Sunday – 4)

Option (a) is correct.

14. A man started from home at 14:30 hours and drove to a village, arriving there when the village clock indicated 15:15 hours. After staying for 25 minutes (min), he drove back by a different route of length (5/4) times the first route at a rate twice as fast, reaching

home at 16:00 hours. As compared to the clock at home, the village clock is
(a) 10 min slow
(b) 5 min slow
(c) 5 min fast
(d) 20 min fast

Solution :

Let the distance from home to the village is x and he drove to the village by v speed.

Therefore, time taken to reach village is x/v.

Now, time taken to come back home = (5x/4)/2v = 5x/8v

Total time = x/v + 5x/8v + 25 = (13/8)(x/v) + 25

- ⇨ (13/8)(x/v) + 25 = (16:00 – 14:30)*60 = 90
- ⇨ (x/v) = 65*8/13 = 40

So, he should reach village at 14:30 + 40 min = 15:10 hours.

Therefore, the village clock is (15:15 – 15:10) = 5 min fast.

Option (c) is correct.

15. If (a + b)/(b + c) = (c + d)/(d + a), then
(a) a = c
(b) either a = c or a + b + c + d = 0
(c) a + b + c + d = 0
(d) a = c and b = d

Solution :

(a + b)/(b + c) = (c + d)/(d + a)

- ⇨ (a + b)/(b + c) – 1 = (c + d)/(d + a) – 1
- ⇨ (a – c)/(b + c) = - (a – c)/(d + a)
- ⇨ (a – c)/(b + c) + (a – c)/(d + a) = 0
- ⇨ (a – c){1/(b + c) + 1/(d + a)} = 0
- ⇨ (a – c)(a + b + c + d)/{(b + c)(d + a)} = 0
- ⇨ (a – c)(a + b + c + d) = 0

⇨ Either a = c or a + b + c + d = 0

Option (b) is correct.

16. The expression $(1 + q)(1 + q^2)(1 + q^4)(1 + q^8)(1 + q^{16})(1 + q^{32})(1 + q^{64})$, $q \neq 1$, equals

(a) $(1 - q^{128})/(1 - q)$

(b) $(1 - q^{64})/(1 - q)$

(c) $\{1 - q\text{^}(2^{1 + 2 + \ldots. + 6})\}/(1 - q)$

(d) None of the foregoing expressions.

Solution :

Let $E = (1 + q)(1 + q^2)(1 + q^4)(1 + q^8)(1 + q^{16})(1 + q^{32})(1 + q^{64})$

⇨ $(1 - q)*E - (1 - q)(1 + q)(1 + q^2)(1 + q^4)(1 + q^8)(1 + q^{16})(1 + q^{32})(1 + q^{64})$ $(q \neq 1)$

⇨ $(1 - q)*E = (1 - q^2)(1 + q)(1 + q^2)(1 + q^4)(1 + q^8)(1 + q^{16})(1 + q^{32})(1 + q^{64})$

⇨ $(1 - q)*E = (1 - q^4)(1 + q^4)(1 + q^8)(1 + q^{16})(1 + q^{32})(1 + q^{64})$

⇨ $(1 - q)*E = (1 - q^8)(1 + q^8)(1 + q1^6)(1 + q^{32})(1 + q^{64})$

⇨ $(1 - q)*E = (1 - q^{16})(1 + q^{16})(1 + q^{32})1 + q^{64})$

⇨ $(1 - q)*E = (1 - q^{32})(1 + q^{32})(1 + q^{64})$

⇨ $(1 - q)*E = (1 - q^{64})(1 + q^{64})$

⇨ $(1 - q)*E = (1 - q^{128})$

⇨ $E = (1 - q^{128})/(1 - q)$ $(q \neq 1)$

Option (a) is correct.

17. In an election 10% of the voters on the voters' list did not cast their votes and 60 voters cast their ballot paper blank. There were only two candidates. The winner was supported by 47% of all voters in the list and he got 308 votes more than his rival. The number of voters on the list was

(a) 3600

(b) 6200

(c) 4575

(d) 6028

Solution :

Let number of voters on the voters' list is x.

Did not cast their vote = 10x/100

Therefore, total number of voters who cast their vote in favor of a candidate = x − 10x/100 − 60 = 90x/100 − 60

Winner got 47x/100 votes.

Rival got 90x/100 − 60 − 47x/100 = 43x/100 − 60

According to question, 47x/100 − (43x/100 − 60) = 308

⇨ 4x/100 + 60 = 308
⇨ x/25 = 308 − 60
⇨ x/25 = 248
⇨ x = 248*25
⇨ x = 6200

Option (b) is correct.

18. A student took five papers in an examination, where the full marks were the same for each paper. His marks in these papers were in the proportion of 6 : 7 : 8 : 9 : 10. He obtained (3/5) part of the total full marks. Then the number of papers in which he got more than 50% marks is
(a) 2
(b) 3
(c)4
(d) 5

Solution :

Let the full mark of each paper is x.

Total full marks = 5x

He got total 5x*(3/5) = 3x marks

In first paper he got 3x*6/(6 + 7 + 8 + 9 + 10) = 9x/20

In second paper he got 3x*7/(6 + 7 + 8 + 9 + 10) = 21x/40

In third paper he got 3x*8/(6 + 7 + 8 + 9 + 10) = 3x/5

In fourth paper he got 3x*9/(6 + 7 + 8 + 9 + 10) = 27x/40

In fifth paper he got 3x*10/(6 + 7 + 8 + 9 + 10) = 3x/4

In first paper percentage of marks = (9x/20)*100/x = 45%

In second paper percentage of marks = (21x/40)*100/x = 52.5%

⇨ He got more than 50% in 4 papers.

Option (c) is correct.

19. Two contestants run in a 3-kilometre race along a circular course of length 300 metres. If their speeds are in the ratio of 4 : 3, how often and where would the winner pass the other? (The initial start-off is not counted as passing.)
(a) 4 times, at the starting point
(b) Twice, at the starting point
(c) Twice, at a distance 225 metres from the starting point
(d) Twice, once at 75 metres and again at 225 metres from the starting point.

Solution :

Let they meet at x metres from the starting point.

Let first runner loops n times and second runner loops m times when they meet for the first time.

Therefore, (300n + x)/4 = (300m + x)/3

⇨ 900n + 3x = 1200m + 4x
⇨ x = 900n – 1200m
⇨ x = 300(3n – 4m)

Let x = 225, then 300(3n – 4m) = 225

⇨ 4(3n – 4m) = 3 which is impossible as 3n – 4m is an integer.

Let, x = 75, then 300(3n – 4m) = 75

⇨ 4(3n – 4m) = 1 which is impossible as 3n – 4m is an integer.

Therefore, none of the options (c), (d) are correct.

So, they will meet at starting point.

- ⇨ $x = 0$
- ⇨ $3n = 4m$

Minimum value of n is 4 and m is 3.

Total there are 3000/300 = 10 loops.

Therefore, they will meet for twice as 12th loop is not in course.

Option (b) is correct.

20. If a, b, c and d satisfy the equations $a + 7b + 3c + 5d = 0$, $8a + 4b + 6c + 2d = -16$, $2a + 6b + 4c + 8d = 16$, $5a + 3b + 7c + d = -16$, then $(a + d)(b + c)$ equals
    (a) 16
    (b) -16
    (c) 0
    (d) None of the foregoing numbers.

Solution :

$a + 7b + 3c + 5d = 0$ ………… (1)

$8a + 4b + 6c + 2d = -16$ ……. (2)

$2a + 6b + 4c + 8d = 16$ ……… (3)

$5a + 3b + 7c + d = -16$ ……….. (4)

Adding equations (2) and (3) we get, $10a + 10b + 10c + 10d = 0$

- ⇨ $a + b + c + d = 0$
- ⇨ $(a + d) = -(b + c)$ ……………. (5)

Adding equations (1) and (4) we get, $6a + 10b + 10c + 6d = -16$

- ⇨ $6(a + d) + 10(b + c) = -16$
- ⇨ $-6(b + c) + 10(b + c) = -16$ (from (5))
- ⇨ $4(b + c) = -16$
- ⇨ $(b + c) = -4$
- ⇨ $(a + d) = 4$
- ⇨ $(a + d)(b + c) = 4*(-4) = -16$

Option (b) is correct.

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21. Suppose x and y are positive integers, $x > y$, and $3x + 2y$ and $2x + 3y$ when divided by 5, leave remainders 2 and 3 respectively. It follows that when $x - y$ is divided by 5, the remainder necessarily equals
(a) 2
(b) 1
(c) 4
(d) None of the foregoing numbers.

Solution :

$3x + 2y \equiv 2 \pmod 5$

$2x + 3y \equiv 3 \pmod 5$

⇨ $(3x + 2y) - (2x + 3y) \equiv 2 - 3 \pmod 5$
⇨ $(x - y) \equiv -1 \pmod 5$
⇨ $(x - y) \equiv 4 \pmod 5$

Option (c) is correct.

22. The number of different solutions (x, y, z) of the equation $x + y + z = 10$, where each of x, y and z is a positive integer, is
(a) 36
(b) 121
(c) $10^3 - 10$
(d) ${}^{10}C_3 - {}^{10}C_2$.

Solution :

Clearly it is ${}^{10-1}C_{3-1} = {}^{9}C_2 = 36$ (For reference please see Number Theory book)

Option (a) is correct.

23. The hands of a clock are observed continuously from 12:45 p.m. onwards. They will be observed to point in the same direction some time between
(a) 1:03 p.m. and 1:04 p.m.

(b) 1:04 p.m. and 1:05 p.m.
(c) 1:05 p.m. and 1:06 p.m.
(d) 1:06 p.m. and 1:07 p.m.

Solution :

Clearly, option (a) and (b) cannot be true.

The hour hand moves $2\pi/12$ angle in 60 minutes

The hour hand moves $2\pi/60$ angle in $60*(2\pi/60)/(2\pi/12) = 12$ minutes.

So, option (d) cannot be true as it takes 12 minutes to move to 1:06 for hour hand.

Option (c) is correct.

24. A, B and C are three commodities. A packet containing 5 pieces of A, 3 of B and 7 of C costs Rs. 24. 50. A packet containing 2, 1 and 3 of A, B and C respectively costs Rs. 17.00. The cost of packet containing 16, 9 and 23 items of A, B and C respectively
(a) is Rs. 55.00
(b) is Rs. 75.50
(c) is Rs. 100.00
(d) cannot be determined from the given information.

Solution :

Clearly, 2*first packet + 3* second packet gives the answer.

Therefore, required cost = 2*24.50 + 3*17 = Rs. 100

Option (c) is correct.

25. Four statements are given below regarding elements and subsets of the set {1, 2, {1, 2, 3}}. Only one of them is correct. Which one is it?
(a) {1, 2} ε {1, 2, {1, 2, 3}}
(b) {1, 2} is proper subset of {1, 2, {1, 2, 3}}
(c) {1, 2, 3} is proper subset of {1. 2, {1, 2, 3}}
(d) 3 ε {1, 2, {1, 2, 3}}

Solution :

{1, 2} is not an element of {1, 2, {1, 2, 3}}. So option (a) cannot be true.

3 is not an element of {1, 2, {1, 2, 3}}. So option (d) cannot be true. (Elements are 1, 2, {1, 2, 3})

{1, 2, 3} is an element of {1, 2, {1, 2, 3}} not a subset, rather {{1, 2, 3}} is a subset containing the element {1, 2, 3}. So, (c) cannot be true.

Option (b) is correct.

26. A collection of non-empty subsets of the set {1, 2, ….., n} is called a *simplex* if, whenever a subset S is included in the collection, any non-empty subset T of S is also included in the collection. Only one of the following collections of subsets of {1, 2, …., n} is a simplex. Which one is it?

(a) The collection of all subsets S with the property that 1 belongs to S.
(b) The collection of all subsets having exactly 4 elements.
(c) The collection of all non-empty subsets which do not contain any even number.
(d) The collection of all non-empty subsets except for the subset {1}.

Solution :

Option (a) cannot be true as {2} is not included in the collection of subsets which is subset of the subset {1, 2}.

Option (b) cannot be true as 3 elements subset are not included in the collection of subsets.

Option (d) cannot be true as {1} is not included which is subset of the subset {1, 2}.

Option (c) is correct.

27. S is the set whose elements are zero and all even integers, positive and negative. Consider the five operations : [1] addition; [2] subtraction; [3] multiplication; [4] division; and [5] finding the

arithmetic mean. Which of these operations when applied to any pair of elements of S, yield only elements of S?
(a) [1], [2], [3], [4]
(b) [1], [2], [3], [5]
(c)[1], [3], [5]
(d) [1], [2], [3]

Solution :

If two even integers are added then an even integer is generated. So [1] is true.

If two even integers are subtracted then an even integer is generated. So [2] is true.

If two even integers are multiplied then an even integer is generated. So [3] is true.

If 6/4 then the generated integer doesn't belong to S. So, [4] cannot be true.

(6 + 4)/2 is an odd integer and doesn't belong to S. So, [5] cannot be true.

Option (d) is correct.

Directions for items 28 to 36 :

For sets P, Q of numbers, define

PUQ : the set of all numbers which belong to at least one of P and Q;

P∩Q : the set of all numbers which belong to both P and Q;

P – Q : the set of all numbers which belong to P but not to Q;

PΔQ = (P – Q)U(Q – P) : the set of all numbers which belong to set P alone or set Q alone, but not to both at the same time. For example, if P = {1, 2, 3}, Q = {2, 3, 4} then PUQ = {1, 2, 3, 4}, P∩Q = {2, 3}, P – Q = {1}, PΔQ = {1, 4}.

28. If X = {1, 2, 3, 4}, Y = {2, 3, 5, 7}, Z = {3, 6, 8, 9}, W = {2, 4, 8, 10}, then (XΔY)Δ(ZΔW) is
(a) {4, 8}
(b) {1, 5, 6, 10}
(c){1, 2, 3, 5, 6, 7, 9, 10}

(d) None of the foregoing sets.

Solution :

$(X\Delta Y) = \{1, 4, 5, 7\}$

$(Z\Delta W) = \{2, 3, 4, 6, 9, 10\}$

$(X\Delta Y)\Delta(Z\Delta W) = \{1, 2, 3, 5, 6, 7, 9, 10\}$

Option (c) is correct.

29. If X, Y, Z are any three sets of numbers, then the set of all numbers which belong to exactly two of the sets X, Y, Z is
(a) $(X\cap Y)\cup(Y\cap Z)\cup(Z\cap X)$
(b) $[(X\cup Y)\cup Z] - [(X\Delta Y)\Delta Z]$
(c) $(X\Delta Y)\cup(Y\Delta Z)\cup(Z\Delta X)$
(d) Not necessarily any of (a) to (c).

Solution :

Option (b) is correct. It can be easily verified by Venn diagram.

30. For any three sets P, Q and R s is an element of $(P\Delta Q)\Delta R$ if s is in
(a) Exactly one of P, Q and R
(b) At least one of P, Q and R, but not in all three of them at the same time
(c) Exactly one of P, Q and R
(d) Exactly one P, Q and R or all the three of them.

Solution :

Option (d) is correct. It can be easily verified by Venn diagram.

31. Let $X = \{1, 2, 3, \ldots, 10\}$ and $P = \{1, 2, 3, 4, 5\}$. The number of subsets Q of X such that $P\Delta Q = \{3\}$ is

(a) $2^4 - 1$
(b) 2^4
(c) 2^5
(d) 1.

Solution :

The only subset Q = {1, 2, 4, 5} then PΔQ = {3}

Option (d) is correct.

32. For each positive integer n, consider the set $P_n = \{1, 2, 3, \ldots, n\}$. Let $Q_1 = P_1$, $Q_2 = P_2 \Delta Q_1 = \{2\}$, and in general $Q_{n+1} = P_{n+1} \Delta Q_n$, for n ≥ 1. Then the number of elements in Q_{2k} is
(a) 1
(b) 2k – 2
(c) 2k – 3
(d) k

Solution :

$Q_3 = \{1, 3\}$

$Q_4 = \{2, 4\}$

$Q_5 = \{1, 3, 5\}$

$Q_6 = \{2, 4, 6\}$

$Q_7 = \{1, 3, 5, 7\}$

$Q_8 = \{2, 4, 6, 8\}$

Clearly, option (d) is correct.

33. For any two sets S and T, SΔT is defined as the set of all elements that belong to either S or T but not both, that is, SΔT = (S∪T) – (S∩T). Let A, B and C be sets such that A∩B∩C = Φ, and the number of elements in each of AΔB, BΔC and CΔA equals 100. Then the number of elements in A∪B∪C equals
(a) 150

(b) 300
(c) 230
(d) 210

Solution :

$A \cup B \cup C = A \cup B + C - (A \cup B) \cap C$

$= A + B - A \cap B + C - [A + B - (A \cap B)] \cap C$

$= A + B + C - A \cap B - A \cap C - B \cap C + A \cap B \cap C$

$= A + B + C - A \cap B - B \cap C - C \cap A$

$= (1/2)[2A + 2B + 2C - 2A \cap B - 2B \cap C - 2C \cap A]$

$= (1/2)[(A + B - A \cap B - A \cap B) + (B + C - B \cap C - B \cap C) + (C + A - C \cap A - C \cap A)]$

$= (1/2)[(A \cup B - A \cap B) + (B \cup C - B \cap C) + (C \cup A - C \cap A)]$

$= (1/2)[A \Delta B + B \Delta C + C \Delta A]$

$= (1/2)(100 + 100 + 100)$

$= 150$

Option (a) is correct.

34. Let A, B, C and D be finite sets such that $|A| < |C|$ and $|B| = |D|$, where $|A|$ stands for the number of elements in the set A. Then
(a) $|A \cup B| < |C \cup D|$
(b) $|A \cup B| \le |C \cup D|$ but $|A \cup B| < |C \cup D|$ need not always be true
(c) $|A \cup B| < 2|C \cup D|$ but $|A \cup B| \le |C \cup D|$ need not always be true
(d) None of the foregoing statements is true.

Solution :

It is given option (c) is correct but I think option (d) is correct.

Let us take an example.

A is inside B and C and D are disjoint.

Then $|A \cup B| = |B|$ and $|C \cup D| = |C| + |D| = |C| + |B|$

⇨ $|C \cup D| > |A \cup B|$

35. The subsets A and B of a set X, define A*B as
$A*B = (A \cap B) \cup ((X - A) \cap (X - B))$.
Then only one of the following statements is true. Which one is it?
(a) $A*(X - B)$ is a subset of $A*B$ and $A*(X - A) \neq A*B$
(b) $A*B = A*(X - B)$
(c) $A*B$ is a subset of $A*(X - B)$ and $A*B \neq A*(X - B)$
(d) $X - (A*B) = A*(X - B)$

Solution :

Option (d) is correct. It can be easily verified by Venn diagram.

36. Suppose that A, B and C are sets satisfying $(A - B)\Delta(B - C) = A\Delta B$. Which of the following statements must be true?
(a) $A = C$
(b) $A \cap B = B \cap C$
(c) $A \cup B = B \cup C$
(d) None of the foregoing statements necessarily follows.

Solution :

Option (b) is correct. This can be easily verified by Venn diagram.

Directions for items 37 to 39 :

A word is a finite string of the two symbols α and β. (An empty string, that is, a string containing no symbols at all, is also considered a word.) Any collection of words is called a language. If P and Q are words, then P.Q is meant the word formed by first writing the string of symbols in P and then following it by that of Q. For example P = αβαα and Q = ββ are words and P.Q = αβααββ. For the languages L_1 and L_2, $L_1.L_2$ denotes the language consisting of all words of the form P.Q with the word P coming from L_1 and Q coming from L_2. We also use abbreviations like α^3 for the word ααα, $\alpha\beta^3\alpha^2$ for αβββαα, $(\alpha^2\beta\alpha)^2$ for $\alpha^2\beta\alpha\alpha^2\beta\alpha$ (= $\alpha^2\beta\alpha^3\beta\alpha$) and α^0 or β^0 for the empty word.

37. If $L_1 = \{\alpha^n : n = 0, 1, 2, \ldots\}$ and $L_2 = \{\beta^n : n = 0, 1, 2, \ldots\}$, then $L_1.L_2$ is
(a) $L_1 \cup L_2$
(b) The language consisting of all words
(c) $\{\alpha^n\beta^m : n = 0, 1, 2, \ldots .\ m = 0, 1, 2, \ldots\}$
(d) $\{\alpha^n\beta^n : n = 0, 1, 2, \ldots\}$

Solution :

Clearly option (c) is correct.

38. Suppose L is a language which contains the empty word and has the property that whenever P is in L, the word α.P.β is also in L. The smallest such L is
(a) $\{\alpha^n\beta^m : n = 0, 1, 2, \ldots .\ m = 0, 1, 2, \ldots\}$
(b) $\{\alpha^n\beta^n : n = 0, 1, 2, \ldots\}$
(c) $\{(\alpha\beta)^n : n = 0, 1, 2, \ldots\}$
(d) The language consisting of all possible words.

Solution :

Option (a) is also true but (b) is smallest.

Therefore, option (b) is correct.

39. Suppose L is a language which contains the empty word, the word α and the word β, and has the property that whenever P and Q are in L, the word P.Q is also in L. The smallest such L is
(a) The language consisting of all possible words.
(b) $\{\alpha^n\beta^n : n = 0, 1, 2, \ldots\}$
(c) The language containing precisely the words of the form $(\alpha^{\wedge}n_1)(\beta^{\wedge}n_1)(\alpha^{\wedge}n_2)(\beta^{\wedge}n_2)\ldots(\alpha^{\wedge}n_k)(\beta^{\wedge}n_k)$
Where k is any positive integer and $n_1, n_2, \ldots, n_k$ are nonnegative integers
(d) None of the foregoing languages.

Solution :

Option (b) and (c) cannot be true as α, β are the words of the language.

Option (a) is correct.

40. A relation denoted by <- is defined as follows : For real numbers x, y, z and w, say that "(x, y) <- (z, w)" is either (i) x < z or (ii) x = z and y > w. If (x, y) <- (z, w) and (z, w) <- (r, s) then which one of the following is always true?
(a) (y, x) <- (r, s)
(b) (y, x) <- (s, r)
(c) (x, y) <- (s, r)
(d) (x, y) <- (r, s)

Solution :

(x, y) <- (z, w)

⇨ $x \le z$ and $y > w$

(z, w) <- (r, s)

⇨ $z \le r$ and $w > s$
⇨ $x \le r$ and $y > s$
⇨ (x, y) <- (r, s)

Option (d) is correct.

41. A subset W of all real numbers is called a *ring* if the following two conditions are satisfied :
(i) 1 ε W and
(ii) If a, b ε W then a – b ε W and ab ε W.

Let S = $\{m/2^n$ | m and n are integers$\}$ and T = {p/q | p and q are integers and q is odd}

Then

(a) Neither S nor T is a ring
(b) S is a ring and T is not
(c) T is a ring and S is not
(d) Both S and T are rings.

Solution :

If m = 2 and n = 1 then 1 ε S.

Now, $e/2^x - f/2^y = (e*2^y - f*2^x)/2^{x+y}$ ε S (if $e*2^y - f*2^x$ is negative then also it is ok as m is integer, so positive and negative both)

Now, $(e/2^x)*(f/2^y) = ef/2^{x+y}$ ε S

So, S is a ring.

If p = q then 1 ε S.

Now, $p_1/q_1 - p_2/q_2 = (p_1q_2 - p_2q_1)/q_1q_2$ ε T as q_1q_2 = odd because q_1 and q_2 both odd.

Now, $(p_1/q_1)*(p_2/q_2) = p_1p_2/(q_1q_2)$ ε T

So, T is a ring.

Option (d) is correct.

42. For a real number a, define $a^+ = \max\{a, 0\}$. For example, $2^+ = 2$, $(-3)^+ = 0$. Then, for two real numbers a and b, the equality $(ab)^+ = (a^+)(b^+)$ holds if and only if
(a) Both a and b are positive.
(b) a and b have the same sign
(c) a = b = 0
(d) at least one of a and b is greater than or equal to 0.

Solution :

Clearly, if a and b are both negative then $(ab)^+ = ab$ and $a^+ = 0$, $b^+ = 0$ and the equality doesn't hold.

If one of a and b are negative then $(ab)^+ = 0$ and either of a^+ or $b^+ = 0$ (whichever is negative) and the equality holds.

If both are positive then the equality holds.

So, option (d) is correct.

43. For any real number x, let [x] denote the largest integer less than or equal to x and <x> = x – [x], that is, the fractional part of x. For arbitrary real numbers x, y and z, only one of the following statements is correct. Which one is it?
(a) [x + y + z] = [x] + [y] + [z]
(b) [x + y + z] = [x + y] + [z] = [x] + [y + z] = [x + z] + [y]
(c) <x + y + z> = y + z – [y + z] + <x>
(d) [x + y + z] = [x + y] + [z + <y + x>].

Solution :

Let x = 2.4, y = 2.5 and z = 2.6

Then [x + y + z] = 7 and [x] + [y] + [z] = 6. So option (a) is not true.

Let, [x + y] = 4 and [z] = 2. So option (b) cannot be true.

<x + y + z> = 0.5 but <y + z> + <x> = 0.9 + 0.6 = 1.5. So, option (c) cannot be true.

Option (d) is correct.

44. Suppose that $x_1, x_2, \ldots, x_n$ ($n > 2$) are real numbers such that $x_i = x_{n-i+1}$ for $1 \le i \le n$. Consider the sum $S = \sum\sum\sum x_i x_j x_k$, where summations are taken over all i, j, k : $1 \le i, j, k \le n$ and i, j, k are all distinct. Then S equals
(a) $n!x_1x_2\ldots.x_n$
(b) $(n - 3)(n - 4)$
(c) $(n - 3)(n - 4)(n - 5)$
(d) None of the foregoing expressions.

Solution :

$S = \sum\sum x_i x_j(P - x_i - x_j)$ $i \neq j$ and $P = x_1 + x_2 + \ldots. + x_n$

$= P\sum\sum x_i x_j - \sum\sum x_i^2 x_j - \sum\sum x_i x_j^2$

$= P\sum x_i(P - x_i) - \sum x_i^2(P - x_i) - \sum x_i(Q - x_i^2)$ where $Q = x_1^2 + x_2^2 + \ldots. + x_n^2$

$= P^3 - 3PQ + 2\sum x_i^3$

If n is even then $P = \sum x_i^3 = 0$

Therefore, S = 0

If n is odd, then P = $x_{(n+1)/2}$ and $\sum x_i^3 = (x_{(n+1)/2})^3$

Therefore, $S = (x_{(n+1)/2})^3 - 3x_{(n+1)/2}Q + 2(x_{(n+1)/2})^3 = 3x_{(n+1)/2}\{(x_{(n+1)/2})^2 - Q\}$

Clearly it doesn't match with any of the expressions in (a), (b), (c)

Option (d) is correct.

45. By an *upper bound* for a set A of real numbers, we mean any real number x such that every number a in A is smaller than or equal to x. If x is an upper bound for a set A and no number is strictly smaller than x is an upper bound for a, then x is called sup A.
Let A and B be two sets of real numbers with x = sup A and y = sup B. Let C be the set of all real numbers of the form a + b where a is in A and b is in B. If z = sup C, then
(a) $z > x + y$
(b) $z < x + y$
(c) $z = x + y$
(d) nothing can be said in general about the relation between x, y and z.

Solution :

x = sup A means all the elements of a are equal and equal to x i.e. if a is in x then a = x

Similarly, if b is in B then b = y

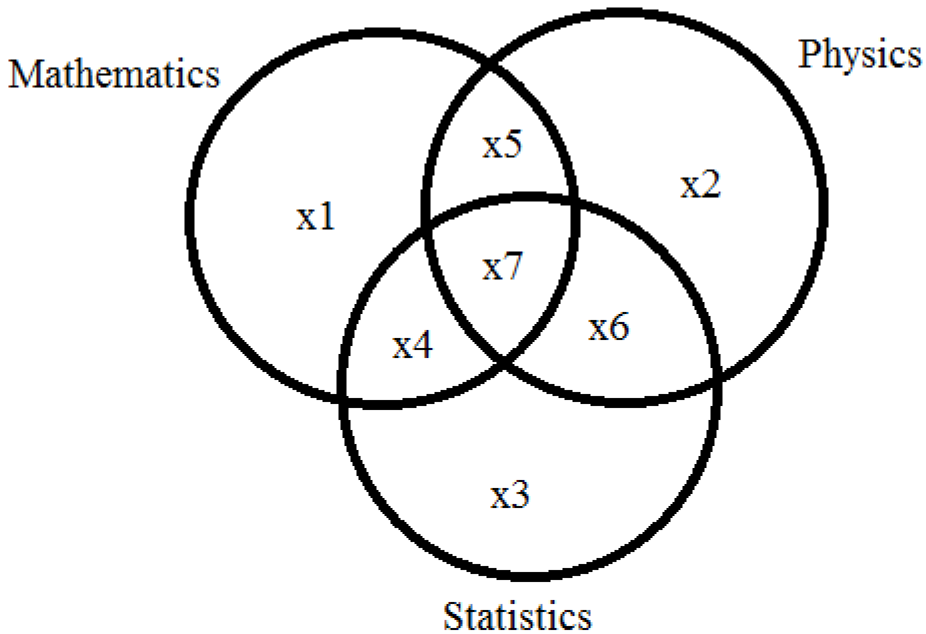
Similarly, z = a + b = x + y

Option (c) is correct.

46. There are 100 students in a class. In an examination, 50 students of them failed in Mathematics, 45 failed in Physics and 40 failed in Statistics, and 32 failed in exactly two of these three subjects. Only one student passed in all the three subjects. The number of students failing all the three subjects
(a) is 12
(b) is 4
(c) is 2

(d) cannot be determined from the given information.

Solution :



Now, $x_1 + x_4 + x_5 + x_7 = 50$ ………… (1)

$x_2 + x_5 + x_6 + x_7 = 45$ ……………… (2)

$x_3 + x_4 + x_6 + x_7 = 40$ ……………….. (3)

$x_4 + x_5 + x_6 = 32$ ………………………. (4)

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + 1 = 100$

⇨ $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 99$ …………… (5)

Doing (5) – (4) we get, $x_1 + x_2 + x_3 = 67 - x_7$ ………… (6)

Adding (1), (2), (3) we get, $(x_1 + x_2 + x_3) + 2(x_4 + x_5 + x_6) + 3x_7 = 135$

⇨ $67 - x_7 + 2*32 + 3x_7 = 135$ (from (4) and (6))
⇨ $2x_7 = 4$
⇨ $x_7 = 2$

Option (c) is correct.

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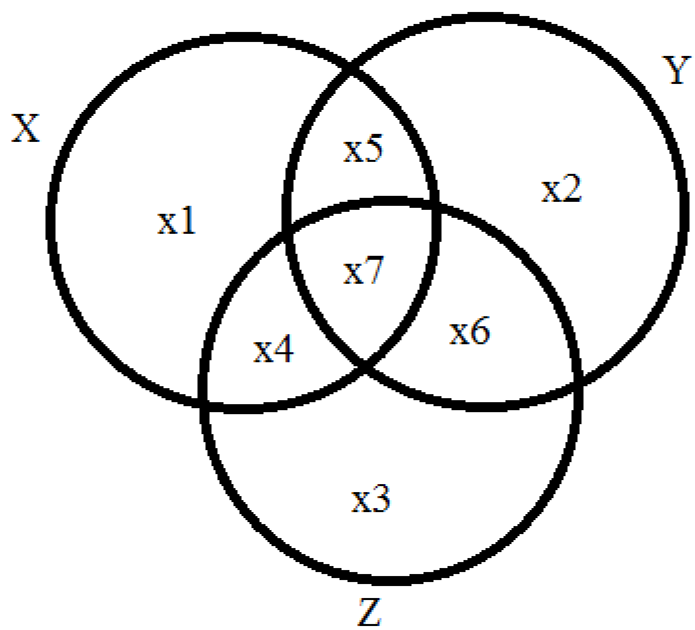
47. A television station telecasts three types of programs X, Y and Z. A survey gives following data on television viewing. Among the people interviewed 60% watch program X, 50% watch program Y, 50% watch program Z, 30% watch programs X and Y, 20% watch programs Y and Z, 30% watch programs X and Z while 10% do not watch any television program. The percentage of people watching all the three programs X, Y and Z is

(a) 90
(b) 50
(c) 10
(d) 20

Solution :



Now, $x_1 + x_4 + x_5 + x_7 = 60\%$ …………. (1)

$x_2 + x_5 + x_6 + x_7 = 50\%$ …………………. (2)

$x_3 + x_4 + x_6 + x_7 = 50\%$ …………………….. (3)

$x_5 + x_7 = 30\%$ ……………………… (4)

$x_6 + x_7 = 20\%$ ………………… (5)

$x_4 + x_7 = 30\%$ ……………. (6)

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + 10\% = 100\%$

⇨ $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 90\%$ ………….. (7)

Adding (4), (5), (6) we get, $x_4 + x_5 + x_6 = 80\% - 3x_7$ …………. (8)

Adding (1), (2), (3) we get, $(x_1 + x_2 + x_3) + 2(x_4 + x_5 + x_6) + 3x_7 = 160\%$

⇨ $(x_1 + x_2 + x_3) = 160\% - 2(80\% - 3x_7) – 3x_7$
⇨ $(x_1 + x_2 + x_3) = 3x_7$ ……………. (9)

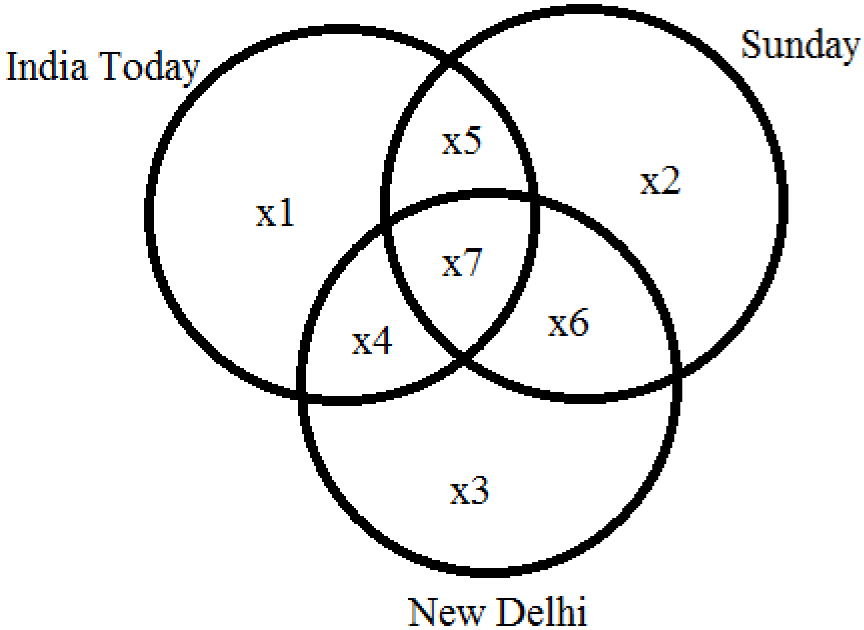
Putting value of (8) and (9) in (7) we get, $3x_7 + 80\% - 3x_7 + x_7 = 90\%$

⇨ $x_7 = 10\%$

Option (c) is correct.

48. In a survey of 100 families, the number of families that read the most recent issues of various magazines was found to be : *India Today* 42, *Sunday* 30, *New Delhi* 28, *India Today* and *Sunday* 10, *India Today* and *New Delhi* 5, *Sunday* and *New Delhi* 8, all three magazines 3. Then the number of families that read none of the three magazines is
(a) 30
(b) 26
(c)23
(d) 20

Solution :



Let, a number of families read none of the three magazines.

Now, $x_1 + x_4 + x_5 + x_7 = 42$ ……….. (1)

$x_2 + x_5 + x_6 + x_7 = 30$ …………. (2)

$x_3 + x_4 + x_6 + x_7 = 28$ ………… (3)

$x_5 + x_7 = 10$ ……….. (4)

$x_4 + x_7 = 5$ ……….. (5)

$x_6 + x_7 = 8$ …………. (6)

$x_7 = 3$

And, $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + a = 100$ …….. (7)

Adding (4), (5), (6) we get, $x_4 + x_5 + x_6 + 3x_7 = 23$

⇨ $(x_4 + x_5 + x_6) = 14$ ($x_7 = 3$) …………. (8)

Adding (1), (2), (3) we get, $(x_1 + x_2 + x_3) + 2(x_4 + x_5 + x_6) + 3x_7 = 100$

⇨ $(x_1 + x_2 + x_3) = 63$ (from (8) and $x_7 = 3$) ……………. (9)

Putting the value from (8), (9) and $x_7 = 3$ in (7) we get, $63 + 14 + 3 + a = 100$

⇨ a = 20

Option (d) is correct.

49. In a survey of 100 families, the number of families that read the most recent issues of various magazines was found to be : *India Today* 42, *Sunday* 30, *New Delhi* 28, *India Today* and *Sunday* 10, *India Today* and *New Delhi* 5, *Sunday* and *New Delhi* 8, all three magazines 3. Then the number of families that read either both or none of the two magazines *Sunday* and *India Today* is

(a) 48
(b) 38
(c) 72
(d) 58

Solution :

From the previous problem's figure we need to find $x_3 + x_5 + x_7 + a$.

From equation (4) $x_5 = 7$, from equation (5) $x_4 = 2$, from equation (6) $x_6 = 5$.

From equation (3), $x_3 = 18$

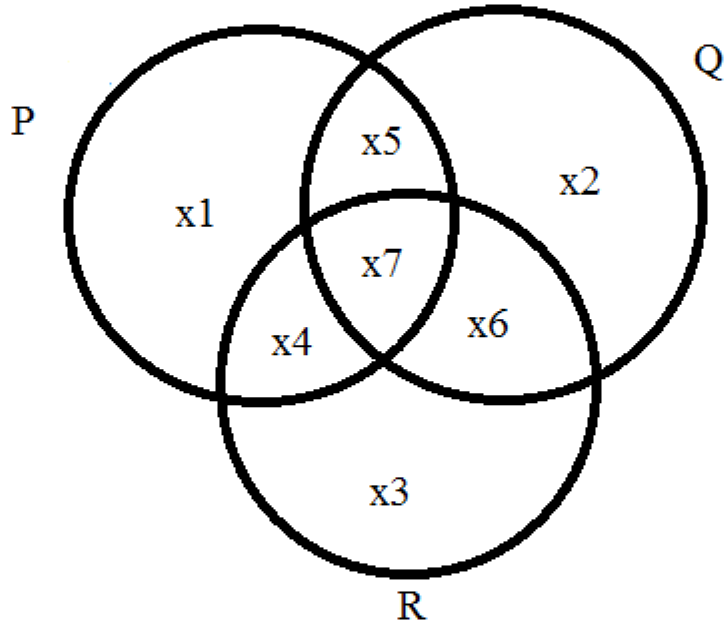
So, $x_3 + x_5 + x_7 + a = 18 + 7 + 3 + 20 = 48$.

Option (a) is correct.

50. In a village of 1000 inhabitants, there are three newspapers P, Q and R in circulation. Each of these papers is read by 500 persons. Papers P and Q are read by 250 persons, papers Q and R are read by 250 persons, papers R and P are read by 250 persons. All the three papers are read by 250 persons. Then the number of persons who read no newspaper at all

(a) is 500
(b) is 250
(c) is 0
(d) cannot be determined from the given information.

Solution :



Let, a number of people read no newspaper at all.

Now, $x_1 + x_4 + x_5 + x_7 = 500$

$x_2 + x_5 + x_6 + x_7 = 500$

$x_3 + x_4 + x_6 + x_7 = 500$

$x_5 + x_7 = 250$

$x_4 + x_7 = 250$

$x_6 + x_7 = 250$

$x_7 = 250$

And, $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + a = 1000$

- ⇨ $x_4 = x_5 = x_6 = 0$
- ⇨ $x_1 = 250, x_2 = 250, x_3 = 250$
- ⇨ $a = 1000 - 250 - 250 - 250 - 0 - 0 - 0 - 250 = 0$

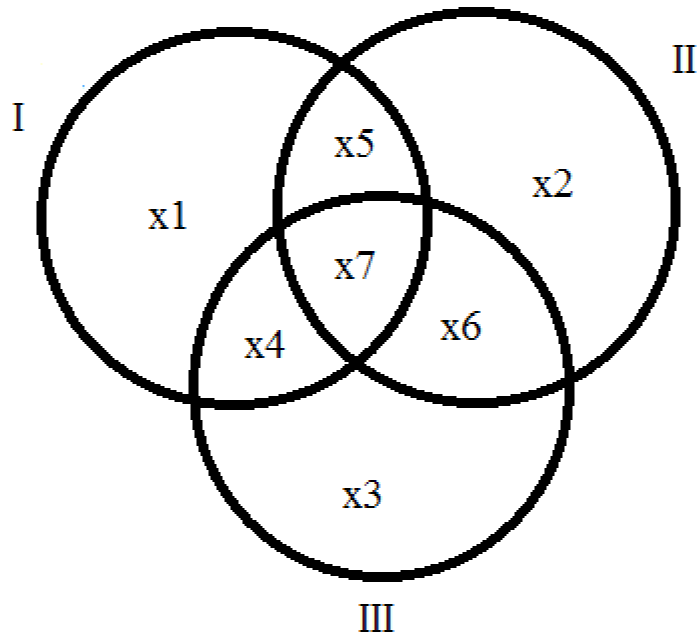
Option (c) is correct.

51. Sixty (60) students appeared in a test consisting of three papers I, II and III. Of these students, 25 passed in Paper I, 20 in Paper II and 8 in Paper III. Further, 42 students passed in at least one of

Papers I and II, 30 in at least one of Papers I and III, 25 in at least one of Papers II and III. Only one student passed in all the three papers. Then the number of students who failed in all the papers is

(a) 15
(b) 17
(c) 45
(d) 33

Solution :



Let, the number of students who failed in all the three papers is a.

Now, $x_1 + x_4 + x_5 + x_7 = 25$ ……….. (1)

$x_2 + x_5 + x_6 + x_7 = 20$ ……………. (2)

$x_3 + x_4 + x_6 + x_7 = 8$ ………… (3)

$x_1 + x_2 + x_4 + x_5 + x_6 + x_7 = 42$ ……….. (4)

$x_1 + x_3 + x_4 + x_5 + x_6 + x_7 = 30$ ……… (5)

$x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 25$ …………. (6)

$x_7 = 1$

And, $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + a = 60$ …………… (7)

(1) + (2) – (4) gives, $x_5 + x_7 = 3 \Rightarrow x_5 = 2$
(2) + (3) – (6) gives, $x_6 + x_7 = 3 \Rightarrow x_6 = 2$
(3) + (1) – (5) gives, $x_4 + x_7 = 3 \Rightarrow x_4 = 2$

From (1), $x_1 - x_7 + (x_4 + x_7) + (x_5 + x_7) = 25$

⇨ $x_1 - x_7 = 19 \Rightarrow x_1 = 20$

Similarly, from (2), $x_2 - x_7 = 14 \Rightarrow x_2 = 15$

And from (3), $x_3 - x_7 = 2 \Rightarrow x_3 = 3$

Putting all the values in (7) we get, a = 60 – 20 – 15 – 3 – 2 – 2 – 2 – 1 = 15

Option (a) is correct.

52. A student studying the weather for d days observed that (i) it rained on 7 days, morning or afternoon; (ii) when it rained in the afternoon, it was clear in the morning; (iii) there were five clear afternoons; and (iv) there were six clear mornings. Then d equals
(a) 7
(b) 11
(c) 10
(d) 9

Solution :

There were d – 5 days in which it rained in afternoons.

There were d – 6 days in which it rained in mornings.

No day it rained in morning and afternoon.

Therefore, d – 5 + d – 6 = 11

⇨ d = 9

Option (d) is correct.

53. A club with x members is organized into four committees according to the following rules :
(i) Each member belongs to exactly two committees.

(ii) Each pair of committees has exactly one member in common.

Then

(a) $x = 4$
(b) $x = 6$
(c) $x = 8$
(d) x cannot be determined from the given information.

Solution :

Let the groups be I, II, III, IV.

So, we need to find number of pair-wise combinations of the group.

I and II, I and III, I and IV, II and III, II and IV, III and IV

There are 6 pairs.

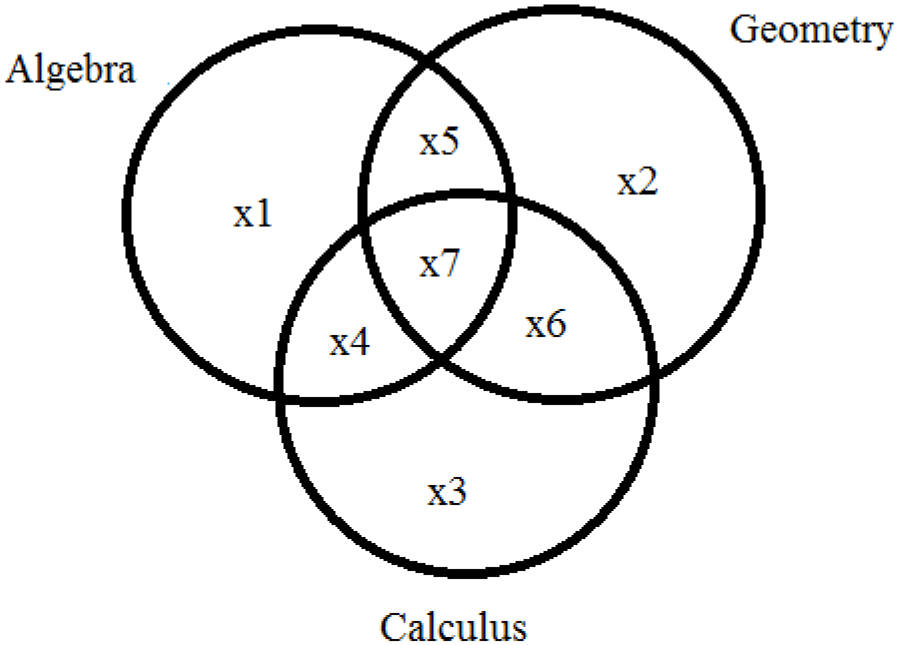
Therefore, $x = 6$

Option (b) is correct.

54. There were 41 candidates in an examination and each candidate was examined in Algebra, Geometry and Calculus. It was found that 12 candidates failed in Algebra, 7 failed in Geometry and 8 failed in Calculus, 2 in Geometry and Calculus, 3 in Calculus and Algebra, 6 in Algebra and Geometry, whereas only 1 failed in all three subjects. Then number of candidates who passed in all three subjects

(a) is 24
(b) is 2
(c) is 14
(d) cannot be determined from the given information.

Solution :



Let, number of candidates who passed in all three subjects is a.

Now, $x_1 + x_4 + x_5 + x_7 = 12$

$x_2 + x_5 + x_6 + x_7 = 7$

$x_3 + x_4 + x_6 + x_7 = 8$

$x_6 + x_7 = 2$

$x_4 + x_7 = 3$

$x_5 + x_7 = 6$

$x_7 = 1$

⇨ $x_6 = 1, x_4 = 2, x_5 = 5$ and $x_1 = 4, x_2 = 0, x_3 = 4$

And, $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + a = 41$

⇨ $a = 41 - 4 - 0 - 4 - 2 - 5 - 1 - 1 = 24$

Option (a) is correct.

55. In a group of 120 persons there are 70 Bengalis, 35 Gujaratis. Further, 70 persons in the group and Muslims and the remaining Hindus. Then the number of Bengali Muslims in the group is
(a) 30 or more
(b) Exactly 20
(c) Between 15 and 25
(d) Between 20 and 25

Solution :

Let all the Gujaratis are Muslims.

Therefore, 30 Bengali Muslims are there.

Option (a) is correct.

56. In a group of 120 persons there are 70 Bengalis, 35 Gujaratis and 15 Maharashtrians. Further, 75 persons in the group are Muslims and remaining are Hindus. Then the number of Bengali Muslims in the group is
(a) Between 10 and 14
(b) Between 15 and 19
(c) Exactly 20
(d) 25 or more.

Solution :

Let, all the Gujaratis and Maharashtrians are Muslims. Then there are 25 Bengali Muslims.

Option (d) is correct.

57. Four passengers in a compartment of the Delhi-Howrah Rajdhani Express discover that they form an interesting group. Two are lawyers and two are doctors. Two of them speak Bengali and the other two Hindi and no two of the same profession speak the same language. They also discover that two of them are Christians and two Muslims, no two of the same religion are of the same profession and no two of the same religion speak the same language. The Hindi speaking doctor

is a Christian. Then only one of the statements below logically follows. Which one is it?

(a) The Bengali-speaking lawyer is a Muslim.
(b) The Christian lawyer speaks Bengali.
(c) The Bengali-speaking doctor is a Christian.
(d) The Bengali-speaking doctor is a Hindu.

Solution :

Clearly, option (d) cannot be true because there is no one whose religion is Hindu.

Also clearly, the another Christian cannot be doctor. (As no two same religion have same profession)

- ⇨ Option (c) cannot be true.

Now, another Christian must speak Bengali and he is a lawyer.

Option (b) is correct.

58. In a football league, a particular team played 60 games in a season. The team never lost three games consecutively and never won five games consecutively in that season. If N is the number of games the team won in that season, then N satisfies

(a) $24 \leq N \leq 50$
(b) $20 \leq N \leq 48$
(c) $12 \leq N \leq 40$
(d) $18 \leq N \leq 42$

Solution :

Let, the team lost 2 games consecutively and won 1 game consecutively.

- ⇨ The team won 1 game in 3 games.
- ⇨ The team won $(1/3)*60 = 20$ games in 60 games.
- ⇨ $N \geq 20$

Let, the team won 4 games consecutively and lost 1 game consecutively.

- ⇨ The team won 4 games in 5 games.
- ⇨ The team won $(4/5)*60 = 48$ games in 60 games.
- ⇨ $N \leq 48$

⇨ $20 \le N \le 48$

Option (b) is correct.

59. A box contains 100 balls of different colors : 28 red, 17 blue, 21 green, 10 white, 12 yellow, 12 black. The smallest number n such that any n balls drawn from the box will contain at least 15 balls of the same color, is
(a) 73
(b) 77
(c) 81
(d) 85

Solution :

Let us take the worst possible scenario.

All the white, yellow and black balls are selected and blue, green and red balls are selected 14 each. If we take 1 more ball then it must be from red, blue of green making any one color at least 15.

Therefore, n = 10 + 12 + 12 + 14*3 + 1 = 77

Option (b) is correct.

60. Let x, y, z, w be positive real numbers, which satisfy the two conditions that
(i) If $x > y$ then $z > w$; and
(ii) If $x > z$ then $y < w$.

Then one of the statements given below is a valid conclusion. Which one is it?

(a) If $x < y$ then $z < w$
(b) If $x < z$ then $y > w$
(c) If $x > y + z$ then $z < y$
(d) If $x > y + z$ then $z > y$

Solution :

Option (a) and (b) cannot be true because there is no such statement that the vice versa will be true.

Option (c) cannot be true as if $x > y$ and $x > z$ then $x > y + z$ but $z > w > y$

So, option (d) is true.

61. Consider the statement : $x(\alpha - x) < y(\alpha - y)$ for all x, y with $0 < x < y < 1$. The statement is true
   (a) If and only if $\alpha \geq 2$
   (b) If and only if $\alpha > 2$
   (c) If and only if $\alpha < -1$
   (d) For no values of α.

Solution :

Now, $x(\alpha - x) < y(\alpha - y)$

- ⇨ $\alpha x - x^2 < \alpha y - y^2$
- ⇨ $y^2 - x^2 - \alpha y + \alpha x < 0$
- ⇨ $(y - x)(y + x) - \alpha(y - x) < 0$
- ⇨ $(y - x)(y + x - \alpha) < 0$

Now, $y - x > 0$

- ⇨ $y + x - \alpha < 0$
- ⇨ $\alpha > x + y$

Now, maximum value of $x + y$ is 2^-

Therefore, $\alpha \geq 2$.

Option (a) is correct. **(For $\alpha = 0.4$, $x = 0.1$, $y = 0.2$ the equation holds good)*

62. In a village, at least 50% of the people read a newspaper. Among those who read a newspaper at the most 25% read more than one paper. Only one of the following statements follows from the statements we have given. Which one is it?
   (a) At the most 25% read exactly one newspaper.
   (b) At least 25% real all the newspapers.
   (c) At the most 37.5% read exactly one newspaper.
   (d) At least 37.5% read exactly one newspaper.

Solution :

Let number of people in the village is x.

Let number of people who read newspaper is y.

Therefore, $y \geq 50x/100 = x/2$

Let, t number of people reads exactly one newspaper.

Therefore, $t \geq y - 25y/100 = 75y/100 \geq (75/100)*(x/2) = 37.5x/100 = 37.5\%$

Option (d) is correct.

63. We consider the relation "a person x shakes hand with a person y". Obviously, if x shakes hand with y, then y shakes hand with x. In a gathering of 99 persons, one of the following statements is *always* true, considering 0 to be an even number. Which one is it?
   (a) There is at least one person who shakes hand exactly with an odd number of persons.
   (b) There is at least one person who shakes hand exactly with an even number of persons.
   (c) There are even number of persons who shake hand exactly with an even number of persons.
   (d) None of the foregoing statements.

Solution :

Let there is one handshake with everybody. Then two people shakes hand will never shake hand with others. Therefore it makes pairs of people. 99 is an odd number. So, it is not possible.

Similarly, to do any odd number of handshakes between n number of people n must be even.

But 99 is odd.

Therefore, there will be always at least one people who will shake hand even number of times considering 0 as even number.

Option (b) is correct.

64. Let P, Q, R, S and T be statements such that if P is true then both Q and R are true, and if both R and S are true then T is false. We then have :
(a) If T is true then both P and R must be true.
(b) If T is true then both P and R must be false.
(c) If T is true then at least one of P and R must be true.
(d) If T is true then at least one of P and R must be false.

Solution :

If T is true then at least one of R and S is false.

If P is false then at least one of Q and R is false.

If R is false then P is false.

If R is true and Q is true then P can be true.

It is given option (d) as answer. *(But consider the case P, Q, R, T all true and S is false; no contradiction found)*

65. Let P, Q, R and S be four statements such that if P is true then Q is true, if Q is true then R is true and if S is true then at least one of Q and R is false. Then it follows that
(a) if S is false then both Q and R are true
(b) if at least one of Q and R is true then S is false
(c) if P is true then S is false
(d) if Q is true then S is true.

Solution :

Clearly, if S is false then Q and R both must be true.

Option (a) is correct.

Also, if P is true then Q and R both true. Implies S is false.

Option (c) is correct.

If Q is true then Q and R are both true then S cannot be true. So option (d) cannot be true.

Option (b) is clearly cannot be true.

So, (a), (c) are correct. But it is given answer (c) only.

66. If A, B, C and D are statements such that if at least one of A and B is true, then at least one of C and D must be true. Further, both A and C are false. Then
(a) if D is false then B is false
(b) both B and D are false
(c) both B and D are true
(d) if D is true then B is true.

Solution :

Clearly option (a) is correct as if D is false then C and D both are false.

⇨ A and B both are false. A is already false means B is also false.

Option (a) is correct.

67. P, Q and R are statements such that if P is true then at least one of the following is correct : (i) Q is true, (ii) R is not true. Then
(a) if both P and Q are true then R is true
(b) if both Q and R are true then P is true
(c) if both P and R are true then Q is true
(d) none of the foregoing statements is correct.

Solution :

If, P and R are true, then as P is true but R is also true so (ii) is not satisfied. Implies (i) must be satisfied. Implies Q is true.

Option (c) is correct.

68. It was a hot day and four couples drank together 44 bottles of cold drink. Anita had 2, Biva 3, Chanchala 4, and Dipti 5 bottles. Mr. Panikkar drank just as many bottles as his wife, but each of the other men drank more than his wife – Mr. Dube twice, Mr. Narayan thrice and Mr. Rao four times as many bottles. Then only one of the following is correct. Which one is it?
(a) Mrs. Panikkar is Chanchala.

(b) Anita's husband had 8 bottles.
(c) Mr. Narayan had 12 bottles.
(d) Mrs. Rao is Dipti.

Solution :

If Dipti is Mrs. Rao then Mr. Rao had 20 bottles.

So, Mr. Panikkar, Mr. Dube and Mr. Narayan had 44 – (20 + 2 + 3 + 4 + 5) = 10 bottles.

So, Mr. Narayan can have maximum 2*3 = 6 bottles. Mr. Panikkar can have maximum 3*2 = 6 bottles which crosses 10.

So, option (d) cannot be true.

If Mrs. Panikkar is Chanchala then Mr. Panikkar had 4 bottles.

So, Mr. Dube, Mr. Narayan and Mr. Rao had 44 – (2 + 3 + 4 + 5 + 4) = 26 bottles.

Dipti is not Mrs. Rao.

Therefore, Mr. Rao can have maximum 3*4 = 12 bottles.

Mr. Narayan can have maximum 5*3 = 15 bottles.

In that case answer doesn't match.

Mr. Narayan had 2*3 = 6 bottles.

And Mr. Dube had 5*2 = 10 bottles.

In that case 12 + 6 + 10 = 28 and not 26.

So, option (a) cannot be true.

If Anita's husband had 8 bottles then Mr. Rao is Anita's husband.

So, Mr. Panikkar, Mr. Dube and Mr. Narayan had 44 – (2 + 3 + 4 + 5 + 8) = 22 bottles.

Now, Chanchala is not Mrs. Panikkar.

So, Mr. Panikkar had either 3 bottles or 5 bottles.

Mr. Dube had 6 bottles, 8 bottles or 10 bottles.

Mr. Narayan had 9 bottles, 12 bottles or 15 bottles.

If Mr. Panikkar had 5 bottles (i.e. Dipti is Mrs. Panikkar), Mr. Dube had 8 bottles (i.e. Chanchala is Mrs. Dube) and Mr. Narayan had 9 bottles (i.e. Biva is Mrs. Narayan) then it is true.

Option (b) is correct.

69. Every integer of the form $(n^3 – n)(n – 2)$, (for n = 3, 4, ….) is
   (a) Divisible by 6 but not always divisible by 12
   (b) Divisible by 12 but not always divisible by 24
   (c) Divisible by 24 but not always divisible by 48
   (d) Divisible by 9.

Solution :

$(n^3 – n)(n – 2) = (n + 1)n(n – 1)(n – 2)$ = multiplication of consecutive four integers which is always divisible by 8 and 3

⇨ It is divisible by 24 as gcd(3, 8) = 1

Option (c) is correct.

70. The number of integers n > 1, such that n, n + 2, n + 4 are all prime numbers, is
   (a) Zero
   (b) One
   (c) Infinite
   (d) More than one, but finite

Solution :

Only one (3, 5, 7)

Option (b) is correct.

71. The number of *ordered* pairs of integers (x, y) satisfying the equation $x^2 + 6x + y^2 = 4$ is
   (a) 2
   (b) 4
   (c) 6

(d) 8

Solution :

If $x = 0$, then $y = \pm 2$

So, two pairs (0, 2); (0, -2)

x needs to be negative as if x is positive then y^2 = negative which is not possible.

Let $x = -1$, $y = \pm 3$

So, two more pairs (-1, 3); (-1, -3)

We have till now 4 pairs.

Let $x = -2$ then $y^2 = 12$ not giving integer solution.

Let $x = -3$ then $y^2 = 13$ not giving integer solution.

Let $x = -4$, then $y^2 = 12$ not giving integer solution.

Let $x = -5$, then $y = \pm 3$

We have two more pairs viz. (-5, 3); (-5, -3)

So, we have 6 pairs till now.

Let $x = -6$ then $y = \pm 2$.

So, we have 8 pairs.

Option (d) is correct.

72. The number of integer (positive, negative or zero) solutions of $xy - 6(x + y) = 0$ with $x \le y$ is
(a) 5
(b) 10
(c) 12
(d) 9

Solution :

$xy - 6(x + y) = 0$

⇨ $y = 6x/(x - 6)$

$x = 0, y = 0$

$x > 6$. (as $x \le y$)

$x = 7, y = 42$

$x = 8, y = 24$

$x = 9, y = 27$

$x = 10, y = 15$

$x = 11$ doesn't give integer solution.

$x = 12, y = 12$

$x = 13, y < x$

$x = -1$ doesn't give integer solution.

$x = -2$ doesn't give integer solution.

$x = -3, y = 2$

$x = -4$ doesn't give integer solution.

$x = -5$ doesn't give any integer solution.

$x = -6, y = 3$

$x = -7$ doesn't give any integer solution

$x = -8$ doesn't give any integer solution

$x = -9$ doesn't give any integer solution

$x = -10$, doesn't give any integer solution.

$x = -11$ doesn't give any integer solution.

$x = -12, y = 4$
No other x will give integer solution.

Option (d) is correct.

73. Let P denote the set of all positive integers and S = {(x, y) : x ε P, y ε P and $x^2 - y^2 = 666$}. The number of distinct elements in the set S is
(a) 0
(b) 1
(c) 2
(d) More than 2

Solution :

$666 = 2*3^2*37$

⇨ $(x + y)(x - y) = 2*3^2*37$

Now, x and y are both even or both odd.

Both cannot be even as 666 is divisible by 2 and not 4.

Again x and y both cannot be odd as x + y and x − y both will be even so 666 must have at least 8 as a factor.

Option (a) is correct.

74. If numbers of the form $3^{4n-2} + 2^{6n-3} + 1$, where n is a positive integer, are divided by 17, the set of all possible remainders is
(a) {1}
(b) {0, 1}
(c) {0, 1, 7}
(d) {1, 7}

Solution :

$3^{4n-2} + 2^{6n-3} + 1 = 9^{2n-1} + 8^{2n-1} + 1 \equiv (-8)^{2n-1} + 8^{2n-1} + 1 \pmod{17} \equiv -8^{2n-1} + 8^{2n-1} + 1 \pmod{7}$ (as 2n-1 is odd) $\equiv 1 \pmod{17}$

Option (a) is correct.

75. Consider the sequence : $a_1 = 101$, $a_2 = 10101$, $a_3 = 1010101$, and so on. Then a_k is a composite number (that is, not a prime number)
(a) if and only if k ≥ 2 and 11 divides $10^{k+1} + 1$

(b) if and only if $k \geq 2$ and 11 divides $10^{k+1} - 1$
(c) if and only if $k \geq 2$ and $k - 2$ is divisible by 3
(d) if and only if $k \geq 2$.

Solution :

$a_k = 101010\ldots k \text{ times}1 = 1*10^{2k} + 1*10^{2k-2} + \ldots + 1 = 1*[\{(10^2)^{k+1} - 1\}/(10^2 - 1)] = (10^{2k+2} - 1)/99$

Now, if k is odd then $10^{2k+2} = 10^{2(k+1)} = 100^{k+1} \equiv (-1)^{k+1} = 1 \pmod{101}$ (k +1 is even)

- ⇨ $10^{2k+2} - 1 \equiv 0 \pmod{101}$

If $k = 6m + 2$ form then $2k + 2 = 12m + 6 = 6(2m + 1)$

Now, $10^3 \equiv 1 \pmod{27}$

- ⇨ $10^{6(2m+1)} - 1 \equiv 0 \pmod{27}$
- ⇨ They are divisible by 3.

None of the (a), (b), (c) are true.

Option (d) is correct.

76. Let n be a positive integer. Now consider all numbers of the form $3^{2n+1} + 2^{2n+1}$. Only one of the following statements is true regarding the *last digit* of these numbers. Which one is it?
(a) It is 5 for some of these numbers but not for all.
(b) It is 5 for all these numbers.
(c) It is always 5 for $n \leq 10$ and it is 5 for some $n > 10$
(d) It is odd for all of these numbers but not necessarily 5.

Solution :

Last digit of 3^{2n+1} is 3 (for n = 1, 5, 9, …) or 7 (for n = 3, 7, 11, …).

Last digit of 2^{2n+1} is 2 (for n = 1, 5, 9, ….) or 8 (for n = 3, 7, 11, ….).

- ⇨ Last digit is always 5

Option (b) is correct.

77. Which one of the following numbers can be expressed as the sum of squares of two integers?
   (a) 1995
   (b) 1999
   (c) 2003
   (d) None of these integers.

Solution :

Let $x^2 + y^2 = 1995, 1999, 2003$ where x is even (say) and y is odd.

Dividing the equation by 4 we get, $0 + 1 \equiv 3, 3, 3 \pmod 4$

Which is impossible,

So, option (d) is correct.

78. If the product of an odd number odd integers is of the form 4n + 1, then
(a) An even number of them must be always of the form 4n + 1
(b) An odd number of them always be of the form 4n + 3
(c) An odd number of them must always be of the form 4n + 1
(d) None of the foregoing statements is true.

Solution :

Option (c) is correct.

79. The two sequences of numbers {1, 4, 16, 64, ….} and {3, 12, 48, 192, ….} are mixed as follows : {1, 3, 4, 12, 16, 48, 64, 192, ….}. One of the numbers in the mixed series is 1048576. Then the number immediately preceding it is
   (a) 786432
   (b) 262144
   (c) 814572
   (d) 786516

Solution :

1048576 is not divisible by 3.

Hence it is from first sequence.

So, $1*4^{n-1} = 1048576$

- ⇨ $4^{n-1} = 4^{10}$
- ⇨ $n = 11$.

Therefore, we need to find 10^{th} term of the second sequence.

It is $3*4^{10-1} = 3*4^9 = 786432$

Option (a) is correct.

80. Let $(a_1, a_2, a_3, ….)$ be a sequence such that $a_1 = 2$ and $a_n - a_{n-1} = 2n$ for all $n \geq 2$. Then $a_1 + a_2 + …. + a_{20}$ is
    (a) 420
    (b) 1750
    (c) 3080
    (d) 3500

Solution :

Now, $a_n - a_{n-1} = 2n$

Putting n = 2, we get, $a_2 - a_1 = 2*2$

Putting n = 3, we get, $a_3 - a_2 = 2*3$

Putting n = 4, we get, $a_4 - a_3 = 2*4$

…

…

Putting n = n, we get, $a_n - a_{n-1} = 2*n$

Adding the above equalities we get, $a_n - a_1 = 2(2 + 3 + …. + n) = 2(1 + 2 + …. + n) - 2 = 2\{ n(n + 1)/2\} - 2$

- ⇨ $a_n = n(n + 1)$ (as $a_1 = 2$)
- ⇨ $\sum a_n$ (n running from 1 to 20) $= \sum n^2 + \sum n = 20*21*41/6 + 20*21/2 = 70*41 + 210 = 3080$

Option (c) is correct.

81. The value of $\sum ij$, where the summation is over all i and j such that $1 \leq i < j \leq 10$, is
(a) 1320
(b) 2640
(c) 3025
(d) None of the foregoing numbers.

Solution :

$\sum ij$, where the summation is over all i and j such that $1 \leq i < j \leq 10$

$= 1*(2 + 3 + \ldots + 10) + 2(3 + 4 + \ldots + 10) + 3(4 + 5 + \ldots + 10) + 4(5 + 6 +\ldots + 10) + 5(6 + 7 + \ldots + 10) + 6(7 + 8 + 9 + 10) + 7(8 + 9 + 10) + 8(9 + 10) + 9*10$

$= (9/2)\{2*2 + (9 - 1)*1\} + 2*(8/2)\{2*3 + (8 - 1)*1\} + 3(7/2)\{2*4 + (7 - 1)*1\} + 4(6/2)\{2*5 + (6 - 1)*1\} + 5(5/2)\{2*6 + (5 - 1)*1\} + 6*34 + 7*27 + 8*19 + 90$

$= 54 + 104 + 147 + 180 + 200 + 204 + 189 + 152 + 90 = 1320$

Option (a) is correct.

82. Let $x_1, x_2, \ldots, x_{100}$ be hundred integers such that the sum of any five of them is 20. Then
(a) The largest x_i equals 5
(b) The smallest x_i equals 3
(c) $x_{17} = x_{83}$
(d) none of the foregoing statements is true.

Solution :

$x_i + x_j + x_k + x_l + x_m = 20$

Again, $x_i + x_j + x_k + x_l + x_n = 20$

- ⇨ $x_m = x_n$
- ⇨ All the integers are equal.
- ⇨ $x_{17} = x_{83}$

Option (c) is correct.

83. The smallest positive integer n with 24 divisors (where 1 and n are also considered as divisors of n) is
(a) 420
(b) 240
(c) 360
(d) 480

Solution :

$240 = 2^4*3*5$, number of divisors = $(4 + 1)*(1 + 1)*(1 + 1) = 20$

$360 = 2^3*3^2*5$, number of divisors = $(3 + 1)*(2 + 1)*(1 + 1) = 24$

Option (c) is correct.

84. The last digit of 2137^{754} is
(a) 1
(b) 3
(c) 7
(d) 9

Solution :

$2137^2 \equiv -1 \pmod{10}$

- ⇨ $(2137^2)^{377} \equiv (-1)^{377} \pmod{10}$
- ⇨ $2137^{754} \equiv -1 \pmod{10} \equiv 9 \pmod{10}$
- ⇨ Last digit is 9.

Option (d) is correct.

85. The smallest integer that produces remainders of 2, 4, 6 and 1 when divided by 3, 5, 7 and 11 respectively is
(a) 104
(b) 1154
(c) 419
(d) None of the foregoing numbers.

Solution :

Now, $n = 3t_1 - 1$

$n = 5t_2 - 1$

$n = 7t_3 - 1$

$n = 11t_4 + 1$

- ⇨ $n = 3*5*7t_5 - 1 = 105t_5 - 1 = 104 + 105t_6$

Now, $104 + 105t_6 \equiv 1 \pmod{11}$

- ⇨ $5 + 6t_6 \equiv 1 \pmod{11}$
- ⇨ $6t_6 \equiv 7 \pmod{11}$
- ⇨ $t_6 = 3 + 11t_7$

Therefore, $n = 104 + 105(3 + 11t_7) = 419 + 1155t_7$

Therefore, least n is 419.

Option (c) is correct.

86. How many integers n are there such that $2 \le n \le 1000$ and the highest common factor of n and 36 is 1?
(a) 166
(b) 332
(c) 361
(d) 416

Solution :

$36 = 2^2*3^2$

Number of positive integers divisible by 2 = 500

Number of positive integers divisible by 3, $3 + (p - 1)*3 = 999$

- ⇨ $p = 333$

Number of positive integers which are divisible by both 2 and 3, $6 + (m - 1)*6 = 996$

- ⇨ $m = 166$.

Therefore number of positive integers divisible by 2 and 3 = 500 + 333 – 166 = 667

Therefore number of n = 999 – 667 = 332

Option (b) is correct.

87. The remainder when 3^{37} is divided by 79 is
   (a) 78
   (b) 1
   (c) 2
   (d) 35

Solution :

$3^4 \equiv 2 \pmod{79}$

- ⇨ $(3^4)^9 \equiv 2^9 \pmod{79}$
- ⇨ $3^{37} \equiv 3*2^9 \pmod{79} \equiv 17*16 \pmod{79} \equiv (-11)*4 \pmod{79} \equiv -44 \pmod{79} \equiv 35 \pmod{79}$

Option (d) is correct.

88. The remainder when 4^{101} is divided by 101 is
   (a) 4
   (b) 64
   (c) 84
   (d) 36

Solution :

By Fermat's little theorem, $4^{100} \equiv 1 \pmod{101}$ (101 is prime)

- ⇨ $4^{101} \equiv 4 \pmod{101}$

Option (a) is correct.

89. The 300-digit number with all digits equal to 1 is
   (a) Divisible by neither 37 nor 101

(b) Divisible by 37 but not by 101
(c) Divisible by 101 but not by 37
(d) Divisible by both 37 and 101.

Solution :

As 300 is divisible by 3 so, the number is divisible by 37 as 37 divides 111.

Now, $1*10^{299} + 1*10^{298} + \ldots.. + 1*10 + 1 = 1*(10^{300} - 1)/(10 - 1) = (10^{300} - 1)/9$

Now, $10^2 \equiv -1 \pmod{101}$

- ⇨ $(10^2)^{150} \equiv (-1)^{150} \pmod{101}$
- ⇨ $10^{300} \equiv 1 \pmod{101}$
- ⇨ $10^{300} - 1 \equiv 0 \pmod{101}$
- ⇨ The number is divisible by 101.

Option (d) is correct.

90. The remainder when $3^{12} + 5^{12}$ is divided by 13 is
(a) 1
(b) 2
(c) 3
(d) 4

Solution :

By Fermat's little theorem, $3^{12}, 5^{12} \equiv 1 \pmod{13}$

- ⇨ $3^{12} + 5^{12} \equiv 1 + 1 \pmod{13} \equiv 2 \pmod{13}$

Option (b) is correct.

91. When $3^{2002} + 7^{2002} + 2002$ s divided by 29 the remainder is
(a) 0
(b) 1
(c) 2
(d) 7

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Solution :

By Fermat's little theorem, 3^{28}, $7^{28} \equiv 1 \pmod{29}$

Now, $2002 \equiv 14 \pmod{28}$ and $2002 \equiv 1 \pmod{29}$

Therefore, $3^{2002} + 7^{2002} + 2002 \equiv 3^{14} + 7^{14} + 1 \pmod{29} \equiv \pm1 \pm1 + 1 \pmod{29}$ (if $a^{(p-1/2)} \equiv \pm1 \pmod{p}$)

If both are +1 then the remainder is 3 which is not option.

If both are -1 then the remainder is -1 which is not option.

So, one is +1 and another is -1 and hence the remainder is $+1 - 1 + 1 = 1$

Option (b) is correct.

92. Let x = 0.101001000100001…. + 0.272727….. Then x
   (a) is irrational.
   (b) Is rational but $\sqrt{x}$ is irrational
   (c) is a root of $x^2 + 0.27x + 1 = 0$
   (d) satisfies none of the above properties.

Solution :

x = 0.101001000100001….. + 0.272727…..

$= 1*10^{-1} + 1*10^{-3} + 1*10^{-6} + 1*10^{-10} + …. + 27*10^{-2} + 27*10^{-4} + 27*10^{-6} + …..$

$= 10^{-1} + 10^{-3} + …… + 27*10^{-2}\{1/(1 - 10^{-2})\}$

$= 10^{-1} + 10^{-3} + 10^{-6} + 10^{-10} + 3/11$

= irrational.

Option (a) is correct.

93. The highest power of 18 contained in ${}^{50}C_{25}$ is
   (a) 3
   (b) 0
   (c) 1
   (d) 2

Solution :

$^{50}C_{25}$ = 50*49*48*……*26/(25*24*….*2*1) = (2^{25}*3^{12}*….)/(2^{22}*3^{10}*….) = 2^3*3^2*….

Therefore, highest power of 18 contained in $^{50}C_{25}$ is 1.

Option (c) is correct.

94. The number of divisors of 2700 including 1 and 2700 equals
(a) 12
(b) 16
(c) 36
(d) 18

Solution :

2700 = 2^2*3^3*5^2

Number of divisors = (2 + 1)(3 + 1)(2 + 1) = 36

Option (c) is correct.

95. The number of different factors of 1800 equals
(a) 12
(b) 210
(c) 36
(d) 18

Solution :

1800 = 2^3*3^2*5^2

Number of factors = (3 + 1)(2 + 1)(2 + 1) = 36

Option (c) is correct.

96. The number of different factors of 3003 is

(a) 2
(b) 15
(c) 7
(d) 16

Solution :

3003 has 3 as a factor. So, 2 cannot be answer. Only square numbers have odd number of factors. So, option (a), (b), (c) cannot be true.

Option (d) is correct.

97. The number of divisors of 6000, where 1 and 6000 are also considered as divisors of 6000, is
(a) 40
(b) 50
(c) 60
(d) 30

Solution :

$6000 = 2^4*3*5^3$

Number of divisors = (4 + 1)(1 + 1)(3 + 1) = 40

Option (a) is correct.

98. The number of positive integers which divide 240 (where 1 and 240 are considered as divisors) is
(a) 18
(b) 20
(c) 30
(d) 24

Solution :

$240 = 2^4*3*5$

Number of positive integers which divide 240 = (4 + 1)(1 + 1)(1 + 1) = 20

Option (b) is correct.

99. The sum of all the positive divisors of 1800 (including 1 and 1800) is
(a) 7201
(b) 6045
(c) 5040
(d) 4017

Solution :

$1800 = 2^3*3^2*5^2$

Sum of divisors = $\{(2^{3+1} - 1)/(2 - 1)\}\{(3^{2+1} - 1)/(3 - 1)\}\{(5^{2+1} - 1)/(5 - 1)\} = 15*13*31 = 6045$

Option (b) is correct.

100. Let $d_1, d_2, \ldots, d_k$ be all the factors of a positive integer n including 1 and n. Suppose $d_1 + d_2 + \ldots + d_k = 72$. Then the value of $1/d_1 + 1/d_2 + \ldots + 1/d_k$
(a) is $k^2/72$
(b) is $72/k$
(c) is $72/n$
(d) cannot be computed from the given information.

Solution :

Now, $1/d_1 + 1/d_2 + \ldots + 1/d_k = (1/n)(n/d_1 + n/d_2 + \ldots + n/d_k)$

Now, n/d_1 will give another factor, n/d_2 will give another factor and so on.

- ⇨ $n/d_1 + n/d_2 + \ldots + n/d_k = d_1 + d_2 + \ldots + d_k = 72$
- ⇨ $1/d_1 + 1/d_2 + \ldots + 1/d_k = 72/n$

Option (c) is correct.

101. The number of ways of distributing 12 identical things among 4 children so that every child gets at least one and no child more than 4 is
(a) 31
(b) 52
(c) 35
(d) 42

Solution :

Let first child gets x_1 oranges, second child gets x_2 oranges, third child gets x_3 oranges, fourth child gets x_4 oranges.

So, $x_1 + x_2 + x_3 + x_4 = 12$ where $1 \le x_1, x_2, x_3, x_4 \le 4$

We fix $x_1 = 1$, then $x_2 + x_3 + x_4 = 11$

Now, x_2 cannot be equal to 1 or 2. So, $x_2 = 3$.

Then only one solution, $x_3 = 4, x_4 = 4$.

Now, let, $x_2 = 4$, two solutions, $x_3 = 3, x_4 = 4$ and $x_3 = 4, x_3 = 3$.

So, we have, 3 solutions.

Now, $x_1 = 2, x_2 = 2, x_3 + x_4 = 8$ one solution, $x_3 = 4, x_4 = 4$.

$x_2 = 3$, then two solutions, $x_3 = 3, x_4 = 4$ and $x_3 = 4, x_4 = 3$

$x_2 = 4, x_3 + x_4 = 6, x_3 = 2, x_4 = 4; x_3 = 3, x_4 = 3; x_3 = 4, x_4 = 2$

So, we have, 6 solutions.

Now, $x_1 = 3, x_2 = 1$, one solution $x_3 = 4, x_4 = 4$

$x_2 = 2$, two solutions, $x_3 = 3, x_4 = 4; x_3 = 4, x_4 = 3$

$x_2 = 3$, three solutions.

$x_2 = 4$, then, $x_3 = 1, x_4 = 4; x_3 = 2, x_4 = 3; x_3 = 3, x_4 = 2; x_3 = 4, x_4 = 1$

So we have, 10 solutions.

Now, $x_1 = 4, x_2 = 1$; two solutions.

$x_2 = 2$, three solutions

$x_2 = 3$, four solutions

$x_2 = 4$, $x_3 = 1$, $x_4 = 3$; $x_3 = 2$, $x_4 = 2$; $x_3 = 3$, $x_4 = 1$ – three solutions

So we have, 12 solutions.

So we have total 3 + 6 + 10 + 12 = 31 solutions.

Therefore, we can distribute the things in 31 ways.

Option (a) is correct.

102. The number of terms in the expansion of $[(a + 3b)^2(a - 3b)^2]^2$, when simplified is
(a) 4
(b) 5
(c) 6
(d) 7

Solution :

$[(a + 3b)^2(a - 3b)^2]^2 = (a^2 - 9b^2)^4$ = 5 terms

Option (b) is correct.

103. The number of ways in which 5 persons P, Q, R, S and T can be seated in a ring so that P sits between Q and R is
(a) 120
(b) 4
(c) 24
(d) 9

Solution :

Let us take (QPR) as unit.

Therefore, total (QPR), S, T – 3 persons will sit in a ring, It can be done in (3 – 1)! = 2 ways.

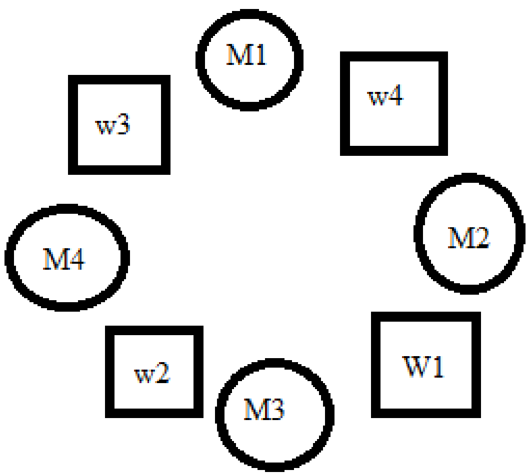
Now, Q and P can move among themselves in 2! = 2 ways.

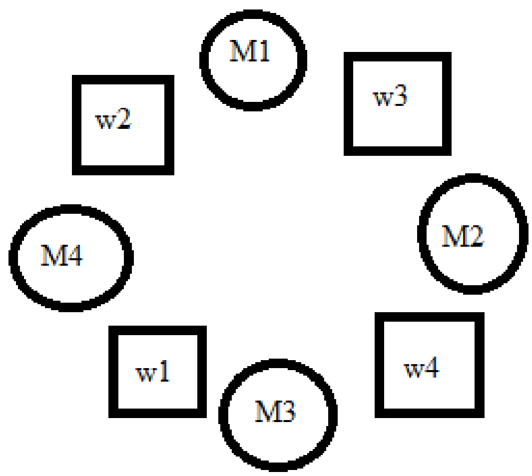
Therefore, the whole arrangement can be done in 2*2 = 4 ways.

Option (b) is correct.

104. Four married couples are to be seated in a merry-go-round with 8 identical seats. In how many ways can they be seated so that
(i) males and females seat alternatively; and
(ii) no husband seats adjacent to this wife?
(a) 8
(b) 12
(c) 16
(d) 20

Solution :





W_1 can sit in two seats either in the seat in left side figure or in the seat in right side figure. In left side figure when W_1 is given seat then W_4 can sit in one seat only as shown and accordingly W_2 and W_3 can also take only one seat. Similarly, right side figure also reveals one possible way to seat. So there are two ways to seat for every combination of Men. Now, Men can arrange themselves in (4 – 1)! = 6 ways. So number of ways = 2*6 = 12.

Option (b) is correct.

105. For a regular polygon with n sides (n > 5), the number of triangles whose vertices are joining non-adjacent vertices of the polygon is

(a) $n(n - 4)(n - 5)$
(b) $(n - 3)(n - 4)(n - 5)/3$
(c) $2(n - 3)(n - 4)(n - 5)$
(d) $n(n - 4)(n - 5)/6$

Solution :

The total number of triangles that can be formed using any three vertices = ${}^nC_3$.

Now, taking two consecutive vertices and one other vertices total number of triangles that can be formed = $n*(n - 4)$

Now, taking three consecutive vertices total number of triangles that can be formed = n

Therefore, number of required triangles = ${}^nC_3 - n(n - 4) - n = n(n - 1)(n - 2)/6 - n(n - 4) - n$

$= (n/6)(n^2 - 3n + 2 - 6n + 24 - 6) = (n/6)(n^2 - 9n + 20) = n(n - 4)(n - 5)/6$

Option (d) is correct.

106. The term that is independent of x in the expansion of $(3x^2/2 - 1/3x)^9$ is
(a) ${}^9C_6(1/3)^3(3/2)^6$
(b) ${}^9C_5(3/2)^5(-1/3)^4$
(c) ${}^9C_3(1/6)^3$
(d) ${}^9C_4(3/2)^4(-1/3)^5$

Solution :

$t_r = {}^9C_r(3x^2/2)^r(-1/3x)^{9 - r} = {}^9C_r(3/2)^r(-1/3)^{9-r}x^{2r - 9 + r}$

Now, $2r - 9 + r = 0$

- ⇨ $r = 3$
- ⇨ The term independent of x is ${}^9C_3(3/2)^3(-1/3)^6 = {}^9C_3(1/6)^3$

Option (c) is correct.

107. The value of $({}^{50}C_0)({}^{50}C_1) + ({}^{50}C_1)({}^{50}C_2) + \ldots.. + ({}^{50}C_{49})({}^{50}C_{50})$ is
(a) ${}^{100}C_{50}$
(b) ${}^{100}C_{51}$
(c) ${}^{50}C_{25}$
(d) $({}^{50}C_{25})^2$

Solution :

Now, $(1 + x)^{50} = {}^{50}C_0 + {}^{50}C_1x + {}^{50}C_2x^2 + \ldots.. + {}^{50}C_{49}x^{49} + {}^{50}C_{50}x^{50}$

$(x + 1)^{50} = {}^{50}C_0x^{50} + {}^{50}C_1x^{49} + \ldots.. + {}^{50}C_{50}$

Now, $(1 + x)^{50}(x + 1)^{50} = ({}^{50}C_0 + {}^{50}C_1x + {}^{50}C_2x^2 + \ldots.. + {}^{50}C_{49}x^{49} + {}^{50}C_{50}x^{50})({}^{50}C_0x^{50} + {}^{50}C_1x^{49} + \ldots.. + {}^{50}C_{50})$

⇨ $(1 + x)^{100} = ({}^{50}C_0{}^{50}C_1 + {}^{50}C_1{}^{50}C_2 + \ldots\ldots + {}^{50}C_{49}{}^{50}C_{50})x^{49} + \ldots\ldots$

Now, coefficient of x^{49} in the expansion of $(1 + x)^{100} = {}^{100}C_{49} = {}^{100}C_{51}$

Option (b) is correct.

108. The value of $({}^{50}C_0)^2 + ({}^{50}C_1)^2 + \ldots\ldots + ({}^{50}C_{50})^2$ is
(a) ${}^{100}C_{50}$
(b) 50^{50}
(c) 2^{100}
(d) 2^{50}

Solution :

From the previous question's solution we get, the coefficient of $({}^{50}C_0)^2 + ({}^{50}C_1)^2 + \ldots. + ({}^{50}C_{50})^2$ is x^{50}

Therefore, coefficient of x^{50} in the expansion of $(1 + x)^{100}$ is ${}^{100}C_{50}$.

Option (a) is correct.

109. The value of $({}^{100}C_0)({}^{200}C_{150}) + ({}^{100}C_1)({}^{200}C_{151}) + \ldots\ldots + ({}^{100}C_{50})({}^{200}C_{200})$ is
(a) ${}^{300}C_{50}$
(b) $({}^{100}C_{50})({}^{200}C_{150})$
(c) $({}^{100}C_{50})^2$

(d) None of the foregoing numbers.

Solution :

$(1 + x)^{100} = {}^{100}C_0 + {}^{100}C_1x + {}^{100}C_2x^2 + ... + {}^{100}C_{100}x^{100}$

$(x + 1)^{200} = {}^{200}C_0x^{200} + ... + {}^{200}C_{150}x^{150} + {}^{200}C_{151}x^{49} + ... + {}^{200}C_{200}$

Now, $(1 + x)^{100}(x + 1)^{200} = ({}^{100}C_0{}^{200}C_{150} + {}^{100}C_1{}^{200}C_{151} + ... + {}^{100}C_{50}{}^{200}C_{200})x^{50} + ...$

Now, in LHS i.e. $(1 + x)^{100}(x + 1)^{200} = (1 + x)^{300}$ coefficient of x^{50} is ${}^{300}C_{50}$

Therefore, ${}^{100}C_0{}^{200}C_{150} + {}^{100}C_1{}^{200}C_{151} + ... + {}^{100}C_{50}{}^{200}C_{200} = {}^{300}C_{50}$

Option (a) is correct.

110. The number of four-digit numbers strictly greater than 4321 that can be formed from the digits 0, 1, 2, 3, 4, 5 allowing for repetition of digits is
(a) 310
(b) 360
(c) 288
(d) 300

Solution :

The first digit can be either 4 or 5

Case 1 : first digit is 4.

Second digit can be 3, 4, 5

So, we can choose second digit when 4 or 5 in 2 ways.

Third digit, fourth digit can be anything when second digit is 4 or 5.

So, number of numbers = 2*6*6 = 72

When second digit is 3, third digit can be 2, 3, 4, 5

When third digit is 3, 4, or 5 fourth digit can be anything.

So, number of numbers = 3*6 = 18

When third digit is 2, fourth digit can be 2, 3, 4, 5. So number of numbers = 4

Therefore, total number of numbers when first digit is 4 is 72 + 18 + 4 = 94

When first digit is 5 then number of numbers = 6^3 = 216

So, total number of required numbers = 94 + 216 = 310

Option (a) is correct.

111. The sum of all the distinct four-digit numbers that can be formed using the digits 1, 2, 3, 4 and 5, each digit appearing at most once, is
(a) 399900
(b) 399960
(c) 390000
(d) 360000

Solution :

Now, 1 will appear as the first digit in 4! = 24 numbers.

Similar thing goes for other digits.

Similar case goes for other digit of the numbers i.e. second digit, third digit, fourth digit.

Therefore sum = 24{1000(1 + 2 + 3 + 4 + 5) + 100(1 + 2 + 3 + 4 + 5) + 10(1 + 2 + 3 + 4 + 5) + (1 + 2 + 3 + 4 + 5)}

= 24*(1 + 2 + 3 + 4 + 5)(1000 + 100 + 10 + 1)

= 24*15*1111

= 399960

Option (b) is correct.

112. The number of integers lying between 3000 and 8000 (including 3000 and 8000) which have at least two digits equal is
(a) 2481
(b) 1977
(c) 4384

(d) 2755

Solution :

Let us consider the number from 3000 – 3999

Let the number is 3xyz

x = 3, and y, z ≠ 3, there are 9*9 = 81 numbers.

Similarly, for y = 3, x, z ≠ 3 and z = 3, x, y ≠ 3 there are 81 + 81 = 162 numbers.

Now, x = y there are $^{10}C_1$*10 = 100 numbers.

Now, y = z, there are $^{10}C_1$*10 = 100 numbers.

Now, z = x, there are $^{10}C_1$*10 = 100 numbers.

Now, x = y = z, there are 10 numbers.

Now, x = y = 3, z ≠ 3, there are 9 numbers.

Similarly, y = z = 3, x ≠ 3 there are 9 numbers and x = z = 3, y ≠ 3 there are 9 numbers.

So, number of required numbers between 3000 and 3999 is 81 + 162 + 100 + 100 + 100 – 3*9 – 2*10 = 496.

So, number of required numbers between 3000 and 7999 is 496*5 = 2480

So including 8000 there are 2480 + 1 = 2481 numbers.

Option (a) is correct.

113. The greatest integer which, when dividing the integers 13511, 13903 and 14593 leaves the same remainder is
(a) 98
(b) 56
(c)2
(d) 7

Solution :

Let the remainder is a and the number is x.

Therefore, 13511 – a = xm_1, 13903 – a = xm_2, 14593 – a = xm_3

Now, 13903 – a – (13511 – a) = $xm_2 - xm_1$

⇨ $392 = x(m_2 - m_1)$

Now, 392 is not divisible by 56, therefore option (b) cannot be true.

Now, 14953 – a –(13903 – a) = $xm_3 - xm_2$

⇨ $1050 = x(m_3 - m_2)$

Now, 1050 is not divisible by 98 therefore option (a) cannot be true.

Now, 14593 – a – (13511 – a) = $xm_3 - xm_1$

⇨ $1082 = x(m_3 - m_1)$

Now, 1082 is not divisible by 7 therefore option (d) cannot be true.

Option (c) is correct.

114. An integer has the following property that when divided by 10, 9, 8, …., 2, it leaves remainders 9, 8, 7, …., 1 respectively. A possible value of n is
(a) 59
(b) 419
(c) 1259
(d) 2519

Solution :

$n = 10m_1 - 1$,

$n = 9m_2 - 1$,

…

..

$n = 2m_9 - 1$

Therefore, $n = 10*9*4*7m_{10} - 1 = 2520m_{11} + 2519$

Option (d) is correct.

115. If n is a positive integer such that 8n + 1 is a perfect square, then
(a) n must be odd
(b) n cannot be a perfect square
(c) n must be a prime number
(d) 2n cannot be a perfect square

Solution :

Let n = odd.

$8n + 1 = 8(2m + 1) + 1 = 16m + 9$

But, $(\text{an odd number})^2 \equiv 1, 9 \pmod{16}$ so, (a) cannot be true.

If n is prime then 8n + 1 is sometimes perfect square and sometimes not. For example n = 3, n = 5.

Option (c) cannot be true.

If n is perfect square then the statement is always not true because $8m^2 + 1 = p^2$ cannot hold true. If we divide the equation by 9 then we can find contradiction as $-m^2 + 1 \equiv p^2 \pmod 9$ and $m^2, p^2 \equiv 0, 3, 6 \pmod 9$

Option (b) cannot be true.

If, 2n is not a perfect square, $8n + 1 = 4*(2n) + 1$ which is always true.

Option (d) is correct.

116. For any two positive integers a and b, define $a \equiv b$ if $a - b$ is divisible by 7. Then $(1512 + 121)*(356)*(645) \equiv$
(a) 4
(b) 5
(c) 3
(d) 2

Solution :

$1512 \equiv 0$, $121 \equiv 2$, $356 \equiv 6$, $645 \equiv 1$

Therefore, (1512 + 121)*(356)*(645) ≡ (0 + 2)*6*1 = 12 ≡ 5

Option (b) is correct.

117. The coefficient of x^2 in the binomial expansion of $(1 + x + x^2)^{10}$ is
(a) $^{10}C_1 + {}^{10}C_2$
(b) $^{10}C_2$
(c) $^{10}C_1$
(d) None of the foregoing numbers.

Solution :

$(1 + x + x^2)^{10} = {}^{10}C_0(1 + x)^{10} + {}^{10}C_1(1 + x)^{10}x^2 + \ldots = (1 + {}^{10}C_1x + {}^{10}C_2x^2 + \ldots) + {}^{10}C_1(1 + \ldots)x^2 + \ldots = ({}^{10}C_2 + {}^{10}C_1)x^2 + \ldots$

Option (a) is correct.

118. The coefficient of x^{17} in the expansion of $\log_e(1 + x + x^2)$, where $|x| < 1$, is
(a) 1/17
(b) -1/17
(c) 3/17
(d) None of the foregoing quantities.

Solution :

$\log_e\{(1 + x + x^2)(1 - x)/(1 - x)\} = \log_e\{(1 - x^3)/(1 - x) = \log_e(1 - x^3) - \log_e(1 - x)$

$= (-x^3 - x^6/2 - \ldots) - (-x - x^2/2 - \ldots - x^{17}/17 - \ldots)$

$= (1/17)x^{17} + \ldots$

Option (a) is correct.

119. Let $a_1, a_2, \ldots, a_{11}$ be an arbitrary arrangement (i.e. permutation) of the integers 1, 2, …, 11. Then the number $(a_1 - 1)(a_2 - 2)\ldots(a_{11} - 11)$ is

(a) Necessarily ≤ 0
(b) Necessarily 0
(c) Necessarily even
(d) Not necessarily ≤ 0, 0 or even.

Solution :

There are 6 odd numbers and 5 even numbers. If we subtract all the odd numbers from even numbers then also one odd number remains which when subtracted from the odd number given an even number.

So, it is necessarily even

Option (c) is correct.

120. Three boys of class I, 4 boys of class II and 5 boys of class III sit in a row. The number of ways they can sit, so that boys of the same class sit together is
(a) 3!4!5!
(b) 12!/(3!4!5!)
(c) $(3!)^2 4!5!$
(d) 3*4!5!

Solution :

Let us take the boys of each class as an unit.

Therefore there are 3 units which can be permutated in 3! ways.

Now, class I boys can permutate among themselves in 3! ways, class II boys in 4! And class III boys in 5! ways.

Therefore, total number of ways is $3!*3!*4!*5! = (3!)^2 4!5!$

Option (c) is correct.

121. For each positive integer n consider the set S_n defined as follows : $S_1 = \{1\}$, $S_2 = \{2, 3\}$, $S_3 = \{4, 5, 6\}$, …, and, in general, S_{n+1} consists of n + 1 consecutive integers the smallest of which is one more than the largest integer in S_n. Then the sum of all the integers in S_{21} equals

(a) 1113
(b) 53361
(c) 5082
(d) 4641

Solution :

The largest integer in S_n is the triangular numbers i.e. n(n + 1)/2.

Now, the largest number of S_{20} = 20*21/2 = 210

Therefore, required summation = 211 + 212 + …. 21 terms = (21/2){2*211 + (21 – 1)*1} = 21*(211 + 10) = 4641.

Option (d) is correct.

122. If the constant term in the expansion of $(\sqrt{x} - k/x^2)^{10}$ is 405, then k is
(a) $\pm(3)^{1/4}$
(b) ±2
(c) $\pm(4)^{1/3}$
(d) ±3

Solution :

Any term = $^{10}C_r(\sqrt{x})^r(k/x^2)^{10-r} = {}^{10}C_r k^{10-r} x^{r/2 - 20 + 2r}$

Now, r/2 – 20 + 2r = 0

⇨ 5r/2 = 20
⇨ r = 8

So, $^{10}C_8 * k^{10-8} = 405$

⇨ $(10*9/2)*k^2 = 405$
⇨ $k^2 = 9$
⇨ k = ±3

Option (d) is correct.

123. Consider the equation of the form $x^2 + bx + c = 0$. The number of such equations that have real roots and have coefficients b and c in the set {1, 2, 3, 4, 5, 6}, (b may be equal to c), is
(a) 20
(b) 18
(c) 17
(d) 19

Solution :

Roots are real.

⇨ $b^2 - 4c \geq 0$
⇨ $b^2 \geq 4c$

b cannot be 1

if b = 2, c = 1

if b = 3, c = 1, 2

if b = 4, c = 1, 2, 3, 4

if b = 5, c = 1, 2, 3, 4, 5, 6

if b = 6, c = 1, 2, 3, 4, 5, 6

Therefore, total number of equations = 1 + 2 + 4 + 6 + 6 = 19

Option (d) is correct.

124. The number of polynomials of the form $x^3 + ax^2 + bx + c$ which is divisible by $x^2 + 1$ and where a, b and c belong to {1, 2, …., 10} is
(a) 1
(b) 10
(c) 11
(d) 100

Solution :

Now, $x^2 \equiv -1 \pmod{x^2 + 1}$

⇨ $x^3 + ax^2 + bx + c \equiv x(1 - b) + (c - a) \pmod{x^2 + 1}$

But remainder must be 0.

⇨ b = 1 and a = c

Now, a = c can be done in 10 ways and b = 1 can be done in 1 way.

Therefore, total number of polynomials = 10*1 = 10

Option (b) is correct.

125. The number of distinct 6-digit numbers between 1 and 300000 which are divisible by 4 and are obtained by rearranging the digits 112233, is
(a) 12
(b) 15
(c) 18
(d) 90

Solution :

Last digit needs to be 2.

First digit can be 1 or 2.

First digit is 1, last digit is 2 and fifth digit cannot be 2 as the five-digit number up to fifth digit from left then congruent to 2 or 0 modulus 4 and in both the cases the six digit number is not divisible by 4.

So, fifth digit is 1 or 3.

First digit 1, fifth digit 1, last digit 2.

The digits left for 3 digits are 3, 3, 2

Number of numbers = 3!/2! = 3

First digit 1, fifth digit 3, last digit 2.

The digits left are 1, 3, 2.

Number of numbers = 3! = 6

Total numbers in this case = 3 + 6 = 9

First digit 2, fifth digit 1, last digit 2.

Digits left are 1, 3, 3

Number of numbers = 3!/2! = 3

First digit 2, fifth digit 3, last digit 2.

Digits left = 1, 1, 3

Number of numbers = 3!/2! = 3

Total number of numbers in this case = 3 + 3 = 6

So, number of required numbers = 9 + 6 = 15.

Option (b) is correct.

126. The number of odd positive integers smaller than or equal to 10000 which are divisible by neither 3 nor by 5 is
(a) 3332
(b) 2666
(c) 2999
(d) 3665

Solution :

Let n number of odd numbers divisible by 3.

Then 3 + (n – 1)*6 = 9999

- ⇨ 2n – 1 = 3333
- ⇨ n = 1667

Let m number of odd numbers are divisible by 5.

Then, 5 + (m – 1)*10 = 9995

- ⇨ 2m – 1 = 1999
- ⇨ m = 1000

Let p number of odd numbers are divisible by 3*5 = 15

Then 15 + (p – 1)*30 = 9975

- ⇨ 2p – 1 = 665
- ⇨ p = 333

- ⇨ Number of odd numbers divisible by 3 or 5 = 1667 + 1000 − 333 = 2334
- ⇨ Number of odd numbers which are neither divisible by 3 nor by 5 is 5000 – 2334 = 2666.

Option (b) is correct.

127. The number of ways one can put three balls numbered 1, 2, 3 in three boxes labeled a, b, c such that at the most one box is empty is equal to
   (a) 6
   (b) 24
   (c) 42
   (d) 18

Solution :

No box is empty – number of ways = 3! = 6

Box a is empty – combinations are – 1, 2 in b, 3 in c; 1, 3 in b, 2 in c; 2, 3 in b; 1 in c; 1 in b, 2, 3 in c; 2 in b, 1, 3 in c; 3 in b, 1, 2 in c – 6 ways

Similarly, box b is empty – 6 ways and box c is empty – 6 ways.

Therefore total number of ways = 6 + 6 + 6 + 6 = 24.

Option (b) is correct.

128. A bag contains colored balls of which at least 90% are red. Balls are drawn from the bag one by one and their color noted. It is found that 49 of the first 50 balls drawn are red. Thereafter 7 out of every 8 balls are red. The number of balls in the bag CAN NOT BE
   (a) 170
   (b) 210
   (c) 250
   (d) 194

Solution :

Let number of balls in the bag is n.

Let, m number of times 8 balls are drawn.

Therefore, n = 50 + 8m

Red balls = 49 + 7m

Percentage of red balls = {(49 + 7m)/(50 + 8m)}*100 ≥ 90

- ⇨ 49 + 7m ≥ 50*0.9 + 8m*0.9
- ⇨ 49 + 7m ≥ 45 + 7.2m
- ⇨ 0.2m ≤ 4
- ⇨ m ≤ 20
- ⇨ n ≤ 50 + 8*20 = 210

Option (c) is correct.

129. There are N boxes, each containing at most r balls. If the number of boxes containing at least i balls is N_i, for i = 1, 2, …., r, then the total number of balls in these boxes
(a) Cannot be determined from the given information
(b) Is exactly equal to $N_1 + N_2 + \ldots + N_r$
(c) Is strictly larger than $N_1 + N_2 + \ldots + N_r$
(d) Is strictly smaller than $N_1 + N_2 + \ldots + N_r$

Solution :

Option (b) is correct.

130. For all n, the value of ${}^{2n}C_n$ is equal to
(a) ${}^{2n}C_0 - {}^{2n}C_1 + {}^{2n}C_2 - {}^{2n}C_3 + \ldots + {}^{2n}C_{2n}$
(b) $({}^{2n}C_0)^2 + ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 + \ldots + ({}^{2n}C_n)^2$
(c) $({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - ({}^{2n}C_3)^2 + \ldots + ({}^{2n}C_{2n})^2$
(d) None of the foregoing expressions.

Solution :

Now, the expression in option (a) is equal to zero.

The expression is option (b) is obviously greater than ${}^{2n}C_n$.

Let is check the expression in option (c).

$(1 + x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1x + {}^{2n}C_2x^2 + ... + {}^{2n}C_{2n}x^{2n}$

$(x - 1)^{2n} = {}^{2n}C_0x^{2n} - {}^{2n}C_1x^{2n-1} + ... + {}^{2n}C_{2n}$

Now, $(1 + x)^{2n}(x - 1)^{2n} = \{({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ... + ({}^{2n}C_{2n})^2\}x^{2n} + ...$

Now, $(1 + x)^{2n}(x - 1)^{2n} = (x^2 - 1)^{4n}$ in this coefficient of x^{2n} is $(-1)^n*{}^{4n}C_n$.

So, option (d) is correct.

131. The coefficients of three consecutive terms in the expansion of $(1 + x)^n$ are 165, 330 and 462. Then the value of n is
   (a) 10
   (b) 12
   (c) 13
   (d) 11

Solution :

Let ${}^nC_{r-1} = 165$, ${}^nC_r = 330$, ${}^nC_{r+1} = 462$

Now, ${}^nC_r/{}^nC_{r-1} = 330/165$

- ⇨ $[n!/\{(n - r)!r!\}]/[n!/\{(n - r + 1)!(r - 1)!\} = 2$
- ⇨ $(n - r + 1)/r = 2$
- ⇨ $n - r + 1 = 2r$
- ⇨ $n + 1 = 3r$ ……. (i)

Now, ${}^nC_{r+1}/{}^nC_r = 462/330$

- ⇨ $[n!/\{(n - r - 1)!(r + 1)!]/[n!/\{(n - r)!r!\}] = 7/5$
- ⇨ $(n - r)/(r + 1) = 7/5$
- ⇨ $5n - 5r = 7r + 7$
- ⇨ $5n = 12r + 7$
- ⇨ $5n = 4(n + 1) + 7$ (From (i))
- ⇨ $n = 11$

Option (d) is correct.

132. The number of ways in which 4 persons can be divided into two equal groups is
   (a) 3
   (b) 12

(c) 6
(d) None of the foregoing numbers.

Solution :

Let the persons are A, B, C, D.

We can choose any two persons in $^4C_2$ ways = 6 ways. This is the answer as we can place them in 1 way.

Option (c) is correct.

133. The number of ways in which 8 persons numbered 1, 2, ……, 8 can be seated in a ring so that 1 always sits between 2 and 3 is
(a) 240
(b) 360
(c) 72
(d) 120

Solution :

Let us take 213 as unit.

So there are 6 units.

They can sit in (6 – 1)! = 5! ways.

Now, 2 and 3 can interchange their position in 2! ways.

Therefore, total number of sitting arrangement is 5!*2! = 240.

Option (a) is correct.

134. There are seven greeting cards, each of a different color, and seven envelops of the same seven colors. The number of ways in which the cards can be put in the envelops so that exactly 4 cards go into the envelops into the right colors is
(a) $^7C_3$
(b) $2*^7C_3$
(c) $(3!)^4C_3$
(d) $(3!)*^7C_3*^4C_3$

Solution :

We can choose any 4 greeting card which go to correct color envelop in ${}^7C_4$ = ${}^7C_3$ ways.

Now, let the color of the rest three envelops are red, green, blue and the greeting card of the same color go to different color envelop in 2 ways, as given below :

Red envelop	green envelop	blue envelop
Green card	blue card	red card
Blue card	red card	green card

So, number of ways = $2*{}^7C_3$.

Option (b) is correct.

135. The number of distinct positive integers that can be formed using 0, 1, 2, 4 where each integer is used at most once is equal to
(a) 48
(b) 84
(c) 64
(d) 36

Solution :

One digit number can be formed = 3

Two digit number can be formed = ${}^4C_2*2! - {}^3C_1*1! = 9$ (we need to subtract the numbers which begin with 0)

Three digit number that can be formed = ${}^4C_3*3! - {}^3C_2*2! = 18$

Four digit number that can be formed = $4! - 3! = 18$

Total number of such numbers = 3 + 9 + 18 + 18 = 48

Option (a) is correct.

136. A class contains three girls and four boys. Every Saturday five students go on a picnic, a different group being sent each week. During the picnic, each girl in the group is given a doll by the accompanying teacher. After all possible groups of five have gone once, the total number of dolls the girls have got is
(a) 27
(b) 11
(c) 21
(d) 45

Solution :

Number of picnics, in which 1 girl, 4 boys went = ${}^4C_4 * {}^3C_1 = 3$

Number of picnics in which 2 girls, 3 boys went = ${}^3C_2 * {}^4C_3 = 12$

Number of picnics in which 3 girls, 2 boys went = ${}^3C_3 * {}^4C_2 = 6$

So, number of dolls = 3*1 + 12*2 + 6*3 = 3 + 24 + 18 = 45

Option (d) is correct.

137. From a group of seven persons, seven committees are formed. Any two committees have exactly one member in common. Each person is in exactly three committees. Then
(a) At least one committee must have more than three members.
(b) Each committee must have exactly three members.
(c) Each committee must have more than three members.
(d) Nothing can be said about the sizes of the committees.

Solution :

First committee contains A_1, A_2, A_3

Second committee contains A_1, A_4, A_5

Third committee contains A_1, A_6, A_7

Fourth committee contains A_2, A_4,A_6

Fifth committee contains A_2, A_5, A_7

Sixth committee contains A_3, A_5, A_6

Seventh committee contains A_3, A_7, A_4

Option (b) is correct.

138. Three ladies have each brought a child for admission to a school. The head of the school wishes to interview the six people one by one, taking care that no child is interviewed before its mother. In how many different ways can the interviews be arranged?
(a) 6
(b) 36
(c) 72
(d) 90

Solution :

$M_1M_2C_1C_2M_3C_3$ – in this combination there are 2!*2! = 4 ways. (mothers can interchange among them in 2! Ways and children can in 2! Ways)

Now, there are 3 such combinations with change of position of mothers and children. So 4*3 = 12 ways.

Now, $M_1C_1M_2M_3C_2C_3$ – in this combination there are again 12 ways.

So, total 12 + 12 = 24 ways.

Now, take $M_1C_1M_2C_2M_3C_3$ – taken the mother child combination as unit then there are 3! = 6 ways.

Now, take the combination $M_1M_2M_3C_1C_2C_3$ – in this combination 3!*3! = 6*6 = 36 ways.

Take this combination $M_1M_2C_1M_3C_2C_3$

We can choose any 2 mother in ${}^3C_2$ ways, any 1 child from 2 children of the two interviewed mother in ${}^2C_1$ ways and the two mother can interchange among themselves in 2! Ways and the two child at the end can interchange among themselves in 2! Ways. So total ${}^3C_2*{}^2C_1*2!*2!$ = 24 ways.

Therefore, total number of ways = 24 + 6 + 36 + 24 = 90.

Option (d) is correct.

139. The coefficient of x^4 in the expansion of $(1 + x - 2x^2)^7$ is

(a) -81
(b) -91
(c) 81
(d) 91

Solution :

$(1 + x - 2x^2)^7 = {}^7C_0 + {}^7C_1(x - 2x^2) + {}^7C_2(x - 2x^2)^2 + {}^7C_3(x - 2x^2)^3 + {}^7C_4(x - 2x^2)^4 + ...$

$= \{{}^7C_2*2^2 + {}^7C_3*3*1^2*(-2) + {}^7C_4\}x^4 + ...$

So, coefficient of $x^4 = (7*6/2)*4 - (7*6*5/3*2)*3*2 + 7*6*5/3*2$

$= 84 - 35*6 + 35$

$= 84 - 35*5$

$= 84 - 175$

$= - 91$

Option (b) is correct.

140. The coefficient of $a^3b^4c^5$ in the expansion of $(bc + ca + ab)^6$ is
(a) 12!/(3!4!5!)
(b) ${}^6C_3*3!$
(c) 33
(d) $3*{}^6C_3$

Solution :

$(bc + ca + ab)^6 = {}^6C_0(bc)^6 + {}^6C_1(bc)^5(ca + ab) + {}^6C_2(bc)^4(ca + ab)^2 + {}^6C_3(bc)^3(ca + ab)^3 + ...$ *(power of a is more than 3)*

$= {}^6C_3(a^3b^3c^3)(c + b)^3 + ...$

$= {}^6C_3(a^3b^3c^3)(c^3 + 3c^2b + 3cb^2 + b^3) + ...$

$= 3*{}^6C_3a^3b^4c^5 + ...$

Option (d) is correct.

141. The coefficient of t^3 in the expansion of $\{(1 - t^6)/(1 - t)\}^3$ is
(a) 10
(b) 12
(c) 18
(d) 0

Solution :

$\{(1 - t^6)/(1 - t)\}^3 = (1 + t + t^2 + t^3 + t^4 + t^5)^3 = (1 + t + t^2 + t^3 + t^4 + t^5)(1 + t + t^2 + t^3 + t^4 + t^5)(1 + t + t^2 + t^3 + t^4 + t^5)$

$= \{1 + 2t + 3t^2 + 4t^3 + \ldots)(1 + t + t^2 + t^3 + t^4 + t^5)$

Coefficient of $t^3 = 1 + 2 + 3 + 4 = 10$.

Option (a) is correct.

142. The value of $({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \ldots.. - ({}^{2n}C_{2n-1})^2 + ({}^{2n}C_{2n})^2$ is
(a) ${}^{4n}C_{2n}$
(b) ${}^{2n}C_n$
(c) 0
(d) $(-1)^n * {}^{2n}C_n$

Solution :

The answer I am getting is $(-1)^n * {}^{4n}C_n$ (see solution of problem 130)

But, the answer is given as option (d).

143. There are 14 intermediate stations between Dusi and Visakhapatnam on the South Eastern Railway. A train is to be arranged from Dusi to Visakhapatnam so that it halts at exactly three intermediate stations, no two of which are consecutive. Then number of ways of doing this is
(a) ${}^{14}C_3 - ({}^{13}C_1)({}^{12}C_1) + {}^{12}C_1$
(b) 10*11*12/6
(c) ${}^{14}C_3 - {}^{14}C_2 - {}^{14}C_1$
(d) ${}^{14}C_3 - {}^{14}C_2 + {}^{14}C_1$

Solution :

Any three stations can be selected in $^{14}C_3$ ways.

Two stations consecutive and one station any station in $(^{13}C_1)(^{12}C_1)$ ways.

Now, 3 stations consecutive will appear twice so add $(^{12}C_1)$ i.e. 3 stations consecutive.

So, total number of ways = $^{14}C_3 - (^{13}C_1)(^{12}C_1) + {}^{12}C_1$

Option (a) is correct.

144. The letters of the word "MOTHER" are permutated, and all the permutations so formed are arranged in alphabetical order as in a dictionary. Then the number of permutations which come before the word "MOTHER" is
(a) 503
(b) 93
(c) $6!/2 - 1$
(d) 308

Solution :

MOTHER

Alphabetically, E, H, M, O, R, T

First letter will be E, there will be 5! = 120 words.

First letter H, there are 5! = 120 words.

Now, comes M.

Second letter E, there are 4! = 24 words.

Second letter H, there are 4! = 24 words.

Now, comes O.

Third letter E, there are 3! = 6 words.

Third letter H, there are 3! = 6 words.

Third letter R, there are 3! = 6 words.

Now comes T.

Fourth letter E, there are 2! = 2 words.

Now comes H.

Now comes fifth letter H and sixth letter R.

So, there are, 120 + 120 + 24 + 24 + 6 + 6 + 6 + 2 = 308 words before MOTHER.

Option (d) is correct.

145. All the letters of the word PESSIMISTIC are to be arranged so that no two S's occur together, no two I's occur together, and S, I do not occur together. The number of such arrangement is
(a) 2400
(b) 5480
(c) 48000
(d) 50400

Solution :

S – S – S – I – I – I in the five places between S's and I's 5 letters viz. P, E, M, T, C will be placed.

SSSIII will get permutated among themselves in 6!/(3!*3!) = 20 ways.

5 letters in the gaps can get permutated among themselves in 5! ways.

So, number of arrangement = 5!*6!/(3!*3!) = 2400

Option (a) is correct.

146. Suppose that x is irrational number and a, b, c, d are rational numbers such that (ax + b)/(cx + d) is rational. Then it follows that
(a) a = c = 0
(b) a = c and b = d
(c) a + b = c + d
(d) ad = bc

Solution :

Let, (ax + b)/(cx + d) = m where m is rational.

- ⇨ ax + b = mcx + dm
- ⇨ a = mc and b = dm (equating the rational and irrational coefficients from both sides)
- ⇨ a/c = b/d = m
- ⇨ ad = bc

Option (d) is correct.

147. Let p, q and s be integers such that $p^2 = sq^2$. Then it follows that
(a) p is an even number
(b) if s divides p, then s is a perfect square
(c) s divides p
(d) q^2 divides p

Solution :

Now, if s divides p, then p^2/s is a perfect square and p^2 already a perfect square.

- ⇨ s must be a perfect square.

Option (b) is correct.

148. The number of pairs of positive integers (x, y) where x and y are prime numbers and $x^2 - 2y^2 = 1$ is
(a) 0
(b) 1
(c) 2
(d) 8

Solution :

Now, $x^2 - 2y^2 = 1$

If x and y are both odd primes then dividing the equation by 4 we get, $1 - 2*1 \equiv 1 \pmod 4$

Which is impossible so x, y both cannot be odd.

x cannot be even to hold the equality.

Let y is even prime = 2.

Therefore, $x^2 - 2*2^2 = 1$

- ⇨ $x^2 = 9$
- ⇨ $x = 3$
- ⇨ Only one solution x = 3, y = 2.

Option (b) is correct.

149. A point P with coordinates (x, y) is said to be *good* if both x and y are positive integers. The number of good points on the curve xy = 27027 is
(a) 8
(b) 16
(c) 32
(d) 64

Solution :

Now, $27027 = 3^3*7*11*13$

So, number of factors = (3 + 1)(1 + 1)(1 + 1)(1 + 1) = 32

Option (c) is correct.

150. Let p be an odd prime number. Then the number of positive integers k with 1 < k < p, for which k^2 leaves a remainder of 1 when divided by p, is
(a) 2
(b) 1
(c) p – 1
(d) (p – 1)/2

Solution :

$k^2 \equiv 1 \pmod{p}$

- ⇨ $k^2 - 1 \equiv 0 \pmod p$
- ⇨ $(k - 1)(k + 1) \equiv 0 \pmod p$
- ⇨ $k - 1$ or $k + 1$ is divisible by p as p is prime.
- ⇨ Only one solution $k = p - 1$.

Option (b) is correct.

151. Let $n = 51! + 1$. Then the number of primes among $n + 1$, $n + 2$, …., $n + 50$ is
   (a) 0
   (b) 1
   (c) 2
   (d) More than 2.

Solution :

Now, $n + i = 51! + (i + 1) = (i + 1)(51*50*\ldots(i + 2)*i! + 1)$

This can be factorized in this way when $i = 1, 2, \ldots, 50$.

Therefore no prime numbers.

Option (a) is correct.

152. If three prime numbers, all greater than 3, are in A.P., then their common difference
   (a) must be divisible by 2 but not necessarily by 3
   (b) must be divisible by 3 but not necessarily by 2
   (c) must be divisible by both 2 and 3
   (d) need not be divisible by any of 2 and 3

Solution :

As primes great than 3 so all are odd. Hence the common difference must be divisible by 2.

Let the primes are p, p + d, p + 2d

Let $p \equiv 1 \pmod 3$

And $d \equiv 1 \pmod 3$

- ⇨ p + 2d is divisible by 3. Which is not possible as p + 2d is prime.
- ⇨ $d \equiv 2 \pmod 3$
- ⇨ p + d is divisible by 3. Which is not possible as p + d is prime.

Let, $p \equiv 2 \pmod 3$ and $d \equiv 1 \pmod 3$

Then p + d is divisible by 3 which is not possible as p + d is prime.

Let, $d \equiv 2 \pmod 3$

- ⇨ p + 2d is divisible by 3 which is not possible as p +2d is prime.
- ⇨ d is divisible by 3.

Option (c) is correct.

153. Let N be a positive integer not equal to 1. Then note that none of the numbers 2, 3, …., N is a divisor of (N! – 1). From this, we can conclude that
 (a) (N! – 1) is a prime number
 (b) At least one of the numbers N + 1, N + 2, …., N! – 2 is a divisor of (N! – 1).
 (c) The smallest number between N and N! which is a divisor of (N! – 1), is a prime number.
 (d) None of the foregoing statements is necessarily correct.

Solution :

(N! – 1) may be a prime or may not be a prime number. So, option (a) and (b) not necessarily correct.

The smallest number between N and N! which divides (N! – 1) must be a prime because if it is not prime then it has a factor of primes between 1 and N. But no primes between 1 and N divides (N! – 1).

Option (c) is correct.

154. The number 1000! = 1*2*3*….*1000 ends exactly with
 (a) 249 zeros
 (b) 250 zeros
 (c) 240 zeros
 (d) 200 zeros.

Solution :

Number of zeros at the end of 1000! = $[1000/5] + [1000/5^2] + [1000/5^3] + [1000/5^4]$ where [x] denotes the greatest integer less than or equal to x.

= 200 + 40 + 8 + 1 = 249

Option (a) is correct.

155. Let A denote the set of all prime numbers, B the set of all prime numbers and the number 4, and C the set of all prime numbers and their squares. Let D be the set of positive integers k, for which (k – 1)!/k is not an integer. Then
(a) D = A
(b) D = B
(c) D = C
(d) B subset of D subset of C.

Solution :

Now, (k – 1)!/k is not an integer if k is prime. If k can be factored then the factors will come from 1 to k. Therefore k will divide (k – 1)! except 4.

Therefore, D = B.

Option (b) is correct.

156. Let n be any integer. Then n(n + 1)(2n + 1)
(a) is a perfect square
(b) is an odd number
(c) is an integral multiple of 6
(d) does not necessarily have any foregoing properties.

Solution :

n(n + 1) is divisible by 2 as they are consecutive integers.

Now, let n ≡ 1 (mod 3)

Then 2n + 1 is divisible by 3.

Let n ≡ 2 (mod 3)

Then n + 1 is divisible by 3

Now, if n is divisible by 3, then we can say that n(n + 1)(2n + 1) is always divisible by 2*3 = 6

Option (c) is correct.

157. The numbers 12n + 1 and 30n + 2 are relatively prime for
   (a) any positive integer n
   (b) infinitely many, but not all, integers n
   (c) for finitely many integers n
   (d) none of the above.

Solution :

Let p divides both 12n + 1 and 30n + 2

⇨ 12n + 1 ≡ 0 (mod p)
⇨ 12n ≡ -1 (mod p)

Also, 30n + 2 ≡ 0 (mod p)

⇨ 60n + 4 ≡ 0 (mod p) (p is odd prime as 12n + 1 is odd)
⇨ 5*(12n) + 4 ≡ 0 (mod p)
⇨ 5(-1) + 4 ≡ 0 (mod p)
⇨ -1 ≡ 0 (mod p) which is impossible.

Option (a) is correct.

158. The expression 1 + (1/2)($^nC_1$) + (1/3)($^nC_2$) + …. + {1/(n + 1)}(^nC_n) equals
   (a) (2^{n+1} – 1)/(n + 1)
   (b) 2(2^n – 1)/(n + 1)
   (c) (2^n – 1)/n
   (d) 2(2^{n+1} – 1)/(n + 1)

Solution :

Now, $\{1/(r + 1)\}(^{n}C_{r}) = \{1/(r + 1)\}*n!/\{(n - r)r!\} = \{1/(n + 1)\}(n + 1)!/\{(n - r)!(r + 1)!\} = \{1/(n + 1)\}(^{n+1}C_{r+1})$

So, the expression becomes, $\{1/(n + 1)\}[^{n+1}C_{1} + {}^{n+1}C_{2} + \ldots.. + {}^{n+1}C_{n+1}] = (2^{n+1} - 1)/(n + 1)$

Option (a) is correct.

159. The value of $^{30}C_{1}/2 + {}^{30}C_{3}/4 + {}^{30}C_{5}/6 + \ldots.. + {}^{30}C_{29}/30$ is
   (a) $2^{31}/30$
   (b) $2^{30}/31$
   (c) $(2^{31} - 1)/31$
   (d) $(2^{30} - 1)/31$

Solution :

Now, $^{30}C_{2r-1}/2r = 30!/\{(30 - 2r + 1)!(2r - 1)!2r\} = (1/31)[31!/\{(31 - 2r)!(2r)!\}] = {}^{31}C_{2r}/31$

Now, the expression becomes, $(1/31)[^{31}C_{2} + {}^{31}C_{4} + \ldots. + {}^{31}C_{30}] = (2^{30} - {}^{31}C_{0})/31 = (2^{30} - 1)/31$.

Option (c) is correct.

160. The value of $[\sum(^{k}C_{i})(^{M-k}C_{100-i})\{(k - i)/(M - 100)\}]/^{M}C_{100}$, where $M - k > 100$, $k > 100$ and $^{m}C_{n} = m!/\{(m - n)!n!\}$ equals (summation running from $i = 0$ to $i = 100$)
   (a) k/M
   (b) M/k
   (c) k/M^{2}
   (d) M/k^{2}

Solution :

Option (a) is correct.

161. The remainder obtained when 1! + 2! + …. + 95! Is divided by 15 is
   (a) 14

(b) 3
(c) 1
(d) None of the foregoing numbers.

Solution :

From 5! all the numbers are divisible by 15.

So, it is required to find the remainder when 1! + 2! + 3! + 4! is divided by 15

1! + 2! + 3! + 4! = 1 + 2 + 6 + 24 = 33 ≡ 3 (mod 15)

Option (b) is correct.

162. Let x_1, x_2, ….., x_{50} be fifty integers such that the sum of any six of them is 24. Then
(a) The largest of x_i equals 6
(b) The smallest of x_i equals 3
(c) $x_{16} = x_{34}$
(d) none of the foregoing statements is correct.

Solution :

All x_i's are equal and = 4. (See solution of problem 82)

Thus, option (c) is correct.

163. Let x_1, x_2, ….., x_{50} be fifty nonzero numbers such that $x_i + x_{i+1} = k$ for all i, $1 \leq i \leq 49$. If $x_{14} = a$, $x_{27} = b$ then $x_{20} + x_{37}$ equals
(a) 2(a + b) – k
(b) k + a
(c) k + b
(d) none of the foregoing expressions.

Solution :

Now, $x_i + x_{i+1} = k$ and $x_{i+1} + x_{i+2} = k$

⇨ $x_i = x_{i+2}$
⇨ $x_{14} = x_{16} = x_{18} = x_{20} = a$

And, $x_{27} = x_{29} = x_{31} = x_{33} = x_{35} = x_{37} = b$

Therefore, $x_{20} + x_{37} = a + b$

Now, $x_{14} + x_{15} = k$

$x_{15} = x_{17} = x_{19} = x_{21} = x_{23} = x_{25} = x_{27} = b$

⇨ $a + b = k$
⇨ $x_{20} + x_{37} = 2(a + b) - k$

Option (a) is correct.

164. Let S be the set of all numbers of the form $4^n - 3n - 1$, where n = 1, 2, 3, …. Let T be the set of all numbers of the form $9(n - 1)$, where n = 1, 2, 3, ….. Only one of the following statements is correct. Which one is it?
(a) Each number in S is also in T
(b) Each number in T is also in S
(c) Every number in S in T and every number in T is in S.
(d) There are numbers in S which are not in T and there are numbers in T which are not in S.

Solution :

Now, $4^n = (1 + 3)^n = 1 + 3n + {}^nC_2(3)^2 + ….. + (3)^n$

⇨ $4^n - 3n - 1 = 9({}^nC_2 + …. + 3^{n-2})$

Clearly, option (a) is correct.

165. The number of four-digit numbers greater than 5000 that can be formed out of the digits 3, 4, 5, 6 and 7, no digit being repeated, is
(a) 52
(b) 61
(c) 72
(d) 80

Solution :

First digit can be 5, 6 or 7.

If first digit is 5 number of such numbers = $^4C_3*3!$ = 24

Similarly, if first digit is 6 or 7 in each case number of such numbers = 24

Therefore, total number of such numbers = 24*3 = 72

Option (c) is correct.

166. The number of positive integers of 5 digits such that each digit is 1, 2 or 3, and all three of the digits appear at least once, is
(a) 243
(b) 150
(c) 147
(d) 193

Solution :

Number of combinations = $^{5-1}C_{3-1} = {}^4C_2$ = 6.

Three 1, one 2, one 3, number of numbers = 5!/3! = 20

Two 1, one 2, two 3, number of numbers = 5!/(2!*2!) = 30

Two 1, two 2, one 3, number of such numbers = 5!/(2!*2!) = 30

One 1, one 2, three 3, number of such numbers = 5!/3! = 20

One 1, two 2, two 3, number of such numbers = 5!/(2!*2!) = 30

One 1, three 2, one 3, number of such numbers = 5!/(3!) = 20

Therefore, total number of such numbers = 20 + 30 + 30 + 20 + 30 + 20 = 150

Option (b) is correct.

167. In a chess tournament, each of the 5 players plays against every other player. No game results in a draw and the winner of each game gets one point and loser gets zero. Then which one of the following sequences *cannot* represent the scores of the five players?

(a) 3, 3, 2, 1, 1
(b) 3, 2, 2, 2, 1
(c) 2, 2, 2, 2, 2
(d) 4, 4, 1, 1, 0

Solution :

As in option (d) we see that first two players have won all the games.

It cannot be true because the game in between them one must lose and one must win.

So, it is not possible.

Option (d) is correct.

168. Ten (10) persons numbered 1, 2, …, 10 play a chess tournament, each player playing against every other player exactly one game. Assume that each game results in a win for one of the players (that is, there is no draw). Let $w_1, w_2, ..., w_{10}$ be the number of games won by players 1, 2, …, 10 respectively. Also, let $l_1, l_2, ..., l_{10}$ be the number of games lost by players 1, 2, …, 10 respectively. Then
(a) $w_1^2 + w_2^2 + ... + w_{10}^2 = 81 - (l_1^2 + l_2^2 + ... + l_{10}^2)$
(b) $w_1^2 + w_2^2 + ... + w_{10}^2 = 81 + (l_1^2 + l_2^2 + ... + l_{10}^2)$
(c) $w_1^2 + w_2^2 + ... + w_{10}^2 = l_1^2 + l_2^2 + ... + l_{10}^2$
(d) none of the foregoing equalities is necessarily true.

Solution :

Now, $w_1 + w_2 + ... + w_{10} = l_1 + l_2 + ... + l_{10}$ = number of games.

And, $w_i + l_i$ = constant = one player playing number of games for i = 1, 2, …, 10

⇨ $(w_1 - l_1) + (w_2 - l_2) + ... + (w_{10} - l_{10}) = 0$
⇨ $(w_1 + l_1)(w_1 - l_1) + (w_2 + l_2)(w_2 - l_2) + ... + (w_{10} + l_{10})(w_{10} - l_{10}) = 0$
⇨ $w_1^2 - l_1^2 + w_2^2 - l_2^2 + ... + w_{10}^2 - l_{10}^2 = 0$
⇨ $w_1^2 + w_2^2 + ... + w_{10}^2 = l_1^2 + l_2^2 + ... + l_{10}^2$

Option (c) is correct.

169. A game consisting of 10 rounds is played among three players A, B and C as follows : Two players play in each round and the loser is replaced by the third player in the next round. If the only rounds when A played against B are the first, fourth and tenth rounds, the number of games won by C

(a) is 5
(b) is 6
(c) is 7
(d) cannot be determined from the above information.

Solution :

First round between A and B.

One of them lost and C joined in 2^{nd} round.

C won as A and B did not play in 3^{rd} round. (**So win 1)**

C lost in 3^{rd} round as A and B played fourth round.

C joined in 5^{th} round and won as A and B did not play 6^{th} round (**So win 1**)

6^{th}, 7^{th}, 8^{th} round won by C (**So 3 wins**), in 9^{th} round C lost as A and B played 10^{th} round.

So, number of games won by C = 5.

Option (a) is correct.

170. An nxn chess board is a square of side n units which has been sub-divided into n^2 unit squares by equally-spaced straight lines parallel to the sides. The total number of squares of all sizes on the nxn board is

(a) $n(n + 1)/2$
(b) $1^2 + 2^2 + …. + n^2$
(c) $2*1 + 3*2 + 4*3 + …. + n*(n - 1)$
(d) Given by none of the foregoing expressions.

Solution :

If we take all the lines then there are n^2 squares.

If we take 2 unit squares together then there are $(n - 1)^2$ squares.

If we take 3 unit squares together then there are $(n - 2)^2$ squares.

…

…

If we take all the n unit squares then there are 1^2 unit squares.

Therefore, total number of squares = $1^2 + 2^2 + \ldots. + n^2$.

Option (b) is correct.

171. Given any five points in the square $I^2 = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$, only one of the following statements is true. Which one is it?
(a) The five points lie on a circle.
(b) At least one square can be formed using four of the five points.
(c) At least three of the five points are collinear.
(d) There are at least two points such that distance between them does not exceed $1/\sqrt{2}$.

Solution :

The farthest four points may be at the four corners of the square.

So, one of them must be nearer to some point wherever they are located, the distance less than or equal to half of the diagonal. If the fifth point is on the diagonal then the distance is $1/\sqrt{2}$, otherwise it is less.

Option (d) is correct.

172. The quantities l, c, h and m are measured in the units mentioned against each l : centimetre; c : centimetre per second; h : ergs*second; mc^2 : ergs. Of the expressions $\alpha = (ch/ml)^{1/2}$; $\beta = (mc/hl)^3$; $\gamma = h/mcl$, which ones are pure numbers, that is, do not involve any unit?
(a) Only α
(b) Only β
(c) Only γ
(d) None

Solution :

Clearly, option (c) is correct.

173. The number of distinct rearrangements of the letters of the word "MULTIPLE" that can be made preserving the order in which the vowels (U, I, E) occur and not containing the original arrangement is
(a) 6719
(b) 3359
(c) 6720
(d) 3214

Solution :

U, I, E can get permutated among themselves in 3! = 6 ways.

Out of them only one permutation is required.

Therefore in this permutation number of arrangements = total number of arrangement/6 = 8!/{(2!)6} = 3360.

Excluding the original arrangement 3360 – 1 = 3359

Option (b) is correct.

174. The number of terms in the expansion of $(x + y + z + w)^{10}$ is
(a) ${}^{10}C_4$
(b) ${}^{13}C_3$
(c) ${}^{14}C_4$
(d) 11^4

Solution :

Any term = (coefficient)$x^r y^s z^t w^u$ where r + s + t + w = 10 and r, s, t, w are non-negative integers.

Number of solution of this equation is ${}^{n + r - 1}C_{r - 1}$ where r = number of variables and n is the sum.

Here n = 10, r = 4. (See number theory note for proof)

Therefore, number of terms = ${}^{10 + 4 - 1}C_{4 - 1} = {}^{13}C_3$.

Option (b) is correct.

175. The number of ways in which three non-negative integers n_1, n_2, n_3 can be chosen such that $n_1 + n_2 + n_3 = 10$ is
(a) 66
(b) 55
(c) 10^3
(d) 10!/(3!2!1!)

Solution :

Number of solution of this equation is ${}^{10+3-1}C_{3-1} = {}^{12}C_2$ (see number theory note for proof) = 12*11/2 = 66.

Option (a) is correct.

176. In an examination, the score in each four languages – Bengali, Hindi, Urdu and Telegu – can be integers between 0 and 10. Then the number of ways in which a student can secure a total score of 21 is
(a) 880
(b) 760
(c) 450
(d) 1360

Solution :

Let B be the score in Bengali, H be the score in Hindi, U be the score in Urdu and T be the score in Telegu.

Therefore, B + H + U + T = 21 and B, H, U, T are non-negative integers and less than or equal to 10.

Let B = 0, H = 1, U = 10, T = 10 (one solution)

B = 0, H = 2, two solutions

B = 0, H = 3, three solutions.

…

…

B = 0, H = 10, ten solutions.

Number of solutions for B = 0 is 1 + 2 + … + 10 = 10*11/2 = 55

B = 1, H = 0, one solution

B = 1, H = 1, two solutions

…

..

B = 1, H = 10, eleven solutions

Number of solutions for B = 1 is 1 + 2 + … + 11 = 11*12/2 = 66

B = 2, H = 0, two solutions.

..

..

B = 2, H = 8, ten solutions.

B = 2, H = 9, eleven solutions.

B = 2, H = 10, ten solutions.

Number of solutions for B = 2 is 2 + 3 + … + 10 + 11 + 10 = 75.

B = 3, H = 0, three solutions.

..

..

B = 3, H = 7, ten solutions.

B = 3, H = 8, eleven solutions.

B = 3, H = 9, ten solutions.

B = 3, H = 10, nine solutions.

Number of solutions for B = 3 is 3 + 4 + … + 10 + 11 + 10 + 9 = 82

Number of solutions for B = 4 is 4 + 5 + … + 10 + 11 + 10 + 9 + 8 = 87

Number of solutions for B = 5 is 5 + 6 + … + 10 + 11 + 10 + … + 7 = 90

Number of solutions for B = 6 is, 6 + 7 + .. + 10 + 11 + 10 + … + 6 = 91

Number of solutions for B = 7 is, 7 + 8 + 9 + 10 + 11 + 10 + … + 5 = 90

Number of solutions for B = 8 is, 8 + 9 + 10 + 11 + 10 + … + 4 = 87

Number of solutions for B = 9 is, 9 + 10 + 11 + 9 + …. + 3 = 82

Number of solutions for B = 10 is 10 + 11 + 10 + 9 + … + 2 = 75

Total number of solutions = 55 + 2(75 + 82 + 87 + 90) + 91 + 66 = 880

Option (a) is correct.

177. The number of ordered pairs (x, y) of positive integers such that x + y = 90 and their greatest common divisor is 6 equals

(a) 15
(b) 14
(c) 8
(d) 10

Solution :

Let $x = 6x_1$ and $y = 6y_1$

⇨ $6(x_1 + y_1) = 90$
⇨ $x_1 + y_1 = 15$ ($\gcd(x_1, y_1) = 1$)
⇨ (1, 14); (2, 13); (4, 11); (7, 8)

So there are 4*2 = 8 pairs.

Option (c) is correct.

178. How many pairs of positive integers (m, n) are there satisfying $m^3 - n^3 = 21$?

(a) Exactly one
(b) None
(c) Exactly three
(d) Infinitely many

Solution :

Now, $m^3 - n^3 = 21$

⇨ $(m - n)(m^2 + mn + n^2) = 3*7$

So, two case can be possible, $m - n = 3$, $m^2 + mn + n^2 = 7$ and $m - n = 1$, $m^2 + mn + n^2 = 21$

First case, $(3 + n)^2 + (3 + n)n + n^2 = 7$

⇨ $3n^2 + 9n + 2 = 0$
⇨ $n = \{-9 \pm \sqrt{(9^2 - 4*3*2)}\}/6$ = not integer solution.

So this case is not possible.

Second case, $(n + 1)^2 + (n + 1)n + n^2 = 21$

⇨ $3n^2 + 3n - 20 = 0$
⇨ $n = \{-3 \pm \sqrt{(9 + 4*3*20)}\}/6$ = not integer solution.
⇨ Option (b) is correct.

179. The number of ways in which three distinct numbers in A.P. can be selected from 1, 2, …, 24 is
(a) 144
(b) 276
(c) 572
(d) 132

Solution :

With 1 common difference we can select A.P.'s = 22

With 2 common difference we can select A.P.'s = 2*10 = 20

With 3 common difference we can select A.P.'s = 3*6 = 18

With 4 common difference we can select A.P.'s = 4*4 = 16

With 5 common difference we can select A.P.'s = 3*4 + 2 = 14

With 6 common difference we can select A.P.'s = 2*6 = 12

With 7 common difference we can select A.P.'s = 3*2 + 4*1 = 10

With 8 common difference we can select A.P.'s = 1*8 = 8

With 9 common difference we can select A.P.'s = (1, 10, 19); (2, 11, 20); (3, 12, 21); (4, 13, 22); (5, 14, 23); (6, 15, 24) = 6

With 10 common difference we can select A.P.'s = (1, 11, 21); (2, 12, 22); (3, 13, 23); (4, 14, 24) = 4

With 11 common difference we can select A.P.'s = (1, 12, 23); (2, 13, 24) = 2

So, total number of A.P.'s = 22 + 20 + … + 2 = (11/2){2*22 + (11 − 1)(-2)} = 11(22 – 10) = 11*12 = 132

Option (d) is correct.

[In general if we need to select A.P.'s from 1, 2, …, n then with common difference 1 we can select n – 2 A.P.'s , with common difference 2 the largest A.P. with first term will be n – 4, so n – 4 A.P.'s, with common difference 3 the largest A.P. with first term will be n – 6, so n – 6 A.P.'s and so on. So, number of A.P.'s will be (n – 2) + (n – 4) + ….. + up to 2 or 1 according to n is even or odd.]

180. The number of ways you can invite 3 of your friends on 5 consecutive days, exactly one friend a day, such that no friend is invited on more than two days is
(a) 90
(b) 60
(c) 30
(d) 10

Solution :

Let the friends are A, B, C.

We need to distribute A, B, C in 5 places such that A, B, C occurs at least once.

Two A, two B, one C = 5!/(2!2!) = 30

Two A, one B, two C = 5!/(2!2!) = 30

One A, two B, two C = 5!/(2!2!) = 30

Total number of ways = 30 + 30 + 30 = 90

Option (a) is correct.

181. Consider three boxes, each containing 10 balls labeled 1, 2, …, 10. Suppose one ball is drawn from each of the boxes. Denote by n_i, the label of the ball drawn from the i-th box, i = 1, 2, 3. Then the number of ways in which the balls can be chosen such that $n_1 < n_2 < n_3$, is
(a) 120
(b) 130
(c) 150
(d) 160

Solution :

If $n_1 = 1$, $n_2 = 2$, n_3 can be 8

If $n_1 = 1$, $n_2 = 3$, n_3 can be 7

So, if $n_1 = 1$ then the possible ways = 8 + 7 + … + 1 = 8*9/2 = 36

If, $n_1 = 2$, $n_2 = 3$, n_3 can be 7

So, if $n_1 = 2$, then the possible ways = 7 + 6 + … + 1 = 7*8/2 = 28

So, if $n_1 = 3$, then the possible ways = 6 + 5 + … + 1 = 6*7/2 = 21

So, these are the triangular numbers.

Therefore, total possible ways = ∑n(n + 1)/2 (summation running from n = 1 to n = 8) = $(1/2)\sum n^2 + (1/2)\sum n$ = (1/2)*8*9*17/6 + (1/2)8*9/2 = 102 + 18 = 120

Option (a) is correct.

182. The number of sequences of length five with 0 and 1 as terms which contain at least two consecutive 0's is
(a) $4*2^3$
(b) ${}^5C_2$
(c) 20
(d) 19

Solution :

Two consecutive 0'sin left means third is 1, fourth and fifth can be put in 2*2 = 4 ways.

Two consecutive 0's in the middle then both side is 1 and another one can be put in 2 ways in both the side. Therefore, 2*2 = 4 sequences.

Two consecutive 0's in the right means third from right is 1. Fourth and fifth can be put in 2*2 = 4 ways.

So, for two consecutive 0's number of sequences = 4 + 4 + 4 = 12

Three consecutive zeros in left means fourth is 1 and fifth can be put in 2 ways.

Three consecutive 0's in middle means 1 way.

Three consecutive 0's in right means 2 ways.

For three consecutive 0's number of sequences = 2 + 1 + 2 = 5.

Four consecutive zeros = 1 + 1 = 2 sequences.

Five consecutive 0's – no sequence as there is no 1.

Total number of sequence = 12 + 5 + 2 = 19.

Option (d) is correct.

183. There are 7 identical white balls and 3 identical black balls. The number of distinguishable arrangements in a row of all the balls, so that no two black balls are adjacent, is
(a) 120
(b) 89(8!)
(c) 56
(d) $42*5^4$

Solution :

Total number of arrangements = 10!/(7!3!) = 10*9*8/6 = 120

Now, take the black balls as unit. So there are 8 units.

Therefore, total number of arrangements = 8!/7! = 8

Now, take 2 black balls as unit. There are 9 units.

Total number of arrangements = 9!/7! = 72

So, number of arrangements in which at least 2 black balls will come together = 72 – 8 = 64

So, number of required arrangements = 120 – 64 = 56.

Option (c) is correct.

184. In a multiple-choice test there are 6 questions. Four alternative answers are given for each question, of which only one answer is correct. If a candidate answers all the questions by choosing one answer for each question, then the number of ways to get 4 correct answers is
(a) $4^6 - 4^2$
(b) 135
(c) 9
(d) 120

Solution :

4 questions can be chosen from 6 questions in $^6C_4$ ways = 15 ways.

Now, rest two questions can be answered wrong in 3 ways each (because 1 is correct).

So number of ways of doing this = 3*3 = 9

Therefore, total number of ways = 15*9 = 135.

Option (b) is correct.

185. In a multiple-choice test there are 8 questions. Each question has 4 alternatives, of which only one is correct.. If a candidate answers all the questions by choosing one alternative for each, the number of ways of doing it so that exactly 4 answers are correct is
(a) 70
(b) 2835
(c) 5670
(d) None of the foregoing numbers.

Solution :

Same question as previous one. Total number of ways = $^8C_4*3^4$ = 5670.

Option (c) is correct.

186. Among the 8! Permutations of the digits 1, 2, 3, …, 8, consider those arrangements which have the following property : if you take any five consecutive positions, the product of the digits in those positions is divisible by 5. The number of such arrangements is
(a) 7!
(b) 2*7!
(c) 8*7!
(d) $4(^7C_4)5!3!4!$

Solution :

So, 5 can be in 4^{th} or 5^{th} place.

In 4^{th} place total number of arrangements = 7!, same goes for 5^{th} place.

Therefore, total number of required permutations = 2*7!

Option (b) is correct.

187. A closet has 5 pairs of shoes. The number of ways in which 4 shoes can be chosen from it so that there will be no complete pair is
(a) 80
(b) 160
(c) 200
(d) None of the foregoing numbers.

Solution :

4 pairs can be chosen from 5 pairs in $^5C_4$ = 5 ways.

Out of these 4 pairs 1 shoe can to be chosen from each pair in $^2C_1*^2C_1*^2C_1*^2C_1$ = 16 ways.

Therefore, total number of ways = 5*16 = 80.

Option (a) is correct.

188. The number of ways in which 4 distinct balls can be put into 4 boxes labeled a, b, c, d so that exactly one box remains empty is
(a) 232
(b) 196
(c) 192
(d) 144

Solution :

Let box d is empty.

Number of ways in which we can put 4 distinct balls into 3 boxes where each box gets at least one ball = $3^4 - {}^3C_1*2^4 + {}^3C_2*1^4 - {}^3C_3*0^4$ (for this formula please see my number theory book)

= 81 – 48 + 3 – 0

= 36

Now, for four boxes there will be 36*4 = 144 ways.

Option (d) is correct.

189. The number of permutations of the letters a, b, c, d such that b does not follow a, and c does not follow b, and d does not follow c, is
(a) 12
(b) 11
(c) 14
(d) 13

Solution :

acbd, adcb, badc, bdac, bdca, cadb, cbad, cbda, dacb, dbac, dcba = 11.

Option (b) is correct.

190. The number of ways of seating three gentlemen and three ladies in a row, such that each gentlemen is adjacent to at least one lady, is
(a) 360

(b) 72
(c) 720
(d) None of the foregoing numbers.

Solution :

Three gentlemen together can seat in 4!*3! = 144 ways.

Now, two gentlemen at left end. Then a lady. Number of arrangement = $^3C_2*^3C_1*3!*2!$ = 108 ways. (2 gentlemen at left end can be chosen from 3 gentlemen in $^3C_2$ ways, they can get permuted among them in 2! ways. One lady from 3 ladies can be chosen in $^3C_1$ ways and the rest one lady and two gentlemen can get permutated among themselves in 3! ways)

Similarly, two gentlemen at right end and then one lady, number of arrangement = 108.

Total number of cancelled arrangement = 144 + 108 + 108 = 360.

Six gentlemen and six ladies can seat in 6! = 720 ways.

Therefore, number of arrangements = 720 – 360 = 360.

191. The number of maps f from the set {1, 2, 3} into the set {1, 2, 3, 4, 5} such that f(i) ≤ f(j), whenever i < j, is
(a) 30
(b) 35
(c) 50
(d) 60

Solution :

1 - > 1, 2 - > 1, 3 can map to 5 numbers.

1 - > 1, 2 - > 2, 3 can map into 4 numbers.

So, for 1 - > 1 number of mapping = 5 + 4 + 3 + 2 + 1 = 15

1 - > 2, 2 - > 2, 3 can map to 4 numbers.

So, number of mapping = 4 + 3 + 2 + 1 = 10 numbers.

In general number of mapping = $\sum n(n + 1)/2$ (summation running from n = 1 to n = 5) = $(1/2)\sum n^2 + (1/2)\sum n = (1/2)*5*6*11/6 + (1/2)*5*6/2 = 55/2 + 15/2 = 35$.

Option (b) is correct.

192. For each integer i, $1 \le i \le 100$; ε_i be either +1 or -1. Assume that $\varepsilon_1 = +1$ and $\varepsilon_{100} = -1$. Say that a sign change occurs at $i \ge 2$ if ε_i, ε_{i-1} are of opposite sign. Then the total number of sign changes
(a) is odd
(b) is even
(c) is at most 50
(d) can have 49 distinct values

Solution :

Now, $\varepsilon_1 = +1$, now it will continue till it gets a -1, if any ε is -1 then the next ε will be +1 again because sign change will occur. So, if it gets again -1 then +1 will occur. So, + to + sign change is even, as + to – and then – to +, now, ε_{100} is -1. So, number of sign changes must be odd.

Option (a) is correct.

193. Let S = {1, 2, …, n}. The number of possible pairs of the form (A, B) with A subset of B for subsets A and B of S is
(a) 2^n
(b) 3^n
(c) $\sum({}^nC_k)({}^nC_{n-k})$ (summation running from k = 0 to k = n)
(d) n!

Solution :

We can choose r elements for B from n elements in ${}^nC_r$ ways. Now, for r elements number of subsets = 2^r.

Therefore, number of pairs = ${}^nC_r*2^r$.

Therefore, total number of pairs = $\sum({}^nC_r)*2^r$ (r running from 0 to n) = 3^n.

Option (b) is correct.

194. There are 4 pairs of shoes of different sizes. Each of the shoes can be colored with one of the four colors : black, brown, white and red. In how many ways can one color the shoes so that in at least three pairs, the left and the right shoes do not have the same color?
(a) 12^4
(b) $28*12^3$
(c) $16*12^3$
(d) $4*12^3$

Solution :

3 pairs different color + 4 pairs different color. (at least 3 pairs different color)

3 pairs different color :

We can choose any 3 pairs of shoes out of 4 pairs in $^4C_3$ = 4 ways. We can paint first shoe in $^4C_1$ and second shoe in $^3C_1$ i.e. one pair in $^4C_1*^3C_1$ =12 ways. So, three pairs in 12^3 ways. And the last have same color and we can choose any one color from 4 colors in $^4C_1$ = 4 ways.

So, total number of ways = $4*12^3*4 = 16*12^3$

4 pairs different color :

In this case clearly, total number of ways = $12*12^3$

So, at least 3 pair of shoes are of different color the number of ways of painting = $12*12^3 + 16*12^3 = 28*12^3$

Option (b) is correct.

195. Let S = {1, 2, …., 100}. The number of nonempty subsets A of S such that the product of elements in A is even is
(a) $2^{50}(2^{50} – 1)$
(b) $2^{100} – 1$
(c) $2^{50} – 1$
(d) None of these numbers.

Solution :

We can select at least one even numbers in $^{50}C_1 + {}^{50}C_2 + {}^{50}C_3 + \ldots + {}^{50}C_{50} = 2^{50} - 1$

We can select any number of odd numbers in $^{50}C_0 + {}^{50}C_1 + \ldots + {}^{50}C_{50} = 2^{50}$

So, total number of subsets = $2^{50}(2^{50} - 1)$

Option (a) is correct.

196. The number of functions f from {1, 2, …., 20} onto {1, 2, …, 20} such that f(k) is a multiple of 3 whenever k is a multiple of 4 is
(a) 5!*6!*9!
(b) 5^6*15!
(c) 6^5*14!
(d) 15!*6!

Solution :

{4, 8, 12, 16, 20} - > {3, 6, 9, 12, 15, 18}

We can select any 5 numbers from 6 numbers of the later set in $^6C_5$ ways and they will get permutated in 5! ways. So, in this case number of permutation = $^6C_5$*5! = 6!

Rest 15 numbers will map to 15 numbers in 15! ways.

Therefore, total number of functions = 6!*15!

Option (d) is correct.

197. Let X = {a_1, a_2, …., a_7} be a set of seven elements and Y = {b_1, b_2, b_3} a set of three elements. The number of functions f from X to Y such that (i) f is onto and (ii) there are exactly three elements x in X such that f(x) = b_1, is
(a) 490
(b) 558
(c) 560
(d) 1680

Solution :

We can select any 3 elements from 7 elements to be mapped to b_1 in $^7C_3$ ways.

Now, rest 4 elements needs to be distributed among b_2 and b_3 so that b_2 and b_3 gets at least one element.

Now, we can distribute the 4 elements in b_2 and b_3 in $2*2*2*2 = 2^4$ ways out of which in 2 ways one is for b_1 gets none and b_2 gets every elements and other is b_2 gets none and b_1 gets all elements. So, total number of ways $= 2^4 - 2 = 14$.

Therefore, total mapping = $^7C_3*14 = 490$.

Option (a) is correct.

198. Consider the quadratic equation of the form $x^2 + bx + c = 0$. The number of such equations that have real roots and coefficients b and c from the set {1, 2, 3, 4, 5} (b and c may be equal) is
(a) 18
(b) 15
(c) 12
(d) None of the foregoing quantities.

Solution :

Now, $b^2 > 4c$

b cannot be equal to 1.

If b = 2, c = 1

If b = 3, c = 1, 2

If b = 4, c = 1, 2, 3, 4

If b = 5, c = 1, 2, 3, 4, 5

Total number of equations = 1 + 2 + 4 + 5 = 12

Option (c) is correct.

199. Let A_1, A_2, A_3 be three points on a straight line. Let B_1, B_2, B_3, B_4, B_5 be five points on a straight line parallel to the first line. Each of

the three points on the first line is joined by a straight line to each of the five points on the second straight line. Further, no three or more of these joining lines meet at a point except possibly at the A's or the B's. Then the number of points of intersections of the joining lines lying between the two given straight lines is

(a) 30
(b) 25
(c) 35
(d) 20

Solution :

We first calculate number of intersection points when a straight line from A_1 meets other straight lines from A_2, A_3.

A_2, A_3, -> B_1 – 2 +2 + 2 + 2 = 8 points (A_1 – B_2, B_3, B_4, B_5)

A_2, A_3 -> B_2 – 2 + 2 + 2 = 6 points (A_1 – B_3, B_4, B_5)

A_2, A_3 -> B_3 – 2 + 2 = 4 points (A_1 – B_4, B_5)

A_2, A_3 -> B_4 – 2 points (A_1 – B_5)

A_2 -> B_5 – 4 points (A_3 – B_4, B_3, B_2, B_1)

A_2 -> B_4 – 3 points (A_3 – B_3, B_2, B_1)

A_2 -> B_3 – 2 points (A_3 – B_2, B_1)

A_2 -> B_2 – 1 point (A_3 – B_1)

So, total = 8 + 6 + 4 + 2 + 4 + 3 + 2 + 1 = 30 points.

Option (a) is correct.

200. There are 11 points on a plane with 5 lying on one straight line and another 5 lying on second straight line which is parallel to the first line. The remaining point is not collinear with any two of the previous 10 points. The number of triangles that can be formed with vertices chosen from these 11 points is

(a) 85
(b) 105
(c) 125
(d) 145

Solution :

We can choose 1 point from first straight line, 1 point from second straight line and 1 the single point. Number of ways of doing this $^5C_1*^5C_1*1 = 25$.

We can choose 2 points from first straight line and 1 single point in $^5C_2*1 = 10$, same goes for the second straight line, so number of triangles = 10*2 = 20.

We can choose 1 point from first straight line and 2 points from second straight line and vice versa. Number of triangles = $2*^5C_2*^5C_1 = 2*10*5 = 100$.

Total number of triangles = 25 + 20 + 100 = 145.

Option (d) is correct.

201. Let $a_1, a_2, a_3, ….$ be a sequence of real numbers such that lim $a_n = \infty$ as n -> ∞. For any real number x, define an integer-valued function f(x) as the smallest positive integer n for which $a_n \geq x$. Then for any integer $n \geq 1$ and any real number x,
(a) $f(a_n) \leq n$ and $a_{f(x)} \geq x$
(b) $f(a_n) \leq n$ and $a_{f(x)} \leq x$
(c) $f(a_n) \geq n$ and $a_{f(x)} \geq x$
(d) $f(a_n) \geq n$ and $a_{f(x)} \leq x$

Solution :

Option (a) is correct.

202. There are 25 points in a plane, of which 10 are on the same line. Of the rest, no three number are collinear and no two are collinear with any of the first ten points. The number of different straight lines that can be formed joining these points is
(a) 256
(b) 106
(c) 255
(d) 105

Solution :

From 15 non-collinear points number of straight lines can be formed = $^{15}C_2$ = 105

Taking 1 point from 10 collinear points and taking 1 point from 15 non-collinear points number of straight lines can be formed = $^{10}C_1 * ^{15}C_1$ = 150.

One straight line joining the ten points.

Therefore total number of straight lines = 105 + 150 + 1 = 256.

Option (a) is correct.

203. If $f(x) = \sin(\log_{10}x)$ and $h(x) = \cos(\log_{10}x)$, then $f(x)f(y) - (1/2)[h(x/y) - h(xy)]$ equals
(a) $\sin[\log_{10}(xy)]$
(b) $\cos[\log_{10}(xy)]$
(c) $\sin[\log_{10}(x/y)]$
(d) none of the foregoing expressions.

Solution :

$f(x)f(y) - (1/2)[h(x/y) - h(xy)] = \sin(\log_{10}x)\sin(\log_{10}y) - (1/2)[\cos\log_{10}(x/y) - \cos\log_{10}(xy)]$

$= (1/2)[2\sin(\log_{10}x)\sin(\log_{10}y)] - (1/2)[\cos\log_{10}(x/y) - \cos\log_{10}(xy)]$

$= (1/2)[\cos(\log_{10}x - \log_{10}y) - \cos(\log_{10}x + \log_{10}y) - \cos\log_{10}(x/y) + \cos\log_{10}(xy)]$

$= (1/2)[\cos\log_{10}(x/y) - \cos\log_{10}(xy) - \cos\log_{10}(x/y) + \cos\log_{10}(xy)]$

$= 0$

Option (d) is correct.

204. The value of $\log_5(125)(625)/25$ is
(a) 725
(b) 6
(c) 3125
(d) 5

Solution :

$\log_5(125)(625)/25 = \log_5 5*5^4 = \log_5 5^5 = 5\log_5 5 = 5$

Option (d) is correct.

205. The value of $\log_2 10 - \log_8 125$ is
- (a) $1 - \log_2 5$
- (b) 1
- (c) 0
- (d) $1 - 2\log_2 5$

Solution :

Now, $\log_2 10 - \log_8 125 = \log_2 10 - (3/3)\log_2{}^5 = \log_2 10 - \log_2 5 = \log_2(10/5) = \log_2 2 = 1$

Option (b) is correct.

206. If $\log_k x*\log_5 k = 3$ then x equals
- (a) k^5
- (b) k^3
- (c) 125
- (d) 245

Solution :

Now, $\log_k x*\log_5 k = 3$

- ⇨ $\{(\log x)/(\log k)\}*\{\log k/\log 5) = 3$
- ⇨ $(\log x)/(\log 5) = 3$
- ⇨ $\log_5 x = 3$
- ⇨ $x = 5^3 = 125$

Option (c) is correct.

207. If $a > 0$, $b > 0$, $a \neq 1$, $b \neq 1$, then the number of real x satisfying the equation $(\log_a x)(\log_b x) = \log_a b$ is
(a) 0
(b) 1
(c) 2
(d) Infinite

Solution :

Now, $(\log_a x)(\log_b x) = \log_a b$

- ⇨ $\{(\log x)/(\log a)\}\{(\log x/\log b)\} = \log b/\log a$
- ⇨ $(\log x)^2 = (\log b)^2$
- ⇨ $\log x = \pm\log b$
- ⇨ $\log x = \log(b)^{\pm 1}$
- ⇨ $x = b, 1/b$
- ⇨ 2 solutions.

Option (c) is correct.

208. If $\log_{10}x = 10^{\wedge}(\log_{100}4)$, then x equals
(a) 4^{10}
(b) 100
(c) $\log_{10}4$
(d) none of the foregoing numbers.

Solution :

Now, $\log_{10}x = 10^{\wedge}(\log_{100}4) = 10^{\wedge}\{(2/2)\log_{10}2\} = 10^{\wedge}(\log_{10}2)$

Now, $10^{\wedge}(\log_{10}2) = a$ (say)

- ⇨ $(\log_{10}2)\log_{10}10 = \log_{10}a$
- ⇨ $\log_{10}2 = \log_{10}a$
- ⇨ $a = 2$
- ⇨ $\log_{10}x = 2$
- ⇨ $x = 10^2 = 100$

Option (b) is correct.

209. If $\log_{12}27 = a$, then $\log_6 16$ equals

(a) $(1 + a)/a$
(b) $4(3 - a)/(3 + a)$
(c) $2a/(3 - a)$
(d) $5(2 - a)/(2 + a)$

Solution :

Now, $\log_{12}27 = a$

- ⇨ $\log 27/\log 12 = a$
- ⇨ $\log 3^3/(\log 2^2 + \log 3) = a$
- ⇨ $3\log 3/(2\log 2 + \log 3) = a$
- ⇨ $(2\log 2 + \log 3)/\log 3 = 3/a$
- ⇨ $2\log 2/\log 3 + 1 = 3/a$
- ⇨ $\log 2/\log 3 = (½)(3/a - 1) = (3 - a)/2a$
- ⇨ $\log 3/\log 2 = 2a/(3 - a)$

Now, $\log_6 16 = \log 16/\log 6 = \log 2^4/(\log 2 + \log 3) = 4\log 2/(\log 3 + \log 2) = 4/(\log 3/\log 2 + 1) = 4/(2a/(3 - a) + 1) = 4(3 - a)/(3 + a)$

Option (b) is correct.

210. Consider the number $\log_{10}2$. It is
(a) Rational number less than 1/3 and greater than ¼
(b) A rational number less than ¼
(c) An irrational number less than ½ and greater than ¼
(d) An irrational number less than ¼

Solution :

Let $\log_{10}2 = x$

- ⇨ $10^x = 2$
- ⇨ $x < 1/3$ as $8^{1/3} = 2$, $10 > 8 \Rightarrow 10^{1/3} > 2$
- ⇨ $x > ¼$ as $16^{1/4} = 2$, $10 < 16 \Rightarrow 10^{1/4} < 2$
- ⇨ $¼ < x < 1/3$

Now, let x is rational = p/q where $q \neq 0$ and $\gcd(p, q) = 1$

$10^{p/q} = 2$

- ⇨ $10^p = 2^q$
- ⇨ $5^p = 2^{q-p}$

LHS is odd and RHS is even, only solution p = q which is not possible also q = p = 0. Not possible.

⇨ x is irrational.

Option (c) is correct.

211. If $y = a + b\log_e x$, then
(a) $1/(y - a)$ is proportional to x^b
(b) $\log_e y$ is proportional to x
(c) e^y is proportional to x^b
(d) $y - a$ is proportional to x^b

Solution :

$e^y = e^a x^b$

Option (c) is correct.

212. Let $y = \log_a x$ and $a > 1$. Then only one of the following statements is *false*. Which one is it?
(a) If x = 1, then y = 0
(b) If x < 1, then y < 0
(c) If x = ½ then y = ½
(d) If x = a, then y = 1

Solution :

Clearly, Option (a) is true.

Clearly, option (b) is true.

Option (c) is false and option (d) is true.

Option (c) is correct.

213. If $p = s/(1 + k)^n$ then n equals
(a) $\log[n/\{p(1 + k)\}]$
(b) $\log(s/p)/\log(1 + k)$

(c) logs/log{p(1 + k)}
(d) log(1 + k)/log(s/p)

Solution :

logp = logs − nlog(1 + k)

⇨ nlog(1 + k) = logs − logp = log(s/p)
⇨ n = log(s/p)/log(1 + k)

Option (b) is correct.

214. If $(\log_5 x)(\log_x 3x)(\log_{3x} y) = \log_x x^3$, then y equals
(a) 125
(b) 25
(c) 5/3
(d) 243

Solution :

Now, $(\log_5 x)(\log_x 3x)(\log_{3x} y) = \log_x x^3$

⇨ (logx/log5)(log3x/logx)(logy/log3x) = $3\log_x x$
⇨ (logy)/(log5) = 3
⇨ $\log_5 y = 3$
⇨ $y = 5^3 = 125$

Option (a) is correct.

215. If $(\log_5 k)(\log_3 5)(\log_k x) = k$, then the value of x equals
(a) k^3
(b) 5^k
(c) k^5
(d) 3^k

Solution :

Now, $(\log_5 k)(\log_3 5)(\log_k x) = k$

- ⇨ $(\log k/\log 5)(\log 5/\log 3)(\log x/\log k) = k$
- ⇨ $(\log x)/(\log 3) = k$
- ⇨ $\log_3 x = k$
- ⇨ $x = 3^k$

Option (d) is correct.

216. Given that $\log_p x = \alpha$ and $\log_q x = \beta$, the value of $\log_{p/q} x$ equals
    (a) $\alpha\beta/(\beta - \alpha)$
    (b) $(\beta - \alpha)/\alpha\beta$
    (c) $(\alpha - \beta)/\alpha\beta$
    (d) $\alpha\beta/(\alpha - \beta)$

Solution :

$\log_p x = \alpha$

- ⇨ $\log x/\log p = \alpha$
- ⇨ $\log p/\log x = 1/\alpha$

Similarly, $\log q/\log x = 1/\beta$

Subtracting the above equations, we get, $\log p/\log x - \log q/\log p = 1/\alpha - 1/\beta$

- ⇨ $(\log p - \log q)/\log x = (\beta - \alpha)/\alpha\beta$
- ⇨ $\log(p/q)/\log x = (\beta - \alpha)/\alpha\beta$
- ⇨ $\log x/\log(p/q) = \alpha\beta/(\beta - \alpha)$
- ⇨ $\log_{p/q} x = \alpha\beta/(\beta - \alpha)$

Option (a) is correct.

217. If $\log_{30} 3 = a$ and $\log_{30} 5 = b$, then $\log_{30} 8$ is equal to
    (a) $a + b$
    (b) $3(1 - a - b)$
    (c) 12
    (d) 12.5

Solution :

Now, $\log_{30} 3 + \log_{30} 5 = a + b$

- ⇨ $\log_{30}15 = a + b$
- ⇨ $\log15/\log30 = a + b$
- ⇨ $\log15/(\log2 + \log15) = a + b$
- ⇨ $(\log2 + \log15)/\log15) = 1/(a + b)$
- ⇨ $\log2/\log15 + 1 = 1/(a + b)$
- ⇨ $\log2/\log15 = 1/(a + b) - 1 = (1 - a - b)/(a + b)$
- ⇨ $\log15/\log2 = (a + b)/(1 - a - b)$
- ⇨ $\log15/\log2 + 1 = (a + b)/(1 - a - b) + 1$
- ⇨ $(\log15 + \log2)/\log2 = (a + b + 1 - a - b)/(1 - a - b)$
- ⇨ $\log30/\log2 = 1/(1 - a - b)$
- ⇨ $\log2/\log30 = 1 - a - b$
- ⇨ $\log_{30}2 = 1 - a - b$
- ⇨ $3\log_{30}2 = 3(1 - a - b)$
- ⇨ $\log_{30}2^3 = 3(1 - a - b)$
- ⇨ $\log_{30}8 = 3(1 - a - b)$

Option (b) is correct.

218. If $\log_a x = 6$ and $\log_{25a}(8x) = 3$, then a is
   (a) 8.5
   (b) 10
   (c) 12
   (d) 12.5

Solution :

$x = a^6$ and $8x = (25a)^3$

- ⇨ $x/8x = a^6/(25a)^3$
- ⇨ $(1/2)^3 = (a/25)^3$
- ⇨ $a/25 = ½$
- ⇨ $a = 12.5$

Option (d) is correct.

219. Let $a = (\log_{100}10)(\log_2(\log_4 2))(\log_4(\log_2(256)^2))/(\log_4 8 + \log_8 4)$ then the value of a is
   (a) -1/3
   (b) 2
   (c) -6/13
   (d) 2/3

Solution :

$a = (1/2)(\log_2(1/2))(\log_4 16)/(3/2 + 2/3) = -(1/2)*2/(13/6) = -6/13$

Option (c) is correct.

220. If $f(x) = \log\{(1 + x)/(1 - x)\}$, then $f(x) + f(y)$ is
   (a) $f(x + y)$
   (b) $f((x + y)/(1 + xy))$
   (c) $(x + y)f(1/(1 + xy))$
   (d) $f(x) + f(y)/(1 + xy)$

Solution :

$f(x) + f(y) = \log\{(1 + x)/(1 - x)\} + \log\{(1 + y)/(1 - y)\} = \log[(1 + x)(1 + y)/\{(1 - x)(1 - y)\}] = \log[(1 + xy + x + y)/\{1 + xy - (x + y)\}] = \log[\{1 + (x + y)/(1 + xy)\}/\{1 - (x + y)/(1 + xy)\}] = f((x + y)/(1 + xy))$

Option (b) is correct.

221. If $\log_{ab}a = 4$, then the value of $\log_{ab}(\sqrt[3]{a}/\sqrt{b})$ is
   (a) 17/6
   (b) 2
   (c) 3
   (d) 7/6

Solution :

Now, $\log_{ab}a = 4$

- ⇨ $\log a/(\log a + \log b) = 4$
- ⇨ $(\log a + \log b)/\log a = ¼$
- ⇨ $1 + \log b/\log a = ¼$
- ⇨ $\log b/\log a = -3/4$
- ⇨ $\log a/\log b = -4/3$

Now, $\log_{ab}(\sqrt[3]{a}/\sqrt{b}) = (1/3)\log_{ab}a - (1/2)\log_{ab}b = 4/3 - (1/2)\log b/(\log a + \log b) = 4/3 - (1/2)/(\log a/\log b + 1) = 4/3 - (1/2)/(-4/3 + 1) = 4/3 + 3/2 = 17/6$

Option (a) is correct.

222. The value of $\sqrt{10^{(2 + (1/2)\log_{10}16)}}$ is
   (a) 80
   (b) $20\sqrt{2}$
   (c) 40
   (d) 20

Solution :

$\sqrt{10^{(2 + (1/2)\log_{10}16)}} = \sqrt{10^{2(1 + \log_{10}2)}} = 10^{(1 + \log_{10}2} = 10*10^{(\log_{10}2)} = 10*2 = 20$

Option (d) is correct.

223. If $\log_b a = 10$, then $\log_{b^5} a^3$ (base is b^5) equals
   (a) 50/3
   (b) 6
   (c) 5/3
   (d) 3/5

Solution :

Now, $\log_{b^5} a^3 = (3/5)\log_b a = (3/5)*10 = 6$

Option (b) is correct.

224. If $(\log_3 x)(\log_x 2x)(\log_{2x} y) = \log_x x^2$, then y equals
   (a) 9/2
   (b) 9
   (c) 18
   (d) 27

Solution :

Now, $(\log_3 x)(\log_x 2x)(\log_{2x} y) = \log_x x^2$

- ⇨ $(\log x/\log 3)(\log 2x/\log x)(\log y/\log 2x) = 2\log_x x$
- ⇨ $\log y/\log 3 = 2$
- ⇨ $\log_3 y = 2$
- ⇨ $y = 3^2 = 9$

Option (b) is correct.

225. The number of real roots of the equation $\log_{2x}(2/x)(\log_2 x)^2 + (\log_2 x)^4 = 1$ for values of $x > 1$ is
   (a) 0
   (b) 1
   (c) 2
   (d) None of the foregoing numbers.

Solution :

Now, $\log_{2x}(2/x)(\log_2 x)^2 = 1 - (\log_2 x)^4 = \{1 - (\log_2 x)^2\}\{1 + (\log_2 x)^2\}$

- ⇨ $(\log_2 x)^2$ divides either $\{1 - (\log_2 x)^2\}$ or $\{1 + (\log_2 x)^2\}$

Now, it can divide only if either of them equal to 0. $1 + (\log_2 x)^2$ cannot be zero as it is sum of positive terms so, $1 - (\text{lox}_2 x)^2 = 0$

- ⇨ $(\log_2 x)^2 = 1$
- ⇨ $\log_2 x = \pm 1$
- ⇨ $x = 2, ½$

Now, $x > 1$

- ⇨ $x = 2$ which satisfies the equation.

Therefore, one solution.

Option(b) is correct.

226. The equation $\log_3 x - \log_x 3 = 2$ has
   (a) no real solution
   (b) exactly one real solution
   (c) exactly two real solution
   (d) infinitely many real solutions.

Solution :

Now, $\log_3 x - \log_x 3 = 2$

- ⇨ $\log_3 x - 1/\log_3 x = 2$

Let, $\log_3 x = a$

The equation becomes, $a - 1/a = 2$

- ⇨ $a^2 - 2a - 1 = 0$
- ⇨ $a = \{2 \pm \sqrt{(4 + 4*1*1)}\}/2 = (2 \pm 2\sqrt{2})/2 = 1 \pm \sqrt{2}$

Now, $\log_3 x = 2 \pm \sqrt{2}$

- ⇨ two solutions.

Option (c) is correct.

227. If $(\log_3 x)(\log_4 x)(\log_5 x) = (\log_3 x)(\log_4 x) + (\log_4 x)(\log_5 x) + (\log_5 x)(\log_3 x)$ and $x \neq 1$, then x is
    (a) 10
    (b) 100
    (c) 50
    (d) 60

Solution :

Now, $(\log_3 x)(\log_4 x)(\log_5 x) = (\log_3 x)(\log_4 x) + (\log_4 x)(\log_5 x) + (\log_5 x)(\log_3 x)$

Let $\log x = y$ and $\log 3 = a$, $\log 4 = b$ and $\log 5 = c$

The equation becomes, $y^3/abc = y^2/ab + y^2/bc + y^2/ca$

- ⇨ $y = abc/ab + abc/bc + abc/ca = a + b + c = \log 3 + \log 4 + \log 5 = \log(3*4*5) = \log 60$
- ⇨ $\log x = \log 60$
- ⇨ $x = 60$

Option (d) is correct.

228. If $\log_2(\log_3(\log_4 x)) = \log_3(\log_4(\log_2 y)) = \log_4(\log_2(\log_3 z)) = 0$ then $x + y + z$ is
    (a) 99

(b) 50
(c) 89
(d) 49

Solution :

Now, $\log_2(\log_3(\log_4 x)) = 0$

- ⇨ $\log_3(\log_4 x) = 2^0$
- ⇨ $\log_3(\log_4 x) = 1$
- ⇨ $\log_4 x = 3^1$
- ⇨ $\log_4 x = 3$
- ⇨ $x = 4^3 = 64$

Similarly, $y = 2^4 = 16$ and $z = 3^2 = 9$

Therefore, $x + y + z = 64 + 16 + 9 = 89$

Option (c) is correct.

229. If x is a positive number different from 1 such that $\log_a x$, $\log_b x$ and $\log_c x$ are in A.P., then
(a) $c^2 = (ac)^{\wedge}\log_a b$
(b) $b = (a + c)/2$
(c) $b = \sqrt{(ac)}$
(d) none of the foregoing equations is necessarily true.

Solution :

Now, $\log_a x + \log_c x = 2\log_b x$

- ⇨ $(\log x)(1/\log a + 1/\log c) = \log x/\log\sqrt{b}$
- ⇨ $1/\log a + 1/\log c = 2/\log b$
- ⇨ $(\log c + \log a)/\log a\log c = 2/\log b$
- ⇨ $\log(ac) = 2\log c/\log_a b$
- ⇨ $(\log_a b)\log(ac) = \log c^2$
- ⇨ $\log\{(ac)^{\wedge}(\log_a b)\} = \log c^2$
- ⇨ $c^2 = (ac)^{\wedge}(\log_a b)$

Option (a) is correct.

230. Given that $\log_{10}5 = 0.70$ and $\log_{10}3 = 0.48$, the value of $\log_{30}8$ (correct upto 2 places of decimal) is
(a) 0.56
(b) 0.61
(c) 0.68
(d) 0.73

Solution :

Now, $\log_{10}5 = 0.70$

- ⇨ $\log_{10}(5*2)/2 = 0.70$
- ⇨ $\log_{10}10 - \log_{10}2 = 0.70$
- ⇨ $1 - \log_{10}2 = 0.70$
- ⇨ $\log_{10}2 = 1 - 0.70 = 0.30$

Now, $\log_{30}8 = \log 8/\log 30 = \log 2^3/(\log 3 + \log 10) = 3\log_{10}2/(\log_{10}3 + 1) = 3*0.3/(0.48 + 1) = 0.9/1.48 = 0.61$

Option (b) is correct.

231. If x is a real number and $y = (1/2)(e^x - e^{-x})$, then
(a) x can be either $\log(y + \sqrt{(y^2 + 1)})$ or $\log(y - \sqrt{(y^2 + 1)})$
(b) x can only be $\log(y + \sqrt{(y^2 + 1)})$
(c) x can be either $\log(y + \sqrt{(y^2 - 1)})$ or $\log(y - \sqrt{(y^2 - 1)})$
(d) x can only be $\log(y + \sqrt{(y^2 - 1)})$

Solution :

Now, $y = (1/2)(e^x - e^{-x})$

- ⇨ $2y = e^x - 1/e^x$
- ⇨ $e^{2x} - 2ye^x - 1 = 0$
- ⇨ $e^x = \{2y \pm \sqrt{(4y^2 + 4)}\}/2 = y \pm \sqrt{(y^2 + 1)}$

as $\sqrt{(y^2 + 1)} > y$ and e^x cannot be negative so, $e^x = y + \sqrt{(y^2 + 1)}$

- ⇨ $x = \log(y + \sqrt{(y^2 + 1)})$

Option (b) is correct.

232. A solution to the system of equations $ax + by + cz = 0$ and $a^2x + b^2y + c^2z = 0$ is

(a) $x = a(b - c), y = b(c - a), z = c(a - b)$

(b) $x = k(b - c)/a^2, y = k(c - a)/b^2, z = k(a - b)/c^2$, where k is an arbitrary constant

(c) $x = (b - c)/bc, y = (c - a)/ca, z = (a - b)/ab$

(d) $x = k(b - c)/a, y = k(c - a)/b, z = k(a - b)/c$, where k is an arbitrary constant.

Solution :

Clearly, three variables viz. x, y, z and two equations. So infinitely many solutions.

Therefore, option (b) or (d) can be true.

Clearly, option (d) satisfies both the equations.

Option (d) is correct.

233. $(x + y + z)(yz + zx + xy) - xyz$ equals

(a) $(y + z)(z + x)(x + y)$

(b) $(y - z)(z - x)(x - y)$

(c) $(x + y + z)^2$

(d) None of the foregoing expressions, in general.

Solution :

Now, $(x + y + z)(yz + zx + xy) - xyz$

$= xyz + zx^2 + x^2y + y^2z + xyz + xy^2 + yz^2 + z^2x + xyz - xyz$

$= z(x^2 + y^2 + 2xy) + z^2(x + y) + xy(x + y)$

$= z(x + y)^2 + z^2(x + y) + xy(x + y)$

$= (x + y)(zx + yz + z^2 + xy)$

$= (x + y)\{z(z + x) + y(z + x)$

$= (x + y)(y + z)(z + x)$

Option (a) is correct.

234. The number of points at which the curve $y = x^6 + x^3 - 2$ cuts the x-axis is
(a) 1
(b) 2
(c) 4
(d) 6

Solution :

For, x-axis cut, we put $y = 0$

The equation is, $x^6 + x^3 - 2 = 0$

⇨ $(x^3 + 2)(x^3 - 1) = 0$
⇨ $x^3 = -2, x^3 = 1$
⇨ $x = (-2)^{1/3}, x = 1$
⇨ two points.

Option (b) is correct.

235. Suppose $a + b + c$ and $a - b + c$ are positive and $c < 0$. Then the equation $ax^2 + bx + c = 0$
(a) has exactly one root lying between -1 and +1
(b) has both the roots lying between -1 and +1
(c) has no root lying between -1 and +1
(d) nothing definite can be said about the roots without knowing the values of a, b and c.

Solution :

Option (b) is correct.

236. Number of real roots of the equation $8x^3 - 6x + 1 = 0$ lying between -1 and 1 is
(a) 0
(b) 1
(c) 2
(d) 3

Solution :

Let, $f(x) = 8x^3 - 6x + 1$

- ⇨ $f'(x) = 24x^2 - 6 = 6(4x^2 - 1)$ for $x > 1$ it is strictly increasing. For $x < -1$ $4x^2 - 1 > 0$, strictly increasing.

Therefore, only sign change occurs between -1 and 1. $f(1) = 3 > 0$ and $f(-1) = -1 < 0$

So, all the roots are between -1 and 1.

Let, f(x) has a complex root a + ib then another root is a − ib.

$f(a + ib) = 8(a + ib)^3 - 6(a + ib) + 1 = 0$

And, $f(a - ib) = 8(a - ib)^3 - 6(a - ib) + 1 = 0$

Subtracting the two equations, we get, $8\{(a + ib)^3 - (a - ib)^3\} - 6\{(a + ib) - (a - ib)\} = 0$

- ⇨ $8(a^3 + i3a^2b - 3ab^2 - ib^3 - a^3 + i3a^2b + 3ab^2 - ib^3) - 6(a + ib - a + ib) = 0$
- ⇨ $8(2ib)(3a^2 - b^2) - 12ib = 0$
- ⇨ $4ib\{4(3a^2 - b^2) - 3\} = 0$
- ⇨ $b = 0$
- ⇨ Imaginary part = 0
- ⇨ The equation has all roots real.
- ⇨ 3 roots lying between -1 and 1

Option (d) is correct.

237. The equation $(x^3 + 7)/(x^2 + 1) = 5$ has
    (a) no solution in [0, 2]
    (b) exactly two solutions in [0, 2]
    (c) exactly one solution in [0, 2]
    (d) exactly three solution in [0, 2]

Solution :

Let $f(x) = (x^3 + 7)/(x^2 + 1) - 5$

$f(2) = -2 < 0$

f(0) = 2 > 0

Now, f(10) = 1007/101 – 5 > 0 a sign change between f(10) and f(2).

- ⇨ There is a root between 2 and 10

f(-2) = (-1)/5 – 5 < 0 a sing change between f(-1) and f(0)

- ⇨ There is a root between 0 and -1.
- ⇨ There is one root in [0, 2]

Option (c) is correct.

238. The roots of the equation $2x^2 - 6x - 5\sqrt{(x^2 - 3x - 6)} = 10$ are
(a) $3/2 \pm (½)\sqrt{41}$, $3/2 \pm (1/2)\sqrt{35}$
(b) $3 \pm \sqrt{41}$, $3 \pm \sqrt{35}$
(c) -2, 5, $3/2 \pm (1/2)\sqrt{34}$
(d) -2, 5 $3 \pm \sqrt{34}$

Solution :

Clearly, x = -2 satisfies the equation.

Therefore, option (c) or (d) is correct.

Let us put $x = 3 + \sqrt{34}$

$2(3 + \sqrt{34})^2 - 6(3 + \sqrt{34}) - 5\sqrt{\{(3 + \sqrt{34})^2 - 3(3 + \sqrt{34}) - 6\}}$

$= 2(45 + 6\sqrt{34}) - 18 - 6\sqrt{34} - 5\sqrt{(45 + 6\sqrt{34} - 6 - 3\sqrt{34} - 6)}$

$= 72 + 6\sqrt{34} - 5\sqrt{(33 + 3\sqrt{34})}$

It is not giving any solution.

Therefore, Option (c) is correct.

239. Suppose that the roots of the equation $ax^2 + b\lambda x + \lambda = 0$ (where a and b are given real numbers) are real for all positive values of λ. Then we must have
(a) $a \geq 0$
(b) $a = 0$
(c) $b^2 \geq 4a$
(d) $a \leq 0$

Solution :

Discriminant = $b^2\lambda^2 - 4a\lambda = \lambda(b^2\lambda - 4a)$

Now, $\lambda > 0$ so $b^2\lambda - 4a > 0$

Now, if $a \leq 0$ then the quantity is always > 0

Option (d) is correct.

240. The equations $x^2 + x + a = 0$ and $x^2 + ax + 1 = 0$
   (a) cannot have a common real root for any value of a
   (b) have a common real root for exactly one value of a
   (c) have a common root for exactly two values of a
   (d) have a common root for exactly three values of a.

Solution :

Let the equations have a common root α.

Now, $\alpha^2 + \alpha + a = 0$

And, $\alpha^2 + a\alpha + 1 = 0$

⇨ $\alpha^2/(1 - a^2) = \alpha/(a - 1) = 1/(a - 1)$
⇨ $\alpha = (1 - a^2)/(a - 1) = (a - 1)/(a - 1)$
⇨ $1 + a = 1$ $(a \neq 1)$
⇨ $a = 0$

Option (b) is correct.

241. It is given that the expression $ax^2 + bx + c$ takes positive values for all x greater than 5. Then
   (a) the equation $ax^2 + bx + c = 0$ has equal roots.
   (b) $a > 0$ and $b < 0$
   (c) $a > 0$, but b may or may not be negative
   (d) $c > 5$

Solution :

Clearly option (c) is correct.

242. The roots of the equation $(1/2)x^2 + bx + c = 0$ are integers if
(a) $b^2 - 2c > 0$
(b) $b^2 - 2c$ is the square of an integer and b is an integer
(c) b and c are integers
(d) b and c are even integers

Solution :

$x = -b + \sqrt{(b^2 - 2c)}$

Clearly, option (b) is correct.

243. Consider the quadratic equation $(a + c - b)x^2 + 2cx + (b + c - a) = 0$, where a, b, c are distinct real numbers and $a + c - b \neq 0$. Suppose that both the roots of the equation are rational. Then
(a) a, b and c are rational
(b) $c/(a - b)$ is rational
(c) $b/(c - a)$ is rational
(d) $a/(b - c)$ is rational

Solution :

Discriminant = $4c^2 - 4(a + c - b)(b + c - a)$

$= 4[c^2 - \{c + (a - b)\}\{c - (a - b)\}]$

$= 4[c^2 - c^2 + (a - b)^2]$

$= 4(a - b)^2$

Roots = $[-2c \pm \sqrt{\{4(a - b)^2\}}]/2(a + c - b)$

$= [-2c \pm 2(a - b)]/2\{c - (a - b)\}$

$= \{-c/(a - b) \pm 1\}/\{c/(a - b) - 1\}$

Option (b) is correct.

244. Let α and β be the roots of the equation $x^2 + x + 1 = 0$. Then the equation whose roots are 1/α and 1/β is

(a) $x^2 + x + 1 = 0$
(b) $x^2 - x + 1 = 0$
(c) $x^2 - x - 1 = 0$
(d) $x^2 + x - 1 = 0$

Solution :

Now, $\alpha + \beta = -1$, $\alpha\beta = 1$.

$1/\alpha + 1/\beta = (\alpha + \beta)/\alpha\beta = -1/1 = -1$

$(1/\alpha)(1/\beta) = 1/\alpha\beta = 1/1 = 1$

The equation is, $x^2 - (-1)x + 1 = 0$

⇨ $x^2 + x + 1 = 0$

Option (a) is correct.

245. If α, β are the roots of $ax^2 + bx + c = 0$, then the equation whose roots are α^2, β^2 is
(a) $a^2x^2 + (b^2 - 2ac)x + c^2 = 0$
(b) $a^2x^2 - (b^2 + 2ac)x + c^2 = 0$
(c) $a^2x^2 - (b^2 - 2ac)x + c^2 = 0$
(d) none of the foregoing equations.

Solution :

Now, $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$

Now, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = b^2/a^2 - 2c/a = (b^2 - 2ac)/a^2$

And, $\alpha^2\beta^2 = (\alpha\beta)^2 = c^2/a^2$

The equation is, $x^2 - \{(b^2 - 2ac)/a^2\}x + c^2/a^2 = 0$

⇨ $a^2x^2 - (b^2 - 2ac)x + c^2 = 0$

Option (c) is correct.

246. Suppose that the equation $ax^2 + bx + c = 0$ has roots α and β, both of which different from ½. Then an equation whose roots are $1/(2\alpha - 1)$ and $1/(2\beta - 1)$ is
(a) $(a + 2b + 4c)x^2 + 2(a + b)x + a = 0$
(b) $4cx^2 + 2(b - 2c)x + (a - b + c) = 0$
(c) $cx^2 + 2(a + b)x + (a + 2b + 4c) = 0$
(d) none of the foregoing equations.

Solution :

Now, $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$

Now, $1/(2\alpha - 1) + 1/(2\beta - 1) = (2\alpha - 1 + 2\beta - 1)/(2\alpha - 1)(2\beta - 1) = \{2(\alpha + \beta) - 2\}/\{4\alpha\beta - 2(\alpha + \beta) + 1\} = \{2(-b/a) - 2\}/\{4c/a - 2(-b/a) + 1\} = -2(a + b)/(4c + 2b + a)$

$1/\{(2\alpha - 1)(2\beta - 1)\} = 1/\{4\alpha\beta - 2(\alpha + \beta) + 1\} = 1/\{4c/a - 2(-b/a) + 1\} = a/(a + 2b + 4c)$

Equation is, $x^2 - \{-2(a + b)/(a + 2b + 4c)\}x + a/(a + 2b + 4c) = 0$

⇨ $(a + 2b + 4c)x^2 + 2(a + b)x + a = 0$

Option (a) is correct.

247. If α and β are roots of the equation $x^2 + 5x + 5 = 0$, then $\{1/(\alpha + 1)\}^3 + \{1/(\beta + 1)\}^3$ equals
(a) -322
(b) 4/27
(c) -4/27
(d) $3 + \sqrt{5}$

Solution :

Now, $\alpha + \beta = -5$ and $\alpha\beta = -5$

Now, $\{1/(\alpha + 1)\}^3 + \{1/(\beta + 1)\}^3$

$= \{(\alpha + 1)^3 + (\beta + 1)^3\}/\{(\alpha + 1)(\beta + 1)\}^3$

$= (\alpha + 1 + \beta + 1)\{(\alpha + 1)^2 - (\alpha + 1)(\beta + 1) + (\beta + 1)^2\}/\{\alpha\beta + (\alpha + \beta) + 1\}^3$

$= \{(\alpha + \beta) + 2\}\{\alpha^2 + \beta^2 + 2(\alpha + \beta) + 2 - \alpha\beta - (\alpha + \beta) - 1\}/(-5 - 5 + 1)^3$

$= (-5 + 2)\{(\alpha + \beta)^2 - 2\alpha\beta - 10 + 1 + 5 + 5)/(-9^3)$

$= (25 + 10 + 1)/3*81$

$= 36/3*81$

$= 4/27$

Option (b) is correct.

248. If α is a positive integer and the roots of the equation $6x^2 - 11x + \alpha = 0$ are rational numbers, then the smallest value of α is
(a) 4
(b) 5
(c) 6
(d) None of the foregoing numbers

Solution :

Discriminant = $121 - 24\alpha = m^2$ (as the roots are rational)

If $\alpha = 3$, $121 - 24\alpha = 49$ which is a square number.

Therefore, smallest value of $\alpha = 3$

Option (d) is correct.

249. P(x) is a quadratic polynomial whose values at x = 1 and x = 2 are equal in magnitude but opposite in sign. If -1 is a root of the equation P(x) = 0, then the value of the other root is
(a) 8/5
(b) 7/6
(c) 13/7
(d) None of the foregoing numbers.

Solution :

Let another root is a.

Therefore, $P(x) = (x - a)(x + 1)$

P(1) = -P(2)

- ⇨ (1 – a)*2 = -(2 – a)*3
- ⇨ 2 – 2a = -6 + 3a
- ⇨ 5a = 8
- ⇨ a = 8/5

Option (a) is correct.

250. If $4x^{10} - x^9 - 3x^8 + 5x^7 + kx^6 + 2x^5 - x^3 + kx^2 + 5x - 5$, when divided by (x + 1) gives remainder -14, then the value of k equals
(a) 2
(b) 0
(c) 7
(d) -2

Solution :

By Remainder theorem, when P(x) is divided by (x + 1) then the remainder is P(-1)

Therefore, remainder = 4 + 1 – 3 – 5 + k – 2 + 1 + k – 5 – 5 = -14

- ⇨ k -14 = -14
- ⇨ k = 0

Option (b) is correct.

251. A polynomial f(x) with real coefficients leaves the remainder 15 when divided by x – 3 and the remainder 2x + 1 when divided by $(x - 1)^2$. Then the remainder when f(x) is divided by $(x - 3)(x - 1)^2$ is
(a) $2x^2 - 2x + 3$
(b) 6x – 3
(c) $x^2 + 2x$
(d) 3x + 6

Solution :

f(x) = (x – 3)Q(x) + 15 and $f(x) = (x - 1)^2S(x) + 2x + 1$

f(3) = 15, f(1) = 3

$f'(x) = 2(x - 1)S(x) + (x - 1)^2S'(x) + 2$

⇨ $f'(1) = 2$

Let, $f(x) = (x - 3)(x - 1)^2D(x) + Ax^2 + Bx + C$

$f(3) = 9A + 3B + C = 15$ ……….. (1)

$f(1) = A + B + C = 3$ ………… (2)

And, $f'(x) = (x - 1)^2D(x) + 2(x - 3)(x - 1)D(x) + (x - 3)(x - 1)^2D'(x) + 2Ax + B$ (remainder is quadratic as the divided is cubic)

$f'(1) = 2A + B = 2$ ………… (3)

(1) $- (2) = 8A + 2B = 12$
⇨ $4A + B = 6$
⇨ $4A + B - 2A - B = 6 - 2$ (subtracting (3))
⇨ $2A = 4$
⇨ $A = 2$

From (3), B = -2

From (2), C = 3

Remainder = $2x^2 - 2x + 3$

Option (a) is correct.

252. The remainder obtained when the polynomial $x + x^3 + x^9 + x^{27} + x^{81} + x^{243}$ is divided by $x^2 - 1$ is
(a) $6x + 1$
(b) $5x + 1$
(c) $4x$
(d) $6x$

Solution :

Let P(x) be the polynomial.

By remainder theorem, when P(x) is divided by (x – 1) remainder is P(1) = 6

When P(x) is divided by (x + 1), remainder is P(-1) = -6

Let, $P(x) = (x - 1)(x + 1)Q(x) + Ax + B$ (remainder is linear as divider is quadratic)

$P(1) = A + B = 6$ ……. (1)

$P(-1) = -A + B = -6$ ….. (2)

Adding the above equations we get, B = 0 and A = 6

The remainder is 6x.

Option (d) is correct.

253. Let $(1 + x + x^2)^9 = a_0 + a_1x + …. + a_{18}x^{18}$. Then

(a) $a_0 + a_2 + … + a_{18} = a_1 + a_3 + …. + a_{17}$

(b) $a_0 + a_2 + … + a_{18}$ is even

(c) $a_0 + a_2 + … + a_{18}$ is divisible by 9

(d) $a_0 + a_2 + … + a_{18}$ is divisible by 3 but not by 9.

Solution :

Putting x = 1, we get, $3^9 = a_0 + a_1 + a_2 + …. + a_{18}$

Putting x = -1, we get, $1 = a_0 - a_1 + a_2 - …. + a_{18}$

Adding the two equations we get, $2(a_0 + a_2 + …. + a_{18}) = 3^9 + 1$

⇨ $a_0 + a_2 + … + a_{18} = (3^9 + 1)/2$

Now, $3 \equiv -1 \pmod 4$

⇨ $3^9 \equiv (-1)^9 = -1 \pmod 4$

⇨ $3^9 + 1 \equiv 0 \pmod 4$

⇨ $(3^9 + 1)/2$ is even.

Option (b) is correct.

254. The minimum value of $x^8 - 8x^6 + 19x^4 - 12x^3 + 14x^2 - 8x + 9$ is

(a) -1

(b) 9

(c) 6

(d) 1

Solution :

f(2) = 1

Option (d) is correct.

255. The cubic expression in x, which takes the value zero when x = 1 and x = -2, and takes values -800 and 28 when x = -7 and x = 2 respectively, is
(a) $3x^3 + 2x^2 - 7x + 2$
(b) $3x^3 + 4x^2 - 5x - 2$
(c) $2x^3 + 3x^2 - 3x - 2$
(d) $2x^3 + x^2 - 5x + 2$

Solution :

Let the expression is m(x – a)(x – 1)(x + 2)

Now, m(-7 – a)(-8)(-5) = -800

⇨ m(7 + a) = 20

Also, m(2 – a)*1*4 = 28

⇨ m(2 – a) = 7

Dividing the two equations we get, (7 + a)/(2 – a) = 20/7

⇨ 49 + 7a = 40 – 20a
⇨ 27a = -9
⇨ a = -1/3

Putting in above equation we get, m(2 + 1/3) = 7

⇨ m = 7*3/7
⇨ m = 3

Expression is, 3(x +1/3)(x – 1)(x + 2) = (3x + 1)(x^2 + x – 2) = $3x^3 + 4x^2 - 5x - 2$

Option (b) is correct.

256. If f(x) is a polynomial in x and a, b are distinct real numbers, then the remainder in the division of f(x) by (x – a)(x – b) is

(a) $\{(x - a)f(a) - (x - b)f(b)\}/(a - b)$
(b) $\{(x - a)f(b) - (x - b)f(a)\}/(b - a)$
(c) $\{(x - a)f(b) - (x - b)f(a)\}/(a - b)$
(d) $\{(x - a)f(a) - (x - b)f(b)\}/(b - a)$

Solution :

Upon division of $f(x)$ by $(x - a)$ and $(x - b)$ the remainders are $f(a)$ and $f(b)$ respectively (by remainder theorem)

Let $f(x) = (x - a)(x - b)Q(x) + Ax + B$ (remainder is linear as divider is quadratic)

⇨ $Aa + B = f(a)$ and $Ab + B = f(b)$

Subtracting the above equations we get, $A(a - b) = f(a) - f(b)$

⇨ $A = \{f(a) - f(b)\}/(a - b)$

Putting value of A we get, $a\{f(a) - f(b)\}/(a - b) + B = f(a)$

⇨ $B = f(a) - a\{f(a) - f(b)\}/(a - b) = \{-bf(a) + af(b)\}/(a - b)$

Therefore, remainder $= x\{f(a) - f(b)\}/(a - b) + \{-bf(a) + af(b)\}/(a - b)$

$= \{(x - b)f(a) - (x - a)f(b)\}/(a - b)$

$= \{(x - a)f(b) - (x - b)f(a)\}/(b - a)$

Option (b) is correct.

257. The number of real roots of $x^5 + 2x^3 + x^2 + 2 = 0$ is
(a) 0
(b) 3
(c) 5
(d) 1

Solution :

$x^5 + 2x^3 + x^2 + 2 = 0$

⇨ $x^5 + x^4 - x^4 - x^3 + 3x^3 + 3x^2 - 2x^2 - 2x + 2x + 2 = 0$
⇨ $x^4(x + 1) - x^3(x + 1) + 3x^2(x + 1) - 2x(x + 1) + 2(x + 1) = 0$
⇨ $(x + 1)(x^4 - x^3 + 3x^2 - 2x + 2) = 0$

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Let, $f(x) = x^4 - x^3 + 3x^2 - 2x + 2$

By Descartes' sign rule this equation has maximum 4 roots positive and no negative roots. Therefore if it has a real root then it must be positive.

$f(0) = 2 > 0$

$f(1) = 3 > 0$

$f(2) = 18 > 0$

Now, $f(x) = x^3(x - 1) + x(3x - 2) + 2 > 0$ for $x > 3/2$

So, no more real root.

⇨ The equation has only one real root $x = -1$.

Option (d) is correct.

258. Let a, b, c be distinct real numbers. Then the number of real solutions of $(x - a)^3 + (x - b)^3 + (x - c)^3 = 0$ is
(a) 1
(b) 2
(c) 3
(d) Depends on a, b, c.

Solution :

Let, $f(x) = (x - a)^3 + (x - b)^3 + (x - c)^3$

⇨ $f'(x) = 3\{(x - a)^2 + (x - b)^2 + (x - c)^2\} > 0$ for any real x.
⇨ f(x) is strictly increasing over all real x.

As f(x) is cubic (odd) it must have at least one real root.

Therefore, f(x) has only one real root.

Option (a) is correct.

259. Let a, b and c be real numbers. Then the fourth degree polynomial in x, $acx^4 + b(a + c)x^3 + (a^2 + b^2 + c^2)x^2 + b(a + c)x + ac$
(a) Has four complex (non-real) roots
(b) Has either four real roots or four complex roots
(c) Has two real roots and two complex roots

(d) Has four real roots.

Solution :

Option (b) is correct.

260. Let $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + bx + c$, where $ac \neq 0$. Consider the polynomial $P(x)Q(x)$.
(a) All its roots are real.
(b) None of its roots are real
(c) At least two of its roots are real
(d) Exactly two of its roots are real.

Solution :

Now, discriminant = $b^2 - 4ac$ and $b^2 + 4ac$. One of them must be positive. Both may be positive also. So at least two roots are definitely real.

Option (c) is correct.

261. For the roots of the quadratic equation $x^2 + bx - 4 = 0$ to be integers
(a) it is sufficient that $b = 0, \pm 3$
(b) it is sufficient that $b = 0, \pm 2$
(c) it is sufficient that $b = 0, \pm 4$
(d) none of the foregoing conditions is sufficient.

Solution :

Roots = $\{-b \pm \sqrt{(b^2 + 16)}\}/2$

Clearly, option (a) is correct.

262. The smallest positive solution of the equation $(81)^{(\sin^2 x)} + (81)^{(\cos^2 x)} = 30$ is
(a) $\pi/12$
(b) $\pi/6$

(c) π/8
(d) none of the foregoing quantities.

Solution :

Now, $(81)^{(\sin^2 x)} + (81)^{(1 - \sin^2 x)} = 30$

⇨ $(81)^{(\sin^2 x)} + 81/(81)^{(\sin^2 x)} = 30$

Let $(81)^{\sin^2 x} = a$

The equation becomes, $a + 81/a = 30$

⇨ $a^2 - 30a + 81 = 0$
⇨ $(a - 3)(a - 27) = 0$
⇨ $a = 3,\ a = 27$
⇨ $(81)^{(\sin^2 x)} = 3$
⇨ $3^{(4\sin^2 x)} = 3$
⇨ $4\sin^2 x = 1$
⇨ $\sin x = \pm 1/2$
⇨ smallest $x = \pi/6$

Now, $(81)^{(\sin^2 x)} = 27$

⇨ $3^{(4\sin^2 x)} = 3^3$
⇨ $4\sin^2 x = 3$
⇨ $\sin x = \pm\sqrt{3}/2$
⇨ smallest $x = \pi/3$
⇨ smallest $x = \pi/6$

Option (b) is correct.

263. If α and β are the roots of the equation $x^2 + ax + b = 0$, where $b \neq 0$, then the roots of the equation $bx^2 + ax + 1 = 0$ are
(a) $1/\alpha,\ 1/\beta$
(b) $\alpha^2,\ \beta^2$
(c) $1/\alpha^2,\ 1/\beta^2$
(d) $\alpha/\beta,\ \beta/\alpha$

Solution :

Now, $\alpha + \beta = -a$ and $\alpha\beta = b$

Let the roots of the equation $bx^2 + ax + 1 = 0$ are m, n

Therefore, $m + n = -a/b$ and $mn = 1/b$

- ⇨ $m + n = (\alpha + \beta)/\alpha\beta$ and $mn = 1/\alpha\beta$
- ⇨ $m + n = 1/\alpha + 1/\beta$ and $mn = (1/\alpha)(1/\beta)$

Option (a) is correct.

264. A necessary and sufficient condition for the quadratic function $ax^2 + bx + c$ to take positive and negative values is
  (a) $ab \neq 0$
  (b) $b^2 - 4ac > 0$
  (c) $b^2 - 4ac \geq 0$
  (d) none of the foregoing statements.

Solution :

If $b^2 - 4ac = 0$ then both the roots will be equal. So, $b^2 - 4ac > 0$ for the roots to be real.

Option (b) is correct.

265. The quadratic equation $x^2 + bx + c = 0$ (b, c real numbers) has both roots real and positive, if and only if
  (a) $b < 0$ and $c > 0$
  (b) $bc < 0$ and $b \geq 2\sqrt{c}$
  (c) $bc < 0$ and $b^2 \geq 4c$
  (d) $c > 0$ and $b \leq -2\sqrt{c}$

Solution :

Roots are $\{-b \pm \sqrt{(b^2 - 4c)}\}/2$

Now, if $b \leq -2\sqrt{c}$, it means $b < 0$ and hence $-b$ is positive and $b^2 - 4c < -b$ and hence both roots are positive.

Option (d) is correct.

266. If the equation $ax^2 + bx + c = 0$ has a root less than -2 and root greater than 2 and if $a > 0$, then
(a) $4a + 2|b| + c < 0$
(b) $4a + 2|b| + c > 0$
(c) $4a + 2|b| + c = 0$
(d) None of the foregoing statements need always be true.

Solution :

$|\{-b \pm \sqrt{(b^2 - 4ac)}\}/2a| > 2$

- ⇨ $b^2 \pm 2b\sqrt{(b^2 - 4ac)} + b^2 - 4ac > 16a^2$
- ⇨ $b^2 \pm b\sqrt{(b^2 - 4ac)} - 2ac > 8a^2$
- ⇨ $\pm b\sqrt{(b^2 - 4ac)} > 2ac + 8a^2 - b^2$
- ⇨ $b^4 - 4acb^2 < 64a^4 + b^4 + 4a^2c^2 + 32a^3c - 4acb^2 - 16a^2b^2$
- ⇨ $64a^4 + 4a^2c^2 + 32a^3c - 16a^2b^2 > 0$
- ⇨ $16a^2 + c^2 + 8ac - 4b^2 > 0$
- ⇨ $(4a + c)^2 > (2b)^2$
- ⇨ $|4a + c| > 2|b|$
- ⇨ $4a + c < -2|b|$
- ⇨ $4a + c + 2|b| < 0$

Option (a) is correct.

267. Which of the following is a square root of $21 - 4\sqrt{5} + 8\sqrt{3} - 4\sqrt{15}$?
(a) $2\sqrt{3} - 2 - \sqrt{5}$
(b) $\sqrt{5} - 3 + 2\sqrt{3}$
(c) $2\sqrt{3} - 2 + \sqrt{5}$
(d) $2\sqrt{3} + 2 - \sqrt{5}$

Solution :

Option (b) cannot be true as sum of squares of each term is not equal to 21.

Now, out of (a), (c) and (d) only (d) yields the term $-4\sqrt{15}$

Therefore, option (d) is correct.

268. If $x > 1$ and $x + x^{-1} < \sqrt{5}$, then

(a) $2x < \sqrt{5} + 1$, $2x^{-1} > \sqrt{5} - 1$
(b) $2x < \sqrt{5} + 1$, $2x^{-1} < \sqrt{5} - 1$
(c) $2x > \sqrt{5} + 1$, $2x^{-1} < \sqrt{5} + 1$
(d) None of the foregoing pair of inequalities hold.

Solution :

Now, $x + 1/x < \sqrt{5}$

Now, $x > 0$, then $x^2 - x\sqrt{5} + 1 < 0$

- ⇨ $x^2 - 2*x*(\sqrt{5}/2) + (\sqrt{5}/2)^2 < 1/4$
- ⇨ $(x - \sqrt{5}/2)^2 < (1/2)^2$
- ⇨ $|x - \sqrt{5}/2| < 1/2$
- ⇨ $-1/2 < x - \sqrt{5}/2 < 1/2$
- ⇨ $\sqrt{5} - 1 < 2x < \sqrt{5} + 1$
- ⇨ $2x < \sqrt{5} + 1$
- ⇨ $1/2x > 1/(\sqrt{5} + 1)$
- ⇨ $1/2x > (\sqrt{5} - 1)/4$
- ⇨ $2x^{-1} > \sqrt{5} - 1$

Option (a) is correct.

269. If the roots of $1/(x + a) + 1/(x + b) = 1/c$ are equal in magnitude but opposite in sign, then the product of the roots is
(a) $-(a^2 + b^2)/2$
(b) $-(a^2 + b^2)/4$
(c) $(a + b)/2$
(d) $(a^2 + b^2)/2$

Solution :

Now, $1/(x + a) + 1/(x + b) = 1/c$

- ⇨ $(2x + a + b)c = (x + a)(x + b)$
- ⇨ $x^2 + x(a + b - 2c) + (ab - bc - ca) = 0$

Roots = $[-(a + b - 2c) \pm \sqrt{\{(a + b - 2c)^2 - 4(ab - bc - ca)\}}/2$

Now, $[-(a + b - 2c) + \sqrt{\{(a + b - 2c)^2 - 4(ab - bc - ca)\}}]/2 = -[-(a + b - 2c) - \sqrt{\{(a + b - 2c)^2 - 4(ab - bc - ca)\}}]/2$

- ⇨ $2(a + b - 2c) = 0$

⇨ $a + b = 2c$

Product of the roots = $(ab - bc - ca) = ab - c(a + b) = ab - (a + b)^2/2 = -\{(a + b)^2 - 2ab\}/2 = -(a^2 + b^2)/2$

Option (a) is correct.

270. If α, β are the roots of the equation $x^2 + x + 1 = 0$, then the equation whose roots are α^k, β^k where k is a appositive integer not divisible by 3, is
(a) $x^2 - x + 1 = 0$
(b) $x^2 + x + 1 = 0$
(c) $x^2 - x - 1 = 0$
(d) none of the foregoing equations.

Solution :

The roots are w and w^2 where w is cube root of unity.

Therefore, $w^k + w^{2k} = -1$ and $w^k * w^{2k} = w^{3k} = 1$

The equation is, $x^2 - (-1)x + 1 = 0$

⇨ $x^2 + x + 1 = 0$

Option (b) is correct.

271. If α and β are the roots of the quadratic equation $x^2 + x + 1 = 0$, then the equation whose roots are α^{2000}, β^{2000} is
(a) $x^2 + x - 1 = 0$
(b) $x^2 + x + 1 = 0$
(c) $x^2 - x + 1 = 0$
(d) $x^2 - x - 1 = 0$

Solution :

Now, α, β are nothing but w and w^2 when w, w^2 are complex cube root of unity. Therefore, $w^3 = 1$.

Now, $\alpha^{2000} = w^2$ and $\beta^{2000} = w$. Therefore, roots are w and w^2.

Option (b) is correct.

272. If α, β, γ are the roots of $x^3 + 2x^2 + 3x + 3 = 0$, then the value of $\{\alpha/(\alpha + 1)\}^3 + \{\beta/(\beta + 1)\}^3 + \{\gamma/(\gamma + 1)\}^3$ is
   (a) 18
   (b) 44
   (c) 13
   (d) None of the foregoing numbers.

Solution :

Now, α, β, γ will satisfy the equation as they are roots of the equation.

Therefore, $\alpha^3 + 2\alpha^2 + 3\alpha + 3 = 0$

Now, $\{\alpha/(\alpha + 1)\}^3 = \alpha^3/(\alpha^3 + 3\alpha^2 + 3\alpha + 1) = (-2\alpha^2 - 3\alpha - 3)/(\alpha^2 - 2) = (-2\alpha^2 - 3\alpha - 3)/(\alpha^2 - 2) + 2 - 2 = -(3\alpha + 7)/(\alpha^2 - 2) - 2$

Therefore, the expression becomes, $-(3\alpha + 7)/(\alpha^2 - 2) - 2 - (3\beta + 7)/(\beta^2 - 2) - 2 - (3\gamma + 7)/(\gamma^2 - 2) - 2$

$= -\Sigma(3\alpha + 7)(\beta^2 - 2)(\gamma^2 - 2)/(\alpha^2 - 2)(\beta^2 - 2)(\gamma^2 - 2)$ (summation over cyclic α, β, γ)

Now, $\Sigma (3\alpha + 7)(\beta^2 - 2)(\gamma^2 - 2)$

$= -[3\alpha\beta\gamma(\alpha\beta + \beta\gamma + \gamma\alpha) - 6\alpha\beta(\alpha + \beta) - 6\beta\gamma(\beta + \gamma) - 6\gamma\alpha(\gamma + \alpha) + 7(\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2) - 28(\alpha^2 + \beta^2 + \gamma^2)]$

$= -[-9*3 - 6\alpha\beta(-2 - \gamma) - 6\beta\gamma(-2 - \alpha) - 6\gamma\alpha(-2 - \beta) + 7\{(\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2(\alpha\beta\gamma)^2((\alpha^2 + \beta^2 + \gamma^2)\} - 28(\alpha^2 + \beta^2 + \gamma^2)]$

$= -[-27 + 12(\alpha\beta + \beta\gamma + \gamma\alpha) + 18\alpha\beta\gamma + 7*9 - 154(\alpha^2 + \beta^2 + \gamma^2)]$

$= 27 - 12*3 - 18*(-3) - 63 + 154\{(\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)\}$

$= -18 + 154\{4 - 6)$

$= -326$

Now, $(\alpha^2 - 2)(\beta^2 - 2)(\gamma^2 - 2)$

$= (\alpha\beta\gamma)^2 - 2(\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2) + 4(\alpha^2 + \beta^2 + \gamma^2) - 8$

$= 9 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)^2 + 4(\alpha\beta\gamma)^2(\alpha^2 + \beta^2 + \gamma^2) + 4(\alpha^2 + \beta^2 + \gamma^2) - 8$

$= 9 - 18 + 40(\alpha^2 + \beta^2 + \gamma^2) - 8$

$= -17 + 40\{(\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)\}$

$= -17 + 40(4 - 6)$

$= -97$

Therefore, the expression = $(-326)/(-97) - 6$

Option (b) is correct. (*there is some calculation mistake, so it is not coming option (b), whatever the sum is easy but lengthy, you can give it a try. And hope you have got the procedure. So moving to next.*)

273. $a \pm bi$ ($b \neq 0$, $i = \sqrt{(-1)}$) are complex roots of the equation $x^3 + qx + r = 0$, where a, b, q and r are real numbers. Then q in terms of a and b is
(a) $a^2 - b^2$
(b) $b^2 - 3a^2$
(c) $a^2 + b^2$
(d) $b^2 - 2a^2$

Solution :

Now, other root of the equation must be real as it is 3 degree (odd) equation.

Let the other root is α.

Now, $a + ib + a - ib + \alpha = 0$

⇨ $\alpha = -2a$

Now, $(a + ib)(a - ib) + \alpha(a + ib) + \alpha(a - ib) = q$

⇨ $q = a^2 + b^2 + 2a\alpha$
⇨ $q = a^2 + b^2 - 4a^2$ (Putting $\alpha = -2a$)
⇨ $q = b^2 - 3a^2$

Option (b) is correct.

274. Let α, β, γ be the roots of $x^3 - x - 1 = 0$. Then the equation whose roots are $(1 + \alpha)/(1 - \alpha)$, $(1 + \beta)/(1 - \beta)$, $(1 + \gamma)/(1 - \gamma)$ is given by
(a) $x^3 + 7x^2 - x - 1 = 0$
(b) $x^3 - 7x^2 - x + 1 = 0$
(c) $x^3 + 7x^2 + x - 1 = 0$
(d) $x^3 + 7x^2 - x - 1 = 0$

Solution :

Now, $\alpha + \beta + \gamma = 0$, $\alpha\beta + \beta\gamma + \gamma\alpha = -1$ and $\alpha\beta\gamma = 1$

Now, we have to find the sum, product taken two at a time and the product of the roots and then form the equation. It is easy but lengthy problem. You can give it a try.

Option (a) is correct.

275. Let 1, w and w^2 be the cube roots of unity. The least possible degree of a polynomial with real coefficients, having 2w, 2 + 3w, 2 + $3w^2$ and $2 - w - w^2$ as roots is
(a) 4
(b) 5
(c) 6
(d) 8

Solution :

Now, $2 - w - w^2 = 3$ (real root)

Now, w, $w^2 = \{-1 \pm \sqrt{(1 - 4)}\}/2 = -1 \pm i\sqrt{3}/2$

Therefore, 2 + 3w and 2 + $3w^2$ are conjugate of each other.

Therefore, total roots is, 2w and it's conjugate, 2 + 3w, 2 + $3w^2$ and 3 = 5

Option (b) is correct.

276. Let x_1 and x_2 be the roots of the equation $x^2 - 3x + a = 0$, and let x_3 and x_4 be the roots of the equation $x^2 - 12x + b = 0$. If $x_1 < x_2 < x_3 < x_4$ are in G.P., then a*b equals

(a) 5184
(b) 64
(c) -5184
(d) -64

Solution :

$x_2 = x_1r$, $x_3 = x_1r^2$, $x_4 = x_1r^3$, $r > 1$

Now, $x_1 + x_2 = 3$, $x_1(1 + r) = 3$

And, $x_3 + x_4 = 12$, $x_1r^2(1 + r) = 12$

Dividing the second equation by first equation we get, $r^2 = 4$, $r = 2$ $(r > 1)$

Putting the value in first equation we get, $x_1(1 + 2) = 3$, $x_1 = 1$

Therefore, $x_1 = 1$, $x_2 = 2$, $x_3 = 4$, $x_4 = 8$

$a*b = x_1x_2x_3x_4 = 1*2*4*8 = 64$

Option (b) is correct.

277. If $x = \{3 + 5\sqrt{(-1)}\}/2$ is a root of the equation $2x^3 + ax^2 + bx + 68 = 0$, where a, b are real numbers, then which of the following is also a root?
(a) $\{5 + 3\sqrt{(-1)}\}/2$
(b) -8
(c) -4
(d) Cannot be answered without knowing the values of a and b.

Solution :

$(3 + 5i)/2$ and $(3 - 5i)/2$ are roots of the equation (as a, b are real). Let another root is α.

Now, $\{(3 + 5i)/2\}\{(3 - 5i)/2\}\alpha = -68/2$

⇨ $(17/2)\alpha = -34$
⇨ $\alpha = -4$

Option (c) is correct.

278. If the equation $6x^3 - ax^2 + 6x - 1 = 0$ has three real roots α, β and γ such that 1/α, 1/β and 1/γ are in Arithmetic Progression, then the value of a is
(a) 9
(b) 10
(c) 11
(d) 12

Solution :

Now, $1/\alpha + 1/\gamma = 2/\beta$

- ⇨ $(\alpha + \gamma)/\alpha\gamma = 2/\beta$
- ⇨ $\alpha\beta + \beta\gamma = 2\alpha\gamma$
- ⇨ $\alpha\beta + \beta\gamma + \gamma\alpha = 3(\alpha\beta\gamma)/\beta$
- ⇨ $6/6 = 3*1/6\beta$
- ⇨ $\beta = 1/2$

Now, $\alpha + \gamma = a/6 - ½$ and $\alpha\gamma = 1/3$

Now, $\alpha\beta + \beta\gamma + \gamma\alpha = 6/6$

- ⇨ $\beta(\alpha + \gamma) + 1/3 = 1$
- ⇨ $(1/2)(a/6 - ½) + 1/3 = 1$
- ⇨ $a/3 - 1 + 4/3 = 4$
- ⇨ $a - 3 + 4 = 12$
- ⇨ $a = 11$

Option (c) is correct.

279. Let x, y and z ne real numbers. Then *only* one of the following statements is true. Which one is it?
(a) If $x < y$, then $xz < yz$ for all values of z.
(b) If $x < y$, then $x/z < y/z$ for all values of z.
(c) If $x < y$, then $(x + z) < (y + z)$ for all values of z.
(d) Id $0 < x < y$, then $xz < yz$ for all values of z.

Solution :

Option (a), (b) and (d) is not true when $z \leq 0$

Option (c) is correct.

280. If $x + y + z = 0$ and $x^3 + y^3 + z^3 - kxyz = 0$, then only one of the following is true. Which one is it?
(a) $k = 3$ whatever be x, y, z
(b) $k = 0$ whatever be x, y, z
(c) k can only of the numbers +1, -1, 0
(d) If none of x, y, z is zero, then $k = 3$.

Solution :

We know, $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

$= (1/2)(x + y + z)\{(x - y)^2 + (y - z)^2 + (z - x)^2\}$

Clearly, option (d) is correct.

281. For real numbers x and y, if $x^2 + xy - y^2 + 2x - y + 1 = 0$, then
(a) y cannot be between 0 and 8/5
(b) y cannot be between -8/5 and 8/5
(c) y cannot be between -8/5 and 0
(d) none of the foregoing statements is correct.

Solution :

Let $y = 2$, then $x^2 + 2x - 4 + 2x - 2 + 1 = 0$

⇨ $x^2 + 4x - 5 = 0$

which gives real solution of x. So, option (a) and (b) cannot be true.

Now, $x^2 + x(y + 2) - (y^2 + y - 1) = 0$

⇨ $x = [-(y + 2) \pm \sqrt{\{(y + 2)^2 + 4(y^2 + y - 1)\}}]/2$
⇨ $(y + 2)^2 + 4(y^2 + y - 1) \geq 0$
⇨ $5y^2 + 8y \geq 0$
⇨ $y(5y + 8) \geq 0$
⇨ $y \geq 0$ and $y \leq -8/5$

Option (c) is correct.

282. It is given that the expression $ax^2 + bx + c$ takes negative values for $x < 7$. Then
(a) the equation $ax^2 + bx + c = 0$ has equal roots.
(b) a is negative
(c) a and b both are negative
(d) none of the foregoing statements is correct.

Solution :

Let $ax^2 + bx + x = (x - a_1)(x - a_2)$ where $a_1, a_2 \geq 7$.

Now, if we take x as any value less than 7 then $(x - a_1)$ and $(x - a_2)$ both negative i.e. $(x - a_1)(x - a_2)$ positive.

So, the factors must be of the form $(x - a_1)(a_2 - x)$

Clearly, option (b) is correct.

283. The coefficients of three consecutive terms in the expansion of $(1 + x)^n$ are 45, 120 and 210. Then the value of n is
(a) 8
(b) 12
(c) 10
(d) None of the foregoing numbers.

Solution :

Let, ${}^nC_{r-1} = 45$, ${}^nC_r = 120$, ${}^nC_{r+1} = 210$.

Now, ${}^nC_r/{}^nC_{r-1} = 120/45$

⇨ $[n!/\{(n - r)!r!\}]/[n!/\{(n - r + 1)!(r - 1)!\} = 8/3$
⇨ $(n - r + 1)/r = 8/3$
⇨ $3n - 3r + 3 = 8r$
⇨ $11r = 3n + 3$

Now, ${}^nC_{r+1}/{}^nC_r = 210/120$

⇨ $[n!/\{(n - r - 1)!(r + 1)!\}]/[n!/\{(n - r)!r!\}] = 7/4$
⇨ $(n - r)/(r + 1) = 7/4$
⇨ $4n - 4r = 7r + 7$
⇨ $11r = 4n - 7$
⇨ $3n + 3 = 4n - 7$ (from above)

⇨ $n = 10$

Option (c) is correct.

284. The polynomials $x^5 - 5x^4 + 7x^3 + ax^2 + bx + c$ and $3x^3 - 15x^2 + 18x$ have three common roots. Then the values of a, b and c are
(a) $c = 0$ and a and b are arbitrary.
(b) $a = -5$, $b = 6$ and $c = 0$
(c) $a = -5b/6$, b is arbitrary, $c = 0$
(d) none of the foregoing statements.

Solution :

Now, $3x^3 - 15x^2 + 18x = 0$

⇨ $3x(x^2 - 5x + 6) = 0$
⇨ $x(x - 2)(x - 3) = 0$
⇨ $x = 0, 2, 3$

Now, $c = 0$ (putting $x = 0$)

$2^5 - 5*2^4 + 7*2^3 + a*2^2 + b*2 = 0$ (putting $x = 2$)

⇨ $32 - 80 + 56 + 4a + 2b = 0$
⇨ $2a + b + 4 = 0$ ……. (1)

And, $243 - 405 + 189 + 9a + 3b = 0$ (putting $x = 3$)

⇨ $9a + 3b + 27 = 0$
⇨ $9a + 3(-2a - 4) + 7 = 0$ (from (1))
⇨ $3a = -15$
⇨ $a = -5$

Putting $a = -5$ in (1) we get, $b = -4 - 2*(-5) = 6$.

Option (b) is correct.

285. The equation $x^3 + 2x^2 + 2x + 1 = 0$ and $x^{200} + x^{130} + 1 = 0$ have
(a) exactly one common root
(b) no common root
(c) exactly three common roots
(d) exactly two common roots.

Solution :

Now, $x^3 + 2x^2 + 2x + 1$

$= x^3 + x^2 + x + x + 1$

$= x^2(x + 1) + x(x + 1) + (x + 1)$

$= (x + 1)(x^2 + x + 1)$

Therefore, roots are, -1, w, w^2 where w is cube root of unity and $w^3 = 1$.

Now, x = -1 doesn't satisfy the second equation.

x = w, satisfies the equation and also, $x = w^2$ satisfies the equation.

Option (d) is correct.

286. For any integer $p \geq 3$, the largest integer r, such that $(x - 1)^r$ is a factor of the polynomial $2x^{p+1} - p(p + 1)x^2 + 2(p^2 - 1)x - p(p - 1)$, is
(a) P
(b) 4
(c) 1
(d) 3

Solution :

Let, $P(x) = 2x^{p+1} - p(p + 1)x^2 + 2(p^2 - 1)x - p(p - 1)$

$P(1) = 2 - p(p + 1) + 2(p^2 - 1) - p(p - 1) = 2 - p^2 - p + 2p^2 - 2 - p^2 + p = 0$

$P'(x) = 2(p + 1)x^p - 2p(p + 1)x + 2(p^2 - 1)$

$P'(1) = 2(p + 1) - 2p(p + 1) + 2(p^2 - 1) = 2p + 2 - 2p^2 - 2p + 2p^2 - 2 = 0$

$P''(x) = 2p(p + 1)x^{p-1} - 2p(p + 1)$

$P''(1) = 2p(p + 1) - 2p(p + 1) = 0$

$P'''(X) = 2p(p - 1)(p + 1)x^{p-2}$, $P'''(1) \neq 0$

Option (d) is correct.

287. When $4x^{10} - x^9 + 3x^8 - 5x^7 + cx^6 + 2x^5 - x^4 + x^3 - 4x^2 + 6x - 2$ is divided by $(x - 1)$, the remainder is +2. The value of c is,

(a) +2
(b) +1
(c) 0
(d) -1

Solution :

Remainder = 4 − 1 + 3 − 5 + c + 2 − 1 + 1 − 4 + 6 − 2 (By remainder theorem if P(x) is divided by x − 1 then the remainder is P(1))

⇨ 3 + c = 2
⇨ c = -1

Option (d) is correct.

288. The remainder R(x) obtained by dividing the polynomial x^{100} by the polynomial $x^2 - 3x + 2$ is

(a) $2^{100} - 1$
(b) $(2^{100} - 1)x - 2(2^{99} - 1)$
(c) $2^{100}x - 3*2^{100}$
(d) $(2^{100} - 1)x + 2(2^{99} - 1)$

Solution :

Now, $x^2 - 3x + 2 = (x - 1)(x - 2)$

Let, $x^{100} = (x - 1)(x - 2)Q(x) + Ax + B$ (as the divider is quadratic, remainder is linear, Q(x) is quotient)

Putting x = 1 we get, A + B = 1 ………. (1)

Putting x = 2, 23 get, $2A + B = 2^{100}$

⇨ $2A + 1 - A = 2^{100}$ (from (1))
⇨ $A = 2^{100} - 1$

Putting value of A in equation (1) we get, $B = 1 - 2^{100} + 1 = -2(2^{99} - 1)$

Remainder = $(2^{100} - 1)x - 2(2^{99} - 1)$

Option (b) is correct.

289. If $3x^4 - 6x^3 + kx^2 - 8x - 12$ is divided by $x - 3$ then it is also divisible by
(a) $3x^2 - 4$
(b) $3x^2 + 4$
(c) $3x^2 + x$
(d) $3x^2 - x$

Solution :

Let $P(x) = 3x^4 - 6x^3 + kx^2 - 8x - 12$

By remainder theorem, if we divide it by $x - 3$ then the remainder is $P(3)$.

Therefore, $P(3) = 0$ (as it is divisible by $x - 3$)

⇨ $243 - 162 + 9k - 24 - 12 = 0$
⇨ $9k + 45 = 0$
⇨ $k = -5$

Therefore, $P(x) = 3x^4 - 6x^3 - 5x^2 - 8x - 12$

$= 3x^4 - 9x^3 + 3x^3 - 9x^2 + 4x^2 - 12x + 4x - 12$

$= 3x^3(x - 3) + 3x^2(x - 3) + 4x(x - 3) + 4(x - 3)$

$= (x - 3)(3x^3 + 3x^2 + 4x + 4)$

$= (x - 3)\{3x^2(x + 1) + 4(x + 1)\}$

$= (x - 3)(x + 1)(3x^2 + 4)$

Option (b) is correct.

290. The number of integers x such that $2^{2x} - 3(2^{x+2}) + 2^5 = 0$ is
(a) 0
(b) 1
(c) 2
(d) None of the foregoing numbers.

Solution :

Let $2^x = a$

The equation becomes, $a^2 - 12a + 32 = 0$

- ⇨ $(a - 4)(a - 8) = 0$
- ⇨ $a = 4, 8$
- ⇨ $2^x = 2^2, 2^3$
- ⇨ $x = 2, 3$

Option (c) is correct.

291. If the roots of the equation $(x - a)(x - b) + (x - b)(x - c) + (x - c)(x - a) = 0$ (where a, b, c are real numbers) are equal, then
    (a) $b^2 - 4ac = 0$
    (b) $a = b = c$
    (c) $a + b + c = 0$
    (d) none of the foregoing statements is correct.

Solution :

Now, the equation is, $3x^2 - 2x(a + b + c) + (ab + bc + ca) = 0$

So, $4(a + b + c)^2 - 12(ab + bc + ca) = 0$

- ⇨ $a^2 + b^2 + c^2 - ab - bc - ca = 0$
- ⇨ $(1/2)\{(a - b)^2 + (b - c)^2 + (c - a)^2\} = 0$
- ⇨ $a = b = c$

Option (b) is correct.

292. Suppose that a, b, c are three distinct real numbers. The expression $(x - a)(x - b)/\{(c - a)(c - b) + (x - b)(x - c)/\{(a - b)(a - c)\} + (x - c)(x - a)/\{(b - c)(b - a)\} - 1$ takes the value zero for
    (a) no real x
    (b) exactly two distinct real x
    (c) exactly three distinct real x
    (d) more than three real x.

Solution :

Option (d) is correct.

293. If $|x^2 - 7x + 12| > x^2 - 7x + 12$, then
(a) $x \leq 3$ or $x \geq 4$
(b) $3 \leq x \leq 4$
(c) $3 < x < 4$
(d) x can take any value except 3 and 4.

Solution :

$|(x - 3)(x - 4)| > (x - 3)(x - 4)$

Clearly, option (c) is correct.

294. The real numbers x such that $x^2 + 4|x| - 4 = 0$ are
(a) $-2 \pm \sqrt{8}$
(b) $2 \pm \sqrt{8}$
(c) $-2 \pm \sqrt{8}, 2 \pm \sqrt{8}$
(d) $\pm(\sqrt{8} - 2)$

Solution :

$|x|^2 + 4|x| - 4 = 0$

⇨ $|x|^2 + 4|x| + 4 = 8$
⇨ $(|x| + 2)^2 = 8$
⇨ $|x| + 2 = \pm \sqrt{8}$
⇨ $|x| = -2 \pm \sqrt{8}$
⇨ $x = \pm(-2 \pm \sqrt{8}) = -2 \pm \sqrt{8}, 2 \pm \sqrt{8}$

Option (c) is correct.

295. The number of distinct real roots of the equation $|x^2 + x - 6| - 3x + 7 = 0$
(a) 0
(b) 2
(c) 3
(d) 4

Solution :

$x^2 + x - 6 - 3x + 7 = 0$

- ⇨ $x^2 - 2x + 1 = 0$
- ⇨ $(x - 1)^2 = 0$
- ⇨ $x = 1$

where, $x^2 + x - 6 > 0$

- ⇨ $(x + 3)(x - 2) > 0$
- ⇨ $x = 1$ is not a solution.

Now, $-x^2 - x + 6 - 3x + 7 = 0$

- ⇨ $x^2 + 4x - 13 = 0$
- ⇨ $x = \{-4 \pm \sqrt{(16 + 52)}\}/2$
- ⇨ $x = -2 \pm \sqrt{17}$

Where, $x^2 + x - 6 < 0$

- ⇨ $(x + 3)(x - 2) < 0$

Both are no solution.

Option (a) is correct.

296. If a is strictly negative and is not equal to -2, then the equation $x^2 + a|x| + 1 = 0$
    (a) cannot have any real roots
    (b) must have either four real roots or no real roots
    (c) must have exactly two real roots
    (d) must have either two real roots or no real roots.

Solution :

Now, $x^2 - ax + 1 = 0$ and $x^2 + ax + 1 = 0$ both have same discriminant = $a^2 - 4$.

So, either both have real roots or both have imaginary roots.

Option (b) is correct.

297. The angles of a triangle are in A.P. and the ratio of the greatest to the smallest angle is 3 : 1. Then the smallest angle is
(a) $\pi/6$
(b) $\pi/3$
(c) $\pi/4$
(d) none of the foregoing angles.

Solution :

Let angles are $A - d, A, A + d$.

$A - d + A + A + d = \pi$

⇨ $A = \pi/3$

Now, $(A + d)/(A - d) = 3/1$

⇨ $A + d = 3A - 3d$
⇨ $d = A/2 = \pi/6$

Smallest angle = $\pi/3 - \pi/6 = \pi/6$

Option (a) is correct.

298. Let x_1, x_2, …. Be positive integers in A.P., such that $x_1 + x_2 + x_3 = 12$ and $x_4 + x_6 = 14$. Then x_5 is
(a) 7
(b) 1
(c) 4
(d) None of the foregoing numbers.

Solution :

Let common difference is d.

$x_1 + x_1 + d + x_1 + 2d = 12$

⇨ $x_1 + d = 4$ …… (1)

Now, $x_1 + 3d + x_1 + 5d = 14$

⇨ $x_1 + 4d = 7$
⇨ $4 - d + 4d = 7$ (from (1))

- ⇨ $3d = 3$
- ⇨ $d = 1$.
- ⇨ $x_1 = 3$ (from (1))
- ⇨ $x_5 = x_1 + 4d = 3 + 4 = 7$

Option (a) is correct.

299. The sum of the first m terms of an Arithmetic Progression is n and the sum of the first n terms is m, where $m \neq n$. Then the sum of first m + n terms is
(a) 0
(b) $m + n$
(c) $-mn$
(d) $-m - n$

Solution :

Let, first term is a and common difference is d.

$(m/2)\{2a + (m - 1)d\} = n$ and $(n/2)\{2a + (n - 1)d\} = m$

- ⇨ $2a + (m - 1)d = 2n/m$ and $2a + (n - 1)d = 2m/n$

Subtracting we get, $(m - 1 - n + 1)d = 2n/m - 2m/n$

- ⇨ $(m - n)d = 2(n - m)(n + m)/mn$
- ⇨ $d = -2(m + n)/mn$
- ⇨ $2a - 2(m - 1)(m + n)/mn = 2n/m$
- ⇨ $a = n/m + (m - 1)(m + n)/mn$

Sum of m + n terms = $\{(m + n)/2\}\{2a + (m + n - 1)d\}$

$= \{(m + n)/2\}\{2n/m + 2(m - 1)(m + n)/mn - 2(m + n - 1)(m + n)/mn\}$

$= (m + n)\{n/m + (m - 1)(m + n)/mn - (m - 1)(m + n)/mn - n(m + n)/mn\}$

$= (m + n)(-1)$

$= -m - n$

Option (d) is correct.

300. In an A.P., suppose that, for some $m \neq n$, the ratio of the sum of the first m terms to the sum of the first n terms is m^2/n^2. If the 13th term of the A.P. is 50, then the 26th term of the A.P. is

(a) 75
(b) 76
(c) 100
(d) 102

Solution :

Now, $(m/2)\{2a + (m - 1)d\}/[(n/2)\{2a + (n - 1)d\} = m^2/n^2$

- ⇨ $\{2a + (m - 1)d\}/\{2a + (n - 1)d\} = m/n$
- ⇨ $\{2a + (m - 1)d\}/\{2a + (n - 1)d\} - 1 = m/n - 1$
- ⇨ $(m - n)d/\{2a + (n - 1)d\} = (m - n)/n$
- ⇨ $2a + (n - 1)d = nd$
- ⇨ $2a = d$

Now, $a + 12d = 50$

- ⇨ $a + 24a = 50$ (from above)
- ⇨ $a = 2$, $d = 4$
- ⇨ 26th term = $a + 25d = 102$

Option (d) is correct.

301. Let S_n, $n \geq 1$, be the set defined as follows :
$S_1 = \{0\}$, $S_2 = \{3/2, 5/2\}$, $S_3 = \{8/3, 11/3, 14/3\}$, $S_4 = \{15/4, 19/4, 23/4, 27/4\}$, and so on. Then, the sum of the elements of S_{20} is

(a) 589
(b) 609
(c) 189
(d) 209

Solution :

First term of $S_{20} = (20^2 - 1)/20$ and common difference = 1

Therefore, sum = $(20/2)[2*\{(20^2 - 1)/20\} + (20 - 1)*1] = 10(399/10 + 19) = 10(39.9 + 19) = 10*58.9 = 589$

Option (a) is correct.

302. The value of 1*2 + 2*3 + 3*4 + …. + 99*100 equals
   (a) 333000
   (b) 333300
   (c) 30330
   (d) 33300

Solution :

$\sum n(n + 1)$ (summation running from n = 1 to n = 99)

$= \sum(n^2 + n) = \sum n^2 + \sum n = 99*100*199/6 + 99*100/2 = 333300$

Option (b) is correct.

303. The value of 1*2*3 + 2*3*4 + 3*4*5 + …. + 20*21*22 equals
   (a) 51330
   (b) 53130
   (c) 53310
   (d) 35130

Solution :

$\sum n(n + 1)(n + 2)$ (summation running from n =1 to n = 20)

$= \sum n(n^2 + 3n + 2)$

$= \sum(n^3 + 3n^2 + 2n)$

$= \sum n^3 + 3\sum n^2 + 2\sum n$

$= \{20*21/2\}^2 + 3*20*21*41/6 + 2*20*21/2$

$= 53130$

Option (b) is correct.

304. Six numbers are in A.P. such that their sum is 3. The first number is four times the third number. The fifth number is equal to
   (a) -15
   (b) -3

(c) 9
(d) -4

Solution :

a = 4(a + 2d) (first term = a, common difference = d)

⇨ 3a + 8d = 0

(6/2){2a + (6 – 1)d} = 3

⇨ 2a + 5d = 1
⇨ d = -3, a = 8 (solving above two equations)
⇨ 5^{th} term = 8 + 4(-3) = -4

Option (d) is correct.

305. The sum of the first n terms (n > 1) of an A.P. is 153 and the common difference is 2. If the first term is an integer, the number of possible values of n is
(a) 3
(b) 4
(c) 5
(d) 6

Solution :

(n/2){2a + (n – 1)*2} = 153

⇨ n(a + n – 1) = 3^2*17

Number of factors of 153 excluding 1 is (2 + 1)(1 + 1) – 1 = 5

Option (c) is correct.

306. Six numbers are in G.P. such that their product is 512. If the fourth number is 4, then the second number is
(a) ½
(b) 1
(c) 2
(d) None of the foregoing numbers.

Solution :

$ar^3 = 4$ (a = first term, r = common ratio)

$a*ar*ar^2*ar^3*ar^4*ar^5 = 512$

- ⇨ $a^6r^{15} = 512$
- ⇨ $a^2r^5 = 8$
- ⇨ $(ar^3)^2/r = 8$
- ⇨ $16/r = 8$ (from above)
- ⇨ $r = 2$.
- ⇨ $a = ½$
- ⇨ $ar = 1$

Option (b) is correct.

307. Let a and b be positive integers with no common factors. Then
  (a) a + b and a − b have no common factor other than3
  (b) a + b and a − b have no common factor greater than 2, whatever be a and b
  (c) a + b and a − b have a common factor, whatever be a and b
  (d) none of the foregoing statements is correct.

Solution :

If a and b both odd then a + b and a − b have 2 as common factor. So option (a) cannot be true.

Let us consider 20 and 23. Then 43 and 3 doesn't have any common factor. So, option (c) cannot be true.

Option (b) is correct. (Because 4 cannot be a factor of a + b and a − b)

308. If positive numbers a, b, c, d are in harmonic progression and a ≠ b, then
  (a) a + d > b + c is always true
  (b) a + b > c + d is always true
  (c) a + c > b + d is always true
  (d) none of the foregoing statements is always true.

Solution :

Now, 1/a + 1/d = 1/b + 1/c

- ⇨ (a + d)/ad = (b + c)/bc
- ⇨ (a + d)/(b + c) = ad/bc = (1/b)(1/c)/{(1/a)(1/d)} = (1/a + d)(1/a + 2d)/{(1/a + 3d)(1/a)}

Let 1/a = a_1

- ⇨ (a + d)/(b + c) = (a_1 + r)(a_1 + 2r)/{(a_1 + 3r)a_1} = (a_1^2 + 3ra_1 + 2dr^2)/(a_1^2 + 3ra_1)
- ⇨ (a + d)/(b + c) – 1 = 2r^2/(a_1^2 + 3ra_1) = 2r^2/(1/a)(1/d) = 2r^2ad > 0 (as a, d > 0 and r^2 > 0)
- ⇨ (a + d)/(b + c) > 1
- ⇨ (a + d) > (b + c)

Option (a) is correct.

309. The sum of the series 1 + 11 + 111 + ….. to n terms is
   (a) (1/9)[(10/9)(10^n – 1) + n]
   (b) (1/9)[(10/9)(10^n -) – n]
   (c)(10/9)[(1/9)(10^n – 1) – n]
   (d) (10/9)[(1/9)(10^n – 1) + n]

Solution :

1 + 11 + 111 + …. To ne terms

= (1/9)(9 + 99 + 999 + … to n terms)

= (1/9)(10 – 1 + 10^2 – 1 + 10^3 – 1 + …. To n terms)

= (1/9)[(10 + 10^2 + …. + 10^n) – n]

= (1/9)[10*(10^n – 1)/(10 – 1) – n]

= (1/9)[(10/9)(10^n – 1) – n]

Option (b) is correct.

310. Two men set out at the same time to walk towards each other from points A and B, 72 km apart. The first man walks at the rate of 4 km per hour. The second man walks 2 km the first hour, 2.5 km the second hour, 3 km the third hour, and so on. Then the men will meet

(a) in 7 hours
(b) nearer A than B
(c) nearer B than A
(d) midway between A and B

Solution :

Let, they meet after n hour.

So, the first man goes = 4n km

Second man goes = 2 + 2.5 + 3 + 3.5 + …. To n terms = (n/2){2*2 + (n – 1)*0.5} = (n/2){4 + (n – 1)0.5}

Now, 4n + (n/2){4 + (n – 1)0.5} = 72

- ⇨ (n/2)(8 + 4 + 0.5n – 0.5) = 72
- ⇨ $0.5n^2 + 11.5n - 144 = 0$
- ⇨ $n^2 + 23n - 288 = 0$
- ⇨ (n + 32)(n – 9) = 0
- ⇨ n = 9

First person goes, 4*9 = 36 km

Therefore they meet at midway between A and B.

Option (d) is correct.

311. The second term of a geometric progression (of positive numbers) is 54and the fourth term is 24. Then the fifth term is
(a) 12
(b) 18
(c) 16
(d) None of the foregoing numbers.

Solution :

$ar = 54$ and $ar^3 = 24$ (a = first term and r = common ratio)

- ⇨ $ar^3/ar = 24/54$
- ⇨ $r^2 = 4/9$
- ⇨ r = 2/3 (as G.P. is of positive terms)
- ⇨ a = 81

⇨ fifth term = fourth term*r = 24*(2/3) = 16

Option (c) is correct.

312. Consider an arithmetic progression whose first term is 4 and the common difference is -0.1. Let S_n stand for the sum of the first n terms. Suppose r is a number such that S_n = r for some n. Then the number of *other* values of n for which S_n = r is
(a) 0 or 1
(b) 0
(c) 1
(d) > 1

Solution :

Option (a) is correct.

313. The three sides of a right-angles triangle are in G.P. The tangents of the two acute angles are
(a) (√5 + 1)/2 and (√5 – 1)/2
(b) √{(√5 + 1)/2} and √{(√5 – 1)/2}
(c) √5 and 1/√5
(d) None of the foregoing pairs of numbers.

Solution :

Now, b^2 = ac (Right angle at A)

⇨ $\sin^2 B = \sin A \sin C$
⇨ $\sin^2 B = \sin C$ ($\sin A = 1$)
⇨ $\sin^2 B = \sin(90 - B)$ ($C + B = 90$)
⇨ $\sin^2 B = \cos B$
⇨ $1 - \cos^2 B = \cos B$
⇨ $\cos^2 B + \cos B - 1 = 0$
⇨ $\cos B = \{-1 + \sqrt{(1 + 4)}\}/2$ (As B is acute)
⇨ $\cos B = (\sqrt{5} - 1)/2$
⇨ $\tan B = \sqrt{\{2^2 - (\sqrt{5} - 1)^2\}}/(\sqrt{5} - 1) = \sqrt{(4 - 5 - 1 + 2\sqrt{5})}/(\sqrt{5} - 1) = \sqrt{\{2(\sqrt{5} - 1)\}}/(\sqrt{5} - 1)$
⇨ $\tan B = \sqrt{\{2/(\sqrt{5} - 1)\}} = \sqrt{\{(\sqrt{5} + 1)/2\}}$
⇨ $\tan C = 1/\tan B = \sqrt{\{2/(\sqrt{5} + 1)\}} = \sqrt{\{(\sqrt{5} - 1)/2\}}$

Option (b) is correct.

314. The m^{th} term of an arithmetic progression is x and n^{th} term is y. Then the sum of the first (m + n) teerms is

(a) $\{(m + n)/2\}[(x + y) + (x - y)/(m - n)]$
(b) $\{(m + n)/2\}[(x - y) + (x + y)/(m - n)]$
(c) $(1/2)[(x + y)/(m + n) + (x - y)/(m - n)]$
(d) $(1/2)[(x + y)/(m + n) - (x - y)/(m - n)]$

Solution :

$(m/2)\{2a + (m - 1)d\} = x$

$(n/2)\{2a + (n - 1)d\} = y$

Two equations, two unknowns – a, d. Solve them and put in $\{(m + n)/2\}[2a + (m + n - 1)d]$ and check which answer is correct. It is a long calculation based problem.

Option (a) is correct.

315. The time required for any initial amount of a radioactive substance to decrease to half amount is called the *half-life* of that substance. For example, radium has a half-life of 1620 years. If 1 gm of radium is taken in a capsule, then after 4860 years, the amount of radium left in the capsule will be, in gm,

(a) 1/3
(b) ¼
(c) 1/6
(d) 1/8

Solution :

Now, 4860/1620 = 3

Third term of the G.P. = ½ , ¼, 1/8

Option (d) is correct.

316. The sum of all the number s between 200 and 400 which are divisible by 7 is
(a) 9872
(b) 7289
(c) 8729
(d) 8279

Solution :

Now, 200 ≡ 4 (mod 7)

First term = 203.

400 ≡ 1 (mod 7)

Last term = 399

Let 203 + (n – 1)7 = 399

- ⇨ (n – 1)*7 = 196
- ⇨ n – 1 = 28
- ⇨ n = 29.

Sum = (29/2)(203 + 399) = (29/2)*602 = 29*301 = 8729

Option (c) is correct.

317. The sum of the series $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots. - 100^2$ is
(a) -10100
(b) -5050
(c) -2525
(d) None of the foregoing numbers.

Solution :

$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots. - 100^2$

$= (1^2 + 2^2 + 3^2 + \ldots. + 99^2 + 100^2) - 2*2^2(1^2 + 2^2 + \ldots. + 50^2)$

= 100*101*201/6 – 8*50*51*101/6

= 101*50(67 – 68)

= - 5050

Option (b) is correct.

318. x_1, x_2, x_3, …. Is an infinite sequence of positive integers in G.P., such that $x_1x_2x_3x_4 = 64$. Then the value of x_5 is
 (a) 4
 (b) 64
 (c) 128
 (d) 16

Solution :

Let, common ratio = r

So, $x_1*(x_1r)*(x_1r^2)*(x_1r^3) = 64$

- ⇨ $x_1^4r^6 = 64$
- ⇨ $x_1^2r^3 = 8$
- ⇨ $x_1 = 1, r = 2$ (as $r \neq 1$)
- ⇨ $x_5 = 1*2^4 = 16$

Option (d) is correct.

319. The value of 100[1/(1*2) + 1/(2*3) + 1/(3*4) + …. + 1/(99*100)]
 (a) is 99
 (b) lies between 50 and 98
 (c) is 100
 (d) is different from values specified in the foregoing statements.

Solution :

100[1 – ½ + ½ - 1/3 + 1/3 – ¼ + …. + 1/99 – 1/100] = 100(1 – 1/100) = 100(99/100) = 99

Option (a) is correct.

320. The value of 1*3*5 + 3*5*7 + 5*7*9 + …. + 17*19*21 equals
 (a) 12270
 (b) 17220

(c) 12720
(d) 19503

Solution :

$\sum(2n - 1)(2n + 1)(2n + 3)$ (summation running from n = 1 to n = 9)

$= \sum(4n^2 - 1)(2n + 3)$

$= \sum(8n^3 + 12n^2 - 2n - 3)$

$= 8\sum n^3 + 12\sum n^2 - 2\sum n - \sum 3$

$= 8*(9*10/2)^2 + 12(9*10*19/6) - 2*(9*10/2) - 3*9$

$= 8*45^2 + 12*15*19 - 90 - 27$

$= 4*15(270 + 57) - 117$

$= 19503$

Option (d) is correct.

321. The sum 1*1! + 2*2! + 3*3! + …. + 50*50! Equals
(a) 51!
(b) 2*51!
(c) 51! – 1
(d) 51! + 1

Solution :

Now, n*n! = (n + 1 – 1)n! = (n + 1)*n! – n! = (n + 1)! – n!

The sum is, 2! – 1! + 3! – 2! + 4! – 3! + …. + 51! – 50!

= 51! – 1! = 51! – 1

Option (c) is correct.

322. The value of 1/(1*3*5) + 1/(3*5*7) + 1/(5*7*9) + 1/(7*9*11) + 1/(9*11*13) equals
(a) 70/249

(b) 53/249
(c) 35/429
(d) 35/249

Solution :

Now, 1/(3*5*7) = (1/2)(5 – 3)/(3*5*7) = (1/2){1/(3*7) – 1/(5*7)} = (1/8)(7 – 3)/(3*7) – (1/4)(7 – 5)/(7*5)

= (1/8)(1/3) – (1/8)(1/7) – (1/4)(1/5) + (1/4)(1/7)

= (1/8)(1/3) – (1/4)(1/5) + (1/8)(1/7)

Similarly doing for other terms the sum becomes,

(1/8)(1/1) – (1/4)(1/3) + (1/8)(1/5) + (1/8)(1/3) – (1/4)(1/5) + (1/8)(1/7) + (1/8)(1/5) – (1/4)(1/7) + (1/8)(1/9) + …. + (1/8)(1/9) – (1/4)(1/11) + (1/8)(1/13)

= 1/8 – 1/(8*3) – 1/(8*11) + 1/(8*13)

= {1/(8*3*11*13)}(3*11*13 – 11*13 – 39 + 33)

= (2*11*13 – 6)/(8*3*11*13)

= (11*13 – 3)/(4*3*11*13)

= 140/(4*3*11*13)

= 35/429

Option (c) is correct.

323. The value of 1/(1*2*3*4) + (1/2*3*4*5) + 1/(3*4*5*6) + …. + 1/(9*10*11*12) is
(a) 73/1320
(b) 733/11880
(c) 73/440
(d) 1/18

Solution :

Same process as the previous one.

Option (a) is correct.

324. The value of $1/(1*3*5) + 1/(3*5*7) + \ldots + 1/(11*13*15)$ equals
(a) 32/195
(b) 16/195
(c) 64/195
(d) None of the foregoing numbers.

Solution :

Same process as previous one.

Option (b) is correct.

325. The value of $(1*2)/3! + (2*2^2)/4! + (3*2^3)/5! + \ldots + (15*2^{15})/17!$ Equals
(a) $2 - (16*2^{17})/17!$
(b) $2 - 2^{17}/17!$
(c) $1 - (16*2^{17})/17!$
(d) $1 - 2^{16}/17!$

Solution :

Now, $n*2^n/(n + 2)! = (n + 2 - 2)*2^n/(n+2)! = (n + 2)2^n/(n + 2)! - 2^{n+1}/(n + 2)! = 2^n/(n + 1)! - 2^{n+1}/(n + 2)!$

Putting n = 1, we get, $2^1/2! - 2^2/3!$

Putting n = 2, we get, $2^2/3! - 2^3/4!$

Putting n = 3, we get, $2^3/4! - 2^4/5!$

…

..

Putting n = 15 we get, $2^{15}/16! - 2^{16}/17!$

Adding the above equalities we get, the sum $= 2^1/2! - 2^{16}/17! = 1 - 2^{16}/17!$

Option (d) is correct.

326. The value of $4^2 + 2*5^2 + 3*6^2 + ... + 27*30^2$ is
(a) 187854
(b) 187860
(c) 187868
(d) 187866

Solution :

Now, $\sum n*(n + 3)^2$ (summation running from n = 1 to n = 27)

$= \sum n(n^2 + 6n + 9)$

$= \sum(n^3 + 6n^2 + 9n)$

$= \sum n^3 + 6\sum n^2 + 9\sum n$

$= (27*28/2)^2 + 6*27*28*55/6 + 9*27*28/2$

= 142884 + 41580 + 3402

= 187866

Option (d) is correct.

327. The distances passed over by a pendulum bob in successive swings are 16, 12, 9, 6.75, …. Cm. Then the total distance traversed by the bob before it comes to rest is (in cm)
(a) 60
(b) 64
(c) 65
(d) 67

Solution :

First term = 16, r = ¾ (r = common ratio)

This is sum of an infinite G.P. = a/(1 – r) = 16/(1 – ¾) = 64

Option (b) is correct.

328. In a sequence a_1, a_2, …. of real numbers it is observed that $a_p = \sqrt{2}$, $a_q = \sqrt{3}$ and $a_r = \sqrt{5}$, where $1 \le p < q < r$ are positive integers. Then a_p, a_q, a_r can be terms of
(a) an arithmetic progression
(b) a harmonic progression
(c) an arithmetic progression if and only if p, q, r are perfect squares
(d) neither an arithmetic progression nor an harmonic progression.

Solution :

Clearly, option (d) is correct.

329. Suppose a, b, c are in G.P. and $a^p = b^q = c^r$. Then
(a) p, q, r are in G.P.
(b) p, q, r are in A.P.
(c) 1/p, 1/q, 1/r are in A.P.
(d) None of the foregoing statements is true.

Solution :

$b^2 = ac$

⇨ 2logb = loga + logc

Now, $a^p = b^q = c^r = k$

Ploga = k

⇨ loga = k/p, similarly, logb = k/q and logc = k/r

Putting values in above equation we get, 2k/q = k/p + k/r

⇨ 1/p + 1/r = 2/q

Option (c) is correct.

330. Three real numbers a, b, c are such that a^2, b^2, c^2 are terms of an arithmetic progression. Then
(a) a, b, c are terms of a geometric progression
(b) (b + c), (c + a), (a + b) are terms of an arithmetic progression
(c) (b + c), (c + a), (a + b) are terms of an harmonic progression

(d) None of the foregoing statements is necessarily true.

Solution :

$2b^2 = c^2 + a^2$

Option (a) cannot be true.

If, option (b) is correct, then $2(c + a) = 2b + c + a$

- ⇨ $(c + a) = 2b$
- ⇨ Option (b) cannot be true.

If option (c) is correct then $2/(c + a) = 1/(b + c) + 1/(a + b)$

- ⇨ $2(b + c)(a + b) = (c + a)(c + a + 2b)$
- ⇨ $2b^2 + 2ab + 2bc + 2ca = c^2 + a^2 + 2ca + 2ab + 2bc$
- ⇨ $2b^2 = c^2 + a^2$

Option (c) is correct.

331. If a, b, c, d and p are distinct real numbers such that $(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)p + (b^2 + c^2 + d^2) \leq 0$, then
(a) a, b, c and d are in H.P.
(b) ab, bc and cd are in A.P.
(c) a, b, c and d are in A.P.
(d) a, b, c and d are in G.P.

Solution :

Now, $(a^2p^2 - 2abp + b^2) + (b^2p^2 - 2bcp + c^2) + (c^2p^2 - 2cdp + d^2) \leq 0$

- ⇨ $(ap - b)^2 + (bp - c)^2 + (cp - d)^2 \leq 0$
- ⇨ Sum of squares less than or equal to zero.
- ⇨ Individually all equal to zero.
- ⇨ $a/b = c/b = d/c = p$

Option (d) is correct.

332. Let n quantities be in A.P., d being the common difference. Let the arithmetic mean of the squares of these quantities exceed the

square of the arithmetic mean of these quantities by a quantity p. Then o
(a) is always negative
(b) equals $\{(n^2 - 1)/12\}d^2$
(c) equals $d^2/12$
(d) equals $(n^2 - 1)/12$

Solution :

$(a_1^2 + a_2^2 + \ldots + a_n^2)/n - \{(a_1 + a_2 + \ldots + a_n)/n\}^2 = p$

- ⇨ $p = (a_1^2 + a_1^2 + 2a_1d + d^2 + a_1^2 + 4a_1d + 4d^2 + a_1^2 + 6a_1d + 9d^2 + \ldots)/n - [a_1 + \{(n - 1)/2\}d]^2$
- ⇨ $p = a_1^2 + 2a_1d(n - 1)/2 + d^2(n - 1)(2n - 1)/6 - a_1^2 - (n - 1)a_1d - (n - 1)^2d^2/4$
- ⇨ $p = (d^2/12)(4n^2 - 6n + 2 - 3n^2 + 6n - 3)$
- ⇨ $p = \{(n^2 - 1)/12\}d^2$

Option (b) is correct.

333. Suppose that $F(n + 1) = (2F(n) + 1)/2$ for $n = 1, 2, 3, \ldots$ and $F(1) = 2$. Then $F(101)$ equals
(a) 50
(b) 52
(c) 54
(d) None of the foregoing quantities.

Solution :

Now, $F(n + 1) - F(n) = ½$

Putting $n = 1$, we get, $F(2) - F(1) = ½$

Putting $n = 2$, we get, $F(3) - F(2) = ½$

Putting $n = 3$, we get, $F(4) - F(3) = ½$

…

…

Putting $n = 100$, we get, $F(101) - F(100) = ½$

Adding the above equalities we get, F(101) – F(1) = 100*(1/2)

⇨ F(101) = 52

Option (b) is correct.

334. Let $\{F_n\}$ be the sequence of numbers defined by $F_1 = 1 = F_2$; $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$. Let f_n be the remainder left when F_n is divided by 5. Then f_{2000} equals
(a) 0
(b) 1
(c) 2
(d) 3

Solution :

Fibonacci numbers are, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …..

So, F_5, F_{10}, … i.e. whose index is divisible by 5 are divisible by 5.

Therefore, $f_{2000} = 0$ (as 2000 is divisible by 5)

Option (a) is correct.

335. Consider the two arithmetic progressions 3, 7, 11, …., 407 and 2, 9, 16, …, 709. The number of common terms of these two progressions is
(a) 0
(b) 7
(c) 15
(d) 14

Solution :

First A.P.. first term = 3, common difference = 4

Second A.P. first term = 2, common difference = 7.

So, common term will come after 7^{th} term of the first A.P. and 4^{th} term of second A.P.

First common term is 23.

Number of terms = n of first A.P. (say)

- ⇨ $3 + (n - 1)*4 = 407$
- ⇨ $n = 102$.
- ⇨ $23 = 6^{th}$ term.
- ⇨ $6 + (m - 1)*7 \leq 102$
- ⇨ $7(m - 1) \leq 96$
- ⇨ $m - 1 = 13$
- ⇨ $m = 14$

Option (d) is correct.

336. The arithmetic mean of two positive numbers is 18 + ¾ and their geometric mean is 15. The larger of the two numbers is
    (a) 24
    (b) 25
    (c) 20
    (d) 30

Solution :

$a > b$

$(a + b)/2 = 18 + ¾$, $a + b = 37.5$

$ab = 15^2 = 225$

$(a - b)^2 = (a + b)^2 - 4ab = (37.5)^2 - 4*225$

- ⇨ $a - b = 22.5$
- ⇨ $a + b = 37.5$
- ⇨ $a = 25$

Option (b) is correct.

337. The difference between roots of the equation $6x^2 + \alpha x + 1 = 0$ is 1/6. Further, α is a positive number. Then the value of α is
    (a) 3
    (b) 4
    (c) 5
    (d) 2 + 1/3

Solution :

Let roots are a and b.

$a - b = 1/6$, $a + b = -\alpha/6$ and $ab = 1/6$

$(a + b)^2 = (a - b)^2 + 4ab$

- ⇨ $\alpha^2/36 = 1/36 + 4/6$
- ⇨ $\alpha^2 = 1 + 24$
- ⇨ $\alpha = 5$

Option (c) is correct.

338. If $4^x - 4^{x-1} = 24$, then $(2x)^x$ equals
   (a) $5\sqrt{5}$
   (b) $25\sqrt{5}$
   (c) 125
   (d) 25

Solution :

Let, $4^x = a$

The equation becomes, $a - a/4 = 24$

- ⇨ $3a/4 = 24$
- ⇨ $a = 32$
- ⇨ $4^x = 2^5$
- ⇨ $2^{2x} = 2^5$
- ⇨ $x = 5/2$

$(2x)^x = 5^{5/2} = 25\sqrt{5}$

Option (b) is correct.

339. The number of solutions f the simultaneous equations $y = 3\log_e x$, $y = \log_e(3x)$ is
   (a) 0
   (b) 1
   (c) 3

(d) Infinite

Solution :

Now, $3\log_e x = \log_e(3x)$

- ⇨ $\log_e x^3 = \log_e(3x)$
- ⇨ $x^3 = 3x$
- ⇨ $x^2 = 3$ $(x \neq 0)$
- ⇨ $x = \sqrt{3}$ (x cannot be negative)

Option (b) is correct.

340. The number of solutions to the system of simultaneous equations $|z + 1 - i| = \sqrt{2}$ and $|z| = 3$ is
(a) 0
(b) 1
(c) 2
(d) > 2

Solution :

Let $z = x + iy$

Now $|z + 1 - i| = \sqrt{2}$

- ⇨ $|x + iy + 1 - i| = \sqrt{2}$
- ⇨ $|(x + 1) + i(y - 1)| = \sqrt{2}$
- ⇨ $(x + 1)^2 + (y - 1)^2 = 2$ …….. (1)

And, $|z| = 3$

- ⇨ $x^2 + y^2 = 9$ ……… (2)

Doing (1) – (2) we get, $2x + 1 - 2y + 1 = -7$

- ⇨ $x - y = -9/2$
- ⇨ $y = 9/2 + x$ ………. (3)

Putting value of (3) in (1) we get, $x^2 + (9/2 + x)^2 = 9$

- ⇨ $2x^2 + 9x + 81/4 = 9$
- ⇨ $2x^2 + 9x + 45/4 = 0$

- ⇨ $8x^2 + 36x + 45 = 0$
- ⇨ $x = \{-36 \pm \sqrt{(36^2 - 4*8*45)}\}/16$
- ⇨ $x = (-36 \pm \sqrt{(-144)}/16$

So, no real value of x. So no solution at all.

Option (a) is correct.

341. The number of pairs (x, y) of real numbers that satisfy $2x^2 + y^2 + 2xy - 2y + 2 = 0$ is
   (a) 0
   (b) 1
   (c) 2
   (d) None of the foregoing numbers.

Solution :

Now, $2x^2 + 2xy + (y^2 - 2y + 2) = 0$

x is real. Therefore, discriminiant ≥ 0

- ⇨ $4y^2 - 4*2*(y^2 - 2y + 2) \geq 0$
- ⇨ $-4y^2 + 16y - 16 \geq 0$
- ⇨ $y^2 - 4y + 4 \leq 0$
- ⇨ $(y - 2)^2 \leq 0$
- ⇨ $y = 2$

Putting y = 2 we get, $2x^2 + 4x + (4 - 4 + 2) = 0$

- ⇨ $x^2 + 2x + 1 = 0$
- ⇨ $(x + 1)^2 = 0$
- ⇨ $x = -1$

One solution (-1, 2)

Option (b) is correct.

342. Consider the following equation in x and y : $(x - 2y - 1)^2 + (4x + 3y - 4)^2 + (x - 2y - 1)(4x + 3y - 4) = 0$. How many solutions to (x, y) with x, y real, does the equation have?
   (a) none
   (b) exactly one
   (c) exactly two

(d) more than two

Solution :

$(x - 2y - 1)^2 - w(x - 2y - 1)(4x + 3y - 4) - w^2(x - 2y - 1)(4x + 3y - 4) + w^3(4x + 3y - 4)^2 = 0$ (where w is cube root of unity)

⇨ $(x - 2y - 1)\{(x - 2y - 1) - w(4x + 3y - 4)\} - w^2(4x + 3y - 4)\{(x - 2y - 1) - w(4x + 3y - 4)\} = 0$
⇨ $\{(x - 2y - 1) - w(4x + 3y - 4)\}\{(x - 2y - 1) - w^2(4x + 3y - 4)\} = 0$
⇨ $(x - 2y - 1) - w(4x + 3y - 4) = 0$
⇨ $(x - 2y - 1) - \{(-1 + i\sqrt{3})/2\}(4x + 3y - 4) = 0$

Equating the real and imaginary parts from both sides we get,

$x - 2y - 1 + (1/2)(4x + 3y - 4) = 0$ and $4x + 3y - 4 = 0$

⇨ $x - 2y - 1 = 0$
⇨ $4x - 8y - 4 = 0$

Subtracting we get, $3y + 8y - 4 + 4 = 0$

⇨ $y = 0, x = 1$

Now, equating the real and imaginary part of the second equation we get same solution.

Therefore, option (b) is correct.

343. Let x and y be positive numbers and let a and b be real numbers, positive or negative. Suppose that $x^a = y^b$ and $y^a = x^b$. Then we can conclude that
(a) $a = b$ and $x = y$
(b) $a = b$ but x need not be equal to y
(c) $x = y$ but a need not be equal to b
(d) $a = b$ if $x \neq y$

Solution :

Dividing the two equations we get, $(x/y)^a = (y/x)^b$

⇨ $(x/y)^{(a - b)} = 1$
⇨ $a - b = 0$

⇨ a = b if $x \neq y$ because if x = y then a may not be equal to b.

Option (d) is correct.

344. On a straight road XY, 100 metres long, 15 heavy stones are placed one metre apart beginning at the end X. A worker, starting at X, has to transport all the stones to Y, by carrying only one stone at a time. The minimum distance he has to travel is (in km)
(a) 1.395
(b) 2.79
(c) 2.69
(d) 1.495

Solution :

First stone carried which is at X, 100 metre distance covered.

Second stone carried, 2*99 metre distance covered.

Third stone carried 2*98 metre distance covered.

…

15th stone carried 2*86 metre distance covered.

Therefore, total distance = 100 + 2(99 + 98 + … + 86)

= 100 + 2*(14/2){2*99 + (14 – 1)*(-1)}

= 100 + 14(198 – 13)

= 100 + 14*185

= 2690 metre = 2.69 km

Option (c) is correct.

345. lim [1/(1*3) + 1/(2*4) + 1/(3*5) + …. + 1/{n(n + 2)}] as n - > ∞ is
(a) 0
(b) 3/2
(c) ½
(d) ¾

Solution :

Now, [1/(1*3) + 1/(2*4) + 1/(3*5) + …. + 1/{n(n + 2)}] = (1/2)[1/1 – 1/3 + ½ - ¼ + 1/3 – 1/5 + … + 1/n – 1/(n + 2)]

= (1/2)[1 + ½ - 1/(n + 1) – 1/(n + 2)]

= (1/2)(1 + ½ - 0 – 0] as n - > ∞

= ¾

Option (d) is correct.

346. lim [1*3/2n^3 + 3*5/2n^3 + …. + (2n – 1)(2n + 1)/2n^3] as n - > ∞ is
(a) 2/3
(b) 1/3
(c) 0
(d) 2

Solution :

Now, ∑(2r – 1)(2r + 1) (summation running from r = 1 to r = n)

= ∑(4r^2 – 1)

= 4∑r^2 - ∑1

= 4n(n + 1)(2n + 1)/6 – n

Now, ∑(2r – 1)(2r + 1)/2n^3 = 4n(n + 1)(2n + 1)/(6*2n^3) – n/2n^3

= (1 + 1/n)(2 + 1/n)/3 – 1/2n^2

= (1 + 0)(2 + 0)/3 – 0 as n - > ∞

= 2/3

Option (a) is correct.

347. The coefficient of x^n in the expansion of (2 – 3x)/(1 – 3x + 2x^2) is

(a) $(-3)^n - (2)^{n/2-1}$
(b) $2^n + 1$
(c) $3(2)^{n/2-1} - 2(3)^n$
(d) None of the foregoing numbers.

Solution :

Now, $1 - 3x + 2x^2 = (1 - x)(1 - 2x)$

Now, $(1 - x)^{-1} = 1 + (-1)(-x) + \{(-1)(-1 -1)/2!\}(-x)^2 + \{(-1)(-1-1)(-1-2)/3!\}(-x)^3 + \ldots$

$= 1 + x + x^2 + x^3 + \ldots.$

Now, $(1 - 2x)^{-1} = 1 + (-1)(-2x) + \{(-1)(-1 - 1)/2!\}(-2x)^2 + \{(-1)(-1-1)(-1-2)/3!\}(-2x)^3 + \ldots$

$= 1 + 2x + (2x)^2 + (2x)^3 + \ldots..$

Coefficient of x^n in $(1 - x)^{-1}(1 - 2x)^{-1} = 2^n + 2^{n-1} + 2^{n-2} + \ldots. + 2 + 1 = 2^{n+1} - 1$

Coefficient of x^{n-1} in $(1 - x)^{-1}(1 - 2x)^{-1} = 2^n - 1$

Now, coefficient of x^n in $(2 - 3x)(1 - x)^{-1}(1 - 2x)^{-1} = 2^{n+2} - 2 - 3(2^n - 1) = 2^{n+2} - 3(2)^n + 1 = 4*2^n - 3*2^n + 1 = 2^n + 1$.

Option (b) is correct.

348. The infinite sum $1 + 1/3 + (1*3)/(3*6) + (1*3*5)/(3*6*9) + (1*3*5*7)/(3*6*9*12) + \ldots.$ is
(a) $\sqrt{2}$
(b) $\sqrt{3}$
(c) $\sqrt{(3/2)}$
(d) $\sqrt{(1/3)}$

Solution :

Now, $(1 - x)^{-1/2} = 1 + (-1/2)(-x) + \{(-1/2)(-3/2)/2!\}\}(-x)^2 + \{(-1/2)(-3/2)(-5/2)/3!\}(-x)^3 + \ldots..$

$= 1 + (1/2)x + [(1*3)/\{2^2(2!)\}]x^2 + [(1*3*5)/\{2^3(3!)\}]x^3 + \ldots..$

Putting x = 2/3 we get,

$(1 - 2/3)^{-1/2} = 1 + 1/3 + (1*3)/(3*6) + (1*3*5)/(3*6*9) + ….$

Therefore, required sum = $(1 - 2/3)^{-1/2} = (1/3)^{-1/2} = \sqrt{3}$

Option (b) is correct.

349. The sum of the infinite series 1 + (1 + 2)/2! + (1 + 2 + 3)/3! + (1 + 2 + 3 + 4)/4! + … is
(a) 3e/2
(b) 3e/4
(c) $3(e + e^{-1})/2$
(d) $e^2 - e$

Solution :

General term = (1 + 2 + …. + n)/n! = n(n + 1)/2(n!) = (n + 1)/2(n − 1)! = (n − 1 + 2)/2(n − 1)! = 1/{2(n − 2)!} + 1/(n − 1)!

Now, $e^x = \sum(x^n/n!)$ (summation running from n = 0 to n = ∞)

e = ∑(1/n!) (summation running from n = 0 to n = ∞)

Therefore, required sum = e/2 + e = 3e/2

Option (a) is correct.

350. For a nonzero number x, if $y = 1 - x + x^2/2! - x^3/3! + ….$ and $z = -y - y^2/2 - y^3/3 - …..$ then the value of $\log_e\{1/(1 - e^z)\}$ is
(a) 1 − x
(b) 1/x
(c) 1 + x
(d) x

Solution :

$z = -\sum(y^n/n)$ (summation running from n = 1 to n = ∞) = $\log_e(1 - y)$

Now, $y = \sum(-x)^n/n!$ (summation running from n = 0 to n = ∞) = e^{-x}

Now, $\log_e\{1/(1 - e^z)\} = \log_e\{1/(1 - (1 - y)\} = \log_e(1/y) = \log_e e^x = x$

Option (d) is correct.

351. For a given real number $\alpha > 0$, define $a_n = (1^\alpha + 2^\alpha + \ldots. + n^\alpha)$ and $b_n = n^n(n!)^\alpha$, for n = 1, 2, … Then

(a) $a_n < b_n$ for all $n > 1$

(b) there exists an integer $n > 1$ such that $a_n < b_n$

(c) $a_n > b_n$ for all $n > 1$

(d) there exists integers n and m both larger than one such that $a_n > b_n$ and $a_m < b_m$.

Solution :

Now, $(1^\alpha + 2^\alpha + \ldots. + n^\alpha)n > \{(1^\alpha)(2^\alpha)\ldots(n^\alpha)\}^{1/n}$ (A.M. > G.M. for unequal quantities)

- ⇨ $a_n > n^n(n!)^\alpha$
- ⇨ $a_n > b_n$

Option (c) is correct.

352. Let $a_n = (10^{n+1} + 1)/(10^n + 1)$ for n = 1, 2, ….. Then

(a) for every n, $a_n \geq a_{n+1}$

(b) for every n, $a_n \leq a_{n+1}$

(c) there is an integer k such that $a_{n+k} = a_n$ for all n

(d) none of the above holds.

Solution :

Now, $a_{n+1} - a_n = (10^{n+2} + 1)/(10^{n+1} + 1) - (10^{n+1} + 1)/(10^n + 1)$

$= \{(10^{n+2} + 1)(10^n + 1) - (10^{n+1} + 1)^2\}/(10^{n+1} + 1)(10^n + 1)$

Numerator $= 10^{2n+2} + 10^{n+2} + 10^n + 1 - 10^{2n+2} - 2*10^{n+1} - 1 = 8*10^{n+1} + 10^n$

- ⇨ $a_{n+1} > a_n$

Option (b) is correct.

353. Let a, b and c be fixed positive real numbers. Let $u_n = na/(b + nc)$ for $n \geq 1$. Then as n increases

(a) u_n increases
(b) u_n decreases
(c) u_n increases first and then decreases
(d) none of the foregoing statements is necessarily true

Solution :

$u_n = na/(b + nc) = a/(b/n + c)$

As n increases b/n decreases and b/n + c decreases, so u_n increases.

Option (a) is correct.

354. Suppose n is a positive integer. Then the least value of N for which $|(n^2 + n + 1)/(3n^2 + 1) - 1/3| < 1/10$, when $n \geq N$, is

(a) 4
(b) 5
(c) 100
(d) 1000

Solution :

$|(n^2 + n + 1)/(3n^2 + 1) - 1/3| < 1/10$

- ⇨ $|(3n^2 + 3n + 3 - 3n^2 - 1)/3(3n^2 + 1)| < 1/10$
- ⇨ $|(3n + 2)|/3(3n^2 + 1) < 1/10$ (as $3n^2 + 1$ is always positive)
- ⇨ $10|3n + 2| < 9n^2 + 3$
- ⇨ $10(3n + 2) < 9n^2 + 3$ (as $n > 0$)
- ⇨ $9n^2 - 30n - 17 > 0$
- ⇨ $9n^2 - 30n + 25 > 25 + 17$
- ⇨ $(3n - 5)^2 > 42$
- ⇨ $3n - 5 > \sqrt{42}$
- ⇨ $3n - 5 > 6 + f$ $(0 < f < 1)$
- ⇨ $3n > 11 + f$
- ⇨ $n > 11/3 + f/3$
- ⇨ $n > 3 + f_1$ $(0 < f_1 < 1)$
- ⇨ $N = 4$

Option (a) is correct.

355. The maximum value of xyz for positive x, y, z, subject to the condition xy + yz + zx = 12 is

(a) 9
(b) 6
(c)8
(d) 12

Solution :

Now, $(xy + yz + zx)/3 \geq (xy*yz*zx)^{1/3}$ (A.M. ≥ G.M.)

⇨ $12/3 \geq (xyz)^{2/3}$
⇨ $xyz \leq (4)^{3/2} = 8$

Option (c) is correct.

356. If a, b are positive real numbers satisfying $a^2 + b^2 = 1$, then the minimum value of a + b +] 1/ab is

(a) 2
(b) $2 + \sqrt{2}$
(c)3
(d) $1 + \sqrt{2}$

Solution :

Now, $(a^2 + b^2)/2 \geq \{(a + b)/2\}^2$

⇨ $(a + b) \leq \sqrt{2}$ $(a^2 + b^2 = 1)$

Now, $ab \leq (a^2 + b^2)/2$ (GM ≤ AM)

⇨ $1/ab \geq 2$

Now, $a + b \leq \sqrt{2}$ and $1/ab \geq 2$

The rate of increase of 1/ab is more than rate of decrease of a + b. So minimum value will occur when 1/ab is minimum i.e. $a = b = 1/\sqrt{2}$

So, minimum value of $a + b + 1/ab = 1/\sqrt{2} + 1/\sqrt{2} + 2 = \sqrt{2} + 2$

Option (b) is correct.

357. Let M and m be, respectively the maximum and the minimum of n arbitrary real numbers $x_1, x_2, \ldots, x_n$. Further, let M′ and m′ denote the maximum and the minimum, respectively, of the following numbers :
$x_1, (x_1 + x_2)/2, (x_1 + x_2 + x_3)/3, \ldots, (x_1 + x_2 + \ldots + x_n)/n$
Then
(a) $m \le m' \le M \le M'$
(b) $m \le m' \le M' \le M$
(c) $m' \le m \le M' \le M$
(d) $m' \le m \le M \le M'$

Solution :

The minimum value of $(x_1 + x_2 + \ldots + x_r)/r$ occurs when $x_1 = x_2 = \ldots = x_r$. In that case $m' = m$

Otherwise the minimum value is > the minimum of $x_1, x_2, \ldots, x_r$

- ⇨ $m' > m$
- ⇨ $m \le m'$

Now, maximum value of the above expression ≤ maximum of $x_1, x_2, \ldots, x_r$

- ⇨ $M' \le M$
- ⇨ $m \le m' \le M' \le M$

Option (b) is correct.

358. A stick of length 20 units is to be divided into n parts so that the product of the lengths of the part is greater than unity. The maximum possible value of n is
(a) 18
(b) 20
(c) 19
(d) 21

Solution :

Now, $(x_1x_2\ldots x_n)^{1/n} \le (x_1 + x_2 + \ldots + x_n)/n$ (GM ≤ AM) where $x_1, x_2, \ldots, x_n$ are the lengths of n parts.

- ⇨ $(x_1x_2....x_n) \le (20/n)^n$
- ⇨ Maximum n = 19.

Option (c) is correct.

359. It is given that the numbers $a \ge 0$, $b \ge 0$, $c \ge 0$ are such that $a + b + c = 4$ and $(a + b)(b + c)(c + a) = 24$. Then only one of the following statements is correct. Which one is it?
   (a) More information is needed to determine the values of a, b and c.
   (b) Even when a is given to be 1, more information needed to determine the value of b and c.
   (c) These two equations are inconsistent.
   (d) There exist values of a and b from which value of c could be determined.

Solution :

$(a + b)(4 – a)(4 – b) = 24$

- ⇨ $(a + b)\{16 – 4(a + b) + ab\} = 24$
- ⇨ $(4 – c)\{16 – 4(a + b) + ab\} = 24$
- ⇨ $64 – 16(a + b) + 4ab – 16c + 4c(a + b) – abc = 24$
- ⇨ $64 – 16(a + b + c) + 4ab – 16c + 4c(a + b) – abc = 24$
- ⇨ $64 – 64 + 4ab – 16c + 4c(a + b) – abc = 24$
- ⇨ $4ab – 4c(4 – a – b) – abc = 24$
- ⇨ $4ab – 4c^2 – abc = 24$
- ⇨ $ab(4 – c) – 4c^2 = 24$
- ⇨ $ab(a + b) = 24 + 4c^2$

Now, minimum value of c = 0, maximum value of a + b = 4 and maximum value of ab occurs when a = b = 2.

Therefore, maximum value of ab(a + b) = 2*2*4 = 16 whereas minimum RHS = 24

So, the equations are inconsistent.

Option (c) is correct.

360. Let a, b, c be any real numbers such that $a^2 + b^2 + c^2 = 1$, Then the quantity $(ab + bc + ca)$ satisfies the condition(s)

(a) (ab + bc + ca) is constant
(b) $-1/2 \le (ab + bc + ca) \le 1$
(c) $-1/4 \le (ab + bc + ca) \le 1$
(d) $-1 \le (ab + bc + ca) \le ½$

Solution :

Now, $(a + b + c)^2 \ge 0$

- ⇨ $(a^2 + b^2 + c^2) + 2(ab + bc + ca) \ge 0$
- ⇨ $(ab + bc + ca) \ge - (a^2 + b^2 + c^2)/2 = -1/2$

Now, $(a - b)^2 \ge 0$

- ⇨ $ab \le (a^2 + b^2)/2$
- ⇨ $bc \le (b^2 + c^2)/2$
- ⇨ $ca \le (c^2 + a^2)/2$
- ⇨ $ab + bc + ca \le a^2 + b^2 + c^2$ (adding the above inequalities) = 1

Option (b) is correct.

361. Let x, y, z be positive numbers. The least value of $\{x(1 + y) + y(1 + z) + z(1 + x)\}/\sqrt{(xyz)}$ is
(a) $9/\sqrt{2}$
(b) 6
(c) $1/\sqrt{6}$
(d) None of these numbers.

Solution :

Now, $x(1 + y) + y(1 + z) + z(1 + x) = x + y + z + xy + yz + zx$

Now, $(x + y + z + xy + yz + zx)/6 \ge \{x*y*z*xy*yz*zx\}^{1/6} = \sqrt{xyz}$ (AM ≥ GM)

- ⇨ $(x + y + z + xy + yz + zx)/\sqrt{(xyz)} \ge 6$
- ⇨ Least value is 6

Option (b) is correct.

362. Let a, b and c be such that $a + b + c = 0$ and $l = a^2/(2a^2 + bc) + b^2/(2b^2 + ca) + c^2/(2c^2 + ab)$ is defined. Then the value of l is

(a) 1

(b) -1

(c) 0

(d) None of the foregoing numbers.

Solution :

Now, $a^2/(2a^2 + bc) + b^2/(2b^2 + ca)$

$= a^2/(2a^2 - ab - b^2) + b^2/(2b^2 - ab - a^2)$ (Putting $c = -a - b$)

$= a^2/\{(2a + b)(a - b)\} + b^2/\{(2b + a)(b - a)$

$= \{1/(a - b)\}\{a^2/(2a + b) - b^2/(2b + a)\}$

$= \{1/(a - b)\}[(2a^2b + a^3 - 2ab^2 - b^3)/\{(2b + a)(2a + b)\}]$

$= \{1/(a - b)\}[\{2ab(a - b) + (a - b)(a^2 + ab + b^2)\}/\{(2b + a)(2a + b)\}]$

$= (a^2 + 3ab + b^2)/\{(2b + a)(2a + b)\}$

$= (1/2)\{2a^2 + 6ab + 2b^2 - (2b + a)(2a + b)\}/\{(2b + a)(2a + b) + 1$

$= (1/2)[ab/\{(2b + a)(2a + b)\} + 1$

Now, $(1/2)[ab/\{(2b + a)(2a + b)\}] + 1 + c^2/(2c^2 + ab)$

$= (1/2)[ab/\{(2b + a)(a - c)\}] + 1 + c^2/\{(2c + a)(c - a)\}$

$= \{1/(c - a)\}[(4bc^2 + 2ac^2 - 2abc - a^2b)/\{2(2c + a)(2b + a)\}] + 1$

$= \{1/(c - a)\}[\{b(4c^2 - a^2) + 2ac(c - b)\}/\{2(2c + a)(2b + a)\}] + 1$

$= \{1/(c - a)\}[\{b(2c + a)(2c - a) + 2ac(2c + a)\}/\{2(2c + a)(2b + a)\}] + 1$

$= \{1/(c - a)\}[(2bc - ab + 2ac)/\{2(2b + a)\}] + 1$

$= \{1/(c - a)\}[(-2c^2 - 2ac + ac + a^2 + 2ac)/\{2(2b + a)\}] + 1$

$= \{1/(c - a)\}[(a^2 + ac - 2c^2)/\{2(2b + a)\} + 1$

$= \{1/(c - a)\}[(a - c)(a + 2c)/\{2(2b + a)\}] + 1$

$= -(a + 2c)/\{2(2b + a)\} + 1$

$= -(a + 2c)/\{2(-2c - 2a + a)\} + 1$

= (a + 2c)/{2(a + 2c)} + 1

= ½ + 1 = 3/2

I think there is some calculation mistake. Whatever the problem is easy and evolves just calculation. Nothing else. So, you can give it a try.

Option (a) is correct.

363. Let a, b and c be distinct real numbers such that $a^2 - b = b^2 - c = c^2 - a$. Then (a + b)(b + c)(c + a) equals
   (a) 0
   (b) 1
   (c) -1
   (d) None of the foregoing numbers.

Solution :

Now, $a^2 - b = b^2 - c$

⇨ $a^2 - b^2 = b - c$
⇨ $(a + b)(a - b) = (b - c)$
⇨ $(a + b) = (b - c)/(a - b)$

Similarly, $(b + c) = (c - a)/(b - c)$ and $(c + a) = (a - b)/(c - a)$

Multiplying the three above equalities we get, the answer is 1.

Option (b) is correct.

364. Let a and y be real numbers such that $x + y \neq 0$. Then there exists an angle θ such that $\sec^2\theta = 4xy/(x + y)^2$ if and only if
   (a) $x + y > 0$
   (b) $x + y > 1$
   (c) $xy > 0$
   (d) $x = y$

Solution :

Now, $(x - y)^2 \geq 0$

- ⇨ $(x + y)^2 \geq 4xy$
- ⇨ $4xy/(x + y)^2 \leq 1$
- ⇨ $\sec^2\theta \leq 1$

But, $\sec^2\theta \geq 1$

So, it holds if and only if, $\sec^2\theta = 1$ i.e. $x = y$ (the equality holds when $x = y$)

Option (d) is correct.

365. Consider the equation $\sin\theta = (a^2 + b^2 + c^2)/(ab + bc + ca)$, where a, b, c are fixed non-zero real numbers. The equation has a solution for θ
(a) whatever be a, b, c
(b) if and only if $a^2 + b^2 + c^2 < 1$
(c) if and only if a, b and c all lie in the interval (-1, 1)
(d) if and only if $a = b = c$

Solution :

Now, $(1/2)\{(a - b)^2 + (b - c)^2 + (c - a)^2\} \geq 0$

- ⇨ $a^2 + b^2 + c^2 - ab - bc - ca \geq 0$
- ⇨ $a^2 + b^2 + c^2 \geq ab + bc + ca$
- ⇨ $(a^2 + b^2 + c^2)/(ab + bc + ca) \geq 1$
- ⇨ $\sin\theta \geq 1$

But, $\sin\theta \leq 1$

- ⇨ The equation holds if $\sin\theta = 1$ i.e. $(a - b)^2 + (b - c)^2 + (c - a)^2 = 0$ i.e. if and only if $a = b = c$

Option (d) is correct.

366. Consider the real-valued function f. defined over the set of real numbers, as $f(x) = e^{\sin(x^2 + px + q)}$, $-\infty < x < \infty$, where p, q are arbitrary real numbers. The set of real numbers y for which the equation $f(x) = y$ has a solution depends on
(a) p and not on q
(b) q and not on p
(c) both p and q

(d) neither p nor q

Solution :

Clearly, $-1 \leq \sin(x^2 + px + q) \leq 1$

So, the range of the function is $e^{-1} \leq f(x) \leq e$. So it doesn't depend on p and q.

Option (d) is correct.

367. The equation $x - \log_e(1 + e^x) = c$ has a solution
(a) for every $c \geq 1$
(b) for every $c < 1$
(c) for every $c < 0$
(d) for every $c > -1$

Solution :

Now, $x - \log_e(1 + e^x) = c$

- ⇨ $\log_e(1 + e^x) = x - c$
- ⇨ $1 + e^x = e^{x-c}$
- ⇨ $1 = e^x(e^{-c} - 1)$
- ⇨ $e^{-c} = 1/e^x + 1$
- ⇨ $e^c = e^x/(1 + e^x) < 1$
- ⇨ $c < 0$

Option (c) is correct.

368. A real value of $\log_e(6x^2 - 5x + 1)$ can be determined if and only if x lies in the subset of the real numbers defined by
(a) $\{x : 1/3 < x < ½\}$
(b) $\{x : x < 1/3\} \cup \{x : x > ½\}$
(c) $\{x : x \leq 1/3\} \cup \{x : x \geq ½\}$
(d) all the real numbers.

Solution :

Now, $6x^2 - 5x + 1 > 0$

⇨ $(3x - 1)(2x - 1) > 0$
⇨ $x > 1/3$ and $x > ½$ or $x < 1/3$ and $x < ½$
⇨ $x > ½$ or $x < 1/3$

Option (b) is correct.

369. The domain of definition of the function $f(x) = \sqrt{[\log_{10}\{(3x - x^2)/2\}]}$ is
(a) (1, 2)
(b) (0, 1]U[2, ∞)
(c) [1, 2]
(d) (0, 3)

Solution :

Now, $(3x - x^2)/2 > 0$

⇨ $x(3 - x) > 0$
⇨ $x > 0$ and $x < 3$ or $x < 0$ and $x > 3$
⇨ $0 < x < 3$

Now, $\log_{10}\{(3x - x^2)/2\} \geq 0$

⇨ $(3x - x^2)/2 \geq 1$
⇨ $3x - x^2 \geq 2$
⇨ $x^2 - 3x + 2 \leq 0$
⇨ $(x - 1)(x - 2) \leq 0$
⇨ $x \leq 1$ and $x \geq 2$ or $x \geq 1$ and $x \leq 2$
⇨ $1 \leq x \leq 2$

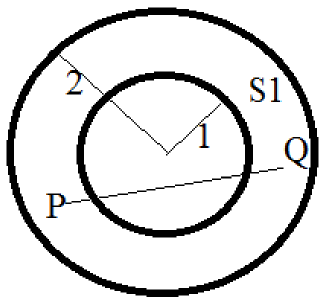
Now, the intersection of $0 < x < 3$ and $1 \leq x \leq 2$ is $1 \leq x \leq 2$

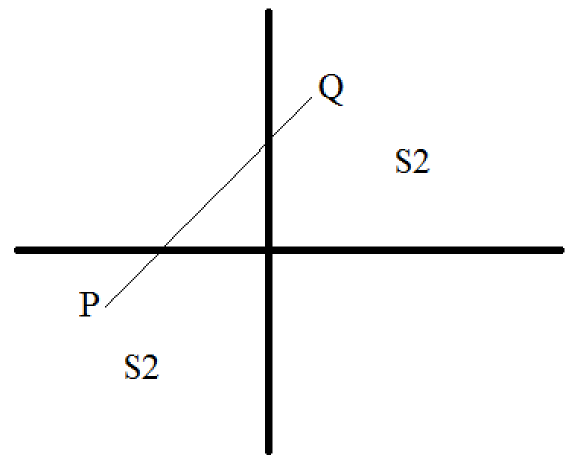
Option (c) is correct.

370. A collection S of points (x, y) of the plane is said to be *convex*, if whenever two points P = (u, v) and Q = (s, t) belong to S, every point on the line segment PQ also belongs to S. Let S_1 be the collection of all points (x, y) for which $1 < x^2 + y^2 < 2$ and let S_2 be the collection of all points (x, y) for which x and y have the same sign. Then
(a) S_1 is convex and S_2 is not convex

(b) S_1 and S_2 are both convex
(c) neither S_1 nor S_2 is convex
(d) S_1 is not convex and S_2 is convex.

Solution :

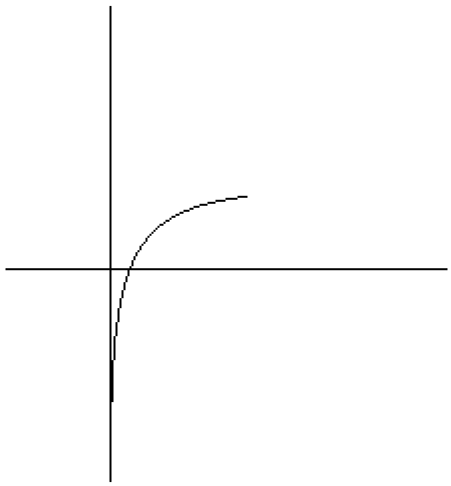




Clearly, option (c) is correct.

371. A function $y = f(x)$ is said to be *convex* if the *line segment* joining any two points $A = (x_1, f(x_1))$ and $B = (x_2, f(x_2))$ on the graph of the function *lies above the graph*. Such a line may also touch the graph at some or all points. Only one of the following four functions is not *convex*. Which one is it?
(a) $f(x) = x^2$
(b) $f(x) = e^x$
(c) $f(x) = \log_e x$
(d) $f(x) = 7 - x$

Solution :



This is graph of $y = \log_e x$

Clearly, option (c) is correct.

372. If S is the set of all real numbers x such that $|1 - x| - x \geq 0$, then
(a) $S = (-\infty, -1/2)$
(b) $S = [-1/2, ½]$
(c) $S = (-\infty, 0]$
(d) $S = (-\infty, ½]$

Solution :

Now, $|1 - x| - x \geq 0$

- ⇨ $|1 - x| \geq x$
- ⇨ $(1 - x)^2 \geq x^2$
- ⇨ $1 - 2x + x^2 \geq x^2$
- ⇨ $1 - 2x \geq 0$
- ⇨ $2x \leq 1$
- ⇨ $x \leq ½$

Option (d) is correct.

373. The inequality $\sqrt{(x + 2)} \geq x$ is satisfied if and only if
(a) $-2 \leq x \leq 2$
(b) $-1 \leq x \leq 2$

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(c) $0 \le x \le 2$
(d) None of the foregoing conditions.

Solution :

Clearly, Option (a) is correct.

374. If $l^2 + m^2 + n^2 = 1$ and $l'^2 + m'^2 + n'^2 = 1$, then the value of $ll' + mm' + nn'$
(a) is always greater than 2
(b) is always greater than 1, but less than 2
(c) is always less than or equal to 1
(d) doesn't satisfy any of the foregoing conditions

Solution :

Now, $ll' \le (l^2 + l'^2)/2$, $mm' \le (m^2 + m'^2)/2$, $nn' \le (n^2 + n'^2)/2$

⇨ $(ll' + mm' + nn') \le (1/2)\{(l^2 + m^2 + n^2) + (l'^2 + m'^2 + n'^2)\} = 1$

Option (c) is correct.

375. If a and b are positive numbers and c and d are real numbers, positive or negative, then $a^c \le b^d$
(a) if $a \le b$ and $c \le d$
(b) if either $a \le b$ or $c \le d$
(c) if $a \ge 1$, $b \ge 1$, $d \ge c$
(d) is not implied by any of the foregoing conditions.

Solution :

The condition should be, $a \ge 1$, $b \ge 1$, $a \le b$ and $d \ge c$.

Option (d) is correct.

376. For all x such that $1 \le x \le 3$, the inequality $(x - 3a)(x - a - 3) < 0$ holds for

(a) no value of a
(b) all a satisfying $2/3 < a < 1$
(c) all a satisfying $0 < a < 1/3$
(d) all a satisfying $1/3 < a < 2/3$

Solution :

Taking x = 1 and a = ¾ we get, $(1 - 9/4)(1 - 3/4 - 3) > 0$

Option (b) cannot be true.

Taking x = 1, a = ¼ we get, $(1 - 3/4)(1 - 1/4 - 3) < 0$

So, option (a) cannot be true.

Taking x = 1, a = 2/5 we get, $(1 - 6/5)(1 - 2/5 - 3) > 0$

Option (c) is correct.

377. Given that x is a real number satisfying $(3x^2 - 10x + 3)(2x^2 - 5x + 2) < 0$, it follows that
(a) $x < 1/3$
(b) $1/3 < x < 1/2$
(c) $2 < x < 3$
(d) $1/3 < x < 1/2$ or $2 < x < 3$

Solution :

Putting x = 0, we get, $3*2 > 0$

Option (a) cannot be true.

Taking x = 2/5, we get, $(12/25 - 4 + 3)(8/25 - 2 + 2) < 0$

Taking x = 5/2, we get, $(75/4 - 25 + 3)(25/2 - 25/2 + 2) < 0$

So, option (d) is correct.

378. If x, y, z are arbitrary real numbers satisfying the condition $xy + yz + zx < 0$ and if $u = (x^2 + y^2 + z^2)/(xy + yz + zx)$, then only one of the following is always correct. Which one is it?
(a) $-1 \le u < 0$

(b) u takes all negative real values
(c) $-2 < u < -1$
(d) $u \leq -2$

Solution :

$(x + y + z)^2 \geq 0$

- ⇨ $(x^2 + y^2 + z^2) \geq -2(xy + yz + zx)$
- ⇨ $(x^2 + y^2 + z^2)/(xy + yz + zx) \leq -2$ (as $xy + yz + zx < 0$ so the sing changes)
- ⇨ $u \leq -2$

Option (d) is correct.

379. The inequality $(|x|^2 - |x| - 2)/(2|x| - |x|^2 – 2) > 2$ holds if and only if
(a) $-1 < x < -2/3$ or $2/3 < x < 1$
(b) $-1 < x < 1$
(c) $2/3 < x < 1$
(d) $x > 1$ or $x < -1$ or $-2/3 < x < 2/3$

Solution :

Taking $x = 5/6$ we get, $(25/36 – 5/6 – 2)/(5/3 – 25/36 – 2) = (25 – 30 – 72)/(60 – 25 – 72) = 77/37 > 2$

Taking $x = -5/6$, we get same result.

Option (a) is correct.

380. The set of all real numbers x satisfying the inequality $|x^2 + 3x| + x^2 – 2 \geq 0$ is
(a) all the real numbers x with either $x \leq -3$ or $x \geq 2$
(b) all the real numbers x with either $x \leq -3/2$ or $x \geq ½$
(c) all the real numbers x with either $x \leq -2$ or $x \geq ½$
(d) described by none of the foregoing statements.

Solution :

Now, $x^2 + 3x + x^2 - 2 \geq 0$ where $x^2 + 3x \geq 0$ i.e. $x \geq 0$ or $x \leq -3$

- ⇨ $2x^2 + 3x - 2 \geq 0$
- ⇨ $(2x - 1)(x + 2) \geq 0$
- ⇨ $x \geq ½$ and $x \geq -2$ or $x \leq ½$ and $x \leq -2$
- ⇨ $x \geq ½$ or $x \leq -2$

The intersection is, $x \geq ½$ or $x \leq -3$

Now, $-(x^2 + 3x) + x^2 - 2) \geq 0$ where $x^2 + 3x \leq 0$ i.e. $-3 \leq x \leq 0$

- ⇨ $3x \leq -2$
- ⇨ $x \leq -2/3$

Intersection is $-3 \leq x \leq -2/3$

Now, $x \geq ½$ or $x \leq -3$ U $-3 \leq x \leq -2/3$ = $x \geq ½$ or $x \leq -2/3$

Option (b) is correct.

381. The least value of $1/x + 1/y + 1/z$ for positive a, y, z satisfying the condition $x + y + z = 9$ is
(a) 15/7
(b) 1/9
(c) 3
(d) 1

Solution :

Now, AM ≥ HM on $1/x$, $1/y$ and $1/z$ we get, $(1/x + 1/y + 1/z)/3 \geq 3/(x + y + z) = 1/3$

- ⇨ $1/x + 1/y + 1/z \geq 1$

Option (d) is correct.

382. The smallest value of α satisfying the conditions that α is a positive integer and that α/540 is a square of a rational number is
(a) 15
(b) 5
(c) 6
(d) 3

Solution :

Now, 540 = $2^2*3^3*5 = (2*3)^2*(3*5)$

So, ɑ = 15 so that the non square term in the denominator cancels.

Option (a) is correct.

383. The set of all values of x satisfying the inequality $(6x^2 + 5x + 3)/(x^2 + 2x + 3) > 2$ is
(a) $x > ¾$
(b) $|x| > 1$
(c) either $x > ¾$ or $x < -1$
(d) $|x| > ¾$

Solution :

Taking x = -2, we get, (24 – 10 + 3)/(4 – 4 + 2) = 17/2 > 2

Taking x = 5/6 we get, (25/6 + 25/6 + 3)/(25/36 + 5/3 + 3) = 68*6/193 = 408/193 > 2

Taking x = -5/6 we get, (25/6 – 25/6 + 3)/(25/36 – 5/3 + 3) = 3*36/73 = 108/73 < 2

Option (c) is correct.

384. The set of all x satisfying $|x^2 – 4| > 4x$ is
(a) $x < 2(\sqrt{2} – 1)$ or $x > 2(\sqrt{2} + 1)$
(b) $x > 2(\sqrt{2} + 1)$
(c) $x < -2(\sqrt{2} – 1)$ or $x > 2(\sqrt{2} + 1)$
(d) none of the foregoing sets

Solution :

Now, $x^2 – 4 > 4x$ where $x^2 > 4$ i.e. $|x| > 2$ i.e. $x > 2$ or $x < -2$

⇨ $x^2 – 4x + 4 > 8$
⇨ $(x – 2)^2 > 8$

- ⇨ $|x - 2| > 2\sqrt{2}$
- ⇨ $x > 2 + 2\sqrt{2}$ or $x < 2 - 2\sqrt{2}$
- ⇨ $x > 2(\sqrt{2} + 1)$ or $x < 2(\sqrt{2} - 1)$

Intersection is $x > 2(\sqrt{2} + 1)$ or $x < -2$

Now, $x^2 - 4 < -4x$ where $x^2 < 4$ i.e. $-2 < x < 2$

- ⇨ $x^2 + 4x + 4 < 8$
- ⇨ $(x + 2)^2 < 8$
- ⇨ $|x + 2| < 2\sqrt{2}$
- ⇨ $-2\sqrt{2} < x + 2 < 2\sqrt{2}$
- ⇨ $-2(\sqrt{2} + 1) < x < 2(\sqrt{2} - 1)$

Intersection is $-2 < x < 2(\sqrt{2} - 1)$

Therefore, required set = $x < 2(\sqrt{2} - 1)$ or $x > 2(\sqrt{2} + 1)$

Option (a) is correct.

385. If a, b, c are positive real numbers and $\alpha = (b^2 + c^2)/(b + c) + (c^2 + a^2)/(c + a) + (a^2 + b^2)/(a + b)$ then only one of the following statements is *always* true. Which one is it?

(a) $0 \le \alpha < a$
(b) $a \le \alpha < a + b$
(c) $a + b \le \alpha < a + b + c$
(d) $a + b + c \le \alpha < 2(a + b + c)$

Solution :

Now, $(b^2 + c^2)/2 \ge \{(b + c)/2\}^2$

- ⇨ $(b^2 + c^2)/(b + c) \ge (b + c)/2$

Similarly, $(c^2 + a^2)/(c + a) \ge (c + a)/2$ and $(a^2 + b^2)/(a + b) \ge (a + b)/2$

Adding the above inequalities we get, $\alpha \ge a + b + c$

Now, $2bc > 0$

- ⇨ $b^2 + c^2 + 2bc > b^2 + c^2$
- ⇨ $(b^2 + c^2) < (b + c)^2$
- ⇨ $(b^2 + c^2)/(b + c) < b + c$

Similarly, $(c^2 + a^2)/(c + a) < c + a$ and $(a^2 + b^2)/(a + b) < a + b$

Adding the above inequalities we get, ɑ < 2(a + b + c)

Option (d) is correct.

386. Suppose a, b, c are real numbers such that $a^2b^2 + b^2c^2 + c^2a^2 = k$, where k is constant. Then he set of all possible values of abc(a + b + c) is precisely the interval

(a) [-k, k]
(b) [-k/2, k/2]
(c) [-k/2, k]
(d) [-k, k/2]

Solution :

Now, $(a^2b^2 + b^2c^2 + c^2a^2)/3 \geq (abc)^{4/3}$ (AM ≥ GM)

⇨ $abc \leq (k/3)^{3/4}$

Maximum value of abc occurs when $a = b = c = (k/3)^{1/4}$

So, maximum value of $abc(a + b + c) = (k/3)^{3/4}*3(k/3)^{1/4} = 3*(k/3) = k$

Let us take, abc(a + b + c) = -3k/4

Now, $abc(a + b + c) = a^2bc + ab^2c + abc^2 \geq 3abc$ (AM ≥ GM)

$(ab + bc + ca)^2 = a^2b^2 + b^2c^2 + c^2a^2 + 2(a^2bc + ab^2c + abc^2) = a^2b^2 + b^2c^2 + c^2a^2 + 2abc(a + b + c)$

Now, $(ab + bc + ca)^2 \geq 0$

⇨ $k + 2abc(a + b + c) \geq 0$
⇨ $abc(a + b + c) \leq -k/2$

Option (c) is correct.

387. If a, b, c, d are real numbers such that b > 0, d > 0 and a/b < c/d, then only one of the following statements is *always* true. Which one is it?

(a) a/b < (a – c)/(b – d) < c/d
(b) a/b < (a + c)/(b + d) < c/d
(c) a/b < (a – c)/(b + d) < c/d
(d) a/b < (a + c)/(b – d) < c/d

Solution :

Now, a/b < c/d

- ⇨ ad < bc (as b > 0 and d > 0)
- ⇨ ad + cd < bc + cd
- ⇨ d(a + c) < c(b + d)
- ⇨ (a + c)/(b + d) < c/d (as b > 0, d > 0)

Now, ad < bc

- ⇨ ad + ab < ab + bc
- ⇨ a(b + d) < b(a + c)
- ⇨ a/b < (a + c)/(b + d)
- ⇨ a/b < (a + c)/(b + d) < c/d

Option (b) is correct.

388. If x, y, z are arbitrary positive numbers satisfying the equation 4xy + 6yz + 8zx = 9, then the maximum possible value of the product xyz is
(a) 1/2√2
(b) √3/4
(c) 3/8
(d) None of the foregoing values.

Solution :

Now, $(4xy + 6yz + 8zx)/3 \geq (4xy*6yz*8zx)^{1/3}$

- ⇨ $4*3^{1/3}(xyz)^{2/3} \leq 3$
- ⇨ $64*3(xyz)^2 \leq 3^3$
- ⇨ $(xyz)^2 \leq 3^2/64$
- ⇨ $xyz \leq 3/8$

Option (c) is correct.

389. Let P and Q be the subsets of the X-Y plane defined as : $P = \{(x, y) : x > 0, y > 0 \text{ and } x^2 + y^2 = 1\}$, and $Q = \{(x, y) : x > 0, y > 0 \text{ and } x^8 + y^8 < 1\}$. Then P∩Q is
(a) The empty set Φ

(b) P
(c) Q
(d) None of the foregoing sets.

Solution :

Clearly, Option (b) is correct.

390. The minimum value of the quantity $(a^2 + 3a + 1)(b^2 + 3b + 1)(c^2 + 3c + 1)/abc$, where a, b and c are positive real numbers, is
(a) $11^3/2^3$
(b) 125
(c) 25
(d) 27

Solution :

Now, $(a^2*1 + a*3 + 1*1)/(1 + 3 + 1) \geq \{(a^2)(a)^3*1\}^{1/(1 + 3 + 1)}$ (weighted AM ≥ weighted GM) = a

⇨ $(a^2 + 3a + 1)/a \geq 5$

Similarly, others, hence minimum value = 5*5*5 = 125

Option (b) is correct.

391. The smallest integer greater than the real number $(\sqrt{5} + \sqrt{3})^{2n}$ (for nonnegative integer n) is
(a) 8^n
(b) 4^{2n}
(c) $(\sqrt{5} + \sqrt{3})^{2n} + (\sqrt{5} - \sqrt{3})^{2n} - 1$
(d) $(\sqrt{5} + \sqrt{3})^{2n} + (\sqrt{5} - \sqrt{3})^{2n}$

Solution :

Option (d) is correct.

392. The set of all values of m for which $mx^2 - 6mx + 5m + 1 > 0$ for all real x is
(a) $0 \le m \le 1/4$
(b) $m < 1/4$
(c) $m \ge 0$
(d) $0 \le m < 1/4$

Solution :

Let, $m = 1/4$

$x^2/4 - 3x/2 + 5/4 + 1 > 0$

- ⇨ $x^2 - 6x + 5 + 4 > 0$
- ⇨ $(x - 3)^2 > 0$

This is not true for x = 3.

Therefore, m = ¼ is not a solution.

Therefore, option (a) and (c) cannot be true.

Let m = -1/4

$-x^2/4 + 3x/2 - 5/4 + 1 > 0$

=> $-x^2 + 6x - 1 > 0$

=> $x^2 - 6x + 1 < 0$

=> $(x - 3)^2 < 8$

This is not true for x = 10.

So, option (b) cannot be true.

Option (d) is correct.

393. The value of $(1^r + 2^r + \ldots + n^r)^n$, where r is a real number, is
(a) greater than or equal to $n^n*(n!)^r$
(b) less than $n^n*(n!)^{2r}$
(c) less than or equal to $n^{2n}*(n!)^r$
(d) greater than $n^n*(n!)^r$

Solution :

See solution of problem 351.

Option (d) is correct.

394. The value of $(\sqrt{3}/2 + i*1/2)^{165}$ is
(a) -1
(b) $\sqrt{3}/2 - i*1/2$
(c) i
(d) –i

Solution :

Now, $(\sqrt{3}/2 + i*1/2)^{165} = (\cos\pi/6 + i\sin\pi/6)^{165} = e^{(i\pi/6)*165} = e^{i55\pi/2} = \cos(55\pi/2) + i\sin(55\pi/2) = \cos(28\pi - \pi/2) + i\sin(28\pi - \pi/2) = \cos(\pi/2) - i\sin(\pi/2) = -i$

Option (d) is correct.

395. The value of the expression $[\{-1 + \sqrt{(-3)}\}/2]^n + [\{-1 - \sqrt{(-3)}\}/2]^n$ is
(a) 3 when n is positive multiple of 3, and 0 when n is any other positive integer
(b) 2 when n is a positive multiple of 3, and -1 when n is any other positive integer
(c) 1 when n is a positive multiple of 3 and -2 when n is any other positive integer
(d) None of the foregoing numbers.

Solution :

$\{-1 + \sqrt{(-3)}\}/2 = w$ and $\{-1 - \sqrt{(-3)}\}/2 = w^2$ where w is cube root of unity.

Therefore, it is $w^n + w^{2n} = -1$ when n is not a multiple of 3; $= 1 + 1 = 2$ when n is positive multiple of 3.

Option (b) is correct.

396. How many integers k are there for which $(1 - i)^k = 2^k$? (here $i = \sqrt{(-1)}$)
(a) One
(b) None
(c) Two
(d) More than two

Solution :

$(1 - i)^k = 2^k$

- ⇨ $\{(1 - i)/2\}^k = 1$
- ⇨ $k = 0$

Option (a) is correct.

397. If n is a multiple of 4, the sum $S = 1 + 2i + 3i^2 + \ldots + (n + 1)i^n$, where $i = \sqrt{(-1)}$ is
(a) $1 - i$
(b) $(n + 2)/2$
(c) $(n^2 + 8 - 4ni)/8$
(d) $(n + 2 - ni)/2$

Solution :

$S = 1 + 2i + 3i^2 + \ldots + (n + 1)i^n$

$Si = \quad 1 + 2i^2 + \ldots + ni^n + (n + 1)i^{n+1}$

- ⇨ $(S - Si) = 1 + i + i^2 + \ldots + i^n - (n + 1)i^{n+1}$
- ⇨ $S(1 - i) = (i^{n+1} - 1)/(i - 1) - (n + 1)i$ (As n is multiple of 4)
- ⇨ $S(1 - i) = (i - 1)/(i - 1) - (n + 1)i$ (as n is multiple of 4)
- ⇨ $S(1 - i) = 1 - (n + 1)i$
- ⇨ $S = \{1 - (n + 1)i\}/(1 - i)$
- ⇨ $S = (1 + i)\{1 - (n + 1)i\}/2$
- ⇨ $S = \{1 + i - (n + 1)i + (n + 1)\}/2$
- ⇨ $S = (n + 2 - ni)/2$

Option (d) is correct.

398. If $a_0, a_1, \ldots, a_n$ are real numbers such that $(1 + z)^n = a_0 + a_1z + a_2z^2 + \ldots + a_nz^n$, for all complex numbers z, then the value of $(a_0 - a_2 + a_4 - a_6 + \ldots)^2 + (a_1 - a_3 + a_5 - a_7 + \ldots)^2$ equals
(a) 2^n
(b) $a_0^2 + a_1^2 + \ldots + a_n^2$
(c) $2^{\wedge}(n^2)$
(d) $2n^2$

Solution :

Putting z = i we get, $(1 + i)^n = a_0 + a_1i - a_2 - a_3i + a_4 + a_5i - \ldots = (a_0 - a_2 + a_4 - \ldots) + i(a_1 - a_3 + a_5 - \ldots)$

Putting z = -i we gwt, $(1 - i)^n = a_0 - a_1i - a_2 + a_3i + a_4 - a_5i + \ldots = (a_0 - a_2 + a_4 - \ldots) - i(a_1 - a_3 + a_5 - \ldots)$

Multiplying the above two equations, we get,

$\{(1 + i)(1 - i)\}^n = (a_0 - a_2 + a_4 - \ldots)^2 + (a_1 - a_3 + a_5 - \ldots)^2 = 2^n$

Option (a) is correct.

399. If $t_k = {}^{100}C_kx^{100-k}$, for k = 0, 1, 2, …, 100, then $(t_0 - t_2 + t_4 - \ldots + t_{100})^2 + (t_1 - t_3 + t_5 - \ldots - t_{99})^2$ equals
(a) $(x^2 - 1)^{100}$
(b) $(x + 1)^{100}$
(c) $(x^2 + 1)^{100}$
(d) $(x - 1)^{100}$

Solution :

$(xi + 1)^{100} = t_0 - it_1 - t_2 + \ldots + t_{100} = (t_0 - t_2 + t_4 - \ldots + t_{100}) - i(t_1 - t_3 + t_5 - \ldots - t_{99})$

$(-xi + 1)^{100} = t_0 + it_1 - t_2 + \ldots + t_{100} = (t_0 - t_2 + t_4 - \ldots + t_{100}) + i(t_1 - t_3 + t_5 - \ldots - t_{99})$

Multiplying the above two equalities we get, $(t_0 - t_2 + t_4 - \ldots + t_{100})^2 + (t_1 - t_3 + t_5 - \ldots - t_{99})^2 = (1 + x^2)^{100}$

Option (c) is correct.

400. The expression $(1 + i)^n/(1 - i)^{n-2}$ equals
  (a) $-i^{n+1}$
  (b) i^{n+1}
  (c) $-2i^{n+1}$
  (d) 1

Solution :

$(1 + i)^n/(1 - i)^{n-2} = \{(1 + i)^{2n-2}/2^{n-2} = (1 +2i + i^2)^{n-1}/2^{n-2} = 2^{n-1}i^{n-1}/2^{n-2} = 2i^{n-1} = -2i^{n+1}$

Option (c) is correct.

401. The value of the sum $\cos(\pi/1000) + \cos(2\pi/1000) + \ldots. + \cos(999\pi/1000)$ equals
  (a) 0
  (b) 1
  (c) 1/1000
  (d) An irrational number.

Solution :

Now, $\cos(\pi/1000) + \cos(999\pi/1000) = \cos(\pi/2)\cos(499\pi/1000) = 0$ (as $\cos(\pi/2) = 0$)

Similarly, $\cos(2\pi/1000) + \cos(998\pi/1000) = 0$

…

…

$\cos(500\pi/1000) = \cos(\pi/2) = 0$

Therefore, the sum equals 0

Option (a) is correct.

402. The sum $1 + {}^nC_1\cos\theta + {}^nC_2\cos2\theta + \ldots. + {}^nC_n\cos n\theta$ equals
  (a) $(2\cos(\theta/2))^n\cos(n\theta/2)$
  (b) $(2\cos^2(\theta/2))^n$
  (c) $(2\cos^2(n\theta/2))^n$

(d) None of the foregoing quantities.

Solution :

Now, $(1 + x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_nx^n$

Putting $x = \cos\theta + i\sin\theta$ we get,

$(1 + \cos\theta + i\sin\theta)^n = 1 + {}^nC_1(\cos\theta + i\sin\theta) + {}^nC_2(\cos\theta + i\sin\theta)^2 + \dots + {}^nC_n(\cos\theta + i\sin\theta)^n$

- ⇨ $\{2\cos^2(\theta/2) + i*2\sin(\theta/2)\cos(\theta/2)\}^n = 1 + {}^nC_1(\cos\theta + i\sin\theta) + {}^nC_2(\cos2\theta + i\sin2\theta) + \dots + {}^nC_n(\cos n\theta + i\sin n\theta)$
- ⇨ $(2\cos(\theta/2))^n\{\cos(\theta/2) + i\sin(\theta/2)\}^n = (1 + {}^nC_1\cos\theta + {}^nC_2\cos2\theta + \dots + {}^nC_n\cos n\theta) + i(\sin\theta + \dots + {}^nC_n\sin n\theta)$
- ⇨ $(2\cos(\theta/2))^n\{\cos(n\theta/2) + i\sin(n\theta/2)\} = (1 + {}^nC_1\cos\theta + {}^nC_2\cos2\theta + \dots + {}^nC_n\cos n\theta) + i(\sin\theta + \dots + {}^nC_n\sin n\theta)$

Equating the real part from both sides of the equation we get,

$(1 + {}^nC_1\cos\theta + {}^nC_2\cos2\theta + \dots + {}^nC_n\cos n\theta) = (2\cos(\theta/2))^n\cos(n\theta/2)$

Option (a) is correct.

403. Let $i = \sqrt{(-1)}$. Then
(a) i and $-i$ each has exactly one square root
(b) i has two square roots but $-i$ doesn't have any
(c) neither i nor $-i$ has any square root
(d) i and $-i$ each has exactly two square roots.

Solution :

Let $z^2 = i = \cos(\pi/2) + i\sin(\pi/2) = e^{i\pi/2}$

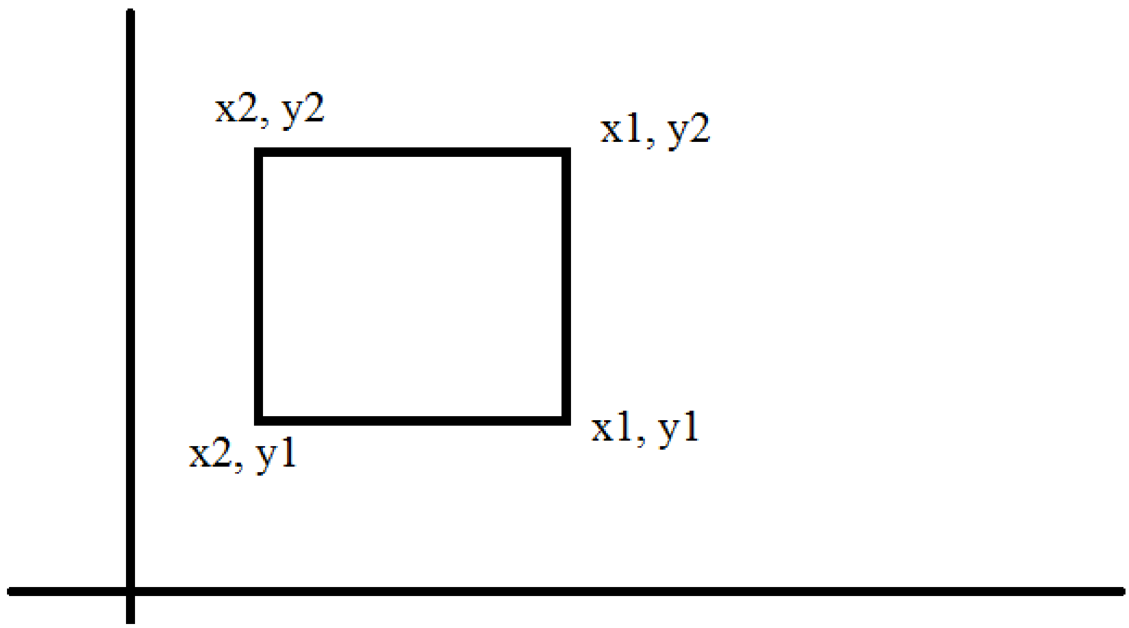
- ⇨ $z = \pm e^{i\pi/4} = \pm(\cos(\pi/4) + i\sin(\pi/4)) = \pm(1/\sqrt{2} + i/\sqrt{2})$

Similarly, $-i$ has exactly two square roots $= \pm e^{i3\pi/4} = \pm (\cos(3\pi/4) + i\sin(3\pi/4)) = \pm (-1/\sqrt{2} + i/\sqrt{2})$

Option (d) is correct.

404. If the complex numbers w and z represent two diagonally opposite vertices of a square, then the other two vertices are given by the complex numbers
(a) w + iz and w – iz
(b) (1/2)(w + iz) + (1/2)(w – iz) and (1/2)(w + z) – (1/2)i(w + z)
(c)(1/2)(w – z) + (1/2)i(w – z) and (1/2)(w – z) – (1/2)i(w – z)
(d) (1/2)(w + z) + (1/2)i(w – z) and (1/2)(w + z) – (1/2)i(w – z).

Solution :



Option (d) is correct. (Self-explanatory)

405. Let A = {a + b√(-1) a, b are integers} and U = {x Ɛ A | 1/x Ɛ A}. Then the number of elements of U is
(a) 2
(b) 4
(c)6
(d) 8

Solution :

a = 0, b = 1 and a = 0, b = -1 i.e. i and –i belong to U.

a = 1, b = 0 i.e. 1 belong to U.

a = -1, b = 0 i.e. -1 belong to U.

Option (b) is correct.

406. Let $i = \sqrt{(-1)}$. Then the number of distinct elements in the set S = $\{i^n + i^{-n} : n \text{ an integer}\}$ is
(a) 3
(b) 4
(c) greater than 4 but finite
(d) infinite

Solution :

Let, n = 1, $i + 1/i = 0$

n = 2, $i^2 + 1/i^2 = -1 -1 = -2$

n = 3, $i^3 + 1/i^3 = -i + i = 0$

n = 4, $i^4 + 1/i^4 = 1 + 1 = 2$

Option (a) is correct.

407. Let $i = \sqrt{(-1)}$ and p be a positive integer. A necessary and sufficient condition for $(-i)^p = i$ is
(a) p is one of 3, 11, 19, 27, ….
(b) p is an odd integer
(c) p is not divisible by 4
(d) none of the foregoing conditions.

Solution :

Clearly, p = 3, 7, 11, 15, 19, 23, 27, ….

Option (d) is correct.

408. Recall that for a complex number $z = x + iy$, where $i = \sqrt{(-1)}$, $z' = x - iy$ and $|z| = (x^2 + y^2)^{1/2}$. The set of all pairs of complex numbers (z_1, z_2) which satisfy $|(z_1 - z_2)/(1 - z_1'z_2)| < 1$ is

(a) all possible pairs (z_1, z_2) of complex numbers
(b) all pairs of complex numbers (z_1, z_2) for which $|z_1| < 1$ and $|z_2| < 1$
(c) all pairs of complex numbers (z_1, z_2) for which at least one of the following statements is true :
(i) $|z_1| < 1$ and $|z_2| > 1$
(ii) $|z_1| > 1$ and $|z_2| < 1$
(d) all pairs of complex numbers (z_1, z_2) for which at least one of the following statements is true :
(i) $|z_1| < 1$ and $|z_2| < 1$
(ii) $|z_1| > 1$ and $|z_2| > 1$

Solution :

$|(z_1 - z_2)/(1 - z_1'z_2)| < 1$

- ⇨ $|z_1 - z_2| < |1 - z_1'z_2|$
- ⇨ $|z_1 - z_2|^2 < |1 - z_1'z_2|^2$
- ⇨ $(z_1 - z_2)(z_1' - z_2') < (1 - z_1'z_2)(1 - z_1z_2')$
- ⇨ $|z_1|^2 - z_1z_2' - z_1'z_2 + |z_2|^2 < 1 - z_1z_2' - z_1'z_2 + |z_1|^2|z_2|^2$
- ⇨ $1 - |z_1|^2 - |z_2|^2 + |z_1|^2|z_2|^2 > 0$
- ⇨ $(1 - |z_1|^2)(1 - |z_2|^2) > 0$
- ⇨ $1 - |z_1|^2 > 0$ and $1 - |z_2|^2 > 0$ or $1 - |z_1|^2 < 0$ and $1 - |z_2|^2 < 0$
- ⇨ $|z_1| < 1$ and $|z_2| < 1$ or $|z_1| > 1$ and $|z_2| > 1$

Option (d) is correct.

409. Suppose z_1, z_2 are complex numbers satisfying $z_2 \neq 0$, $z_1 \neq z_2$ and $|(z_1 + z_2)/(z_1 - z_2)| = 1$. Then z_1/z_2 is
(a) real and negative
(b) real and positive
(c) purely imaginary
(d) not necessarily any of these.

Solution :

$|(z_1 + z_2)/(z_1 - z_2)| = 1$

- ⇨ $|z_1 + z_2| = |z_1 - z_2|$
- ⇨ $|z_1 + z_2|^2 = |z_1 - z_2|^2$
- ⇨ $(z_1 + z_2)(z_1' + z_2') = (z_1 - z_2)(z_1' - z_2')$

- ⇨ $|z_1|^2 + z_1z_2' + z_1'z_2 + |z_2|^2 = |z_1|^2 - z_1z_2' - z_1'z_2 + |z_2|^2$
- ⇨ $2z_1z_2' = -2z_1'z_2$
- ⇨ $z_1/z_2 = -z_1'/z_2'$

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

Therefore, $z_1/z_2 = (x_1 + iy_1)/(x_2 + iy_2) = (-x_1 + iy_1)/(x_2 - iy_2) = (x_1 + iy_1 - x_1 + iy_1)/(x_2 + iy_2 + x_2 - iy_2) = iy_1/x_2$ = purely imaginary.

Option (c) is correct.

410. The modulus of the complex number $\{(2 + i\sqrt{5})/(2 - i\sqrt{5})\}^{10} + \{(2 - i\sqrt{5})/(2 + i\sqrt{5})\}^{10}$ is
(a) $2\cos(20\cos^{-1}2/3)$
(b) $2\sin(10\cos^{-1}2/3)$
(c) $2\cos(10\cos^{-1}2/3)$
(d) $2\sin(20\cos^{-1}2/3)$

Solution :

$\{(2 + i\sqrt{5})/(2 - i\sqrt{5})\}^{10} + \{(2 - i\sqrt{5})/(2 + i\sqrt{5})\}^{10}$

$= \{(2 + i\sqrt{5})^2/9\}^{10} + \{(2 - i\sqrt{5})/9\}^{10}$

$= (2/3 + i\sqrt{5}/3)^{20} + (2/3 - i\sqrt{5}/3)^{20}$

Let, $\cos A = 2/3$, then $\sin A = \sqrt{5}/3$

The expression becomes, $(\cos A + i\sin A)^{20} + (\cos A - i\sin A)^{20}$

$= \cos 20A + i\text{son}20A + \cos 20A - i\sin 20A$

$= 2\cos 20A$

Now, $|2\cos 20A| = 2\cos 20A = 2\cos(20\cos^{-1}2/3)$

Option (a) is correct.

411. For any complex number $z = x + iy$ with x and y real, define $<z> = |x| + |y|$. Let z_1 and z_2 be any two complex numbers. Then
(a) $<z_1 + z_2> \le <z_1> + <z_2>$
(b) $<z_1 + z_2> = <z_1> + <z_2>$
(c) $<z_1 + z_2> \ge <z_1> + <z_2>$

(d) None of the foregoing statements need always be true.

Solution :

$|x_1 + x_2| \le |x_1| + |x_2|$ and $|y_1 + y_2| \le |y_1| + |y_2|$

Option (a) is correct.

412. Recall that for a complex number $z = x + iy$, where $i = \sqrt{(-1)}$, $|z| = (x^2 + y^2)^{1/2}$ and $\arg(z)$ = principal value of $\tan^{-1}(y/x)$. Given complex numbers $z_1 = a + ib$, $z_2 = (a/\sqrt{2})(1 - i) + (b/\sqrt{2})(1 + i)$, $z_3 = (a/\sqrt{2})(i - 1) - (b/\sqrt{2})(i + 1)$, where a and b are real numbers, only one of the following statements is true. Which one is it?
(a) $|z_1| = |z_2|$ and $|z_2| > |z_3|$
(b) $|z_1| = |z_3|$ and $|z_1| < |z_2|$
(c) $\arg(z_1) = \arg(z_2)$ and $\arg(z_1) - \arg(z_3) = \pi/4$
(d) $\arg(z_2) - \arg(z_1) = -\pi/4$ and $\arg(z_3) - \arg(z_2) = \pm\pi$

Solution :

Clearly, option (d) is correct. (*Very easy problem but lengthy)*

413. If $a_0, a_1, \ldots, a_{2n}$ are real numbers such that $(1 + z)^{2n} = a_0 + a_1z + a_2z^2 + \ldots + a_{2n}z^{2n}$, for all complex numbers z, then
(a) $a_0 + a_1 + a_2 + \ldots + a_{2n} = 2^n$
(b) $(a_0 - a_2 + a_4 - \ldots)^2 + (a_1 - a_3 + a_5 - \ldots)^2 = 2^{2n}$
(c) $a_0^2 + a_1^2 + a_2^2 + \ldots + a_{2n}^2 = 2^{2n}$
(d) $(a_0 + a_2 + \ldots)^2 + (a_1 + a_3 + a_5 + \ldots)^2 = 2^{2n}$

Solution :

See solution of problems 398, 399.

Option (b) is correct.

414. If z is a nonzero complex number and $z/(1 + z)$ is purely imaginary, then z

(a) can be neither real nor purely imaginary
(b) is real
(c) is purely imaginary
(d) satisfies none of the above properties

Solution :

Now, $z/(1 + z)$ = purely imaginary

- ⇨ $(x + iy)/\{(1 + x) + iy)\}$ = purely imaginary
- ⇨ $(x + iy)\{(1 + x) - iy\}/\{(1 + x)^2 + y^2\}$ = purely imaginary
- ⇨ $\{x(1 + x) + y^2\}/\{(1 + x)^2 + y^2\} = 0$ (real part = 0)
- ⇨ $x = -(x^2 + y^2)$

Option (a) is correct.

415. Let a and b be any two nonzero real numbers. Then the number of complex numbers z satisfying the equation $|z|^2 + a|z| + b = 0$ is
(a) 0 or 2 and both these values are possible
(b) 0 or 4 and both these values are possible
(c) 0, 2 or 4 and all these values are possible
(d) 0 or infinitely many and both these values are possible

Solution :

$|z| = \{-a \pm \sqrt{(a^2 - 4b)}\}/2$

Obviously option (d) is correct.

416. Let C denote the set of complex numbers and define A and B by $A = \{(z, w) : z, w \in C \text{ and } |z| = |w|\}$; $B = \{(z, w) : z, w \in C \text{ and } z^2 = w^2\}$. Then
(a) A = B
(b) A is a subset of B
(c) B is a subset of A
(d) None of the foregoing statements is correct.

Solution :

Let, $z = x_1 + iy_1$ and $w = x_2 + iy_2$

$|z| = |w| \Rightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2$

And, $z^2 = w^2 \Rightarrow x_1^2 - y_1^2 + i(2x_1y_1) = x_2^2 - y_2^2 + i(2x_2y_2)$ i.e. $x_1^2 - y_1^2 = x_2^2 - y_2^2$ and $x_1y_1 = x_2y_2$

Now, $(x_1^2 + y_1^2)^2 = (x_1^2 - y_1^2)^2 + (2x_1y_1)^2 = (x_2^2 - y_2^2)^2 + (2x_2y_2)^2 = (x_2^2 + y_2^2)^2$

⇨ $(x_1^2 + y_1^2) = (x_2^2 + y_2^2)$

Option (c) is correct.

417. Among the complex numbers z satisfying $|z - 25i| \leq 15$, the number having the least argument is
(a) 10i
(b) -15 + 25i
(c) 12 + 16i
(d) 7 + 12i

Solution :

Put each of the complex numbers of the option in the given equation and see them if satisfy the equation. Those who satisfy, find arguments of them and check the least one.

Option (c) is correct.

418. The minimum possible value of $|z|^2 + |z - 3|^2 + |z - 6i|^2$, where z is a complex number and $i = \sqrt{(-1)}$, is
(a) 15
(b) 45
(c) 30
(d) 20

Solution :

Let $z = x + iy$,

$|z|^2 + |z - 3|^2 + |z - 6i|^2 = x^2 + y^2 + (x - 3)^2 + y^2 + x^2 + (y - 6)^2 = 3(x^2 + y^2) - 6x - 12y + 45 = 3(x^2 + y^2 - 2x - 4y + 15) = 3(x - 1)^2 + 3(y - 2)^2 + 30 \geq 30$

Option (c) is correct.

419. The curve in the complex plane given by the equation Re(1/z) = ¼ is a
(a) vertical straight line at a distance of 4 from the imaginary axis
(b) circle with radius unity
(c) circle with radius 2
(d) straight line not passing through the origin

Solution :

Let, $z = x + iy$

$1/z = 1/(x + iy) = (x - iy)/(x^2 + y^2)$

$\text{Re}(1/z) = x/(x^2 + y^2) = ¼$

- ⇨ $x^2 + y^2 = 4x$
- ⇨ $x^2 + y^2 - 4x + 4 = 4$
- ⇨ $(x - 2)^2 + y^2 = 2^2$

Option (c) is correct.

420. The set of all complex numbers z such that $\arg\{(z - 2)/(z + 2)\} = \pi/3$ represents
(a) part of a circle
(b) a circle
(c) an ellipse
(d) part of an ellipse

Solution :

Now, $\arg\{(z - 2)/(z + 2)\} = \pi/3$

- ⇨ $\arg[\{(x - 2) + iy\}/\{(x + 2) + iy\}] = \pi/3$
- ⇨ $\arg\{(x - 2) + iy\} - \arg\{(x + 2) + iy\} = \pi/3$
- ⇨ $\tan^{-1}\{y/(x - 2)\} - \tan^{-1}\{y/(x + 2)\} = \pi/3$

- ⇨ $\tan^{-1}[\{y/(x - 2) - y/(x + 2)\}/\{1 + y^2/(x^2 - 4)\}] = \pi/3$
- ⇨ $\{y(x + 2) - y(x - 2)\}/(x^2 - 4 + y^2) = \sqrt{3}$
- ⇨ $4y = \sqrt{3}(x^2 + y^2 - 4)$
- ⇨ $\sqrt{3}(x^2 + y^2) - 4y - 4\sqrt{3} = 0$

Option (a) is correct. (As arg represents principal value)

421. Let $z = x + iy$ where x and y are real and $i = \sqrt{(-1)}$. The points (x, y) in the plane, for which $(z + i)/(z - i)$ is purely imaginary (that is, it is of the form ib when b is a real number), lie on
(a) a straight line
(b) a circle
(c) an ellipse
(d) a hyperbola

Solution :

$(z + i)/(z - i) = \{x + i(y + 1)\}/\{x + i(y - 1)\} = \{x^2 + (y^2 - 1) - ix(y - 1) + ix(y + 1)\}/\{x^2 + (y - 1)^2\}$

Real part = 0

- ⇨ $\{x^2 + (y^2 - 1)\}/\{x^2 + (y - 1)^2\} = 0$
- ⇨ $x^2 + y^2 = 1$

Option (b) is correct.

422. If the point z in the complex plane describes a circle of radius 2 with centre at the origin, then the point $z + 1/z$ describe
(a) a circle
(b) a parabola
(c) an ellipse
(d) a hyperbola

Solution :

$|z| = 2 \Rightarrow x^2 + y^2 = 4$

$z + 1/z = x + iy + 1/(x + iy) = x + iy + (x - iy)/(x^2 + y^2) = x + iy + (x - iy)/4 = (5x + 3iy)/4$

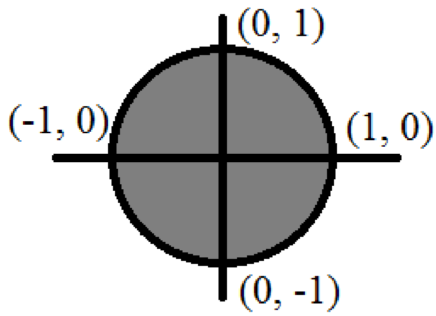
$|z + 1/z| = \sqrt{(25x^2 + 9y^2)}/4 = a$

⇨ $25x^2 + 9y^2 = 16a^2$

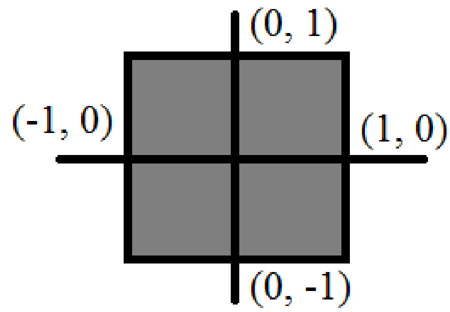
Option (c) is correct.

423. The set $\{(x, y) : |x| + |y| \leq 1\}$ is represented by the shaded region in one of the four figures. Which one is it?

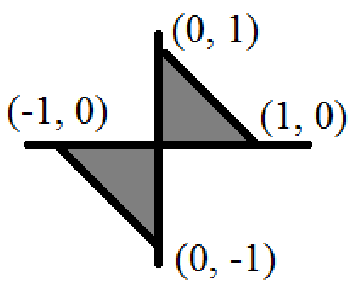
(a)



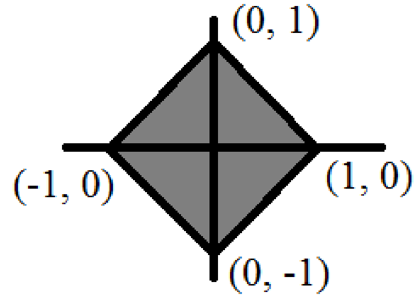
(b)



(c)



(d)



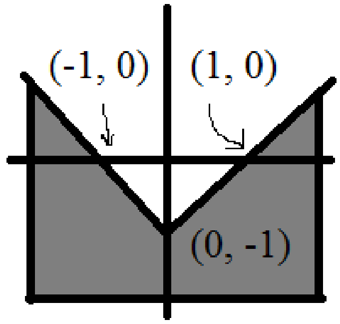
Solution :

The curves are $x + \leq 1$, $- x - y \leq 1$, $x - y \leq 1$ and $-x + y \leq 1$ all are straight lines.

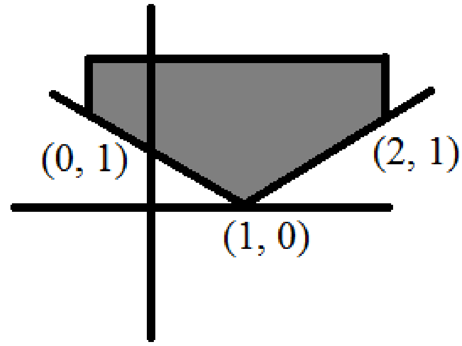
Clearly, option (d) is correct.

424. The sets $\{(x, y) : |y - 1| - x \geq 1\}$; $\{(x, y) : |x| - y \geq 1\}$; $\{(x, y) : x - |y| \leq 1\}$; $\{(x, y) : y - |x - 1| \geq 0\}$ are represented by the shaded regions in the figures given below in some order.

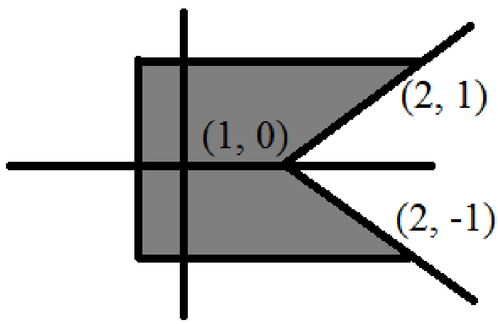
F1



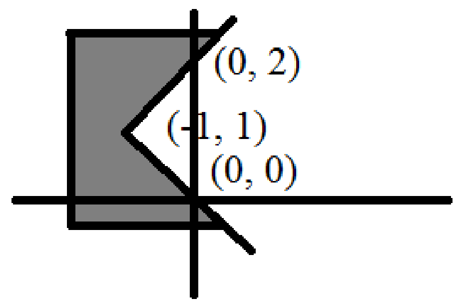
F2



F3



F4



Then the correct order of the figures is

(a) F_4, F_1, F_2, F_3

(b) F_4, F_2, F_3, F_1

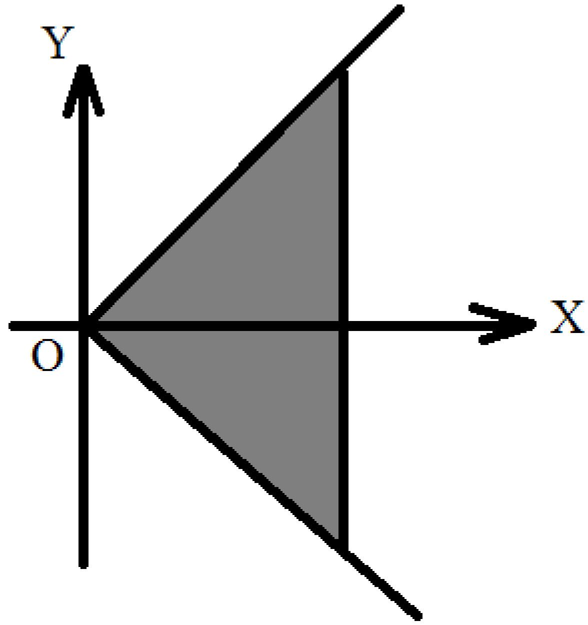
(c) F_1, F_4, F_3, F_2

(d) F_4, F_1, F_3, F_2

Solution :

Option (d) is correct.

425. The shaded region in the diagram represents the relation



(a) $y \le x$
(b) $|y| \le |x|$
(c) $y \le |x|$
(d) $|y| \le x$

Solution :

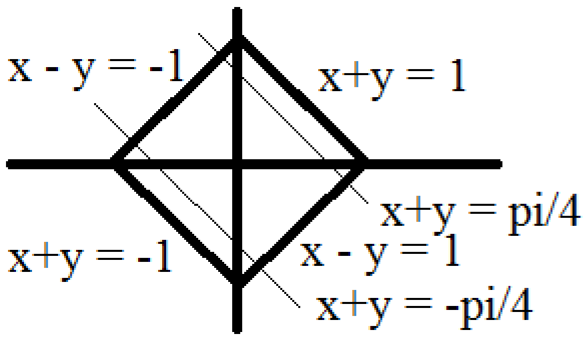
Option (d) is correct.

426. The number of points (x, y) in the plane satisfying the two equations $|x| + |y| = 1$ and $\cos\{2(x + y)\} = 0$ is
(a) 0
(b) 2
(c) 4
(d) Infinitely many

Solution :

Now, $\cos\{2(x + y)\} = 0 = \cos(\pm\pi/2)$

⇨ $x + y = \pm\pi/4$



4 points of intersection.

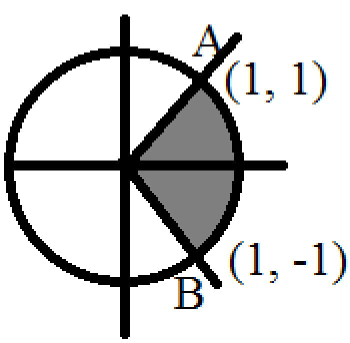
Option (c) is correct.

Directions for Items 427 and 428 :

Let the diameter of a subset S of the plane be defined as the maximum of the distances between arbitrary pairs of points of S.

427. Let S = $\{(x, y) : (y - x) \leq 0, (x + y) \geq 0, x^2 + y^2 \leq 2\}$. Then the diameter of S is
(a) 4
(b) 2
(c) $2\sqrt{2}$
(d) $\sqrt{2}$

Solution :

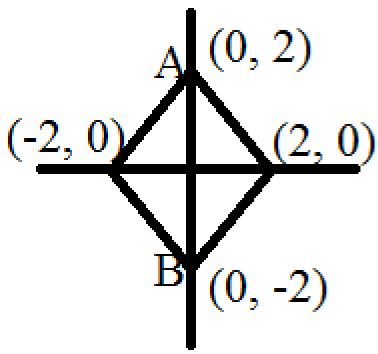


Clearly, diameter = AB = 2

Option (b) is correct.

428. Let $S = \{(x, y) : |x| + |y| = 2\}$. Then the diameter of S is
(a) 2
(b) $4\sqrt{2}$
(c) 4
(d) $3\sqrt{2}$

Solution :



Clearly, diameter = AB = 4

Option (c) is correct.

429. The points (2, 1), (8, 5) and (x, 7) lie on a straight line. The value of x is
(a) 10
(b) 11
(c) 12
(d) 11 + 2/3

Solution :

Now, the area formed by the triangle by the points = 0

⇨ $(1/2)\{2(5 - 7) + 8(7 - 1) + x(1 - 5)\} = 0$
⇨ $4x = 44$
⇨ $x = 11$

Option (b) is correct.

430. In a parallelogram PQRS, P is the point (-1, -1), Q is (8, 0) and R is (7, 5). Then S is the point
(a) (-1, 4)
(b) (-2, 4)
(c)(-2, 3.5)
(d) (-1.5, 4)

Solution :

Now, (P + R)/2 = (Q + S)/2 (as in a parallelogram the diagonals bisects each other)

⇨ (Q + S)/2 = (3, 2)

Clearly, option (b) is correct.

431. The equation of the line passing through the point of intersection of the lines $x - y + 1 = 0$ and $3x + y - 5 = 0$ and is perpendicular to the line $x + 3y + 1 = 0$ us
(a) $x + 3y - 1 = 0$
(b) $x - 3y + 1 = 0$
(c)$3x - y + 1 = 0$
(d) $3x - y - 1 = 0$

Solution :

The equation of the straight line which is perpendicular to the line $x + 3y + 1 = 0$ is, $3x - y + c = 0$

Now, $x - y + 1 = 0$ and $3x + y - 5 = 0$

Solving them we get, $x = 1$, $y = 2$

Putting the values in the equation we get, $3 - 2 + c = 0 \Rightarrow c = -1$

Option (d) is correct.

432. A rectangle PQRS joins the points P = (2, 3), Q = (x_1, y_1), R = (8, 11), S = (x_2, y_2). The line QS is known to be parallel to the y-axis. Then the coordinates of Q and S respectively
(a) (0, 7) and (10, 7)
(b) (5, 2) and (5, 12)
(c) (7, 6) and (7, 10)
(d) None of the foregoing pairs

Solution :

Now, as QS is parallel to y-axis, so x-coordinate of both Q and S are same.

Option (a) cannot be true.

Now, diagonals of a rectangle bisects each other.

Therefore, (P + R)/2 = (Q + S)/2

⇨ Option (c) cannot be true.

Slope of PQ, (2 – 3)/(5 – 2) = -1/3

Slope of QR = (11 – 2)/(8 – 5) = 3

Therefore, PQ and QR are perpendicular.

So, option (b) is correct.

433. The sum of the interior angles of a polygon is equal to 56 right angles. Then the number of sides of the polygon is
(a) 12
(b) 15
(c) 30
(d) 25

Solution :

Now, (n – 2)*π = 56*(π/2)

⇨ n = 30

Option (c) is correct.

434. The ration of a circumference of a circle to the perimeter of the inscribed regular polygon with n sides is

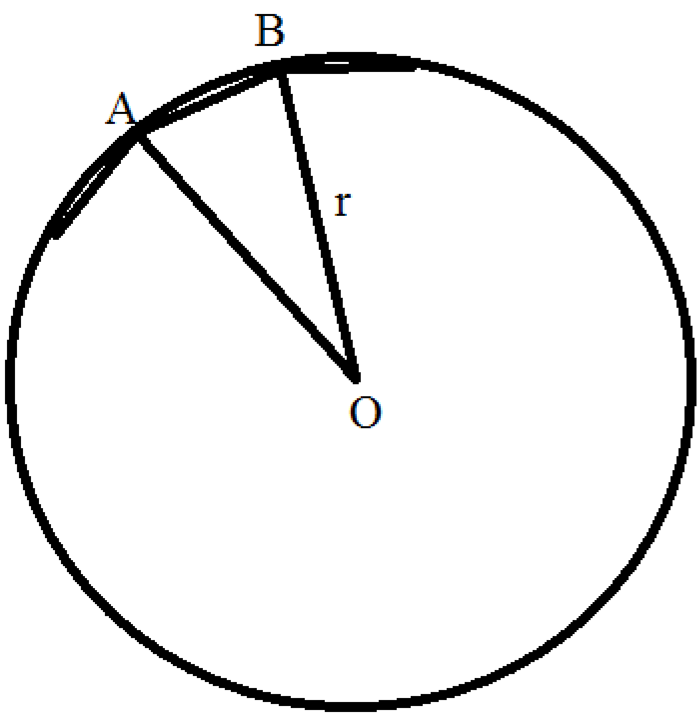
(a) $2\pi : 2n\sin(\pi/n)$

(b) $2\pi : n\sin(\pi/n)$

(c) $2\pi : 2n\sin(2\pi/n)$

(d) $2\pi : n\sin(2\pi/n)$

Solution :



Angle OAB = $(1/2)(n - 2)\pi/n$ = Angle OBA

Angle AOB = $\pi - (n - 2)\pi/n = 2\pi/n$

Now, in triangle OAB we get, $OA/\sin\{(n - 2)\pi/2n\} = AB/\sin(2\pi/n)$

⇨ $AB = r\sin(2\pi/n)/\sin(\pi/2 - \pi/n)$
⇨ $AB = r*2\sin(\pi/n)\cos(\pi/n)/\cos(\pi/n)$
⇨ $AB = 2r\sin(\pi/n)$

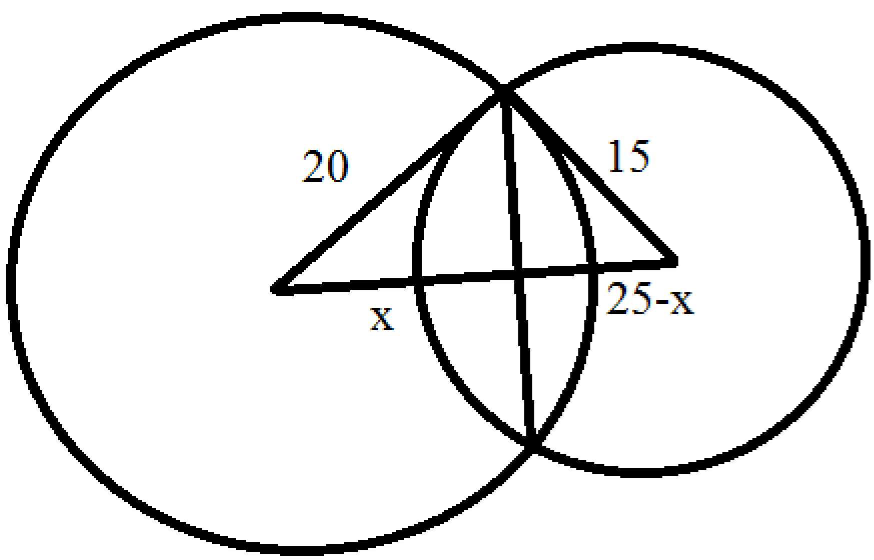
Perimeter = $r*2n\sin(\pi/n)$

Therefore, required ratio = $2\pi r : r*2n\sin(\pi/n) = 2\pi : 2n\sin(\pi/n)$

Option (a) is correct.

435. The length of the common chord of two circles of radii 15 cm and 20 cm, whose centres are 25 cm apart, is (in cm)
(a) 24
(b) 25
(c) 15
(d) 20

Solution :



Now, $(20^2 - x^2) = 15^2 - (25 - x)^2$

⇨ $400 - x^2 = 225 - 625 + 50x - x^2$
⇨ $50x = 800$
⇨ $x = 16$

Length of the chord = $2\sqrt{(20^2 - 16^2)} = 2*12 = 24$ cm

Option (a) is correct.

436. A circle of radius $\sqrt{3} - 1$ units with both coordinates of the centre negative, touches the straight line $y - \sqrt{3}x = 0$ and $x - \sqrt{3}y = 0$. The equation of the circle is
(a) $x^2 + y^2 + 2(x + y) + (\sqrt{3} - 1)^2 = 0$

(b) $x^2 + y^2 + 2(x + y) + (\sqrt{3} + 1)^2 = 0$
(c) $x^2 + y^2 + 4(x + y) + (\sqrt{3} - 1)^2 = 0$
(d) $x^2 + y^2 + 4(x + y) + (\sqrt{3} + 1)^2 = 0$

Solution :

Let the coordinate of the centre of the circle = (-g, -f).

Now, $|-f + \sqrt{3}g|/2 = \sqrt{3} - 1$

⇨ $\sqrt{3}g - f = 2(\sqrt{3} - 1)$

And, $\sqrt{3}f - g = 2(\sqrt{3} - 1)$

$3f - \sqrt{3}g = 2\sqrt{3}(\sqrt{3} - 1)$

Adding we get, $2f = 2(\sqrt{3} - 1)(\sqrt{3} + 1)$

⇨ $f = 2$
⇨ $g = 2$

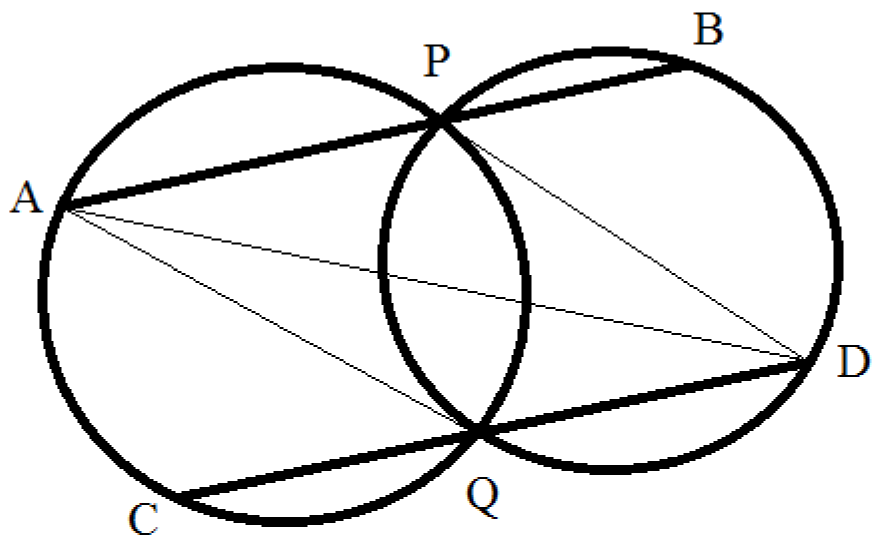
Equation of circle is, $(x + 2)^2 + (y + 2)^2 = (\sqrt{3} - 1)^2$

⇨ $x^2 + y^2 + 4(x + y) + 8 = 3 - 2\sqrt{3} + 1$
⇨ $x^2 + y^2 + 4(x + y) + (\sqrt{3} + 1)^2 = 0$

Option (d) is correct.

437. Two circles APQC and PBDQ intersect each other at the points P and Q and APB and CQD are two parallel straight lines. Then only one of the following statements is *always* true. Which one is it?
(a) ABDC is a cyclic quadrilateral
(b) AC is parallel to BD
(c) ABDC is a rectangle
(d) Angle ACQ is right angle

Solution :



We join A and Q, P and D, A and D.

Now, Angle PAQ and PDQ are on the same arc PQ. So, Angle PAQ = Angle PDQ.

Now, Angle PDQ = Angle BPD (AB||CD and PD is intersector)

- ⇨ Angle PAQ = Angle BPD
- ⇨ PD||AQ
- ⇨ PAQD is a parallelogram.
- ⇨ PA = DQ

Similarly, PB = CQ

- ⇨ AB = CD

Now, AB = CD and AB||CD

- ⇨ ACDB is a parallelogram.

Therefore, AC||BD

Option (b) is correct.

438. The area of the triangle whose vertices are (a, a), (a + 1, a + 1), (a + 2, a) is
   (a) a^2
   (b) 2a
   (c) 1

(d) $\sqrt{2}$

Solution :

Area = $|(1/2)\{a(a + 1 - a) + (a + 1)(a - a) + (a + 2)(a - a - 1)\}|$

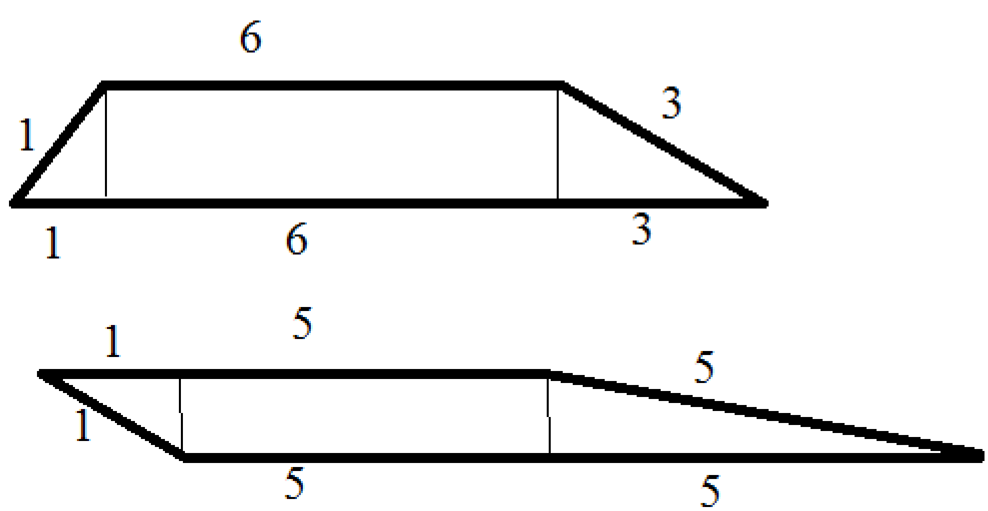
= $|(1/2)\{a - a - 2\}|$

= $|-1| = 1$

Option (c) is correct.

439. In a trapezium, the lengths of two parallel sides are 6 and 10 units. If one of the oblique sides has length 1 unit, then the length of the other oblique side must be
(a) greater than 3 units but less than 4 units
(b) greater than 3 units but less than 5 units
(c) less than or equal to 3 units
(d) greater than 5 units but less than 6 units

Solution :



Between 3 and 5 units.

Option (b) is correct.

440. If in a triangle, the radius of the circumcircle is double the radius of the inscribed circle, then the triangle is
(a) equilateral
(b) isosceles
(c) right-angled
(d) not necessarily any of the foregoing types

Solution :

So, R = 2r

Distance between circumcentre and incentre = $R^2 - 2Rr = 4r^2 - 4r^2 = 0$

- ⇨ Incentre and circumcentre is same.
- ⇨ Triangle is equilateral.

Option (a) is correct.

441. If in a triangle ABC with a, b, c denoting sides opposite to angles A, B and C respectively, a = 2b and A = 3B, then the triangle
(a) is isosceles
(b) is right-angled but not isosceles
(c) is right-angled and isosceles
(d) need not necessarily be any of the above types

Solution :

a = 2b

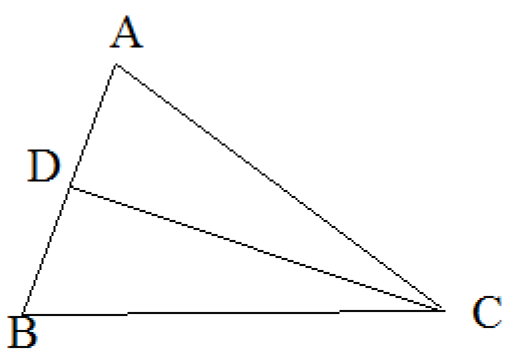
- ⇨ sinA = 2sinB
- ⇨ sin3B = 2sinB
- ⇨ $3\sin B - 4\sin^3 B - 2\sin B = 0$
- ⇨ $1 - 4\sin^2 B = 0$
- ⇨ sinB = ½
- ⇨ B = 30
- ⇨ A = 90
- ⇨ C = 60

Option (b) is correct.

442. Let the bisector of the angle at C of a triangle ABC intersect the side AB in a point D. Then the geometric mean of CA and CB

(a) is less than CD
(b) is equal to CD
(c) is greater than CD
(d) doesn't always satisfy any one of the foregoing properties

Solution :



Angle BDC = 180 – B – C/2 = 180 – B – C + C/2 = A + C/2

Similarly, CDA = B + C/2

In triangle BCD, CB/sin(A + C/2) = CD/sin(C/2)

- ⇨ CB = CDsin(A + C/2)/sinB
- ⇨ CA = CDsin(B + C/2)/sinA
- ⇨ CA*CB = CD^2sin(A + C/2)sin(B + C/2)/sinAsinB
- ⇨ CA*CB = CD^2{2sin(A + C/2)sin(B + C/2)/2sinAsinB
- ⇨ CA*CB = CD^2{cos(A – B) – cos(A + B + C)}/{cos(A – B) – cos(A + B)}
- ⇨ CA*CB = CD^2{cos(A – B) + 1}/{cos(A – B) + cosC}

Now, cosC < 1

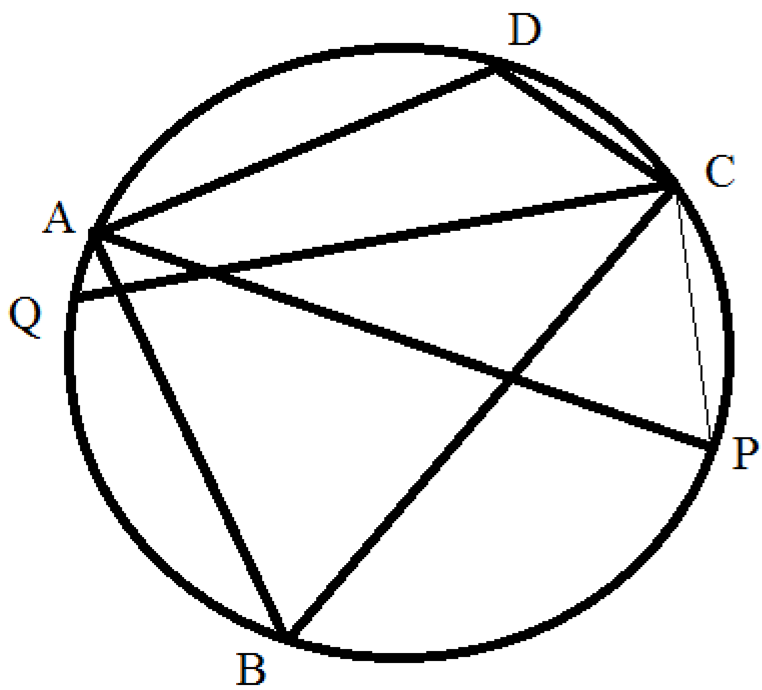
- ⇨ cos(A – B) + cosC < cos(A – B) + 1
- ⇨ {cos(A – B) + 1}/{cos(A – B) + cosC) > 1
- ⇨ CA*CB > CD^2
- ⇨ √(CA*CB) > CD

Option (c) is correct.

443. Suppose ABCD is a cyclic quadrilateral within a circle of radius r. The bisector of the angle A cuts the circle at a point P and the bisector of angle C cuts the circle at point Q. Then

(a) AP = 2r
(b) PQ = 2r
(c) BQ = DP
(d) PQ = AP

Solution :



Now, Angle A + Angle C = 180

- ⇨ Angle A/2 + Angle C/2 = 90
- ⇨ Angle BAP + Angle BQC = 90

Now, Angle BAP = Angle BCP (On the samw arc BP)

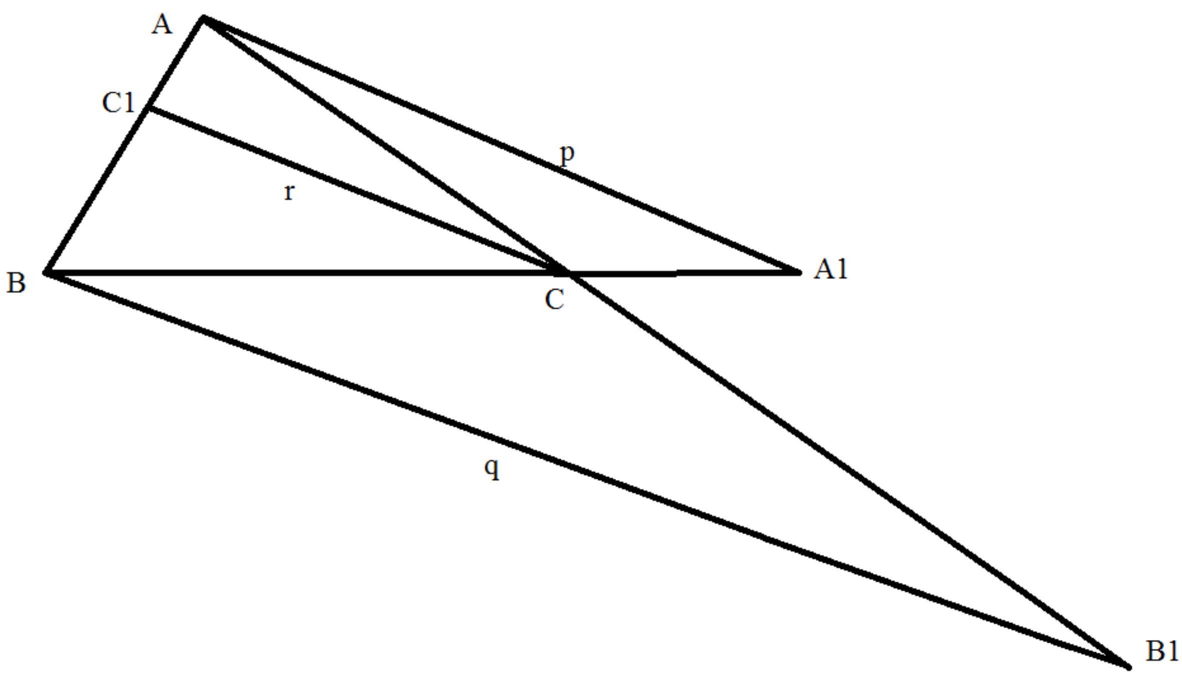
- ⇨ Angle BCP + Angle BQC = 90
- ⇨ Angle PQC = 90
- ⇨ PQ = diameter (As semicircular angle is right-angle)
- ⇨ PQ = 2r

Option (b) is correct.

444. In a triangle ABC, let C_1 be any point on the side AB other than A or B. Join CC_1. The line passing through A and parallel to CC_1 intersects the line BC extended at A_1. The line passing through B and parallel to CC_1 intersects the line AC extended at B_1. The lengths AA_1, BB_1, CC_1 are given to be p, q, r units respectively. Then

(a) $r = pq/(p + q)$
(b) $r = (p + q)/4$
(c) $r = \sqrt{(pq)}/2$
(d) none of the foregoing statements is true.

Solution :



Triangle AC_1C and triangle ABB_1 are similar,

Therefore, $r/q = AC_1/AB$

Triangle BC_1C and triangle ABA_1 are similar.

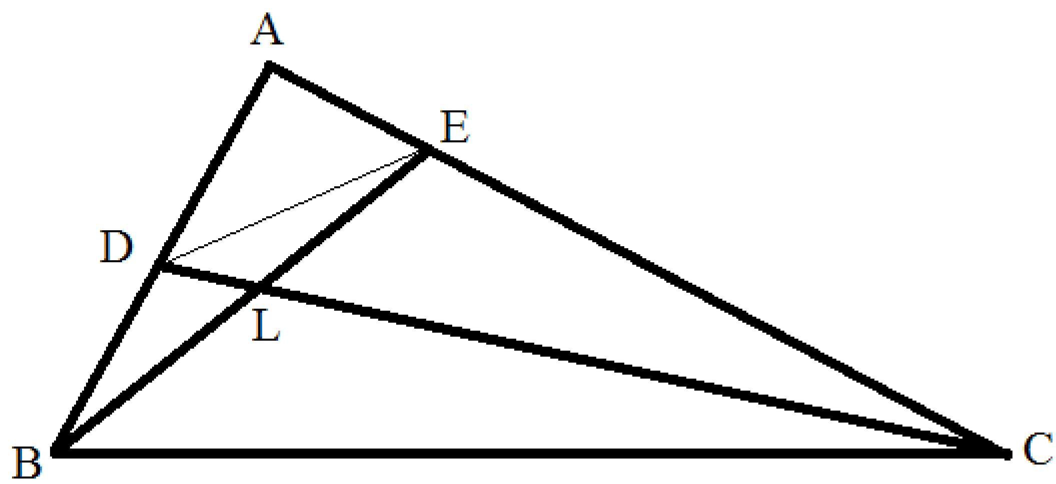
Therefore, $r/p = BC_1/AB$

- ⇨ $r/q + r/p = (AC_1 + BC_1)/AB$
- ⇨ $r(p + q)/pq = 1$
- ⇨ $r = pq/(p + q)$

Option (a) is correct.

445. In a triangle ABC, D and E are the points on AB and AC respectively such that Angle BDC = Angle BEC. Then
(a) Angle BED = Angle BCD
(b) Angle CBE = Angle BED
(c) Angle BED + Angle CDE = Angle BAC
(d) Angle BED + Angle BCD = Angle BAC

Solution :



In triangle BDL and triangle LEC, Angle DLB = Angle CLE (opposite angle)

Angle BDL = Angle CEL (given)

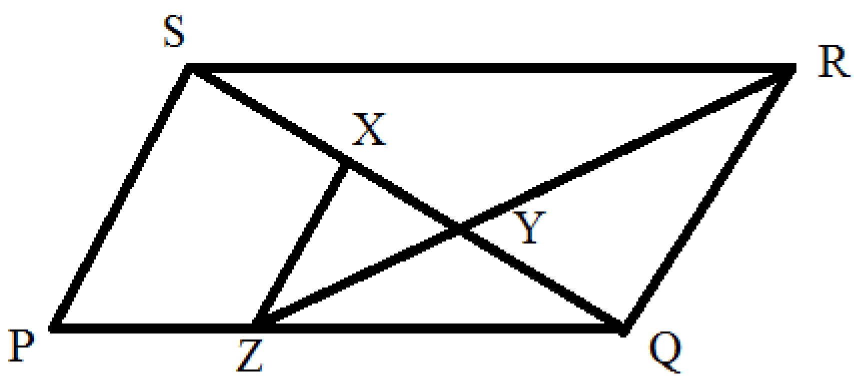
Therefore, Angle DBL = Angle LCE i.e. Angle DBE = Angle DCE

Therefore, BCED is a cyclic quadrilateral (Angle DBE and Angle DCE are on the same arc DE and same)

⇨ Angle BED = Angle BCD (on same arc BD)

Option (a) is correct.

446. In the picture, PQRS is a parallelogram. PS is parallel to ZX and PZ/ZQ equals 2/3. Then XY/SQ equals



(a) ¼
(b) 9/40
(c) 1/5
(d) 9/25

Solution :

Triangles QXZ and QSP are similar.

Therefore, QX/SQ = ZX/PS = ZQ/PQ

Now, in triangles XYZ and YRQ, Angle XYZ = Angle RYQ (opposite angle)

Angle YXZ = Angle YQR and Angle YZX = Angle YRQ (ZX||RQ)

Triangles XYZ and YRQ are similar.

Therefore, XY/YQ = ZX/QR = ZX/PS = ZQ/PQ

Let XY/SQ = a.

Now, YQ/XY = PQ/ZQ

- ⇨ (YQ + XY)/XY = PQ/ZQ + 1
- ⇨ QX/XY = PQ/ZQ + 1
- ⇨ (SQ/XY)(QX/SQ) = PQ/ZQ + 1
- ⇨ QX/SQ = a(PQ/ZQ + 1)
- ⇨ ZQ/PQ = a(PQ/ZQ + 1)

Now, PZ/ZQ = 2/3

- ⇨ (PZ + ZQ)/ZQ = (2 + 3)/3
- ⇨ ZQ/PQ = 3/5

Therefore, the above equation becomes, 3/5 = a(5/3 + 1)

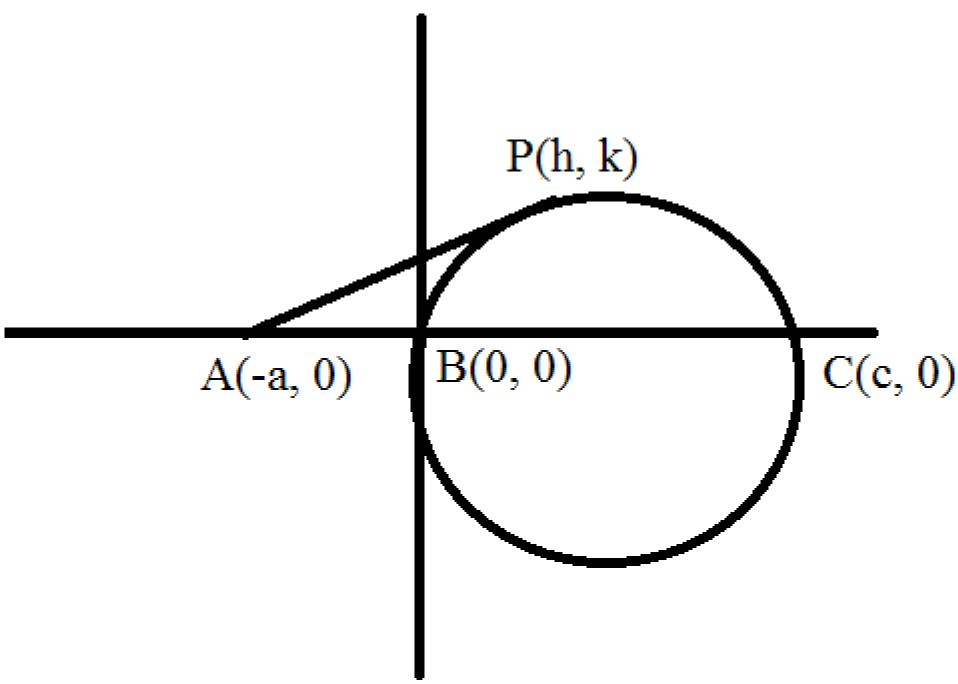
⇨ 3/5 = 8a/3
⇨ a = 9/40

Option (b) is correct.

447. Let A, B, C be three points on a straight line, B lying between A and C. Consider all circles passing through B and C. The points of contact of the tangents from A to these circles lie on

(a) a straight line
(b) a circle
(c) a parabola
(d) a curve of none of the foregoing types

Solution :



Let, centre of the circle = (-g, -f)

Let equation of the circle = $x^2 + y^2 + 2gx + 2fy + c_1 = 0$

The circle passes through (0, 0), so $c_1 = 0$

Now, the circle passes through (c, 0), so, $c^2 + 2gc = 0$, $g = -c/2$

The circle passes through (h, k), so $h^2 + k^2 + 2(-c/2)h + 2fk = 0$

⇨ $f = -(h^2 + k^2 + ch)/2k$

Now, slope of AP = $(k - 0)/(h + a) = k/(h + a)$

Slope of OP (O being the centre) = $\{k - (h^2 + k^2 + ch)/2k\}/(h - c/2) = (k^2 - h^2 - ch)/(2kh - kc)$

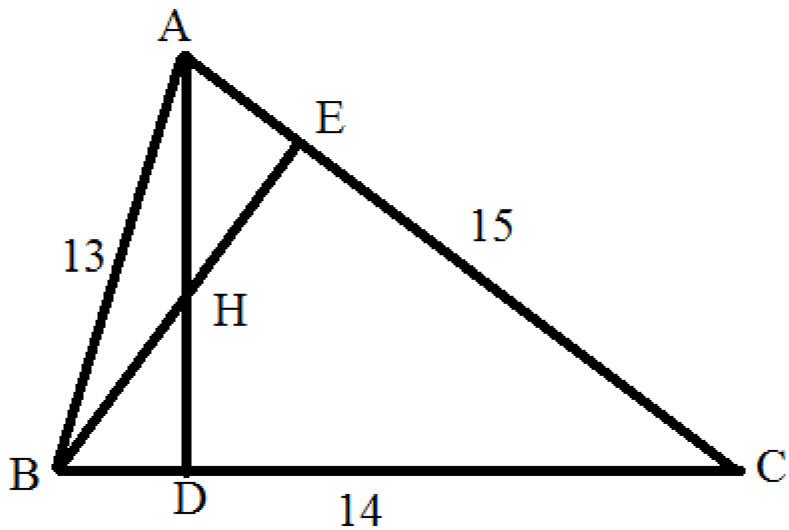
Now, slope of AP*slope of OP = -1 (perpendicular as AP is tangent)

⇨ $\{(k^2 - h^2 - ch)/(2kh - kc)\}\{k/(h + a) = -1$
⇨ $k(k^2 - h^2 - ch) = -2kh^2 + hkc - 2kha + kac$
⇨ $k^3 - h^2k - hkc = -2kh^2 + hkc - 2kha + kac$
⇨ $k^3 + kh^2 - 2khc + 2kha - kac = 0$
⇨ $h^2 + k^2 - 2h(c - a) - ac = 0$
⇨ a circle.

Option (b) is correct.

448. ABC be a triangle with AB = 13; BC = 14 and CA = 15. AD and BE are the altitudes from A and B to BC and AC respectively. H is the point of intersection of AD and BE. Then the ratio HD/HB is
(a) 3/5
(b) 12/13
(c)4/5
(d) 5/9

Solution :



Now, Angle ABE = 180 – (90 + A) = 90 – A (from triangle ABE)

Angle HBD = B – (90 – A) = A + B – 90

sin(HBD) = sin(A + B – 90) = sin(180 – C – 90) = sin(90 – C) = cosC = $(a^2 + b^2 - c^2)/2ab = (14^2 + 15^2 - 13^2)/(2*14*15) = (196 + 225 - 169)/(2*14*15) = 252/(2*14*15) = 3/5$

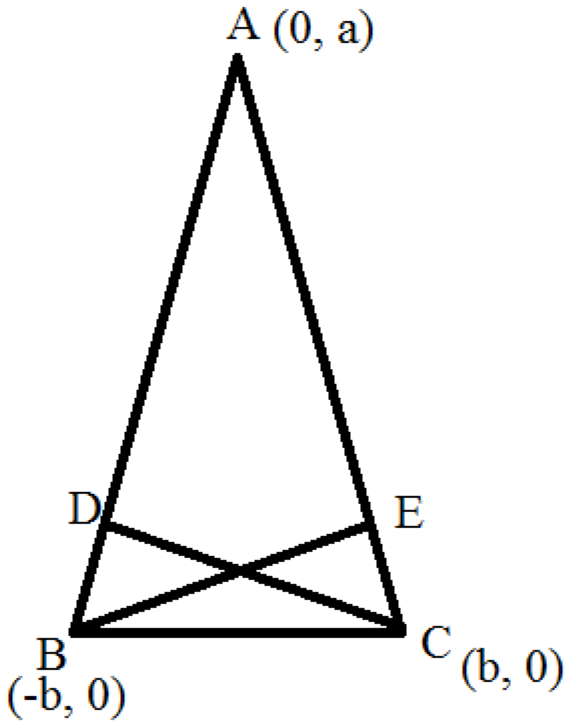
⇨ HD/HB = 3/5 (from triangle HBD)

Option (a) is correct.

449. ABC is a triangle such that AB = AC. Let D be the foot of the perpendicular from C to AB and E the foot of the perpendicular from B to AC. Then

(a) $BC^3 < BD^3 + BE^3$
(b) $BC^3 = BD^3 + BE^3$
(c) $BC^3 > BD^3 + BE^3$
(d) None of the foregoing statements need always be true.

Solution :



Equation of AC is, $x/b + y/a = 1$ i.e. $ax + by = ab$, slope $= - a/b$

Slope of BE $= b/a$ (as perpendicular on AC)

Equation of BE is, $y - 0 = (b/a)(x + b)$ i.e. $bx - ay + b^2 = 0$

Solving them we get, $x = b(a^2 - b^2)/(a^2 + b^2)$, $y = 2ab^2/(a^2 + b^2)$

Therefore, $E = \{b(a^2 - b^2)/(a^2 + b^2), 2ab^2/(a^2 + b^2)\}$

$BE = \sqrt{[\{2ab^2/(a^2 + b^2)\}^2 + \{b(a^2 - b^2)/(a^2 + b^2) + b\}^2]}$

$BE^3 = [\{2ab^2/(a^2 + b^2)\}^2 + \{2a^2b/(a^2 + b^2)\}^2]^{3/2}$

$BE^3 = [\{2ab/(a^2 + b^2)\}^2(a^2 + b^2)]^{3/2}$

$BE^3 = (2ab)^3/(a^2 + b^2)^{3/2}$

$CE = [\{b(a^2 - b^2)/(a^2 + b^2) - b\}^2 + \{2ab^2/(a^2 + b^2)\}^2]^{1/2}$

$CE = [\{2b^3/(a^2 + b^2)\}^2 + \{2ab^2/(a^2 + b^2)\}^2]^{1/2}$

$CE = [\{2b^2/(a^2 + b^2)\}^2(b^2 + a^2)]^{1/2}$

$CE = 2b^2/(a^2 + b^2)^{1/2}$

$CE^3 = (2b^2)^3/(a^2 + b^2)^{3/2}$

$BE^3 + CE^3 = \{(2ab)^3 + (2b^2)^3\}/(a^2 + b^2)^{3/2}$

$= (2b)^3(a^3 + b^3)/(a^2 + b^2)^{3/2}$

$= (BC)^3\{(a^3 + b^3)/(a^2 + b^2)^{3/2}\} < (BC)^3$

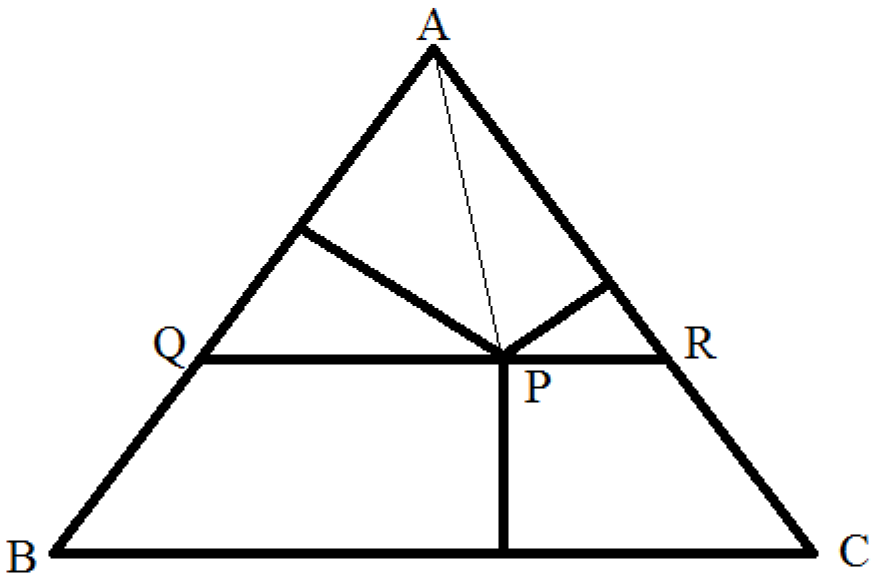
Now, let, $(a^3 + b^3)^2 \geq (a^2 + b^2)^3$

- ⇨ $a^6 + b^6 + 2a^3b^3 \geq a^6 + b^6 + 3a^2b^2(a^2 + b^2)$
- ⇨ $ab/3 \geq (a^2 + b^2)/2$ (But AM ≥ GM says, $(a^2 + b^2)/2 \geq ab > ab/3$)
- ⇨ $(a^3 + b^3) < (a^2 + b^2)^{3/2}$
- ⇨ $BC^3 > BE^3 + BD^3$ (As CE = BD)

Option (c) is correct.

450. Through the centroid of an equilateral triangle, a line parallel to the base is drawn. On this line, an arbitrary point P is taken inside the triangle. Let h denote the distance of P from the base of the triangle. Let h_1 and h_2 be the distances of P from the other two sides of the triangle. Then
(a) $h = (h_1 + h_2)/2$
(b) $h = \sqrt{(h_1h_2)}$
(c) $h = 2h_1h_2/(h_1 + h_2)$
(d) none of the foregoing conditions is necessarily true.

Solution :



AQ = AR = 2a/3 (where a is side of the equilateral triangle)

Area of triangle APQ = $(1/2)h_1*(2a/3) = ah_1/3$

Similarly, area of triangle APR = $ah_2/3$

Area of triangle APQ = area of triangle APQ + area of triangle APR = $(a/3)(h_1 + h_2)$

Now, height of the triangle AQR = 2H/3 where H is height of triangle ABC.

h = H/3

Therefore, height of the triangle AQR = 2h

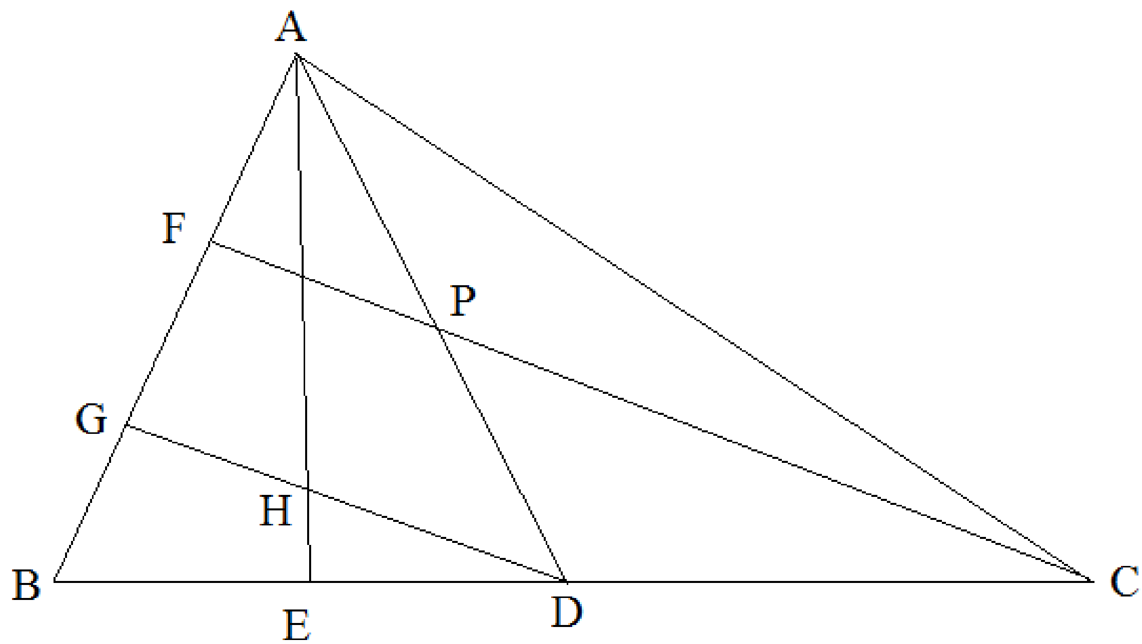
Area of triangle AQR = (1/2)(2a/3)*2h = 2ah/3

Therefore, $2ah/3 = (a/3)(h_1 + h_2)$

⇨ $h = (h_1 + h_2)/2$

Option (a) is correct.

451. In the figure that follows, BD = CD, BE = DE, AP = PD and DG||CF. Then (area of triangle ADH)/(area of triangle ABC) is equal to



(a) 1/6
(b) ¼
(c) 1/3
(d) None of the foregoing quantities.

Solution :

P is mid-point of AD and PF||DG.

Therefore, F is mid-point of AG.

In triangle BCE, GD||CF and D is mid-point of BC.

Therefore, G is mid-point of BF.

Therefore, AF = FG = BG

- ⇨ DGB = BGF = DFA
- ⇨ DGB = (1/3)ABD

In triangle AGD, GD||PF and P is mid-point of AD. Therefore, H is

Option (c) is correct.

452. Let A be the fixed point (0, 4) and B be a moving point (2t, 0). Let M be the mid-point of AB and let the perpendicular bisector of AB meet the y-axis at R. The locus of the mid-point P of MR is
(a) $y + x^2 = 2$
(b) $x^2 + (y - 2)^2 = 1/4$
(c) $(y - 2)^2 - x^2 = 1/4$
(d) None of the foregoing curves.

Solution :

Mid-point of AB = (t, 2)

Slope of AB = (4 – 0)/(0 – 2t) = -2/t

Slope of perpendicular bisector of AB = t/2

Equation of perpendicular bisector of AB is, y – 2 = (t/2)(x – t)

Putting x = 0, we get, $y = -t^2/2 + 2$

So, $R = (0, -t^2/2 + 2)$

Mid-point of MR, $P = (t/2, -t^2/4 + 2)$

$h = t/2$ and $k = -t^2/4 + 2$

⇨ $k + h^2 = 2$

Locus is $y + x^2 = 2$

Option (a) is correct.

453. Let l_1 and l_2 be a pair of intersecting lines in the plane. Then the locus of the points P such that the distance of P from l_1 is twice the distance of P from l_2 is
(a) an ellipse
(b) a parabola
(c) a hyperbola
(d) a pair of straight lines

Solution :

Let $l_1 \Rightarrow a_1x + b_1y + c_1 = 0$ and $l_2 \Rightarrow a_2x + b_2y + c_2 = 0$

$P = (h, k)$

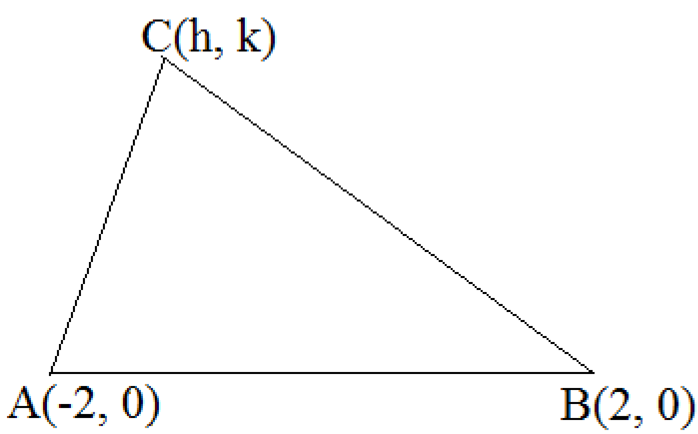
So, $|(a_1h + b_1k + c_1)/\sqrt{(a_1^2 + b_1^2)}| = 2|(a_2h + b_2k + c_2)/\sqrt{(a_2^2 + b_2^2)}|$

⇨ Locus is pair of straight lines. (One for + and one straight line for -)

Option (d) is correct.

454. A triangle ABC has fixed base AB and the ratio of the other two unequal sides is a constant. The locus of the vertex C is
(a) A straight line parallel to AB
(b) A straight line which is perpendicular to AB
(c) A circle with AB as a diameter
(d) A circle with centre on AB

Solution :



Now, CA/CB = constant = c

⇨ $\sqrt{\{(h + 2)^2 + k^2\}}/\sqrt{\{(h - 2)^2 + k^2\}} = c$
⇨ $(h + 2)^2 + k^2 = c^2(h - 2)^2 + c^2k^2$
⇨ $(1 - c^2)(h^2 + k^2) + 4h(1 + c^2) - 4c^2 = 0$

Locus is, $(1 - c^2)(x^2 + y^2) + 4x(1 + c^2) - 4c^2 = 0$

Option (d) is correct.

455. P is a variable point on a circle C and Q is a fixed point outside of C. R is a point on PQ dividing it in the ratio p : q, where p > 0 and q > 0 are fixed. Then the locus of R is
(a) a circle
(b) an ellipse
(c) a circle if p = q and an ellipse otherwise
(d) none of the foregoing curves

Solution :

Let C is $x^2 + y^2 = a^2$.

Let P = (b, c)

Q = (m, n)

Therefore, $b^2 + c^2 = a^2$.

R = (h, k)

So, h = (pm + qb)/(p + q) and k = (pn + qc)/(p + q)

- ⇨ qb = (p + q)h – pm and qc = (p + q)k – pn
- ⇨ $q^2(b^2 + c^2) = (p + q)^2h^2 + (p + q)^2k^2 - 2pm(p + q)h + p^2m^2 - 2pn(p + q)k + p^2n^2$
- ⇨ $q^2a^2 = (p + q)^2(h^2 + k^2) - 2p(p + q)(mh + nk) + p^2(m^2 + n^2)$
- ⇨ $(p + q)^2(h^2 + k^2) - 2p(p + q)(mh + nk) + p^2(m^2 + n^2) - q^2a^2 = 0$

Locus is, $(p + q)^2(x^2 + y^2) - 2p(p + q)(mx + ny) + p^2(m^2 + n^2) - q^2a^2 = 0$

A circle.

Option (a) is correct.

456. Let r be the length of then chord intercepted by the ellipse $9x^2 + 16y^2 = 144$ on the line 3x + 4y = 12. Then
(a) r = 5
(b) r > 5
(c) r = 3
(d) r = √7

Solution :

Solving the two equations,

$3x = 12 - 4y$

$(3x)^2 + 16y^2 = 144$

- ⇨ $(12 - 4y)^2 + 16y^2 = 144$
- ⇨ $144 - 96y + 16y^2 + 16y^2 = 144$
- ⇨ $32y(y - 3) = 0$
- ⇨ $y = 0$, $y = 3$
- ⇨ $x = 4$, $x = 0$

Points are (4, 0) and (0, 3)

Distance $= \sqrt{\{(4 - 0)^2 + (0 - 3)^2\}} = 5$

Option (a) is correct.

457. The angles A, B and C of a triangle ABC are in arithmetic progression. AB = 6 and BC = 7. Then AC is
   (a) 5
   (b) 7
   (c) 8
   (d) None of the foregoing numbers.

Solution :

Let $A = B - d$ and $C = B + d$

$A + B + C = 180$

- ⇨ $B - d + B + B + d = 180$
- ⇨ $B = 60$

Now, $AB/\sin C = BC/\sin A = AC/\sin B$

- ⇨ $\sin A = 7/(AC*\sqrt{3}/2) = 14/AC\sqrt{3}$
- ⇨ $\sin C = 12/AC\sqrt{3}$

$\sin(A + C) = \sin A\cos C + \cos A\sin C$

- ⇨ $\sin 120 = \{14/AC\sqrt{3})\cos C + \cos A\{12/AC\sqrt{3}\}$
- ⇨ $\sqrt{3}/2 = \{14/AC\sqrt{3}\}\cos C + \cos A\{12/AC\sqrt{3}\}$
- ⇨ $3/4 = 7\cos C/AC + 6\cos A/AC$
- ⇨ $3/4 = 7\sqrt{(3AC^2 - 144)}/AC^2\sqrt{3} + 6\sqrt{(3AC^2 - 196)}/AC^2\sqrt{3}$

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⇨ $3\sqrt{3}AC^2/4 = 7\sqrt{(3AC^2 - 144)} + 6\sqrt{(AC^2 - 196)}$

Clearly, none of (a), (b), (c) satisfies the equation.

Option (d) is correct.

458. ABC is a triangle. P, Q and R are respectively the mid-points of AB, BC and CA. The area of the triangle ABC is 20. Then the area of the triangle PQR is
(a) 4
(b) 5
(c) 6
(d) 8

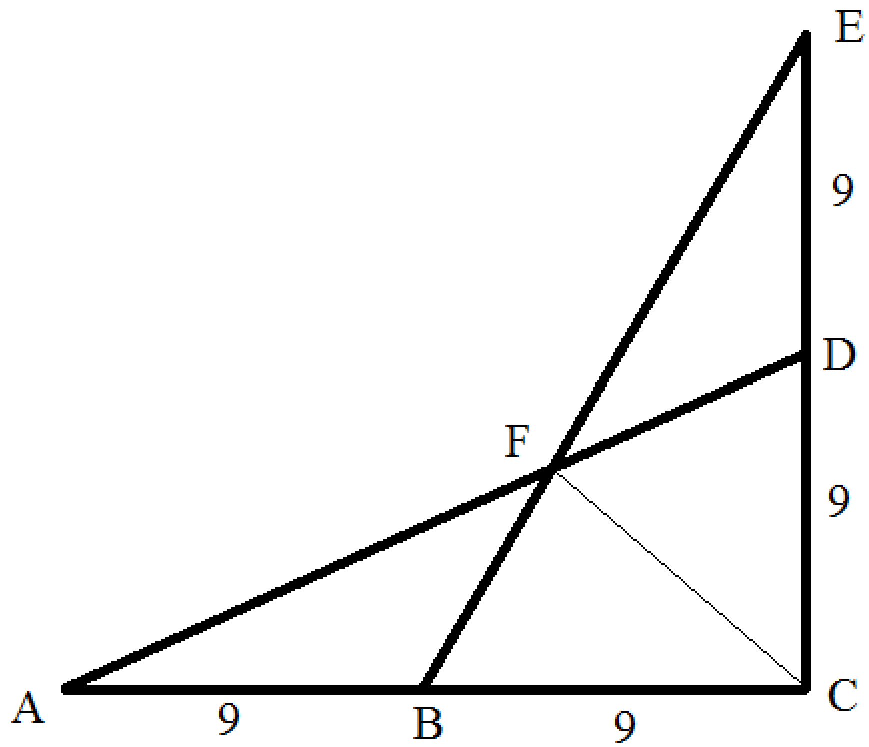
Solution :

Area of triangle PQR = (1/4)*area of triangle ABC = 5

Option (b) is correct.

459. Let AC and CE be perpendicular line segments, each of length 18. Suppose B and D are the mid-points of AC and CE, respectively. If F is the point of intersection of EB and AD, then the area of the triangle DEF is
(a) 18
(b) $18\sqrt{2}$
(c) 27
(d) $5\sqrt{85}/2$

Solution :



Now, area of triangle DEF = area of triangle DFC (as base is same and height is same)

= area of triangle BFC

Therefore, area of triangle DEF = (1/3)*area of triangle EBC

Area of triangle EBC = (1/2)*9*18 = 81

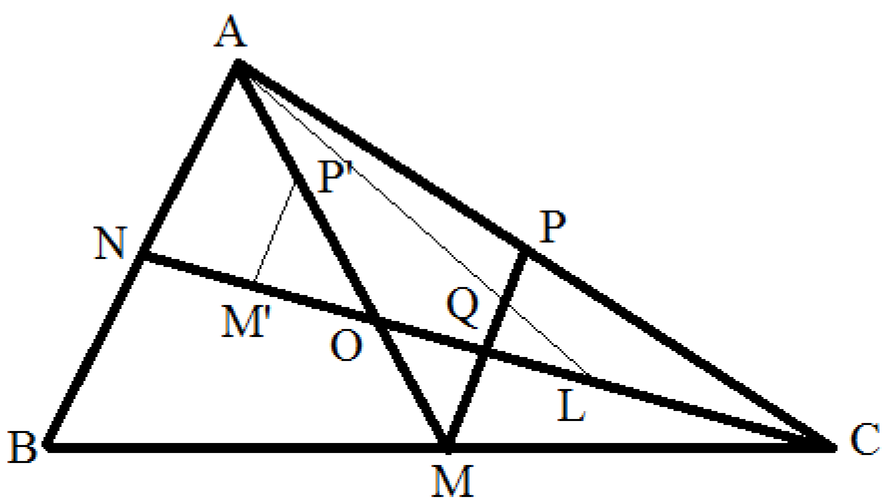
Area of triangle DEF = 81/3 = 27

Option (c) is correct.

460. In a triangle ABC, the medians AM and CN to the sides BC and AB respectively, intersect at the point O. Let P be the mid-point of AC and let MP intersect CN at Q. If the area of the triangle OMQ is s square units, the area of ABC is

(a) 16s
(b) 18s
(c) 21s
(d) 24s

Solution :



In triangles OMQ and ANO, Angle QOM = Angle AON (opposite angle)

Angle OMQ = Angle OAN and Angle OQM = Angle ANO (PM||AN as P and M are mid-points of AC and BC respectively)

Now, OQ/ON = OM/OA = QM/AN = ½ (QM = (1/2)BN (In triangle BNC Q and M are mid-points of CN and BC)

So, we draw P'OM' where OP' = OM = ½ OA and OM' – OQ = ½ ON.

Therefore, OMQ = OM'P'= (1/4)AON

L is mid-point of OC and we know medians intersects at 2 : `1 ratio.

Therefore, ON = OL = LC

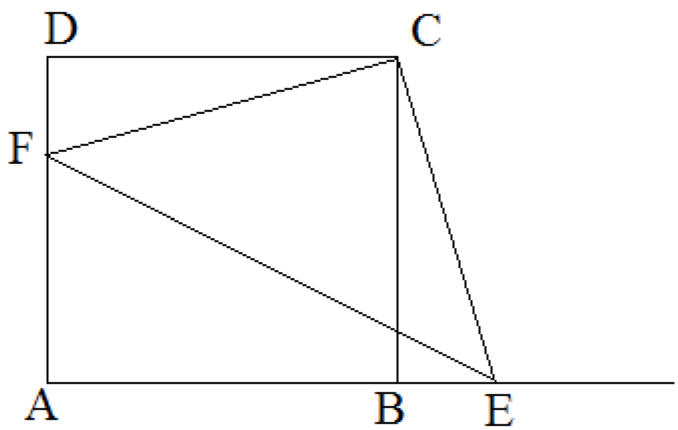
- ⇨ AON = AOL = ALC
- ⇨ AON = (1/3)ANC
- ⇨ OMQ = (1/4)AON = (1/4)(1/3)ANC = (1/12)(1/2)ABC
- ⇨ ABC = 24OMQ = 24s

Option (d) is correct.

461. Let F be a point on the side AD of a square ABCD of area 256. Suppose the perpendicular to the line FC at C meets the line segment AB extended at E. If the area of the triangle CEF is 200, then the length of BE is
(a) 12

(b) 14
(c) 15
(d) 20

Solution :



In triangles CFD and BEC, Angle FDC = Angle CBE (both right angles)

CD = CB (both sides of square ABCD)

Angle DCF = Angle BCE (Angle DCB – Angle FCB = Angle FCE – Angle FCB)

So, CF = CE

Area of triangle CEF = (1/2)*CF*CE = (1/2)CE^2 = 200

⇨ CE = 20

CB^2 = 256

⇨ CB = 16

BE = $\sqrt{(20^2 - 16^2)}$ = 12

Option (a) is correct.

462. Consider the circle with centre C = (1, 2) which passes through the points P = (1, 7) and Q = (4, -2). If R is the point of intersection of the tangents to the circle drawn at P and Q, then the area of the quadrilateral CPRQ is

(a) 50
(b) $50\sqrt{2}$
(c) 75
(d) 100

Solution :

Slope of CP = (7 – 2)/(1 – 1) = 5/0

Slope of tangent at P = 0

Therefore, equation of tangent at P is, y – 7 = 0(x – 1)

⇨ y = 7

Slope of CQ = (2 + 2)/(1 – 4) = -4/3

Slope of tangent at Q = ¾

Equation of tangent at Q is, y + 2 = (3/4)(x – 4)

⇨ 3x – 4y = 20

Solving them we get, x = 16

R = (16, 7)

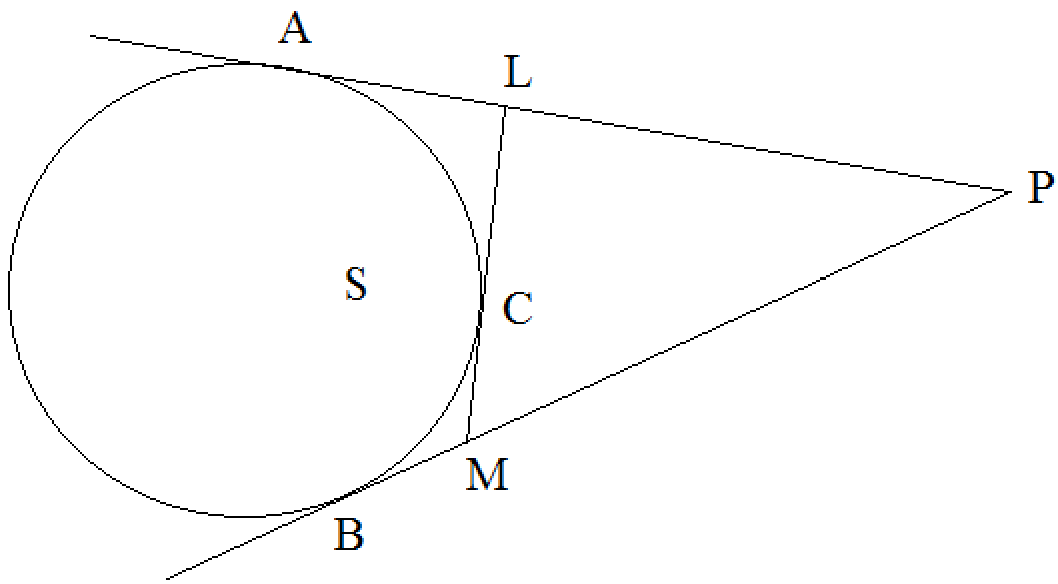
Area of triangle CPR = |(1/2){1(7 – 7) + 1(7 – 2) + 16(2 – 7)}| = 75/2

Area of triangle CRQ = |(1/2){1(7 + 2) + 16(-2 – 2) + 4(2 – 7)| = 75/2

Area of quadrilateral CPRQ = 75/2 + 75/2 = 75

Option (c) is correct.

463. PA and PB are tangents to a circle S touching S at points A and B. C is a point on S in between A and B as shown in the figure. LCM is a tangent to S intersecting PA and PB in points L and M, respectively. Then the perimeter of the triangle PLM depends on



(a) A, B, C and P
(b) P, but not on C
(c) P and C only
(d) the radius of S only

Solution :

Now, LA = LC (tangents from same point L)

PA – PL = LC

⇨ PA = PL + LC

MB = MC

PB – PM = MC

⇨ PB = PM + MC
⇨ PA + PB = PL + LC + PM + MC = PL + PM + LM = constant.
⇨ Therefore, it depends on only P.

Option (b) is correct.

464. A and B are two points lying outside a plane ∏, but on the same side of it. P and Q are, respectively, the feet of perpendiculars from A

and B on ∏. Let X be any point on ∏. Then (AX + XB) is minimum when X
(a) lies on PQ and Angle AXP = Angle BXQ
(b) is the mid-point of PQ
(c) is any point of ∏ with Angle AXP = Angle BXQ
(d) is any point on the perpendicular bisector of PQ in ∏

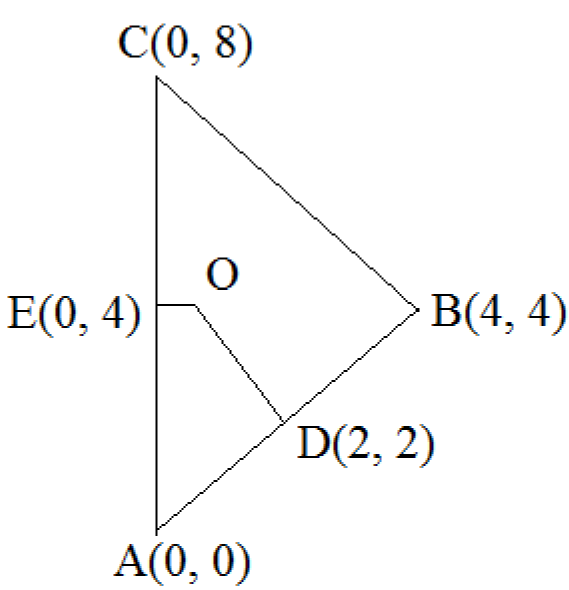
Solution :

This is self-explanatory.

Option (a) is correct.

465. The vertices of a triangle are the points (0, 0), (4, 4) and (0, 8). The radius of the circumcircle of the triangle is
(a) $3\sqrt{2}$
(b) $2\sqrt{2}$
(c) 3
(d) 4

Solution :



Slope of AB = (4 – 0)/(4 – 0) = 1

Slope of OD = -1

Equation of OD is, $y - 2 = (-1)(x - 2)$

- ⇨ $x + y = 4$

Equation of OE is, $y = 4$

Solving them we get, $x = 0$

Therefore, O = (0, 4)

Circumradius = 4

Option (d) is correct.

466. The number of different angles θ satisfying the equation $\cos\theta + \cos 2\theta = -1$ and at the same time satisfying the condition $0 < \theta < 360$ is
(a) 0
(b) 4
(c) 2
(d) 3

Solution :

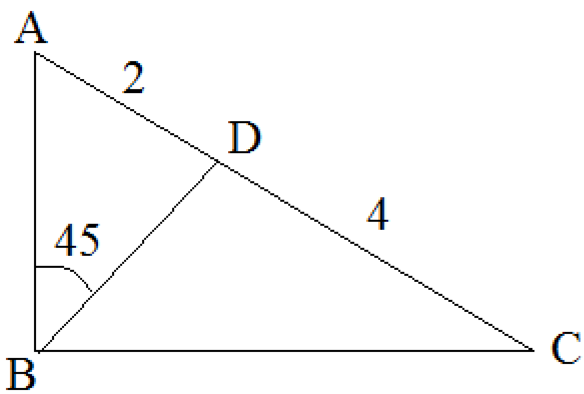
$\cos\theta + 2\cos^2\theta - 1 = -1$

- ⇨ $\cos\theta(2\cos\theta + 1) = 0$
- ⇨ $\cos\theta = 0$, $\theta = 90, 270$
- ⇨ $2\cos\theta + 1 = 0$
- ⇨ $\cos\theta = -1/2$
- ⇨ $\theta = 120, 240$

Option (b) is correct.

467. ABC is a right-angled triangle with right angle at B. D is a point on AC such that Angle ABD = 45. If AC = 6 cm and AD = 2 cm then AB is
(a) $6/\sqrt{5}$ cm
(b) $3\sqrt{2}$ cm
(c) $12/\sqrt{5}$ cm
(d) 2 cm

Solution :



Now, in triangle ABD, 2/sin45 = AB/sin(ADB) = AB/sin(180 – BDC) = AB/sin(BDC)

In triangle BDC, 4/sin45 = BC/sin(BDC)

Dividing both the equation we get, ½ = AB/BC

⇨ BC = 2AB

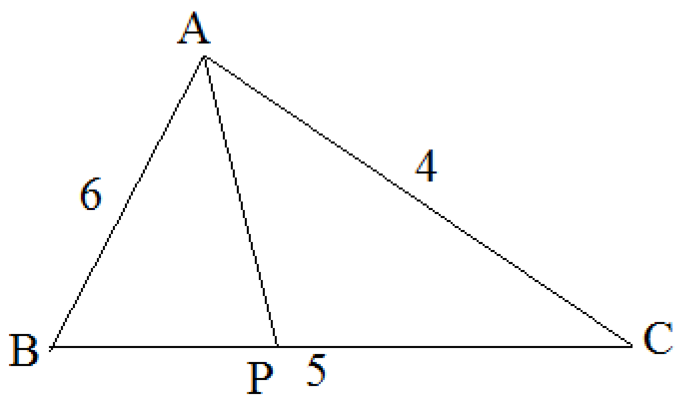
Now, $AB^2 + BC^2 = 6^2$

⇨ $AB^2 + 4AB^2 = 6^2$
⇨ $5AB^2 = 6^2$
⇨ $AB = 6/\sqrt{5}$

Option (a) is correct.

468. In the triangle ABC, AB = 6, BC = 5, CA = 4. AP bisects the angle A and P lies on BC. Then BP equals
(a) 3
(b) 3.1
(c) 2.9
(d) 4.5

Solution :



In triangle ABP, BP/sin(A/2) = 6/sin(APB) = 6/sin(180 – APC) = 6/sin(APC)

In triangle ACP, PC/sin(A/2) = 4/sin(APC)

Dividing the two equations we get, BP/PC = 3/2

BP = 5*(3/5) = 3

Option (a) is correct.

469. In a triangle ABC, the internal bisector of the angle A meets BC at D. If AB = 4, AC = 3 and A = 60, then length of AD is

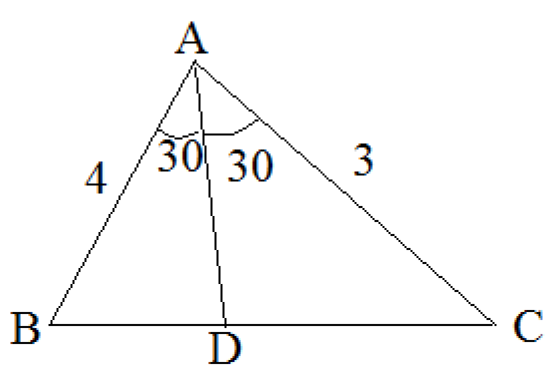
(a) $2\sqrt{3}$

(b) $12\sqrt{3}/7$

(c) $15\sqrt{3}/8$

(d) None of these numbers.

Solution :



Now, cos60 = $(4^2 + 3^2 - BC^2)/(2*4*3)$

⇨ $BC = \sqrt{13}$

Now, BD/CD = 4/3 (See previous problem)

⇨ $BD = \sqrt{13}(4/7) = 4\sqrt{13}/7$

Now, in triangle ABD, $\cos 30 = \{4^2 + AD^2 - (4\sqrt{13}/7)^2\}/(2*4*AD)$

⇨ $4\sqrt{3}AD = 16 - 16*13/49 + AD^2$
⇨ $AD^2 - 4\sqrt{3}AD + 16*36/49 = 0$
⇨ $AD = \{4\sqrt{3} \pm \sqrt{(48 - 4*16*36/49)}\}/2 = \{4\sqrt{3} \pm 4\sqrt{3}/7\}/2 = 2\sqrt{3}(1 \pm 1/7) = 16\sqrt{3}/7, 12\sqrt{3}/7$
⇨ $AD = 12\sqrt{3}/7$ (as $16\sqrt{3}/7 > 4 + 4\sqrt{13}/7$)

Option (b) is correct.

470. ABC is a triangle with BC = a, CA = b and Angle BCA = 120. CD is the bisector of Angle BCA meeting AB at D. Then length of CD is
(a) $(a + b)/4$
(b) $ab/(a + b)$
(c) $(a^2 + b^2)/2(a + b)$
(d) $(a^2 + ab + b^2)/3(a + b)$

Solution :

Same problem as the previous one.

Option (b) is correct.

471. The diagonal of the square PQRS is a + b. The perimeter of a square with twice the area of PQRS is
(a) $2(a + b)$
(b) $4(a + b)$
(c) $\sqrt{8}(a + b)$
(d) $8ab$

Solution :

Area of PQRS = $(a + b)^2/2$

Area of required square = $(a + b)^2$

Side of the required square = (a + b)

Perimeter = 4(a + b)

Option (b) is correct.

472. A string of length 12 inches is bent first into a square PQRS and then into a right-angled triangle PQT by keeping the side PQ of the square fixed. Then the area of PQRS equals
(a) area of PQT
(b) 2(area + PQT)
(c) 3(area of PQT)/2
(d) None of the foregoing numbers.

Solution :

PQ = 12/4 = 3

QT + TP = 12 – 3 = 9

TP = (9 – QT)

Now, $TP^2 = QT^2 + PQ^2$ (right angle at Q)

⇨ $(9 - QT)^2 = QT^2 + 9$
⇨ $81 - 18QT + QT^2 = QT^2 + 9$
⇨ 18QT = 72
⇨ QT = 4

Area of triangle PQT = (1/2)*PQ*QT = (1/2)*3*4 = 6

Area of PQRS = $3^2 = 9$

Option (c) is correct.

473. Instead of walking along two adjacent sides of a rectangular field, a boy took a short-cut along the diagonal and saved a distance equal to half the longer side. Then the ratio of the shorter side to the longer side is
(a) ½
(b) 2/3
(c) ¼
(d) ¾

Solution :

Longer side = b, shorter side = a.

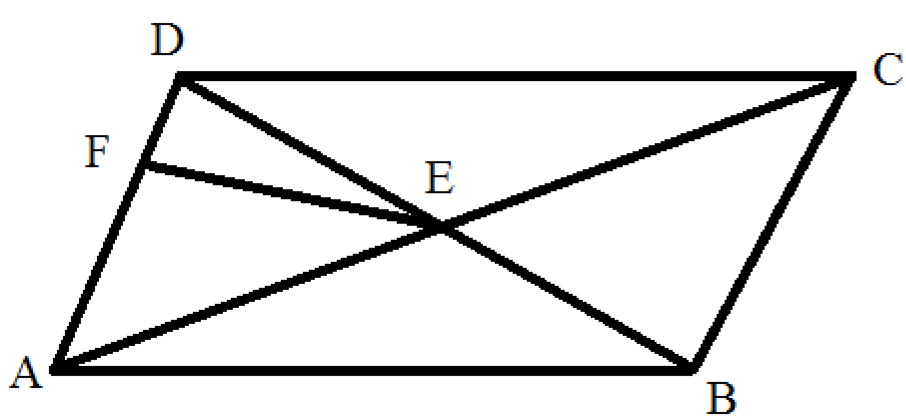
$a + b - \sqrt{(a^2 + b^2)} = b/2$

- ⇨ $\sqrt{(a^2 + b^2)} = a + b/2$
- ⇨ $a^2 + b^2 = a^2 + ab + b^2/4$
- ⇨ $3b^2/4 = ab$
- ⇨ $a/b = 3/4$

Option (d) is correct.

474. Consider a parallelogram ABCD with E as the midpoint of its diagonal BD. The point E is connected to a point F on DA such that DF = (1/3)DA. Then, the ratio of the area of the triangle DEF to the area of the quadrilateral ABEF is
(a) 1 : 2
(b) 1 : 3
(c) 1 : 5
(d) 1 : 4

Solution :



Now, DF = (1/3)DA

- ⇨ Area of DEF = (1/3)(area of AED) = (1/6)(area of ABD) = (1/6)(area of DEF + area of ABEF)

- ⇨ (5/6)(area of DEF) = (1/6)(area of ABEF)
- ⇨ (area of DEF)/(area of ABEF) = 1/5

Option (c) is correct.

475. The external length, breadth and height of a closed box are 10 cm, 9 cm and 7 cm respectively. The total inner surface area of the box is 262 sq cm. If the walls of the box are of uniform thickness d cm, then d equals
(a) 1.5
(b) 2
(c) 2.5
(d) 1

Solution :

Inner length = 10 – 2d, inner breadth = 9 – 2d, inner height = 7 – 2d

Now, 2{(10 – 2d)(9 – 2d) + (9 – 2d)(7 – 2d) + (10 – 2d)(7 – 2d)} = 262

- ⇨ $90 - 38d + 4d^2 + 63 - 32d + 4d^2 + 70 - 34d + 4d^2 = 131$
- ⇨ $12d^2 - 104d + 92 = 0$
- ⇨ $3d^2 - 26d + 23 = 0$
- ⇨ $3d^2 - 3d - 23d + 23 = 0$
- ⇨ 3d(d – 1) – 23(d – 1) = 0
- ⇨ (d – 1)(3d – 23) = 0
- ⇨ d = 1 (d = 23/3 > 7)

Option (d) is correct.

476. A hollow spherical ball whose inner radius 4 cm is full of water. Half of the water is transferred to a conical cup and it completely fills the cup. If the height of the cup is 2 cm, then the radius of the base of the cone, in cm, is
(a) 4
(b) 8π
(c) 8
(d) 16

Solution :

Volume of the sphere = $(4/3)\pi*4^3 = 256\pi/3$

Volume of the cone = $(1/2)*256\pi/3 = 128\pi/3$

Volume of cone = $(1/3)\pi r^2h = 128\pi/3$

- ⇨ $r^2*2 = 128$
- ⇨ $r = 8$

Option (c) is correct.

477. PQRS is a trapezium with PQ and RS parallel, PQ = 6 cm, QR = 5 cm, RS = 3 cm, PS = 4cm. The area of PQRS
(a) is 27 cm^2
(b) 12 cm^2
(c) 18cm^2
(d) cannot be determined from the given information

Solution :

Let, the distance between PQ and RS is h.

Therefore, $\sqrt{(4^2 - h^2)} + \sqrt{(5^2 - h^2)} + 3 = 6$

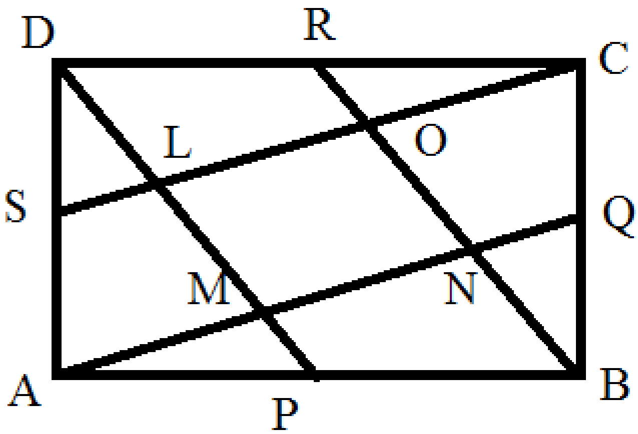
- ⇨ $\sqrt{(16 - h^2)} = 3 - \sqrt{(25 - h^2)}$
- ⇨ $16 - h^2 = 9 - 6\sqrt{(25 - h^2)} + 25 - h^2$
- ⇨ $\sqrt{(25 - h^2)} = 3$
- ⇨ $25 - h^2 = 9$
- ⇨ $h = 4$

Area = $(1/2)(5 + 4)*4 = 18$ cm^2

Option (c) is correct.

478. Suppose P, Q, R and S are the midpoints of the sides AB, BC, CD and DA, respectively, of a rectangle ABCD. If the area of the rectangle is Δ, then the area of the figure bounded by the straight lines AQ, BR, CS and DP is
(a) Δ/4
(b) Δ/5
(c) Δ/8
(d) Δ/2

Solution :



SL = (1/2)AM (S is mid-point of AD and SL||AM) = (1/2)OC = (1/2)OL

Height of triangle DSL = distance between SL and AM

- ⇨ Area of DSL = (1/2)(area of OML) = (1/4)(area of LMNO)
- ⇨ Area of OCR = (1/2)(area of ONM) = (1/4)(area of LMNO)

Now, Area of DSL = (1/4)(area of DAM)

Similarly, area of BOC = (1/4)(area of OCB)

Area of OCR = (1/4)(area of CLD)

Now, area of AMD + area of ANB + area of BOC + area of CLD + area of LMNO = Δ

- ⇨ 4(area of DSL) + 4(area of AMP) + 4(area of BNQ) + 4(area of OCR) + area of LMNO = Δ
- ⇨ 8(area of DSL) + 8(area of OCR) +area of LMNO = Δ
- ⇨ 2(area of LMNO) + 2(area + LMNO) + area of LMNO = Δ
- ⇨ Area of LMNO = Δ/5

Option (b) is correct.

479. The ratio of the area of a triangle ABC to the area of the triangle whose sides are equal to the medians of the triangle is
   (a) 2 : 1
   (b) 3 : 1

(c)4 : 3
(d) 3 : 2

Solution :

Take an equilateral triangle of side a and calculate the ratio.

Area of ABC = $(\sqrt{3}/4)a^2$

Median = $(\sqrt{3}/2)a$

Area of triangle formed by medians = $(\sqrt{3}/4)(3a^2/4)$

Ratio = $(\sqrt{3}/4)a^2 : (\sqrt{3}/4)(3a^2/4) = 4 : 3$

Option (c) is correct.

480. Let C_1 and C_2 be the inscribed and circumscribed circles of a triangle with sides 3 cm, 4 cm and 5 cm. Then (area of C_1)/(area of C_2) is
(a) 16/25
(b) 4/25
(c)9/25
(d) 9/16

Solution :

R = abc/4Δ and r = Δ/s

Now, S = (3 + 4 + 5)/2 = 6

$\Delta = \sqrt{\{6(6 - 3)(6 - 4)(6 - 5)\}} = 6$

r = 1, R = 3*4*5/(4*6) = 5/2

(area of C_1)/(area of C_2) = $(\pi*1^2)/(\pi*25/4) = 4/25$

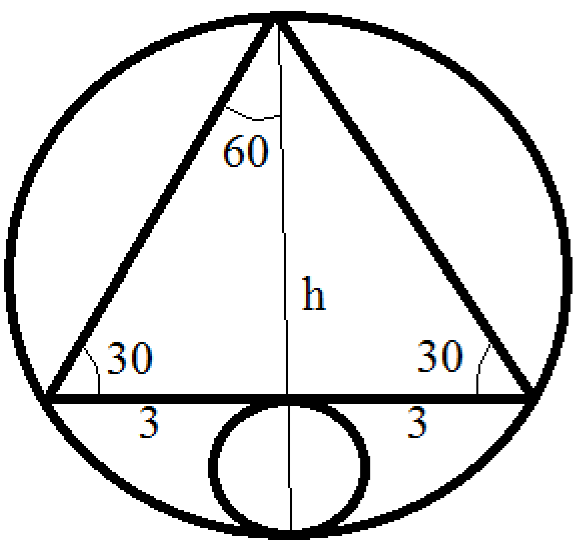
Option (b) is correct.

481. An isosceles triangle with base 6 cm and base angles 30 each is inscribed in a circle. A second circle touches the first circle and also

touches the base of the triangle at its midpoint. If the second circle is situated outside the triangle, then its radius (in cm) is

(a) $3\sqrt{3}/2$
(b) $\sqrt{3}/2$
(c) $\sqrt{3}$
(d) $4/\sqrt{3}$

Solution :



Now, $h/\sin 30 = 3/\sin 60$

- ⇨ $h = 3*(1/2)/(\sqrt{3}/2) = \sqrt{3}$
- ⇨ r = radius of big circle = $6/(2\sin 120) = 3/(\sqrt{3}/2) = 2\sqrt{3}$
- ⇨ Radius of small circle = $(2r - h)/2 = r - h/2 = 2\sqrt{3} - \sqrt{3}/2 = 3\sqrt{3}/2$

Option (a) is correct.

482. In an isosceles triangle ABC, $A = C = \pi/6$ and the radius of its circumcircle is 4. The radius of its incircle is

(a) $4\sqrt{3} - 6$
(b) $4\sqrt{3} + 6$
(c) $2\sqrt{3} - 2$
(d) $2\sqrt{3} + 2$

Solution :

$B = \pi - 2\pi/6 = 2\pi/3$

$a = 2*4*\sin(\pi/6) = 4$

$b = 2*4*\sin(2\pi/3) = 4\sqrt{3}$

$c = 4$

$S = (a + b + c)/2 = 4 + 2\sqrt{3}$

$\Delta = \sqrt{(4 + 2\sqrt{3})(4 + 2\sqrt{3} - 4)(4 + 2\sqrt{3} - 4\sqrt{3})(4 + 2\sqrt{3} - 4)} = (2\sqrt{3})\sqrt{(4 + 2\sqrt{3})(4 - 2\sqrt{3})} = (2\sqrt{3})\sqrt{(16 - 12)} = 4\sqrt{3}$

$r = \Delta/s = 4\sqrt{3}/(4 + 2\sqrt{3}) = 4\sqrt{3}(4 - 2\sqrt{3})/4 = 4\sqrt{3} - 6$

Option (a) is correct.

483. PQRS is a quadrilateral in which PQ and SR are parallel (that is, PQRS is a trapezium). Further, PQ = 10, QR = 5, RS = 4, SP = 5. Then area of the quadrilateral is
(a) 25
(b) 28
(c) 20
(d) $10\sqrt{10}$

Solution :

See solution of problem 477. Same problem.

Option (b) is correct.

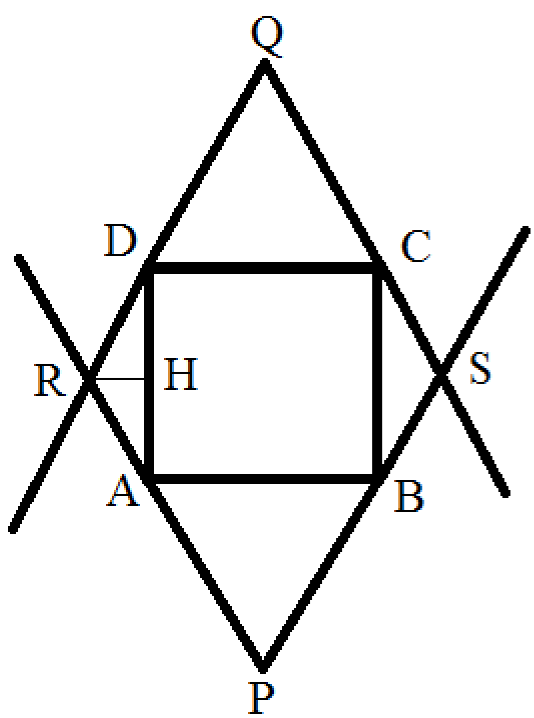
484. The area of quadrilateral ABCD with sides a, b, c, d is given by the formula $\{(s - a)(s - b)(s - c)(s - d) - abcd\cos^2\theta]^{1/2}$, where 2s is the perimeter and 2θ is the sum of opposite angles A and C. Then the area of the quadrilateral circumscribing a circle is given by
(a) $\tan\theta\sqrt{(abcd)}$
(b) $\cos\theta\sqrt{(abcd)}$
(c) $\sin\theta\sqrt{(abcd)}$
(d) none of the foregoing formula

Solution :

Option (c) is correct.

485. Consider a unit square ABCD. Two equilateral triangles PAB and QCD are drawn so that AP, DQ intersect at R, and BP, CQ intersect in S. The area of the quadrilateral PRQS is equal to
(a) $(2 - \sqrt{3})/6$
(b) $(2 - \sqrt{3})/3$
(c) $(2 + \sqrt{3})/6\sqrt{3}$
(d) $(2 - \sqrt{3})/\sqrt{3}$

Solution :



Now, Angle QDC = 60 (equilateral triangle)

Angle CDA = 90

Therefore, Angle RDA = 180 – (90 + 60) = 30

DH = ½

RH/DH = tan30, RH = (1/2)(1/√3) = 1/2√3

Area of RAD = (1/2)(1/2√3)*1= 1/4√3

Area of RAD + Area of CSB = 2(1/4√3) = 1/2√3

Area of equilateral triangles = $(\sqrt{3}/4)*1^2*2 = \sqrt{3}/2$

Area of the square = $1^2 = 1$

Therefore area of PRQS = 1 + √3/2 + 1/2√3 = 1 + 4/2√3 = (4 + 2√3)/2√3) = (2 + √3)/√3

It is given option (d) is correct.

486. Through an arbitrary point lying inside a triangle, three straight lines parallel to its sides are drawn. These lines divide the triangle into six parts, three of which are triangles. If the area of these triangles are S_1, S_2 and S_3, then the area of the given triangle equals

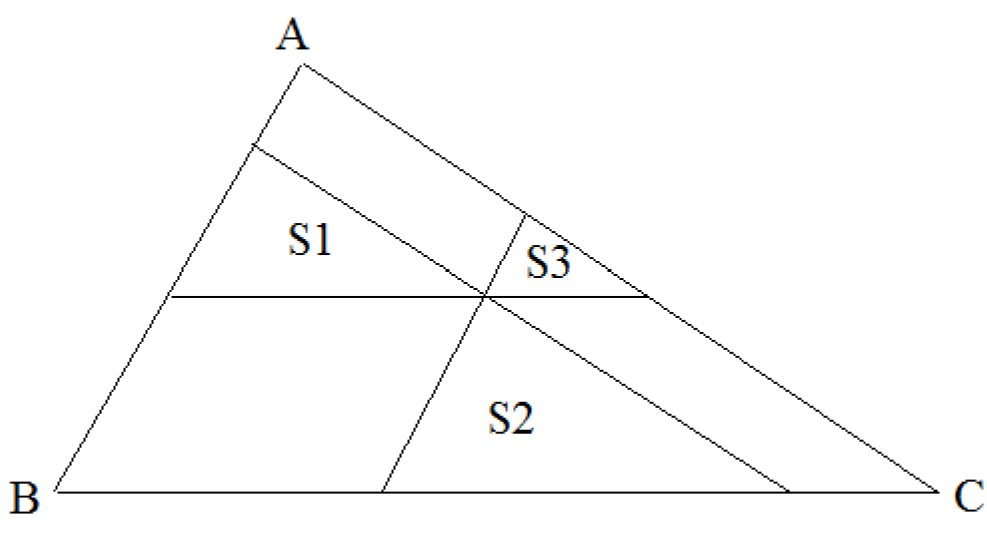
(a) $3(S_1 + S_2 + S_3)$

(b) $(\sqrt{(S_1S_2)} + \sqrt{(S_2S_3)} + \sqrt{(S_3S_1)})^2$

(c) $(\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3})^2$

(d) None of the foregoing quantities.

Solution :



Option (c) is correct.

487. The sides of a triangle are given by $\sqrt{(b^2 + c^2)}$, $\sqrt{(c^2 + a^2)}$ and $\sqrt{(a^2 + b^2)}$, where a, b, c are positive. Then the area of the triangle equals
(a) $(1/2)\sqrt{(b^2c^2 + c^2a^2 + a^2b^2)}$
(b) $(1/2)\sqrt{(a^4 + b^4 + c^4)}$
(c) $(\sqrt{3}/2)\sqrt{(b^2c^2 + c^2a^2 + a^2b^2)}$
(d) $(\sqrt{3}/2)(bc + ca + ab)$

Solution :

$\cos A = (c^2 + a^2 + a^2 + b^2 - b^2 - c^2)/2\sqrt{\{(c^2 + a^2)(a^2 + b^2)\}} = a^2/\sqrt{\{(c^2 + a^2)(a^2 + b^2)}$

$\sin A = \sqrt{\{(c^2 + a^2)(a^2 + b^2) - a^4\}}/\sqrt{\{(c^2 + a^2)(a^2 + b^2)\}} = \sqrt{(b^2c^2 + a^2b^2 + c^2a^2)}/\sqrt{\{(c^2 + a^2)(a^2 + b^2)\}}$

Area $= (1/2)\sqrt{(c^2 + a^2)}\sqrt{(a^2 + b^2)}\sqrt{(b^2c^2 + c^2a^2 + a^2b^2)}/\sqrt{\{(c^2 + a^2)(a^2 + b^2)\}}$

$= (1/2)\sqrt{(b^2c^2 + c^2a^2 + a^2b^2)}$

Option (a) is correct.

488. Two sides of a triangle are 4 and 5. Then, for the area of the triangle which one of the following bounds is the sharpest?
(a) < 10
(b) ≤ 10
(c) ≤ 8
(d) > 5

Solution :

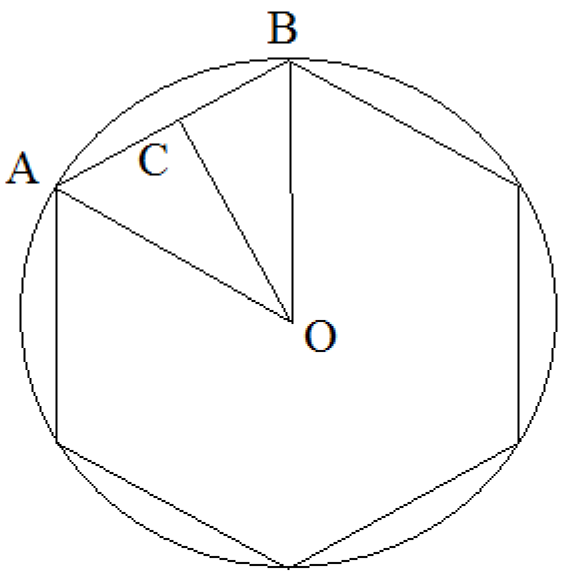
Area = $(1/2)*4*5*\sin A = 10\sin A \leq 10$

Option (b) is correct.

489. The area of a regular hexagon (that is, six-sided polygon) inscribed in a circle of radius 1 is
(a) $3\sqrt{3}/2$

(b) 3
(c) 4
(d) $2\sqrt{3}$

Solution :



Angle OAB = $\{(6 - 2)\pi/6\}/2 = \pi/3$

OC/AO = $\sin(\pi/3)$

- ⇨ OC = $1*(\sqrt{3}/2) = \sqrt{3}/2$
- ⇨ AC = OCcos($\pi/3$) = ½
- ⇨ AB = 1

Area of AOB = $(1/2)*(\sqrt{3}/2)*1 = \sqrt{3}/4$

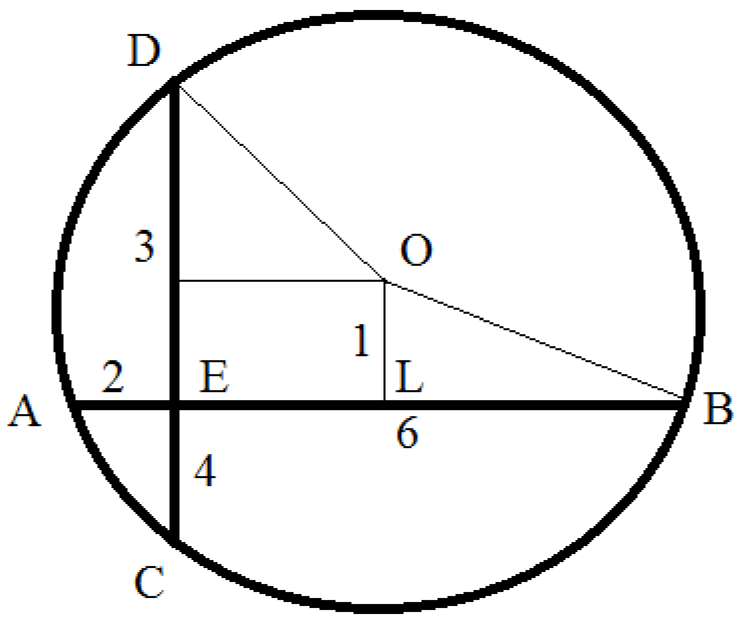
Area of hexagon = $6*(\sqrt{3}/4) = 3\sqrt{3}/2$

Option (a) is correct.

490. Chords AB and CD of a circle intersect at a point E at right angles to each other. If the segments AE, EB and ED are of lengths 2, 6 and 3 units respectively, then the diameter of the circle is
(a) $\sqrt{65}$
(b) 12
(c) $\sqrt{52}$

(d) $\sqrt{63}$

Solution :



Now, if we join B, C and A, D then triangles ADE and BCE are similar. (Angle ECB = Angle EAD; on same arc BD and Angle DEA = Angle CEB (right angles))

So, CE/AE = EB/ED

⇨ CE = 4

Now, LB = (6 + 2)/2 = 4

OL = DC/2 – 3 = (3 + 4)/2 – 3 = ½ (figure is not drawn to the scale)

In triangle OLB, $OB^2 = r^2 = (½)^2 + 4^2 = 65/4$

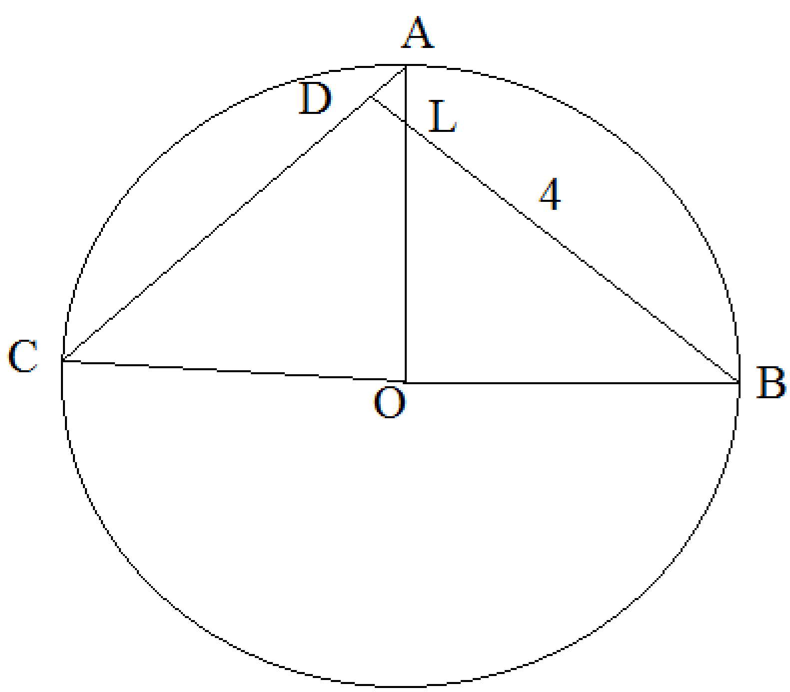
⇨ $r = \sqrt{65}/2$
⇨ $2r = \sqrt{65}$

Option (a) is correct.

491. In a circle with centre O, OA and OB are two radii perpendicular to each other. Let AC be a chord and D the foot of the perpendicular

drawn from B to AC. If the length of BD is 4 cm then the length of CD (in cm) is
(a) 4
(b) $2\sqrt{2}$
(c) $2\sqrt{3}$
(d) $3\sqrt{2}$

Solution :



In triangle ACO, Angle OAC = Angle OCA (OA = OC both radius)

In triangles ADL and OLB, Angle DLA = Angle OLB (opposite angle)

Angle LDA = Angle LOB (both right angles)

⇨ Angle DAL = Angle LBO
⇨ Angle OCA = Angle DBO

Now, in triangles ODC and OBD,

OD is common, OC = OB (both radius) and Angle OCD = Angle OBD

⇨ ODC and OBD are equal triangles.
⇨ CD = DB = 4

Option (a) is correct.

492. ABC is a triangle and P is a point inside it such that Angle BPC = Angle CPA = Angle APB. Then P is
(a) the point of intersection of medians
(b) the incentre
(c) the circumcentre
(d) none of the foregoing points

Solution :

Clearly, none of the foregoing points satisfy this.

Therefore, option (d) is correct.

493. Suppose the circumcentre of a triangle ABC lies on BC. Then the orthocentre of the triangle is
(a) the point A
(b) the incentre of the triangle
(c) the mid-point of the line segment joining the mid-points of AB and AC
(d) the centroid of the triangle

Solution :

It means ABC is right-angled triangle with right angle at A.

And circumcentre is the mid-point of BC.

Obviously, orthicentre of the triangle is point A.

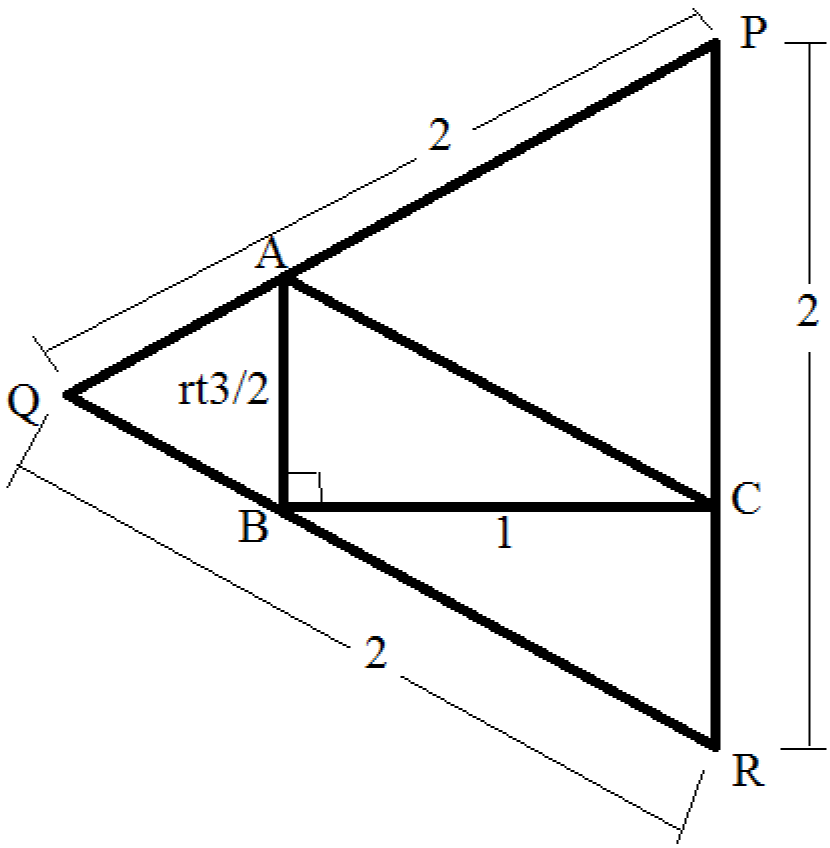
Option (a) is correct.

494. ABC is a triangle inscribed in a circle. AD, AE are straight lines drawn from the vertex A to the base BC parallel to the tangents at B and C respectively. If AB = 5 cm, AC = 6 cm, and CE = 9 cm, then the length of BD (in cm) equals
(a) 7.5
(b) 10.8
(c) 7.0
(d) 6.25

Solution :

Option (d) is correct.

495. ABC is a triangle with AB = √3/2, BC = 1 and B = 90. PQR is an equilateral triangle with sides PQ, QR, RP passing through the points A, B, C respectively and each having length 2. Then the length of the segment BR is



(a) (2/√3)sin75
(b) 4/(2 + √3)
(c) either 1 or 15/13
(d) 2 – sin75

Solution :

Option (c) is correct.

496. The equation $x^2y - 2xy + 2y = 0$ represents
(a) a straight line
(b) a circle
(c) a hyperbola
(d) none of the foregoing curves

Solution

Now, $x^2y - 2xy + 2y = 0$

⇨ $y(x^2 - 2x + 2) = 0$
⇨ $y\{(x - 1)^2 + 1\} = 0$

Now, $(x - 1)^2 + 1 > 0$ (always)

⇨ $y = 0$

Option (a) is correct.

497. The equation $r = 2a\cos\theta + 2b\sin\theta$, in polar coordinates, represents
(a) a circle passing through the origin
(b) a circle with the origin lying outside it
(c) a circle with radius $2\sqrt{(a^2 + b^2)}$
(d) a circle with centre at the origin.

Solution :

$r = 2a\cos\theta + 2b\sin\theta = 2ax/r + 2by/r$

⇨ $r^2 = 2ax + 2by$
⇨ $x^2 + y^2 - 2ax - 2by = 0$

Option (a) is correct.

498. The curve whose equation in polar coordinates is $r\sin^2\theta - \sin\theta - r = 0$, is
(a) an ellipse
(b) a parabola
(c) a hyperbola
(d) none of the foregoing curves

Solution :

$r\sin^2\theta - \sin\theta - r = 0$

- ⇨ $r(y^2/r^2) - y/r - r = 0$
- ⇨ $y^2 - y - r^2 = 0$
- ⇨ $y^2 - y - x^2 - y^2 = 0$
- ⇨ $x^2 = -y$
- ⇨ a parabola

Option (b) is correct.

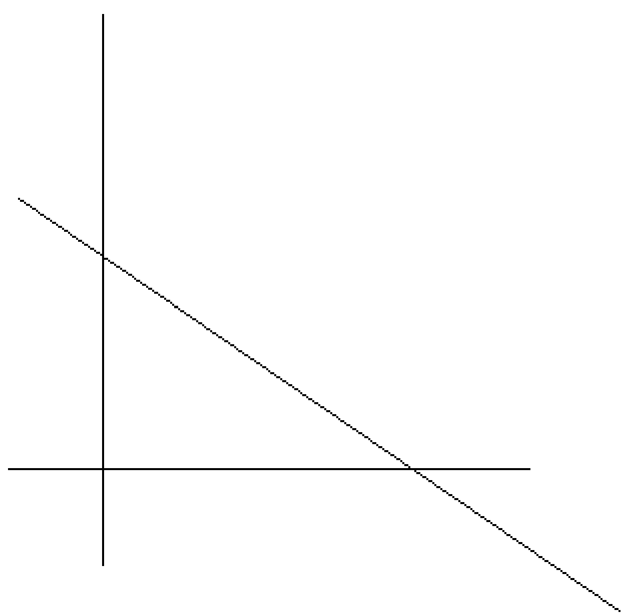
499. A point P on the line $3x + 5y = 15$ is equidistant from the coordinate axes. P can lie in

(a) quadrant I only
(b) quadrant I or quadrant II
(c) quadrant I or quadrant III
(d) any quadrant

Solution :

$3x + 5y = 15$

- ⇨ $x/5 + y/3 = 5$



Now, P cannot lie on quadrant III as the straight line is not there.

So, option (c) and (d) cannot be true.

Now, P can lie on quadrant II as well as the straight line is bent along x-axis in quadrant II. So, somewhere we will find a point on the line which is equidistant from both the axis.

Let co-ordinate of P is (h, -h) i.e. considering in quadrant IV.

Then $3h - 5h = 15$

⇨ $h = -15/2$ but h is positive. So it cannot stay on quadrant IV.

Let coordinate of P is (-h, h) i.e. considering it in quadrant II

Then $-3h + 5h = 15$

⇨ $h = 15/2 > 0$ (so possible)

Option (b) is correct.

500. The set of all points (x, y) in the plane satisfying the equation $5x^2y - xy + y = 0$ forms
(a) a straight line
(b) a parabola
(c) a circle
(d) none of the foregoing curves

Solution :

$5x^2y - xy + y = 0$

⇨ $y(5x^2 - x + 1) = 0$
⇨ $5y(x^2 - x/5 + 1/5) = 0$
⇨ $5y(x^2 - 2*(1/10)*x + 1/100 + 1/5 - 1/100) = 0$
⇨ $5y\{(x - 1/10)^2 + 19/100\} = 0$

Now, $(x - 1/10)^2 + 19/100 > 0$ (always)

Therefore, $y = 0$

Option (a) is correct.

501. The equation of the line passing through the intersection of the lines $2x + 3y + 4 = 0$ and $3x + 4y - 5 = 0$ and perpendicular to the line $7x - 5y + 8 = 0$ is
(a) $5x + 7y - 1 = 0$
(b) $7x + 5y + 1 = 0$
(c) $5x - 7y + 1 = 0$
(d) $7x - 5y - 1 = 0$

Solution :

$2x + 3y + 4 = 0$ …. (1)

$3x + 4y - 5 = 0$ …. (2)

Doing (1)*3 – (2)*2 we get, $6x + 9y + 12 - 6x - 8y + 10 = 0$

⇨ $y = -22$
⇨ $x = (-4 + 3*22)/2 = 31$

Slope of $7x - 5y + 8 = 0$ is $7/5$

Slope of required straight line is $(-5/7)$

Equation of the required straight line is, $y + 22 = (-5/7)(x - 31)$

⇨ $7y + 154 = -5x + 155$
⇨ $5x + 7y - 1 = 0$

Option (a) is correct.

502. Two equal sides of an isosceles triangle are given by the equations $y = 7x$ and $y = -x$ and its third side passes through $(1, -10)$. Then the equation of the third side is
(a) $3x + y + 7 = 0$ or $x - 3y - 31 = 0$
(b) $x + 3y + 29 = 0$ or $-3x + y + 13 = 0$
(c) $3x + y + 7 = 0$ or $x + 3y + 29 = 0$
(d) $X = 3y - 31 = 0$ or $-3x + y + 13 = 0$

Solution :

$m_1 = 7$ and $m_2 = -1$

Now, $(7 - m)/(1 + 7m) = (m + 1)/(1 - m)$

⇨ $(7 - m)(1 - m) = (1 + 7m)(m + 1)$
⇨ $7 - 8m + m^2 = 1 + 8m + 7m^2$
⇨ $6m^2 + 16m - 6 = 0$
⇨ $3m^2 + 8m - 3 = 0$
⇨ $3m^2 + 9m - m - 3 = 0$
⇨ $3m(m + 3) - (m + 3) = 0$
⇨ $(m + 3)(3m - 1) = 0$
⇨ $m = -3, 1/3$

Equation is, $y + 10 = -3(x - 1)$ or $y + 10 = (1/3)(x - 1)$

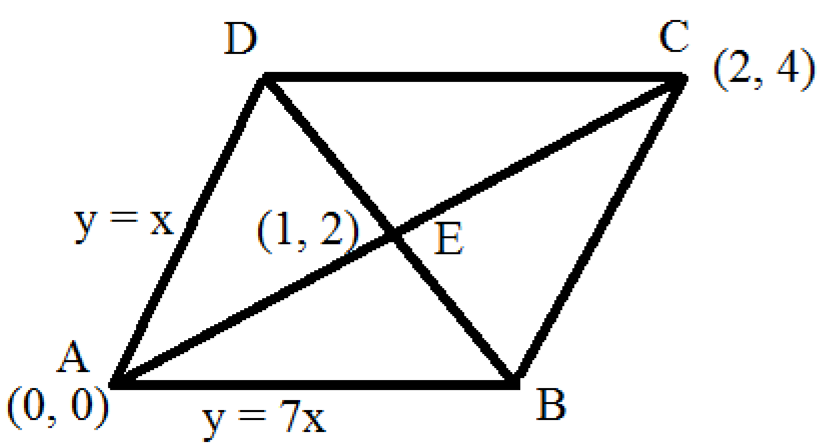
⇨ $y + 10 = -3x + 3$ or $3y + 30 = x - 1$
⇨ $3x + y + 7 = 0$ or $x - 3y - 31 = 0$

Option (a) is correct.

503. The equation of two adjacent sides of a rhombus are given by $y = x$ and $y = 7x$. The diagonals of the rhombus intersect each other at the point (1, 2). The area of the rhombus is

(a) 10/3
(b) 20/3
(c) 50/3
(d) None of the foregoing quantities

Solution :



Now, Equation of BC is, $y - 4 = 1(x - 2)$

⇨ $x - y + 2 = 0$

Solving $x - y + 2 = 0$ and $y = 7x$, we get, $x = 1/3$, $y = 7/3$

Therefore, B = (1/3, 7/3)

BE = $\sqrt{(1 - 1/3)^2 + (2 - 7/3)^2} = \sqrt{5}/3$

AC = $\sqrt{(2 - 0)^2 + (4 - 0)^2} = 2\sqrt{5}$

Area of triangle ABC = $(1/2)(\sqrt{5}/3)(2\sqrt{5}) = 5/3$

Area of rhombus = 2*(5/3) = 10/3

Option (a) is correct.

504. It is given that three distinct points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear. Then a necessary and sufficient condition for (x_2, y_2) to lie on the line segment joining (x_3, y_3) to (x_1, y_1) is

(a) either $x_1 + y_1 < x_2 + y_2 < x_3 + y_3$ or $x_3 + y_3 < x_2 + y_2 < x_1 + y_1$

(b) either $x_1 - y_1 < x_2 - y_2 < x_3 - y_3$ or $x_3 - y_3 < x_2 - y_2 < x_1 - y_1$

(c) either $0 < (x_2 - x_3)/(x_1 - x_3) < 1$ or $0 < (y_2 - y_3)/(y_1 - y_2) < 1$

(d) none of the foregoing statements

Solution :

The ratio $(x_2 - x_3)/(x_1 - x_3)$ says that the distance between x-coordinate between x_2 and x_3 and the distance between the x-coordinate between x_1 and x_3 are of same sign and the modulus of the previous is smaller than the latter i.e. x_2 lie between x_1 and x_3.

Let (x_2, y_2) divides (x_1, y_1) and (x_3, y_3) in the ratio m : n.

Therefore, $x_2 = (mx_1 + nx_3)/(m + n)$

⇨ $mx_2 + nx_2 = mx_1 + nx_3$
⇨ $m(x_2 - x_1) = n(x_3 - x_2)$
⇨ $(x_2 - x_1)/(x_3 - x_2) = n/m$
⇨ $(x_2 - x_1 + x_3 - x_2)/(x_3 - x_2) = (n + m)/m$
⇨ $(x_3 - x_2)/(x_3 - x_1) = m/(n + m)$
⇨ $(x_2 - x_3)/(x_1 - x_3) = m/(n + m)$

Now, $0 < m/(n + m) < 1$

Follows, $0 < (x_2 - x_3)/(x_1 - x_3) < 1$

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Option (c) is correct.

505. Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, $D(x_4, y_4)$ be four points such that x_1, x_2, x_3, x_4 and y_1, y_2, y_3, y_4 are both in A.P. If Δ denotes the area of the quadrilateral ABCD, then
(a) $\Delta = 0$
(b) $\Delta > 1$
(c) $\Delta < 1$
(d) Δ depends on the coordinates of A, B, C and D.

Solution :

Let the common difference of the A.P. x_1, x_2, x_3, y_3 is d and the common difference of the A.P. y_1, y_2, y_3, y_4 is d_1.

Now, $AB = \sqrt{\{(x_2 - x_1)^2 + (y_2 - y_1)^2\}} = \sqrt{(d^2 + d_1^2)} = a$ (say) > 0

$BC = CD = a$

$DA = \sqrt{\{(x_4 - x_1)^2 + (y_4 - y_1)^2\}} = 3a$

$AC = \sqrt{\{(x_3 - x_1)^2 + (y_3 - y_1)^2\}} = 2a$

Now, AB + BC = AC

⇨ A, B, C are collinear.

Now, AC + CD = DA

⇨ A, C, D are collinear.

Therefore, $\Delta = 0$

Option (a) is correct.

506. The number of points (x, y) satisfying (i) $3y - 4x = 20$ and (ii) $x^2 + y^2 \leq 16$ is
(a) 0
(b) 1
(c) 2
(d) Infinite

Solution :

Let us see the intersection point of 3y – 4x = 20 with the circle $x^2 + y^2 = 16$

Now, y = (4x + 20/3

$\{(4x + 20)/3\}^2 + x^2 = 16$

- ⇨ $16x^2 + 160x + 400 + 9x^2 = 144$
- ⇨ $25x^2 + 160x + 256 = 0$
- ⇨ $(5x)^2 + 2*5x*16 + (16)^2 = 0$
- ⇨ $(5x + 16)^2 = 0$
- ⇨ x = -16/5

One solution. Therefore it touches the circle.

So, no intersection in the inside of the circle.

Therefore, 1 solution.

Option (b) is correct.

507. The equation of the line parallel to the line 3x + 4y = 0 and touching the circle $x^2 + y^2 = 9$ in the first quadrant is
(a) 3x + 4y = 9
(b) 3x + 4y = 45
(c) 3x + 4y = 15
(d) None of the foregoing equations

Solution :

Equation of the required line is, 3x + 4y = c

Distance of the line from (0, 0) is $|-c/\sqrt{(3^2 + 4^2)}|$ = radius = 3

- ⇨ c = 15

Option (c) is correct.

508. The distance between the radii of the largest and smallest circles, which have their centres on the circumference of the circle $x^2 + 2x + y^2 + 4y = 4$ and pass through the point (a, b) lying outside the given circle, is

(a) 6
(b) $\sqrt{(a + 1)^2 + (b + 2)^2}$
(c) 3
(d) $\sqrt{(a + 1)^2 + (b + 2)^2} - 3$.

Solution :

Option (a) is correct.

509. The perimeter of the region bounded by $x^2 + y^2 \le 100$ and $x^2 + y^2 - 10x - 10(2 - \sqrt{3})y \le 0$ is
(a) $(5\pi/3)(5 + \sqrt{6} - \sqrt{2})$
(b) $(5\pi/3)(1 + \sqrt{6} - \sqrt{2})$
(c) $(5\pi/3)(1 + 2\sqrt{6} - 2\sqrt{2})$
(d) $(5\pi/3)(5 + 2\sqrt{6} - 2\sqrt{2})$

Solution :

Subtracting the equations we get, $10x + 10(2 - \sqrt{3})y = 100$

⇨ $x = 10 - (2 - \sqrt{3})y$

Putting in first equation we get, $\{10 - (2 - \sqrt{3})y\}^2 + y^2 = 100$

⇨ $100 - 20(2 - \sqrt{3})y + y^2 = 0$
⇨ $y = 0, 20(2 - \sqrt{3})$
⇨ $x = 10, 10 - 20(2 - \sqrt{3})^2 = 80\sqrt{3} - 130$

So, the points are $(10, 0)$ and $(80\sqrt{3} - 130, 20(2 - \sqrt{3}))$

So, $m_1 = (0 - 0)/(10 - 0) = 0$ and $m_2 = \{20(2 - \sqrt{3}) - 0\}/(80 - 130\sqrt{3} - 0) = 2(2 - \sqrt{3})/(8 - 13\sqrt{3})$

So, $\theta = \tan^{-1}\{2(2 - \sqrt{3})/(8 - 13\sqrt{3})\}$

Now, $s = r\theta = 10\tan^{-1}\{2(2 - \sqrt{3})/(8 - 13\sqrt{3})\}$

Centre of second circle = $(5, 5(2 - \sqrt{3}))$

Radius = $\sqrt{[5^2 + \{5(2 - \sqrt{3})\}^2]} = 5\sqrt{(1 + 4 + 3 - 4\sqrt{3})} = 5\sqrt{(8 - 4\sqrt{3})} = 5(\sqrt{6} - \sqrt{2})$

Now, $m_3 = (0 - 5)/(10 - 5(2 - \sqrt{3}))$, $m_4 = \{20(2 - \sqrt{3}) - 5\}/(80\sqrt{3} - 130 - 5(2 - \sqrt{3}))$

$\tan\theta = (m_3 - m_4)/(1 + m_3m_4)$

From this, $s_1 = r\theta = 5(\sqrt{6} - \sqrt{2})\tan^{-1}\{(m_3 - m_4)/(1 + m_3m_4)\}$

Perimeter = $s + s_1$

After simplification, option (c) will be the answer.

510. The equation of the circle which has both coordinate axes as its tangents and which touches the circle $x^2 + y^2 = 6x + 6y - 9 - 4\sqrt{2}$ is
(a) $x^2 + y^2 = 2x + 2y + 1$
(b) $x^2 + y^2 = 2x - 2y + 1$
(c) $x^2 + y^2 = 2x - 2y - 1$
(d) $x^2 + y^2 = 2x + 2y - 1$

Solution :

Centre is (r, r) where r is radius.

Centre of second circle = (3, 3) and radius = $2\sqrt{2} - 1$

Distance between centres = sum of radius

- ⇨ $\sqrt{\{(r - 3)^2 + (r - 3)^2\}} = r + 2\sqrt{2} - 1$
- ⇨ $|r - 3|\sqrt{2} = r + 2\sqrt{2} - 1$
- ⇨ $(r - 3)\sqrt{2} = r + 2\sqrt{2} - 1$
- ⇨ $r(\sqrt{2} - 1) = 5\sqrt{2} - 1$
- ⇨ $r = (5\sqrt{2} - 1)/(\sqrt{2} - 1)$

Also, $-\sqrt{2}(r - 3) = r + 2\sqrt{2} - 1$

- ⇨ $r(\sqrt{2} + 1) = (\sqrt{2} + 1)$
- ⇨ $r = 1$

Equation is, $(x - 1)^2 + (y - 1)^2 = 1^2$

- ⇨ $x^2 + y^2 = 2x + 2y - 1$

Option (d) is correct.

511. A circle and a square have the same perimeter. Then

(a) their areas are equal
(b) the area of the circle is larger
(c) the area of the square is larger
(d) the area of the circle is π times the area of the square

Solution :

Now, $2\pi r = 4a$

- ⇨ $\pi r = 2a$
- ⇨ $(\pi r)^2 = 4a^2$
- ⇨ $(\pi r^2)/a^2 = 4/\pi > 1$
- ⇨ area of circle > area of square

Option (b) is correct.

512. The equation $x^2 + y^2 - 2xy - 1 = 0$ represents
(a) two parallel straight lines
(b) two perpendicular straight lines
(c) a circle
(d) a hyperbola

Solution :

$x^2 + y^2 - 2xy - 1 = 0$

- ⇨ $(x - y)^2 = 1$
- ⇨ $x - y = \pm 1$

Pair of parallel straight lines.

Option (a) is correct.

513. The equation $x^3 - yx^2 + x - y = 0$ represents
(a) a straight line
(b) a parabola and two straight lines
(c) a hyperbola and two straight lines
(d) a straight line and a circle

Solution :

Now, $x^3 - yx^2 + x - y = 0$

⇨ $x^2(x - y) + x - y = 0$
⇨ $(x - y)(x^2 + 1) = 0$
⇨ $x - y = 0$ as $x^2 + 1 > 0$ (always)

Option (a) is correct.

514. The equation $x^3y + xy^3 + xy = 0$ represents
(a) a circle
(b) a circle and a pair of straight lines
(c) a rectangular hyperbola
(d) a pair of straight lines

Solution :

Now, $x^3y + xy^3 + xy = 0$

⇨ $xy(x^2 + y^2 + 1) = 0$
⇨ $xy = 0$ as $x^2 + y^2 + 1 > 0$ (always)
⇨ $x = 0, y = 0$

Option (d) is correct.

515. A circle of radius r touches the parabola $x^2 + 4ay = 0$ $(a > 0)$ at the vertex of the parabola. The centre of the circle lies below the vertex and the circle lies entirely within parabola. Then the largest possible value of r is
(a) a
(b) 2a
(c) 4a
(d) None of the foregoing expressions

Solution :

Any point on the parabola $(2at, -at^2)$

Centre of circle is $(0, -r)$

Distance = $\sqrt{(2at - 0)^2 + (-at^2 + r)^2} \geq r$

- ⇨ $4a^2t^2 + a^2t^4 - 2at^2r + r^2 \geq r^2$
- ⇨ $4a^2t^2 + a^2t^4 - 2at^2r \geq 0$
- ⇨ $4a + at^2 - 2r \geq 0$
- ⇨ $r \leq 2a + at^2/2$

Maximum value will occur when t = 0 i.e. r = 2a

Option (b) is correct.

516. The equation $16x^4 - y^4 = 0$ represents
   (a) a pair of straight lines
   (b) one straight line
   (c) a point
   (d) a hyperbola

Solution :

$16x^4 - y^4 = 0$

- ⇨ $(4x^2 - y^2)(4x^2 + y^2) = 0$
- ⇨ $(2x - y)(2x + y(4x^2 + y^2) = 0$

A pair of straight lines as $4x^2 + y^2 > 0$

Option (a) is correct.

517. The equation of the straight line which passes through the point of intersection of the lines x + 2y + 3 = 0 and 3x + 4y + 7 = 0 and is perpendicular to the straight line y − x = 8 is
   (a) 6x + 6y − 8 = 0
   (b) x + y + 2 = 0
   (c) 4x + 8y + 12 = 0
   (d) 3x + 3y − 6 = 0

Solution :

Solving x + 2y + 3 = 0 and 3x + 4y + 7 = 0 we get, (-1, -1)

Equation of the required straight line is x + y + c = 0

$-1 -1 + c = 0$

$\Rightarrow c = 2$

$x + y + 2 = -$

Option (b) is correct.

518. Two circles with equal radii are intersecting at the points (0, 1) and (0, -1). The tangent at (0, 1) to one of the circle passes through the centre of the other circle. Then the centres of the two circles are at
(a) (2, 0) and (-2, 0)
(b) (0.75, 0) and (-0.75, 0)
(c)(1, 0) and (-1, 0)
(d) None of the foregoing pairs of points.

Solution :

Let the centre of the two circles are at (a, 0) and (-a, 0)

$(C_1C_2)^2 = r^2 + r^2$

$4a^2 = a^2 + 1 + a^2 + 1$

⇨ $2a^2 = 2$
⇨ $a = \pm 1$

Option (c) is correct.

519. The number of distinct solutions (x, y) of the system of equations $x^2 = y^2$ and $(x - a)^2 + y^2 = 1$, where a is any real number, can only be
(a) 0, 1, 2, 3, 4 or 5
(b) 0, 1 or 3
(c)0, 1, 2 or 4
(d) 0, 2, 3 or 4

Solution :

Problem incomplete.

520. The number of distinct points common to the curves $x^2 + 4y^2 = 1$ and $4x^2 + y^2 = 4$ is
(a) 0
(b) 1
(c) 2
(d) 4

Solution :

Now, $4(x^2 + 4y^2) - (4x^2 + y^2) = 4*1 - 4$

⇨ $y = 0$
⇨ $x = \pm 1$

Two points (1, 0), (-1, 0)

Option (c) is correct.

521. The centres of the three circles $x^2 + y^2 - 10x + 9 = 0$, $x^2 + y^2 - 6x + 2y + 1 = 0$ and $x^2 + y^2 - 9x - 4y + 2 = 0$
(a) lie on the straight line $x - 2y = 5$
(b) lie on the straight line $y - 2x = 5$
(c) lie on the straight line $2y - x - 5 = 0$
(d) do not lie on a straight line

Solution :

Centres are (5, 0); (3, -1); (9/2, 2)

Area = $(1/2)[5(-1 - 2) + 3(2 - 0) + (9/2)(0 + 1)] = (1/2)[-15 + 6 + 9/2] \neq 0$

Option (d) is correct.

522. In a parallelogram ABCD, A is the point (1, 3), B is the point (5, 6), C is the point (4, 2). Then D is the point
(a) (0, -1)
(b) (-1, 0)
(c) (-1, 1)

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(d) (1, -1)

Solution :

Clearly, (A + C)/2 = (B + D)/2

⇨ {(1, 3) + (4, 2)}/2 = {(5, 6) + D}/2
⇨ D = (0, -1)

Option (a) is correct.

523. A square, whose side is 2 metres, has its corners cut away so as to form a regular octagon. Then area of the octagon, in square metres, equals
(a) 2
(b) 8/(√2 + 1)
(c)4(3 - 2√2)
(d) None of the foregoing numbers.

Solution :

Let the length of the sides which is cut out is x.

The length of the side of the octagon = (2 – 2x)

The hypotenuse of the cut triangle = x√2

Now, 2 – 2x = x√2

⇨ x = 2/(2 + √2)

Therefore, x√2 = 2/(√2 + 1)

Area = 2[(1/2){2/(√2 + 1)}*2*{2/(2 + √2)} + 2*2/(√2 + 1) = 8/(√2 + 1)

Option (b) is correct.

524. The equation of the line passing through the intersection of the lines 3x + 4y = -5, 4x + 6y = 6 and perpendicular to 7x – 5y + 3 = 0 is
(a) 5x + 7y – 2 = 0
(b) 5x – 7y + 2 = 0

(c) $7x - 5y + 2 = 0$
(d) $5x + 7y + 2 = 0$

Solution :

$4(3x + 4y) - 3(4x + 6y) = 4(-5) - 3*6$

- ⇨ $-2y = -38$
- ⇨ $y = 19$
- ⇨ $x = -27$

Equation of the line perpendicular to $7x - 5y + 4 = 0$ is $5x + 7y + c = 0$

Therefore, $5(-27) + 7*19 + c = 0$

- ⇨ $c = -133 + 135 = 2$

Equation is, $5x + 7y + 2 = 0$

Option (d) is correct.

525. The area of the triangle formed by the straight lines whose equations are $y = 4x + 2$, $2y = x + 3$, $x = 0$, is
(a) $25/7\sqrt{2}$
(b) $\sqrt{2}/28$
(c) $1/28$
(d) $15/7$

Solution :

Solving $y = 4x + 2$ and $2y = x + 3$, $8x + 4 = x + 3$ i.e. $x = -1/7$, $y = 10/7$

Solving $y = 4x + 2$ and $x = 0$, $x = 0$, $y = 2$

Solving $2y = x + 3$ and $x = 0$, $x = 0$, $y = 3/2$

So, vertices are $(-1/7, 10/7)$, $(0, 2)$, $(0, 3/2)$

Area = $|(1/2)[(-1/7)(2 - 3/2) + 0(3/2 - 10/7) + 0(10/7 - 2)]| = 1/28$

Option (c) is correct.

526. A circle is inscribed in an equilateral triangle and a square in inscribed in the circle. The ratio of the area of the triangle to the area of the square is
(a) √3 : √2
(b) 3√3 : √2
(c) 3 : √2
(d) √3 : 1

Solution :

Let the side of the triangle = a.

Area of the triangle = $(\sqrt{3}/4)a^2$

Radius of the circle = $(1/3)(\sqrt{3}/2)a = (1/2\sqrt{3})a$

Diagonal of the square = $2*(1/2\sqrt{3})*a = a/\sqrt{3}$

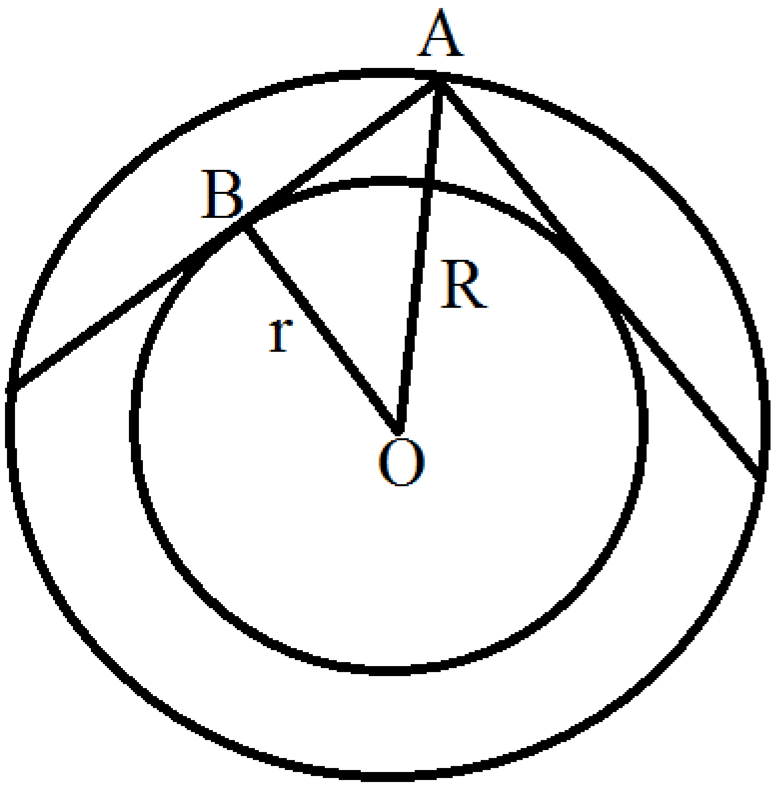
Area of the square = $(1/2)*(a/\sqrt{3})^2 = a^2/6$

Ratio = $(\sqrt{3}/4)a^2 : a^2/6 = 3\sqrt{3} : 2$

Option (b) is correct.

527. If the area of the circumcircle of a regular polygon with n sides is A then the area of the circle inscribed in the polygon is
(a) $A\cos^2(2\pi/n)$
(b) $(A/2)(\cos(2\pi/n) + 1)$
(c) $(A/2)\cos^2(\pi/n)$
(d) $A(\cos(2\pi/n) + 1)$

Solution :



Now, $\pi R^2 = A$

Now, Angle OAB = $\{(n - 2)\pi/n\}/2 = (\pi/2 - \pi/n)$

In triangle, OAB, $OB/OA = \sin(\pi/2 - \pi/n)$

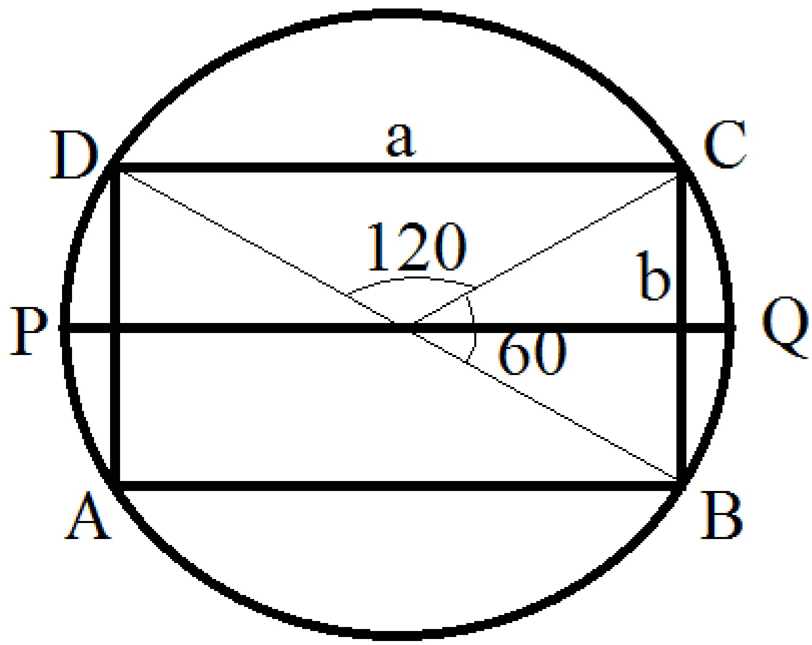
- ⇨ $r/R = \cos(\pi/n)$
- ⇨ $r = R\cos(\pi/n)$
- ⇨ $\pi r^2 = (\pi R^2)\cos^2(\pi/n) = A\cos^2(\pi/n) = (A/2)(\cos(2\pi/n) + 1)$

Option (b) is correct.

528. A rectangle ABCD is inscribed in a circle. Let PQ be the diameter of the circle parallel to the side of AB. If Angle BPC = 30, then the ratio of the area of the rectangle to that of the circle is

(a) $\sqrt{3}/\pi$
(b) $\sqrt{3}/2\pi$
(c) $3/\pi$
(d) $\sqrt{3}/9\pi$

Solution :



Angle BOC = 2*(Angle BPC) = 60 (central angle = 2*peripheral angle)

Let the radius of the circle is r.

From triangle BOC we get, r = b

And from triangle DOC we get, a/sin120 = r/sin30

- ⇨ a = r(√3/2)/(1/2) = r√3
- ⇨ Area of rectangle = r*r√3 = r^2√3

Area of circle = пr^2

Ratio = r^2√3/(пr^2) = √3/п

Option (a) is correct.

529. Consider a circle passing through the points (0, 1 – a), (a, 1) and (0, 1 + a). If a parallelogram with two adjacent sides having lengths a and b and an angle 150 between them has the same area as the circle, then b equals
(a) пa
(b) 2пa

(c)(1/2)πa

(d) None of these numbers

Solution :

Area of parallelogram = (1/2)(a + a)*bsin30 = ab/2

Let, the equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

$0^2 + (1 - a)^2 + 2g*0 + 2f(1 - a) + c = 0$

⇨ $(1 - a)^2 + 2f(1 - a) + c = 0$

Again, $0^2 + (1 + a)^2 + 2g*0 + 2f(1 + a) + c = 0$

⇨ $(1 + a)^2 + 2f(1 + a) + c = 0$

Subtracting we get, $(1 + a)^2 - (1 - a)^2 + 2f(1 + a - 1 + a) = 0$

⇨ $(1 + a + 1 - a)(1 + a - 1 + a) + 2f*2a = 0$
⇨ $2a*2 + 2f*2a = 0$
⇨ $f = -1$
⇨ $c = -\{(1 + a)^2 - 2(1 + a)\} = -\{(1 + a)(1 + a - 2)\} = -(a + 1)(a - 1) = 1 - a^2$

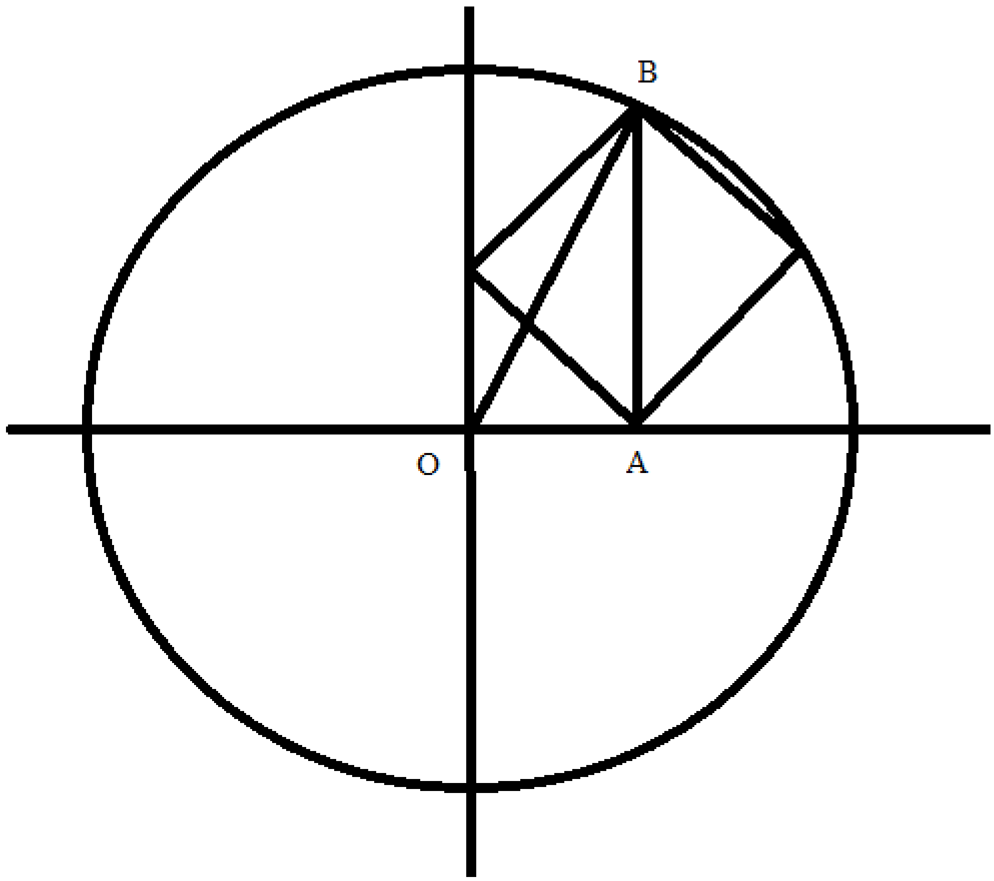
Now, $a^2 + 1^2 + 2g*a + 2f*1 + c = 0$

⇨ $a^2 + 1 + 2ga - 2 + 1 - a^2 = 0$
⇨ $2ga = 0$
⇨ $g = 0$
⇨ $r^2 = g^2 + f^2 - c = (-1)^2 - (1 - a^2) = a^2$
⇨ $\pi r^2 = \pi a^2 = ab/2$
⇨ $b = 2\pi a$

Option (b) is correct.

530. A square is inscribed in a *quarter-circle* in such a manner that two of its adjacent vertices lie on the two radii at an equal distance from the centre, while the other two vertices lie on the circular arc. If the square has sides of length x, then the radius of the circle is
(a) 16x/(π + 4)
(b) 2x/√π
(c)√5x/√2
(d) √2x

Solution :



Now, OA = $x/\sqrt{2}$ and AB = $x\sqrt{2}$

From triangle OAB we get, $r^2 = OA^2 + AB^2 = 5x^2/2$

⇨ $r = \sqrt{5}x/\sqrt{2}$

Option (c) is correct.

531. Let Q = (x_1, y_1) be an exterior point and P is a point on the circle centred at the origin and with radius r. Let θ be the angle which the line joining P to the centre makes with the positive direction of the x-axis. If the line PQ is tangent to the circle, then $x_1\cos\theta + y_1\sin\theta$ equal to

(a) r
(b) r^2
(c) $1/r$
(d) $1/r^2$

Solution :

Equation of OP is, $y = x\tan\theta$

⇨ $y\cos\theta - x\sin\theta = 0$

Therefore, equation of the tangent at P is, $x\cos\theta + y\sin\theta + c = 0$

It passes through (x_1, y_1)

Therefore, $x_1\cos\theta + y_1\sin\theta + c = 0$

⇨ $c = -(x_1\cos\theta + y_1\sin\theta)$

Equation is, $x\cos\theta + y\sin\theta - (x_1\cos\theta + y_1\sin\theta) = 0$

Distance from origin = radius

⇨ $|-(x_1\cos\theta + y_1\sin\theta)/\sqrt{(\cos^2\theta + \sin^2\theta)}| = r$
⇨ $x_1\cos\theta + y_1\sin\theta = r$

Option (a) is correct.

532. A straight line is drawn through the point (1, 2) making an angle θ $0 \leq \theta \leq \pi/3$, with the positive direction of the x-axis to intersect the line $x + y = 4$ at a point P so that the distance of P from the point (1, 2) is $\sqrt{6}/3$. Then the value of θ is
(a) $\pi/18$
(b) $\pi/12$
(c) $\pi/10$
(d) $\pi/3$

Solution :

Equation of the line is, $y - 2 = \tan\theta(x - 1)$

Solving this equation with $x + y = 4$ we get, $x = (2 + \tan\theta)/(1 + \tan\theta)$, $y = (2 + 3\tan\theta)/(1 + \tan\theta)$

Distance of P from (1, 2) is $\sqrt{[\{(2 + \tan\theta)/(1 + \tan\theta) - 1\}^2 + \{(2 + 3\tan\theta)/(1 + \tan\theta) - 2\}^2]}$

$= \sqrt{[\{1/(1 + \tan\theta)\}^2 + \{\tan\theta/(1 + \tan\theta)\}^2]} = \sec\theta/(1 + \tan\theta) = 1/(\sin\theta + \cos\theta) = \sqrt{6}/3$

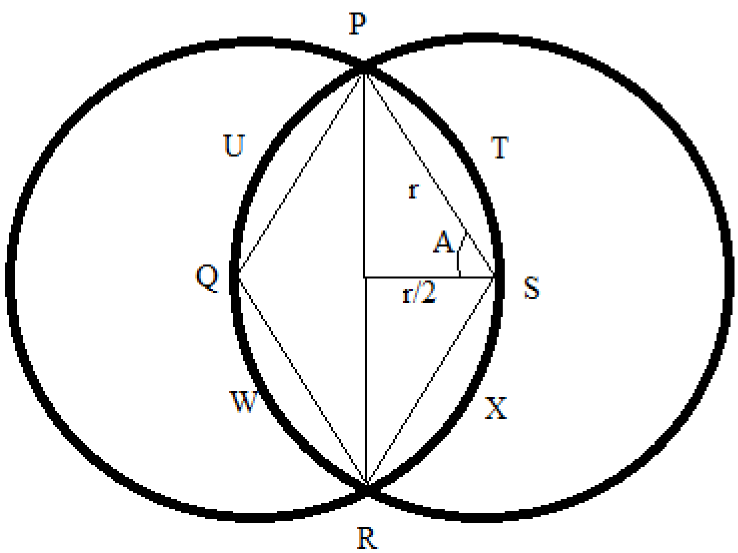
⇨ $\sin\theta + \cos\theta = 3/\sqrt{6}$
⇨ $\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 9/6$
⇨ $1 + \sin 2\theta = 3/2$
⇨ $\sin 2\theta = ½$
⇨ $2\theta = \pi/6$
⇨ $\theta = \pi/12$

Option (b) is correct.

533. The area of intersection of two circular discs each of radius r and with the boundary of each disc passing through the centre of the other is

(a) $\pi r^2/3$
(b) $\pi r^2/6$
(c)$(\pi r^2/4)(2\pi - \sqrt{3}/2)$
(d) $(r^2/6)(4\pi - 3\sqrt{3})$

Solution :



From the figure we get, $(r/2)/r = \cos A$

⇨ $A = \pi/3$
⇨ $2A = 2\pi/3$
⇨ Area of SRWQUPS = $(\pi r^2)(2\pi/3)/2\pi = (\pi/3)r^2$

Similarly, area of QRXSTPQ = $(\pi/3)r^2$

PR = $2*\sqrt{(r^2 - r^2/4)} = r\sqrt{3}$

Now, area of PQRS = $(1/2)r*(r\sqrt{3}) = \sqrt{3}r^2/2$

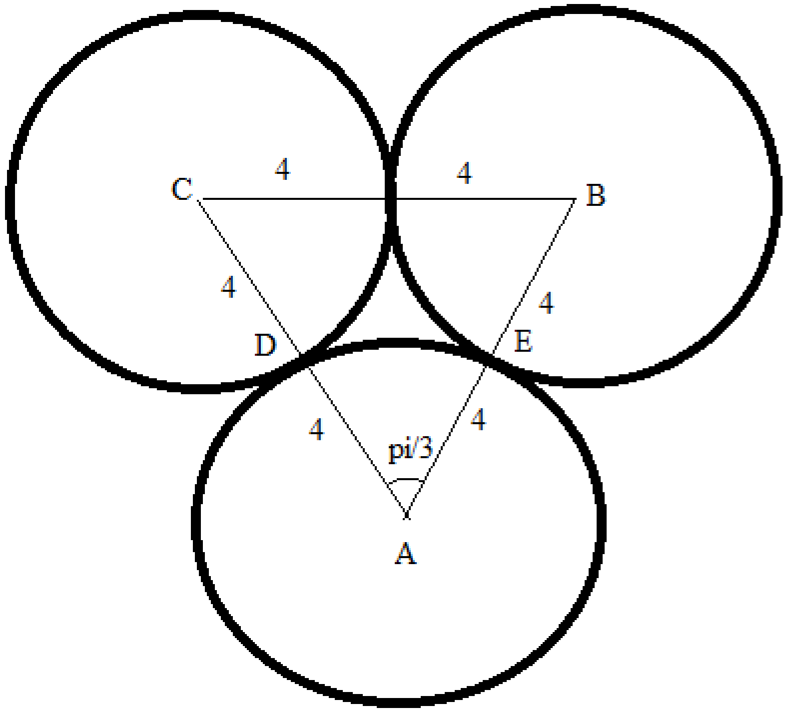
Area of SRWQUPS + area of QRXSTPQ = $(2\pi/3)r^2$

- ⇨ Area of RWQR + area of PUQP + area of PQRS + area of SXRS + area of STPS + area of PQRS = $(2\pi/3)r^2$
- ⇨ Area of RQQR + area of PUQP + area of SXRS + area of STPS + area of PQRS + $\sqrt{3}r^2/2 = (2\pi/3)r^2$
- ⇨ Required area = $(2\pi/3 - \sqrt{3}/2)r^2 = (r^2/6)(4\pi - 3\sqrt{3})$

Option (d) is correct.

534. Three cylinders each of height 16 cm and radius 4 cm are placed on a plane so that each cylinder touches the other two. Then the volume of the region between the three cylinders is, in cm^3,
(a) $98(4\sqrt{3} - \pi)$
(b) $98(2\sqrt{3} - \pi)$
(c) $98(\sqrt{3} - \pi)$
(d) $128(2\sqrt{3} - \pi)$

Solution :



Area of ADE = $(\pi 4^2)(\pi/3)/2\pi = 8\pi/3$

Therefore, area of same three portions = $3*(8\pi/3) = 8\pi$

Area of equilateral triangle ABC = $(\sqrt{3}/4)8^2 = 16\sqrt{3}$

Base area = $16\sqrt{3} - 8\pi = 8(2\sqrt{3} - \pi)$

Volume = $16*8(2\sqrt{3} - \pi) = 128(2\sqrt{3} - \pi)$

Option (d) is correct.

535. From a solid right circular cone made of iron with base of radius 2 cm and height 5 cm, a hemisphere of diameter 2 cm and centre coinciding with the centre of the base of the cone is scooped out. The resultant object is then dropped in a right circular cylinder whose inner diameter is 6 cm and inner height is 10 cm. Water is then poured into the cylinder to fill it up to brim. The volume of water required is

(a) 80π cm^3
(b) $250\pi/3$ cm^3
(c) $270\pi/4$ cm^3
(d) 84π cm^3

Solution :

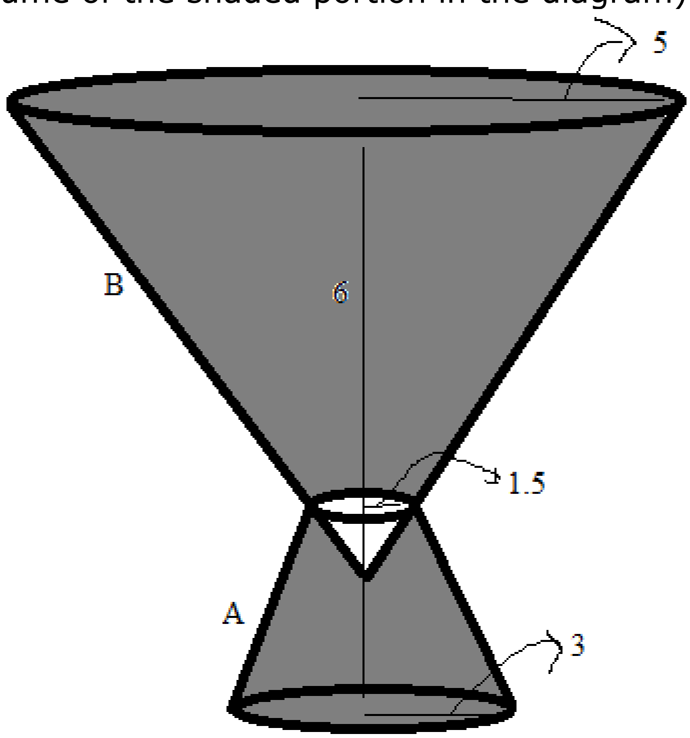
Volume of resultant object after cutting the hemisphere = (1/3)π*2^2*5 – (1/2)(4/3)π*1^3 = 20π/3 – 2π/3 = 6π.

Volume of cylinder = π*3^2*10 = 90π

Volume of water required = 90π - 6π = 84π

Option (d) is correct.

536. A right-circular cone A with base radius 3 units and height 5 units is truncated in such a way that the radius of the circle at the top is 1.5 units and the top parallel to the base. A second right-circular cone B with base radius 5 units and height 6 units is placed vertically inside the cone A as shown in the diagram. The total volume of the portion of the cone B that is outside cone A and the portion of the cone A excluding the portion of cone B that is inside A (that is, the total volume of the shaded portion in the diagram) is



(a) 1867π/40
(b) 1913π/40
(c) 2417π/40
(d) 2153π/40

Solution :

Let the height of the portion of cone B that is inside A is h

Therefore, h/6 = 1.5/5

⇨ h = 9/5

Volume of the portion of cone B that is inside A is $(1/3)\pi(1.5)^2*(9/5) = 27\pi/20$

Volume of the portion of cone B that is outside cone A = $(1/3)\pi*5^2*6 - 27\pi/20 = 50\pi - 27\pi/20 = 973\pi/20$

Let the height of the portion of cone A that is excluded is h_1.

Therefore, $h_1/5 = 1.5/3$

⇨ $h_1 = 5/2$

Volume of the portion of cone A that is excluded = $(1/3)\pi*(1.5)^2*(5/2) = 15\pi/8$

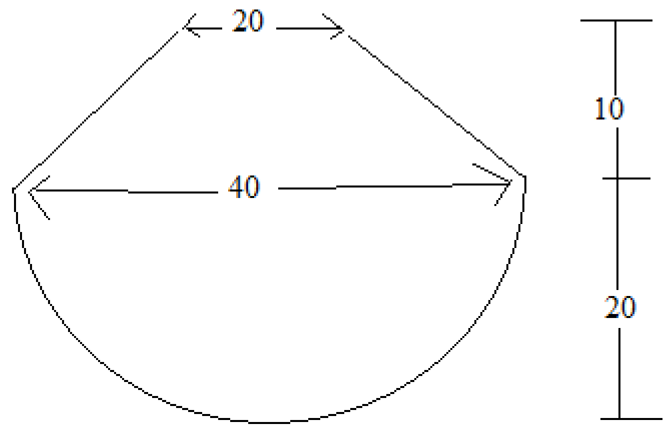
Volume of the portion of cone A that is there (including volume of portion B that is inside A) = $(1/3)\pi(3)^2*5 - 15\pi/8 = 15\pi - 15\pi/8 = 105\pi/8$

Volume of cone A excluding portion of cone B that is inside A = $105\pi/8 - 27\pi/20 = 471\pi/40$

Required volume = $973\pi/20 + 471\pi/40 = (1946\pi + 471\pi)/40 = 2417\pi/40$

Option (c) is correct.

537. A cooking pot has a spherical bottom, while the upper part is a truncated cone. Its vertical cross-section is shown in the figure. If the volume of food increases by 15% during cooking, the maximum initial volume of food that can be cooked without spilling is, in, cc,



(a) $14450(\pi/3)$
(b) $19550(\pi/3)$
(c) $(340000/23)(\pi/3)$
(d) $20000(\pi/3)$

Solution :

Let the height of the portion of the cone that is truncated is h

Therefore, $h/(h + 10) = 20/40$

- ⇨ $2h = h + 10$
- ⇨ $h = 10$

Volume of the truncated portion = $(1/3)\pi*10^2*10 = 1000(\pi/3)$

Volume of the cone = $(1/3)\pi*20^2*20 = 8000(\pi/3)$

Therefore, volume of the cone portion of the pot = $8000(\pi/3) - 1000(\pi/3) = 7000(\pi/3)$

Now, volume of the hemispherical portion = $(1/2)(4/3)\pi*20^3 = 16000(\pi/3)$

Total volume of the pot = $16000(\pi/3) + 7000(\pi/3) = 23000(\pi/3)$

Let the volume of the initial food = x

Volume during cooking = $115x/100$

Now, $115x/100 = 23000(\pi/3)$

- ⇨ $x = 20000(\pi/3)$

Option (d) is correct.

538. A sealed cylinder drum of radius r is 90% filled with paint. If the drum is tilted to rest on its side, the fraction of its *curved* surface area (not counting the flat sides) that will be under the paint is

(a) less than 1/12

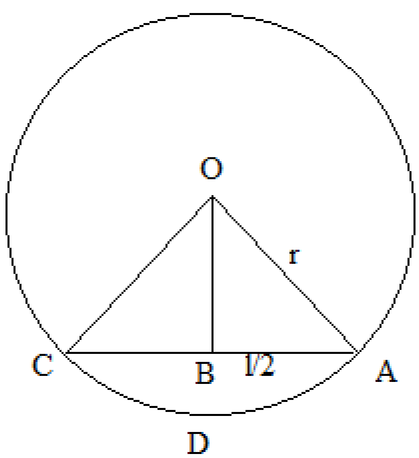
(b) between 1/12 and 1/6

(c) between 1/6 and ¼

(d) greater than ¼

Solution :

Let the length of the upper surface of paint is l.



Angle AOB = $\sin^{-1}(l/2r)$

Angle COA = $2\sin^{-1}(l/2r)$

Now, Area of OADCA = $(\pi r^2)\{2\sin^{-1}(l/2r)\}/2\pi = r^2\sin^{-1}(l/2r)$

Area of triangle OAB = $(1/2)*(l/2)*\sqrt{(r^2 - l^2/4)}$

Area of triangle OAC = $l\sqrt{(r^2 - l^2/4)}/2$

Area of ACDA = $r^2\sin^{-1}(l/2r) - (l/2)\sqrt{(r^2 - l^2/4)}$

Volume of paint = $\{ r^2\sin^{-1}(l/2r) - (l/2)\sqrt{(r^2 - l^2/4)}\}*h$ (where h is height of the cylindrical drum)

Now, $\{r^2\sin^{-1}(l/2r) - (l/2)\sqrt{(r^2 - l^2/4)}\}*h = (9/100)*(\pi r^2 h)$ ……. (1)

Now, s = r*$\sin^{-1}$(l/2r)

Curved surface area that is under paint = r*$\sin^{-1}$(l/2r)*h

Total curved surface area = 2πrh

We have to find the ratio = r*$\sin^{-1}$(l/2r)*h/2πrh = (1/2π)$\sin^{-1}$(l/2r)

Manipulating equation (1) we have to find a range of the ratio.

Option (b) is correct.

539. The number of tangents that can be drawn from the point (2, 3) to the parabola y^2 = 8x is
(a) 1
(b) 2
(c) 0
(d) 3

Solution :

Now, $y^2 - 8x = 0$

$3^2 - 8*2 < 0$

⇨ The point is within parabola.

Option (c) is correct.

540. A ray of light passing through the point (1, 2) is reflected on the x-axis at a point P, and then passes through the point (5, 3). Then the abscissa of the point P is
(a) 2 + 1/5
(b) 2 + 2/5
(c) 2 + 3/5
(d) 2 + 4/5

Solution :

Let co-ordinate of point P is (x, 0)

Now, slope of the incident ray is, m_1 = (2 – 0)/(1 – x) = 2/(1 – x)

So, $\theta_1 = \tan^{-1}\{2/(1 - x)\}$

And, $\theta_2 = \tan^{-1}\{(3 - 0)/(5 - x)\} = \tan^{-1}\{3/(5 - x)\}$

Now, $\tan^{-1}\{2/(1 - x)\} - \pi/2 = \pi/2 - \tan^{-1}\{3/(5 - x)\}$

- ⇨ $\tan^{-1}\{2/(1 - x)\} + \tan^{-1}\{3/(5 - x)\} = \pi$
- ⇨ $\{2/(1 - x) + 3/(5 - x)\}/[1 - \{2/(1 - x)\}\{3/(5 - x)\} = 0$
- ⇨ $2(5 - x) + 3(1 - x) = 0$
- ⇨ $10 - 2x + 3 - 3x = 0$
- ⇨ $x = 13/5 = 2 + 3/5$

Option (c) is correct.

541. If P, Q and R are three points with coordinates (1, 4), (4, 2) and (m, 2m – 1) respectively, then the value of m for which PR + RQ is minimum is
(a) 17/8
(b) 5/2
(c) 7/2
(d) 3/2

Solution :

$PR = \sqrt{\{(m - 1)^2 + (2m - 1 - 4)^2\}} = \sqrt{(5m^2 - 22m + 26)}$

$RQ = \sqrt{\{(m - 4)^2 + (2m - 1 - 2)^2\}} = \sqrt{(5m^2 - 20m + 25)}$

Let $S = PR + RQ = \sqrt{(5m^2 - 22m + 26)} + \sqrt{(5m^2 - 20m + 25)}$

- ⇨ $dS/dm = (10m - 22)/2\sqrt{(5m^2 - 22m + 26)} + (10m - 20)/2\sqrt{(5m^2 - 20m + 25)} = 0$
- ⇨ $(5m - 11)\sqrt{(5m^2 - 20m + 25)} = -(5m - 10)\sqrt{(5m^2 - 22m + 26)}$
- ⇨ $(5m - 11)^2(5m^2 - 20m + 25) = (5m - 10)^2(5m^2 - 22m + 26)$
- ⇨ $(25m^2 - 110m + 121)/(25m^2 - 100m + 100) = (5m^2 - 22m + 26)/(5m^2 - 20m + 25)$
- ⇨ $(-10m + 21)/(25m^2 - 100m + 100) = (-2m + 1)/(5m^2 - 20m + 25)$
- ⇨ $(-10m + 21)/(-2m + 1) = (25m^2 - 100m + 100)/(5m^2 - 20m + 25)$
- ⇨ $16/(-2m + 1) = -25/(5m^2 - 20m + 25)$
- ⇨ $(5m^2 - 20m + 25)/(2m - 1) = 25/16$
- ⇨ $(m^2 - 4m + 5)/(2m - 1) = 5/16$
- ⇨ $16m^2 - 64m + 80 = 10m - 5$
- ⇨ $16m^2 - 74m + 85 = 0$
- ⇨ $m = \{74 \pm \sqrt{(74^2 - 4*16*85)}\}/2*16 = (74 \pm 6)/32 = 5/2, 17/8$

Now, we have to find d^2S/dm^2 and check that for m = 17/8 it is > 0.

Option (a) is correct.

542. Let A be the point (1, 2) and L be the line x + y = 4. Let M be the line passing through A such that the distance between A and the point of intersection of L and M is √(2/3). Then the angle which M makes with L is
(a) 45
(b) 60
(c) 75
(d) 30

Solution :

M is, y – 2 = m(x – 1)

⇨ y = 2 + m(x – 1)

Now, x + y = 4

⇨ x + 2 + m(x – 1) = 4
⇨ x(m + 1) = 2 + m
⇨ x = (m + 2)/(m + 1)
⇨ y = 2 + m{(m + 2)/(m + 1) – 1} = 2 + m/(m + 1) = (3m + 2)/(m + 1)

Now, $\sqrt{[\{(m + 2)/(m + 1) - 1\}^2 + \{(3m + 2)/(m + 1) - 2\}^2]} = \sqrt{(2/3)}$

⇨ $1/(m + 1)^2 + m^2/(m + 1)^2 = 2/3$
⇨ $3(m^2 + 1) = 2(m^2 + 2m + 1)$
⇨ $m^2 - 4m + 1 = 0$
⇨ m = {4 ± √(16 – 4)}/2 = 2 ± √3

Angle which M makes with L = $\tan^{-1}|\{(2 + \sqrt{3} + 1)/1 - (2 + \sqrt{3})\}| = \tan^{-1}|\{(3 + \sqrt{3})/(-\sqrt{3} - 1)\}| = \tan^{-1}|(-\sqrt{3})| = -\tan^{-1}(\sqrt{3}) = 60$

Option (b) is correct.

543. The equation $x^2 + y^2 - 2x - 4y + 5 = 0$ represents
(a) a circle
(b) a pair of straight lines
(c) an ellipse

(d) a point

Solution :

Now, $x^2 + y^2 - 2x - 4y + 5 = 0$

⇨ $(x - 1)^2 + (y - 2)^2 = 0$
⇨ $x = 1, y = 2$
⇨ a point

Option (d) is correct.

544. The line $x = y$ is tangent at (0, 0) to a circle of radius 1. The centre of the circle is
(a) (1, 0)
(b) either $(1/\sqrt{2}, 1/\sqrt{2})$ or $(-1/\sqrt{2}, -1/\sqrt{2})$
(c) either $(1/\sqrt{2}, -1/\sqrt{2})$ or $(-1/\sqrt{2}, 1/\sqrt{2})$
(d) none of the foregoing points

Solution :

Let centre = (h, k)

Therefore, $h^2 + k^2 = 1$

$|(h - k)/\sqrt{2}| = 1$

⇨ $(h - k)^2 = 2$
⇨ $h^2 + k^2 - 2hk = 2$
⇨ $2hk = -1$
⇨ $hk = -1/2$
⇨ $k = -1/2h$

Putting in first equation, $h^2 + 1/4h^2 = 1$

⇨ $4h^4 - 4h^2 + 1 = 0$
⇨ $(2h^2 - 1) = 0$
⇨ $h = \pm 1/\sqrt{2}$

Therefore centre is either $(1/\sqrt{2}, -1/\sqrt{2})$ or $(-1/\sqrt{2}, 1/\sqrt{2})$

Option (c) is correct.

545. Let C be the circle $x^2 + y^2 + 4x + 6y + 9 = 0$. The point (-1, -2) is
(a) inside C but not the centre of C
(b) outside C
(c) on C
(d) the centre of C

Solution :

$(x + 2)^2 + (y + 3)^2 = 2^2$

C is not centre.

Now, $(-1)^2 + (-2)^2 + 4(-1) + 6(-2) + 9 = 1 + 4 - 4 - 12 + 9 < 0$

Inside circle but not centre.

Option (a) is correct.

546. The equation of the circle circumscribing the triangle formed by the points (0, 0), (1, 0), (0, 1) is
(a) $x^2 + y^2 + x + y = 0$
(b) $x^2 + y^2 + x - y + 2 = 0$
(c) $x^2 + y^2 + x - y - 2 = 0$
(d) $x^2 + y^2 - x - y = 0$

Solution :

Triangle is right-angled.

Therefore, centre = (1/2, ½)

Radius = $\sqrt{(1 - ½)^2 + (0 - ½)^2} = 1/\sqrt{2}$

Equation is, $(x - ½)^2 + (y - ½)^2 = ½$

⇨ $x^2 + y^2 - x - y = 0$

Option (d) is correct.

547. The equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy = 0$ at the origin is
(a) $fx + gy = 0$
(b) $gx + fy = 0$
(c) $x = 0$
(d) $y = 0$

Solution :

Centre = (-g, -f)

Slope of the normal at (0, 0) = f/g

Hence slope of the tangent at (0, 0) = -g/f

Equation is $y = (-g/f)x$ i.e. $gx + fy = 0$

Option (b) is correct.

548. The equation of the circle circumscribing the triangle formed by the points (3, 4), (1, 4) and (3, 2) is
(a) $x^2 - 4x + y^2 - 6y + 11 = 0$
(b) $x^2 + y^2 - 4x - 4y + 3 = 0$
(c) $8x^2 + 8y^2 - 16x - 13y = 0$
(d) None of the foregoing equations.

Solution :

Let the equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

Now, $3^2 + 4^2 + 6g + 8f + c = 0$ i.e. $6g + 8f + c + 25 = 0$ …. (1)

$1 + 16 + 2g + 8f + c = 0$ i.e. $2g + 8f + c + 17 = 0$ …… (2)

Doing (1) – (2) we get, $4g + 8 = 0$, i.e. $g = -2$

$9 + 4 + 6g + 4f + c = 0$, i.e. $6g + 4f + c + 13 = 0$ …. (3)

Doing (1) – (3) we get, $4f + 12 = 0$, i.e. $f = -3$

Putting these values in (3) we get, $-12 - 12 + c + 13 = 0$, i.e. $c = 11$

Equation is, $x^2 + y^2 -4x - 6y + 11 = 0$

Option (a) is correct.

549. The equation of the diameter of the circle $x^2 + y^2 + 2x - 4y + 4 = 0$ that is parallel to $3x + 5y = 4$ is
(a) $3x + 5y = 7$
(b) $3x - 5y = 7$
(c) $3x + 5y = -7$
(d) $3x - 5y = -7$

Solution :

Equation of line which is parallel to $3x + 5y = 4$ is $3x + 5y = c$

Now, centre = (-1, 2)

The diameter passes through centre. Thus, $-3 + 10 = c$, i.e. $c = 7$

Option (a) is correct.

550. Let C_1 and C_2 be the circles given by the equations $x^2 + y^2 - 4x - 5 = 0$ and $x^2 + y^2 + 8y + 7 = 0$. Then the circle having the common chord of C_1 and C_2 as its diameter has
(a) centre at (-1, 1) and radius 2
(b) centre at (1. -2) and radius $2\sqrt{3}$
(c) centre at (1, -2) and radius 2
(d) centre at (3, -3) and radius 2

Solution :

Centre of first circle = (2, 0), centre of second circle = (0, -4)

Mid-point is centre of the circle.

So, centre = (1, -2)

Common chord, $8y + 7 + 4x + 5 = 0$

- ⇨ $4x + 8y + 12 = 0$
- ⇨ $x + 2y + 3 = 0$

Distance from centre of C_1 is $(2 + 3)/\sqrt{(1^2 + 2^2)} = \sqrt{5}$

Radius of $C_1 = \sqrt{(4 + 5)} = 3$

Radius of required circle is, $r = \sqrt{(3^2 - 5)} = 2$

Option (c) is correct.

551. The equation of a circle which passes through the origin, whose radius is a and for which $y = mx$ is a tangent is
(a) $\sqrt{(1 + m^2)}(x^2 + y^2) + 2max + 2ay = 0$
(b) $\sqrt{(1 + m^2)}(x^2 + y^2) + 2ax - 2may = 0$
(c) $\sqrt{(1 + m^2)}(x^2 + y^2) - 2max + 2ay = 0$
(d) $\sqrt{(1 + m^2)}(x^2 + y^2) + 2ax + 2may = 0$

Solution :

Let, the centre of the circle is $(-g, -f)$

Therefore, $|(-f + gm)/\sqrt{(1 + m^2)}| = a$

⇨ $gm - f = a\sqrt{(1 + m^2)}$

Now, $g^2 + f^2 = a^2$

$g^2 + \{gm - a\sqrt{(1 + m^2)}\}^2 = a^2$

⇨ $g^2(1 + m^2) - 2gam\sqrt{(1 + m^2)} + a^2m^2 = 0$
⇨ $g = 2am\sqrt{(1 + m^2)} + \sqrt{\{4a^2m^2(1 + m^2) - 4a^2m^2(1 + m^2)\}}/2(1 + m^2) = am/\sqrt{(1 + m^2)}$
⇨ $f = am^2/\sqrt{(1 + m^2)} - a\sqrt{(1 + m^2)} = -a/\sqrt{(1 + m^2)}$

Equation is, $x^2 + y^2 + 2amx/\sqrt{(1 + m^2)} - 2ay/\sqrt{(1 + m^2)} = 0$

⇨ $\sqrt{(1 + m^2)}(x^2 + y^2) + 2amx - 2ay = 0$

There is no such option. So, the previous equation should be $f - gm = a\sqrt{(1 + m^2)}$

And it will come out to be option (c).

552. The circles $x^2 + y^2 + 4x + 2y + 4 = 0$ and $x^2 + y^2 - 2x = 0$
(a) intersect at two points
(b) touch at one point
(c) do not intersect

(d) satisfy none of the foregoing properties.

Solution :

Subtracting we get, $4x + 2y + 4 + 2x = 0$

- ⇨ $6x + 2y + 4 = 0$
- ⇨ $2x + y + 2 = 0$
- ⇨ $y = -(2x + 2)$

Putting in second equation we get, $x^2 + \{-(2x + 2)\}^2 - 2x = 0$

- ⇨ $5x^2 + 6x + 4 = 0$

Now, discriminant = $6^2 - 4*4*5 < 0$

They do not intersect or touch.

Option (c) is correct.

553. Let P be the point of intersection of the lines $ax + by - a = 0$ and $bx - ay + b = 0$. A circle with centre (1, 0) passes through P. The tangent to this circle at P meets the x-axis at the point (d, 0). Then the value of d is

(a) $2ab/(a^2 + b^2)$
(b) 0
(c) -1
(d) None of the foregoing values.

Solution :

Now, $ax + by - a = 0$ ….. (1)

And, $bx - ay + b = 0$ ……. (2)

Doing (1)*a + (2)*b we get, $(a^2 + b^2)x = a^2 - b^2$

- ⇨ $x = (a^2 - b^2)/(a^2 + b^2)$

$ax + by - a = 0$

- ⇨ $a(a^2 - b^2)/(a^2 + b^2) - a + by = 0$
- ⇨ $by = -2ab^2/(a^2 + b^2)$
- ⇨ $y = -2ab/(a^2 + b^2)$

Slope of normal at P is, $-2ab/(a^2 + b^2)/\{(a^2 - b^2)/(a^2 + b^2) - 1\} = -2ab/(-2b^2) = a/b$

Slope of tangent at P is $-(b/a)$

Equation of tangent at P is $y + 2ab/(a^2 + b^2) = -(b/a)\{x - (a^2 - b^2)/(a^2 + b^2)\}$

Putting y = 0, we get, $x - (a^2 - b^2)/(a^2 + b^2) = -2a^2/(a^2 + b^2)$

⇨ $x = -1$

Option (c) is correct.

554. The circles $x^2 + y^2 = 1$ and $x^2 + y^2 - 8x - 6y + c = 0$ touch each other externally. That is, the circles are mutually tangential and they lie outside each other. Then value of c is
(a) 9
(b) 8
(c) 6
(d) 4

Solution :

Subtracting we get, $8x + 6y - c = 1$

⇨ $y = \{(c + 1) - 8x\}/6$

Putting in first equation we get, $x^2 + [\{(c + 1) - 8x\}/6]^2 = 1$

⇨ $36x^2 + 64x^2 - 16(c + 1)x + (c + 1)^2 - 36 = 0$
⇨ $100x^2 - 16(c + 1)x + \{(c + 1)^2 - 36\} = 0$

As the circles touch each other so, roots are equal. Therefore, discriminant = 0

⇨ $256(c + 1)^2 - 4*100\{c + 1)^2 - 36\} = 0$
⇨ $16(c + 1)^2 - 25(c + 1)^2 = -25*36$
⇨ $9(c + 1)^2 = (5*6)^2$
⇨ $3(c + 1) = \pm 30$
⇨ $(c + 1) = \pm 10$
⇨ $c = 9, -11$

To check for what value of c the circles touch internally and externally apply $C_1C_2 = r_1 - r_2$ and $C_1C_2 = r_1 + r_2$ respectively.

Option (a) is correct.

555. The circles $x^2 + y^2 + 2ax + c^2 = 0$ and $x^2 + y^2 + 2by + c^2 = 0$ will touch if
(a) $1/a^2 + 1/b^2 = 1/c^2$
(b) $a^2 + b^2 = c^2$
(c) $a + b = c$
(d) $1/a + 1/b = 1/c$

Solution :

Subtracting we get, $2ax – 2by = 0$

⇨ $y = ax/b$

Putting in first equation we get, $x^2 + a^2x^2/b^2 + 2ax + c^2 = 0$

⇨ $x^2(1 + a^2/b^2) + 2ax + c^2 = 0$

The circles will touch if discriminant = 0

⇨ $4a^2 – 4c^2(1 + a^2/b^2) = 0$
⇨ $a^2/c^2 = 1 + a^2/b^2$
⇨ $1/a^2 + 1/b^2 = 1/c^2$

Option (a) is correct.

556. Two circles are said to cut *each other orthogonally* if the tangents at a point of intersection are perpendicular to each other. The locus of the center of a circle that cuts the circle $x^2 + y^2 = 1$ orthogonally and touch the line $x = 2$ is
(a) a pair of straight lines
(b) an ellipse
(c) a hyperbola
(d) a parabola

Solution :

Let the centre of the circle is $(-g, -f)$

Equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

Therefore, 2*g*0 + 2*f*0 = c – 1

⇨ c = 1

Now, putting x = 2 in the equation of the circle we get, $4 + y^2 + 4g + 2fy + 1 = 0$

⇨ $y^2 + 2fy + (4g + 5) = 0$

As the circle touches the line x = 2, so roots are equal i.e. discriminant = 0

⇨ $4f^2 - 4(4g + 5) = 0$
⇨ $f^2 - 4g - 5 = 0$
⇨ $y^2 + 4x + 5 = 0$
⇨ a parabola

Option (d) is correct.

557. The equation of the circle circumscribing the triangle formed by the lines y = 0, y = x and 2x + 3y = 10 is
(a) $x^2 + y^2 + 5x - y = 0$
(b) $x^2 + y^2 - 5x - y = 0$
(c) $x^2 + y^2 - 5x + y = 0$
(d) $x^2 + y^2 - x + 5y = 0$

Solution :

Vertex are (0, 0), (5, 0), (2, 2)

Let the equation of the circle is $x^2 + y^2 + 2gx + 2fy = 0$ (c = 0 as passes through (0, 0))

Now, 25 + 10g = 0

⇨ g = -5/2

Now, 4 + 4 + 4g + 4f = 0

⇨ f = -2 – g = -2 + 5/2 = 1/2

Equation is $x^2 + y^2 - 5x + y = 0$

Option (c) is correct.

558. Two gas companies X and Y, where X is situated at (40, 0) and Y at (0, 30) (unit = 1 km), offer to install equally priced gas furnaces in buyers' houses. Company X adds a charge of Rs. 40 per km of distance (measured along a straight line) between its location and the buyers' house, while company Y charges Rs. 60 per km of distance in the same way. Then the region where it is cheaper to have furnace installed by company X is

(a) the inside of circle $(x - 54)^2 + (y + 30)^2 = 3600$
(b) the inside of circle $(x - 24)^2 + (y + 30)^2 = 2500$
(c) the outside of the circle $(x + 32)^2 + (y - 54)^2 = 3600$
(d) the outside of the circle $(x + 24)^2 + (y - 12)^2 = 2500$

Solution :

Distance between X and Y = 50.

Let at a distance x from company X it is same to have any company's furnace installed.

So, $40*x = 60(50 - x)$

- ⇨ $2x = 150 - 3x$
- ⇨ $x = 30$

So, it divides the line joining company X and Y in 30 : 20 = 3 : 2

The coordinate = $(2*40 + 3*0)/(2 + 3) = 16$ and $(2*0 + 3*30)/(2 + 3) = 18$ i.e. (16, 18)

Let u be the distance from y in the far end from X such that there both the company's cost is same.

$40*(50 + u) = 60*u$

- ⇨ $100 + 2u = 3u$
- ⇨ $u = 100$
- ⇨ $50 + u = 150$
- ⇨ Diameter = 150 − 30 = 120
- ⇨ Radius = 120/2 = 60

Let the other side of the diameter is (x_1, y_1)

(0, 30) divides the line joining (x_1, y_1) and (40, 0) in 100 : 50 = 2 : 1

Therefore, $0 = (x_1 + 2*40)/(1 + 2)$

- ⇨ $x_1 = -80$

And, $30 = (y_1 + 2*0)/(1 + 2)$

- ⇨ $y_1 = 90$

Centre = (-80 + 16)/2 = -32 and (90 + 18/2) = 54 i.e. (-32, 54)

So, outside the circle $(x + 32)^2 + (y - 54)^2 = 3600$

Option (c) is correct.

559. Let C be the circle $x^2 + y^2 - 4x - 4y - 1 = 0$. The number of points common to C and the sides of the rectangle by the lines x = 2, x = 5, y = -1 and y = 5, equals
(a) 5
(b) 1
(c)2
(d) 3

Solution :

Put x = 2, $4 + y^2 - 8 - 4y - 1 = 0$

- ⇨ $y^2 - 4y - 5 = 0$
- ⇨ $(y - 2)^2 = 9$
- ⇨ y = 5, -1

Points are (2, 5), (2, -1)

Put x = 5 we get, $25 + y^2 - 20 - 4y - 1 = 0$

- ⇨ $y^2 - 4y + 4 = 0$
- ⇨ $(y - 2)^2 = 0$
- ⇨ y = 2
- ⇨ point is (5, 2)

Put y = -1, $x^2 + 1 - 4x + 4 - 1 = 0$

- ⇨ $(x - 2)^2 = 0$
- ⇨ x = 2
- ⇨ point is (2, -1) which is evaluated earlier.

Put y = 5, $x^2 + 25 - 4x - 20 - 1 = 0$

- ⇨ $x^2 - 4x + 4 = 0$
- ⇨ $(x - 2)^2 = 0$

⇨ $x = 2$
⇨ point is (2, 5) which is evaluated earlier.
⇨ Therefore, 3 points.

Option (d) is correct.

560. A circle of radius a with both coordinates of centre positive, touches the x-axis and also the line $3y = 4x$. Then its equation is
(a) $x^2 + y^2 - 2ax - 2ay + a^2 = 0$
(b) $x^2 + y^2 - 6ax - 4ay + 12a^2 = 0$
(c) $x^2 + y^2 - 4ax - 2ay + 4a^2 = 0$
(d) none of the foregoing equations

Solution :

Centre = (h, a)

Now, $(4h - 3a)/\sqrt{(4^2 + 3^2)} = a$

⇨ $h = 2a$

Equation is $(x - 2a)^2 + (y - a)^2 = a^2$

⇨ $x^2 + y^2 - 4ax - 2ay + 4a^2 = 0$

Option (c) is correct.

561. The equation of the circle with centre in the first quadrant and radius ½ such that the line $15y = 8x$ and the X-axis are both tangents to the circle, is
(a) $x^2 + y^2 - 8x - y + 16 = 0$
(b) $x^2 + y^2 - 4x - y + 4 = 0$
(c) $x^2 + y^2 - x - 4y + 4 = 0$
(d) $x^2 + y^2 - x - 8y + 16 = 0$

Solution :

Centre = (h, ½)

Now, $(8h - 15/2)/\sqrt{(8^2 + 15^2)} = ½$

⇨ $h = 2$

Equation is, $(x - 2)^2 + (y - ½)^2 = (1/2)^2$

⇨ $x^2 + y^2 - 4x - y + 4 = 0$

Option (b) is correct.

562. The centre of the circle $x^2 + y^2 - 8x - 2fy - 11 = 0$ lies on the straight line which passes through the point (0, -1) and makes an angle of 45 with the positive direction of the horizontal axis. The circle
(a) touches the vertical axis
(b) touches the horizontal axis
(c) passes through origin
(d) meets the axes at four points

Solution :

Equation of straight line is, $y + 1 = 1(x - 0)$

⇨ $x - y - 1 = 0$

Centre of the circle is (4, -f)

So, $4 + f - 1 = 0$

⇨ $f = -3$

Radius = $\sqrt{(4^2 + 3^2 + 11)} = 6$

Therefore, option (d) is correct.

563. Let P and Q be any two points on the circles $x^2 + y^2 - 2x - 3 = 0$ and $x^2 + y^2 - 8x - 8y + 28 = 0$, respectively. If d is the distance between P and Q, then the set of all possible values of d is
(a) $0 \leq d \leq 9$
(b) $0 \leq d \leq 8$
(c) $1 \leq d \leq 8$
(d) $1 \leq d \leq 9$

Solution :

Subtracting we get, $-2x - 3 + 8x + 8y - 28 = 0$

- ⇨ $6x + 8y = 31$
- ⇨ $y = (31 - 6x)/8$

Putting in first equation we get, $x^2 + \{(31 - 6x)/8\}^2 - 2x - 3 = 0$

- ⇨ $64x^2 + 36x^2 - 372x + 961 - 128x - 192 = 0$
- ⇨ $100x^2 - 500x + 769 = 0$

Discriminant = $500^2 - 4*100*769 = 400(125 - 769) < 0$ so both the circle does not meet.

Centres = (1, 0) and (4, 4)

Now, we need to find the equation of the line joining the centres and then solve with the two circles, you will get 4 points, then calculate minimum and maximum distance.

But, here we will go by short-cut method. According to options minimum distance cannot be 0 and hence minimum distance = 1.

Now, radius of the circles = $\sqrt{(1^1 + 3)} = 2$ and $\sqrt{(4^2 + 4^2 - 28)} = 2$

Therefore, maximum distance = 1 + 2(2 + 2) = 9

Option (d) is correct.

564. All points whose distance from the nearest point on the circle $(x - 1)^2 + y^2 = 1$ is half the distance from the line $x = 5$ lie on

(a) an ellipse
(b) a pair of straight lines
(c) a parabola
(d) a circle

Solution :

Let the point is (h, k).

Centre of the circle (1, 0) and radius = 1

Nearest distance from circle = $\sqrt{\{(h - 1)^2 + k^2\}} - 1$

Distance from the line is $|h - 5|$

So, $\sqrt{\{(h - 1)^2 + k^2\}} - 1 = (1/2)|h - 5|$

⇨ $4(h - 1)^2 + 4k^2 = (h - 5)^2 + 2|h - 5| + 1$

Option (a) is correct.

565. If P = (0, 0), Q = (1, 0) and R = (1/2, √3/2), then the centre of the circle for which the lines PQ, QR and RP are tangents, is
(a) (1/2, ¼)
(b) (1/2, √3/4)
(c)(1/2, 1/2√3)
(d) (1/2, -1/√3)

Solution :

PQ is x-axis.

So, centre = (h, r)

Equation of RP is, y = √3x

So, |(r - √3h)/2| = r

⇨ (√3h – r) = 2r (otherwise r and h will be of opposite sign but the centre is in first quadrant)
⇨ h = √3r

Equation of QR is, (y – 0)/(√3/2 – 0) = (x – 1)/(1/2 – 1)

⇨ 2y/√3 = -2(x – 1)
⇨ √3x + y - √3 = 0

So, |(√3h + r - √3)/2| = r

⇨ -4r + √3 = 2r
⇨ r = √3/6 = 1/2√3 (because r < √3/2 which you will get if you take 4r - √3 = 2r)
⇨ h = ½

Option (c) is correct.

566. The equations of the pair of straight lines parallel to the x-axis and tangent to the curve $9x^2 + 4y^2 = 36$ are
(a) y = -3, y = 9
(b) y = 3, y = -6

(c) $y = \pm 6$
(d) $y = \pm 3$

Solution :

Let us say, the equation of the tangent is $y = a$

So, putting $y = a$ in the equation of ellipse we get, $9x^2 + 4a^2 = 36$

- ⇨ $9x^2 = 4(9 - a^2) = 0$ (because x must have one solution)
- ⇨ $a = \pm 3$

Option (d) is correct.

567. If the parabola $y = x^2 + bx + c$ is tangent to the straight line $x = y$ at the point (1, 1) then
(a) $b = -1, c = +1$
(b) $b = +1, c = -1$
(c) $b = -1$, c arbitrary
(d) $b = 0, c = -1$

Solution :

Putting $y = x$ we get, $x^2 + x(b - 1) + c = 0$

- ⇨ $(b - 1)^2 - 4c = 0$ …… (1) (roots are equal as tangent)

Now, the parabola passes through (1, 1)

- ⇨ $1 = 1^2 + 1*b + c$
- ⇨ $b + c = 0$
- ⇨ $c = -b$
- ⇨ $(b - 1)^2 + 4b = 0$ (from (1))
- ⇨ $(b + 1)^2 = 0$
- ⇨ $b = -1, c = +1$

Option (a) is correct.

568. The condition that the line $x/a + y/b = 1$ be a tangent to the curve $x^{2/3} + y^{2/3} = 1$ is
(a) $a^2 + b^2 = 2$

(b) $a^2 + b^2 = 1$
(c) $1/a^2 + 1/b^2 = 1$
(d) $a^2 + b^2 = 2/3$

Solution :

Any point on the curve is $(\cos^3\theta, \sin^3\theta)$

Now, $x^{2/3} + y^{2/3} = 1$

- ⇨ $(2/3)x^{-1/3} + (2/3)y^{-1/3}(dy/dx) = 0$
- ⇨ $dy/dx = -y^{1/3}/x^{1/3}$
- ⇨ (dy/dx) at $(\cos^3\theta, \sin^3\theta) = -\tan\theta$

Now, $-\tan\theta = -b/a$

- ⇨ $a\sin\theta = b\cos\theta$

Now, the line passes through $(\cos^3\theta, \sin^3\theta)$

So, $\cos^3\theta/a + \sin^3\theta/b = 1$

- ⇨ $\cos^2\theta\sin\theta/b + \sin^3\theta/b = 1$
- ⇨ $\sin\theta/b = 1$
- ⇨ $\sin\theta = b$
- ⇨ $\cos\theta = a$
- ⇨ $a^2 + b^2 = 1$

Option (b) is correct.

569. If the two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles, then the locus of P is
(a) $x - 1 = 0$
(b) $2x + 1 = 0$
(c) $x + 1 = 0$
(d) $2x - 1 = 0$

Solution :

Vertex = (0, 0) and a = 1

Therefore, equation of directrix is, x = -1 i.e. x + 1 = 0

If two tangents to a parabola from a given point are at right angles then the point lies on the directrix.

Option (c) is correct.

570. Let A be the point (0, 0) and let B be the point (1, 0). A point P moves so that the angle APB measures $\pi/6$. The locus of P is
   (a) a parabola
   (b) arcs of two circles with centres $(1/\sqrt{2}, 1/\sqrt{2})$ and $(1/\sqrt{2}, -1/\sqrt{2})$
   (c) arcs of two circles each of radius 1
   (d) a pair of straight lines

Solution :

Let P = (h, k)

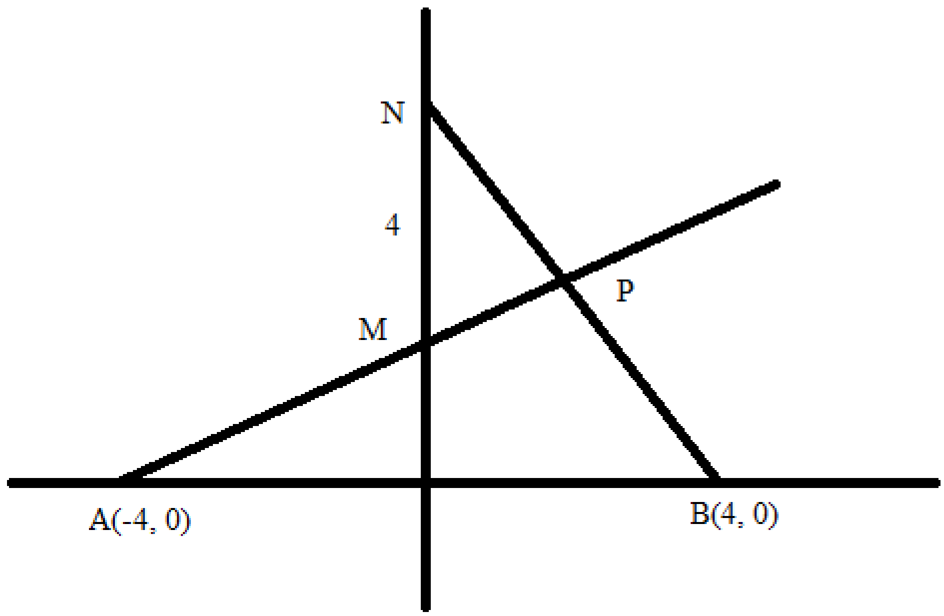
Slope of AP = k/h and slope of BP = k/(h – 1)

$\tan(APB) = |\{k/h - k/(h - 1)\}/\{1 + k^2/h(h - 1)\}|$

- ⇨ $1/\sqrt{3} = |k(h - 1 - h)/(h^2 + k^2 - h)|$
- ⇨ $k/(h^2 + k^2 - h) = \pm 1/\sqrt{3}$
- ⇨ $h^2 + k^2 - h \pm \sqrt{3}k = 0$
- ⇨ Therefore, radius = $\sqrt{\{(1/2)^2 + (\sqrt{3}/2)^2\}} = 1$

Option (c) is correct.

571. Let A = (-4, 0) and B = (4, 0). Let M and N be points on the y-axis, with MN = 4. Let P be the point of intersection of AM and BN. This is illustrated in the figure. Then the locus of P is



(a) $x^2 - 2xy = 16$
(b) $x^2 + 2xy = 16$
(c) $x^2 + 2xy + y^2 = 64$
(d) $x^2 - 2xy + y^2 = 64$

Solution :

Let, M = (0, a), N = (0, 4 + a)

Equation of AM is, $(y - 0)/(a - 0) = (x + 4)/(0 + 4)$

⇨ $4y = ax + 4a$

Equation of BN is, $(y - 0)/(4 + a - 0) = (x - 4)/(0 - 4)$

⇨ $-4y = (4 + a)x - 4(4 + a)$

Adding the equations we get, $0 = ax + 4a + (4 + a)x - 4(4 + a)$

⇨ $x(4 + 2a) = 16$
⇨ $x = 8/(2 + a)$
⇨ $4y = 8a/(2 + a) + 4a = (16a + 4a^2)/(2 + a)$
⇨ $y = a(4 + a)/(2 + a)$

Let, P = (h, k)

So, $h = 8/(2 + a)$ and $k = a(4 + a)/(2 + a)$

$k/h = a(4 + a)/8$

Now, $2 + a = 8/h$

- ⇨ $a = 8/h - 2 = (8 - 2h)/h$

$k/h = \{(8 - 2h)/h\}(4 + (8 - 2h)/h)/8$

- ⇨ $8k/h = (64 - 4h^2)/h^2$
- ⇨ $2kh = 16 - h^2$
- ⇨ $h^2 + 2kh = 16$
- ⇨ $x^2 + 2xy = 16$

Option (b) is correct.

572. Consider a circle in the XY plane with diameter 1, passing through the origin O and through the point A(1, 0). For any point B on the circle, let C be the point of intersection of the line OB with the vertical line through A. If M is the point on the line OBC such that OM and BC are of equal length, then the locus of the point M as B varies is given by the equation
(a) $y = \sqrt{x(x^2 + y^2)}$
(b) $y^2 = x$
(c) $(x^2 + y^2)x - y^2 = 0$
(d) $y = x\sqrt{(x^2 + y^2)}$

Solution :

Let the equation of the circle is $x^2 + y^2 + 2gx + 2fy = 0$ (c = 0 as passes through origin)

Now, $1^2 + 0 + 2g*1 + 0 = 0$

- ⇨ $g = -1/2$

Now, $g^2 + f^2 = (1/2)^2$ (radius = ½)

- ⇨ $f = 0$

Equation of the circle is, $x^2 + y^2 - x = 0$

Let $B = (x_1, y_1)$

So, $x_1^2 + y_1^2 - x_1 = 0$ …………….. (1)

Equation of OB is, $y = (y_1/x_1)x$

Equation of vertical line through A is, $x = 1$.

Putting $x = 1$, we get, $y = y_1/x_1$

So, $C = (1, y_1/x_1)$

Now, $BC^2 = (x_1 - 1)^2 + (y_1 - y_1/x_1)^2 = (x_1 - 1)^2 + y_1^2(x_1 - 1)^2/x_1^2 = (x_1 - 1)^2(1 + y_1^2/x_1^2) = (x_1 - 1)^2(x_1^2 + y_1^2)/x_1^2 = (x_1 - 1)^2/x_1$ (from (1))

Let $M = (h, k)$

$OM^2 = h^2 + k^2$

Now, $h^2 + k^2 = (x_1 - 1)^2/x_1$

Now, $k = (y_1/x_1)h$ (as M lies on OB)

- ⇨ $k^2/h^2 = y_1^2/x_1^2$
- ⇨ $(h^2 + k^2)/h^2 = (x_1^2 + y_1^2)/x_1^2 = 1/x_1$ (from (1))
- ⇨ $x_1 = h^2/(h^2 + k^2)$

Putting value in above equation we get, $h^2 + k^2 = \{(h^2/(h^2 + k^2) - 1)^2/h^2\}(h^2 + k^2)$

- ⇨ $h^2 = \{k^2/(h^2 + k^2)\}^2$
- ⇨ $h = k^2/(h^2 + k^2)$
- ⇨ $h(h^2 + k^2) = k^2$
- ⇨ $y^2 = x(x^2 + y^2)$

Option (c) is correct.

573. The locus of the foot of the perpendicular from any focus upon any tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ is
   (a) $x^2/b^2 + y^2/a^2 = 1$
   (b) $x^2 + y^2 = a^2 + b^2$
   (c) $x^2 + y^2 = a^2$
   (d) none of the foregoing curves

Solution :

Let the foot of the perpendicular from focus = (h, k)

Focus = (ae, 0)

Slope of focus joining (h, k) = (k – 0)/(h – ae) = k/(h – ae)

Therefore, slope of tangent = -(h – ae)/k

Equation of tangent is, y – k = -{(h – ae)/k}(x – h)

⇨ y = k – (h – ae)(x – h)/k

Putting in the equation of ellipse we get, $x^2/a^2 + \{k - (h - ae)(x - h)/k\}^2/b^2 = 1$

Now, equate the discriminant of this equation to zero and use $(a^2 - b^2)/a^2 = e^2$ and reduce the equation of (h, k) and then put (x, y) in place of (h, k) and you get the locus.

Option (c) is correct.

574. The area of the triangle formed by a tangent of slope m to the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the two coordinate axes is
(a) $\{|m|/2\}(a^2 + b^2)$
(b) $\{1/2|m|\}(a^2 + b^2)$
(c) $\{|m|/2\}(a^2m^2 + b^2)$
(d) $\{1/2|m|\}(a^2m^2 + b^2)$

Solution :

Let the tangent is at the point (acosθ, bsinθ)

Now, $x^2/a^2 + y^2/b^2 = 1$

⇨ $2x/a^2 + (2y/b^2)(dy/dx) = 0$
⇨ (dy/dx) at (acosθ, bsinθ) = $-(a\cos\theta/a^2)/(b\sin\theta/b^2) = -b\cos\theta/a\sin\theta = m$
⇨ tanθ = -b/am

Now, equation of the tangent is, y – bsinθ = m(x – acosθ)

⇨ mx – y – amcosθ + bsinθ = 0

Putting x = 0 we get, y = amcosθ – bsinθ and putting y = 0 we get, x = (amcosθ – bsinθ)/m

Area = $|(1/2)(am\cos\theta - b\sin\theta)^2/m| = \{1/2|m|\}(a^2m^2\cos^2\theta + b^2\sin^2\theta - 2amb\cos\theta\sin\theta)$

$= \{1/2|m|\}(a^2m^2 - a^2m^2\sin^2\theta + b^2\sin^2\theta - 2amb\cos\theta\sin\theta)$

$\{1/2|m|\}(a^2m^2 - b^2\cos^2\theta + b^2\sin^2\theta + 2b^2\cos^2\theta)$ (from $\tan\theta = -b/am$)

$= \{1/2|m|\}(a^2m^2 + b^2)$

Option (d) is correct.

575. Consider the locus of a moving point $P = (x, y)$ in the plane which satisfies the law $2x^2 = r^2 + r^4$, where $r^2 = x^2 + y^2$. Then only one of the following statements is true. Which one is it?
(a) For every positive real number d, there is a point (x, y) on the locus such that r = d.
(b) For every value d, $0 < d < 1$, there are exactly four points on the locus, each of which is at a distance d from the origin.
(c) The point P always lies in the first quadrant.
(d) The locus of P is an ellipse.

Solution :

Clearly, option (b) is correct.

Because let r = 50, $r^2 = 2500$, $r^4 = 6250000$

So, $2x^2 = 2500 + 6250000$

⇨ $x^2 > r^2$

So, option (a) cannot be true. And option (c) cannot be true because P may be anywhere. It doesn't matter if x or y is negative. And (d) is not true because it is not the equation of an ellipse.

576. Let A be any variable point on the X-axis and B the point (2, 3). The perpendicular at A to the line AB meets the Y-axis at C. Then the locus of the mid-point of the segment AC as A moves is given by the equation
(a) $2x^2 - 2x + 3y = 0$
(b) $3x^2 - 3x + 2y = 0$
(c) $3x^2 - 3x - 2y = 0$
(d) $2x - 2x^2 + 3y = 0$

Solution :

Let A = (a, 0)

Now, slope of AB = (3 – 0)/(2 – a) = 3/(2 – a)

Slope of perpendicular on AB = (a – 2)/3

Equation of perpendicular to AB at A is, y – 0 = {(a – 2)/3}(x – a)

Putting x = 0, we get, y = a(2 – a)/3

C = (0, a(2 – a)/3)

Let, mid-point of AC = (h, k)

Therefore, h = a/2, k = a(2 – a)/6

- ⇨ k = h(2 – 2h)/3
- ⇨ $3k = 2h - 2h^2$
- ⇨ $2h^2 - 2h + 3k = 0$

Locus is , $2x^2 - 2x + 3y = 0$

Option (a) is correct.

577. A straight line segment AB of length a moves with its ends on the axes. Then the locus of the point P such that AP : BP = 2 : 1 is
(a) $9(x^2 + y^2) = 4a^2$
(b) $9(x^2 + 4y^2) = 4a^2$
(c) $9(y^2 + 4x^2) = 4a^2$
(d) $9x^2 + 4y^2 = a^2$

Solution :

Let A = (0, q) and B = (p, 0)

Let P = (h, k)

Therefore, h = 2p/3 and k = q/3

- ⇨ p = 3h/2 an q = 3k

Now, $p^2 + q^2 = a^2$

- ⇨ $9h^2/4 + 9k^2 = a^2$
- ⇨ $9(h^2 + 4k^2) = 4a^2$

Locus is, $9(x^2 + 4y^2) = 4a^2$

Option (b) is correct.

578. Let P be a point moving on the straight line $\sqrt{3}x + y = 2$. Denote the origin by O. Suppose now that the line-segment OP is rotated. With O fixed, by an angle 30 in anti-clockwise direction, to get OQ. The locus of Q is
(a) $\sqrt{3}x + 2y = 2$
(b) $2x + \sqrt{3}y = 2$
(c) $\sqrt{3}x + 2y = 1$
(d) $x + \sqrt{3}y = 2$

Solution :

Let coordinate of P = (a, b)

So, $\sqrt{3}a + b = 2$

Now, slope of OP = b/a

Let Q = (h, k)

Slope of OQ = k/h

Now, $\tan 30 = \{(k/h) – (b/a)\}/(1 + (k/h)(b/a))$

- ⇨ $1/\sqrt{3} = (ak – bh)/(ah + bk)$
- ⇨ $ah + bk = \sqrt{3}ak - \sqrt{3}bh$
- ⇨ $a(\sqrt{3}k –h) = b(k + \sqrt{3}h)$

Now, $\sqrt{3}a + b = 2$

- ⇨ $\sqrt{3}(k + \sqrt{3}h)b/(\sqrt{3}k – h) + b = 2$
- ⇨ $b\{\sqrt{3}k + 3h + \sqrt{3}k – h) = 2(\sqrt{3}k – h)$
- ⇨ $b = 2(\sqrt{3}k – h)/2(\sqrt{3}k + h) = (\sqrt{3}k – h)/(\sqrt{3}k + h)$
- ⇨ $a = (k + \sqrt{3}h)/(\sqrt{3}k + h)$

And we have, $h^2 + k^2 = a^2 + b^2$

- ⇨ $h^2 + k^2 = \{(k + \sqrt{3}h)/(\sqrt{3}k + h)\}^2 + \{(\sqrt{3}k – h)/(\sqrt{3}k + h)\}^2 = (k^2 + 3h^2 + 2\sqrt{3}hk + 3k^2 + h^2 - 2\sqrt{3}hk)/(\sqrt{3}k + h)^2$
- ⇨ $h^2 + k^2 = 4(h^2 + k^2)/(\sqrt{3}k + h)^2$
- ⇨ $(\sqrt{3}k + h)^2 = 4$
- ⇨ $\sqrt{3}k + h = 2$

Locus is, $x + \sqrt{3}y = 2$

Option (d) is correct.

579. Consider an ellipse with centre at the origin. From any arbitrary point P on the ellipse, perpendiculars PA and PB are dropped on the axes of the ellipse. Then the locus of point Q that divides AB in the fixed ratio m : n is
(a) a circle
(b) an ellipse
(c) a hyperbola
(d) none of the foregoing curves

Solution :

Let the equation of the ellipse is $x^2/a^2 + y^2/b^2 = 1$

Any point on the ellipse = (acosθ, bsinθ)

Now, A = (acosθ, 0) and B = (0, bsinθ)

Let Q = (h, k)

Therefore, h = nacosθ/(m + n) and k = mbsinθ/(m + n)

⇨ h/na = cosθ/(m + n) and k/mb = sinθ/(m + n)
⇨ $(h/na)^2 + (k/mb)^2 = 1/(m + n)^2$

Locus is, $x^2/n^2a^2 + y^2/m^2b^2 = 1/(m + n)^2$

⇨ An ellipse.

Option (b) is correct.

580. Let A and C be two distinct points in the plane and B a point on the line segment AC such that AB = 2BC. Then, the locus of the point P lying in the plane and satisfying $AP^2 + CP^2 = 2BP^2$ is
(a) a straight line parallel to the line AC
(b) a straight line perpendicular to the line AC
(c) a circle passing through A and C
(d) none of the foregoing curves

Solution :

Let A = (a, 0) and C = (c, 0)

B = ((2c + a)/3, 0)

Let P = (h, k)

Therefore, $(h - a)^2 + k^2 + (h - c)^2 + k^2 = 2\{h - (2c + a)/3\}^2 + 2k^2$

- ⇨ $2h^2 - 2h(a + c) + a^2 + c^2 = 2h^2 - 2h(2c + a)/3 + \{(2c + a)/3\}^2$
- ⇨ $2h(2c + a - 3a - 3c)/3 = (4c^2 + a^2 - 9c^2 - 9a^2)/9$
- ⇨ $2h(2a + c)/3 = (8a^2 + 5c^2)/9$
- ⇨ $h = (8a^2 + 5c^2)/\{6(2a + c)\}$

Locus is, x = b

So, straight line perpendicular to AC.

Option (b) is correct.

581. Let C be a circle and L is a line on the same plane such that C and L do not intersect. Let P be a moving point such that the circle drawn with centre at P to touch L also couches C. Then the locus of P is

(a) A straight line parallel to L not intersecting C

(b) A circle concentric with C

(c) A parabola whose focus is centre of C and whose directrix is L

(d) A parabola whose focus is the centre of C and whose directrix is a straight line parallel to L.

Solution :

Let P = (h, k)

Let C is $x^2 + y^2 = 4$ and L is y = 3.

Let the radius of the circle is r.

Therefore, (3 – k) = r

And $\sqrt{(h^2 + k^2)} = 2 + r$ (as the circle touches the circle C) = 2 + 3 – k = 4 – k

- ⇨ $h^2 + k^2 = 16 - 8k + k^2$

⇨ $h^2 = -8(k - 2)$
Locus is $x^2 = -4*2(y - 2)$

So, vertex = (0, 2) focus = (0, 0)

Option (d) is correct.

582. A right triangle with sides 3, 4 and 5 lies inside the circle $2x^2 + 2y^2 = 25$. The triangle is moved inside the circle in such a way that its hypotenuse always forms a chord of the circle. The locus of the vertex opposite to the hypotenuse is
(a) $2x^2 + 2y^2 = 1$
(b) $x^2 + y^2 = 1$
(c) $x^2 + y^2 = 2$
(d) $2x^2 + 2y^2 = 5$

Solution :

Option (a) is correct.

583. Let P be the point (-3, 0) and Q be a moving point (0, 3t). Let PQ be trisected to R so that R is nearer to Q. RN is drawn perpendicular to PQ meeting the x-axis at N. The locus of the mid-point of RN is
(a) $(x + 3)^2 - 3y = 0$
(b) $(y + 3)^2 - 3x = 0$
(c) $x^2 - y = 1$
(d) $y^2 - x = 1$

Solution :

PR : RQ = 2 : 1

R = (-3/3, 6t/3) = (-1, 2t)

Slope of PQ = (3t – 0)/(0 + 3) = t

Slope of perpendicular to PQ = -1/t

Equation of RN is, y – 2t = (-1/t)(x + 1)

Putting y = 0, we get, $x = 2t^2 - 1$

So, $N = (2t^2 - 1, 0)$

Let mid-point of RN = (h, k)

Therefore, $h = (2t^2 - 1 - 1)/2$ and $k = 2t/2$

- ⇨ $h = t^2 - 1$ and $k = t$
- ⇨ $h = k^2 - 1$

Locus is, $y^2 - x = 1$

Option (d) is correct.

584. The maximum distance between two points of the unit cube is
   (a) $\sqrt{2} + 1$
   (b) $\sqrt{2}$
   (c) $\sqrt{3}$
   (d) $\sqrt{2} + \sqrt{3}$

Solution :

Maximum distance = $\sqrt{(1^2 + 1^2 + 1^2)} = \sqrt{3}$ (between two opposite vertex along space diagonal)

Option (c) is correct.

585. Each side of a cube is increased by 50%. Then the surface area of the cube is increased by
   (a) 50%
   (b) 100%
   (c) 125%
   (d) 150%

Solution :

Let side of cube = a.

Surface area = $6a^2$

New side = $3a/2$

New surface area = $6(3a/2)^2 = 27a^2/2$

Increase = $27a^2/2 - 6a^2 = 15a^2/2$

% increase = $\{(15a^2/2)/6a^2\}*100 = 125$

Option (c) is correct.

586. A variable plane passes through a fixed point (a, b, c) and cuts the coordinate axes at P, Q, R. Then the coordinates (x, y, z) of the centre of the sphere passing through P, Q, R and the origin satisfy the equation
(a) $a/x + b/y + c/z = 2$
(b) $x/a + y/b + z/c = 3$
(c) $ax + by + cz = 1$
(d) $ax + by + cz = a^2 + b^2 + c^2$

Solution :

Option (a) is correct.

587. Let A = (0, 10) and B = (30, 20) be two points in the plane and let P = (x, 0) be a moving point on the x-axis. The value of x for which the sum of the distances of P from A and B is minimum equals
(a) 0
(b) 10
(c) 15
(d) 20

Solution :

$D = \sqrt{(x^2 + 100)} + \sqrt{\{(x - 30)^2 + 400\}}$

$dD/dx = 2x/2\sqrt{(x^2 + 100)} + 2(x - 30)/2\sqrt{\{(x - 30)^2 + 400\}} = 0$

- ⇨ $x\sqrt{\{(x - 30)^2 + 400\}} = -(x - 30)\sqrt{(x^2 + 100)}$
- ⇨ $x^2(x - 30)^2 + 400x^2 = (x - 30)^2x^2 + 100(x - 30)^2$
- ⇨ $4x^2 = x^2 - 60x + 900$
- ⇨ $3x^2 + 60x - 900 = 0$
- ⇨ $x^2 + 20x - 300 = 0$
- ⇨ $(x + 30)(x - 10) = 0$

⇨ $x = 10$

Option (b) is correct.

588. The number of solutions to the pair of equations $\sin\{(x + y)/2\} = 0$ and $|x| + |y| = 1$ is
(a) 2
(b) 3
(c) 4
(d) 1

Solution :

$\sin\{(x + y)/2\} = 0$

⇨ $(x + y)/2 = 0$
⇨ $x + y = 0$
⇨ $x = ½$ and $y = -1/2$ and $x = -1/2$ and $y = ½$

Two solutions.

Option (a) is correct.

589. The equation $r^2\cos\theta + 2ar\sin^2(\theta/2) - a^2 = 0$ (a positive) represents
(a) a circle
(b) a circle and a straight line
(c) two straight lines
(d) none of the foregoing curves

Solution :

Now, $r^2x\text{os}\theta + 2ar\sin^2(\theta/2) - a^2 = 0$

⇨ $rx + ar(1 - \cos\theta) - a^2 = 0$
⇨ $rx + ar - ax - a^2 = 0$
⇨ $r(x + a) - a(x + a) = 0$
⇨ $(x + a)(r - a) = 0$
⇨ $x + a = 0$, $r = a$ i.e. $x^2 + y^2 = a^2$
⇨ a circle and a straight line

Option (b) is correct.

590. The number of distinct solutions of $\sin5\theta\cos3\theta = \sin9\theta\cos7\theta$, in the interval $0 \le \theta \le \pi/2$ is
(a) 5
(b) 4
(c) 8
(d) 9

Solution :

Now, $\sin5\theta\cos3\theta = \sin9\theta\cos7\theta$

- ⇨ $2\sin5\theta\cos3\theta = 2\sin9\theta\cos7\theta$
- ⇨ $\sin8\theta + \sin2\theta = \sin16\theta + \sin2\theta$
- ⇨ $\sin8\theta = \sin16\theta$
- ⇨ $\sin8\theta - 2\sin8\theta\cos8\theta = 0$
- ⇨ $\sin8\theta(1 - 2\cos8\theta) = 0$
- ⇨ $\sin8\theta = 0$ or $\cos8\theta = ½$
- ⇨ $8\theta = 0, \pi, 2\pi, 3\pi, 4\pi$
- ⇨ $\theta = 0, \pi/8, \pi/4, 3\pi/8, \pi/2$

Now, $\cos8\theta = ½$

$8\theta = \pi/3, 2\pi - \pi/3, 2\pi + \pi/3, 4\pi - \pi/3$

- ⇨ $\theta = \pi/24, \pi/4 - \pi/24, \pi/4 + \pi/24, \pi/2 - \pi/24$
- ⇨ 9 solutions.

Option (d) is correct.

591. The value of sin15 is
(a) $(\sqrt{6} - \sqrt{2})/4$
(b) $(\sqrt{6} + \sqrt{2})/4$
(c) $(\sqrt{5} + 1)/2$
(d) $(\sqrt{5} - 1)/2$

Solution :

$\cos30 = \sqrt{3}/2$

$1 - 2\sin^2 15 = \sqrt{3}/2$

$\sin^2 15 = (2 - \sqrt{3})/4 = (4 - 2\sqrt{3})/8 = \{(\sqrt{3} - 1)/2\sqrt{2}\}^2$

⇨ $\sin 15 = (\sqrt{3} - 1)/2\sqrt{2} = (\sqrt{6} - \sqrt{2})/4$

Option (a) is correct.

592. The value of sin25sin35sin85 is equal to

(a) $\sqrt{3}/4$

(b) $\sqrt{(2 - \sqrt{3})}/4$

(c) $5\sqrt{3}/9$

(d) $\sqrt{(1/2 + \sqrt{3}/4)}/4$

Solution :

$\sin 25\sin 35\sin 85 = (1/2)(2\sin 25\sin 35)\sin 85 = (1/2)(-\cos 60 + \cos 10)\sin 85 = (1/2)(-\sin 85/2 + \cos 10\sin 85) = (1/2)\{-\sin 85/2 + (1/2)2\cos 10\cos 5)\} = (1/2)\{-\cos 5/2 + (1/2)\cos 15 + \cos 5/2\} = (1/4)\cos 15 = -(1/4)\sqrt{\{(1 + \cos 30)/2\}} = (1/4)\sqrt{(1/2 + \sqrt{3}/4)}$

Option (d) is correct.

593. The angle made by the complex number $1/(\sqrt{3} + i)^{100}$ with the positive real axis is

(a) 135

(b) 120

(c) 240

(d) 180

Solution :

$1/(\sqrt{3} + i)^{100} = (\sqrt{3} - i)^{100}/2^{100} = \{\sqrt{3}/2 - i(1/2)\}^{100} = \{\cos(\pi/6) - i\sin(\pi/6)\}^{100} = \cos(100\pi/6) - i\sin(100\pi/6) = \cos(16\pi + 4\pi/6) - i\sin(16\pi + 4\pi/6) = \cos(2\pi/3) - i\sin(2\pi/3) = \cos(2\pi - 2\pi/3) + i\sin(2\pi - 2\pi/3) = \cos(4\pi/3) + i\sin(4\pi/3)$

Option (c) is correct.

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594. The value of $\tan\{(\pi/4)\sin^2 x\}$, $-\infty < x < \infty$, lies between

(a) -1 and +1
(b) 0 and 1
(c) 0 and ∞
(d) -∞ and +∞

Solution :

$0 \le \sin^2 x \le 1$

⇨ $0 \le (\pi/4)\sin^2 x \le (\pi/4)$
⇨ $0 \le \tan\{(\pi/4)\sin^2 x\} \le 1$

Option (b) is correct.

595. If $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$, then the value of $\cos(\theta - \pi/4)$ is

(a) $\pm 1/2\sqrt{2}$
(b) $\pm 1/2$
(c) $\pm 1/\sqrt{2}$
(d) 0

Solution :

Now, $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$

⇨ $\pi\cos\theta = \pi/2 - \pi\sin\theta$
⇨ $\cos\theta + \sin\theta = ½$
⇨ $(1/\sqrt{2})\cos\theta + (1/\sqrt{2})\sin\theta = 1/2\sqrt{2}$
⇨ $\cos(\pi/4)\cos\theta + \sin(\pi/4)\sin\theta = 1/2\sqrt{2}$
⇨ $\cos(\theta - \pi/4) = 1/2\sqrt{2}$

Option (a) is correct.

596. If $f(x) = (1 - x)/(1 + x)$. then $f(f(\cos x))$ equals

(a) x
(b) cosx
(c) $\tan^2(x/2)$
(d) none of the foregoing expressions

Solution :

f(cosx) = (1 − cosx)/(1 + cosx) = $\tan^2(x/2)$

f(fcosx) = $\{1 - \tan^2(x/2)\}/\{1 + \tan^2(x/2)\}$ = cosx

Option (b) is correct.

597. If cosx/cosy = a/b, then atanx + btany equals
(a) (a + b)cot{(x + y)/2}
(b) (a + b)tan{(x + y)/2}
(c)(a + b){tan(x/2) + tan(y/2)}
(d) (a + b){cot(x/2) + cot(y/2)}

Solution :

As there is a factor (a + b) in every option so we start with

(atanx + btany)/(a + b)

= (bsinx/cosy + bsiny/cosy)/(bcosx/cosy + b)

= (sinx + siny)/(cosx + cosy)

= 2sin{x + y)/2}cos{(x − y)/2}/[2cos{(x + y)/2}cos{(x − y)/2}]

= tan{(x + y)/2}

Option (b) is correct.

598. Let θ be an angle in the second quadrant (that is 90 ≤ θ ≤ 180) with tanθ = -2/3. Then the value of {tan(90 + θ) + cos(180 + θ)}/{sin(270 − θ) − cot(-θ)} is
(a) (2 + √13)/(2 - √13)
(b) (2 - √13)/(2 + √13)
(c)(2 + √39)/(2 - √39)
(d) 2 + 3√13

Solution :

Now, {tan(90 + θ) + cos(180 + θ)}/{sin(270 − θ) − cot(-θ)}

$= (-\cot\theta - \cos\theta)/(-\cos\theta + \cot\theta)$

$= (\cos\theta + \cot\theta)/(\cos\theta - \cot\theta)$

$= (-3/\sqrt{13} - 3/2)/(-3/\sqrt{13} + 3/2)$

$= (2 + \sqrt{13})/(2 - \sqrt{13})$

Option (a) is correct.

599. Let P be a moving point such that if PA and PB are the two tangents drawn from P to the circle $x^2 + y^2 = 1$ (A, B being the points of contact), then Angle AOB = 60, where O is origin. Then the locus of P is
(a) a circle of radius $2/\sqrt{3}$
(b) a circle of radius 2
(c) a circle of radius $\sqrt{3}$
(d) none of the foregoing curves

Solution :

P = (h, k)

Let A = (cosA, sinA) and B = (cosB, sinB)

Now, $(\cos A/\sin A)\{(\cos A - k)/(\sin A - h)\} = -1$

- ⇨ $\cos^2 A - k\cos A = -\sin^2 A + h\sin A$
- ⇨ $h\sin A + k\cos A = 1$
- ⇨ $h\tan A + k = \sec A$
- ⇨ $(h\tan A + k)^2 = \sec^2 A$
- ⇨ $h^2\tan^2 A + 2hk\tan A + k^2 = 1 + \tan^2 A$
- ⇨ $\tan^2 A(h^2 - 1) + 2hk\tan A + (k^2 - 1) = 0$
- ⇨ $\tan A + \tan B = -2hk/(h^2 - 1)$ and $\tan A\tan B = (k^2 - 1)/(h^2 - 1)$

Now, $\tan 60 = (\tan A - \tan B)/(1 + \tan A\tan B)$

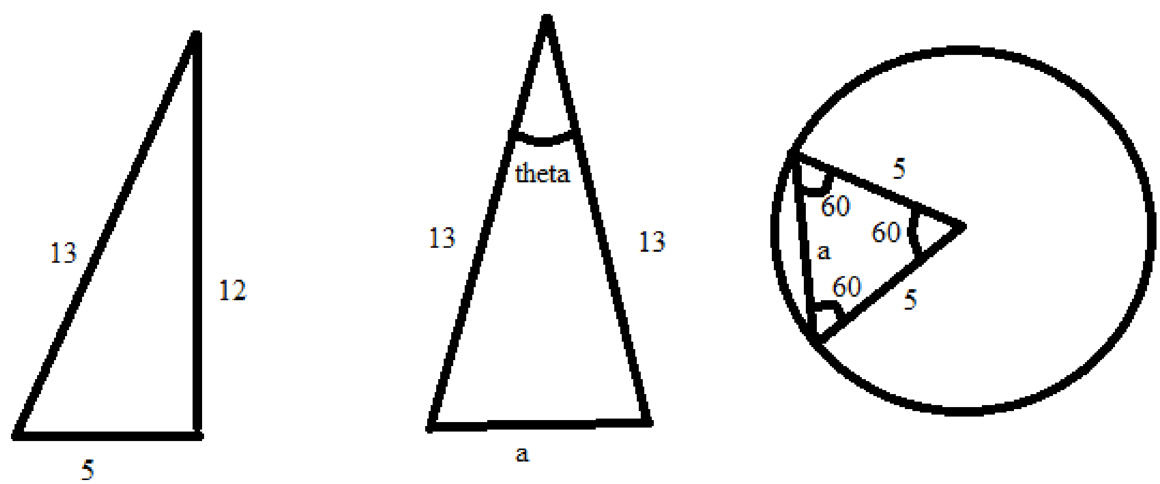
- ⇨ $3 = \{(\tan A + \tan B)^2 - 4\tan A\tan B\}/(1 + \tan A\tan B)^2$
- ⇨ $3\{1 + (k^2 - 1)/(h^2 - 1)\}^2 = \{4h^2k^2/(h^2 - 1)^2 - 4(k^2 - 1)/(h^2 - 1)\}$
- ⇨ $3(h^1 + k^2 - 2)^2 = 4\{h^2k^2 - (h^2 - 1)(k^2 - 1)\}$
- ⇨ $3(h^2 + k^2 - 2)^2 = 4(h^2k^2 - h^2k^2 + h^2 + k^2 - 1\}$
- ⇨ $3(h^2 + k^2 - 2)^2 = 4(h^2 + k^2 - 1)$
- ⇨ $3(h^2 + k^2)^2 - 12(h^2 + k^2) + 12 = 4(h^2 + k^2) - 4$
- ⇨ $3(h^2 + k^2)^2 - 16(h^2 + k^2) + 16 = 0$

- ⇨ $(h^2 + k^2) = \{16 \pm \sqrt{(256 - 4*3*16)}\}/6 = (16 \pm 8)/6 = 4, 4/3$
- ⇨ $h^2 + k^2 = 4/3$
- ⇨ Locus is $x^2 + y^2 = (2/\sqrt{3})^2$

Option (a) is correct.

600. A ring of 10 cm in diameter is suspended from a point 12 cm vertically above the centre by six equal strings. The strings are attached to the circumference of the ring at equal intervals, thus keeping the ring in a horizontal plane. The cosine of the angle between two adjacent strings is
   (a) $2/\sqrt{13}$
   (b) 313/338
   (c) $5/\sqrt{26}$
   (d) $5\sqrt{651}/338$

Solution :

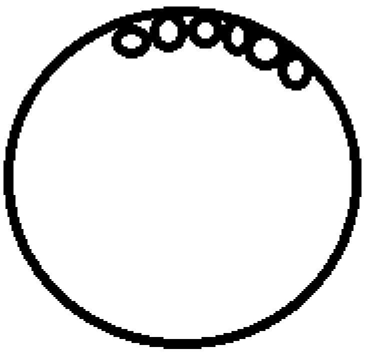


Now, from third figure, a = 5

From second figure, $\cos\theta = (13^2 + 13^2 - 5^2)/2*13*13 = 313/338$

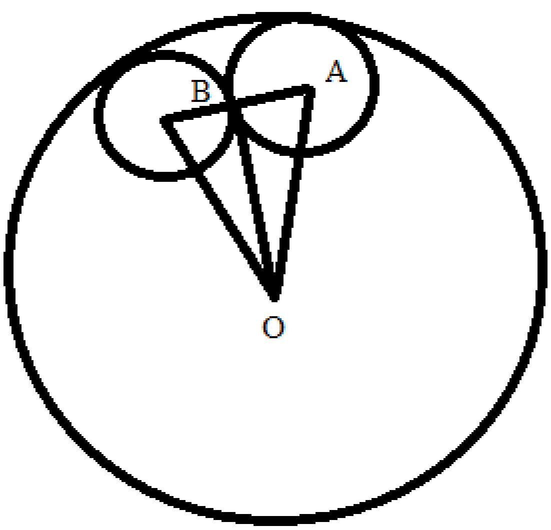
Option (b) is correct.

601. If, inside a big circle, exactly n ($n \geq 3$) small circles, each of radius r, can be drawn in such a way that each small circle touches the big circle and also touches both its adjacent circles (as shown in the picture), then the radius of the big circle is



(a) $r\text{cosec}(\pi/n)$
(b) $r(1 + \text{cosec}(2\pi/n))$
(c) $r(1 + \text{cosec}(\pi/2n))$
(d) $r(1 + \text{cosec}(\pi/n))$

Solution :



Angle AOB = $(2\pi/n)/2 = (\pi/n)$

Now, $\sin(\pi/n) = AB/OA$

- ⇨ $OA = AB\text{cosec}(\pi/n) = r\text{cosec}(\pi/n)$
- ⇨ Radius = $r + r\text{cosec}(\pi/n) = r\{1 + \text{cosec}(\pi/n)\}$

Option (d) is correct.

602. The range of values taken by $4\cos^3A - 3\cos A$ is
(a) all negative values
(b) all positive and negative values between -4/3 and +4/3
(c) all positive and negative values between -1 and +1
(d) all positive values

Solution :

$4\cos^3A - 3\cos A = \cos 3A$

Option (c) is correct.

603. If $-\pi/4 < \theta < \pi/4$ then $\cos\theta - \sin\theta$ is
(a) always negative
(b) sometimes zero
(c) always positive
(d) sometimes positive, sometimes negative

Solution :

$\cos\theta - \sin\theta = \sqrt{2}\{(1/\sqrt{2})\cos\theta - (1/\sqrt{2})\sin\theta)\} = \sqrt{2}\{\sin(\pi/4)\cos\theta - \cos(\pi/4)\sin\theta\} = \sqrt{2}\sin(\pi/4 - \theta) > 0$ as $(\pi/4 - \theta) > 0$

Option (c) is correct.

604. For all angles A $\sin 2A\cos A/(1 + \cos 2A)(1 + \cos A)$ equals
(a) $\sin A/2$
(b) $\cos A/2$
(c) $\tan A/2$
(d) $\sin A$

Solution :

$\sin 2A\cos A/(1 + \cos 2A)(1 + \cos A)$

$= 2\sin A\cos^2 A/\{(2\cos^2 A)(2\cos^2 A/2)\}$

$= 2\sin(A/2)\cos(A/2)/2\cos^2(A/2)$

$= \tan A/2$

Option (c) is correct.

605. If the angle θ with $0 < \theta < \pi/2$ is measured in radians, then cosθ always lies between
   (a) 0 and $1 - \theta^2/2$
   (b) $1 - \theta^2/2 + \theta$ and 1
   (c) $1 - \theta^2/3$ and 1
   (d) $1 - \theta^2/2$ and 1

Solution :

$\cos\theta = 1 - \theta^2/2! + \theta^4/4! - \ldots\ldots$

For small values of θ neglecting the higher power terms we get, $\cos\theta = 1 - \theta^2/2$

So, cosθ always lies between $1 - \theta^2/2$ and 1.

Option (d) is correct.

606. All possible values of x in [0, 2π] satisfying the inequality $\sin 2x < \sin x$, are given by
   (a) $\pi/3 < x < 5\pi/3$
   (b) $\pi/3 < x < 2\pi/3$ and $4\pi/3 < x < 5\pi/3$
   (c) $\pi/3 < x < \pi$ and $4\pi/3 < x < 2\pi$
   (d) $\pi/3 < x < \pi$ and $5\pi/3 < x < 2\pi$

Solution :

$\sin 2x - \sin x < 0$

⇨ $2\cos(3x/2)\sin(x/2) < 0$

Now, $\sin(x/2) > 0$ (always)

Therefore, we need to find the values of x for which $\cos(3x/2) < 0$

⇨ $\pi/2 < 3x/2 < 3\pi/2$ and $5\pi/2 < 3x/2 < 3\pi$
⇨ $\pi/3 < x < \pi$ and $5\pi/3 < x < 2\pi$

Option (d) is correct.

607. If $0 \le \alpha \le \pi/2$, then which of the following is true?
(a) $\sin(\cos\alpha) < \cos(\sin\alpha)$
(b) $\sin(\cos\alpha) \le \cos(\sin\alpha)$ and equality holds for some $\alpha \in [0, \pi/2]$
(c) $\sin(\cos\alpha) > \cos(\sin\alpha)$
(d) $\sin(\cos\alpha) \ge \cos(\sin\alpha)$ and equality holds for some $\alpha \in [0, \pi/2]$

Solution :

Clearly, option (a) is correct. Because equality will never hold. To hold the equality $\cos\alpha = \pi/4$ and $\sin\alpha = \pi/4$ and $\cos^2\alpha + \sin^2\alpha \neq 1$. Here is the contradiction.

608. The value of $\cos^4(\pi/8) + \cos^4(3\pi/8) + \cos^4(5\pi/8) + \cos^4(7\pi/8)$ is
(a) ¾
(b) $1/\sqrt{2}$
(c) $3/2$
(d) $\sqrt{3}/2$

Solution :

Now, $\cos^4(\pi/8) + \cos^4(3\pi/8) + \cos^4(5\pi/8) + \cos^4(7\pi/8) = 2\{\cos^4(\pi/8) + \cos^4(3\pi/8)\} = 2\{\cos^4(\pi/8) + \sin^4(\pi/8)\}$ (as $\pi/2 - \pi/8 = 3\pi/8$) $= 2\{1 - 2\cos^2(\pi/8)\sin^2(\pi/8)\} = 2\{1 - (1/2)\sin^2(\pi/4)\} = 2\{1 - ¼) = 2(3/4) = 3/2$

Option (c) is correct.

609. The expression $\tan\theta + 2\tan(2\theta) + 2^2\tan(2^2\theta) + \ldots\ldots + 2^{14}\tan(2^{14}\theta) + 2^{15}\cot(2^{15}\theta)$ is equal to
(a) $2^{16}\tan(2^{16}\theta)$
(b) $\tan\theta$
(c) $\cot\theta$
(d) $2^{16}[\tan(2^{16}\theta) + \cot(2^{16}\theta)]$

Solution :

Now, $2^{14}\tan(2^{14}\theta) + 2^{15}\cot(2^{15}\theta)$

$= 2^{14}\{\tan(2^{14}\theta) + 2/\tan(2^{15}\theta)\}$

$= 2^{14}[\tan(2^{14}\theta) + \{1 - \tan^2(2^{14}\theta)\}/\tan(2^{14}\theta)]$ (writing $\tan 2\theta = 2\tan\theta/(1 - \tan^2\theta)$

$= 2^{14}[1/\tan(2^{14}\theta)]$

$= 2^{14}\cot(2^{14}\theta)$

So, again $2^{13}\tan(2^{13}\theta) + 2^{14}\cot(2^{14}\theta) = 2^{13c}\cot(2^{13}\theta)$

..

..

The expression becomes, $\tan\theta + 2\cot 2\theta = \tan\theta + 2/\tan 2\theta = \tan\theta + (1 - \tan^2\theta)/\tan\theta = 1/\tan\theta = \cot\theta$

Option (c) is correct.

610. If α and β are two different solutions, lying between $-\pi/2$ and $+\pi/2$, of the equation $2\tan\theta + \sec\theta = 2$, then $\tan\alpha + \tan\beta$ is
(a) 0
(b) 1
(c) 4/3
(d) 8/3

Solution :

Now, $\sec\theta = 2(1 - \tan\theta)$

⇨ $\sec^2\theta = 2(1 - \tan\theta)^2$
⇨ $1 + \tan^2\theta = 4(1 - 2\tan\theta + \tan^2\theta)$
⇨ $3\tan^2\theta - 8\tan\theta + 3 = 0$
⇨ $\tan\alpha + \tan\beta = -(-8/3) = 8/3$ (sum of roots $= -b/a$)

Option (d) is correct.

611. Given that $\tan\theta = b/a$, the value of $a\cos 2\theta + b\sin 2\theta$ is
(a) $a^2(1 - b^2/a^2) + 2b^2$
(b) $(a^2 + b^2)/a$
(c) a
(d) $(a^2 + b^2)/a^2$

Solution :

$a\cos 2\theta + b\sin 2\theta$

$= a(1 - \tan^2\theta)/(1 + \tan^2\theta) + b2\tan\theta/(1 + \tan^2\theta)$

$= a(1 - b^2/a^2)/(1+ b^2/a^2) + b*2(b/a)/(1 + b^2/a^2)$

$= a(a^2 - b^2)/(a^2 + b^2) + 2b^2a/(a^2 + b^2)$

$= \{a/(a^2 + b^2)\}(a^2 - b^2 + 2b^2)$

$= \{a/(a^2 + b^2)\}(a^2 + b^2)$

$= a$

Option (c) is correct.

612. If $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$, then the value of $\cos(\theta - \pi/4)$ is
(a) ½
(b) $\pm 1/2\sqrt{2}$
(c) $-1/2\sqrt{2}$
(d) $1/2\sqrt{2}$

Solution :

See solution of problem 595.

Option (b) is correct.

613. If $\tan x = 2/5$, then $\sin 2x$ equals
(a) 20/29
(b) $\pm 10/\sqrt{29}$
(c) $-20/29$
(d) None of the foregoing numbers

Solution :

$\sin 2x = 2\tan x/(1 + \tan^2 x) = 2(2/5)/\{1 + (2/5)^2\} = 4*5/(5^2 + 2^2) = 20/29$

Option (a) is correct.

614. If $x = \tan 15$, then
(a) $x^2 + 2\sqrt{3}x - 1 = 0$
(b) $x^2 + 2\sqrt{3}x + 1 = 0$
(c) $x = 1/2\sqrt{3}$
(d) $x = 2/\sqrt{3}$

Solution :

Now, $\tan 30 = 2\tan 15/(1 - \tan^2 15)$

⇨ $1 - \tan^2 15 = 2\sqrt{3}\tan 15$
⇨ $\tan^2 15 + 2\sqrt{3}\tan 15 - 1 = 0$

Option (a) is correct.

615. The value of $2\sin(\theta/2)\cos(3\theta/2) + 4\sin\theta\sin^2(\theta/2)$ equals
(a) $\sin(\theta/2)$
(b) $\sin(\theta/2)\cos\theta$
(c) $\sin\theta$
(d) $\cos\theta$

Solution :

$2\sin(\theta/2)\cos(3\theta/2) + 4\sin\theta\sin^2(\theta/2)$

$= \sin 2\theta - \sin\theta + 2\sin\theta(1 - \cos\theta)$

$= \sin 2\theta - \sin\theta + 2\sin\theta - 2\sin\theta\cos\theta$

$= \sin 2\theta + \sin\theta - \sin 2\theta$

$= \sin\theta$

Option (c) is correct.

616. If a and b are given positive numbers, then the values of c and θ with $0 \le \theta \le \pi$ for which $a\sin x + b\cos x = c\sin(x + \theta)$ is true for all x are given by
(a) $c = \sqrt{(a^2 + b^2)}$ and $\tan\theta = a/b$
(b) $c = -\sqrt{(a^2 + b^2)}$ and $\tan\theta = b/a$
(c) $c = a^2 + b^2$ and $\tan\theta = b/a$
(d) $c = \sqrt{(a^2 + b^2)}$ and $\tan\theta = b/a$

Solution :

Now, $a\sin x + b\cos x = [\sqrt{(a^2 + b^2)}][\{a/\sqrt{(a^2 + b^2)}\}\sin x + \{b/\sqrt{(a^2 + b^2)}\}\cos x] = [\sqrt{(a^2 + b^2)}][\cos\theta\sin x + \sin\theta\cos x]$ (where $\cos\theta = a/\sqrt{(a^2 + b^2)}$ and $\sin\theta = b/\sqrt{(a^2 + b^2)}$ i.e. $\tan\theta = b/a$)

$= \sqrt{(a^2 + b^2)}\sin(x + \theta)$

⇨ $c = \sqrt{(a^2 + b^2)}$

Option (d) is correct.

617. The value of $\sin 330 + \tan 45 - 4\sin^2 120 + 2\cos^2 135 + \sec^2 180$ is
(a) ½
(b) $\sqrt{3}/2$
(c) $-\sqrt{3}/2$
(d) -1/2

Solution :

$\sin 330 + \tan 45 - 4\sin^2 120 + 2\cos^2 135 + \sec^2 180$

$= -\sin 30 + 1 - 4(3/4) + 2(1/2) + 1$

$= -1/2 + 1 - 3 + 1 + 1$

$= -½$

Option (d) is correct.

618. Given that $\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$, then the value of $\tan(5\pi/8)$ is
(a) $-(\sqrt{2} + 1)$
(b) $-1/(\sqrt{2} + 1)$
(c) $1 - \sqrt{2}$
(d) $1/(\sqrt{2} - 1)$

Solution :

$\tan(5\pi/8) = \tan(\pi/2 + \pi/8) = -\cot(\pi/8) = -\cos(\pi/8)/\sin(\pi/8) = -2\cos^2(\pi/8)/\{2\sin(\pi/8)\cos(\pi/8)\} = -(1 + \cos(\pi/4))/\sin(\pi/4) = -(1 + 1/\sqrt{2})/(1/\sqrt{2}) = -(\sqrt{2} + 1)$

Option (a) is correct.

619. $\sin^6(\pi/49) + \cos^6(\pi/49) - 1 + 3\sin^2(\pi/49)\cos^2(\pi/49)$ equals
(a) 0
(b) $\tan^6(\pi/49)$
(c) ½
(d) None of the foregoing numbers

Solution :

Now, $\sin^6(\pi/49) + \cos^6(\pi/49) - 1 + 3\sin^2(\pi/49)\cos^2(\pi/49)$

$= \{\sin^2(\pi/49)\}^3 + \{\cos^2(\pi/49)\}^3 + 3\sin^2(\pi/49)\cos^2(\pi/49)\{\sin^2(\pi/49) + \cos^2(\pi/49)\} - 1$

$= \{\sin^2(\pi/49) + \cos^2(\pi/49)\}^3 - 1$

$= 1^3 - 1 = 0$

Option (a) is correct.

620. If $a\sin\theta = b\cos\theta$, then the value of $\sqrt{\{(a - b)/(a + b)\}} + \sqrt{\{(a + b)/(a - b)\}}$ equals
(a) $2\cos\theta$
(b) $2\cos\theta/\sqrt{\cos 2\theta}$
(c) $2\sin\theta/\sqrt{\cos 2\theta}$
(d) $2/\sqrt{\cos 2\theta}$

Solution :

$a\sin\theta = b\cos\theta$

⇨ $\tan\theta = b/a$

Now, $\sqrt{\{(a - b)/(a + b)} + \sqrt{\{(a + b)/(a - b)\}}$

$= \sqrt{\{(1 - b/a)/(1 + b/a)\}} + \sqrt{\{(1 + b/a)/(1 - b/a)\}}$

$= \sqrt{\{(1 - \tan\theta)/(1 + \tan\theta)\}} + \sqrt{\{(1 + \tan\theta)/(1 - \tan\theta)\}}$

$= (1 - \tan\theta + 1 + \tan\theta)/\sqrt{\{(1 - \tan\theta)(1 + \tan\theta)\}}$

$= 2/\sqrt{(1 - \tan^2\theta)}$

$= 2\cos\theta/\sqrt{\cos 2\theta}$

Option (b) is correct.

621. The sides of a triangle are given to be $x^2 + x + 1$, $2x + 1$ and $x^2 - 1$. Then the largest of the three angles of the triangle is
(a) 75
(b) $\{x/(1 + x)\}\pi$
(c) 120
(d) 135

Solution :

$\cos A = \{(x^2 - 1)^2 + (2x + 1)^2 - (x^2 + x + 1)^2\}/\{2(x^2 - 1)(2x + 1)\}$

$= \{x^4 - 2x^2 + 1 + 4x^2 + 4x + 1 - x^4 - x^2 - 1 - 2x^3 - 2x^2 - 2x\}/2\{(x^2 - 1)(2x + 1)\}$

$= -(2x^3 + x^2 - 2x - 1)/2\{(x^2 - 1)(2x + 1)\}$

$= -(2x + 1)(x^2 - 1)/2\{(x^2 - 1)(2x + 1)\}$

$= -1/2$

⇨ $A = 120$

Option (c) is correct.

622. If A, B, C are angles of a triangle, then the value of $1 - \{\sin^2(A/2) + \sin^2(B/2) + \sin^2(C/2)\}$ equals

(a) 2sinAsinBsinC

(b) 2sin(A/2)sin(B/2)sin(C/2)

(c) 4sin(A/2)sin(B/2)sin(C/2)

(d) 4sinAsinBsinC

Solution :

$1 - \{\sin^2(A/2) + \sin^2(B/2) + \sin^2(C/2)\}$

$= 1 - (1/2)\{1 - \cos A + 1 - \cos B + 1 - \cos C\}$

$= -1/2 + (1/2)[2\cos\{(A + B)/2\}\cos\{(A - B)/2\} + \cos C]$

$= -1/2 + (1/2)[2\sin(C/2)\cos\{(A - B)/2\} + 1 - 2\sin^2(C/2)]$

$= \{\sin(C/2)\}[\cos\{(A - B)/2\} - \cos\{(A + B)/2\}]$

$= \{\sin(C/2)\}*2\sin(A/2)\sin(B/2)$

$= 2\sin(A/2)\sin(B/2)\sin(C/2)$

Option (b) is correct.

623. In any triangle if tan(A/2) = 5/6, tan(B/2) = 20/37, and tan(C/2) = 2/5, then

(a) a + c = 2b

(b) a + b = 2c

(c) b + c = 2a

(d) none of these holds

Solution :

$\sin A = 2\tan(A/2)/\{1 + \tan^2(A/2)\} = 2(5/6)/\{1 + 25/36\} = 60/61$

$\sin B = 2(20/37)/\{1 + 400/1369\} = 1480/1769$

$\sin C = 2(2/5)/(1 + 4/25) = 20/29$

Now, $\sin A + \sin C = (60*29 + 61*20)/(29*61) = 2960/1769 = 2(1480/1769) = 2\sin B$

⇨ a + c = 2b

Option (a) is correct.

624. Let $\cos(\alpha - \beta) = -1$. Then only one of the following statements is *always* true. Which one is it?
(a) α is not less than β
(b) $\sin\alpha + \sin\beta = 0$ and $\cos\alpha + \cos\beta = 0$
(c) Angles α and β are both positive
(d) $\sin\alpha + \sin\beta = 0$ but $\cos\alpha + \cos\beta$ may not be zero

Solution :

$\sin\alpha + \sin\beta = 2\sin\{(\alpha + \beta)/2\}\cos\{(\alpha - \beta)/2\} = 2\sin\{(\alpha + \beta)/2\}\sqrt{\{1 + \cos(\alpha - \beta)\}} = 0$

$\cos\alpha + \cos\beta = 2\cos\{(\alpha + \beta)/2\}\cos\{(\alpha - \beta)/2\} = 0$

Option (b) is correct.

625. If the trigonometric equation $1 + \sin^2 x\theta = \cos\theta$ has a nonzero solution in θ, then x must be
(a) an integer
(b) a rational number
(c) an irrational number
(d) strictly between 0 and 1

Solution :

Now, $1 + \sin^2 x\theta = \cos\theta$

⇨ $(1 - \cos\theta) + \sin^2 x\theta = 0$
⇨ $2\sin^2(\theta/2) + \sin^2 x\theta = 0$
⇨ $\sin(\theta/2) = 0$ and $\sin x\theta = 0$
⇨ $\theta = 2n\pi$ and $\theta = m\pi/x$
⇨ $m\pi/x = 2n\pi$
⇨ $x = m/2n$

Option (b) is correct.

626. It is given that tanA and tanB are the roots of the equation $x^2 - bx + c = 0$. Then value of $\sin^2(A + B)$ is
(a) $b^2/\{b^2 + (1 - c)^2\}$
(b) $b^2/(b^2 + c^2)$
(c) $b^2/(b + c)^2$
(d) $b^2/\{c^2 + (1 - b)^2\}$

Solution :

Now, $\tan A + \tan B = b$ and $\tan A\tan B = c$

$\sin^2(A + B) = (\sin A\cos B + \cos A\sin B)^2 = \cos^2 A\cos^2 B(\tan A + \tan B)^2 = b^2/\sec^2 A\sec^2 B = b^2/(1 + \tan^2 A)(1 + \tan^2 B) = b^2/(1 + \tan^2 A + \tan^2 B + \tan^2 A\tan^2 B) = b^2/\{1 + (\tan A + \tan B)^2 - 2\tan A\tan B + c^2\} = b^2/\{1 + b^2 - 2c + c^2) = b^2/\{b^2 + (1 - c)^2\}$

Option (a) is correct.

627. If $\cos x + \cos y + \cos z = 0$, $\sin x + \sin y + \sin z = 0$, then $\cos\{(x - y)/2\}$ is
(a) $\pm\sqrt{3}/2$
(b) $\pm 1/2$
(c) $\pm 1/\sqrt{2}$
(d) 0

Solution :

Now, $\cos x + \cos y = -\cos z$

⇨ $(\cos x + \cos y)^2 = \cos^2 z$ and $(\sin x + \sin y)^2 = \sin^2 z$

Adding we get, $\cos^2 x + \cos^2 y + 2\cos x\cos y + \sin^2 x + \sin^2 y + 2\sin x\sin y = \cos^2 z + \sin^2 z$

⇨ $1 + 1 + 2\cos(x - y) = 1$
⇨ $2\{1 + \cos(x - y)\} = 1$
⇨ $2\cos^2\{(x - y)/2\} = ½$
⇨ $\cos^2\{(x - y)/2\} = ¼$
⇨ $\cos\{(x - y)/2\} = \pm 1/2$

Option (b) is correct.

628. If x, y, z are in G.P. and $\tan^{-1}x$, $\tan^{-1}y$, $\tan^{-1}z$ are in A.P., then
(a) $x = y = z$ or $y = \pm 1$
(b) $z = 1/x$
(c) $x = y = z$ but their common value is not necessarily zero
(d) $x = y = z = 0$

Solution :

$\tan^{-1}x + \tan^{-1}z = 2\tan^{-1}y$

- ⇨ $\tan^{-1}\{(x + z)/(1 - zx)\} = \tan^{-1}\{2y/(1 - y^2)\}$
- ⇨ $(x + z)/(1 - y^2) = 2y/(1 - y^2)$ $(zx = y^2)$
- ⇨ $x + z = 2y$ or $y = \pm 1$ (if $y = \pm 1$ then both sides are undefined mean $\tan^{-1}$(undefined) $= \pi/2$)
- ⇨ $(x + z)^2 = 4y^2$
- ⇨ $(x + z)^2 - 4zx = 0$ $(y^2 = zx)$
- ⇨ $(z - x)^2 = 0$
- ⇨ $z = x$
- ⇨ $x = y$
- ⇨ $x = y = z$

Option (a) is correct.

629. If α and β satisfy the equation $\sin\alpha + \sin\beta = \sqrt{3}(\cos\alpha - \cos\beta)$, then
(a) $\sin3\alpha + \sin3\beta = 1$
(b) $\sin3\alpha + \sin3\beta = 0$
(c) $\sin3\alpha - \sin3\beta = 0$
(d) $\sin3\alpha - \sin3\beta = 1$

Solution :

$\sin\alpha + \sin\beta = \sqrt{3}(\cos\alpha - \cos\beta)$

- ⇨ $2\sin\{(\alpha + \beta)/2\}\cos\{(\alpha - \beta)/2\} = 2\sqrt{3}\sin\{(\alpha + \beta)/2\}\sin\{(\beta - \alpha)/2\}$
- ⇨ $\sin\{(\alpha + \beta)/2\} = 0$ or $\tan\{(\beta - \alpha)/2\} = 1/\sqrt{3}$
- ⇨ $\alpha + \beta = 0$ or $\beta - \alpha = \pi/3$

Now, $\sin3\alpha + \sin3\beta$

$= 2\sin\{3(\alpha + \beta)/2\}\cos\{3(\alpha - \beta)/2\}$

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If $\alpha + \beta = 0$ then it is equal to 0. Also if $\beta - \alpha = \pi/3$, then $\cos\{3(\beta - \alpha)/2\} = \cos(\pi/2) = 0$ ($\cos(-x) = \cos x$)

Option (b) is correct.

630. If $\cos 2\theta = \sqrt{2}(\cos\theta - \sin\theta)$, then $\tan\theta$ is
   (a) $1/\sqrt{2}$, $-1/\sqrt{2}$ or 1
   (b) 1
   (c) 1 or -1
   (d) None of the foregoing values

Solution :

Now, $\cos 2\theta = \sqrt{2}(\cos\theta - \sin\theta)$

- ⇨ $\cos^2 2\theta = 2(\cos^2\theta + \sin^2\theta - 2\sin\theta\cos\theta)$
- ⇨ $1 - \sin^2 2\theta = 2(1 - \sin 2\theta)$
- ⇨ $\sin^2 2\theta - 2\sin 2\theta + 1 = 0$
- ⇨ $(\sin 2\theta - 1)^2 = 0$
- ⇨ $\sin 2\theta = 1$
- ⇨ $2\tan\theta/(1 + \tan^2\theta) = 1$
- ⇨ $\tan^2\theta - 2\tan\theta + 1 = 0$
- ⇨ $(\tan\theta - 1)^2 = 0$
- ⇨ $\tan\theta = 1$

Option (b) is correct.

631. The number of roots between 0 and π of the equation $2\sin^2 x + 1 = 3\sin x$ equals
   (a) 2
   (b) 4
   (c) 1
   (d) 3

Solution :

Now, $2\sin^2 x + 1 = 3\sin x$

- ⇨ $2\sin^2 x - 3\sin x + 1 = 0$
- ⇨ $(2\sin x - 1)(\sin x - 1) = 0$

- ⇨ $\sin x = ½$ or $\sin x = 1$
- ⇨ $x = π/6, π – π/6, x = π/2$

Option (d) is correct.

632. The equation in θ given by $\operatorname{cosec}^2θ – (2\sqrt{3}/3)\operatorname{cosec}θ\secθ – \sec^2θ = 0$ has solutions
   (a) only in the first and third quadrants
   (b) only in the second and fourth quadrants
   (c) only in the third quadrant
   (d) in all the four quadrants

Solution :

Now, $\operatorname{cosec}^2θ – (2\sqrt{3}/3)\operatorname{cosec}θ\secθ – \sec^2θ = 0$

- ⇨ $3\operatorname{cosec}^2θ/\sec^2θ - 2\sqrt{3}\operatorname{cosec}θ/\secθ – 3 = 0$
- ⇨ $3\cot^2θ - 2\sqrt{3}\cotθ – 3 = 0$
- ⇨ $\cotθ = \{2\sqrt{3} ± \sqrt{(12 + 36)}\}/6 = \sqrt{3}, -1/\sqrt{3}$

Option (d) is correct.

633. If $\tanθ + \cotθ = 4$, then θ, for some integer n, is
   (a) $nπ/2 + (-1)^n(π/12)$
   (b) $nπ + (-1)^n(π/12)$
   (c) $nπ + π/12$
   (d) $nπ – π/12$

Solution :

Now, $\tanθ + \cotθ = 4$

- ⇨ $(\sin^2θ + \cos^2θ)/(\sinθ\cosθ) = 4$
- ⇨ $1/(2\sinθ\cosθ) = 2$
- ⇨ $\sin2θ = ½$
- ⇨ $\sin2θ = \sin(π/6)$
- ⇨ $2θ = nπ + (-1)^n(π/6)$
- ⇨ $θ = nθ/2 + (-1)^n(π/12)$

Option (a) is correct.

634. The equation sinx(sinx + cosx) = k has real solutions if and only if k is a real number such that

(a) $0 \leq k \leq (1 + \sqrt{2})/2$

(b) $2 - \sqrt{3} \leq k \leq 2 + \sqrt{3}$

(c) $0 \leq k \leq 2 - \sqrt{3}$

(d) $(1 - \sqrt{2})/2 \leq k \leq (1 + \sqrt{2})/2$

Solution :

Now, sinx(sinx + cosx) = k

⇨ $2\sin^2 x + 2\sin x \text{coxs} = 2k$
⇨ $1 - \cos 2x + \sin 2x = 2k$
⇨ $\sin 2x - \cos 2x = 2k - 1$
⇨ $\sin^2 2x + \cos^2 2x - 2\sin 2x\cos 2x = (2k - 1)^2$
⇨ $1 - \sin 4x = 4k^2 - 4k + 1$
⇨ $\sin 4x = 4k - 4k^2$

Now, $\sin 4x \leq 1$

⇨ $4k - 4k^2 \leq 1$
⇨ $4k^2 - 4k + 1 \geq 0$
⇨ $(2k - 1)^2 \geq 0$ which is obvious

Now, $-1 \leq \sin 4x$

⇨ $-1 \leq 4k - 4k^2$
⇨ $4k^2 - 4k - 1 \leq 0$
⇨ $(2k - 1)^2 \leq 2$
⇨ $|2k - 1| \leq \sqrt{2}$
⇨ $-\sqrt{2} \leq 2k - 1 \leq \sqrt{2}$
⇨ $(1 - \sqrt{2})/2 \leq k \leq (1 + \sqrt{2})/2$

Option (d) is correct.

635. The number of solutions of the equation $2\sin\theta + 3\cos\theta = 4$ for $0 \leq \theta \leq 2\pi$ is

(a) 0

(b) 1

(c) 2

(d) More than 2

Solution :

$2\sin\theta + 3\cos\theta = 4$

- ⇨ $2\tan\theta + 3 = 4\sec\theta$
- ⇨ $(2\tan\theta + 3)^2 = 16\sec^2\theta$
- ⇨ $4\tan^2\theta + 12\tan\theta + 9 = 16 + 16\tan^2\theta$
- ⇨ $12\tan^2\theta - 12\tan\theta + 7 = 0$

Now, discriminate = 144 – 4*12*7 < 0

- ⇨ No real solution.

Option (a) is correct.

636. The number of values of x satisfying the equation $\sqrt{\sin x} - 1/\sqrt{\sin x} = \cos x$ is
(a) 1
(b) 2
(c)3
(d) More than 3

Solution :

Now, $\sqrt{\sin x} - 1/\sqrt{\sin x} = \cos x$

- ⇨ $\sin x - 1 = \cos x\sqrt{\sin x}$
- ⇨ $\sin^2 x - 2\sin x + 1 = \cos^2 x\sin x$
- ⇨ $\sin^2 x - 2\sin x + 1 = \sin x - \sin^3 x$
- ⇨ $\sin^3 x + \sin^2 x - 3\sin x + 1 = 0$
- ⇨ $(\sin x - 1)(\sin^2 x + 2\sin x - 1) = 0$
- ⇨ $\sin x = 1$ or $\sin^2 x + 2\sin x - 1 = 0$
- ⇨ $\sin x = \{-2 \pm \sqrt{(4 + 4)}\}/2 = -1 \pm \sqrt{2}$
- ⇨ $\sin x = \sqrt{2} - 1$, $\sin x \neq -1 - \sqrt{2}$ as $\sin x > -1$
- ⇨ 2 values in $0 < x \leq \pi/2$

Option (d) is correct. (as there is no boundary for x specified)

637. The number of times the function $f(x) = |\text{minimum}\{\sin x, \cos x\}|$ takes the value 0.8 between $20\pi/3$ and $43\pi/6$ is
(a) 2
(b) More than 2

(c)0
(d) 1

Solution :

It never can happen because if sinx > 0.5 then cosx < 0.5 or if cosx > 0.5, then sinx < 0.5

Option (a) is correct.

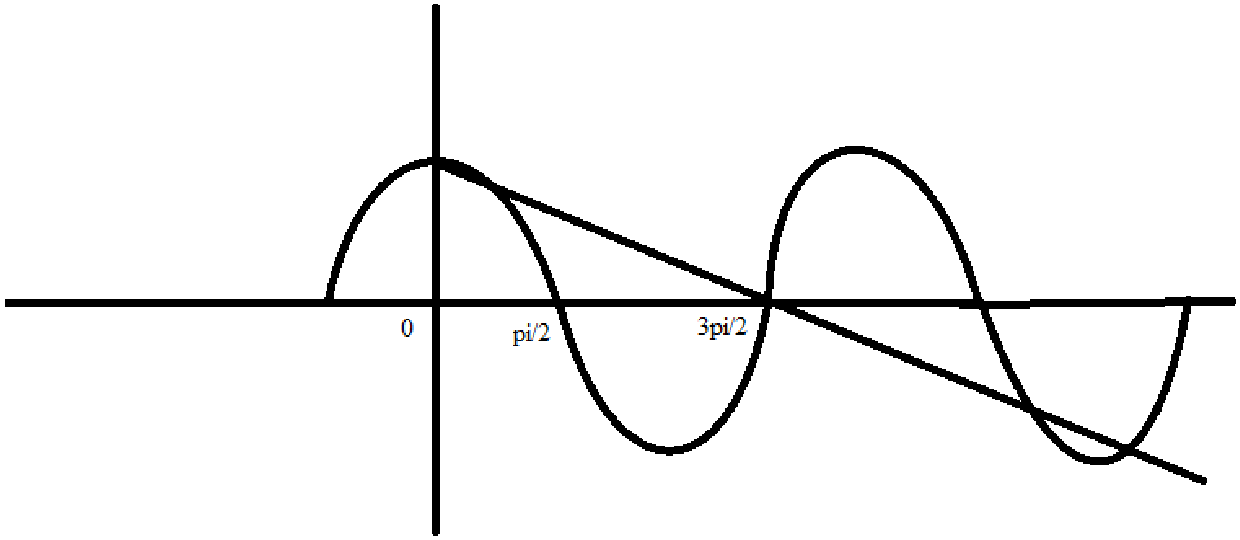
638. The number of roots of the equation $2x = 3\pi(1 - \cos x)$, where x is measured in radians, is
(a) 3
(b) 5
(c)4
(d) 2

Solution :

Now, $2x = 3\pi(1 - \cos x)$

⇨ $\cos x = 1 - 2x/3\pi$

Now, we will draw the graph of $y = \cos x$ and $y = 1 - 2x/3\pi$ and see the number of intersection point. That will give number of solutions.



Option (b) is correct.

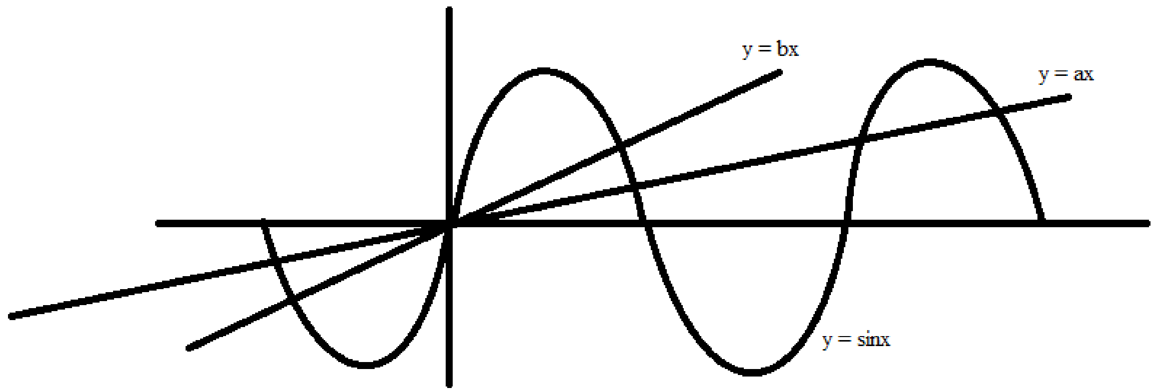
639. Let $f(x) = \sin x - ax$ and $g(x) = \sin x - bx$, where $0 < a, b < 1$. Suppose that the number of real roots of $f(x) = 0$ is greater than that of $g(x) = 0$. Then
(a) $a < b$
(b) $a > b$
(c) $ab = \pi/6$
(d) none of the foregoing relations hold

Solution :

$f(x) = 0$

⇨ $\sin x = ax$

Now to see number of real roots of this equation we will draw curves of $y = \sin x$ and $y = ax$ and see number of intersection point that will give number of solutions.



Now, this must be the scenario to have f(x) = 0 more roots than g(x) = 0. So a < b.

Option (a) is correct.

640. The number of solutions $0 < \theta < \pi/2$ of the equation $\sin7\theta - \sin\theta = \sin3\theta$ is
   (a) 1
   (b) 2
   (c) 3
   (d) 4

Solution :

Now, $\sin7\theta - \sin\theta = \sin3\theta$

- ⇨ $2\cos4\theta\sin3\theta - \sin3\theta = 0$
- ⇨ $\sin3\theta(2\cos4\theta - 1) = 0$
- ⇨ $\sin3\theta = 0$ or $\cos4\theta = ½$
- ⇨ $3\theta = \pi$ or $4\theta = \pi/6, 2\pi - \pi/6$
- ⇨ $\theta = \pi/3$ or $\theta = \pi/24, \pi/2 - \pi/24$
- ⇨ 3 solutions

Option (c) is correct.

641. The number of solutions of the equation $\tan5\theta = \cot2\theta$ such that $0 \le \theta \le \pi/2$ is

(a) 1
(b) 2
(c) 3
(d) 4

Solution :

tan5θ = cot2θ

- ⇨ tan5θ = tan{(2n – 1)π/2 - 2θ}
- ⇨ 5θ = (2n – 1)π/2 - 2θ
- ⇨ 7θ = (2n – 1)π/2
- ⇨ θ = (2n – 1)π/14
- ⇨ θ = π/14, 3π/14, 5π/14, 7π/14
- ⇨ 4 solutions.

Option (d) is correct.

642. If $\sin^{-1}(1/\sqrt{5})$ and $\cos^{-1}(3/\sqrt{10})$ are angles in [0, π/2], then their sum is equal to
(a) π/6
(b) π/4
(c) π/3
(d) $\sin^{-1}(1/\sqrt{50})$

Solution :

Let $\sin^{-1}(1/\sqrt{5})$ = A

- ⇨ sinA = 1/√5
- ⇨ cosA = 2/√5

Let, $\cos^{-1}(3/\sqrt{10})$ = B

- ⇨ cosB = 3/√10
- ⇨ sinB = 1/√10

Now, sin(A + B) = sinAcosB + cosAsinB = (1/√5)(3/√10) + (2/√5)(1/√10) = 5/√50 = 1/√2

- ⇨ A + B = π/4

Option (b) is correct.

643. If $\cot(\sin^{-1}\sqrt{(13/17)}) = \sin(\tan^{-1}\alpha)$, then α is

(a) $4/17$
(b) $\sqrt{\{(17^2 - 13^2)/(17*13)\}}$
(c) $\sqrt{\{(17^2 - 13^2)/(17^2 + 13^2)\}}$
(d) $2/3$

Solution :

Now, $\cot(\sin^{-1}\sqrt{(13/17)}) = \sin(\tan^{-1}\alpha)$

⇨ $\cot(\cot^{-1}2/\sqrt{13}) = \sin[\sin^{-1}\{\alpha/\sqrt{(1 + \alpha^2)}\}]$
⇨ $2/\sqrt{13} = \alpha/\sqrt{(1 + \alpha^2)}$
⇨ $4/13 = \alpha^2/(1 + \alpha^2)$
⇨ $1 - 4/13 = 1 - \alpha^2/(1 + \alpha^2)$
⇨ $9/13 = 1/(1 + \alpha^2)$
⇨ $1 + \alpha^2 = 13/9$
⇨ $\alpha^2 = 4/9$
⇨ $\alpha = 2/3$

Option (d) is correct.

644. The minimum value of $\sin2\theta - \theta$ for $-\pi/2 \le \theta \le \pi/2$ is

(a) $-\sqrt{3}/2 + \pi/6$
(b) $-\pi$
(c) $\sqrt{3}/2 - \pi/6$
(d) $-\pi/2$

Solution :

Let $f(\theta) = \sin2\theta - \theta$

⇨ $f'(\theta) = 2\cos2\theta - 1 = 0$
⇨ $\cos2\theta = ½$
⇨ $2\theta = \pi/3, -\pi/3$
⇨ $\theta = \pi/6, -\pi/6$
⇨ $f''(\theta) = -4\sin2\theta > 0$ for $\theta = -\pi/6$

Minimum value of $f(\theta) = f(-\pi/6) = -\sin(\pi/3) + \pi/6 = -\sqrt{3}/2 + \pi/6$

Option (a) is correct.

645. The number of solutions θ in the range $-\pi/2 < \theta < \pi/2$ and satisfying the equation $\sin^3\theta + \sin^2\theta + \sin\theta - \sin\theta\sin2\theta - \sin2\theta - 2\cos\theta = 0$ is

(a) 0
(b) 1
(c) 2
(d) 3

Solution :

Now, $\sin^3\theta + \sin^2\theta + \sin\theta - \sin\theta\sin2\theta - \sin2\theta - 2\cos\theta = 0$

⇨ $\sin\theta(\sin^2\theta + \sin\theta + 1) - 2\cos\theta(\sin^2\theta + \sin\theta + 1) = 0$
⇨ $(\sin^2\theta + \sin\theta + 1)(\sin\theta - 2\cos\theta) = 0$
⇨ $\sin\theta - 2\cos\theta = 0$ (as $\sin^2\theta + \sin\theta + 1 = 0$ has imaginary roots)
⇨ $\tan\theta = 2$
⇨ 1 solution.

Option (b) is correct.

646. The number of roots of the equation $\cos^8\theta - \sin^8\theta = 1$ in the interval $[0, 2\pi]$ is

(a) 4
(b) 8
(c) 3
(d) 6

Solution :

Now, $\cos^8\theta - \sin^8\theta = 1$

⇨ $(\cos^4\theta - \sin^4\theta)(\sin^4\theta + \cos^4\theta) = 1$
⇨ $(\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)\{(\cos^2\theta + \sin^2\theta)^2 - 2\cos^2\theta\sin^2\theta) = 1$
⇨ $\cos2\theta(1 - \sin^22\theta/2) = 1$
⇨ $\cos2\theta(1 + \cos^22\theta) = 2$
⇨ $\cos^32\theta + \cos2\theta - 2 = 0$
⇨ $(\cos2\theta - 1)(\cos^22\theta + \cos2\theta + 2) = 0$
⇨ $\cos2\theta = 1$ (as $\cos^22\theta + \cos2\theta + 2 = 0$ gives imaginary root)
⇨ $2\theta = 0, 2\pi, 4\pi$
⇨ $\theta = 0, \pi, 2\pi$

⇨ 3 solutions.

Option (c) is correct.

647. If $\sin6\theta = \sin4\theta - \sin2\theta$, then θ must be, for some integer n, equal to
(a) $n\pi/4$
(b) $n\pi \pm \pi/6$
(c) $n\pi/4$ or $n\pi \pm \pi/6$
(d) $n\pi/2$

Solution :

Now, $\sin6\theta = \sin4\theta - \sin2\theta$

⇨ $\sin6\theta + \sin2\theta = \sin4\theta$
⇨ $2\sin4\theta\cos2\theta - \sin4\theta = 0$
⇨ $\sin4\theta(2\cos2\theta - 1) = 0$
⇨ $\sin4\theta = 0$ or $\cos2\theta = ½$
⇨ $4\theta = n\pi$ or $2\theta = 2n\pi \pm \pi/3$
⇨ $\theta = n\pi/4$ or $\theta = n\pi \pm \pi/6$

Option (c) is correct.

648. Consider the solutions of the equation $\sqrt{2}\tan^2x - \sqrt{10}\tan x + \sqrt{2} = 0$ in the range $0 \le x \le \pi/2$. Then only one of the following statements is true. Which one is it?
(a) No solutions for x exist in the given range
(b) Two solutions x_1 and x_2 exist with $x_1 + x_2 = \pi/4$
(c) Two solutions x_1 and x_2 exist with $x_1 - x_2 = \pi/4$
(d) Two solutions x_1 and x_2 exist with $x_1 + x_2 = \pi/2$

Solution :

Now, $\tan x = \{\sqrt{10} \pm \sqrt{(10 - 8)}\}/2\sqrt{2} = (\sqrt{5} \pm 1)/2$

⇨ Two solutions exist.

Now, $\tan x_1 + \tan x_2 = \sqrt{5}$ and $\tan x_1\tan x_2 = 1$

Now, $\tan(x_1 + x_2) = (\tan x_1 + \tan x_2)/(1 - \tan x_1\tan x_2) = \sqrt{5}/(1 - 1)$

⇨ $x_1 + x_2 = \pi/2$

Option (d) is correct.

649. The set of all values of θ which satisfy the equation $\cos 2\theta = \sin\theta + \cos\theta$ is
(a) $\theta = 0$
(b) $\theta = n\pi + \pi/2$, where n is any integer
(c) $\theta = 2n\pi$ or $\theta = 2n\pi - \pi/2$ or $\theta = n\pi - \pi/4$, where n is any integer
(d) $\theta = 2n\pi$ or $\theta = n\pi + \pi/4$, where n is any integer

Solution :

$\cos 2\theta = \sin\theta + \cos\theta$

Clearly the values of option (c) satisfies the equation.

Therefore, option (c) is correct.

650. The equation $2x = (2n + 1)\pi(1 - \cos x)$, where n is a positive integer, has
(a) infinitely many real solutions
(b) exactly $2n + 1$ real roots
(c) exactly one real root
(d) exactly $2n + 3$ real roots

Solution :

If we take $x = (2n + 1)\pi$ then the equation gets satisfied where n is any positive integer.

So, it should have infinitely many real solutions.

But option (d) is given as correct.

651. The number of roots of the equation $\sin 2x + 2\sin x - \cos x - 1 = 0$ in the range $0 \le x \le 2\pi$ is
(a) 1
(b) 2

(c) 3
(d) 4

Solution :

$\sin 2x + 2\sin x - \cos x - 1 = 0$

- ⇨ $2\sin x(\cos x + 1) + (\cos x + 1) = 0$
- ⇨ $(\cos x + 1)(2\sin x + 1) = 0$
- ⇨ $\cos x = -1$ or $\sin x = -1/2$
- ⇨ $x = \pi$ or $x = \pi + \pi/6,\ 2\pi - \pi/6$
- ⇨ 3 solutions

Option (c) is correct.

652. If $2\sec 2\alpha = \tan\beta + \cot\beta$, then one possible value of $\alpha + \beta$ is
(a) $\pi/2$
(b) $\pi/4$
(c) $\pi/3$
(d) 0

Solution :

$2\sec 2\alpha = \tan\beta + \cot\beta$

- ⇨ $2\sec 2\alpha = \tan\beta + 1/\tan\beta = (1 + \tan^2\beta)/\tan\beta$
- ⇨ $\sec 2\alpha = 1/\{2\tan\beta/(1 + \tan^2\beta)\}$
- ⇨ $\sec 2\alpha = 1/\sin 2\beta$
- ⇨ $\sin 2\beta = \cos 2\alpha$
- ⇨ $\sin 2\beta - \sin(\pi/2 - 2\alpha) = 0$
- ⇨ $2\cos(\beta - \alpha + \pi/4)\sin(\alpha + \beta - \pi/4) = 0$
- ⇨ $\alpha + \beta = \pi/4$

Option (b) is correct.

653. The equation $[3\sin^4\theta - 2\cos^6\theta + y - 2\sin^6\theta + 3\cos^4\theta]^2 = 9$ is true
(a) for any value of θ and $y = 2$ or -4
(b) only for $\theta = \pi/4$ or π and $y = -2$ or 4
(c) only for $\theta = \pi/2$ or π and $y = 2$ or -4

(d) only for $\theta = 0$ or $\pi/2$ and $y = 2$ or -2

Solution :

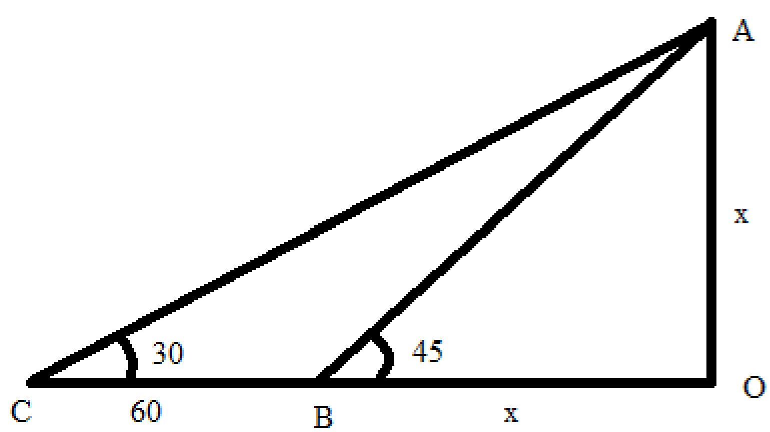
$[3\{(\cos^2\theta + \sin^2\theta)^2 - 2\sin^2\theta\cos^2\theta\} + y - 2\{(\cos^2\theta + \sin^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\cos^2\theta + \sin^2\theta)\}]^2 = 9$

- ⇨ $[3(1 - \sin^2 2\theta/2) + y - 2(1 - 3\sin^2 2\theta/4]^2 = 9$
- ⇨ $(1 + y)^2 = 9$
- ⇨ For any value of θ and $y = 2$ or -4

Option (a) is correct.

654. If the shadow of a tower standing on the level plane is found to be 60 feet (ft) longer when the sun's altitude is 30 than when it is 45, then the height of the tower is, in ft,
(a) $30(1 + \sqrt{3}/2)$
(b) 45
(c) $30(1 + \sqrt{3})$
(d) 30

Solution :



From triangle OAB, we get, $x/(60 + x) = \tan 30 = 1/\sqrt{3}$

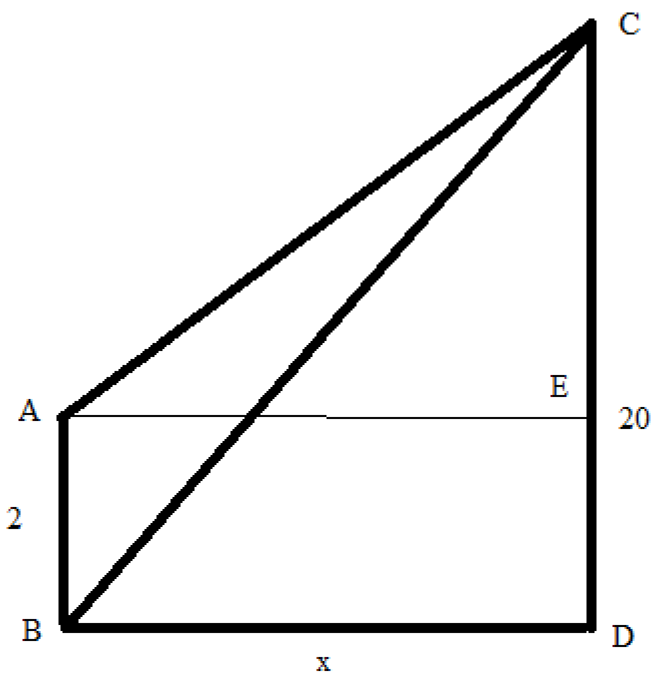
- ⇨ $\sqrt{3}x = 60 + x$
- ⇨ $x(\sqrt{3} - 1) = 60$
- ⇨ $x = 60/(\sqrt{3} - 1) = 60(\sqrt{3} + 1)/2 = 30(\sqrt{3} + 1)$

Option (c) is correct.

655. Two poles, AB of length 2 metres and CD of length 20 metres are erected vertically with bases at B and D. The two poles are at a distance not less than twenty metres. It is observed that tan(ACB) = 2/77. The distance between the two poles, in metres, is

(a) 72
(b) 68
(c) 24
(d) 24.27

Solution :



From triangle BCD we get, tan(BCD) = x/20

From triangle AEC we get, tan(ACE) = x/18

Now, tan(ACB) = tan(ACE – BCD) = {tan(ACE) – tan(BCD)}/{1 + tan(ACE)tan(BCD)}

- ⇨ 2/77 = (x/18 – x/20)/{1 + (x/18)(x/20)}
- ⇨ 2/77 = 2x/(360 + x^2)
- ⇨ 720 + $2x^2$ = 154x

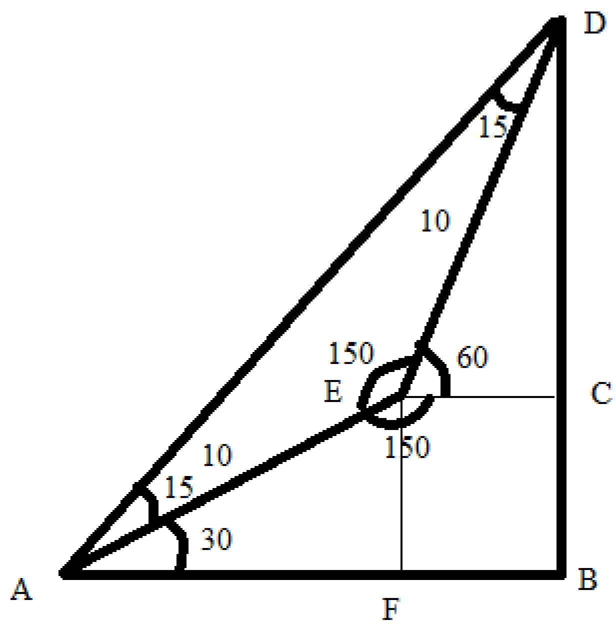
⇨ $2x^2 - 154x + 720 = 0$
⇨ $x^2 - 77x + 360 = 0$
⇨ $(x - 72)(x - 5) = 0$
⇨ $x = 72$ ($x \neq 5$ as distance between the poles greater than 20 metres)

Option (a) is correct.

656. The elevation of the top of a tower from a point A is 45. From A, a man walks 10 metres up a path sloping at an angle of 30. After this the slope becomes steeper and after walking up another 10 metres the man reaches the top. Then the distance of A from the foot of the tower is

(a) $5(\sqrt{3} + 1)$ metres
(b) 5 metres
(c) $10\sqrt{2}$ metres
(d) $5\sqrt{2}$ metres

Solution :



From quadrilateral ABCE, Angle E = 150

From triangle AED, Angle E = 150

Therefore, Angle DEC = 360 – (150 + 150) = 60

Now, from triangle DCE, we get, CE/DE = cos60

⇨ CE = 10*(1/2) = 5

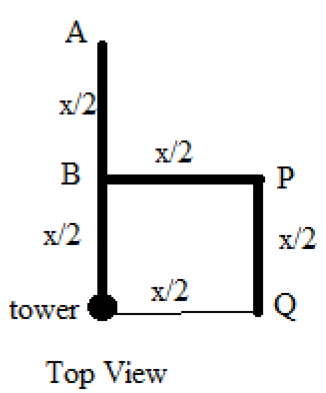
From triangle AEF, we get, AF/AE = cos30

⇨ AF = 10(√3/2) = 5√3
⇨ AB = AF + BF = AF + CE = 5√3 + 5 = 5(√3 + 1)

Option (a) is correct.

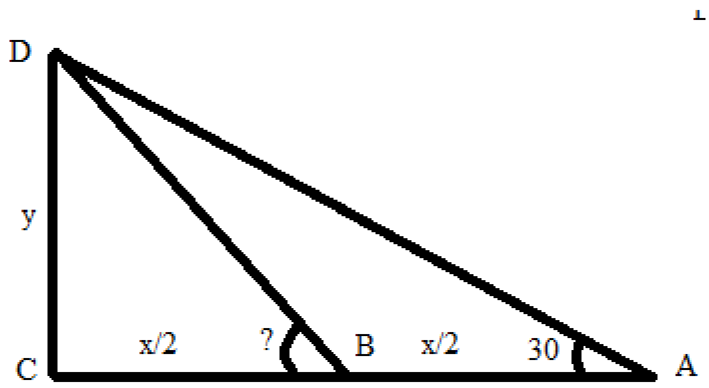
657. A man standing x metres to the north of a tower finds the angle of elevation of its top to be 30. He then starts walking towards the tower. After walking a distance of x/2 metres, he turns east and walks x/2 metres. Then again he turns south and walks x/2 metres. The angle of elevation of the top of the tower from his new position is
(a) 30
(b) $\tan^{-1}\sqrt{(2/3)}$
(c) $\tan^{-1}(2/\sqrt{3})$
(d) none of the foregoing quantities

Solution :



Clearly, from the above figure the angle of elevation at point Q = angle of elevation at point B.

Therefore, we draw the following picture.



From triangle ACD we get, CD/AC = tan30

⇨ $y/x = 1/\sqrt{3}$

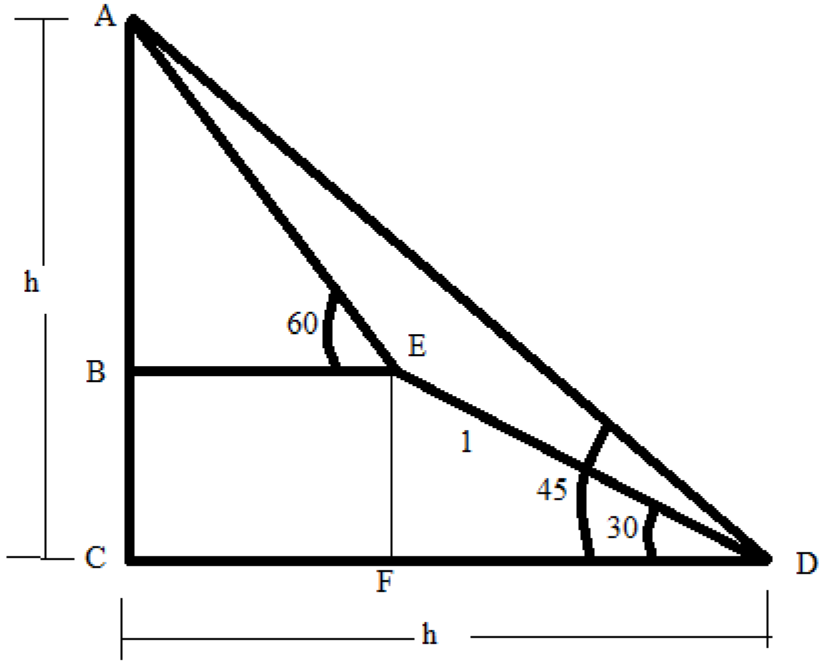
Now, from triangle BCD we get, CD/BC = tanθ

⇨ $\tan\theta = y/(x/2) = 2(y/x) = 2/\sqrt{3}$
⇨ $\theta = \tan^{-1}(2/\sqrt{3})$

Option (c) is correct.

658. The elevation of the summit of mountain is found to be 45. After ascending one km the summit up a slope of 30 inclination, the elevation is found to be 60. Then the height of the mountain is, in km,
(a) $(\sqrt{3} + 1)/(\sqrt{3} - 1)$
(b) $(\sqrt{3} - 1)/(\sqrt{3} + 1)$
(c) $1/(\sqrt{3} - 1)$
(d) $1/(\sqrt{3} + 1)$

Solution :



From triangle DEF we get, EF/DE = sin30

- ⇨ EF = 1*(1/2) = ½
- ⇨ DF = DEcos30 = 1*(√3/2) = √3/2
- ⇨ CF = (h - √3/2) and AB = h – ½

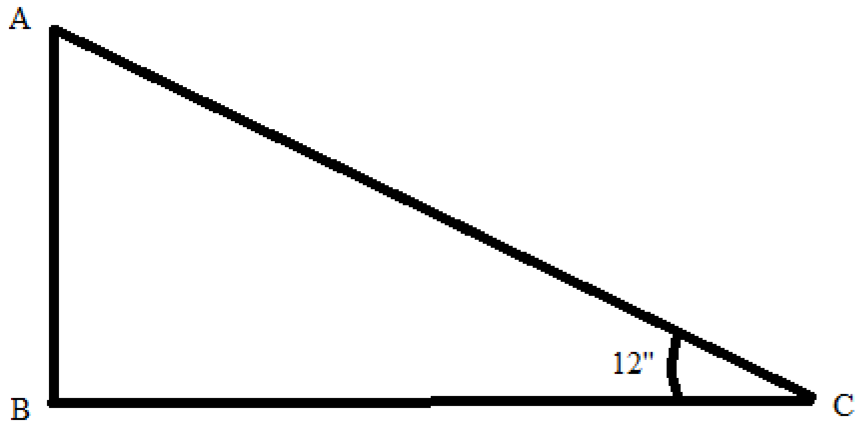
From triangle, ABE we get, AB/BE = tan60

- ⇨ (h – ½)/(h - √3/2) = √3
- ⇨ h – ½ = √3h – 3/2
- ⇨ h(√3 – 1) = 1
- ⇨ h = 1/(√3 – 1)

Option (c) is correct.

659. The distance at which a vertical pillar, of height 33 feet, subtends an angle of 12′′ (that is, 12 seconds) is, approximately in yards (1 yard = 3 feet),
(a) 11000000/6π
(b) 864000/11π
(c) 594000/π
(d) 864000/π

Solution :



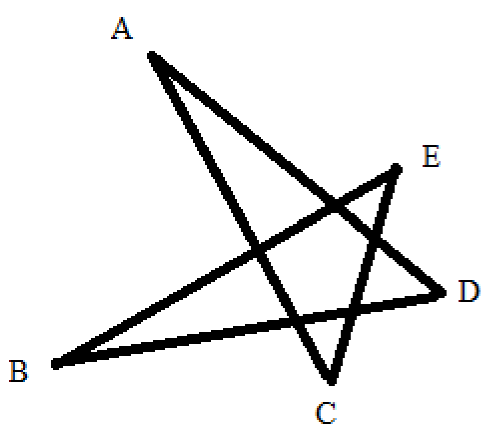
12'' = (12/3600)*(π/180) = π/54000 radian

AB/BC = tan(π/54000)

⇨ BC = AB/tan(π/54000) = 33/(π/54000) (approx.) = 33*54000/π feet = (1/3)(33*54000/π) yard = 584000/π

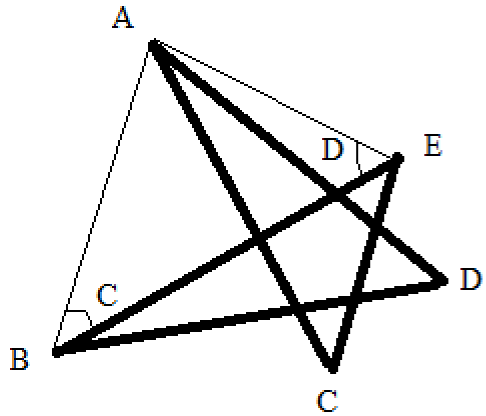
Option (c) is correct.

660. If the points A, B, C, D and E in the figure lie on a circle, then AD/BE



(a) equals sin(A + D)/sin(B + E)
(b) equals sinB/sinD
(c) equals sin(B + C)/sin(C + D)
(d) cannot be found unless the radius of the circle is given

Solution :



Now, angle ABE = C (both are on same arc AE)

Similarly, Angle AEB = D (both are on same arc AB)

From triangle ABE we get, BE/sin{180 – (C + D)} = AC/sinD

⇨ BE/sin(C + D) = AC/sinD

From triangle ABD we get, AD/sin(B + C) = AC/sinD

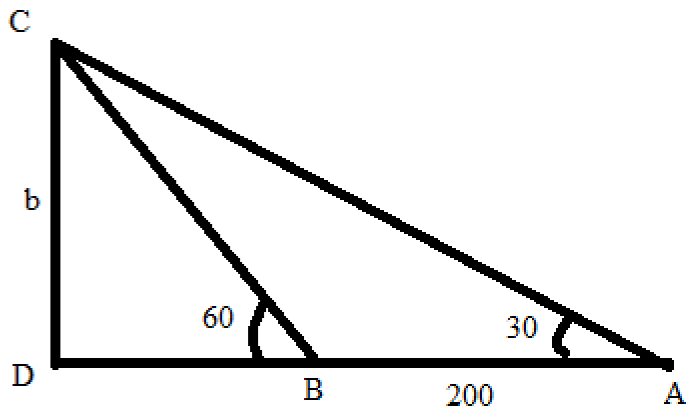
⇨ BE/sin(C + D) = AD/sin(B + C)
⇨ AD/BE = sin(B + C)/sin(C + D)

Option (c) is correct.

661. A man stands at a point A on the bank AB of a straight river and observes that the line joining A to a post C on the opposite bank makes with AB an angle of 30. He then goes 200 metres along the bank to B, finds that BC makes an angle of 60 with the bank. If b is breadth of the river, then

(a) 50√3 is the only possible value of b
(b) 100√3 is the only possible value of b
(c) 50√3 and 100√3 are the only possible values of b
(d) None of the foregoing statements is correct.

Solution :



From triangle ADC we get, CD/AD = tan30

⇨ AD = b√3

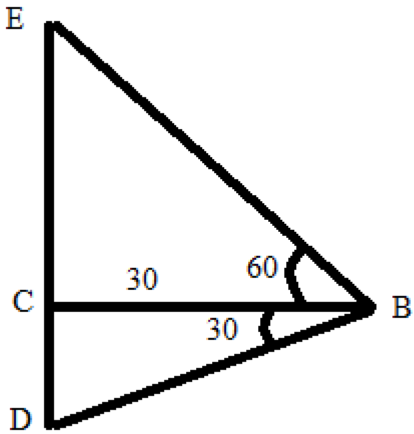
From triangle BDC we get, CD/BD = tan60

⇨ BD = b/√3
⇨ AD – BD = b√3 – b/√3
⇨ AB = b(2/√3)
⇨ 200 = b(2/√3)
⇨ b = 100√3

Option (b) is correct.

662. A straight pole A subtends a right angle at a point B of another pole at a distance of 30 metres from A, the top of A being 60 above the horizontal line joining the point B to the point A. The length of the pole A is, in metres,
(a) 20√3
(b) 40√3
(c) 60√3
(d) 40/√3

Solution :



From triangle BCE we get, EC/BC = tan60

⇨ EC = $30\sqrt{3}$

From triangle BCD we get, CD/BC = tan30

⇨ CD = $30/\sqrt{3} = 10\sqrt{3}$

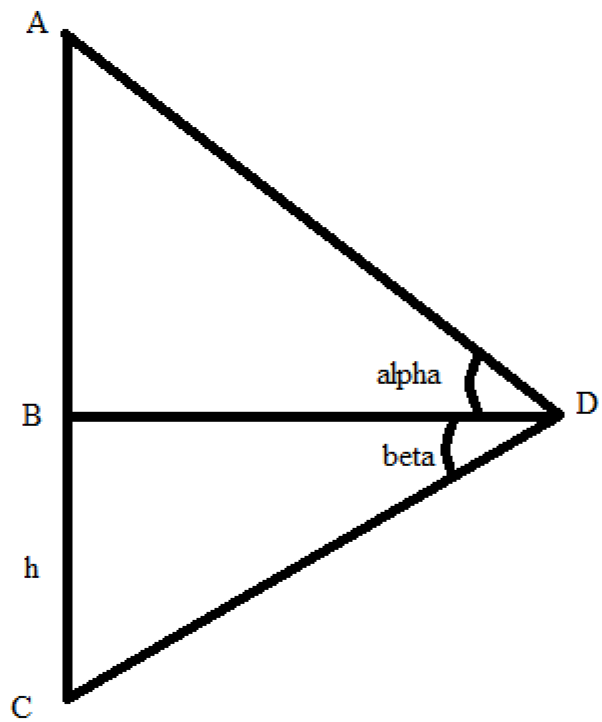
Therefore, length of the pole A = DE = CD + EC = $10\sqrt{3} + 30\sqrt{3} = 40\sqrt{3}$

Option (b) is correct.

663. The angle of elevation of a bird from a point h metres above a lake is α and the angle of depression of its image in the lake from the same point is β. The height of the bird above the lake is, in metres,

(a) $h\sin(\beta - \alpha)/(\sin\beta\cos\alpha)$
(b) $h\sin(\beta + \alpha)/(\sin\alpha\cos\beta)$
(c) $h\sin(\beta - \alpha)/\sin(\alpha + \beta)$
(d) $h\sin(\beta + \alpha)/\sin(\beta - \alpha)$

Solution :



From triangle BCD we get, BC/BD = tanβ

⇨ BD = h/tanβ

From triangle ABD we get, AB/BD = tanα

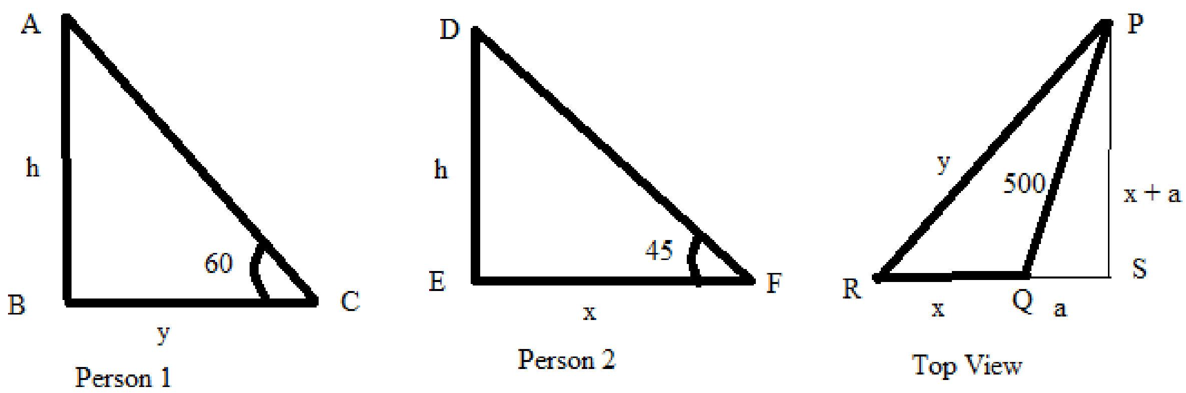
⇨ AB = htanα/tanβ

Height of the bird from lake = AC = AB + BC = htanα/tanβ + h = h{tanα/tanβ + 1) = h{sinαcosβ/cosαsinβ + 1) = h{(sinαcosβ + cosαsinβ)/(cosαsinβ)} = hsin(β + α)/(cosαsinβ)

It is given that option (d) is correct.

664. Two persons who are 500 metres apart, observe the direction and the angular elevation of a balloon at the same instant. One finds the elevation to be 60 and the direction South-West, while the other the elevation to be 45 and the direction West. Then the height of the balloon is, in metres,
(a) 500√{(12 + 3√6)/10}
(b) 500√{(12 - 3√6)/10}
(c) 250√3
(d) None of the foregoing numbers.

Solution :



From triangle ABC we get, AB/BC = tan60

- ⇨ $y = h/\sqrt{3}$

From triangle DEF we get, $x = h$

From triangle SQR we get, $a^2 = 500^2 - (x + a)^2$

- ⇨ $a^2 = 500^2 - x^2 - 2ax - a^2$
- ⇨ $2a^2 + 2ax = 500^2 - x^2$
- ⇨ $2a(x + a) = 500^2 - x^2$

From triangle PSR we get, $y = \sqrt{2}(x + a)$

- ⇨ $a = (y - \sqrt{2}x)/\sqrt{2}$

Putting in above equation we get, $2\{(y - \sqrt{2}x)/\sqrt{2}\}(y/\sqrt{2}) = 500^2 - h^2$

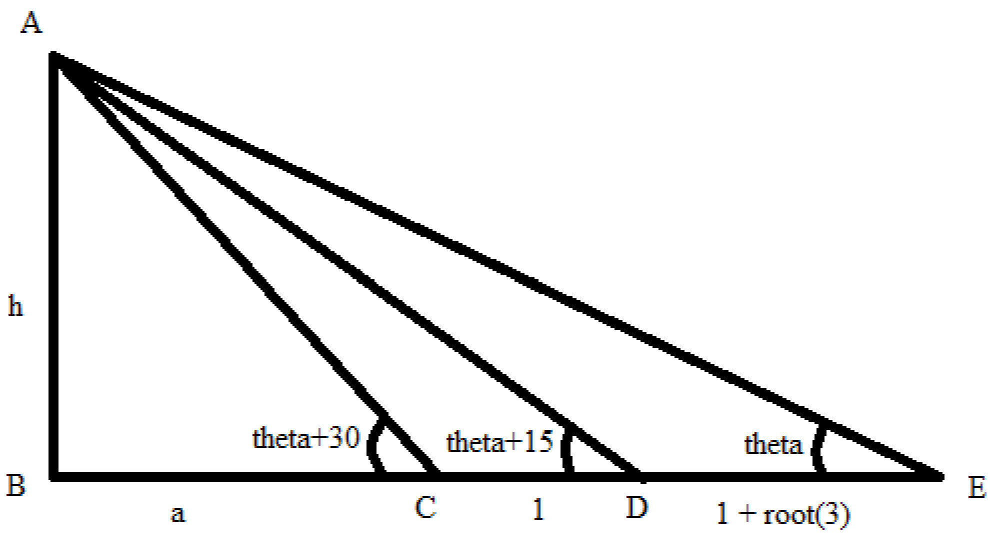
- ⇨ $y^2 - \sqrt{2}xy = 500^2$
- ⇨ $h^2/3 - \sqrt{2}*h*(h/\sqrt{3}) + h^2 = 500^2$
- ⇨ $h^2(4 - \sqrt{6})/3 = 500^2$
- ⇨ $h = 500\sqrt{3}/\sqrt{(4 - \sqrt{6})}$
- ⇨ $h = 500\sqrt{3}(\sqrt{(4 + \sqrt{6})}/\sqrt{10}$
- ⇨ $h = 500\sqrt{\{(12 + 3\sqrt{6})/10\}}$

Option (a) is correct.

665. Standing far from a hill, an observer records its elevation. The elevation increases 15 as he walks 1 + √3 miles towards the hill, and

by a further 15 as he walks another mile in the same direction. Then, the height of the hill is
(a) $(\sqrt{3} + 1)/2$ miles
(b) $(\sqrt{3} - 1)/(\sqrt{2} - 1)$ miles
(c) $(\sqrt{3} - 1)/2$ miles
(d) none of these

Solution :



From triangle ABE we get, $h/(a + 2 + \sqrt{3}) = \tan\theta$

From triangle ABD we get, $h/(a + 1) = \tan(\theta + 15)$

From triangle ABC we get, $h/a = \tan(\theta + 30)$

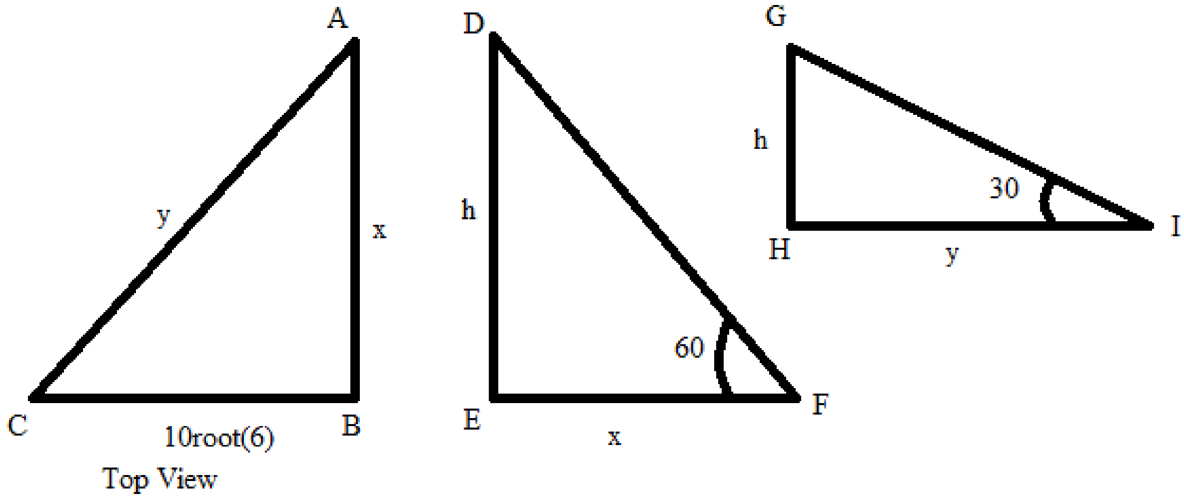
Now, there are three unknowns a, h, θ and three equations so we can solve h.

Option (c) is correct.

666. A man finds that at a point due south of a tower the angle of elevation of the tower is 60. He then walks due west $10\sqrt{6}$ metres on a horizontal plane and finds that the angle of elevation of the tower at that point is 30. Then the original distance of the man from the tower is, in metres,
(a) $5\sqrt{3}$
(b) $15\sqrt{3}$
(c) 15

(d) 180

Solution :



From triangle DEF we get, $h/x = \tan 60$

⇨ $h = x\sqrt{3}$

From triangle GHI we get, $h/y = \tan 30$

⇨ $h = y/\sqrt{3}$

Dividing the two equations we get, $1 = x\sqrt{3}/(y/\sqrt{3})$

⇨ $y = 3x$

Now, from triangle ABC, we get, $y^2 = (10\sqrt{6})^2 + x^2$

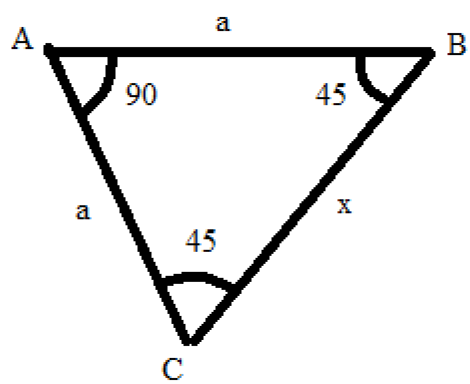
⇨ $(3x)^2 = 600 + x^2$
⇨ $8x^2 = 600$
⇨ $x^2 = 75$
⇨ $x = 5\sqrt{3}$

Option (a) is correct.

667. A man stands a mteres due east of a tower and finds the angle of elevation of the top of the tower to be θ. He then walks x metres north west and finds the angle of elevation to be θ again. Then the value of x is
(a) a

(b) $\sqrt{2}a$
(c) $a/\sqrt{2}$
(d) none of the foregoing expressions

Solution :



Top View

From the data it is clear that AC = a (otherwise at B and C angle of elevation cannot be same)

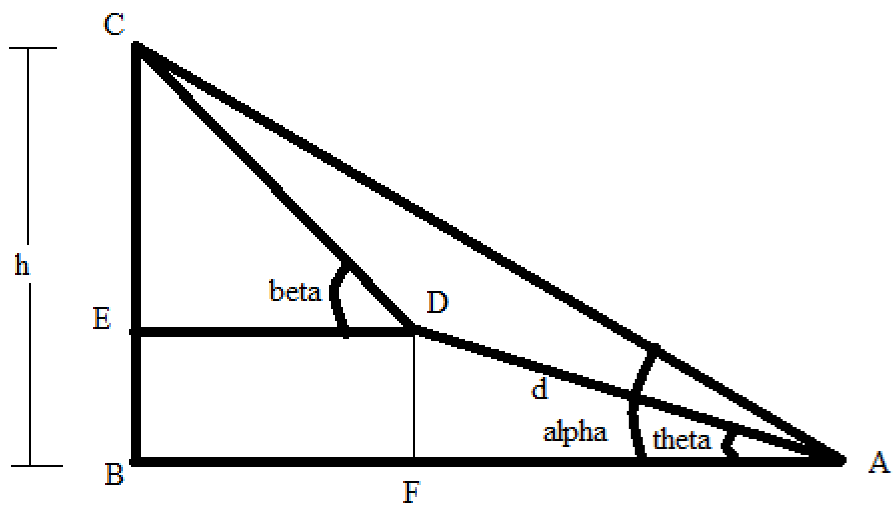
In triangle ABC, AB = a and AC = a, implies Angle ACB = Angle ABC = 45 (as per data)

From the triangle ABC, $x = \sqrt{2}a$

Option (b) is correct.

668. The angle of elevation of the top of a hill from a point is α. After walking a distance d towards the top, up a slope inclined to the horizon at an angle θ, which is less than α, the angle of elevation is β. The height of the hill equals
(a) $d\sin\alpha\sin\theta/\sin(\beta - \alpha)$
(b) $d\sin(\beta - \alpha)\sin\theta/\sin\alpha\sin\beta$
(c) $d\sin(\alpha - \theta)\sin(\beta - \alpha)/\sin(\alpha - \theta)$
(d) $d\sin\alpha\sin(\beta - \theta)/\sin(\beta - \alpha)$

Solution :



From triangle ADF we get, DF/AD = $\sin\theta$

- ⇨ DF = $d\sin\theta$

Again, AF/AD = $\cos\theta$

- ⇨ AF = $d\cos\theta$

CE = $h - d\sin\theta$

Now, from triangle ABC we get, BC/AB = $\tan\alpha$

- ⇨ AB = $h/\tan\alpha$
- ⇨ BF = $h/\tan\alpha - d\cos\theta$ = DE

From triangle BDE we get, CE/DE = $\tan\beta$

- ⇨ $(h - d\sin\theta)/(h/\tan\alpha - d\cos\theta) = \tan\beta$
- ⇨ $h - d\sin\theta = h\tan\beta/\tan\alpha - d\cos\theta\tan\beta$
- ⇨ $h(1 - \tan\beta/\tan\alpha) = d(\sin\theta - \cos\theta\sin\beta/\cos\beta)$
- ⇨ $h(\sin\alpha\cos\beta - \cos\alpha\sin\beta)/(\sin\alpha\cos\beta) = d(\sin\theta\cos\beta - \cos\theta\sin\beta)/\cos\beta$
- ⇨ $h\sin(\alpha - \beta) = d\sin(\theta - \beta)\sin\alpha$
- ⇨ $h = d\sin\alpha\sin(\beta - \theta)/\sin(\beta - \alpha)$

Option (d) is correct.

669. A person observes the angle of elevation of a peak from a point A on the ground to be α. He goes up an incline of inclination β, where $\beta < \alpha$, to the horizontal level towards the top of the peak and observes that the angle of elevation of the peak now is γ. If B is the second

place of observation and AB = y metres, then height of the peak above the ground is
(a) ysinβ + ysin(α – β)cosec(γ – α)sinγ
(b) ysinβ + ysin(β – α)sec(γ – α)sinγ
(c) ysinβ + ysin(α – β)sec(α – γ)sinγ
(d) ysinβ + ysin(α – β)cosec(α – γ)sinγ

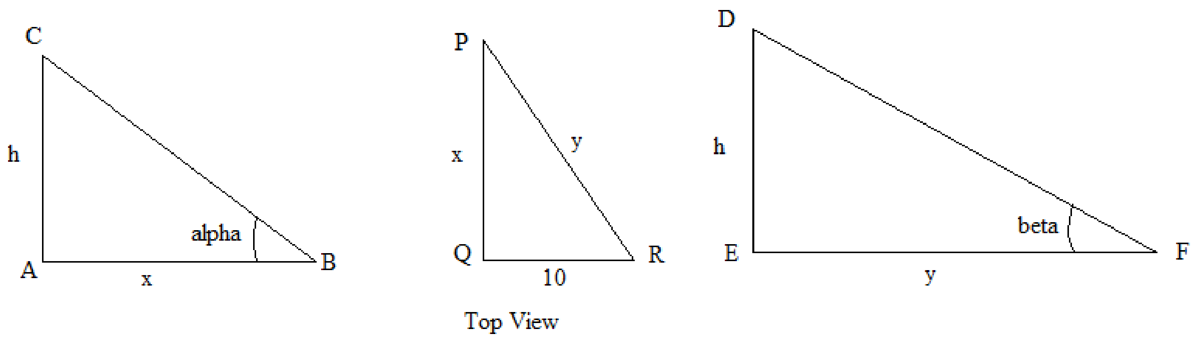
Solution :

Same problem as the previous one.

Option (a) is correct.

670. Standing on one side of a 10 metre wide straight road, a man finds that the angle of elevation of a statue located on the same side of the road is α. After crossing the road by the shortest possible distance, the angle reduces to β. The height of the statue is
(a) $10\tan\alpha\tan\beta/\sqrt{(\tan^2\alpha - \tan^2\beta)}$
(b) $10\sqrt{(\tan^2\alpha - \tan^2\beta)}/(\tan\alpha\tan\beta)$
(c) $10\sqrt{(\tan^2\alpha - \tan^2\beta)}$
(d) $10/\sqrt{(\tan^2\alpha - \tan^2\beta)}$

Solution :



From triangle ABC we get, $h/x = \tan\alpha$ i.e. $x = h/\tan\alpha$

From triangle DEF we get, $h/y = \tan\beta$ i.e. $y = h/\tan\beta$

Now, from triangle PQR we get, $y^2 = x^2 + 10^2$

⇨ $(h/\tan\beta)^2 - (h/\tan\alpha)^2 = 10^2$
⇨ $h^2(\tan^2\alpha - \tan^2\beta)/(\tan^2\alpha\tan^2\beta = 10$

⇨ $h = 10\tan\alpha\tan\beta/\sqrt{(\tan^2\alpha - \tan^2\beta)}$

Option (a) is correct.

671. The complete set of solutions of the equation $\sin^{-1}x = 2\tan^{-1}x$ is
(a) ± 1
(b) 0
(c) ± 1, 0
(d) $\pm 1/2$, ± 1, 0

Solution :

Now, $\sin^{-1}x = 2\tan^{-1}x$

Let $2\tan^{-1}x = A$

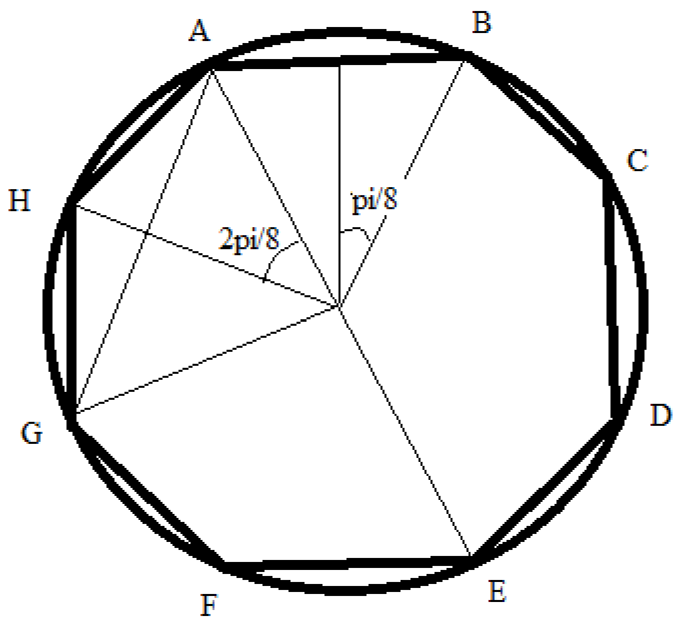
⇨ $\tan(A/2) = x$
⇨ $\sin A = 2\tan(A/2)/(1 + \tan^2(A/2)) = 2x/(1 + x^2)$
⇨ $A = \sin^{-1}\{2x/(1 + x^2)\} = \sin^{-1}x$
⇨ $2x/(1 + x^2) = x$
⇨ $x\{2/(1 + x^2) - 1\} = 0$
⇨ $x = 0$ or $2/(1 + x^2) - 1 = 0$
⇨ $2 = 1 + x^2$
⇨ $x = \pm 1$

Therefore, $x = 0, \pm 1$

Option (c) is correct.

672. For a regular octagon (a polygon with 8 equal sides) inscribed in a circle of radius 1, the product of the distances from a fixed vertex to the other seven vertices is
(a) 4
(b) 8
(c) 12
(d) 16

Solution :



From the figure it is clear that the product of the distances from a fixed vertes to other vertex is

$= \{2r\sin(\pi/8)\}\{2r\sin(2\pi/8)\}(2r\sin(3\pi/8)\}^2\{2r\sin(4\pi/8)\}$

$= 2^6\sin^2(\pi/8)\sin^2(3\pi/8)$

$= 2^4\{1 - \cos(\pi/4)\{1 - \cos(3\pi/4)\}$

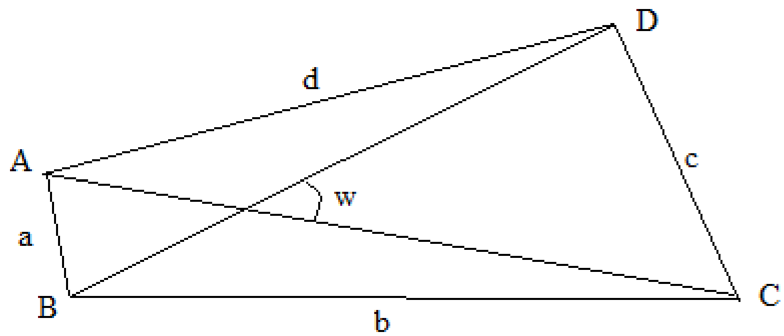
$= 2^4(1 - 1/\sqrt{2})(1 + 1/\sqrt{2})$

$= 2^4(1 - ½)$

$= 2^3 = 8$

Option (b) is correct.

673. In the quadrilateral in the in figure, the lengths of AC and BD are x and y respectively. Then the value of 2xycosw equals



(a) $b^2 + d^2 - a^2 - c^2$
(b) $b^2 + a^2 - c^2 - d^2$
(c) $a^2 + c^2 - b^2 - d^2$
(d) $a^2 + d^2 - b^2 - c^2$

Solution :

Option (a) is correct.

674. In a triangle ABC with sides $a = 5$, $b = 3$, $c = 7$, the value of $3\cos C + 7\cos B$ is
(a) 3
(b) 7
(c) 10
(d) 5

Solution :

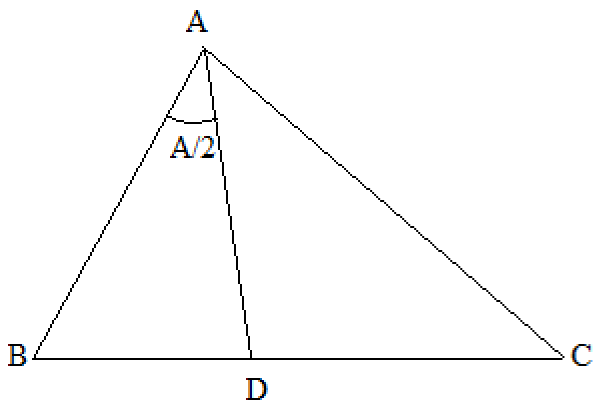
We know, $a = b\cos C + c\cos B$

⇨ $3\cos C + 7\cos B = 5$

Option (d) is correct.

675. If in a triangle ABC, the bisector of the angle A meets the side BC at the point D, then the length of AD equals
(a) $2bc\cos(A/2)/(b + c)$
(b) $bc\cos(A/2)/(b + c)$
(c) $bc\cos A/(b + c)$
(d) $2bc\sin(A/2)/(b + c)$

Solution :



From triangle ABD we get, AD/sinB = BD/sin(A/2)

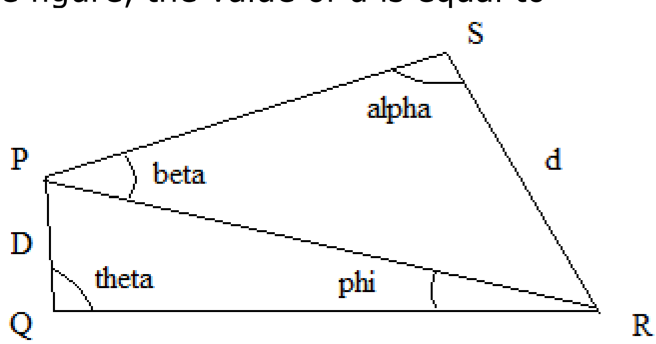
From triangle ACD we get, AD/sinC = CD/sin(A/2)

Adding we get, AD(1/sinB + 1/sinC) = (BD + CD)/sin(A/2)

- ⇨ AD(sinB + sinC)/(sinBsinC) = a/sin(A/2)
- ⇨ AD = absinC/(b + c)sinA/2
- ⇨ AD = 2absinCcos(A/2)/{(b + c)(2sin(A/2)coa(A/2)}
- ⇨ AD = 2absinCcos(A/2)/{(b + c)sinA)
- ⇨ AD = 2abccos(A/2)/{a(b + c)}
- ⇨ AD = 2bccos(A/2)/(b + c)

Option (a) is correct.

676. In an arbitrary quadrilateral with sides and angles as marked in the figure, the value of d is equal to



(a) Dsinθsinα/(sinΦsinβ)

(b) $D\sin\Phi\sin\beta/(\sin\theta\sin\alpha)$
(c) $D\sin\theta\sin\beta/(\sin\Phi\sin\alpha)$
(d) $D\sin\theta\sin\Phi/(\sin\alpha\sin\beta)$

Solution :

From triangle PQR we get, $D/\sin\Phi = PR/\sin\theta$

From triangle PRS we get, $d/\sin\beta = PR/\sin\alpha$

Dividing the equations we get, $(d/\sin\beta)/(D/\sin\Phi) = (PR/\sin\alpha)/(PR/\sin\theta)$

- ⇨ $d\sin\Phi/D\sin\beta = \sin\theta/\sin\alpha$
- ⇨ $d = D\sin\theta\sin\beta/(\sin\Phi\sin\alpha)$

Option (c) is correct.

677. Suppose the internal bisectors of the angles of a quadrilateral form another quadrilateral. Then the sum of the cosines of the angles of the second quadrilateral
(a) is a constant independent of the first quadrilateral
(b) always equals the sum of the sines of the angles of the first quadrilateral
(c) always equals the sum of the cosines of the angles of the first quadrilateral
(d) depends on the angles as well as the sides of the first quadrilateral

Solution :

$S = \pi - (A/2 + D/2)$

$\cos S = -\cos(A/2 + D/2)$

Similarly, $\cos P = -\cos(C/2 + D/2)$, $\cos Q = -\cos(B/2 + C/2)$ and $\cos R = -\cos(A/2 + B/2)$

- ⇨ $\cos P + \cos Q + \cos R + \cos S = -[\cos(C/2 + D/2) + \cos(B/2 + C/2) + \cos(A/2 + B/2) + \cos(D/2 + A/2)]$

$= -[2\cos\{(A + B + C + D)/4\}\cos\{(C + D - A - B)/4\} + 2\cos\{(A + B + C + D)/4\}\cos\{(B + C - D - A)/4\}]$

= -2cos{(A + B + C + D)/4}[cos{(C + D – A – B)/4} + cos{(B + C – A – D)/4}]

= -2cos(π/2)[cos{(C + D – A – B)/4} + cos{(B + C – A – D)/4}] (A + B + C + D = 2π)

= 0

Option (a) is correct.

678. Consider the following two statements :

P : all cyclic quadrilaterals ABCD satisfy tan(A/2)tan(B/2)tan(C/2)tan(D/2) = 1.

Q : all trapeziums ABCD satisfy tan(A/2)tan(B/2)tan(C/2)tan(D/2) = 1.

Then

(a) both P and Q are true
(b) P is true but Q is not true
(c) P is not true and Q is true
(d) Neither P nor Q is true

Solution :

In a cyclic quadrilateral, A + C = B + D = 180

- ⇨ A = 180 – C
- ⇨ A/2 = 90 – C/2
- ⇨ tan(A/2) = tan(90 – C/2) = cot(C/2)
- ⇨ tan(A/2)tan(C/2) = 1

Similarly, tan(B/2)tan(d/2) = 1

Therefore, tan(A/2)tan(B/2)tan(C/2)tan(D/2) = 1 for cyclic quadrilateral

In trapezium with AB||CD, A + D = B + C = 180 (i.e. sum of adjacent angles is 180)

- ⇨ A/2 = 90 – D/2
- ⇨ tan(A/2) = cot(D/2)
- ⇨ tan(A/2)tan(D/2) = 1

Similarly, tan(B/2)tan(C/2) = 1

Therefore, tan(A/2)tan(B/2)tan(C/2)tan(D/2) = 1 for trapezium

Option (a) is correct.

679. Let a, b, c denote the three sides of a triangle and A, B, C the corresponding opposite angles. Only one of the expressions below has the same value for all triangles. Which one is it?
(a) sinA + sinB + sinC
(b) tanAtanB + tanBtanC + tanCtanA
(c)(a + b + c)/(sinA + sinB + sinC)
(d) cotAcotB + cotBcotC + cotCcotA

Solution :

Option (a) and (c) cannot be true because those are function of R (radius of circumcircle)

Let us try cotAcotB + cotBcotC + cotCcotA

= cotB(1/tanA + 1/tanC) + cotCcotA

= cotB(tanA + tanC)/(tanAtanC) + cotCcotA

= cotBtan(A + C)(1 – tanAtanC)/(tanAtanC) + cotCcotA

= cotB(-tanB)(1 – tanAtanC)cotAcotC + cotCcotA

= -(1 – tanAtanC)cotAcotC + cotCcotA

= -cotAcotC + 1 + cotAcotC

= 1

Option (d) is correct.

680. In a triangle ABC, 2sinCcosB = sinA holds. Then one of the following statements is correct. Which one is it?
(a) The triangle must be equilateral.
(b) The triangle must be isosceles but not necessarily equilateral
(c)C must be an obtuse angle
(d) None of the foregoing statements is necessarily true.

Solution :

Now, 2sinCcosB = sinA

- ⇨ $2c(c^2 + a^2 - b^2)/(2ac) = a$
- ⇨ $c^2 + a^2 - b^2 = a^2$
- ⇨ $c^2 = b^2$
- ⇨ $c = b$

Option (b) is correct.

681. If A, B, C are the angles of a triangle and $\sin^2A + \sin^2B = \sin^2C$, then C equals
   (a) 30 degree
   (b) 90 degree
   (c) 45 degree
   (d) None of the foregoing angles

Solution :

$\sin^2A + \sin^2B = \sin^2C$

- ⇨ $a^2 + b^2 = c^2$
- ⇨ c is the hypotenuse of the right-angled triangle ABC
- ⇨ C is 90 degree.

Option (b) is correct.

682. The value of (cos37 + sin37)/(cos37 – sin37) equals
   (a) tan8
   (b) cot8
   (c) sec8
   (d) cosec8

Solution :

(cos37 + sin37)/(cos37 – sin37)

= (1 + tan37)/(1 – tan37)

= (tan45 + tan37)/(1 – tan45tan37)

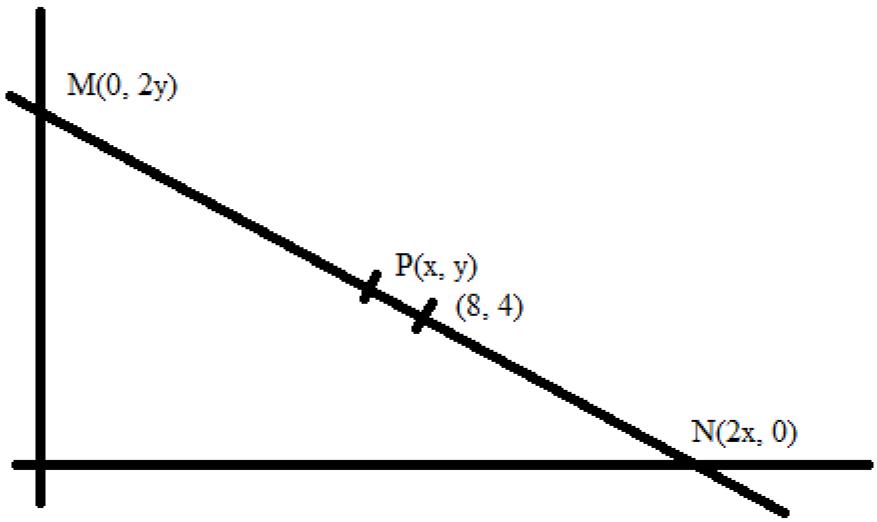
= tan(45 + 37)

= tan82

= cot8

Option (b) is correct.

683. A straight line passes through the fixed point (8, 4) and cuts the y-axis at M and the x-axis at N as in figure. Then the locus of the middle point P of MN is



(a) xy – 4x – 2y + 8 = 0
(b) xy – 2x – 4y = 0
(c) xy + 2x + 4y = 64
(d) xy + 4x + 2y = 72

Solution :

Let the slope of the line is m.

Therefore, equation of the line is y – 4 = m(x – 8)

- ⇨ mx – y = (8m – 4)
- ⇨ x/{(8m – 4)/m} + y/(4 – 8m) = 1

Therefore, 2x = (8m – 4)/m and 2y = 4 – 8m

2y/2x = -m

- ⇨ m = -y/x
- ⇨ 2y = 4 + 8y/x

⇨ $xy = 2x + 4y$
⇨ $xy - 2x - 4y = 0$

Option (b) is correct.

684. In a triangle ABC, a, b and c denote the sides opposite to angles A, B and C respectively. If sinA = 2sinCcosB, then
(a) b = c
(b) c = a
(c) a = b
(d) none of the foregoing statements is true.

Solution :

Now, sinA = 2sinCcosB

⇨ $a = 2c(c^2 + a^2 - b^2)/(2ca)$
⇨ $a^2 = c^2 + a^2 - b^2$
⇨ $c^2 = b^2$
⇨ $c = b$

Option (a) is correct.

685. The lengths of the sides CB and CA of a triangle ABC are given by a and b, and the angle C is $2\pi/3$. The line CD bisects the angle C and meets AB at D. Then the length of CD is
(a) $1/(a + b)$
(b) $(a^2 + b^2)/(a + b)$
(c) $ab/\{2(a + b)\}$
(d) $ab/(a + b)$

Solution :

See solution of problem 675.

Option (d) is correct.

686. Suppose in a triangle ABC, bcosB = ccosC. Then the triangle
(a) is right-angled

(b) is isosceles
(c) is equilateral
(d) need not necessarily be any of the above types

Solution :

Now, bcosB = ccosC

- ⇨ $b(a^2 + c^2 - b^2)/(2ac) = c(a^2 + b^2 - c^2)/(2ab)$
- ⇨ $b^2(a^2 + c^2 - b^2) = c^2(a^2 + b^2 - c^2)$
- ⇨ $a^2b^2 + b^2c^2 - b^4 = c^2a^2 + b^2c^2 - c^4$
- ⇨ $b^4 - c^4 - a^2b^2 + c^2a^2 = 0$
- ⇨ $(b^2 - c^2)(b^2 + c^2) - a^2(b^2 - c^2) = 0$
- ⇨ $(b^2 - c^2)(b^2 + c^2 - a^2) = 0$
- ⇨ $b = c$ or $b^2 + c^2 = a^2$

i.e. it may be isosceles or right-angled.

Option (d) is correct.

687. Let $V_0 = 2$, $V_1 = 3$ and for any natural number $k \geq 1$, let $V_{k+1} = 3V_k - 2V_{k-1}$. Then for any $n \geq 0$, V_n equals
(a) $(1/2)(n^2 + n + 4)$
(b) $(1/6)(n^3 + 5n + 12)$
(c) $2^n + 1$
(d) None of the foregoing expressions.

Solution :

Now, $V_{k+1} = 3V_k - 2V_{k-1}$

- ⇨ $V_{k+1} - V_k = 2V_k - 2V_{k-1}$

Putting k = 1 we get, $V_2 - V_1 = 2V_1 - 2V_0$

Putting k = 2, we get, $V_3 - V_2 = 2V_2 - 2V_1$

…

..

Putting k = n -1 we get, $V_n - V_{n-1} = 2V_{n-1} - 2V_{n-2}$

Summing over we get, $V_n - V_1 = 2V_{n-1} - 2V_0$

- ⇨ $V_n - 3 = 2V_{n-1} - 4$
- ⇨ $V_n = 2V_{n-1} - 1$
- ⇨ $V_n = 2(2V_{n-2} - 1) - 1 = 2^2V_{n-2} - 1 - 2 = 2^2(2V_{n-3} - 1) - 1 - 2 = 2^3V_{n-3} - 1 - 2 - 2^2 = \ldots = 2^nV_0 - (1 + 2 + 2^2 + \ldots + 2^{n-1})$
- ⇨ $V_n = 2^{n+1} - 1(2^n - 1)/(2 - 1) = 2^{n+1} - 2^n + 1 = 2^n + 1$

Option (c) is correct.

688. If $a_n = 1000^n/n!$, for n = 1, 2, 3, …., then the sequence $\{a_n\}$
   (a) doesn't have a maximum
   (b) attains maximum at exactly one value of n
   (c) attains maximum at exactly two values of n
   (d) attains maximum for infinitely many values of n

Solution :

Let, $a_n = a_k$

- ⇨ $1000^n/n! = 1000^k/k!$
- ⇨ $1000^{n-k} = n(n - 1)\ldots(k + 1)$ $(n > k)$

It can be only true if n = 1000 and k = 999.

Now, $1000^{1000}/1000! - 1000^{998}/998! = 1000^{998}/(1000!)(1000^2 - 999*1000) > 0$

Now, $1000^{1000}/1000! - 1000^{1001}/1001! = 1000^{1000}/1001!(1001 - 1000) > 0$

- ⇨ For n = 1000 it is maximum.
- ⇨ For n = 999 it is maximum.

Option (c) is correct.

689. Let f be a function of a real variable such that it satisfies f(r + s) = f(r) + f(s), for all r, s. Let m and n be integers. Then f(m/n) equals
   (a) m/n
   (b) f(m)/f(n)
   (c) (m/n)f(1)
   (d) None of the foregoing expressions, in general.

Solution :

$f(r + s) = f(r) + f(s)$

Putting $s = 0$ we get, $f(r) = f(r) + f(0)$ i.e. $f(0) = 0$

$f(r + s) = f(r) + f(s)$

- ⇨ $f(r + (m - 1)r) = f(r) + f((m - 1)r) = f(r) + f(r) + f((m - 2)r) = \ldots. = mf(r) + f(0) = mf(r)$
- ⇨ $f(mr) = mf(r)$
- ⇨ $f((m/n)r) = (m/n)f(r)$
- ⇨ $f(m/n) = (m/n)f(1)$

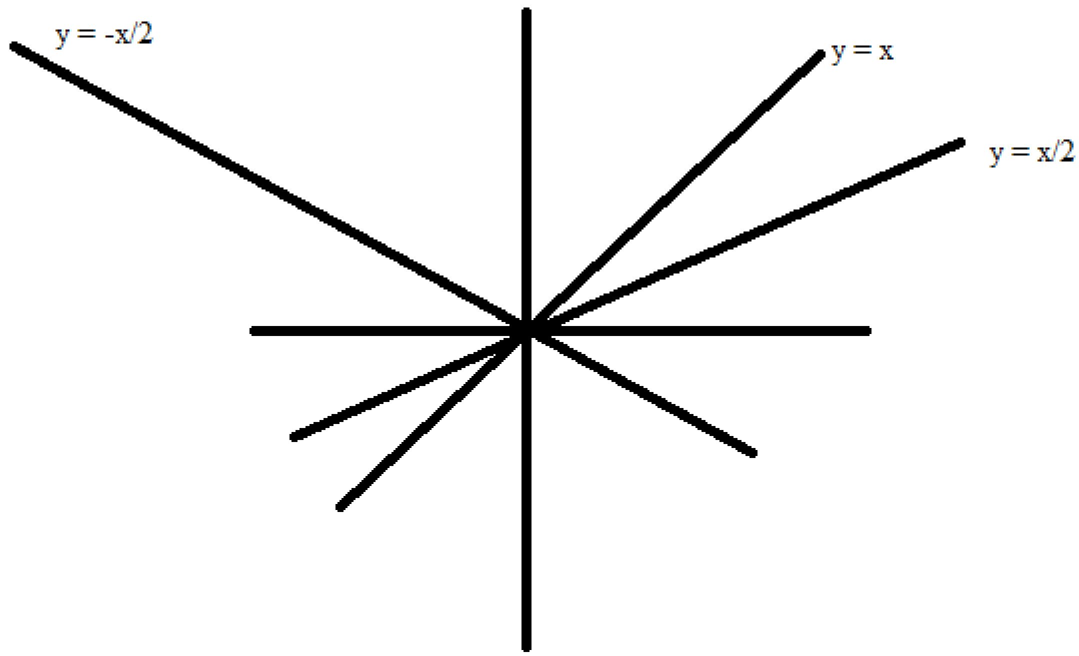
Option (c) is correct.

690. Let $f(x)$ be a real-valued function defined for all real numbers x such that $|f(x) - f(y)| \leq (1/2)|x - y|$ for all x, y. Then the number of points of intersection of the graph of $y = f(x)$ and the line $y = x$ is
(a) 0
(b) 1
(c) 2
(d) None of the foregoing numbers.

Solution :

$|f(x) - f(y)| \leq (1/2)|x - y|$

- ⇨ $\lim |\{f(x) - f(y)\}/(x - y)|$ as $x \to y \leq \lim (1/2)$ as $x \to y$
- ⇨ $|f'(y)| \leq ½$
- ⇨ $-1/2 \leq f'(y) \leq 1/2$
- ⇨ $-y/2 \leq f(y) \leq y/2$ (integrating)
- ⇨ $-x/2 \leq f(x) \leq x/2$



From the figure it is clear that intersection point is 1.

Option (b) is correct.

691. The limit of $(1/n^4)\Sigma k(k + 2)(k + 4)$ (summation running from k = 1 to k = n) as n -> ∞
(a) exists and equals ¼
(b) exists and equals 0
(c) exists and equals 1/8
(d) does not exist

Solution :

Now, $(1/n^4)\Sigma k(k + 2)(k + 4)$ (summation running from k = 1 to k = n)

$= (1/n^4)\Sigma(k^3 + 6k^2 + 8k)$ (summation running from k = 1 to k = n)

$= (1/n^4)[\Sigma k^3 + 6\Sigma k^2 + 8\Sigma k]$ (summation running from k = 1 to k = n)

$= (1/n^4)[\{n(n + 1)/2\}^2 + 6n(n + 1)(2n + 1)/6 + 8n(n + 1)/2]$

$= (1 + 1/n)^2/4 + (1 + 1/n)(2 + 1/n)(1/n) + 4(1 + 1/n)(1/n^2)$

Now, lim of this as n - > ∞ = ¼

Option (a) is correct.

692. The limit of the sequence $\sqrt{2}$, $\sqrt{(2\sqrt{2})}$, $\sqrt{\{2\sqrt{(2\sqrt{2})}\}}$, …. Is

(a) 1
(b) 2
(c) $2\sqrt{2}$
(d) ∞

Solution :

Now, $a_n^2 = 2a_{n-1}$

⇨ $\lim a_n^2 = 2\lim a_{n-1}$ as $n \to \infty$

Let $\lim a_n$ as $n \to \infty = a$

⇨ $\lim a_{n-1}$ as $n \to \infty = a$
⇨ $a^2 = 2a$
⇨ $a = 2$ $(a \neq 0)$

Option (b) is correct.

693. Let $P_n = \{(2^3 - 1)/(2^3 + 1)\}\{(3^3 - 1)/(3^3 + 1)\}\ldots\{(n^3 - 1)/(n^3 + 1)\}$; $n = 2, 3, \ldots$ $\lim P_n$ as $n \to \infty$ is

(a) ¾
(b) 7/11
(c) 2/3
(d) ½

Solution :

Option (c) is correct.

694. Let $a_1 = 1$ and $a_n = n(a_{n-1} + 1)$ for $n = 2, 3, \ldots$. Define $P_n = (1 + 1/a_1)(1 + 1/a_2)\ldots(1 + 1/a_n)$. Then $\lim P_n$ as $n \to \infty$ is

(a) $1 + e$
(b) e
(c) 1
(d) ∞

Solution :

Option (b) is correct.

695. Let x be a real number. Let $a_0 = x$, $a_1 = \sin x$ and, in general, $a_n = \sin a_{n-1}$. Then the sequence $\{a_n\}$
   (a) oscillates between -1 and +1, unless x is a multiple of π
   (b) converges to 0 whatever be x
   (c) converges to 0 if and only if x is a multiple of π
   (d) sometimes converges and sometimes oscillates depending on x

Solution :

Now, for bigger x, sinx < x

- ⇨ $a_2 < a_1$, $a_3 < a_2$, …., $a_n < a_{n-1}$

So, lim a_n as n -> ∞ = small number = b (say)

Now, lim a_{n-1} as n - > ∞ = b (if converges)

- ⇨ b = sinb which is true for small b
- ⇨ The sequence converges.

Option (b) is correct.

696. If k is an integer such that lim $\{\cos^n(k\pi/4) - \cos^n(k\pi/6)\} = 0$, then
   (a) k is divisible neither by 4 nor by 6
   (b) k must be divisible by 12, but not necessarily by 24
   (c) k must be divisible by 24
   (d) either k is divisible by 24 or k is divisible neither by 4 not by 6

Solution :

If k is divisible by 24 then cos(kπ/4) = cos(kπ/6) = 1

- ⇨ The limit exists and equal to RHS i.e. 0

If k is not divisible by 4 or 6 then cos(kπ/4), cos(kπ/6) both < 1

- ⇨ lim $\cos^n(k\pi/4)$, $\cos^n(k\pi/6) = 0$
- ⇨ The equation holds.

Option (d) is correct.

697. The limit of $\sqrt{x}\{\sqrt{(x + 4)} - \sqrt{x}\}$ as $x \to \infty$
   (a) does not exist
   (b) exists and equals 0
   (c) exists and equals ½
   (d) exists and equals 2

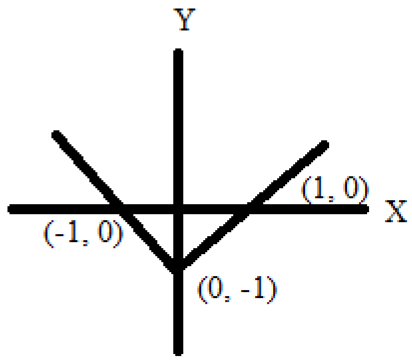
Solution :

Now, $\sqrt{x}\{\sqrt{(x + 4)} - \sqrt{x}\} = \sqrt{x}\{\sqrt{(x + 4)} - \sqrt{x}\}\{\sqrt{(x + 4)} + \sqrt{x}\}/\{\sqrt{(x + 4)} + \sqrt{x}\} = \sqrt{x}(x + 4 - x)/\{\sqrt{(x + 4)} + \sqrt{x}) = 4/\{\sqrt{(1 + 4/x)} + 1\}$

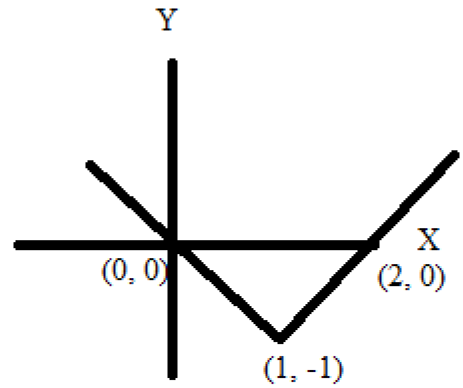
Now, lim $x \to \infty$ this $= 4/(1 + 1) = 2$

Option (d) is correct.

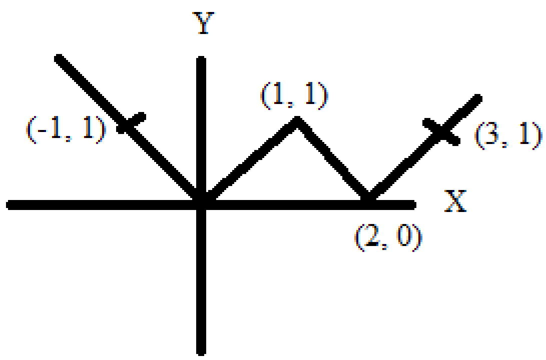
698. Four graphs marked G_1, G_2, G_3 and G_4 are given in the figure which are graphs of the four functions $f_1(x) = |x - 1| - 1$, $f_2(x) = ||x - 1| - 1|$, $f_3(x) = |x| - 1$, $f_4(x) = 1 - |x|$, not necessarily in the correct order.



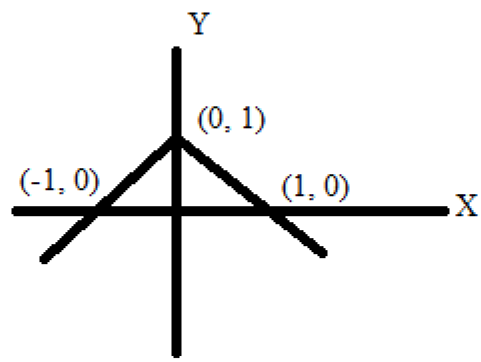
G1



G2



G3



G4

The correct order is
(a) G_2, G_1, G_3, G_4
(b) G_3, G_4, G_1, G_2
(c) G_2, G_3, G_1, G_4
(d) G_4, G_3, G_1, G_2

Solution :

Take the function $f_3(x) = |x| - 1$

If $x > 0$ $y = x - 1$, i.e. $x/1 + y/(-1) = 1$

If $x < 0$ $y = -x - 1$, i.e. $x/(-1) + y/(-1) = 1$

Clearly, G_1 is the graph.

Now, take the function $f_4(x) = 1 - |x|$

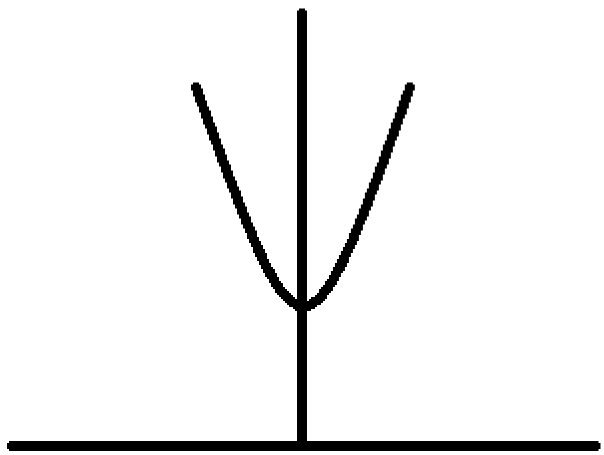
If $x > 0$, $y = 1 - x$ i.e. $x/1 + y/1 = 1$

If $x < 0$ $y = 1 + x$ i.e. $x/(-1) + y/1 = 1$

Clearly, G_4 is the graph.

Hence, option (c) is correct.

699. The adjoining figure is the graph of



(a) $y = 2e^x$
(b) $y = 2e^{-x}$
(c) $y = e^x + e^{-x}$
(d) $y = e^x - e^{-x} + 2$

Solution :

Option (c) is correct.

700. Suppose that the three distinct real numbers a, b, c are in G.P. and a + b + c = xb. Then
(a) $-3 < x < 1$
(b) $x > 1$ or $x < -3$
(c) $x < -1$ or $x > 3$
(d) $-1 < x < 3$

Solution :

Now, $a + b + c = xb$

⇨ $(a/b) + 1 + (c/b) = x$

⇨ $x = r + 1/r + 1$ (where r = common ration of the G.P.)

Let $r > 0$

$(r + 1/r) > 2$ (AM > GM)

⇨ $x > 3$

Let $r < 0$

$(r + 1)^2 > 0$

⇨ $r^2 + 2r + 1 > 0$
⇨ $r + 1/r + 2 < 0$ (as $r < 0$)
⇨ $r + 1/r + 1 < -1$
⇨ $x < -1$

Option (c) is correct.

701. The maximum value attained by the function $y = 10 - |x - 10|$ in the range $-9 \leq x \leq 9$ is
(a) 10
(b) 9
(c) $+\infty$
(d) 1

Solution :

Clearly, $|x - 10|$ is minimum when $x = 9$

⇨ Maximum value of $y = 10 - 1 = 9$

Option (b) is correct.

702. Let f(x) be a real-valued function of a real variable. Then the function is said to be 'one-to-one' if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. The function is said to be 'onto' if it takes all real values. Suppose now $f(x) = x^3 - 3x^2 + 6x - 5$. Then
(a) f is one-to-one and onto
(b) f is one-to-one but not onto
(c) f is onto but not one-to-one
(d) f is neither one-to-one nor onto

Solution :

Now, $f(x) = x^3 - 3x^2 + 6x - 5$

- ⇨ $f'(x) = 3x^2 - 6x + 6 = 3(x^2 - 2x + 2) = 3\{(x-1)^2 + 1\} > 0$ for all real x.
- ⇨ f(x) is increasing

And f(x) is a polynomial so f(x) is continuous everywhere.

- ⇨ f(x) is one-to-one and onto.

Option (a) is correct.

703. Let f be a function from a set X to X such that f(f(x)) = x for all x Ɛ X. Then
(a) f is one-to-one but need not be onto
(b) f is onto but need not be one-to-one
(c) f is both one-to-one and onto
(d) none of the foregoing statements is true

Solution :

Let, $f(x_1) = x_2$

Now, f(f(x) = x

Putting $x = x_1$ we get, $f(f(x_1)) = x_1$

- ⇨ $f(x_2) = x_1$

So, x_1 maps to x_2 and x_2 maps to x_1.

Let $f(x_3) = x_2$

Putting $x = x_3$ we get, $f(f(x_3)) = x_3$

- ⇨ $f(x_2) = x_3$
- ⇨ $x_3 = x_1$
- ⇨ if $x_1 \neq x_3$ then $f(x_1) \neq f(x_3)$
- ⇨ f(x) is one-to-one

And f(x) is onto also because the mapping is from X to X.

Option (c) is correct.

Directions for Items 704 to 706 :

A real-valued function f(x) of a real variable x is said to be periodic if there is a strictly positive number p such that f(x + p) = f(x) for every x. The smallest p satisfying the above property is called the period of f.

704. Only one of the following is not periodic. Which one is it?
   (a) $e^{\sin x}$
   (b) $1/(10 + \sin x + \cos x)$
   (c) $\log_e(\cos x)$
   (d) $\sin(e^x)$

Solution :

Now sinx and cosx are periodic. So, option (a), (b) and (c) are periodic.

Option (d) is correct.

705. Suppose f is periodic with period greater than h. Then
   (a) for all h’ > h and for all x, f(x + h’) = f(x)
   (b) for all x, f(x + h) ≠ f(x)
   (c) for some x, f(x + h) ≠ f(x)
   (d) none of the foregoing statements is true

Solution :

Clearly, option (c) is correct.

706. Suppose f is a function with period a and g is a function with period b. Then the function h(x) = f(g(x))
   (a) may not have any period
   (b) has period a
   (c) has period b
   (d) has period ab

Solution :

Now, h(x + b) = f(g(x + b)) = f(g(x) = h(x)

Option (c) is correct.

707. A function f is said to be odd if f(-x) = -f(x) for all x. Which of the following is not odd?
(a) A function f such that f(x + y) = f(x) + f(y) for all x, y
(b) $f(x) = xe^{x/2}/(1 + e^x)$
(c) f(x) = x – [x]
(d) $f(x) = x^2\sin x + x^3\cos x$

Solution :

Putting y = -x in option (a) we get, f(0) = f(x) + f(-x)

Now, putting y = 0 in option (a) we get, f(x) = f(x) + f(0) i.e. f(0) = 0

- ⇨ f(x) + f(-x) = 0
- ⇨ f(-x) = -f(x)
- ⇨ odd

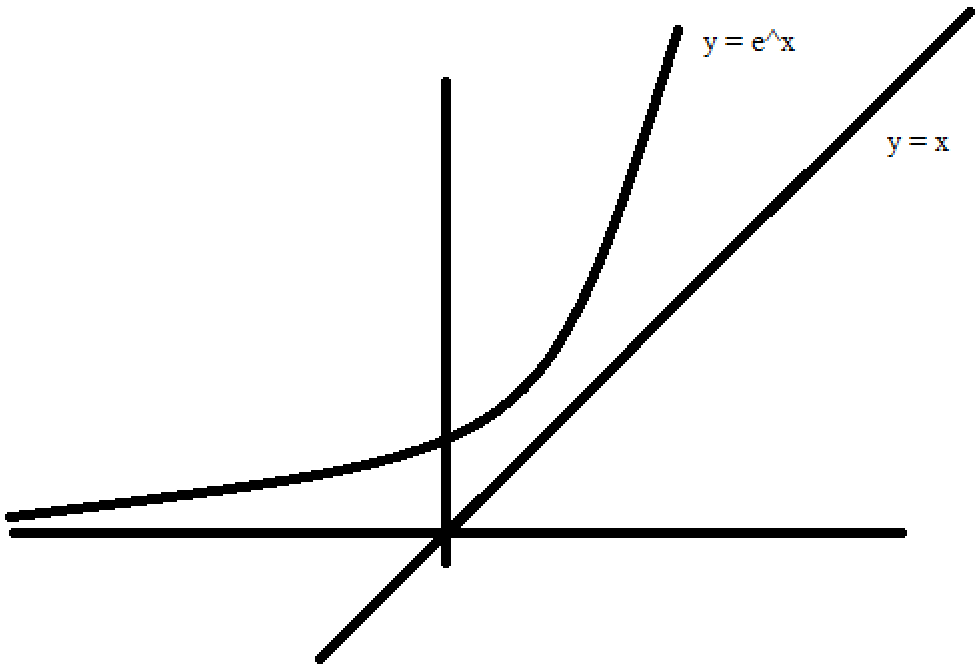
Option (b) and (d) can be proved odd easily.

Option (c) is correct.

708. If n stands for the number of negative roots and p for the number of positive roots of the equation $e^x = x$, then
(a) n = 1, p = 0
(b) n = 0, p = 1
(c) n = 0, p > 1
(d) n = 0, p = 0

Solution :

As $e^x > 0$ (always for all x) so n = 0



From the figure, p = 0

Option (d) is correct.

709. In the interval (-2π, 0) the function $f(x) = \sin(1/x^3)$
(a) never changes sign
(b) changes sign only once
(c) changes sign more than once, but a finite number of times
(d) changes sign infinite number of times

Solution :

As x becomes < 1 and tends to zero then it crosses π, 2π, 3π, ….

So, number of sign changes is infinite.

Option (d) is correct.

710. If $f(x) = a_0 + a_1\cos x + a_2\cos 2x + \dots + a_n\cos nx$, where $a_0, a_1, \dots, a_n$ are nonzero real numbers and $a_n > |a_0| + |a_1| + \dots + |a_{n-1}|$, then the number of roots of $f(x) = 0$ in $0 \le x \le 2\pi$, is
(a) at most n

(b) more than n but less than 2n
(c) at least 2n
(d) zero

Solution :

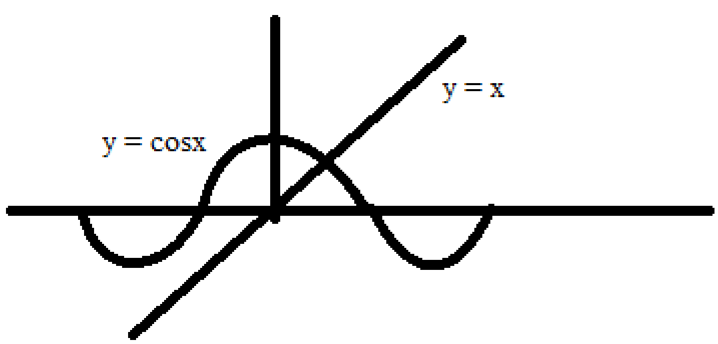
Option (c) is correct.

711. The number of roots of the equation $x^2 + \sin^2 x = 1$ in the closed interval $[0, \pi/2]$ is
(a) 0
(b) 1
(c) 2
(d) 3

Solution :

Now, $x^2 + \sin^2 x = 1$

⇨ $x^2 = 1 - \sin^2 x$
⇨ $x^2 = \cos^2 x$
⇨ $x = \cos x$ (as $\cos x > 0$ in the interval 0 to $\pi/2$)



So, one intersecting point.

⇨ One root

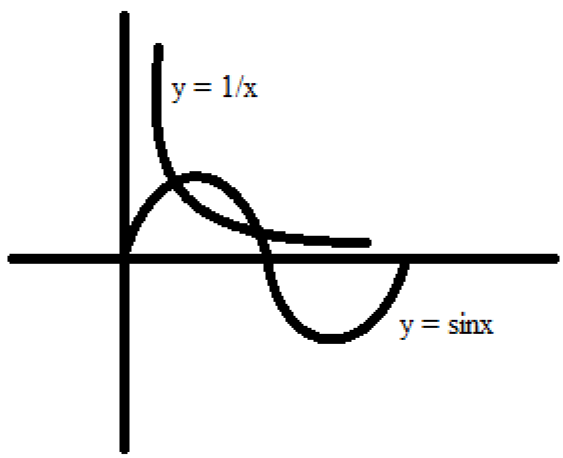
Option (b) is correct.

712. The number of roots of the equation xsinx = 1 in the interval $0 < x \le 2\pi$ is
(a) 0
(b) 1
(c) 2
(d) 4

Solution :

Now, xsinx = 1

⇨ sinx = 1/x



Two intersecting points.

⇨ Two roots.

Option (c) is correct.

713. The number of points in the rectangle {(x, y) | $-10 \le x \le 10$ and $-3 \le y \le 3$} which lie on the curve $y^2 = x + \sin x$ and at which the tangent to the curve is parallel to the x-axis, is
(a) 0
(b) 2
(c) 4
(d) 5

Solution :

Now, $y^2 = x + \sin x$

- ⇨ $2y(dy/dx) = 1 + \cos x$
- ⇨ $(dy/dx) = (1 + \cos x)/y = 0$
- ⇨ $\cos x = -1$
- ⇨ $x = \pm\pi, \pm 3\pi$

For $x = \pi$, $y^2 = \pi + \sin\pi = \pi$

- ⇨ $y = \pm\sqrt{\pi}$
- ⇨ $(\pi, \sqrt{\pi})$ and $(\pi, -\sqrt{\pi})$ both inside the rectangle.

Now, $x = -\pi, -3\pi$ doesn't give any solution.

Now, $x = 3\pi$, $y^2 = 3\pi + \sin 3\pi = 3\pi$

- ⇨ $y = \pm\sqrt{(3\pi)} > 3$
- ⇨ 2 points.

Option (b) is correct.

714. The set of all real numbers x satisfying the inequality $x^3(x + 1)(x - 2) \geq 0$ can be written
   (a) as $2 \leq x \leq \infty$
   (b) as $0 \leq x \leq \infty$
   (c) as $-1 \leq x \leq \infty$
   (d) in none of the foregoing forms

Solution :

$x > 0, x < -1, x < 2$ => no intersection point. So no solution.

$x < 0, x < -1, x > 2$ => no intersection point, So no solution.

$x < 0, x > -1, x < 2$ => $-1 \leq x \leq 0$

$x > 0, x > -1, x > 2$ => $2 \leq x \leq \infty$

Therefore, option (d) is correct.

715. A set S is said to *have a minimum* if there is an element a in S such that $a \leq y$ for all y in S. Similarly, S is said to *have a maximum* if there is an element b in S such that $b \geq y$ for all y in S. If S = {y : y =

(2x + 3)/(x + 2), x ≥ 0}, which one of the following statements is correct?
(a) S has both a maximum and a minimum
(b) S has neither a maximum nor a minimum
(c) S has a maximum but no minimum
(d) S has a minimum but no maximum

Solution :

Let, f(x) = (2x + 3)/(x + 2)

- ⇨ $f'(x) = \{2(x + 2) - (2x + 3)\}/(x + 2)^2 = 1/(x + 2)^2 > 0$
- ⇨ f(x) is increasing.
- ⇨ f(x) doesn't have a maximum
- ⇨ f(x) is minimum at x = 0.
- ⇨ $y_{min} = 3/2$

Option (d) is correct.

716. $\lim (20 + 2\sqrt{x} + 3\sqrt[3]{x})/\{2 + \sqrt{(4x - 3)} + \sqrt[3]{(8x - 4)}\}$ as x - > ∞ is
(a) 10
(b) 3/2
(c) 1
(d) 0

Solution :

$\lim (20/\sqrt{x} + 2 + 3/x^{1/6})/\{2/\sqrt{x} + \sqrt{(4 - 3/x)} + 1/(8x - 4)^{1/6}\}$ as x - > ∞ (dividing numerator and denominator by √x)

= 2/√4 = 1

Option (c) is correct.

717. $\lim [x\sqrt{(x^2 + a^2)} - \sqrt{(x^4 + a^4)}]$ as x - > ∞ is
(a) ∞
(b) $a^2/2$
(c) a^2
(d) 0

Solution :

lim $[x\sqrt{(x^2 + a^2)} - \sqrt{(x^4 + a^4)}][x\sqrt{(x^2 + a^2)} + \sqrt{(x^4 + a^4)}]/ [x\sqrt{(x^2 + a^2)} + \sqrt{(x^4 + a^4)}]$ as $x \to \infty$

= lim $(x^4 + a^2x^2 - x^4 - a^4)/ [x\sqrt{(x^2 + a^2)} + \sqrt{(x^4 + a^4)}]$ as $x \to \infty$

= lim $(a^2 - a^4/x^2)/[\sqrt{(1 + a^2/x^2)} + \sqrt{(1 + a^4/x^4)}]$ as $x \to \infty$

$= a^2/2$

Option (b) is correct.

718. The limit of $x^3[\sqrt{\{x^2 + \sqrt{(x^4 + 1)}\}} - x\sqrt{2}]$ as $x \to \infty$

(a) exists and equals $1/2\sqrt{2}$
(b) exists and equals $1/4\sqrt{2}$
(c) does not exist
(d) exists and equals $3/4\sqrt{2}$

Solution :

lim $x^3[\sqrt{\{x^2 + \sqrt{(x^4 + 1)}\}} - x\sqrt{2}][\sqrt{\{x^2 + \sqrt{(x^4 + 1)}\}} + x\sqrt{2}]/[\sqrt{\{x^2 + \sqrt{(x^4 + 1)}\}} + x\sqrt{2}]$ as $x \to \infty$

$= x^3[x^2 + \sqrt{(x^4 + 1)} - 2x^2]/[\sqrt{\{x^2 + \sqrt{(x^4 + 1)}\}} + x\sqrt{2}]$ as $x \to \infty$

$= x^3[\sqrt{(x^4 + 1)} - x^2]/[\sqrt{\{x^2 + \sqrt{(x^4 + 1)}\}} + x\sqrt{2}]$ as $x \to \infty$

$= x^3[x^4 + 1 - x^4]/[\sqrt{\{x^2 + \sqrt{(x^4 + 1)}\}} + x\sqrt{2}][\sqrt{(x^4 + 1)} + x^2]$ as $x \to \infty$

$= 1/[\sqrt{\{1 + \sqrt{(1 + 1/x^4)}\}} + \sqrt{2}][\sqrt{(1 + 1/x^4)} + 1]$ as $x \to \infty$

$= 1/\{\sqrt{(1 + 1)} + \sqrt{2}\}(1 + 1)$

$= 1/4\sqrt{2}$

Option (b) is correct.

719. If $f(x) = \sqrt{\{(x - \cos^2x)/(x + \sin x)\}}$, then the limit of $f(x)$ as $x \to \infty$ is

(a) 0
(b) 1

(c) ∞
(d) None of 0, 1 or ∞

Solution :

Now, $f(x) = \sqrt{\{(1 - \cos^2 x/x)/(1 + \sin x/x)\}}$

Limit of this as $x \to \infty = \sqrt{(1/1)} = 1$

Option (b) is correct.

720. Consider the function $f(x) = \tan^{-1}\{2\tan(x/2)\}$, where $-\pi/2 \le f(x) \le \pi/2$. (lim $x \to \pi-0$ means limit from the left at π amd lim $x \to \pi+0$ means limit from the right.) Then
(a) lim f(x) as $x \to \pi-0 = \pi/2$, lim f(x) as $x \to \pi+0 = -\pi/2$
(b) lim f(x) as $x \to \pi-0 = -\pi/2$, lim f(x) as $x \to \pi+0 = \pi/2$
(c) lim f(x) as $x \to \pi = \pi/2$
(d) lim f(x) as $x \to \pi = -\pi/2$

Solution :

lim f(x) as $x \to \pi-0$ = lim $\tan^{-1}\{2\tan(x/2)\}$ as $x \to \pi-0 = \pi/2$

Now, lim f(x) as $x \to \pi+0$ = lim $\tan^{-1}\{2\tan(x/2)\} = -\pi/2$ as $\tan(\pi/2 + \text{small value}) = -\tan(\pi/2)$

Option (a) is correct.

721. The value of lim $\{(x\sin a - a\sin x)/(x - a)\}$ as $x \to a$ is
(a) non-existent
(b) $\sin a + a\cos a$
(c) $a\sin a - \cos a$
(d) $\sin a - a\cos a$

Solution :

Now, lim $\{(x\sin a - a\sin x)/(x - a)$ as $x \to a$ = lim $\{(\sin a - a\cos x)/1\}$ as $x \to a$ (applying L'Hospital rule) = $\sin a - a\cos a$

Option (d) is correct.

722. The limit lim [(cosx – secx)/{x^2(x + 1)}] as x -> 0
(a) is 0
(b) is 1
(c) is -1
(d) does not exist

Solution :

Now, lim [(cosx – secx)/{x^2(x + 1)}] as x -> 0

= lim –($\sin^2 x/x^2$)[1/{cosx(x + 1)}] as x - > 0

= -1*[1/{1(0 + 1)}]

= -1

Option (c) is correct.

723. The limit lim {(tanx – x)/(x – sinx)} as x -> 0+ equals
(a) -1
(b) 0
(c) 1
(d) 2

Solution :

Now, lim {(tanx – x)/(x – sinx)} as x -> 0+ = lim {($\sec^2 x$ – 1)/(1 – cosx)} as x -> 0+ (Applying L'Hospital rule) = lim [(1 – cosx)(1 + cosx)/{$\cos^2 x$(1 – cosx)}] as x-> 0+ = lim {(1 + cosx)/$\cos^2 x$} as x -> 0+ = 2

Option (d) is correct.

724. lim [{$(1 + x)^{1/2}$ – 1}/{$(1 + x)^{1/3}$ – 1}] as x-> 0 is
(a) 1
(b) 0
(c) 3/2
(d) ∞

Solution :

Now, lim $[\{(1 + x)^{1/2} - 1\}/\{(1 + x)^{1/3} - 1\}]$

= lim $[\{(1 + x) - 1\}\{(1 + x)^{2/3} + (1 + x)^{1/3} + 1\}/\{(1 + x) - 1\}\{(1 + x)^{1/2} + 1\}]$ as x-> 0

= lim $[\{(1 + x)^{2/3} + (1 + x)^{1/3} + 1\}/\{(1 + x)^{1/2} + 1\}]$ as x -> 0

= (1 + 1 + 1)/(1 + 1)

= 3/2

Option (c) is correct.

725. A right circular cylinder container closed on both sides is to contain a fixed volume of motor oil. Suppose its base has diameter d and its height is h. The overall surface area of the container is minimum when
   (a) h = (4/3)πd
   (b) h = 2d
   (c) h = d
   (d) conditions other that the foregoing are satisfied

Solution :

Surface area = S = $2\pi(d/2)^2 + 2\pi(d/2)h = \pi d^2/2 + \pi dh$

Now, dS/dd = 2πd/2 + πh = 0 => d = -h

Now, $d^2S/dd^2 = \pi > 0$ so minimum.

Condition is, h = d

Option (c) is correct.

726. lim (logx – x) as x -> ∞
   (a) equals +∞
   (b) equals e
   (c) equals -∞
   (d) does not exist

Solution :

lim (logx – x) as x -> ∞ = -∞ (clearly)

Option (c) is correct.

727. lim xtan(1/x) as x -> 0
   (a) equals 0
   (b) equals 1
   (c) equals ∞
   (d) does not exist

Solution :

Option (d) is correct.

728. The limit lim ∫ {h/(h^2 + x^2)}dx (integration running from x =-1 to x = 1) as h -> 0
   (a) equals 0
   (b) equals π
   (c) equals –π
   (d) does not exist

Solution :

Now, ∫{h/(h^2 + x^2)dx (integration running from x = -1 to x = 1)

Let, x = htany

⇨ dx = h$\sec^2$ydy
⇨ x = -1, y = -$\tan^{-1}$(1/h) ansd x = 1, y = $\tan^{-1}$(1/h)

∫{h/(h^2 + x^2)}dx = ∫h(h$\sec^2$ydy)/$h^2\sec^2$y (integration running from y = -$\tan^{-1}$(1/h) to y = $\tan^{-1}$(1/h))

= y (upper limit = $\tan^{-1}$(1/h) and lower limit = -$\tan^{-1}$(1/h)

= 2$\tan^{-1}$(1/h)

Now, lim 2$\tan^{-}$(1/h) as h -> 0 doesn’t exist

Option (d) is correct.

729. If the area of an expanding circular region increases at a constant rate (with respect to time), then the rate of increase of the perimeter with respect to time

(a) varies inversely as the radius
(b) varies directly as the radius
(c) varies directly as the square of the radius
(d) remains constant

Solution :

$A = \pi r^2$

⇨ $dA/dt = 2\pi r(dr/dt) = \text{constant} = k$
⇨ $dr/dt = k/(2\pi r)$

Now, perimeter = $P = 2\pi r$

⇨ $dP/dt = 2\pi(dr/dt) = 2\pi k/(2\pi r) = k/r$
⇨ varies inversely as the radius

Option (a) is correct.

730. Let $y = \tan^{-1}[\{\sqrt{(1 + x^2)} - 1\}/x]$. Then dy/dx equals

(a) $1/\{2(1 + x^2)\}$
(b) $2/(1 + x^2)$
(c) $(-1/2)\{1/(1 + x^2)\}$
(d) $-2/(1 + x^2)$

Solution :

$dy/dx = 1/[1 + \{\sqrt{(1 + x^2)} - 1\}^2/x^2][x^2/\sqrt{(1 + x^2)} - \{\sqrt{(1 + x^2)} - 1\}]/x^2$

$= [1/\{x^2 + 1 + x^2 - 2\sqrt{(1 + x^2)} + 1\}][x^2 - 1 - x^2 + \sqrt{(1 + x^2)}]$

$= (1/[\{2\sqrt{(1 + x^2)}\}\{\sqrt{(1 + x^2)} - 1\}])\{\sqrt{(1 + x^2)} - 1\}$

$= 1/\{2\sqrt{(1 + x^2)}\}$

Option (a) is correct.

731. If θ is an acute angle then the largest value of $3\sin\theta + 4\cos\theta$ is
(a) 4
(b) $3(1 + \sqrt{3}/2)$
(c) $5\sqrt{2}$
(d) 5

Solution :

Now, $3\sin\theta + 4\cos\theta$

$= 5\{(3/5)\sin\theta + (4/5)\cos\theta\}$

$= 5\{\sin\theta\cos\alpha + \cos\theta\sin\alpha)$ where $\cos\alpha = 3/5$ i.e. $\sin\alpha = 4/5$

$= 5\sin(\theta + \alpha)$

Maximum value = 5

Option (d) is correct.

732. Let $f(x) = (x - 1)e^x + 1$. Then
(a) $f(x) \geq 0$ for all $x \geq 0$ and $f(x) < 0$ for all $x < 0$
(b) $f(x) \geq 0$ for all $x \geq 1$ and $f(x) < 0$ for all $x < 1$
(c) $f(x) \geq 0$ for all x
(d) none of the foregoing statements is true

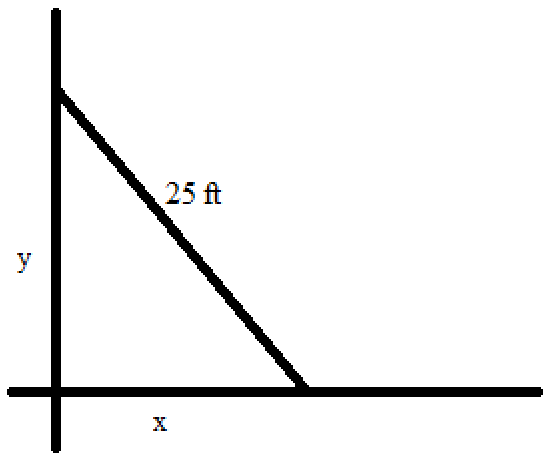
Solution :

Clearly option (c) is correct. You can check by considering values > 0 and < 0 or by drawing graph of f(x).

733. A ladder AB, 25 feet (ft) (1 ft = 12 inches (in)) long leans against a vertical wall. The lower end A, which is at a distance of 7 ft from the bottom of the wall, is being moved away along the ground from the wall at the rate of 2 ft/sec. Then the upper end B will start moving towards the bottom of the wall at the rate of (in in/sec)
(a) 10
(b) 17
(c) 7

(d) 5

Solution :



Now, $x^2 + y^2 = (25*12)^2$

⇨ $2x(dx/dt) + 2y(dy/dt) = 0$
⇨ $(dy/dt) = -(x/y)(dx/dt)$

At $x = 7*12$, $y = \sqrt{(25*12)^2 - (7*12)^2} = 24*7$

$(dy/dt) = -(7*12/24*12)*(2*12) = -7$ in/sec

(-) occurred due to the opposite motion of x and y.

Option (c) is correct.

734. Let $f(x) = ||x - 1| - 1|$ if $x < 1$ and $f(x) = [x]$ if $x \geq 1$, where, for any real number x, [x] denotes the largest integer ≤ x and |y| denotes absolute value of y. Then, the set of discontinuity-points of the function f consists of
(a) all integers ≥ 0
(b) all integers ≥ 1
(c) all integers > 1
(d) the integer 1

Solution :

Let us first check at x = 1.

lim f(x) as x -> 1- = lim ||x – 1| - 1| as x -> 1- = 1

lim f(x) as x-> 1+ = lim [x] as x-> 1+ = 1

So, continuous at x = 1.

Let us now check at x = 0.

lim f(x) as x -> 0- = lim ||x – 1| - 1| as x -> 0- = 0

lim f(x) as x-> 0+ = lim ||x – 1| - 1| as x-> 0+ = 0

So continuous at x = 0.

Let us now check at x= 2.

lim f(x) as x -> 2- = lim [x] as x -> 2- = 1

lim f(x) as x-> 2+ = lim [x] as x -> 2+ = 2

Discontinuous at x = 2.

Option (c) is correct.

735. Let f and g be two functions defined on an interval I such that f(x) ≥ 0 and g(x) ≤ 0 for all x Ɛ I, and f is strictly decreasing on I while g is strictly increasing on I. Then

(a) the product function fg is strictly increasing on I

(b) the product function fg is strictly decreasing on I

(c) the product function fg is increasing but not necessarily strictly increasing on I

(d) nothing can be said about the monotonicity of the product function fg

Solution :

Now, f' < 0 and g ≤ 0 => f'g ≥ 0

And, g' > 0 and f ≥ 0 => fg' ≥ 0

- ⇨ f'g + fg' ≥ 0
- ⇨ (fg)' ≥ 0

Now the equality holds if and only if f and g are zero at same point. But f(x) is decreasing that means f = 0 at final value of I if we order the set I from decreasing to increasing value and g = 0 at first value of I as g is increasing. So they cannot be equal to zero together.

⇨ fg is strictly increasing

Option (a) is correct.

736. Given that f is a real-valued differentiable function such that $f(x)f'(x) < 0$ for all real x, it follows that
(a) f(x) is an increasing function
(b) f(x) is a decreasing function
(c) |f(x)| is an increasing function
(d) |f(x)| is a decreasing function

Solution :

Now, f(x) is differentiable means f(x) is continuous. Now $f(x) \neq 0$ for any x.

Therefore, f(x) either > 0 or < 0

If $f(x) > 0$ then $f'(x) < 0$ and if $f(x) < 0$ then $f'(x) > 0$

⇨ f(x) is either increasing or decreasing function.

Now, $|f(x)| > 0$ so, $|f(x)|' < 0$

⇨ |f(x) is decreasing function.

Option (d) is correct.

737. Let x and y be positive numbers. Which of the following always implies $x^y \geq y^x$?
(a) $x \leq e \leq y$
(b) $y \leq e \leq x$
(c) $x \leq y \leq e$ or $e \leq y \leq x$
(d) $y \leq x \leq e$ or $e \leq x \leq y$

Solution :

Let us take, x, y < e, say x = 2, y = 1

Now, $2^1 > 1^2$

- ⇨ $x^y > y^x$
- ⇨ So if x, y < e then x > y

It is with option (d).

Now, let us check the other part of option (d).

Let, x, y > e, x = 4 and y = 5

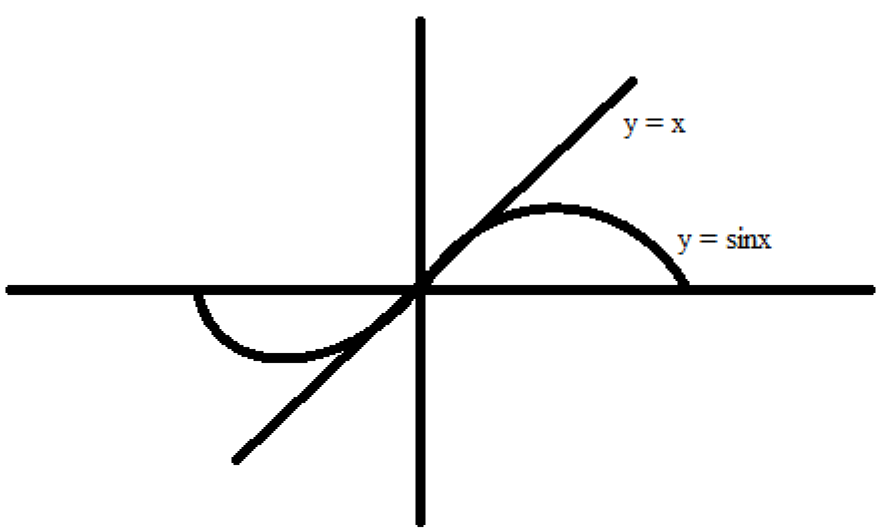
Now, $4^5 > 5^4$

- ⇨ $x^y > y^x$
- ⇨ Option (d) is correct.

738. Let f be a function $f(x) = \cos x - 1 + x^2/2$. Then
   - (a) f(x) is an increasing function on the real line
   - (b) f(x) is a decreasing function on the real line
   - (c) f(x) is an increasing function in the interval $-\infty < x \le 0$ and decreasing in the interval $0 \le x < \infty$
   - (d) f(x) is a decreasing function in the interval $-\infty < x \le 0$ and increasing in the interval $0 \le x < \infty$

Solution :

Now, $f'(x) = x - \sin x$



From the figure it is clear that for $x > 0$, $x - \sin x > 0$ and for $x < 0$ $x - \sin x < 0$

Option (d) is correct.

739. Consider the function $f(n)$ defined for all positive integers as follows :
$f(n) = n + 1$ if n is odd, and $f(n) = n/2$ if n is even.
Let $f^{(k)}$ denote f applied k times; e.g., $f^{(1)}(n) = f(n)$, $f^{(2)}(n) = f(f(n))$ and so on. Then
(a) there exists one integer k_0 such that $f^{(k0)}(n) = 1$
(b) for each $n \geq 2$, there exists an integer k (depending on n) such that $f^{(k)}(n) = 1$
(c) for each $n \geq 2$, there exists an integer k (depending on n) such that $f^{(k)}(n)$ is a multiple of 4
(d) for each n, $f^{(k)}(n)$, $f^{(k)}(n)$ is a decreasing function of k

Solution :

Let us take an odd integer m.

$f(m) = m + 1$

$f((m + 1)) = (m + 1)/2$

if it is odd, then $f((m + 1)/2) = (m + 1)/2 + 1 = (m + 3)/2$ which is even

$f((m + 3)/2) = (m + 3)/4$

Now, $(m + 1)/2 - (m + 1) = -(m + 1)/2$

$(m + 3)/4 - (m + 1)/2 = -(m + 1)/2$

- ⇨ We are getting always a decreased number when the function is applied after two operation when it is odd and when it is even then it is getting halved.
- ⇨ All the function values must come to 1 (the minimum positive integer) but number of application of f may differ.
- ⇨ Option (b) is correct.

740. Let $p_n(x)$, $n = 0, 1, \ldots.$ be a polynomial defined by $p_0(x) = 1$, $p_1(x) = x$ and $p_n(x) = xp_{(n-1)}(x) - p_{(n-2)}(x)$ for $n \geq 2$. Then $p_{10}(0)$ equals

(a) 0
(b) 10
(c) 1
(d) -1

Solution :

Now, $p_n(x) = xp_{(n-1)}(x) - p_{(n-2)}(x)$

- ⇨ $p_{(n)}(0) = -p_{(n-2)}(0)$
- ⇨ $p_{10}(0) = -p_8(0) = p_6(0) = -p_4(0) = p_2(0) = -p_0(0) = -1$

Option (d) is correct.

741. Consider the function $f(x) = x(x - 1)(x + 1)$ from **R** to **R**, where **R** is the set of all real numbers. Then,
(a) f is one-one and onto
(b) f is neither one-one nor onto
(c) f is one-one but not onto
(d) f is not one-one but onto

Solution :

Now, $f(x) = x(x - 1)(x + 1) = 0$ for $x = 0, -1, 1$

So, f(x) is not one-one.

As f(x) is a polynomial function so f(x) is continuous everywhere.

And $f(\infty) = \infty$ and $f(-\infty) = -\infty$

So, f in onto

Option (d) is correct.

742. For all integers $n \geq 2$, define $f_n(x) = (x + 1)^{1/n} - x^{1/n}$, where $x > 0$. Then, as a function of x
(a) f_n is increasing for all n
(b) f_n is decreasing for all n
(c) f_n is increasing for n odd and f_n is decreasing for n even
(d) f_n is decreasing for n odd and f_n is increasing for n even

Solution :

$f_n'(x) = (1/n)(x + 1)^{1/n - 1} - (1/n)x^{1/n - 1}$

- ⇨ $f_n'(x) = (1/n)[1/(1 + x)^{(n - 1)/n} - 1/x^{(n - 1)/n}] = (1/n)[x^{(n - 1)/n} - (1 + x)^{(n - 1)/n}]/\{x(1 + x)\}^{(n - 1)/n} < 0$
- ⇨ f_n is decreasing for all n.

Option (b) is correct.

743. Let $g(x) = \int tf'(t)dt$ (integration running from t = -10 to t = x) for $x \geq -10$, where f is an increasing function. Then
   (a) g(x) is an increasing function of x
   (b) g(x) is a decreasing function of x
   (c) g(x) is increasing for x > 0 and decreasing for -10 < x < 0
   (d) none of the foregoing conclusions is necessarily true

Solution :

Now, $g'(x) = xf'(x) > 0$ for $x > 0$ as f(x) is increasing and < 0 for $x < 0$

Option (c) is correct.

744. Let $f(x) = x^3 - x + 3$ for $0 < x \leq 1$, $f(x) = 2x + 1$ for $1 < x \leq 2$, $f(x) = x^2 + 1$ for $2 < x < 3$. Then
   (a) f(x) is differentiable at x = 1 and at x = 2
   (b) f(x) is differentiable at x = 1 but not at x = 2
   (c) f(x) is differentiable at x = 2 but not at x = 1
   (d) f(x) is differentiable neither at x = 1 nor at x = 2

Solution :

lim $\{f(x) - f(1)\}/(x - 1)$ as x -> 1- = lim $\{x^3 - x + 3 - 3\}/(x - 1)$ as x -> 1- = lim $x(x - 1)(x + 1)/(x - 1)$ as x -> 1- = lim $x(x + 1)$ as x -> 1- = 2

lim $\{f(x) - f(1)\}/(x - 1)$ as x -> 1+ = lim $(2x + 1 - 3)/(x - 1)$ as x -> 1+ = lim $2(x - 1)/(x - 1)$ as x -> 1+ = lim 2 as x -> 1+ = 2

f(x) is differentiable at x = 1.

lim {f(x) – f(2)}/(x – 2) as x -> 2- = lim {2x + 1 – 5}/(x – 2) as x-> 2- = lim 2(x – 2)/(x – 2) as x -> 2- = lim 2 as x-> 2- = 2

lim {f(x) – f(2)}/(x – 2) as x -> 2+ = lim {x^2 + 1 – 5}/(x – 2) as x-> 2+ = lim (x – 2)(x + 2)/(x – 2) as x->2+ = lim (x + 2) as x -> 2+ = 4

The two limits are not same. Hence f(x) is not differentiable at x = 2.

Option (b) is correct.

745. If the function f(x) = (x^2 – 2x + A)/sinx when x ≠ 0, f(x) = B when x = 0, is continuous at x = 0, then
(a) A = 0, B = 0
(b) A = 0, B = -2
(c) A = 1, B = 1
(d) A = 1, B = 0

Solution :

lim f(x) as x-> 0 = lim (x^2 – 2x + A)/sinx as x -> 0

To exist the limit A = 0 (must be)

Therefore, lim (x^2 – 2x)/sinx as x -> 0 = lim (2x – 2)/cosx as x-> 0 (Applying L'Hospital rule) = -2

B = -2

Option (b) is correct.

746. The function f(x) = (1 – cos4x)/x^2 if x < 0, f(x) = a if x = 0, f(x) = 2√x/{√(16 + √x) – 4} if x > 0, is continuous at x = 0 for
(a) a = 8
(b) a = 4
(c) a = 16
(d) no value of a

Solution :

lim f(x) as x ->0- = lim (1 – cos4x)/x^2 as x-> 0- = lim 4sin4x/2x as x-> 0- (Applying L'Hospital rule) = lim 16cos4x/2 as x -> 0-(Again applying L'Hospital rule) = 8

lim f(x) as x-> 0+ = lim 2√x/{√(16 + √x) – 4} as x -> 0+ = lim 2√x{√(16 + √x) + 4}/(16 + √x – 16) as x -> 0+ = lim 2√x{√(16 + √x) + 4}/√x as x -> 0+ = lim 2{√(16 + √x) + 4} as x -> 0+ = 16

Therefore, the limit doesn't exist.

Option (d) is correct.

747. Consider the function f(x) = 0 if x is rational, f(x) = x^2 if x is irrational. Then only one of the following statements is true. Which one is it?
(a) f is differentiable at x = 0 but not continuous at any other point
(b) f is not continuous anywhere
(c) f is continuous but not differentiable at x = 0
(d) None of the foregoing statements is true.

Solution :

Clearly, option (a) is correct.

748. Let f(x) = xsin(1/x) if x ≠ 0, and let f(x) = 0 if x = 0. Then f is
(a) not continuous at 0
(b) continuous but not differentiable at 0
(c) differentiable at 0 and f'(0) = 1
(d) differentiable at 0 and f'(0) = 0

Solution :

lim f(x) as x->0 = lim xsin(1/x) as x->0 = 0

f(0) = 0

So, continuous at x = 0

Now, lim {f(x) – f(0)}/(x – 0) as x -> 0 = lim xsin(1/x)/x as x->0 = lim sin(1/x) as x -> 0 doesn't exist.

So, not differentiable.

Option (b) is correct.

749. Let f(x) be the function defined on the interval (0, 1) by f(x) = x if x is rational, f(x) = 1 – x otherwise. Then f is continuous
(a) at no point in (0, 1)
(b) at exactly one point in (0, 1)
(c) at more than one point, but finitely many points in (0, 1)
(d) at infinitely many points in (0, 1)

Solution :

Clearly, f is continuous at x = ½

Option (b) is correct.

750. The function f(x) = [x] + √(x – [x]), where [x] denotes the largest integer smaller than or equal to x, is
(a) continuous at every real number x
(b) continuous at every real number x except at negative integer values
(c) continuous at every real number x except at integer values
(d) continuous at every real number x except x = 0

Solution :

Let, x = -n where n > 0 i.e. x is a negative integer.

x – [x] = -n – [-n] = - n –(-n) = -n + n = 0

Therefore, f(x) = -n.

lim f(x) as x-> -n- = lim [x] + √(x – [x]) as x -> -n- = -n + √(-n – (-n)) = -n

lim f(x) as x-> -n+ = lim [x] + √(x – [x]) as x -> -n+ = -(n + 1) + √(-n – (-(n + 1))) = -(n + 1) + √(-n + n + 1) = -n – 1 + 1 = -n

So, f(x) is continuous at negative integer values.

So, option (b) and (c) cannot be true.

At x = 0,

lim f(x) as x-> 0- = lim [x] + √(x − [x]) as x-> 0- = -1 + √(0 − (-1)) = -1 + 1 = 0

lim f(x) as x-> 0+ = lim [x] + √(x − [x]) as x-> 0+ = 0 + √(0 − 0) = 0

f(x) is continuous at x = 0

Option (d) cannot be true.

Option (a) is correct.

751. For any real number x and any positive integer n, we can uniquely write x = mn + r, where m is an integer (positive, negative or zero) and 0 ≤ r < n. With this notation we define x mod n = r. For example, 13.2 mod 3 = 1.2. The number of discontinuity points of the function f(x) = $(x \text{ mod } 2)^2$ + (x mod 4) in the interval 0 < x < 9 is
(a) 0
(b) 2
(c)4
(d) 6

Solution :

Now, at x = 2,

lim f(x) as x-> 2- = lim $(x \text{ mod}2)^2$ + (x mod 4) as x-> 2- = 2^2 + 2 = 6

lim f(x) as x-> 2+ = lim $(x \text{ mod } 2)^2$ + (x mod 4) as x -> 2+ = 0^2 + 2 = 2

discontinuous at x = 2.

Similarly, discontinuous at x = 4, 6, 8 i.e. all even numbers.

Option (c) is correct.

752. Let f(x) and g(x) be defined as follows :
f(x) = x if x ≥ 0, f(x) = 0 if x < 0
g(x) = x^2 if x ≥ 0, g(x) = 0 if x < 0
Then

(a) f and g both differentiable at $x = 0$
(b) f is differentiable at $x = 0$ but g is not
(c) g is differentiable at $x = 0$ but f is not
(d) neither f nor g is differentiable at $x = 0$

Solution :

lim (f(x) – f(0)}/(x – 0) as x -> 0- = lim (0 – 0)/(x – 0) as x-> 0- = 0

lim {f(x) – f(0)}/(x – 0) as x-> 0+ lim (x – 0)/x as x ->0+ = lim x/x as x -> 0+ = lim 1 as x-> 0+ = 1

f is not differentiable at x = 0

lim {g(x) – g(0)}/(x – 0) x -> 0- = lim (0 - 0)/(x – 0) as x-> 0- = 0

lim {g(x) – g(0)}/(x – 0) as x-> 0+ = lim (x^2 – 0)/x as x-> 0+ = lim x as x -> 0+ = 0

g(0) = 0

g is differentiable at x = 0

Option (c) is correct.

753. The number of points at which the function f(x) = min{|x|, x^2} if -∞ < x < 1, f(x) = min{2x – 1, x^2} otherwise, is not differentiable is
(a) 0
(b) 1
(c) 2
(d) More than 1

Solution :

At x = 1,

lim {f(x) – f(1)}/(x – 1) as x -> 1- = lim (x^2 – 1)/(x – 1) as x-> 1- = lim (x + 1) as x-> 1- = 2

lim {f(x) – f(1)}/(x – 1) as x-> 1+ = lim (2x – 1 – 1)/(x – 1) as x -> 1+ = lim 2 as x -> 1+ = 2

at x = 1 f(x) is differentiable.

Now, $x^2 > 2x - 1$ i.e. $x^2 - 2x + 1 > 0$ i.e. $(x - 1)^2 > 0$ i.e. $x > 1$.

So, for $x > 1$ $2x - 1$ is always minimum.

So, at every point > 1 f(x) is differentiable.

$f(x) = x^2$ if $x > -1$ and $f(x) = |x|$ if $x < -1$

So, we check differentiability at x = -1.

lim {f(x) – f(-1)}/(x + 1) as x -> -1- = lim (|x| - 1)/(x + 1) as x -> -1- = lim –(x + 1)/(x + 1) as x -> -1- = -1

lim {f(x) – f(-1)}/(x + 1) as x -> -1+ = lim $(x^2 - 1)$/(x + 1) as x -> -1+ = lim (x – 1) as x -> -1+ = - 2

Not differentiable at x = -1

Option (b) is correct.

754. The function f(x) is defined as f(x) = 1/|x|, for |x| > 2, f(x) = a + bx^2 for $|x| \leq 2$, where a and b are known constants. Then, only one of the following statements is true. Which one is it?
 (a) f(x) is differentiable at x = -2 if and only if a = ¾ and b = -1/16
 (b) f(x) is differentiable at x = -2, whatever be the values of a and b
 (c) f(x) is differentiable at x = -2, if b = -1/16 whatever be the value of a
 (d) f(x) is differentiable at x = -2, if b = 1/16 whatever be the value of a

Solution :

lim {f(x) – f(-2)}/(x + 2) as x -> -2- = lim {-1/x – a – 4b)/(x + 2) as x -> -2-

To exist the limit, a + 4b = 1/2

Therefore, the limit is, lim {-1/x – ½}/(x + 2) as x -> -2- = lim -1/(2x) as x -> -2- = 1/4

lim {f(x) – f(-2)}/(x + 2) as x -> -2+ = lim {a + bx^2 – a – 4b}/(x + 2) as x -> -2+ = lim b(x – 2)(x + 2)/(x + 2) as x -> -2+ = lim b(x – 2) as x -> -2+ = -4b

So, -4b = 1/4

- ⇨ b = -1/16
- ⇨ a = 1/2 + 4/16 = 1/2 + ¼ = ¾

Option (a) is correct.

755. The function $f(x) = \sin x^2/x$ if $x \neq 0$, $f(x) = 0$ if $x = 0$
(a) is continuous, but not differentiable at x = 0
(b) is differentiable at x = 0 but the derivative is not continuous at x = 0
(c) is differentiable at x = 0 and the derivative is continuous at x = 0
(d) is not continuous at x = 0

Solution :

lim f(x) as x -> 0 = lim $\sin x^2/x$ as x -> 0 = lim $x(\sin x^2/x^2)$ as x -> 0 = 0*1 = 0

And f(0) = 0.

f(x) is continuous at x = 0.

lim {f(x) – f(0)}/(x – 0) as x -> 0 = lim $\{(\sin x^2/x)/x\}$ as x -> 0 = lim $\sin x^2/x^2$ as x^2 -> 0 = 1

Differentiable at x = 0.

$f'(x) = (2x\sin x^2 * x - 2x\sin x^2)/x^2 = 2\sin x^2 - 2\sin x^2/x$

lim f'(x) as x -> 0 = lim $2\sin x^2 - 2\sin x^2/x$ as x -> 0 = lim $-x(\sin x^2/x^2)$ as x -> 0 = -0*1 = 0

f'(0) = 0

So, f'(x)is continuous at x = 0.

Option (c) is correct.

756. Let f(x) = x[x] where [x] denotes the greatest integer smaller then or equal to x. When x is not an integer, what is f'(x)?
(a) 2x
(b) [x]
(c) 2[x]
(d) It doesn't exist.

Solution :

f(x) = x[x]

⇨ f'(x) = [x] + x*d/dx([x])

Now, d/dx[x] = 0 as it is constant.

Therefore, f(x) = [x]

Option (b) is correct.

757. If f(x) = (sinx)(sin2x)….(sinnx), then f'(x) is
(a) ∑kcoskxf(x) (summation running from k = 1 to k = n)
(b) (cosx)(2cos2x)(3cos3x)…(ncosnx)
(c) ∑(kcoskx)(sinkx) (summation running from k = 1 to k = n)
(d) ∑(kcotkx)f(x) (summation running from k = 1 to k = n)

Solution :

f'(x) = (cosx)(sin2x)(sin3x)….(sinnx) + (sinx)(2cos2x)(sin3x)(sin4x)…(sinnx) + …. + (sinx)(sin2x)…(sin(n – 1)x)(ncosnx)

= (cotx)f(x) + (2cot2x)f(x) + …. + (ncotnx)f(x)

= ∑(kcotkx)f(x) (summation running from k = 1 to k = n)

Option (d) is correct.

758. Let $f(x) = a_0 + a_1|x| + a_2|x|^2 + a_3|x|^3$ where a_0, a_1, a_2 and a_3 are constants. Then only one of the following statements is correct. Which one is it?
(a) f(x) is differentiable at x = 0 whatever be a_0, a_1, a_2, a_3
(b) f(x) is not differentiable at x = 0 whatever be a_0, a_1, a_2, a_3
(c) If f(x) is differentiable at x = 0, then $a_1 = 0$
(d) If f(x) is differentiable at x = 0, then $a_1 = 0$ and $a_3 = 0$

Solution :

lim {f(x) – f(0)}/(x – 0) as x -> 0- = lim {a_0 + a_1|x| + a_2|x|^2 + a_3|x|^3 – a_0}/x as lim x -> 0- = lim (-a_1x + a_2x^2 – a_3x^3)/x as x -> 0- = lim (-a_1 + a_2x – a_3x^2) as x -> 0- = -a_1

lim {f(x) – f(0)}/(x – 0) as x -> 0+ = lim {a_0 + a_1|x| + a_2|x|^2 + a_3|x|^3}/x as x -> 0+ = lim (a_1x + a_2x^2 + a_3x^3)/x as x -> 0+ = lim (a_1 + a_2x + a_3x^2) as x -> 0+ = a_1

Now, if f(x) is differentiable at x = 0 then –a_1 = a_1 i.e. a_1 = 0

Option (c) is correct.

759. Consider the function f(x) = |sinx| + |cosx| defined for x in the interval (0, 2п). Then
(a) f(x) is differentiable everywhere
(b) f(x) is not differentiable at x = п/2 and 3п/2 and differentiable everywhere else
(c) f(x) is not differentiable at x = п/2, п and 3п/2 and differentiable everywhere else
(d) none of the foregoing statements is true

Solution :

At x = п/2

lim {f(x) – f(п/2)}/(x – п/2) as x -> п/2- = lim {|sinx| + |cosx| - 1}/(x – п/2) as x -> п/2- = lim (sinx + cosx – 1)/(x – п/2) as x -> п/2- = lim (cosx – sinx)/1 as x -> п/2- (Applying L'Hospital rule) = -1

lim {f(x) – f(п/2)}/(x – п/2) as x -> п/2+ = lim {|sinx| + |cosx| - 1}/(x – п/2) as x -> п/2+ = lim (sinx – cosx – 1)/(x – п/2) as x -> п/2+ = lim (cosx + sinx)/1 as x -> п/2+ (Applying L'Hispital rule) = 1

Not differentiable at x = п/2

At x = 3п/2

lim {f(x) – f(3п/2)}/(x - 3п/2) as x -> 3п/2- = lim {|sinx| + |cosx| - 1}/(x - 3п/2) as x -> 3п/2- = lim (-sinx – cosx – 1)/(x - 3п/2) as x -> 3п/2- (both sinx and cosx are in 3rd quadrant where only tan is positive) = lim (-cosx + sinx)/1 as x -> 3п/2- (Applying L'Hospital rule) = -1

lim {f(x) – f(3п/2)}/(x - 3п/2) as x -> 3п/2+ = lim {|sinx| + |cosx| - 1}/(x - 3п/2) as x -> 3п/2+ = lim (-sinx + cosx – 1)/(x - 3п/2) as x -> 3п/2+

(sinx is negative and cosx is positive in 4^{th} quadrant) = lim (-cosx – sinx)/1 as x -> 3π/2+ = 1

Not differentiable at x = 3π/2

At x = π

lim {f(x) – f(π)}/(x – π) as x -> π- = lim {|sinx| + |cosx| - 1}/(x – π) as x -> π- = lim (sinx – cosx – 1)/(x – π) as x -> π- (sinx is positive and cosx is nrgative in 2^{nd} quadrant) = lim (cosx + sinx)/1 as x -> π- (Applying L'Hospital rule) = -1

lim {f(x) – f(π)}/(x – π) as x -> π+ = lim {|sinx| + |cosx| - 1}/(x – π) as x -> π+ = lim (-sinx – cosx – 1)/(x – π) as x -> π+ = lim (-cosx + sinx)/1 as x -> π+ (Applying L'Hospital rule) = 1

Not differentiable at x = π

Option (c) is correct.

760. A curve in the XY plane is given by the parametric equations $x = t^2 + t + 1$, $y = t^2 - t + 1$, where the parameter t varies over all nonnegative real numbers. The number of straight line passing through the point (1, 1) which are tangent to the curve, is
(a) 2
(b) 0
(c) 1
(d) 3

Solution :

Now, x – y = 2t

- ⇨ t = (x – y)/2
- ⇨ $x = (x - y)^2/4 + (x - y)/2 + 1$
- ⇨ $4x = x^2 - 2xy + y^2 + 2x - 2y + 4$
- ⇨ $x^2 + y^2 - 2xy - 2x - 2y + 4 = 0$

Equation of the tangent is y – 1 = m(x – 1)

- ⇨ y = mx – (m – 1)

Putting the value of y in the equation of curve we get,

$x^2 + \{mx - (m - 1)\}^2 - 2x\{mx - (m - 1)\} - 2x - 2\{mx - (m - 1)\} + 4 = 0$

⇨ $x^2 + m^2x^2 - 2m(m - 1)x + (m - 1)^2 - 2mx^2 + 2(m - 1)x - 2x - 2mx + 2(m - 1) + 4 = 0$
⇨ $x^2(m^2 - 2m + 1) - 2x(m^2 - m - 2m + 2 + 2 + 2m) + \{(m - 1)^2 + 2(m - 1) + 4\} = 0$
⇨ $x^2(m - 1)^2 - 2x(m^2 - m + 4) + (m^2 + 3) = 0$

Now, discriminant = 0

⇨ $4(m^2 - m + 4)^2 - 4(m - 1)^2(m^2 + 3) = 0$
⇨ $m^4 + m^2 + 16 - 2m^3 - 8m + 8m^2 - (m^2 - 2m + 1)(m^2 + 3) = 0$
⇨ $m^4 - 2m^3 + 9m^2 - 8m + 16 - m^4 - 3m^2 + 6m + 2m^3 - m^2 - 3 = 0$
⇨ $5m^2 - 2m + 13 = 0$
⇨ No solution as discriminant < 0
⇨ No tangents can be drawn.

Option (b) is correct.

761. If $f(x) = \{(a + x)/(b + x)\}^{a + b + 2x}$, then $f'(0)$ equals
(a) $\{(b^2 - a^2)/b^2\}(a/b)^{a + b - 1}$
(b) $\{2\log(a/b) + (b^2 - a^2)/ab\}(a/b)^{a + b}$
(c) $2\log(a/b) + (b^2 - a^2)/ab$
(d) None of the foregoing expressions

Solution :

$\log f(x) = (a + b + 2x)[\log(a + x) - \log(b + x)]$

$f'(x)/f(x) = 2[\log(a + x) - \log(b + x)] + (a + b + 2x)[1/(a + x) - 1/(b + x)$

⇨ $f'(0)/f(0) = 2[\log a - \log b] + (a + b)(1/a - 1/b)$
⇨ $f'(0) = \{(a/b)^{a + b}\}\{2\log(a/b) + (b^2 - a^2)/ab\}$

Option (b) is correct.

762. If $y = 2\sin^{-1}\sqrt{(1 - x)} + \sin^{-1}[2\sqrt{\{x(1 - x)\}}]$ for $0 < x < ½$ then dy/dx equals
(a) $2/\sqrt{\{x(1 - x)\}}$
(b) $\sqrt{\{(1 - x)/x\}}$
(c) $-1/\sqrt{\{x(1 - x)\}}$
(d) 0

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Solution :

Let $x = \cos^2 A$

Now,2 $\sin^{-1}\sqrt{(1 - \cos^2 A)} = 2\sin^{-1}\sin A = 2A$

Now, $\sin^{-1}[2\cos A\sin A] = \sin^{-1}(\sin 2A) = 2A$

Therefore, $y = 2A + 2A = 4A = 4\cos^{-1}\sqrt{x}$

$dy/dx = -4/\sqrt{(1 - x)}$

It is given option (d) is correct.

763. If $y = \sin^{-1}(3x - 4x^3)$ then dy/dx equals
(a) $3x$
(b) 3
(c) $3/\sqrt{(1 - x^2)}$
(d) None of the foregoing expressions.

Solution :

Let $x = \sin A$

$3x - 3x^3 = 3\sin A - \sin^3 A = \sin 3A$

$\sin^{-1}(3x - 3x^3) = \sin^{-1}(\sin 3A) = 3A = 3\sin^{-1}x$

$dy/dx = 3/\sqrt{(1 - x^2)}$

Option (c) is correct.

764. If $y = 3^{\sin ax/\cos bx}$, then dy/dx is
(a) 3^{(acosaxcosbx + bsinaxsinbx)/cox^{2}bx}
(b) $3^{\sin ax/\cos bx}\{(a\cos ax\cos bx + b\sin ax\sin bx)/\cos^2 bx\}\log 3$
(c) $3^{\sin ax/\cos bx}\{(a\cos ax\cos bx - b\sin ax\sin bx)/\cos^2 bx\}\log 3$
(d) $3^{\sin ax/\mathrm{cob}x}\log 3$

Solution :

$\log y = (\sin ax/\cos bx)\log 3$

$(dy/dx)/y = \{(a\cos ax \cos bx + b\sin bx \sin ax)/\cos^2 bx\}\log 3$

Option (b) is correct.

765. $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$, then the value of d^2y/dx^2 at $\theta = \pi/2$ equals
(a) -1/a
(b) -1/4a
(c) –a
(d) None of the foregoing numbers.

Solution :

Now, $d^2y/dx^2 = d/dx(dy/dx) = d/dx\{(dy/d\theta)/(dx/d\theta)\} = d/d\theta\{(dy/d\theta)/(dx/d\theta)\}(d\theta/dx)$

$= \{(d^2y/d\theta^2)(dx/d\theta) - (d^2x/d\theta^2)(dy/d\theta)\}/(dx/d\theta)^3$

Now put the values here and get the answer.

Option (a) is correct.

766. Let $F(x) = e^x$, $G(x) = e^{-x}$ and $H(x) = G(F(x))$, where x is a real number. Then dH/dx at x = 0 is
(a) 1
(b) -1
(c) -1/e
(d) –e

Solution :

$H(x) = e^{\wedge}(-e^x)$

$\log H(x) = -e^x$

$H'(x)/H(x) = -e^x$

$H'(x) = -H(x)e^x$

$H'(0) = -H(0)*1 = -H(0)$

$H(0) = e^{\wedge}(-e^0) = e^{-1} = 1/e$

$H'(0) = -1/e$

Option (c) is correct.

767. Let $f(x) = |\sin^3 x|$ and $g(x) = \sin^3 x$, both being defined for x in the interval $(-\pi/2, \pi/2)$. Then
(a) $f'(x) = g'(x)$ for all x
(b) $f'(x) = -g'(x)$ for all x
(c) $f'(x) = |g'(x)|$ for all x
(d) $g'(x) = |f'(x)|$ for all x

Solution :

$f(x) = \sin^3 x$ for $0 \le x < \pi/2$

$f(x) = -\sin^3 x$ for $-\pi/2 < x < 0$

$g(x) = f(x)$ for $0 \le x < \pi/2$ and $g(x) = -f(x)$ for $-\pi/2 < x < 0$

$g'(x) = f'(x)$ and $g'(x) = -f'(x)$

$g'(x) = |f'(x)|$

Option (d) is correct.

768. Consider the functional equation $f(x - y) = f(x)/f(y)$. If $f'(0) = p$ and $f'(5) = q$, then $f'(-5)$ is
(a) p^2/q
(b) q/p
(c) p/q
(d) q

Solution :

Putting $y = 0$ we get, $f(0) = 1$

Putting $y = 5$ we get, $f(x - 5) = f(x)/f(5)$

⇨ $f'(x - 5) = f'(x)/f(5)$

Putting $x = 0$ we get, $f'(-5) = f'(0)/f(5) = p/f(5)$

Putting x = 5 and y = x we get, f(5 – x) = f(5)/f(x)

- ⇨ f′(5 – x)(-1) = -{f(5)/(f(x))^2}f′(x)

Putting x = 0 we get,- f′(5) = -f(5)f′(0)/(f(0))^2 = -f(5)p

- ⇨ f(5) = q/p
- ⇨ f′(-5) = p/(q/p) = p^2/q

Option (a) is correct.

769. Let f be a polynomial. Then the second derivative of f(e^x) is
   (a) f′′(e^x)*e^x + f′(e^x)
   (b) f′′(e^x)*e^{2x} + f′(x)*e^x
   (c) f′′(e^x)
   (d) f′′(e^x)*e^{2x} + f′(e^x)*e^x

Solution :

Let g(x) = f(e^x)

g′(x) = f′(e^x)*e^x

g′′(x) = f′′(e^x)*e^x*e^x + f′(e^x)*e^x = f′′(e^x)*e^{2x} + f′(e^x)e^x

Option (d) is correct.

770. If A(t) is the area of the region enclosed by the curve y = e$^{-|x|}$ and portion of the x-axis between –t and +t, then lim A(t) as t -> ∞
   (a) is 1
   (b) is ∞
   (c) is 2
   (d) doesn't exist

Solution :

A(t) = ∫e^{-x}dx (integration running from 0 to t) + ∫e^{x}dx (integration running from –t to 0)

= -e^{-x} (upper limit = t and lower limit = 0) + e^x (upper limit = 0 and lower limit = -t)

$= -e^{-t} + 1 + 1 - e^{-t}$

$= 2(1 - e^{-t})$

Lim A(t) as t -> ∞ = 2

Option (c) is correct.

771. lim $\{(e^x - 1)\tan^2 x/x^3\}$ as x -> 0
   (a) doesn't exist
   (b) exists and equals 0
   (c) exists and equals 2/3
   (d) exists and equals 1

Solution :

lim $\{(e^x - 1)/x\}(\tan x/x)^2$ as x -> 0 = 1*1 = 1

Option (d) is correct.

772. If f(x) = sinx, g(x) = x^2 and h(x) = $\log_e x$, and if F(x) = h(g(f(x))), then d^2F/dx^2 equals
   (a) $-2\mathrm{cosec}^2 x$
   (b) $2\mathrm{cosec}^3 x$
   (c) $2\cot(x^2) - 4x^2\mathrm{cosec}^2(x^2)$
   (d) $2x\cot(x^2)$

Solution :

$F(x) = h(g(\sin x)) = h(\sin^2 x) = \log_e \sin^2 x = 2\log_e \sin x$

dF/dx = (2/sinx)cosx

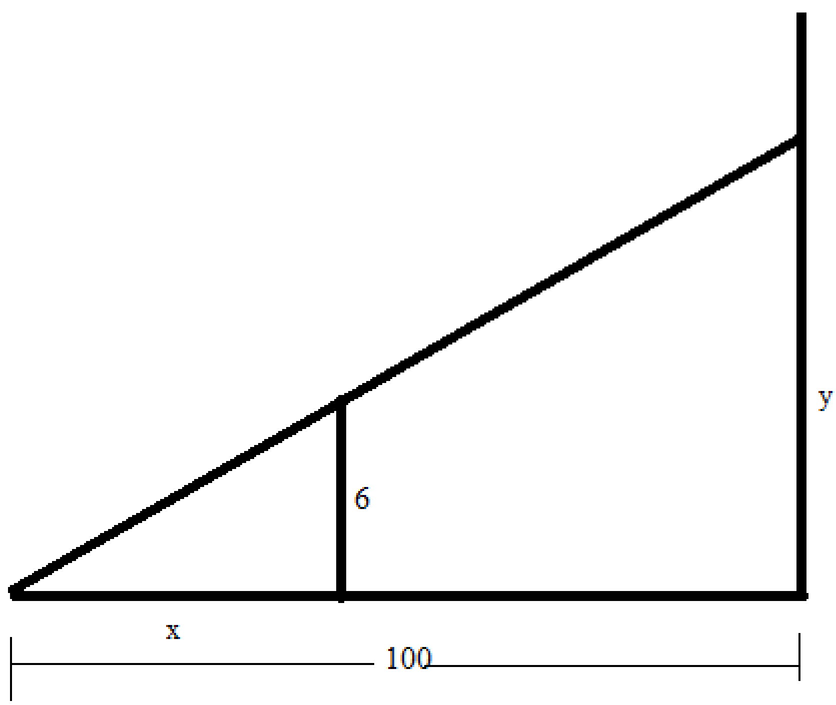
⇨ dF/dx = 2cotx
⇨ $d^2F/dx^2 = -2\mathrm{cosec}^2 x$

Option (a) is correct.

773. A lamp is placed on the ground 100 feet (ft) away from a wall. A man six ft tall is walking at a speed of 10 ft/sec from the lamp to the

nearest point on the wall. When he is midway between the lamp and the wall, the rate of change in the length of his shadow is (in ft/sec)
(a) 2.4
(b) 3
(c) 12
(d) 3.6

Solution :



From the figure it is clear that. 6/x = y/100

⇨ xy = 600
⇨ (dx/dt)y + x(dy/dt) = 0
⇨ (dy/dt) = -(y/x)(dx/dt)

When x = 50, y = 12

(dy/dt) = -(12/50)*10 = - 2.4

Option (a) is correct.

774. A water tank has the shape of a right-circular cone with its vertex down. The radius of the top of the tank is 15 ft and the height

is 10 ft. Water is poured into the tank at a constant rate of C cubic feet per second. Water leaks out from the bottom at a constant rate of one cubic foot per second. The value of C for which the water level will be rising at the rate of four ft per second at the time point when the water is two ft deep, is given by

(a) $C = 1 + 36\pi$

(b) $C = 1 + 9\pi$

(c) $C = 1 + 4\pi$

(d) $C = 1 + 18\pi$

Solution :

$V = (1/3)\pi r^2 h$

Now, r/h = constant.

- ⇨ $r = kh$
- ⇨ $15 = k*10$
- ⇨ $k = 3/2$
- ⇨ $r = 3h/2$
- ⇨ $V = (1/3)\pi(9h^2/4)h$
- ⇨ $V = (3/4)\pi h^3$
- ⇨ $dV/dt = (9/4)\pi h^2(dh/dt)$
- ⇨ $C - 1 = (9/4)\pi*2^2*4$
- ⇨ $C = 1 + 36\pi$

Option (a) is correct.

775. Let $f(x) = a|\sin x| + be^{|x|} + c|x|^3$. If f(x) is differentiable at x = 0, then

(a) $a = b = c = 0$

(b) $a = b = 0$ and c can be any real value

(c) $b = c = 0$ and a can be any real value

(d) $c = a = 0$ and b can be any real value

Solution :

$\lim \{f(x) - f(0)\}/(x - 0)$ as $x \to 0-$ $= \lim \{a|\sin x| + be^{|x|} + c|x|^3 - b\}/x$ as $x \to 0-$ $= \lim \{-a\sin x + be^{-x} - cx^3 - b\}/x$ as $x \to 0-$ $= \lim (-a\cos x - be^{-x} - 3cx^2)/1$ as $x \to 0-$ (Applying L'Hospital rule) $= -a - b$

lim {f(x) – f(0)}/(x – 0) as x -< 0+ = lim {a|sinx| + $be^{|x|}$ + $c|x|^3$ – b}/x as x -> 0+ = lim {asinx + be^x + cx^3 – b)/x as x -> 0+ = lim (acosx + be^x + $3cx^2$)/1 as x -> 0+ (Applying L'Hospital rule) = a + b

Now, -a – b = a + b

⇨ a + b = 0

Option (b) is correct.

776. A necessary and sufficient condition for the function f(x) defined by f(x) = x^2 + 2x if x ≤ 0, f(x) = ax + b if x > 0 to be differentiable at the point x = 0 is that

(a) a = 0 and b = 0
(b) a = 0 while b can be arbitrary
(c) a = 2 while b can be arbitrary
(d) a = 2 and b = 0

Solution :

lim {f(x) – f(0)}/(x – 0) as x -> 0- = lim {x^2 + 2x – 0}/x as x -> 0- = lim x + 2 as x -> 0- = 2

lim {f(x) – f(0)}/(x – 0) as x -> 0+ = lim {ax + b – 0}/x as x -> 0+ = lim (ax + b)/x as x -> 0+

Now, to hold the limit b = 0.

lim ax/x as x -> 0+ = lim a as x -> 0+ = a

And, a = 2

Option (d) is correct.

777. If f(x) = $\log_{x2}(e^x)$ defined for x > 1, then the derivative f'(x) of f(x) is

(a) (logx – 1)/2$(\log x)^2$
(b) (logx – 1)/$(\log x)^2$
(c) (logx + 1)/2$(\log x)^2$
(d) (logx + 1)/$(\log x)^2$

Solution :

$f(x) = \log_{x2}(e^x) = \log(e^x)/(\log x^2) = x/2\log x$

$f'(x) = \{1*(2\log x) - x(2/x)\}/(2\log x)^2 = (\log x - 1)/2(\log x)^2$

Option (a) is correct.

778. For $x > 0$, if $g(x) = x^{\log x}$ and $f(x) = e^{g(x)}$, then $f'(x)$ equals
(a) $\{2x^{(\log x - 1)}\log x\}f(x)$
(b) $\{x^{(2\log x - 1)}\log x\}f(x)$
(c) $(1 + x)e^x$
(d) None of the foregoing expressions

Solution :

$g(x) = x^{\log x}$

$\log g(x) = (\log x)^2$

$g'(x)/g(x) = 2\log x(1/x)$

$g'(x) = (2/x)x^{\log x}\log x = 2x^{(\log x - 1)}\log x$

$f(x) = e^{g(x)}$

$\log f(x) = g(x)$

⇨ $f'(x)/f(x) = g'(x)$
⇨ $f'(x) = \{2x^{(\log x - 1)}\log x\}f(x)$

Option (a) is correct.

779. Suppose f and g are functions having second derivatives f'' and g'' everywhere. If $f(x)g(x) = 1$ for all x and f' and g' are never zero, then $f''(x)/f'(x) - g''(x)/g'(x)$ equals
(a) $-2f'(x)/f(x)$
(b) 0
(c) $-f'(x)/f(x)$
(d) $2f'(x)/f(x)$

Solution :

Now, $f(x)g(x) = 1$

- ⇨ $f'(x)g(x) + f(x)g'(x) = 0$
- ⇨ $f''(x)g(x) + f'(x)g'(x) + f'(x)g'(x) + f(x)g''(x) = 0$
- ⇨ $\{f''(x)/f'(x)\}\{g(x)/g'(x)\} + 2 + \{g''(x)/g'(x)\}\{f(x)/f'(x)\} = 0$ (dividing by $f'(x)g'(x)$)

Now, $f'(x)g(x) + f(x)g'(x) = 0$

- ⇨ $g(x)/g'(x) + f(x)/f'(x) = 0$ (dividing by $f'(x)g'(x)$)
- ⇨ $g(x)/g'(x) = -f(x)/f'(x)$
- ⇨ $\{f''(x)/f'(x)\}\{-f(x)/f'(x)\} + \{g''(x)/g'(x)\}\{f(x)/f'(x)\} = -2$
- ⇨ $-(f(x)/f'(x))\{f''(x)/f'(x) - g''(x)/g'(x)\} = -2$
- ⇨ $f''(x)/f'(x) - g''(x)/g'(x) = 2f'(x)/f(x)$

Option (d) is correct.

780. If $f(x) = a_1e^{|x|} + a_2|x|^5$, where a_1, a_2 are constants, is differentiable at $x = 0$, then
 (a) $a_1 = a_2$
 (b) $a_1 = a_2 = 0$
 (c) $a_1 = 0$
 (d) $a_2 = 0$

Solution :

lim $\{f(x) - f(0)\}/(x - 0)$ as $x \to 0-$ = lim $\{a_1e^{|x|} + a_2|x|^5 - a_1\}/x$ as $x \to 0-$ = lim $\{a_1e^{-x} - a_2x^5 - a_1\}/x$ as $x \to 0-$ = lim $(-a_1e^{-x} - 5a_2x^4)/1$ as $x \to 0-$ (Applying L'Hospital rule) = $-a_1$

lim $\{f(x) - f(0)\}/(x - 0)$ as $x \to 0+$ = lim $\{a_1e^{|x|} + a_2|x|^5 - a_1\}/x$ as $x \to 0+$ = lim $\{a_1e^{x} + a_2x^5 - a_1\}/x$ as $x \to 0+$ = lim $(a_1e^{x} + 5a_2x^4)/1$ as $x \to 0+$ (Applying L'Hospital rule) = a_1

So, $-a_1 = a_1$

- ⇨ $a_1 = 0$

Option (c) is correct.

781. If $y = (\cos^{-1}x)^2$, then the value of $(1 - x^2)d^2y/dx^2 - xdy/dx$ is
 (a) -1
 (b) -2

(c) 1
(d) 2

Solution :

$y = (\cos^{-1}x)^2$

$dy/dx = 2(\cos^{-1}x)\{-1/\surd(1 - x^2)\} = -2\cos^{-1}x/\surd(1 - x^2)$

$d^2y/dx^2 = [\{2/\surd(1 - x^2)\}\surd(1 - x^2) - 2\cos^{-1}x\{2x/2\surd(1 - x^2)\}]/(1 - x^2)$

$(1 - x^2)d^2y/dx^2 = 2 + xdy/dx$

⇨ $(1 - x^2)d^{2y}/dx^2 - xdy/dx = 2$

Option (d) is correct.

782. The n^{th} derivative of the function $f(x) = 1/(1 - x^2)$ at the point $x = 0$, where n is even, is
(a) n^nC_2
(b) 0
(c) n!
(d) none of the foregoing quantities

Solution :

$f'(x) = -(-2x)/(1 - x^2)^2 = 2x/(1 - x^2)^2$

$f''(x) = \{2(1 - x^2)^2 - 2x*2(1 - x^2)(-2x)\}/(1 - x^2)^4 = 2(1 - x^2)(1 - x^2 + 4x^2)/(1 - x^2)^4$

$f''(0) = 2$

$f''(x) = 2(1 + 3x^2)/(1 - x^2)^3$

$f'''(x) = [2(6x)(1 - x^2)^3 - 2(1 + 3x^2)*3(1 - x^2)^2(-2x)]/(1 - x^2)^6 = 2(1 - x^2)^2[6x - 6x^3 + 6x(1 + 3x^2)]/(1 - x^2)^6$

$= 2(12x + 12x^3)/(1 - x^2)^4 = 24x(1 + x^2)/(1 - x^2)^4$

$f^{(4)}(x) = [24(1 + 3x^2)(1 - x^2)^4 - 4(1 - x^2)^3(-2x)24x(1 + x^2)]/(1 - x^2)^8 = 24(1 - x^2)^3[(1 + 3x^2)(1 - x^2) + 8x(1 + x^2)]/(1 - x^2)^8$

$f^{(4)}(0) = 24 = 4!$

Option (c) is correct.

783. Let $f(x) = x^n(1 - x)^n/n!$. Then for any integer $k \geq 0$, the k-th derivative $f^{(k)}(0)$ and $f^{(k)}(1)$
(a) are both integers
(b) are both rational numbers but not necessarily integers
(c) are both integers
(d) do not satisfy any of the foregoing properties

Solution :

$f'(x) = [nx^{n-1}(1 - x)^n + x^n n(1 - x)^{n-1}(-1)]/n!$

$f'(0) = 0$, $f'(1) = 0$

$f'(x) = x^{n-1}(1 - x)^{n-1}(1 - x + x)/(n - 1)! = x^{n-1}(1 - x)^{n-1}/(n - 1)!$

It is obvious that $f^{(k)}(0)$ and $f^{(k)}(1) = 0$ till $k = n$, after that it is an integer.

So, option (c) is correct.

784. Let $f_1(x) = e^x$, $f_2(x) = e^{\wedge}(f_1(x))$, $f_3(x) = e^{\wedge}(f_2(x))$ …. and, in general $f_{n+1}(x) = e^{\wedge}(f_n(x))$ for any $n \geq 1$. Then for any fixed n, the value of $d/dx(f_n(x))$ equals
(a) $f_n(x)$
(b) $f_n(x)f_{n-1}(x)$
(c) $f_n(x)f_{n-1}(x)….f_2(x)f_1(x)$
(d) $f_n(x)f_{n-1}(x)….f_1(x)e^x$

Solution :

$f_n'(x) = e^{\wedge}(f_{n-1}(x))f'_{n-1}(x) = f_n(x)e^{\wedge}(f_{n-2}(x))f_{n-2}'(x) = f_n(x)f_{n-1}(x)f_{n-2}'(x) = … = f_n(x)f_{n-1}(x) ….f_2(x)f_1'(x) = f_n(x)f_{n-1}(x)….f_2(x)f_1(x)$ (as $f_1'(x) = f_1(x)$)

Option (c) is correct.

785. The maximum value of $5\sin\theta + 12\cos\theta$ is
(a) 5
(b) 12

(c) 13
(d) 17

Solution :

Now, $5\sin\theta + 12\cos\theta = 13\{(5/13)\sin\theta + (12/13)\cos\theta\} = 13(\cos\alpha\sin\theta + \sin\alpha\cos\theta)$ where $\cos\alpha = 5/13$ and $\sin\alpha = 12/13$

$= 13\sin(\alpha + \theta)$

Option (c) is correct.

786. Let A and B be the points (1, 0) and (3, 0) respectively. Let P be a variable point on the y-axis. Then the maximum value of the angle APB is
(a) 22.5 degree
(b) 30 degree
(c) 45 degree
(d) None of the foregoing quantities

Solution :

Let p = (0, t)

Slope of AP = (t – 0)/(0 – 1) = -t

Slope of BP = (t – 0)/(0 – 3) = -t/3

$\tan(APB) = (-t/3 + t)/(1 + t^2/3)$

Let $F = (2t/3)/(1 + t^2/3) = 2t/(3 + t^2)$

$dF/dt = \{2(3 + t^2) - 2t*2t\}/(3 + t^2)^2 = 0$

- ⇨ $6 + 2t^2 - 4t^2 = 0$
- ⇨ $t^2 = 3$
- ⇨ $t = \pm\sqrt{3}$

$dF/dt = (6 - 2t^2)/(3 + t^2)^2$

$d^2F/dt^2 = \{-4t(3 + t^2) - 2t(3 + t^2)(6 - 2t^2)\}/(3 + t^2)^4 = (-4t - 12t + 4t^3)/(3 + t^2)^3 = 4t(t^2 - 4)/(3 + t^2)^3 < 0$ at $t = \sqrt{3}$

So, maximum value $= 2\sqrt{3}/(3 + 3) = 1/\sqrt{3}$

Tan(APB) = $1/\sqrt{3}$

⇨ APB = 30 degree

Option (b) is correct.

787. The least value of the expression $(1 + x^2)/(1 + x)$, for values $x \geq 0$, is
(a) $\sqrt{2}$
(b) 1
(c) $2\sqrt{2} - 2$
(d) None of the foregoing numbers.

Solution :

Let $f(x) = (1 + x^2)/(1 + x)$

$f'(x) = \{2x(1 + x) - (1 + x^2)\}/(1 + x)^2 = (x^2 + 2x - 1)/(1 + x)^2 = 0$

⇨ $x = \{-2 \pm \sqrt{(4 + 4)}\}/2 = -1 \pm \sqrt{2}$

$f''(x) = \{(2x + 2)(1 + x)^2 - 2(1 + x)(x^2 + 2x - 1)\}/(1 + x)^4 = 2\{1 + 2x + x^2 - x^2 - 2x + 1)/(1 + x)^3 = 4/(1 + x)^3 > 0$ at $x = \sqrt{2} - 1$

Minimum value = $\{1 + (\sqrt{2} - 1)^2\}/(1 + \sqrt{2} - 1) = \{1 + 2 + 1 - 2\sqrt{2}\}/\sqrt{2} = 2\sqrt{2} - 2$

Option (c) is correct.

788. The maximum value of $3x + 4y$ subject to the condition $x^2y^3 = 6$ and x and y are positive, is
(a) 10
(b) 14
(c) 7
(d) 13

Solution :

Weighted A.M. ≥ Weighted G.M.

⇨ $\{2(3x/2) + 3(4y/3)\}/(2 + 3) \geq \{(3x/2)^2(4y/3)^3\}^{1/5} = \{(16/3)x^2y^3\}^{1/5} = (16*6/3)^{1/5} = 2$

⇨ $3x + 4y \geq 10$

Option (a) is correct.

789. A window is in the form of a rectangle with a semicircular band on the top. If the perimeter of the window is 10 metres, the radius, in metres, of the semicircular band that maximizes the amount of light admitted is

(a) $20/(4 + \pi)$
(b) $10/(4 + \pi)$
(c) $10 - 2\pi$
(d) None of the foregoing numbers.

Solution :

$2r + 2y + \pi r = 10$ where y is height of the rectangular portion and r is the radius of the semicircular portion.

⇨ $y = (10 - 2r - \pi r)/2$

$A = 2ry + \pi r^2/2 = r(10 - 2r - \pi r) + \pi r^2/2 = 10r - 2r^2 - \pi r^2/2$

$dA/dr = 10 - 4r - \pi r = 0$

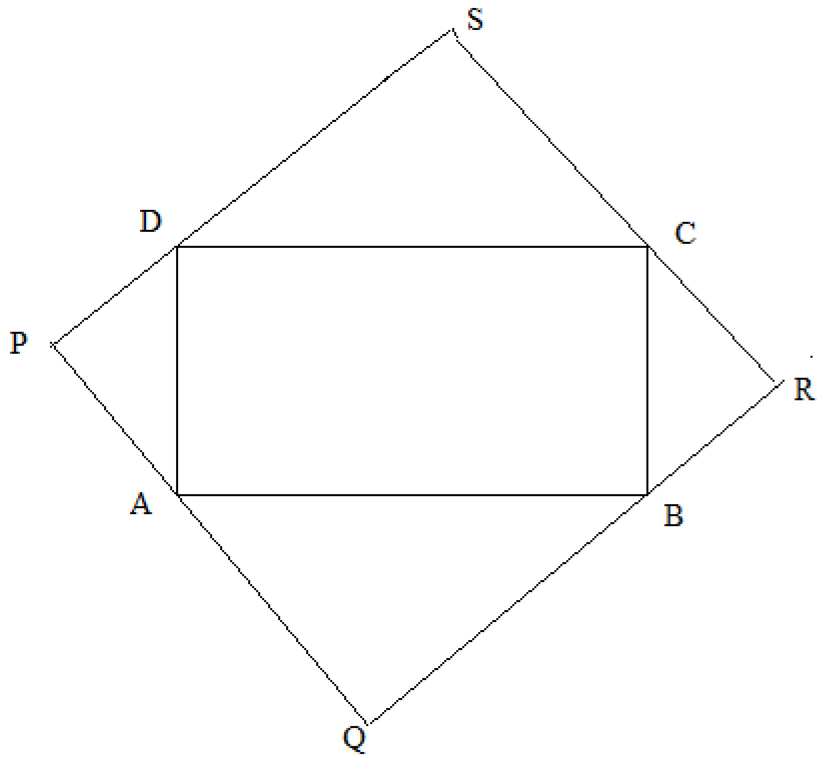
⇨ $r = 10/(4 + \pi)$

Option (b) is correct.

790. ABCD is a fixed rectangle with AB = 2cm and BC = 4 cm. PQRS is a rectangle such that A, B, C and D lie on PQ, QR, RS and SP respectively. Then the maximum possible area of PQRS is

(a) 16 cm^2
(b) 18 cm^2
(c) 20 cm^2
(d) 22 cm^2

Solution :



PQRS rectangle will be maximum when it will be a square.

In that case, $SD^2 + SC^2 = 4^2$ (SD = SC)

⇨ SD = $4/\sqrt{2}$

Similarly, PD = $2/\sqrt{2}$

PS = $4/\sqrt{2} + 2/\sqrt{2} = 6/\sqrt{2} = 3\sqrt{2}$

Area = $PS^2 = 18$

Option (b) is correct.

791. The curve $y = 2x/(1 + x^2)$ has
(a) exactly three points of inflection separated by a point of maximum and a point of minimum
(b) exactly two points of inflection with a point of maximum lying between them
(c) exactly two points of inflection with a point of minimum lying between them
(d) exactly three points of inflection separated by two points of maximum

Solution :

$dy/dx = \{2(1 + x^2) - 2x*2x\}/(1 + x^2)^2 = 2(1 - x^2)/(1 + x^2)^2$

$d^2y/dx^2 = \{2(-2x)(1 + x^2)^2 - 2(1 + x^2)*2x*2(1 - x^2)\}/(1 + x^2)^2 = -4x(1 + x^2 + 2 - 2x^2)/(1 + x^2)^2 = 4x(x^2 - 3)/(1 + x^2)^2$

Now, $d^2y/dx^2 = 0$ gives, three solutions, $x = 0$, $x = \pm \sqrt{3}$

$dy/dx = 0$ gices, $x = \pm 1$ at which $d^2y/dx^2 < 0$ for $x = 1$ and > 0 for $x = -1$ i.e. maximum and minimum points

So, option (a) is correct.

792. As x varies all real numbers, the range of function $f(x) = (x^2 - 3x + 4)/(x^2 + 3x + 4)$ is
   (a) [1/7, 7]
   (b) [-1/7, 7]
   (c) [-7, 7]
   (d) (-∞, 1/7)U(7, ∞)

Solution :

Now, $(x^2 - 3x + 4)/(x^2 + 3x + 4) = f(x)$

- ⇨ $x^2 - 3x + 4 = f(x)x^2 + 3f(x)x + 4f(x)$ where $x^2 + 3x + 4 > 0$
- ⇨ $x^2(1 - f(x)) - 3x(1 + f(x)) + 4(1 - f(x)) = 0$ where $(x + 3/2)^2 + 7/4 > 0$
- ⇨ $9(1 + f(x))^2 - 16(1 - f(x))^2 \geq 0$
- ⇨ $9 + 18f(x) + 9f(x)^2 - 16 - 16f(x)^2 + 32f(x) \geq 0$
- ⇨ $7f(x)^2 - 50f(x) + 7 \leq 0$
- ⇨ $(7f(x) - 1)(f(x) - 7) \leq 0$
- ⇨ $f(x) \leq 1/7$ and $f(x) \geq 1/7$ or $f(x) \geq 1/7$ and $f(x) \leq 7$
- ⇨ $1/7 \leq f(x) \leq 7$

Option (a) is correct.

793. The minimum value of $f(x) = x^8 + x^6 - x^4 - 2x^3 - x^2 - 2x + 9$ is
   (a) 5
   (b) 1
   (c) 0

(d) 9

Solution :

$f'(x) = 8x^7 + 6x^5 - 4x^3 - 6x^2 - 2x - 2 = 0$

⇨ $(x - 1)(8x^6 + 8x^5 + 14x^4 + 10x^3 + 4x + 2) = 0$
⇨ $x = 1$

$f''(x) = 56x^6 + 30x^4 - 12x^2 - 12x - 2 > 0$ at $x = 1$ hence minimum.

Therefore, minimum value = $f(1) = 5$

Option (a) is correct.

794. The number of minima of the polynomial $10x^6 - 24x^5 + 15x^4 + 40x^2 + 108$ is
(a) 0
(b) 1
(c) 2
(d) 3

Solution :

Let $P(x) = 10x^6 - 24x^5 + 15x^4 + 40x^2 + 108$

⇨ $P'(x) = 60x^5 - 120x^4 + 60x^3 + 80x = 0$
⇨ $3x^5 - 4x^4 + 3x^3 + 2x = 0$
⇨ $3x^4 - 4x^3 + 3x^2 + 2 = 0$ or $x = 0$

Clearly, it has no negative roots. For $x < 0$ $3x^4 - 4x^3 + 3x^2 + 2 > 0$

Now, $d/dx(3x^4 - 4x^3 + 3x^2 + 2) = 12x^3 - 12x^2 + 6x = 6x(2x^2 - 2x + 1) = 6x\{x^2 + (x - 1)^2\} > 0$ for $x > 0$

Therefore, $3x^4 - 4x^3 + 3x^2 + 2$ is increasing for $x > 0$ and at $x = 0$ it is 2 and at $x = 1$, it is 4.

So, $P'(x)$ has only one root $x = 0$.

$P''(x) = 20(15x^4 - 16x^3 + 9x^2 + 2) > 0$ at $x = 0$

⇨ At $x = 0$ $P(x)$ is minimum.
⇨ One minimum

Option (b) is correct.

795. The number of local maxima of the function f(x) = x + sinx is
(a) 1
(b) 2
(c) Infinite
(d) 0

Solution :

f'(x) = 1 + cosx = 0

⇨ cosx = -1

f''(x) = -sinx = 0 for cosx = -1

So, no local maxima.

Option (d) is correct.

796. The maximum value of $\log_{10}(4x^3 - 12x^2 + 11x - 3)$ in the interval [2, 3] is
(a) $\log_{10}3$
(b) $1 + \log_{10}5$
(c) $-(3/2)\log_{10}3$
(d) None of these.

Solution :

Let $f(x) = 4x^3 - 12x^2 + 11x - 3$

$f'(x) = 12x^2 - 24x + 11 = 0$

⇨ $x = (24 \pm 4\sqrt{3})/24 = 1 \pm \sqrt{3}/6$

$f''(x) = 24x - 24 = 24(x - 1) < 0$ at $x = (24 - 4\sqrt{3})/24$

Therefore, maximum but $(24 - 4\sqrt{})/24 < 1$

We need to find maximum value in [2, 3]

⇨ Maximum value of f(x) = f(3) = 4*27 – 12*9 + 11*3 – 3 = 30

⇨ $\log_{10}30 = \log_{10}3 + 1$

Option (d) is correct.

797. The maximum value of the function $f(x) = (1 + x)^{0.3}/(1 + x^{0.3})$ in the interval $0 \le x \le 1$ is

(a) 1
(b) $2^{0.7}$
(c) $2^{-0.7}$
(d) None of these.

Solution :

$f'(x) = \{0.3(1 + x)^{-0.7}(1 + x^{0.3}) - 0.3x^{-0.7}(1 + x)^{0.3}\}/(1 + x^{0.3})^2 = 0.3\{x^{0.7}(1 + x^{0,3}) - (1 + x)\}/\{x^{0.7}(1 + x)^{0.7}(1 + x^{0.3})\} = 0.3(x^{0.7} - 1)/\{x^{0.7}(1 + x)^{0.7}(1 + x^{0.3})\} = 0$ gives $x = 1$

$f''(x) = 0.3[0.7x^{-0.3}x^{0.7}(1 + x)^{0.7}(1 + x^{0,3}) - (x^{0.7} - 1)\{0.7x^{-0.3}(1 + x)^{0.7}(1 + x^{0.3}) + 0.7x^{0.7}(1 + x)^{-0.3}(1 + x^{0.3}) + 0.3x^{-0.7}x^{0.7}(1 + x)^{0.7}\}]/\{x^{0.7}(1 + x)^{0.7}(1 + x^{0.3})\}^2 > 0$ for $x = 1$

Therefore, no local maximum value here.

Now, $f(0) = 1$ and $f(1) = 2^{-0.7}$

So, maximum value = $f(0) = 1$

Option (a) is correct.

798. The number of local maxima of the function $f(x) = x - \sin x$ is

(a) Infinitely many
(b) Two
(c) One
(d) Zero

Solution :

$f'(x) = 1 - \cos x = 0$ given $\cos x = 1$

$f''(x) = \sin x = 0$ for $\cos x = 0$

Hence, no local maxima.

Option (d) is correct.

799. From a square tin sheet of side 12 feet (ft) a box with its top open is made by cutting away equal squares at the four corners and then bending the tin sheet so as to form the sides of the box. The side of the removed square for which the box has the maximum possible volume is, in ft,
(a) 3
(b) 1
(c) 2
(d) None of the foregoing numbers.

Solution :

Let the side of the cut square = x.

So, height of the box is x and base area = $(12 - 2x)^2$

Volume = V = $x(12 - 2x)^2$

Now, $dV/dx = (12 - 2x)^2 + x*2(12 - 2x)(-2) = 0 \Rightarrow x = 6, 12 - 2x - 4x = 0, x = 2$

$d^2V/dx^2 = 2(12 - 2x)(-2) + 4(12 - 2x) + 4x(-2) < 0$ as x = 2.

Therefore, maximum value.

Option (c) is correct.

800. A rectangular box of volume 48 cu ft is to be constructed, so that its length is twice its width. The material to be used for the top and the four sides is three times costlier per sq ft than that used for the bottom. Then, the box that minimizes the cost has height equal to (in ft)
(a) 8/27
(b) $8\sqrt[3]{4/3}$
(c) 4/27
(d) 8/3

Solution :

Let the height is h and width is x

Therefore, length = 2x

Volume = $x*2x*h = 2x^2h = 48$

⇨ $x^2 = 24/h$

Cost = $c*x*2x + 3c(x*2x + 2*x*h + 2*2x*h)$ where c is cost per sq ft

Cost = $C = c(48/h) + 3c(48/h + 6h\sqrt{(24/h)}) = 4*48c/h + 36c\sqrt{(6h)}$

$dC/dh = -4*48c/h^2 + 36c\sqrt{6}/2\sqrt{h} = 0$

⇨ $h^{3/2} = 4*48*2/36\sqrt{6} = 16*\sqrt{2}/3^{3/2}$
⇨ $h^{3/2} = (8/3)^{3/2}$
⇨ $h = 8/3$

Option (d) is correct.

801. A truck is to be driven 300 km on a highway at a constant speed of x kmph. Speed rules for highway require that $30 \leq x \leq 60$. The fuel costs Rs. 10 per litre and is consumed at the rate of $2 + x^2/600$ litres per hour. The wages of the driver are Rs. 200 per hour. The most economical speed to drive the truck, in kmph, is
(a) 30
(b) 60
(c) $30\sqrt{3.3}$
(d) $20\sqrt{33}$

Solution :

Time = $300/x$

Cost = $(300/x)*200 + (300/x)(2 + x^2/600)*10 = 60000/x + 6000/x + 5x = 66000/x + 5x$

Now, we have to minimize cost.

Let $C = 66000/x + 5x$

$dC/dx = -66000/x^2 + 5 = 0$

⇨ $x^2 = 66000/5 = 13200$

⇨ $x = 20\sqrt{33}$

$d^2C/dx^2 = -66000*2/x^3 < 0$ at $x = 20\sqrt{33}$, therefore maximum.

So, C at x = 30 is 2200 + 150 = 2350

C at x = 60 is 1100 + 300 = 1400

Therefore, it is economical to drive at x = 60 kmph

Option (b) is correct.

802. Let P be a point in the first quadrant lying on the ellipse $x^2/8 + y^2/18 = 1$. Let AB be the tangent at P to the ellipse meeting the x-axis at A and y-axis at B. If O is the origin, then minimum possible area of the triangle OAB is
(a) 4π
(b) 9π
(c) 9
(d) 12

Solution :

Let P = $(2\sqrt{2}\cos\theta, 3\sqrt{2}\sin\theta)$

Now, $x^2/8 + y^2/18 = 1$

⇨ $x/4 + (y/9)(dy/dx) = 0$
⇨ $dy/dx = - 9x/4y$
⇨ $(dy/dx)_P = -9*2\sqrt{2}\cos\theta/(4*3\sqrt{2}\sin\theta) = -3\cot\theta/2$

Equation of AB is, $y - 3\sqrt{2}\sin\theta = (-3\cot\theta/2)(x - 2\sqrt{2}\cos\theta)$

Putting y = 0 we get, $x = (2\sqrt{2}\cos^2\theta + 2\sqrt{}\sin^2\theta)/\cos\theta = 2\sqrt{2}/\cos\theta$

Putting x = 0 we get, $y = 3\sqrt{2}/\sin\theta$

Therefore, A = $(2\sqrt{2}/\cos\theta, 0)$ and B = $(0, 3\sqrt{2}/\sin\theta)$

Therefore, area of triangle OAB = S = $(1/2)*(2\sqrt{2}/\cos\theta)(3\sqrt{2}/\sin\theta) = 12/\sin 2\theta$

$dS/d\theta = (-12/\sin^2 2\theta)(2\cos 2\theta) = 0 \Rightarrow \theta = \pi/4$

Area = $12/\sin(\pi/2) = 12$

Option (d) is correct.

803. Consider the parabola $y^2 = 4x$. Let P and Q be the points (4, -4) and (9, 6) of the parabola. Let R be a moving point on the arc of the parabola between P and Q. Then area of the triangle RPQ is largest when
(a) Angle PRQ = 90 degree
(b) R = (4, 4)
(c) R = (1/4, 1)
(d) Condition other than the foregoing conditions is satisfied.

Solution :

Let $R = (t^2, 2t)$

Area of triangle RPQ $= A = (1/2)[4(2t - 6) + t^2(6 + 4) + 9(-4 - 2t)] = 5t^2 - 5t - 30$

$dA/dt = 10t - 5 = 0$

⇨ $t = ½$

So, R = (1/4, 1)

Option (c) is correct.

804. Out of a circular sheet of paper of radius a, a sector with central angle θ is cut out and folded into the shape of a conical funnel. The volume of this funnel is maximum when θ equals
(a) $2\pi/\sqrt{2}$
(b) $2\pi\sqrt{(2/3)}$
(c) $\pi/2$
(d) π

Solution :

Length of the arc = $a\theta$

⇨ $2\pi r = a\theta$ (where r is the radius of the base of the funnel)
⇨ $r = a\theta/2\pi$
⇨ h = height of the funnel $= \sqrt{(a^2 - r^2)} = a\sqrt{\{1 - (\theta/2\pi)^2\}}$

Volume = V = $(1/3)\pi(a\theta/2\pi)^2 a\sqrt{\{1 - (\theta/2\pi)^2\}}$

$dV/d\theta = (1/3)(a^2/4\pi)[2\theta\sqrt{\{1 - (\theta/2\pi)^2\}} + \theta^2(-\theta/2\pi^2)/2\sqrt{\{1 - (\theta/2\pi)^2\}} = 0$

- ⇨ $2 - 2(\theta/2\pi)^2 - (\theta/2\pi)^2 = 0$
- ⇨ $\theta/2\pi = \sqrt{(2/3)}$
- ⇨ $\theta = 2\pi\sqrt{(2/3)}$

Option (b) is correct.

805. Let $f(x) = 5 - 4(\sqrt[3]{(x - 2)})^2$. Then at x = 2, the function f(x)
 (a) attains a minimum value
 (b) attains a maximum value
 (c) attains neither a minimum value nor a maximum value
 (d) is undefined

Solution :

x > 2.

For x > 2, $\{\sqrt[3]{(x - 2)}\}^2 > 0$

- ⇨ f(x) attains maximum value at x = 2.

Option (b) is correct.

806. A given circular cone has a volume p, and the largest right circular cylinder that can be inscribed in the given cone has a volume q. Then the ratio p : q equals
 (a) 9 : 4
 (b) 8 : 3
 (c) 7 : 2
 (d) None of the foregoing ratios.

Solution :

Let, the radius of the base of the cone is R and height is H.

Let the height of cylinder is h and base radius is r.

We have, $r/(H - h) = R/H$

⇨ $r = R(H - h)/H$

Now, V = volume of the cylinder = $\pi r^2 h = \pi(R/H)^2(H - h)^2 h$

$dV/dh = \pi(R/H)^2[2(H - h)(-1)h + 1(H - h)^2] = 0$

⇨ $-2h + H - h = 0$
⇨ $h = H/3$
⇨ $r = R(2H/3)/H = 2R/3$
⇨ $q = \pi(2R/3)^2*(H/3) = (1/3)\pi R^2H*(4/9) = p*(4/9)$
⇨ $p/q = 9/4$
⇨ $p : q = 9 : 4$

Option (a) is correct.

807. If [x] stands for the largest integer not exceeding x, then the integral ∫[x]dx (integration running from x = -1 to x = 2) is
(a) 3
(b) 0
(c) 1
(d) 2

Solution :

∫[x]dx (integration running from x = -1 to x = 2)

= ∫[x]dx (integration running from x = -1 to x = 0) + ∫[x]dx (integration running from x = 1 to x = 2) + ∫[x]dx (integration running from x = 1 to x = 2)

= -1(0 + 1) + 0(1 – 0) + 1(2 – 1) = 0

Option (b) is correct.

808. For any real number x, let [x] denote the greatest integer m such that m ≤ x. Then $\int[x^2 - 1]dx$ (integration running from -2 to 2) equals
(a) $2(3 - \sqrt{3} - \sqrt{2})$
(b) $2(5 - \sqrt{3} - \sqrt{2})$
(c) $2(1 - \sqrt{3} - \sqrt{2})$
(d) None of these.

Solution :

Now, $\int[x^2 - 1]dx$ (integration running from -2 to 2) = $2\int[x^2 - 1]dx$ (integration running from 0 to 2) (As $[x^2 - 1]$ is even function)

= $2[\int[x^2 - 1]dx$ (integration running from 0 to 1) + $\int[x^2 - 1]dx$ (integration running from 1 to √2) + $\int[x^2 - 1]dx$ (integration running from √2 to √3) + $\int[x^2 - 1]dx$ (integration running from √3 to 2)

= 2[(-1)(1 – 0) + 0(√2 – 1) + 1(√3 - √2) + 2(2 - √3)]

= 2(-1 + √3 - √2 + 4 - 2√3)

= 2(3 - √3 - √2)

Option (a) is correct.

809. Let f(x) be a continuous function such that its first two derivatives are continuous. The tangents to the graph of f(x) at the points with abscissa x = a and x = b make with X-axis angles π/3 and π/4 respectively. Then the value of the integral $\int f'(x)f''(x)dx$ (integration running from x = a to x = b) equals
(a) 1 - √3
(b) 0
(c) 1
(d) -1

Solution :

Let f'(x) = z

⇨ f''(x)dx = dz

Therefore, $\int f'(x)f''(x)dx = \int zdz = z^2/2 = \{f'(x)\}^2/2|_a^b = [\{f'(b)\}^2 - \{f'(a)\}^2]/2 = \{\tan^2(\pi/4) - \tan^2(\pi/3)\}/2 = (1 - 3)/2 = -1$

Option (d) is correct.

810. The integral $\int e^{x - [x]}dx$ (integration running from 0 to 100) is
(a) $(e^{100} - 1)/100$
(b) $(e^{100} - 1)/(e - 1)$
(c) $100(e - 1)$
(d) $(e - 1)/100$

Solution :

$\int e^{x - [x]}dx$ (integration running from 0 to 100) = $\sum\int e^{x - [x]}dx$ (integration running from i tp i + 1) (Summation running from i = 0 to i = 99)

Now, $\int e^{x - [x]}dx$ (integration running from i to i + 1)

= $\int e^{x - i}dx$ (integration running from i to i + 1)

= $e^{x - i}|_i^{i+1}$ = e – 1

Now, $\sum$(e – 1) (summation running from i = 0 to i = 99) = 100(e – 1)

Option (c) is correct.

811. If S = $\int\{e^t/(t + 1\}dt$ (integration running from 0 to 1) then $\int\{e^{-t}/(t - a - 1)\}dt$ (integration running from a – 1 to a) is
(a) Se^a
(b) Se^{-a}
(c) $-Se^{-a}$
(d) $-Se^a$

Solution :

Now, $\int\{e^{-t}/(t - a - 1)\}dt$ (integration running from a – 1 to a)

= $\int\{e^{-(2a - 1 - t)}/(2a - 1 - t - a - 1)\}dt$ (integration running from a – 1 to a) (As $\int f(x)dx = \int f(a + b - x)dx$ when integration running from a to b)

= $e^{-(2a-1)}\int\{e^t/(a - 2 - t)\}dt$

Let t = z + a – 1

dt = dz and when t = a – 1, z = 0 ; t = a, z = 1

= $e^{-(2a - 1)}\int\{e^{z + a - 1}/(a - 2 - z - a + 1)\}dz = -e^{-(2a - 1) + a - 1}\int\{e^z/(z + 1)\}dz = -e^{-a}S$

Option (c) is correct.

812. If the value of the integral ∫{e^(x^2)}dx (integration running from 1 to 2) is ɑ, then the value of ∫√(logx)dx (integration running from e to e^4) is
(a) e^4 – e – ɑ
(b) $2e^4$ – e – ɑ
(c) 2(e^4 – e) – ɑ
(d) None of the foregoing quantities.

Solution :

∫√(logx)dx (integration running from e to e^4)

= √(logx)x(upper limit = e^4, lower limit = e) – (1/2)∫{x/x√(logx)}dx (integration running from e to e^4)

= $2e^4$ – e – (1/2)∫{1/√(logx)}dx (integration running from e to e^4)

Let √(logx) = z

⇨ (½)dx/x√(logx) = dz
⇨ (1/2)dx/√(logx)dx = {e^(z^2)}dz

When x = e, z = 1 ; x = e^4, z = 2

= $2e^4$ – e - ∫{e^(z^2)}dz

= $2e^4$ – e – ɑ

Option (b) is correct.

813. The value of the integral ∫|1 + 2cosx|dx (integration running from 0 to π) is
(a) π/3 + √3
(b) π/3 + 2√3
(c) π/3 + 4√3
(d) 2π/3 + 4√3

Solution :

∫|1 + 2cosx|dx (integration running from 0 to π

= ∫(1 + 2cosx)dx (integration running from 0 to 2π/3) + ∫-(1 + 2cosx)dx (integration running from 2π/3 to π)

$= x + 2\sin x|_0^{2\pi/3} - (x + 2\sin x)|_{2\pi/3}^{\pi}$

$= 2\pi/3 + 2(\sqrt{3}/2) - (\pi + 2\sin\pi) + (2\pi/3 + 2(\sqrt{3}/2))$

$= 2\pi/3 + \sqrt{3} - \pi + 2\pi/3 + \sqrt{3}$

$= \pi/3 + 2\sqrt{3}$

Option (b) is correct.

814. The value of the integral $\int\sqrt{(1 + \sin(x/2))}dx$ (integration running from 0 to u), where $0 \le u \le \pi$, is
(a) $4 + 4\{\sin(u/4) - \cos(u/4)\}$
(b) $4 + 4\{\cos(u/4) - \sin(u/4)\}$
(c) $4 + (1/4)(\cos(u/4) - \sin(u/4))$
(d) $4 + (1/4)\{\sin(u/4) - \cos(u/4)\}$

Solution :

$\int\sqrt{(1 + \sin(x/2))}dx$ (integration running from 0 to u)

$= \int(\cos(x/4) + \sin(x/4))dx$ (integration running from 0 to u)

$= 4[\sin(x/4) - \cos(x/4)]|_0^u$

$= 4\{\sin(u/4) - \cos(u/4)\} - (-4)$

$= 4 + 4\{\sin(u/4) - \cos(u/4)\}$

Option (a) is correct.

815. The definite integral $\int dx/(1 + \tan^{101}x)$ (integration running from 0 to $\pi/2$) equals
(a) π
(b) $\pi/2$
(c) 0
(d) $\pi/4$

Solution :

Let $I = \int dx/(1 + \tan^{101}x)$ (integration running from 0 to $\pi/2$)

= $\int dx/(1 + \cot^{101}x)$ (integration running from 0 to π/2) (As $\int f(x)dx = \int f(a - x)dx$ when integration is running from 0 to a)

= $\int \tan^{101}x dx/(1 + \tan^{101}x)$ (integration running from 0 to π/2)

I + I = $\int dx$ (integration running from 0 to π/2) = π/2

- ⇨ 2I = π/2
- ⇨ I = π/4

Option (d) is correct.

816. If f(x) is a nonnegative continuous function such that f(x) + f(1/2 + x) = 1 for all x. 0 ≤ x ≤ ½, then $\int f(x)dx$ (integration running from 0 to 1) is equal to
    (a) ½
    (b) ¼
    (c) 1
    (d) 2

Solution :

$\int f(x)dx$ (integration running from 0 to 1)

= $\int f(x)dx$ (integration running from 0 to ½) + $\int f(x)dx$ (integration running from ½ to 1)

= I + J

J = $\int f(x)dx$ integration running from ½ to 1)

Let x = z + ½

- ⇨ dx = dz and x = ½, z = 0 ; x = 1, z = ½

J = $\int f(z + ½)dz$ (integration running from 0 to ½)

= $\int \{1 - f(z)\}dz$ (integration running from 0 to ½) (From the given relation)

= $z|_0^{1/2} - I$

- ⇨ I + J = ½

Option (a) is correct.

817. The value of the integral $\int \log_e(1 + \tan\theta)d\theta$ (integration running from 0 to $\pi/4$) is
(a) $\pi/8$
(b) $(\pi/8)\log_e 2$
(c) 1
(d) $2\log_e 2 - 1$

Solution :

Let $I = \int \log_e(1 + \tan\theta)d\theta$ (integration running from 0 to $\pi/4$)

$= \int \log_e\{1 + \tan(\pi/4 - \theta)\}d\theta$ (integration running from 0 to $\pi/4$)

$= \int \log_e\{1 + (1 - \tan\theta)/(1 + \tan\theta)\}d\theta$ (integration running from 0 to $\pi/4$)

$= \int [\log_e\{2/(1 + \tan\theta)\}]d\theta$ (integration running from 0 to $\pi/4$)

$= \log_e 2 \int d\theta - \int \log_e(1 + \tan\theta)d\theta$ (integration running from 0 to $\pi/4$)

$= \log_e 2(\pi/4 - 0) - I$

⇨ $2I = (\pi/4)\log_e 2$
⇨ $I = (\pi/8)\log_e 2$

Option (b) is correct.

818. Define the real-valued function f on the set of real numbers by $f(x) = \int \{(x^2 + t^2)/(2 - t)\}dt$ (integration running from 0 to 1). Consider the curve $y = f(x)$. It represents
(a) a straight line
(b) a parabola
(c) a hyperbola
(d) an ellipse

Solution :

Let $2 - t = z$

⇨ $dt = -dz$ and $t = 0, z = 2; t = 1, z = 1$

$-\int [\{x^2 + (2 - z)^2\}/z]dz$ (integration running from 2 to 1)

$= \int [\{x^2 + 4 - 4z + z^2\}/z]dz$ (integration running from 1 to 2)

= $(x^2 + 4)\int dz/z - 4\int dz + \int zdz$ (integration running from 1 to 2)

= $(x^2 + 4)\log 2 - 4 + 3/2$

⇨ It is a parabola

Option (b) is correct.

819. lim (1/n)∑cos(rπ/2n) (summation running from 0 to n – 1) as n -> ∞
(a) is 1
(b) is 0
(c) is 2/π
(d) does not exist

Solution :

lim (1/n)∑cos(rπ/2n) (summation running from 0 to n – 1) as n -> ∞

= ∫cos(πx/2)dx (integration running from 0 to 1)

= $(2/\pi)\sin(\pi x/2)|_0^1$

= 2/π

Option (c) is correct.

820. lim (√1 + √2 + …. + √(n – 1))/n√n as n -> ∞ is equal to
(a) ½
(b) 1/3
(c) 2/3
(d) 0

Solution :

lim (1/n)∑√(r/n) (summation running from 0 to n – 1) as n -> ∞

= ∫√xdx (integration running from 0 to 1)

Let $x = z^2$

⇨ dx = 2zdz and x = 0, z = 0; x= 1, z = 1

= $\int 2z^2dz$ (integration running from 0 to 1)

= 2/3

Option (c) is correct.

821. The value of lim $\Sigma(1/n)[\sqrt{(4i/n)}]$ (summation running from i = 1 to i = n) as n -> ∞, where [x] is the largest integer smaller than or equal to x, is

(a) 3
(b) ¾
(c) 4/3
(d) None of the foregoing numbers.

Solution :

lim $(1/n)\Sigma[\sqrt{(4i/n)}]$ (summation running from 1 to n) as n -> ∞

= $\int[\sqrt{(4x)}]dx$ (integration running from 0 to 1)

= $\int[\sqrt{(4x)}]dx$ (integration running from 0 to ¼) + $\int[\sqrt{(4x)}]dx$ (integration running from ¼ to 1)

= 0(1/4 – 0) + 1(1 – ¼) = ¾

Option (b) is correct.

822. Let α = lim $(1^2 + 2^2 +\ldots + n^2)/n^3$ as n -> ∞ and β = lim $\{(1^3 - 1^2) + (2^3 - 2^2) + \ldots + (n^3 - n^2)\}/n^4$ as n -> ∞. Then

(a) α = β
(b) α < β
(c) 4α - 3β = 0
(d) 3α - 4β = 0

Solution :

α = lim $(1/n)\Sigma(r/n)^2$ (summation running from 1 to n) as n -> ∞

= $\int x^2dx$ (integration running from 0 to 1)

= 1/3

$\beta = \lim (1/n)\Sigma\{(r^3 - r^2)/n^3\}$ (summation running from 1 to n) as $n \to \infty$

$= \int x^3 dx$ (integration running from 0 to 1)

$= 1/4$

⇨ $3\alpha - 4\beta = 0$

Option (d) is correct.

823. The value of the integral $\int|x - 3|dx$ (integration running from -4 to 4) is
(a) 13
(b) 8
(c) 25
(d) 24

Solution :

$\int|x - 3|dx$ (integration running from -4 to 4)

$= \int(3 - x)dx$ (integration running from -4 to 3) + $\int(x - 3)dx$ (integration running from 3 to 4)

$= 3x - x^2/2|_{-4}^{3} + (x^2/2 - 3x)|_{3}^{4}$

$= 9 - 9/2 - (-12 - 8) + 8 - 12 - (9/2 - 9)$

$= 9/2 + 20 - 4 + 9/2$

$= 25$

Option (c) is correct.

824. The value of $\int|x(x - 1)|dx$ (integration running from -2 to 2) is
(a) 11/3
(b) 13/3
(c) 16/3
(d) 17/3

Solution :

∫|x(x − 1)|dx (integration running from -2 to 2)

= ∫(x^2 − x)dx (integration running from -2 to 0) + ∫(x − x^2)dx (integration running from 0 to 1) + ∫(x^2 − x)dx (integration running from 1 to 2)

= $(x^3/3 - x^2/2)|_{-2}^{0} + (x^2/2 - x^3/3)|_{0}^{1} + (x^3/3 - x^2/2)|_{1}^{2}$

= -(-8/3 − 2) + (1/2 − 1/3) + (8/3 − 2) − (1/3 − ½)

= 16/3 + 1 − 2/3

= 14/3 + 1

= 17/3

Option (d) is correct.

825. ∫|xsinπx|dx (integration running from -1 to 3/2) is equal to
   (a) $(3\pi + 1)/\pi^2$
   (b) $(\pi + 1)/\pi^2$
   (c) $1/\pi^2$
   (d) $(3\pi - 1)/\pi^2$

Solution :

∫|xsinπx|dx (integration running from -1 to 3/2)

= ∫xsinπxdx (integration running from -1 to 1) + ∫-xsinπxdx (integration running from 1 to 3/2)

= ∫xsinπxdx (integration running from -1 to 1) - ∫xsinπxdx (integration running from 1 to 3/2)

Now, ∫xsinπxdx

= x-cosπx/π - ∫1*{(-cosπx)/π}dx

= $-x\cos\pi x/\pi + \sin\pi x/\pi^2$

Now, $-x\cos\pi x/\pi + \sin\pi x/\pi^2|_{-1}^{1} = 2/\pi$

And, $-x\cos\pi x/\pi + \sin\pi x/\pi^2|_{1}^{3/2} = -1/\pi^2 - 1/\pi$

So, $(2/\pi) - (-1/\pi^2 - 1/\pi) = (3\pi + 1)/\pi^2$

Option (d) is correct.

826. The set of values of a for which the integral $\int(|x - a| - |x - 1|)dx$ (integration running from 0 to 2) is nonnegative, is
(a) all numbers $a \geq 1$
(b) all real numbers
(c) all numbers a with $0 \leq a \leq 2$
(d) all numbers ≤ 1

Solution :

$\int(|x - a| - |x - 1|)dx$ (integration running from 0 to 2)

$= \int|x - a|dx - \int|x - 1|dx$ (integration running from 0 to 2)

$= \int|x - a|dx$ (integration running from 0 to 2) $- \int(1 - x)dx$ (integration running from 0 to 1) $- \int(x - 1)dx$ (integration running from 1 to 2)

$= \int|x - a|dx$ (integration running from 0 to 2) $- (x - x^2/2)|_0^1 - (x^2/2 - x)|_1^2$

$= \int|x - a|dx$(integration running from 0 to 2) $- ½ - ½$

$= \int|x - a|dx$ (integration running from 0 to 2) $- 1$

Let $a \leq 0$

Therefore, $\int|x - a|dx$ (integration running from 0 to 2) $= (x^2/2 - ax)|_0^2 = 2 - 2a$

Which shows that the given integration is positive.

Let $a \geq 2$, therefore $\int|x - a|dx$ (integration running from 0 to 2) $= (ax - x^2/2)|_0^2 = 2a - 2$

Which shows the given integration is positive.

So, option (a), (c), (d) cannot be true.

⇨ Option (b) is correct.

827. The maximum value of a for which the integral $\int e^{-(x - 1)^2}dx$ (integration running from a − 1 to a + 1), where a is a real number, is attained at
(a) $a = 0$
(b) $a = 1$

(c) a = -1
(d) a = 2

Solution :

Let f(a) = ∫e^{-(x – 1)^2}dx (integration running from a – 1 to a + 1)

- ⇨ f′(a) = e^(-a^2) – e^{-(a – 2)^2} = 0
- ⇨ e^{a^2 – (a – 2)^2} = 1
- ⇨ a^2 – (a – 2)^2 = 0
- ⇨ a = 1

Option (b) is correct.

828. Let f(x) = ∫{5 + |1 – y|}dy (integration running from 0 to x) if x > 2, f(x) = 5x + 1 if x ≤ 2. Then
(a) f(x) is continuous but not differentiable at x = 2
(b) f(x) is not continuous at x = 2
(c) f(x) is differentiable everywhere
(d) the tight derivative of f(x) at x = 2 does not exist

Solution :

f(x) = ∫{5 + |1 – y|}dy (integration running from 0 to x) x > 2

= ∫{5 + 1 – y}dy (integration running from 0 to 1) + ∫{5 + y – 1}dy (integration running from 1 to x)

= 6y – y^2/2$|_0^1$ + (4y + y^2/2)$|_1^x$

= 11/2 + 4x + x^2/2 – 4 – ½

= x^2/2 + 4x + 1 for x > 2

lim f(x) as x -> 2- = lim (x^2/2 + 4x + 1) as x -> 2- = 11

lim f(x) as x -> 2+ = lim (5x + 1) as x -> 2+ = 11

f(2) = 11

So, f(x) is continuous at x = 2.

lim {f(x) – f(2)}/(x – 2) as x -> 2- = lim {x^2/2 + 4x + 1 – 11)/(x – 2) as x -> 2- = lim (x + 4)/1 as x -> 2- (Applying L’Hospital rule) = 6 lim {f(x) – f(2)}/(x – 2) as x -> 2+ = lim {5x + 1 – 11}/(x – 2) as x -> 2+ = lim 5(x – 2)/(x – 2) as x -> 2+ = 5

f(x) is not differentiable at x = 2.

Option (a) is correct.

829. Consider the function f(x) = ∫[t]dt (integration running from 0 to x) where x > 0 and [t] denotes the largest integer less than or equal to t. Then

(a) f(x) is not defined for x = 1, 2, 3, ….

(b) f(x) is defined for all x > 0 but is not continuous at x = 1, 2, 3, …..

(c) f(x) is continuous at all x > 0 but is not differentiable at x = 1, 2, 3, ….

(d) f(x) is differentiable at all x > 0

Solution :

f(I) = ∫[t]dt (integration running from 0 to I) where I is any positive integer

= ∑∫[t]dt (summation running from 0 to I – 1) (integration running from r to r + 1)

= ∑r(r + 1 – r) (summation running from 0 to I – 1)

= I(I – 1)/2

So, f(x) is defined for x = 1, 2, 3, ….

lim f(x) as x -> 1- = lim ∫[t]dt (integration running from 0 to x) x -> 1- = lim 0 x -> 1- = 0

lim f(x) as x ->1 + = lim ∫[t]dt (integration running frm 0 to x) x -> 1+ = lim 0 x -> 1+ = 0

f(1) = ∫[t]dt (integration running from 0 to 1) = 0

f(x) is continuous at x = 1, Similarly, f(x) is continuous at x = 2, 3, ….

lim {f(x) – f(1)}/(x – 1) as x -> 1- = lim {∫[t]dt – 0}/(x – 1) (integration running from 0 to x) as x -> 1- = lim 0/(x – 1) as x -> 1- = 0

lim {f(x) – f(1)}/(x – 1) as x -> 1+ = lim {∫[t]dt – 0}/(x – 1) (integration running from 0 to x) as x -> 1+ = lim {∫[t]dt (integration running from 0 to

1) + ∫[t]dt (integration running from 1 to x)}/(x – 1) as x -> 1+ = lim (x – 1)/(x – 1) as x -> 1+ = 1

So, f(x) is not differentiable at x = 1. Similarly, f(x) is not differentiable at x = 2, 3, ….

Option (c) is correct.

830. Let f(x) = 2 if 0 ≤ x ≤ 1, f(x) = 3 if 1 < x ≤ 2. Define g(x) = ∫f(t)dt (integration running from 0 to x), for 0 ≤ x ≤ 2. Then
(a) g is not differentiable at x = 1
(b) g'(1) = 2
(c) g'(1) = 3
(d) none of the above holds

Solution :

g(x) = ∫f(t)dt (integration running from 0 to x) for 0 ≤ x ≤ 2

lim {g(x) – g(1)}/(x – 1)} as x -> 1- = lim {∫f(t)dt – 2}/(x – 1) (integration running from 0 to x) as x -> 1- = lim (2 – 2)/(x – 2) as x -> 2- = 0

lim {g(x) – g(1)}/(x – 1) aas x ->1+ = lim {∫f(t)dt – 2}/(x – 1) (integration running from 0 to x) as x -> 1+ = lim {∫f(t)dt (integration running from 0 to 1) + ∫f(t)dt (integration running from 1 to x) – 2}/(x – 1) as x -> 1+ = lim {2 + (x – 1) – 2}/(x – 1) as x -> 1+ = 1

g is not differentiable at x = 1.

Option (a) is correct.

831. Let [x] denote the greatest integer which is less than or equal to x. Then the value of the integral ∫[3tan^{2}x]dx (integration running from 0 to π/4) is
(a) π/3 – tan^{-1}√(2/3)
(b) π/4 – tan^{-1}√(2/3)
(c) 3 – [3π/4]
(d) [3 - 3π/4]

Solution :

$\int[3\tan^2 x]dx$ (integration running from 0 to π/4)

= $\int[3\tan^2 x]dx$ (integration running from 0 to π/6) + $\int[3\tan^2 x]dx$ (integration running from π/6 to $\tan^{-1}\sqrt{(2/3)}$) + $\int[3\tan^2 x]dx$ (integration running from $\tan^{-1}\sqrt{(2/3)}$ to π/4)

= $0(\pi/6 - 0) + 1(\tan^{-1}\sqrt{(2/3)} - \pi/6) + 2(\pi/4 - \tan^{-1}\sqrt{(2/3)})$

= $\pi/3 - \tan^{-1}\sqrt{(2/3)}$

Option (a) is correct.

832. Consider continuous functions f on the interval [0, 1] which satisfy the following two conditions :
(i) $f(x) \leq \sqrt{5}$ for all x Ɛ [0, 1]
(ii) $f(x) \leq 2/x$ for all x Ɛ [1/2, 1].

Then, the smallest real number ɑ such that inequality $\int f(x)dx$ (integration running from 0 to 1) ≤ ɑ holds for any such f is

(a) $\sqrt{5}$
(b) $\sqrt{5}/2 + 2\log 2$
(c) $2 + 2\log(\sqrt{5}/2)$
(d) $2 + \log(\sqrt{5}/2)$

Solution :

Option (c) is correct.

833. Let $f(x) = \int e^{(-t^2)}dt$ (integration running from 0 to x) for all x > 0. Then for all x > 0,
(a) $xe^{(-x^2)} < f(x)$
(b) $x < f(x)$
(c) $1 < f(x)$
(d) None of the foregoing statements is necessarily true.

Solution :

Integration means sum of the values from lower limit to upper limit.

$f(x) > (x - 0)e^{(-x^2)} = xe^{(-x^2)}$

Option (a) is correct.

834. Let $f(x) = \int\cos\{(t^2 + 2t + 1)/5\}dt$ (integration running from 0 to x), where $0 \le x \le 2$. Then

(a) f(x) increases monotonically as x increases from 0 to 2
(b) f(x) decreases monotonically as x increases from 0 to 2
(c) f(x) has a maximum at $x = \alpha$ such that $2\alpha^2 + 4\alpha = 5\pi - 2$
(d) f(x) has a minimum at $x = \alpha$ such that $2\alpha^2 + 4\alpha = 5\pi - 2$

Solution :

$f'(x) = \cos\{(x^2 + 2x + 1)/5\} = 0$

- ⇨ $(x^2 + 2x + 1)/5 = \pi/2$
- ⇨ $2x^2 + 2x = 5\pi - 2$

$f''(x) = -\sin\{(x^2 + 2x + 1)/5\}(2x + 2) < 0$ at $x = \alpha$ which satisfies the equation $x^2 + 2x = 5\pi - 2$

Option (c) is correct.

835. The maximum value of the integral $\int\{1/(1 + x^8)\}dx$ (integration running from a – 1 to a + 1) is attained

(a) exactly at two values of a
(b) only at one value of a which is positive
(c) only at one value of a which is negative
(d) only at a = 0

Solution :

Let $f(a) = \int\{1/(1 + x^8)\}dx$ (integration running from a -1 to a + 1)

$f'(a) = 1/\{1 + (a + 1)^8\} - 1/\{1 + (a - 1)^8\}$

$= \{1 + (a - 1)^8 - 1 - (a + 1)^8\}/\{(a + 1)(a - 1)\}^8 = 0$

Gives, $(a - 1)^8 = (a + 1)^8$

Clearly, this equation is satisfied only when a = 0

Option (d) is correct.

836. The value of the integral $\int coslogxdx$ is
(a) x[coslogx + sinlogs]
(b) (x/2)[coslogx + sinlogx]
(c) (x/2)[sinlogx – coslogx]
(d) (x/2)[coslogx + sinlogx]

Solution :

$\int coslogxdx$

Let logx = z

- ⇨ $x = e^z$
- ⇨ $dx = e^z dz$

$I = \int e^z coszdz$

$= e^z sinz - \int e^z sinzdx$

$= e^z sinz – e^z(-cosz) + \int e^z(-cosz)dz$

$= e^z(sin + cosz) – I$

- ⇨ $2I = e^z(sinz + cosz)$
- ⇨ $I = (e^z/2)(sinz + cosz) = (x/2)(coslogx + sinlogx)$

Option (d) is correct.

837. If $u_n = \int tan^n xdx$ (integration running from 0 to π/4) for n ≥ 2, then $u_n + u_{n-2}$ equals
(a) 1/(n – 1)
(b) 1/n
(c) 1/(n + 1)
(d) 1/n + 1/(n – 2)

Solution :

Now, $u_n + u_{n-2} = \int tan^{n-2}x(1 + tan^2x)dx = \int tan^{n-2}xsec^2xdx$ (integration running from 0 to π/4)

Let tanx = z, $sec^2xdx = dz$ and x = 0, z = 0, x = π/4, z =1

Therefore, $u_n + u_{n-2} = \int z^{n-2}dz$ (integration running from 0 to 1) = $z^{n-1}/(n-1)$ (upper limit = 1, lower limit = 0) = $1/(n-1)$

Option (a) is correct.

838. $\int \tan^{-1}x\,dx$ (integration running from 0 to 1) is equal to
   (a) $\pi/4 - \log_e\sqrt{2}$
   (b) $\pi/4 + \log_e\sqrt{2}$
   (c) $\pi/4$
   (d) $\log_e\sqrt{2}$

Solution :

$\int \tan^{-1}x\,dx$ (integration running from 0 to 1)

$= \tan^{-1}x * x|_0^1 - \int\{1/(1 + x^2)\}x\,dx$ (integration running from 0 to 1)

$= \pi/4 - (1/2)\int 2x\,dx/(1 + x^2)$ (integration running from 0 to 1)

Let $1 + x^2 = z$

⇨ $2x\,dx = dz$ and $x = 0, z = 1; x = 1, z = 2$

$= \pi/4 - (1/2)\int dz/z$ (integration running from 1 to 2)

$= \pi/4 - (1/2)\log z|_1^2$

$= \pi/4 - \log_e\sqrt{2}$

Option (a) is correct.

839. $\int\{\sin^{100}x/(\sin^{100}x + \text{cox}^{100}x)\}dx$ (integration running from 0 to $\pi/2$) equals
   (a) $\pi/4$
   (b) $\pi/2$
   (c) $3\pi/4$
   (d) $\pi/3$

Solution :

Let, $I = \int\{\sin^{100}x/(\sin^{100}x + \cos^{100}x)dx$ (integration running from 0 to $\pi/2$)

= $\int\{\cos^{100}/(\cos^{100}x + \sin^{100}x)dx$ (integration running from 0 to π/2) (Using the property $\int f(x)dx = \int f(a - x)dx$ when integration is running fro 0 to a)

- ⇨ $I + I = \int\{(\sin^{100}x + \cos^{100}x)/(\sin^{100}x + \cos^{100}x)dx$ (integration running from 0 to π/2)
- ⇨ $2I = \int dx$ (integration running from 0 to π/2)
- ⇨ $2I = \pi/2 - 0$
- ⇨ $I = \pi/4$

Option (a) is correct.

840. The indefinite integral $\int\{\sqrt{x}/\sqrt{(a^3 - x^3)}\}dx$ equals
 (a) $(2/3)\sin^{-1}(x/a)^{3/2} + C$ where C is constant
 (b) $\cos^{-1}(x/a)^{3/2} + C$ where C is constant
 (c) $(2/3)\cos^{-1}(x/a)^{3/2} + C$ where C is constant
 (d) None of the foregoing functions.

Solution :

$\int\{\sqrt{x}/\sqrt{(a^3 - x^3)}\}dx$

Let $x = a\sin^{2/3}z$

- ⇨ $dx = (2/3)a\cos z/\sin^{1/3}z dz$

$= \int\{a^{1/2}\sin^{1/3}z(2/3)a\cos z/\sin^{1/3}z\}dz/a^{3/2}\cos z$

$= (2/3)\int dz$

$= (2/3)z + C$

$= (2/3)\sin^{-1}(x/a)^{3/2} + C$

Option (a) is correct.

841. The value of the integral $\int\{e^x\sqrt{(e^x - 1)/(e^x + 3)}\}dx$ (integration running from 0 to log5) is
 (a) 4π
 (b) 4
 (c) π/2
 (d) 4 – π

Solution :

$\int\{e^x\sqrt{(e^x - 1)/(e^x + 3)}\}dx$ (integration running from 0 to log5)

Let $e^x - 1 = z^2$

⇨ $e^x dx = 2zdz$, $x = 0, z = 0; x = \log 5, z = 2$

$= \int z*2zdz/(z^2 + 4)$ (integration running from 0 to 2)

$= \int(2z^2 + 8 - 8)dz/(z^2 + 4)$ (integration running from 0 to 2)

$= 2\int\{(z^2 + 4)/(z^2+ 4)\}dz - 8\int dz/(z^2 + 4)$ (integration running from 0 to 2)

$= 2z|_0^2 - (8/2)\tan^{-1}(z/2)|_0^2$

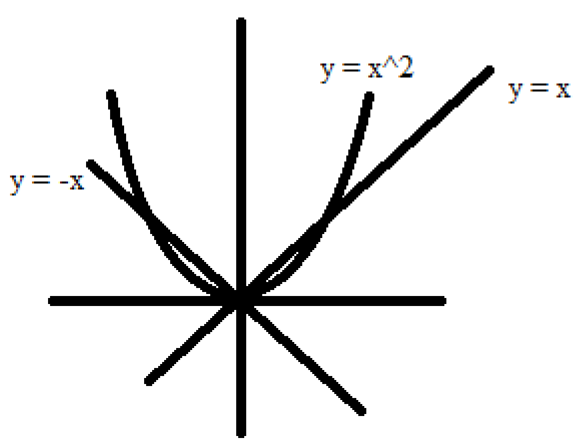
$= 4 - 4(\pi/4 - 0)$

$= 4 - \pi$

Option (d) is correct.

842. The area of the region $\{(x, y) : x^2 \le y \le |x|\}$ is
(a) 1/3
(b) 1/6
(c) ½
(d) 1

Solution :



$x^2 = y$ and $y = x$ solving this we get, $x = 0, x = 1$ and $y = 0, y = 1$

So, the intersection point is (1, 1)

Area =2[∫xdx - ∫x^2dx] (integration running from 0 to 1)

= $2*(x^2/2 – x^3/3)|_0^1$

= 2(1/2 – 1/3)

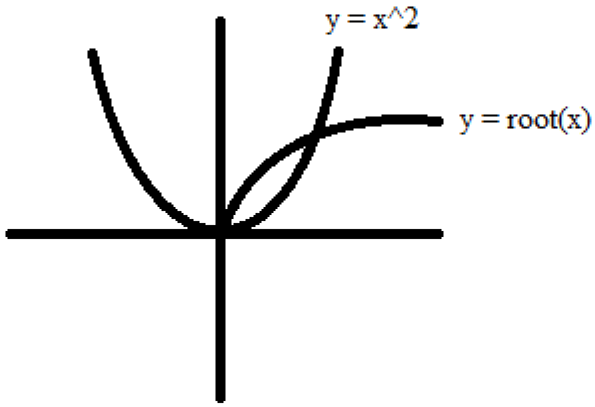
= 2(1/6)

= 1/3

Option (a) is correct.

843. The area bounded by the curves y = √x and y = x^2 is
(a) 1/3
(b) 1
(c) 2/3
(d) None of the foregoing numbers.

Solution :



Solving y = x^2 and y = √x we get, x = 0, x = 1 and y = 0, y = 1

So, the intersection point is (1, 1)

Area = ∫(√x – x^2)dx (integration running from 0 to 1)

= $(2/3)x^{3/2} – x^3/3|_0^1$

= (2/3) – 1/3

= 1/3

Option (a) is correct.

844. The area bounded by the curve $y = \log_e x$, the x-axis and the straight line x = e equals
(a) e
(b) 1
(c) 1 – 1/e
(d) None of the foregoing numbers.

Solution :

Solving y = 0 and $y = \log_e x$ we get, x = 1, y = 0

So, the intersection point is (1, 0)

Solving $y = \log_e x$ and x = e, we get, x = e, y = 1

So, the intersection point is (e, 1)

Area = $\int \log_e x dx$ (integration running fro m1 to e)

$= \log_e x * x|_1^e - \int (1/x)*x dx$ (integration running from 1 to e)

= e – (e – 1)

= 1

Option (b) is correct.

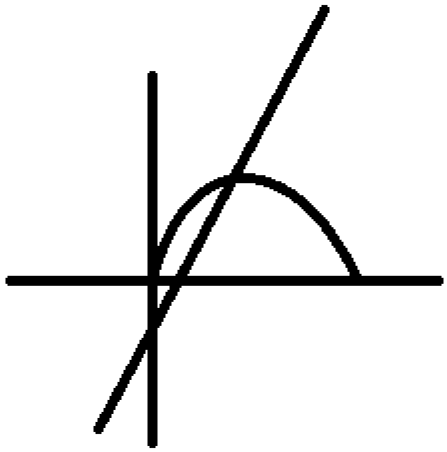
845. The area of the region in the first quadrant bounded by y = sinx and (2y – 1)/(√3 – 1) = (6x – π)/π equals
(a) (√3 – 1)/2 – (π/24)(√3 + 1)
(b) (√3 + 1)/2 – (π/24)(√3 – 1)
(c) {(√3 – 1)/2}(1 – π/12)
(d) None of the above quantities.

Solution :

Now, (2y – 1)/(√3 – 1) = (6x – π)/π

⇨ 2y – 1 = (√3 – 1)(6x/π) – (√3 – 1)
⇨ y = {3(√3 – 1)/π}x – (√3 – 2)/2

Solving the two equations we get, $(\pi/3, \sqrt{3}/2)$



When $y = 0$, the straight line intersects the x-axis at, $x = (\pi/6)(\sqrt{3} - 2)/(\sqrt{3} - 1) = \alpha$

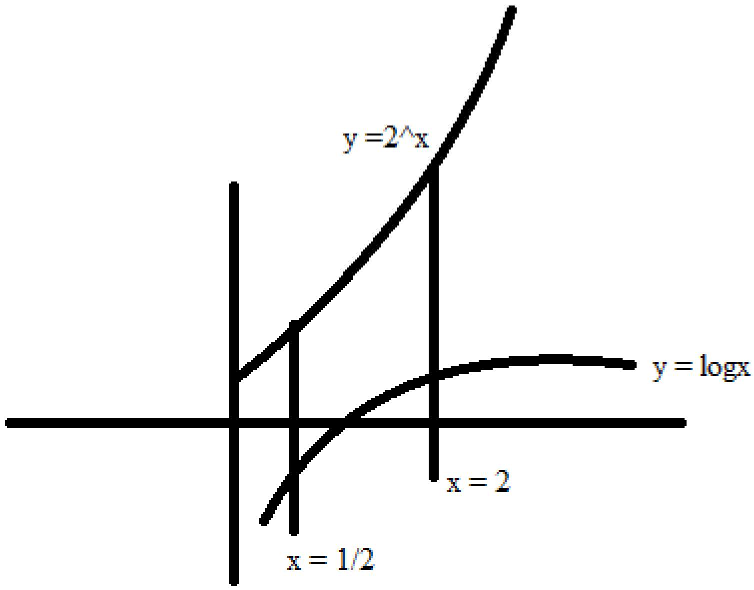
Area = $\int[\{3(\sqrt{3} - 1)/\pi\}x - (\sqrt{3} - 2)/2]dx$ (integration running from α to $\pi/3$) + $\int \sin x dx$ (integration running from $\pi/3$ to π)

Solving this integration you will find the area.

Option (a) is correct.

846. The area of the region bounded by the straight lines $x = ½$ and $x = 2$, and the curves given by the equations $y = \log_e x$ and $y = 2^x$ is
(a) $(1/\log_e 2)(4 + \sqrt{2}) - (5/2)\log_e 2 + 3/2$
(b) $(1/\log_e 2)(4 - \sqrt{2}) - (5/2)\log_e 2$
(c) $(1/\log_e 2)(4 - \sqrt{2}) - (5/2)\log_e 2 + 3/2$
(d) is not equal to any of the foregoing expressions

Solution :

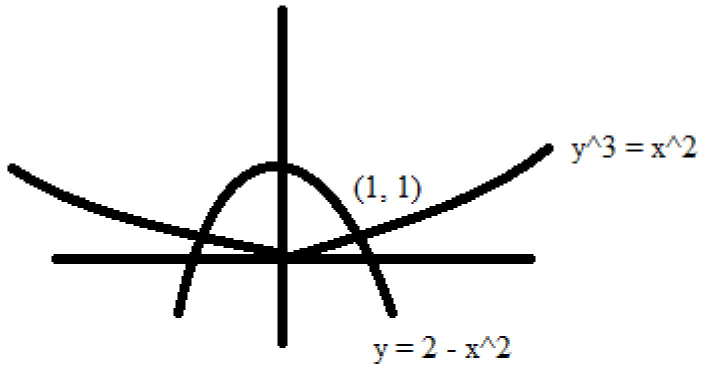


Find the intersection points and write the integrations accordingly and solve them. You will get the area.

Option (c) is correct.

847. The area of the bounded region between the curves $y^3 = x^2$ and $y = 2 - x^2$ is
(a) 2 + 4/15
(b) 1 + 1/15
(c) 2 + 2/15
(d) 2 + 14/15

Solution :



Area = $2\int(2 - x^2 - x^{2/3})dx$ (integration running from 0 to 1)

$= 2(2x - x^3/3 - (3/5)x^{5/3})|_0^1$

$= 2(2 - 1/3 - 3/5)$

$= 2(2 - 14/15)$

$= 2(16/15)$

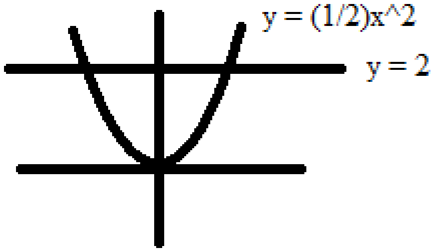
$= 32/15$

$= 2 + 2/15$

Option (c) is correct.

848. The area of the region enclosed between the curve $y = (1/2)x^2$ and the straight line $y = 2$ equals (in sq. units)
(a) 4/3
(b) 8/3
(c) 16/3
(d) 32/3

Solution :



Solving y = $(1/2)x^2$ and y = 2 we get, y = 2, x = ±2

Area = 2[2*2 - ∫$(1/2)x^2$dx] (integration running from 0 to 2)

= 2[4 – (1/2)(8/3)]

= 2(4 – 4/3)

= 8(1 – 1/3)

= 16/3

Option (c) is correct.

849. The value of the integral ∫{e^(-x^2/2)}sinxdx (integration running from -π/2 to π/2) is
(a) π/2 – 1
(b) π/3
(c) √(2π)
(d) None of the foregoing numbers.

Solution :

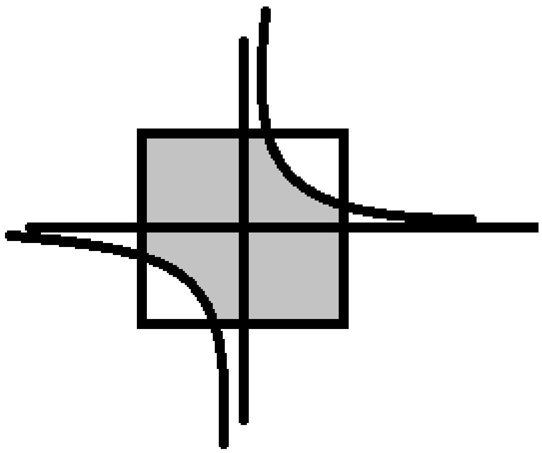
f(x) = {e^(-x^2/2)}sinx = odd function.

So, the integral is zero.

Option (d) is correct.

850. The area of the region of the plane bounded by max(|x|, |y|) ≤ 1 and xy ≤ ½ is
(a) ½ + log2
(b) 3 + log2
(c) 7 + ¾
(d) None of the foregoing numbers.

Solution :



$2\int(1/2x)dx$ (integration running from ½ to 1)

$= -\log(1/2)$

$= \log 2$

Area = $1 + 1 + (1/2)\log 2 + 1*(1/2)*2 = 3 + \log 2$

Option (b) is correct.

851. The largest area of a rectangle which has one side on the x-axis and two vertices on the curve $y = e^{(-x^2)}$ is

(a) $(1/\sqrt{2})e^{-1/2}$

(b) $(1/2)e^{-2}$

(c) $\sqrt{2}e^{-1/2}$

(d) $\sqrt{2}e^{-2}$

Solution :

Let one point is (a, y_1) and another is $(-a, y_1)$

Therefore, area of the rectangle = $2ay_1 = 2a\{e^{(-a^2)}\}$ (as (a, y_1) lies on the curve $e^{(-x^2)}$)

Let, $A = 2ae^{(-a^2)}$

dA/da = $2[e^{(-a^2)} + ae^{(-a^2)}*(-2a)] = 0$

- ⇨ $1 - 2a^2 = 0$
- ⇨ $a = \pm 1/\sqrt{2}$

Therefore, largest area = $2(1/\sqrt{2})e^{(-1/2)}$

= $\sqrt{2}e^{-1/2}$

Option (c) is correct.

852. Approximate value of the integral $I(x) = \int(\cos t)\{e^{(-t^2/10)}\}dt$ (integration running from 0 to x) are given in the following table.

x	π/2	π	3π/2	2π
I(x)	0.95	0.44	0.18	0.22

Which of the following numbers best approximates the value of the integral $\int(\cos t)\{e^{(-t^2/10)}\}dt$ (integration running from 0 to 5π/4)?
(a) 0.16
(b) 0.23
(c) 0.32
(d) 0.40

Solution :

Option (b) is correct.

853. The maximum of the areas of the isosceles triangles lying between the curve $y = e^{-x}$ and the x-axis, with the base on the positive x-axis, is
(a) 1/e
(b) 1
(c) ½
(d) e

Solution :

Let the coordinate of the vertex which lies on the curve is (a, y_1)

Therefore area of the triangle = $(1/2)2a*y_1 = ae^{-a}$

Let, $A = ae^{-a}$

$dA/da = e^{-a} + ae^{-a}(-1) = 0$

⇨ $a = 1$

$A_{max} = 1/e$

Option (a) is correct.

854. The area bounded by the straight lines $x = -1$ and $x = 1$ and the graphs of $f(x)$ and $g(x)$, where $f(x) = x^3$ and $g(x) = x^5$ if $-1 \leq x \leq 0$, $g(x) = x$ if $0 \leq x \leq 1$ is

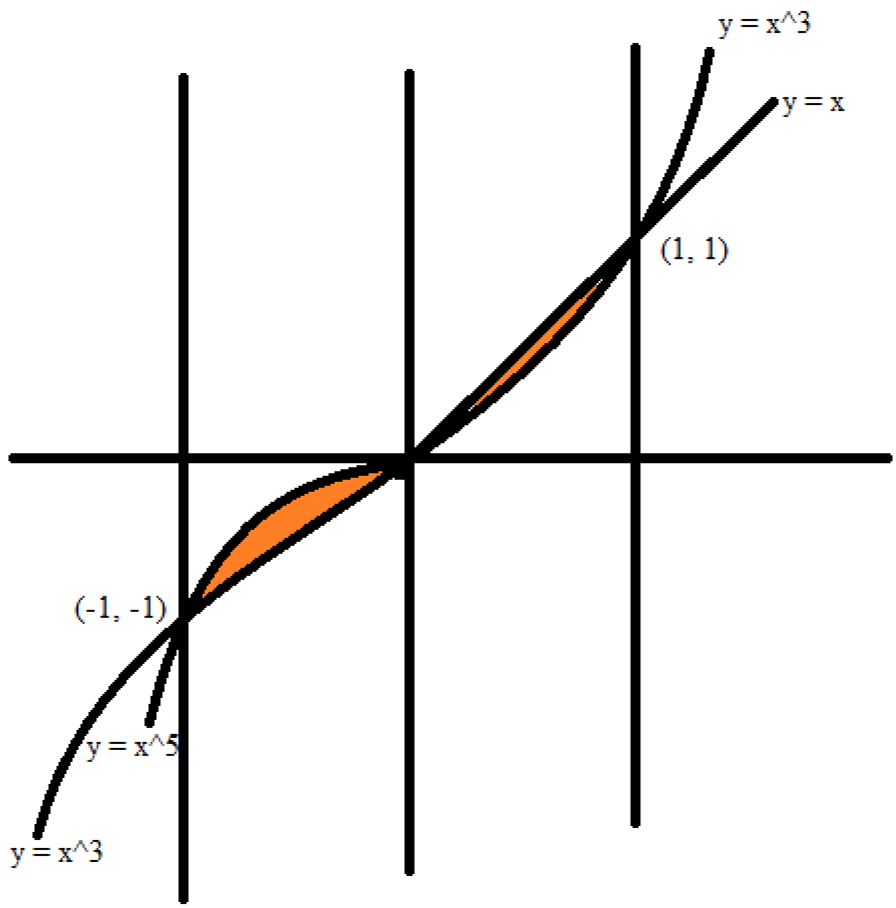
(a) 1/3

(b) 1/8

(c) ½

(d) ¼

Solution :



Area = $\int(x - x^3)dx$ (integration running from 0 to 1) + $|\int(x^3 - x^5)dx|$ (integration running from -1 to 0)

= (1/2 – ¼) + |(1/4 – 1/6)|

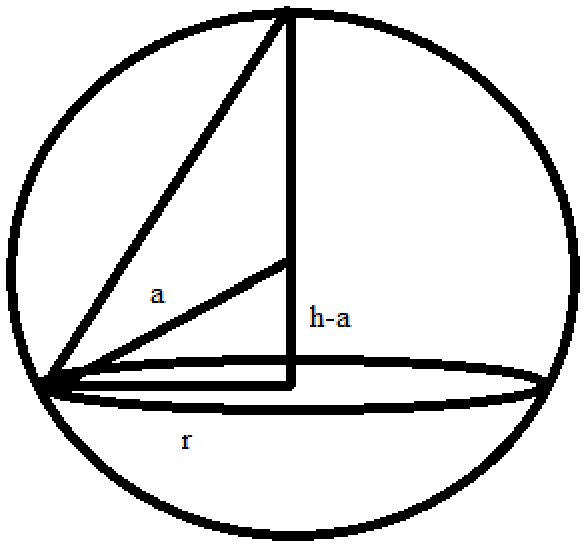
= ¼ + 1/12

= 1/3

Option (a) is correct.

855. A right circular cone is cut from a solid sphere of radius a, the vertex and the circumference of the base lying on the surface of the sphere. The height of the cone when its volume is maximum is
(a) 4a/3
(b) 3a/2
(c) a
(d) 6a/5

Solution :



Let the radius of the cone is r and height is h.

Then we have, $a^2 = (h - a)^2 + r^2$

⇨ $r^2 = a^2 - (h - a)^2 = a^2 - h^2 + 2ha - a^2 = 2ha - h^2$

Volume $= V = (1/3)\pi r^2 h = (1/3)\pi(2ha - h^2)h = (1/3)\pi(2ah^2 - h^3)$

$dV/dh = (1/3)\pi(4ah - 3h^2) = 0$

⇨ $h = 4a/3$

Option (a) is correct.

856. For any choice of *five* distinct points in the unit square (that is, a square with side 1 unit), we can assert that there is a number c such that there are at least two points whose distance is less than or equal to c. The smallest value c for which such an assertion can be made is

(a) $1/\sqrt{2}$

(b) 2/3

(c) ½

(d) None of the foregoing numbers.

Solution :

When the four points are the vertex and the fifth point is the centre of the square.

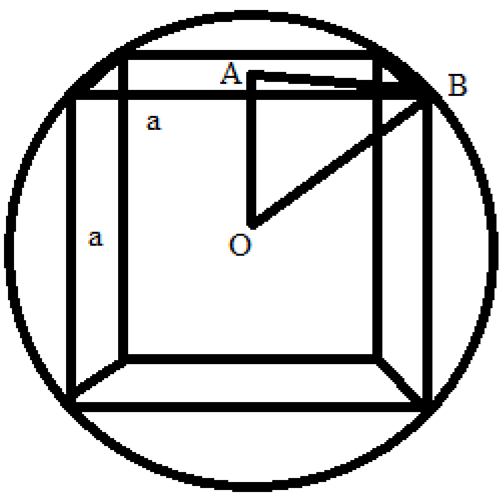
So, $c^2 + c^2 = 1$

⇨ $c = 1/\sqrt{2}$

Option (a) is correct.

857. The largest volume of a cube that can be enclosed in a sphere of diameter 2 cm, is, in cm^3,
(a) 1
(b) $2\sqrt{2}$
(c) π
(d) $8/3\sqrt{3}$

Solution :



$AB^2 = (a/2)^2 + (a/2)^2 = a^2/2$

$OA^2 = (a/2)^2 = a^2/4$

$OB = 2/2 = 1$

Now, $OB^2 = AB^2 + OA^2$

⇨ $1^2 = a^2/2 + a^2/4 = 3a^2/4$
⇨ $a = 2/\sqrt{3}$

⇨ $V = a^3 = 8/3\sqrt{3}$

Option (d) is correct.

858. A lane runs perpendicular to a road 64 feet wide. If it is just possible to carry a pole 125 feet long from the road into the lane, keeping it horizontal, then the minimum width of the lane must be (in feet)

(a) $(125/\sqrt{2} - 64)$
(b) 61
(c) 27
(d) 36

Solution :

Option (c) is correct.