Student's Manual

Essential Mathematics for Economic Analysis

5th edition

Knut Sydsæter
Peter Hammond
Andrés Carvajal
Arne Strøm

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Preface

This Student's solutions manual accompanies *Essential Mathematics for Economic Analysis, 5th Edition*, Pearson, 2016. Its main purpose is to provide more detailed solutions to the problems marked (SM) in the book. The answers provided in this manual should be used in combination with any shorter answers provided in the main test. There are a few cases where only part of the answer is set out in detail, because the rest follows the same pattern.

We would appreciate suggestions for improvements from our readers, as well as help in weeding out inaccuracies and errors.

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Peter Hammond (P.J.Hammond@warwick.ac.uk)
Andrés Carvajal (acarvaes@icloud.com)
Arne Strøm (arne.strom@econ.uio.no)

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1 Essentials of Logic and Set Theory

Review exercises for Chapter 1

3. Consider the Venn diagram for three sets depicted in Fig. SM1.R.3. Let n_k denote the number of students in the set marked (k), for $k = 1, 2, \ldots, 8$. Suppose the sets A, B, and C refer to those who study English, French, and Spanish, respectively. Since 10 students take all three languages, $n_7 = 10$. There are 15 who take French and Spanish, so $15 = n_2 + n_7$, and thus $n_2 = 5$. Furthermore, $32 = n_3 + n_7$, so $n_3 = 22$. Also, $110 = n_1 + n_7$, so $n_1 = 100$. The rest of the information implies that $52 = n_2 + n_3 + n_6 + n_7$, so $n_6 = 52 - 5 - 22 - 10 = 15$. Moreover, $220 = n_1 + n_2 + n_5 + n_7$, so $n_5 = 220 - 100 - 5 - 10 = 105$. Finally, $780 = n_1 + n_3 + n_4 + n_7$, so $n_4 = 780 - 100 - 22 - 10 = 648$. The answers are therefore:

   (a) $n_1 = 100$,

   (b) $n_3 + n_4 = 648 + 22 = 670$,

   (c) $1000 - \sum_{i=1}^{7} n_i = 1000 - 905 = 95$.

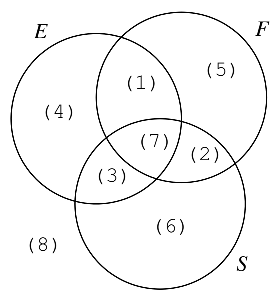

Figure SM1.R.3

4. (a) $\Rightarrow$ is true; $\Leftarrow$ is false, because $x = y = 1$ also solves $x + y = 2$.

   (b) $\Rightarrow$ is false, because $x^2 = 16$ also has the solution $x = -4$; $\Leftarrow$ true, because if $x = 4$, then $x^2 = 16$.

   (c) $\Rightarrow$ is true, because $(x-3)^2 \geq 0$; $\Leftarrow$ false because with $y > -2$ and $x = 3$, one has $(x-3)^2(y+2) = 0$.

   (d) $\Rightarrow$ and $\Leftarrow$ are both true, since the equation $x^3 = 8$ has the solution $x = 2$ and no others.[1]

5. For (a) and (b) see the solutions in the book. For (c), note that when $n = 1$, the inequality is obviously correct.[2] As the induction hypothesis when n equals the arbitrary natural number k, suppose that $(1+x)^k \geq 1 + kx$. Because $1 + x \geq 0$, we then have

$$(1+x)^{k+1} = (1+x)^k(1+x) \geq (1+kx)(1+x) = 1 + (k+1)x + kx^2 \geq 1 + (k+1)x.$$

   where the last inequality holds because $k > 0$. Thus, the induction hypothesis holds for $n = k + 1$. Therefore, by induction, Bernoulli's inequality is true for all natural numbers n.

[1] In the terminology of Section 6.3, function $f(x) = x^3$ is strictly increasing. See Fig. 4.3.7 and Exercise 6.3.3.

[2] And for $n = 2$, it is correct by part (b).

2 Algebra

2.3 Rules of algebra

4. (a) $(2t-1)(t^2-2t+1) = 2t(t^2-2t+1)-(t^2-2t+1) = 2t^3-4t^2+2t-t^2+2t-1 = 2t^3-5t^2+4t-1$.

   (b) $(a+1)^2 + (a-1)^2 - 2(a+1)(a-1) = (a^2+2a+1) + (a^2-2a+1) - 2(a^2-1) = 4$.[3]

   (c) $(x+y+z)^2 = (x+y+z)(x+y+z) = x(x+y+z) + y(x+y+z) + z(x+y+z) = (x^2+xy+xz) + (yx+y^2+yz) + (zx+zy+z^2) = x^2+y^2+z^2+2xy+2xz+2yz$.

   (d) Put $a = x+y+z$ and $b = x-y-z$. Then

$$(x+y+z)^2 - (x-y-z)^2 = a^2 - b^2 = (a+b)(a-b) = 2x(2y+2z) = 4x(y+z)$$

2.4 Fractions

5. (a) $\dfrac{1}{x-2} - \dfrac{1}{x+2} = \dfrac{x+2}{(x-2)(x+2)} - \dfrac{x-2}{(x+2)(x-2)} = \dfrac{x+2-x+2}{(x-2)(x+2)} = \dfrac{4}{x^2-4}$.

   (b) Since $4x+2 = 2(2x+1)$ and $4x^2-1 = (2x+1)(2x-1)$, the lowest common denominator, LCD, is $2(2x+1)(2x-1)$. Then,

$$\frac{6x+25}{4x+2} - \frac{6x^2+x-2}{4x^2-1} = \frac{(6x+25)(2x-1) - 2(6x^2+x-2)}{2(2x+1)(2x-1)} = \frac{42x-21}{2(2x+1)(2x-1)} = \frac{21}{2(2x+1)}.$$

   (c) $\dfrac{18b^2}{a^2-9b^2} - \dfrac{a}{a+3b} + 2 = \dfrac{18b^2 - a(a-3b) + 2(a^2-9b^2)}{(a+3b)(a-3b)} = \dfrac{a(a+3b)}{(a+3b)(a-3b)} = \dfrac{a}{a-3b}$.

   (d) $\dfrac{1}{8ab} - \dfrac{1}{8b(a+2)} = \dfrac{(a+2)-a}{8ab(a+2)} = \dfrac{2}{8ab(a+2)} = \dfrac{1}{4ab(a+2)}$.

   (e) $\dfrac{2t-t^2}{t+2} \cdot \left(\dfrac{5t}{t-2} - \dfrac{2t}{t-2}\right) = \dfrac{t(2-t)}{t+2} \cdot \dfrac{3t}{t-2} = \dfrac{-t(t-2)}{t+2} \cdot \dfrac{3t}{t-2} = \dfrac{-3t^2}{t+2}$.

   (f) Note that $\dfrac{a\left(1-\frac{1}{2a}\right)}{0.25} = \dfrac{a-\frac{1}{2}}{\frac{1}{4}} = 4a-2$, so

$$2 - \frac{a\left(1-\frac{1}{2a}\right)}{0.25} = 2 - (4a-2) = 4 - 4a = 4(1-a).$$

6. (a) $\dfrac{2}{x} + \dfrac{1}{x+1} - 3 = \dfrac{2(x+1) + x - 3x(x+1)}{x(x+1)} = \dfrac{2-3x^2}{x(x+1)}$

   (b) $\dfrac{t}{2t+1} - \dfrac{t}{2t-1} = \dfrac{t(2t-1) - t(2t+1)}{(2t+1)(2t-1)} = \dfrac{-2t}{4t^2-1}$

   (c) $\dfrac{3x}{x+2} - \dfrac{4x}{2-x} - \dfrac{2x-1}{(x-2)(x+2)} = \dfrac{3x(x-2) + 4x(x+2) - (2x-1)}{(x-2)(x+2)} = \dfrac{7x^2+1}{x^2-4}$

   (d) The expression equals $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{1}{xy}} = \dfrac{\left(\dfrac{1}{x}+\dfrac{1}{y}\right)xy}{\dfrac{1}{xy}\cdot xy} = \dfrac{y+x}{1} = x+y$.

[3] Alternatively, apply the quadratic identity $x^2+y^2-2xy = (x-y)^2$ with $x = a+1$ and $y = a-1$ to obtain $(a+1)^2 + (a-1)^2 - 2(a+1)(a-1) = [(a+1)-(a-1)]^2 = 2^2 = 4$.

(e) The expression equals $\dfrac{\dfrac{1}{x^2}-\dfrac{1}{y^2}}{\dfrac{1}{x^2}+\dfrac{1}{y^2}} = \dfrac{\left(\dfrac{1}{x^2}-\dfrac{1}{y^2}\right)\cdot x^2y^2}{\left(\dfrac{1}{x^2}+\dfrac{1}{y^2}\right)\cdot x^2y^2} = \dfrac{y^2-x^2}{y^2+x^2}.$

(f) To clear the fractions within both the numerator and denominator, multiply both by xy to get

$$\frac{a(y-x)}{a(y+x)} = \frac{y-x}{y+x}$$

8. (a) $\dfrac{1}{4}-\dfrac{1}{5}=\dfrac{5}{20}-\dfrac{4}{20}=\dfrac{1}{20}$, so $\left(\dfrac{1}{4}-\dfrac{1}{5}\right)^{-2}=\left(\dfrac{1}{20}\right)^{-2}=20^2=400.$

(b) $n-\dfrac{n}{1-\frac{1}{n}}=n-\dfrac{n\cdot n}{\left(1-\frac{1}{n}\right)\cdot n}=n-\dfrac{n^2}{n-1}=\dfrac{n(n-1)-n^2}{n-1}=-\dfrac{n}{n-1}.$

(c) Let $u=x^{p-q}$. Then $\dfrac{1}{1+x^{p-q}}+\dfrac{1}{1+x^{q-p}}=\dfrac{1}{1+u}+\dfrac{1}{1+1/u}=\dfrac{1}{1+u}+\dfrac{u}{1+u}=1.$

(d) Using $x^2-1=(x+1)(x-1)$, one has

$$\frac{\left(\dfrac{1}{x-1}+\dfrac{1}{x^2-1}\right)(x^2-1)}{\left(x-\dfrac{2}{x+1}\right)(x^2-1)} = \frac{(x+1)+1}{x(x^2-1)-2(x-1)} = \frac{x+2}{(x-1)[x(x+1)-2]},$$

which reduces to

$$\frac{x+2}{(x-1)(x^2+x-2)} = \frac{x+2}{(x-1)[(x+2)(x-1)]} = \frac{1}{(x-1)^2}.$$

(e) Since

$$\frac{1}{(x+h)^2}-\frac{1}{x^2} = \frac{x^2-(x+h)^2}{x^2(x+h)^2} = \frac{-2xh-h^2}{x^2(x+h)^2},$$

it follows that

$$\frac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h} = \frac{-2x-h}{x^2(x+h)^2}.$$

(f) Multiplying both numerator and denominator by $x^2-1=(x+1)(x-1)$ yields $\dfrac{10x^2}{5x(x-1)}$, which reduces to $\dfrac{2x}{x-1}$.

2.5 Fractional powers

5. The answers for each respective part that are given in the book emerge after multiplying both numerator and denominator by the following terms:

(a) $\sqrt{7}-\sqrt{5}$	(b) $\sqrt{5}-\sqrt{3}$	(c) $\sqrt{3}+2$
(d) $x\sqrt{y}-y\sqrt{x}$	(e) $\sqrt{x+h}+\sqrt{x}$	(f) $1-\sqrt{x+1}$

11. (a) $(2^x)^2=2^{2x}$, which equals 2^{x^2} if and only if $2x=x^2$, or if and only if $x=0$ or $x=2$.

(b) Correct because $a^{p-q} = a^p/a^q$.

(c) Correct because $a^{-p} = 1/a^p$.

(d) $5^{1/x} = 1/5^x = 5^{-x}$ if and only if $1/x = -x$ or $-x^2 = 1$, so there is no real x that satisfies the equation.

(e) Put $u = a^x$ and $v = a^y$, which reduces the equation to $uv = u + v$, or $0 = uv - u - v = (u-1)(v-1) - 1$. This is true only for special values of u and v and so for special values of x and y. In particular, the equation is false when $x = y = 1$.

(f) Putting $u = \sqrt{x}$ and $v = \sqrt{y}$ reduces the equation to $2^u \cdot 2^v = 2^{uv}$, which holds if and only if $uv = u + v$, as in (e) above.

2.6 Inequalities

3. (a) This inequality has the same solutions as

$$\frac{3x+1}{2x+4} - 2 > 0, \quad \text{or} \quad \frac{3x+1-2(2x+4)}{2x+4} > 0, \quad \text{or} \quad \frac{-x-7}{2x+4} > 0.$$

A sign diagram reveals that the inequality is satisfied for $-7 < x < -2$. A serious error is to multiply the inequality by $2x+4$, without checking the sign of $2x+4$. If $2x+4 < 0$, mulitiplying by this number will reverse the inequality sign.[4]

(b) The inequality is equivalent to $120/n \leq 0.75 = 3/4$, or $(480 - 3n)/4n \leq 0$. A sign diagram reveals that the inequality is satisfied for $n < 0$ and for $n \geq 160$.[5]

(c) This is easy: $g(g-2) \leq 0$ and so $0 \leq g \leq 2$.

(d) Note that $p^2 - 4p + 4 = (p-2)^2$, so the inequality reduces to $\dfrac{(p-2)+3}{(p-2)^2} = \dfrac{p+1}{(p-2)^2} \geq 0$. The fraction makes no sense if $p = 2$. The conclusion that $p \geq -1$ and $p \neq 2$ follows.

(e) The inequality is equivalent to

$$\frac{-n-2}{n+4} - 2 > 0 \iff \frac{-n-2-2n-8}{n+4} > 0 \iff \frac{-3n-10}{n+4} > 0 \iff -4 < n < -\frac{10}{3}$$

(f) $x^4 - x^2 = x^2(x^2-1) < 0 \iff x \neq 0$ and $x^2 < 1 \iff -1 < x < 0$ or $0 < x < 1$

6. (a) It is easy to see by means of a sign diagram that $x(x+3) < 0$ precisely when x lies in the open interval $(-3, 0)$. Therefore we have $\Rightarrow$, but not $\Leftarrow$: for example, if $x = 10$, then $x(x+3) = 130$.

(b) $x^2 < 9 \iff -3 < x < 3$, so $x^2 < 9 \Rightarrow x < 3$. If $x = -5$, for instance, we have $x < 3$ but $x^2 > 9$, hence we cannot have $\Leftarrow$ here.

(c) If $x > 0$, then $x^2 > 0$, but $x^2 > 0$ also when $x < 0$. So, we have $\Leftarrow$ but not $\Rightarrow$.

(d) As $y^2 \geq 0$ for any real number y, we have that $x > 0$ whenever $x > y^2$. But $x = 1 > 0$ does not imply that $x > y^2$ for $y \geq 1$. That is, we have $\Rightarrow$ but not $\Leftarrow$.

[4] It could be instructive to test the inequality for some values of x. For example, for $x = 0$ it is not true. What about $x = -5$?

[5] Note that for $n = 0$ the inequality makes no sense. For $n = 160$, we have equality.

9. Note that for any $x \geq 0$ and $y \geq 0$, $x - 2\sqrt{xy} + y = (\sqrt{x} - \sqrt{y})^2 \geq 0$, which implies that

$$\frac{1}{2}(x + y) \geq \sqrt{xy}. \tag{$*$}$$

Note also that the inequality is strict unless $x = y$.

To show that $m_A \geq m_G$, simply let $x = a$ and $y = b$, so that

$$m_A = \frac{1}{2}(a + b) \geq \sqrt{ab} = m_G,$$

with strict inequality whenever $a \neq b$.

To show that $m_G \geq m_H$, use $x = 1/a$ and $y = 1/b$, so that, from ($*$),

$$\frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right) \geq \sqrt{\frac{1}{a} \cdot \frac{1}{b}} \geq 0.$$

Rearranging, we get that

$$m_H = \left[\frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)\right]^{-1} \leq \left(\sqrt{\frac{1}{a} \cdot \frac{1}{b}}\right)^{-1} = m_G,$$

where, again, the inequality is strict unless $a = b$.

2.8 Summation

3. (a)–(d): In each case, look at the last term in the sum and replace n by k to get an expression for the k^{th} term. Call it s_k. Then in (a), (b), and (d) the sum is $\sum_{k=1}^{n} s_k$, and in (c) we have $\sum_{k=0}^{n} s_k$.
(e) The coefficients are the powers 3^n for $n = 1, 2, 3, 4, 5$, so the general term is $3^n x^n$.
(f)–(g) See the answers in the Solutions Section of the book.
(h) This is trickier: one has to see that each term is 198 larger than the previous term.[6]

7. (a) Valid: $\sum_{k=1}^{n} ck^2 = c \cdot 1^2 + c \cdot 2^2 + \cdots + c \cdot n^2 = c(1^2 + 2^2 + \cdots + n^2) = c\sum_{k=1}^{n} k^2$.
   (b) Wrong, even for $n = 2$: the left-hand side is $(a_1 + a_2)^2 = a_1^2 + 2a_1a_2 + a_2^2$, but the right-hand side is $a_1^2 + a_2^2$, which is different unless $a_1a_2 = 0$.
   (c) Valid: both sides equal $b_1 + b_2 + \cdots + b_N$.
   (d) Valid: both sides equal $5^1 + 5^2 + 5^3 + 5^4 + 5^5$.
   (e) Valid: both sides equal $a_{0,j}^2 + \cdots + a_{n-1,j}^2$.
   (f) Wrong, even for $n = 2$: the left-hand side is $a_1 + a_2/2$, but the right-hand side is $(1/k)(a_1 + a_2)$.

[6] This problem is related to the story about Gauss in Section 2.9.

2.11 Double sums

1. (a) See the solution in the book.

   (b) Note first that $\left(\dfrac{rs}{r+s}\right)^2 = 0$ when $s = 0$. So $\displaystyle\sum_{s=0}^{2}\sum_{r=2}^{4}\left(\frac{rs}{r+s}\right)^2$ reduces to

   $$\sum_{s=1}^{2}\left[\left(\frac{2s}{2+s}\right)^2 + \left(\frac{3s}{3+s}\right)^2 + \left(\frac{4s}{4+s}\right)^2\right] = \left(\frac{2}{3}\right)^2 + \left(\frac{3}{4}\right)^2 + \left(\frac{4}{5}\right)^2 + \left(\frac{4}{4}\right)^2 + \left(\frac{6}{5}\right)^2 + \left(\frac{8}{6}\right)^2$$

   which equals $\dfrac{4}{9} + \dfrac{9}{16} + \dfrac{16}{25} + 1 + \dfrac{36}{25} + \dfrac{16}{9} = \dfrac{(4+16)\cdot 400 + 9\cdot 225 + (16+36)\cdot 144}{3600} + 1$ or $\dfrac{8000+2025+7488}{3600} + 1 = \dfrac{17\,513}{3600} + 1 = 5 + \dfrac{3113}{3600}$.

   (c) Because $\sum_{i=1}^{m}\sum_{j=1}^{n}(i+j^2) = \sum_{j=1}^{n}\sum_{i=1}^{m} i + \sum_{i=1}^{m}\sum_{j=1}^{n} j^2$, we can use formulae (2.9.4) and (2.9.5) along with the equality $\sum_{k=1}^{p} a = pa$ to write the sum as

   $$\sum_{j=1}^{n}\tfrac{1}{2}m(m+1) + \sum_{i=1}^{m}\tfrac{1}{6}n(n+1)(2n+1) = n\left[\tfrac{1}{2}m(m+1)\right] + m\left[\tfrac{1}{6}n(n+1)(2n+1)\right]$$
   $$= \tfrac{1}{6}mn[3(m+1) + (n+1)(2n+1)] = \tfrac{1}{6}mn(2n^2 + 3n + 3m + 4)$$

   (d) Using formulae (2.9.4) and (2.9.5) again, we get

   $$\sum_{i=1}^{m}\sum_{j=1}^{2} i^j = \sum_{i=1}^{m}(i+i^2) = \sum_{i=1}^{m} i + \sum_{i=1}^{m} i^2 = \tfrac{1}{2}m(m+1) + \tfrac{1}{6}m(m+1)(2m+1)$$
   $$= \tfrac{1}{6}m(m+1)[3 + (2m+1)] = \tfrac{1}{6}m(m+1)(2m+4) = \tfrac{1}{3}m(m+1)(m+2).$$

4. $\bar{a}$ is the mean of the column means $\bar{a}_j$, because

   $$\frac{1}{n}\sum_{j=1}^{n}\bar{a}_j = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{1}{m}\sum_{r=1}^{m} a_{rj}\right) = \frac{1}{mn}\sum_{r=1}^{m}\sum_{j=1}^{n} a_{rj} = \bar{a}.$$

   To prove (∗), note that because $a_{rj} - \bar{a}$ is independent of the summation index s, it is a common factor when we sum over s, so $\sum_{s=1}^{m}(a_{rj}-\bar{a})(a_{sj}-\bar{a}) = (a_{rj}-\bar{a})\sum_{s=1}^{m}(a_{sj}-\bar{a})$ for each r. Next, summing over r gives

   $$\sum_{r=1}^{m}\sum_{s=1}^{m}(a_{rj}-\bar{a})(a_{sj}-\bar{a}) = \left[\sum_{r=1}^{m}(a_{rj}-\bar{a})\right]\left[\sum_{s=1}^{m}(a_{sj}-\bar{a})\right], \qquad (**)$$

   because $\sum_{s=1}^{m}(a_{rj}-\bar{a})$ is a common factor when we sum over r. Using the properties of sums and the definition of $\bar{a}_j$, we have

   $$\sum_{r=1}^{m}(a_{rj}-\bar{a}) = \sum_{r=1}^{m} a_{rj} - \sum_{r=1}^{m}\bar{a} = m\bar{a}_j - m\bar{a} = m(\bar{a}_j - \bar{a}).$$

   Similarly, replacing r with s as the index of summation, one also has $\sum_{s=1}^{m}(a_{sj}-\bar{a}) = m(\bar{a}_j - \bar{a})$. Substituting these values into (∗∗) then confirms (∗).

Review exercises for Chapter 2

5. (a) $(2x)^4 = 2^4x^4 = 16x^4$.

   (b) $2^{-1} - 4^{-1} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$, so $(2^{-1} - 4^{-1})^{-1} = 4$.

   (c) Cancel the common factor $4x^2yz^2$.

   (d) Here $-(-ab^3)^{-3} = -(-1)^{-3}a^{-3}b^{-9} = a^{-3}b^{-9}$,
   so $[-(-ab^3)^{-3}(a^6b^6)^2]^3 = [a^{-3}b^{-9}a^{12}b^{12}]^3 = [a^9b^3]^3 = a^{27}b^9$.

   (e) $\dfrac{a^5 \cdot a^3 \cdot a^{-2}}{a^{-3} \cdot a^6} = \dfrac{a^6}{a^3} = a^3$.

   (f) $\left[\left(\dfrac{x}{2}\right)^3 \cdot \dfrac{8}{x^{-2}}\right]^{-3} = \left(\dfrac{x^3}{8} \cdot 8x^2\right)^{-3} = (x^5)^{-3} = x^{-15}$.

10. Parts (a), (b), (d), (e), and (f) are straightforward; their solutions appear in the book. For the other parts:

   (c) $-\sqrt{3}\,(\sqrt{3} - \sqrt{6}) = -3 + \sqrt{3}\sqrt{6} = -3 + \sqrt{3}\sqrt{3}\sqrt{2} = -3 + 3\sqrt{2}$.

   (g) $(1 + x + x^2 + x^3)(1 - x) = (1 + x + x^2 + x^3) - (1 + x + x^2 + x^3)x = 1 - x^4$.

   (h) $(1 + x)^4 = (1 + x)^2(1 + x)^2 = (1 + 2x + x^2)(1 + 2x + x^2)$, and so on.

12. Parts (a) and (b) are easy, so we focus on the others:

   (c) $ax + ay + 2x + 2y = a(x + y) + 2(x + y) = (a + 2)(x + y)$.

   (d) $2x^2 - 5yz + 10xz - xy = 2x^2 + 10xz - (xy + 5yz) = 2x(x + 5z) - y(x + 5z) = (2x - y)(x + 5z)$.

   (e) $p^2 - q^2 + p - q = (p - q)(p + q) + (p - q) = (p - q)(p + q + 1)$.

   (f) $u^3 + v^3 - u^2v - v^2u = u^2(u - v) + v^2(v - u) = (u^2 - v^2)(u - v) = (u + v)(u - v)(u - v)$, which simplifies to $(u + v)(u - v)^2$.

16. (a) $\dfrac{s}{2s - 1} - \dfrac{s}{2s + 1} = \dfrac{s(2s + 1) - s(2s - 1)}{(2s - 1)(2s + 1)} = \dfrac{2s}{4s^2 - 1}$.

   (b) $\dfrac{x}{3 - x} - \dfrac{1 - x}{x + 3} - \dfrac{24}{x^2 - 9} = \dfrac{-x(x + 3) - (1 - x)(x - 3) - 24}{(x - 3)(x + 3)} = \dfrac{-7x - 21}{(x - 3)(x + 3)} = \dfrac{7}{3 - x}$.

   (c) Multiplying both numerator and denominator by x^2y^2 yields

   $$\frac{y - x}{y^2 - x^2} = \frac{y - x}{(y - x)(y + x)} = \frac{1}{x + y}$$

17. (a) Simply cancel the factor $25ab$.

   (b) Factor $x^2 - y^2 = (x + y)(x - y)$, and then cancel $x + y$.

   (c) The fraction can be written as $\dfrac{(2a - 3b)^2}{(2a - 3b)(2a + 3b)} = \dfrac{2a - 3b}{2a + 3b}$.

   (d) $\dfrac{4x - x^3}{4 - 4x + x^2} = \dfrac{x(4 - x^2)}{(2 - x)^2} = \dfrac{x(2 - x)(2 + x)}{(2 - x)^2} = \dfrac{x(2 + x)}{2 - x}$

25. Let each side have length s. Then the area K of the equilateral triangle ABC shown in Fig. SM2.R.25 is the sum of the areas of the three triangles ABP, BCP, and CAP, which equals $\frac{1}{2}sh_1 + \frac{1}{2}sh_2 + \frac{1}{2}sh_3 = K$. It follows that $h_1 + h_2 + h_3 = 2K/s$, which is independent of where P is placed in the triangle.

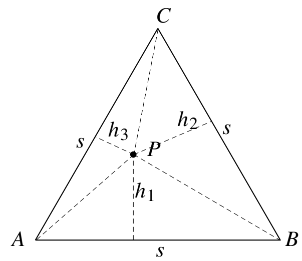

Figure SM2.R.25

29. (a) Using the trick that led to Eq. (2.9.4) in the book,

$$\begin{aligned} R &= \quad 3 + \quad 5 + \quad 7 + \cdots + 197 + 199 + 201 \\ R &= 201 + 199 + 197 + \cdots + \quad 7 + \quad 5 + \quad 3. \end{aligned}$$

Summing vertically term by term, noting that there are 100 terms, gives

$$2R = 204 + 204 + 204 + \cdots + 204 + 204 + 204 = 100 \times 204 = 20400,$$

and thus $R = 10200$.

(b) Here one has

$$\begin{aligned} S &= 1001 + 2002 + 3003 + \cdots + 8008 + 9009 + 10010 \\ &= 1001(1 + 2 + 3 + \cdots + 8 + 9 + 10) = 1001 \cdot 55 = 55055 \end{aligned}$$

3 Solving Equations

3.1 Solving equations

3. (a) We note first that $x = -3$ and $x = -4$ both make the equation absurd. Multiplying the equation by the common denominator $(x+3)(x+4)$ yields $(x-3)(x+4) = (x+3)(x-4)$, i.e. $x^2 + x - 12 = x^2 - x - 12$, and thus $x = 0$.

(b) Multiplying by the common denominator $(x-3)(x+3)$ yields $3(x+3) - 2(x-3) = 9$, from which we get $x = -6$.

(c) Multiplying by the common denominator $15x$, assuming that $x \neq 0$, yields $18x^2 - 75 = 10x^2 - 15x + 8x^2$, from which we get $x = 5$.

5. (a) Multiplying by the common denominator 12 yields $9y - 3 - 4 + 4y + 24 = 36y$, and so $y = 17/23$.

(b) Multiplying by the common denominator $2x(x+2)$ yields $8(x+2) + 6x = 2(2x+2) + 7x$ and so $14x + 16 = 11x + 4$, from which we find that $3x = -12$ and so $x = -4$.

(c) Multiplying both numerator and denominator in the first fraction by $1 - z$ leads to

$$\frac{2 - 2z - z}{(1-z)(1+z)} = \frac{6}{2z+1}.$$

Multiplying each side of the equation by $(1-z^2)(2z+1)$ yields $(2-3z)(2z+1) = 6 - 6z^2$. This simplifies to $2 + z - 6z^2 = 6 - 6z^2$ whose only solution is $z = 4$.

(d) Expanding all the parentheses gives

$$\frac{p}{4}-\frac{3}{8}-\frac{1}{4}+\frac{p}{12}-\frac{1}{3}+\frac{p}{3}=-\frac{1}{3}.$$

Multiplying by the common denominator 24 gives the equation $6p-9-6+2p-8+8p=-8$, whose solution is $p=15/16$.

3.2 Equations and their parameters

2. (a) Multiply both sides by abx to obtain $b+a=2abx$. Hence,

$$x=\frac{b+a}{2ab}=\frac{b}{2ab}+\frac{a}{2ab}=\tfrac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right).$$

(b) Multiply the equation by $cx+d$ to obtain $ax+b=cAx+dA$, or $(a-cA)x=dA-b$, and thus $x=(dA-b)/(a-cA)$ provided that $a\neq cA$.

(c) Multiply the equation by $x^{1/2}$ to obtain $\frac{1}{2}p=wx^{1/2}$. Thus $x^{1/2}=p/2w$ and so, after squaring each side, $x=p^2/4w^2$.

(d) Multiply each side by $\sqrt{1+x}$ to obtain $1+x+ax=0$, so $x=-1/(1+a)$.

(e) $x^2=b^2/a^2$, so $x=\pm b/a$ provided that $a\neq 0$.

(f) We see immediately that $x=0$.

4. (a) $\alpha x-a=\beta x-b$ if and only if $(\alpha-\beta)x=a-b$, so $x=(a-b)/(\alpha-\beta)$ provided that $\alpha\neq\beta$.

(b) Squaring each side of $\sqrt{pq}=3q+5$ yields $pq=(3q+5)^2$, so $p=(3q+5)^2/q$ provided that $q\neq 0$.

(c) $Y=94+0.2[Y-(20+0.5Y)]=94+0.2Y-4-0.1Y$, so $0.9Y=90$, implying that $Y=100$.

(d) Raise each side to the fourth power to obtain $K^2\dfrac{r}{2w}K=Q^4$, so $K^3=2wQ^4/r$, and hence $K=\left(2wQ^4/r\right)^{1/3}$.

(e) Multiplying the numerator and denominator of the left-hand fraction by $4K^{1/2}L^{3/4}$ leads to $2L/K=r/w$, from which we get $L=rK/2w$.

(f) Raise each side to the fourth power to obtain $\frac{1}{16}p^4K^{-1}(r/2w)=r^4$. It follows that $K=\frac{1}{32}p^4r^{-3}w^{-1}$.

3.3 Quadratic equations

3. These results are straightforward and can be found in the book. To illustrate the process, let us consider part (d): First, rewrite the equation as $r^2+(\sqrt{3}-\sqrt{2})r-\sqrt{6}=0$, and then

apply the formula to obtain

$$
\begin{aligned}
r &= \frac{-(\sqrt{3}-\sqrt{2}) \pm \sqrt{(\sqrt{3}-\sqrt{2})^2 - 4\cdot 1\cdot(-\sqrt{6})}}{2\cdot 1} \\
&= \frac{1}{2}\left(-\sqrt{3}+\sqrt{2} \pm \sqrt{3-2\sqrt{6}+2+4\sqrt{6}}\right) \\
&= \frac{1}{2}\left(-\sqrt{3}+\sqrt{2} \pm \sqrt{3+2\sqrt{6}+2}\right) \\
&= \frac{1}{2}\left(-\sqrt{3}+\sqrt{2} \pm \sqrt{(\sqrt{3}+\sqrt{2})^2}\right) \\
&= \frac{1}{2}\left(-\sqrt{3}+\sqrt{2} \pm (\sqrt{3}+\sqrt{2})\right),
\end{aligned}
$$

so the solutions are $r = \frac{1}{2}2\sqrt{2} = \sqrt{2}$ and $r = -\frac{1}{2}2\sqrt{3} = -\sqrt{3}$.

5. (a) See the solution in the book.
   (b) If the smaller of the two natural numbers is n, then the larger is $n+1$, so the requirement is that $n^2 + (n+1)^2 = 13$. This reduces to $2n^2 + 2n - 12 = 0$, i.e. $n^2 + n - 6 = 0$, with solutions $n = -3$ and $n = 2$, so the two numbers are 2 and 3.[7]
   (c) If the shorter side has length x, then the other side has length $x + 14$. According to Pythagoras's Theorem one has $x^2 + (x+14)^2 = 34^2$, or $x^2 + 14x - 480 = 0$. The only positive solution is $x = 16$, and then $x + 14 = 30$.
   (d) If the usual driving speed is s km/h and the usual time spent is t hours, then $st = 80$. Since 16 minutes is $16/60 = 4/15$ hours, driving at the speed $s + 10$ km/h for $s - 4/15$ hours gives $(s+10)(t - 4/15) = 80$. From the first equation, $t = 80/s$. Inserting this into the second equation, we get $(s+10)(80/s - 4/15) = 80$. Rearranging, we obtain $s^2 + 10s - 3000 = 0$, whose only positive solution is $s = 50$. So his usual driving speed is 50 km/h.

3.4 Nonlinear equations

2. (a) The numerator $5 + x^2$ is never 0, so there are no solutions.
   (b) The equation is obviously equivalent to $\dfrac{x^2+1+2x}{x^2+1} = 0$, or $\dfrac{(x+1)^2}{x^2+1} = 0$, so $x = -1$.
   (c) Because $x = -1$ is clearly not a solution, we can multiply the equation by $(x+1)^{2/3}$ to obtain the equivalent equation $(x+1)^{1/3} - \frac{1}{3}x(x+1)^{-2/3} = 0$. Multiplying this equation again by $(x+1)^{2/3}$ yields $x + 1 - \frac{1}{3}x = 0$, whose solution is $x = -3/2$.
   (d) Multiplying by $x - 1$ yields $x + 2x(x-1) = 0$, or $x(2x-1) = 0$. Hence $x = 0$ or $x = 1/2$.

3. (a) $z = 0$ satisfies the equation. If $z \neq 0$, then $z - a = za + zb$, or $(1 - a - b)z = a$. If $a + b = 1$ we have a contradiction. If $a + b \neq 1$, then $z = a/(1 - a - b)$.
   (b) The equation is equivalent to $(1+\lambda)\mu(x - y) = 0$, so $\lambda = -1$, $\mu = 0$, or $x = y$.
   (c) $\mu = \pm 1$ makes the equation meaningless. Otherwise, multiplying the equation by $1 - \mu^2$ yields $\lambda(1-\mu) = -\lambda$, or $\lambda(2-\mu) = 0$, so $\lambda = 0$ or $\mu = 2$.
   (d) The equation is equivalent to $b(1+\lambda)(a-2) = 0$, so $b = 0$, $\lambda = -1$, or $a = 2$.

[7] If we had asked for *integer* solutions, we would have -3 and -2 in addition.

3.5 Using implication arrows

3. (a) If

$$\sqrt{x-4} = \sqrt{x+5} - 9, \tag{i}$$

then squaring each side gives

$$x - 4 = (\sqrt{x+5} - 9)^2. \tag{ii}$$

Expanding the square on the right-hand side of (ii) gives $x - 4 = x + 5 - 18\sqrt{x+5} + 81$, which reduces to $18\sqrt{x+5} = 90$ or $\sqrt{x+5} = 5$, implying that $x + 5 = 25$ and so $x = 20$. This hows that *if* x is a solution of (i), then $x = 20$. No other value of x can satisfy (i). *But if we check this solution*, we find that with $x = 20$ the left-hand side of (i) becomes $\sqrt{16} = 4$, and the right-hand side becomes $\sqrt{25} - 9 = 5 - 9 = -4$. This means that equation (i) actually has no solutions at all.[8]

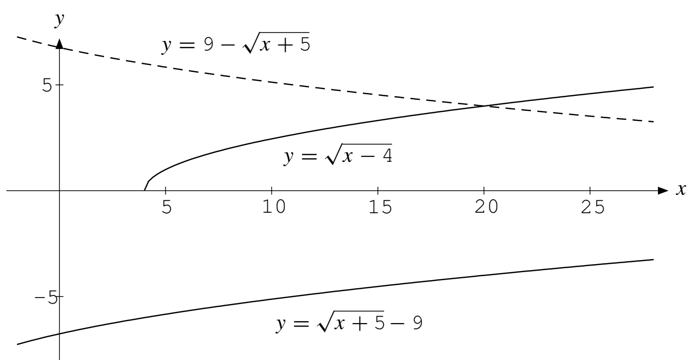

Figure SM3.5.3

(b) If x is a solution of

$$\sqrt{x-4} = 9 - \sqrt{x+5}, \tag{iii}$$

then just as in part (a) we find that x must be a solution of

$$x - 4 = (9 - \sqrt{x+5})^2. \tag{iv}$$

Now, $(9 - \sqrt{x+5})^2 = (\sqrt{x+5} - 9)^2$, so equation (iv) is equivalent to equation (ii) in part (a). This means that (iv) has exactly one solution, namely $x = 20$. Inserting this value of x into equation (iii), we find that $x = 20$ *is* a solution of (iii).

A geometric explanation of the results can be given with reference to Fig. SM3.5.3. We see that the two solid curves in the figure have no point in common, that is, the expressions $\sqrt{x-4}$ and $\sqrt{x+5} - 9$ are not equal for any value of x.[9] This explains why the equation in (a) has no solution. The dashed curve $y = 9 - \sqrt{x+5}$, on the other hand, intersects $y = \sqrt{x+5}$ for $x = 20$ (and only there), and this corresponds to the solution in part (b).[10]

[8] But note that $4^2 = (-4)^2$, i.e. the square of the left-hand side equals the square of the right-hand side. That is how the spurious solution $x = 20$ managed to sneak in.

[9] In fact, $\sqrt{x-4} - (\sqrt{x+5} - 9)$ increases with x, so there is no point of intersection farther to the right, either.

[10] In part (a) it was necessary to check the result, because the transition from (i) to (ii) is only an implication, not an equivalence. Similarly, it was necessary to check the result in part (b), since the transition from (iii) to (iv) also is only an implication — at least, it is not clear that it is an equivalence. (Afterwards, it turned out to be an equivalence, but we could not know that until we had solved the equation.)

3.6 Two linear equations in two unknowns

4. (a) If the two numbers are x and y, then $x + y = 52$ and $x - y = 26$. Adding the two equations gives $2x = 78$, so $x = 39$, and then $y = 52 - 39 = 13$.

   (b) Let the cost of one table be \$$x$ and the cost of one chair \$$y$. Then $5x + 20y = 1800$ and $2x + 3y = 420$. Solving this system yields $x = 120$, $y = 60$.

   (c) Let x and y be the number of units produced of B and P, respectively. This gives the equations $x = \frac{3}{2}y$ and $200x + 300y = 180\,000$. Inserting the expression for x from the first equation into the second equation gives $300y + 300y = 180\,000$, which yields the solution $y = 300$ and then $x = 450$. Thus, 450 units of quality A and 300 units of quality B should be produced.

   (d) Suppose that the person invested \$$x$ at 5% and \$$y$ at 7.2%. Then $x + y = 10\,000$ and $0.05x + 0.072y = 676$. The solution is $x = 2000$ and $y = 8000$.

Review exercises for Chapter 3

2. (a) Assuming $x \neq \pm 4$, multiplying by the lowest common denominator $(x-4)(x+4)$ reduces the equation to $(x-3)(x+4) = (x+3)(x-4)$ or $x = -x$, so $x = 0$.

   (b) The given equation makes sense only if $x \neq \pm 3$. If we multiply the equation by the common denominator $(x+3)(x-3)$ we get $3(x+3)^2 - 2(x^2-9) = 9x+27$ or $x^2+9x+18 = 0$, with the solutions $x = -6$ and $x = -3$. The only solution of the given equation is therefore $x = -6$.

   (c) Subtracting $2x/3$ from each side simplifies the equation to $0 = -1 + 5/x$, whose only solution is $x = 5$.

   (d) Assuming $x \neq 0$ and $x \neq \pm 5$, multiply by the common denominator $x(x-5)(x+5)$ to get $x(x-5)^2 - x(x^2-25) = x^2 - 25 - (11x+20)(x+5)$. Expanding each side of this equation gives $x^3 - 10x^2 + 25x - x^3 + 25x = x^2 - 25 - 11x^2 - 75x - 100$, which simplifies to $50x = -125 - 75x$ with solution $x = -1$.

4. (a) Multiply each side of the equation by $5K^{1/2}$ to obtain $15L^{1/3} = K^{1/2}$. Squaring each side gives $K = 225L^{2/3}$.

   (b) Raise each side to the power $1/t$ to obtain $1 + r/100 = 2^{1/t}$, and so $r = 100(2^{1/t} - 1)$.

   (c) $abx_0^{b-1} = p$, so $x_0^{b-1} = p/ab$. Now raise each side to the power $1/(b-1)$.

   (d) Raise each side to the power $-\rho$ to get $(1-\lambda)a^{-\rho} + \lambda b^{-\rho} = c^{-\rho}$, or $b^{-\rho} = \lambda^{-1}[c^{-\rho} - (1-\lambda)a^{-\rho}]$. Now raise each side to the power $-1/\rho$.

4 Functions of One Variable

4.2 Basic definitions

1. (a) $f(0) = 0^2 + 1 = 1$, $f(-1) = (-1)^2 + 1 = 2$, $f(1/2) = (1/2)^2 + 1 = 1/4 + 1 = 5/4$, and $f(\sqrt{2}) = (\sqrt{2})^2 + 1 = 2 + 1 = 3$.

(b) (i) Since $(-x)^2 = x^2$, $f(x) = f(-x)$ for all x. (ii) $f(x+1) = (x+1)^2+1 = x^2+2x+1+1 = x^2+2x+2$ and $f(x)+f(1) = x^2+1+2 = x^2+3$. Thus equality holds if and only if $x^2+2x+2 = x^2+3$, i.e. if and only if $x = 1/2$. (iii) $f(2x) = (2x)^2+1 = 4x^2+1$ and $2f(x) = 2x^2+2$. Now, $4x^2+1 = 2x^2+2 \iff x^2 = 1/2 \iff x = \pm\sqrt{1/2} = \pm\frac{1}{2}\sqrt{2}$.

13. (a) We require $5 - x \geq 0$, so $x \leq 5$.

(b) The denominator $x^2 - x = x(x-1)$ must be different from 0, so $x \neq 0$ and $x \neq 1$.

(c) To begin with, the denominator must be nonzero, so we require $x \neq 2$ and $x \neq -3$. Moreover, since we can only take the square root of a nonnegative number, the fraction $(x-1)/(x-2)(x+3)$ must be ≥ 0. A sign diagram reveals that $D_f = (-3, 1] \cup (2, \infty)$. Note in particular that the function is defined with value 0 at $x = 1$.

4.4 Linear functions

10. The points that satisfy the inequality $3x + 4y \leq 12$ are those that lie on or below the straight line $3x + 4y = 12$, as explained in Example 4.4.6 for a similar inequality. The points that satisfy the inequality $x - y \leq 1$, or equivalently, $y \geq x - 1$, are those on or above the straight line $x - y = 1$. Finally, the points that satisfy the inequality $3x + y \geq 3$, or equivalently, $y \geq 3 - 3x$, are those on or above the straight line $3x + y = 3$. The set of points that satisfy all these three inequalities simultaneously is the shaded set shown in Fig. A4.4.10 in the book.

4.6 Quadratic functions

9. The key to the argument is part (b). We find that $f(x) = Ax^2 + Bx + C$, with coefficients $A = a_1^2 + a_2^2 + \cdots + a_n^2$, $B = 2(a_1b_1 + a_2b_2 + \cdots + a_nb_n)$, and $C = b_1^2 + b_2^2 + \cdots + b_n^2$. Now, in case $B^2 - 4AC > 0$, then according to formula (2.3.4), the equation $f(x) = Ax^2 + Bx + C = 0$ would have two distinct solutions. This would contradict $f(x) \geq 0$ for all x. Hence $B^2 - 4AC \leq 0$ and the conclusion follows.

4.7 Polynomials

3. (a) The answer is $2x^2 + 2x + 4 + \dfrac{3}{x-1}$, because

$$
\begin{array}{llll}
(2x^3 & & + 2x - 1) \div (x-1) = 2x^2 + 2x + 4 & \\
\underline{2x^3 - 2x^2} & & & \\
 & 2x^2 + 2x - 1 & & \\
 & \underline{2x^2 - 2x} & & \\
 & & 4x - 1 & \\
 & & \underline{4x - 4} & \\
 & & 3 & \text{remainder}
\end{array}
$$

(b) The answer is x^2+1, because

$$
\begin{array}{l}
(x^4+x^3+x^2+x) \div (x^2+x) = x^2+1 \\
\underline{x^4+x^3 \qquad\qquad\quad} \\
\qquad\qquad\quad x^2+x \\
\qquad\qquad\quad \underline{x^2+x} \\
\qquad\qquad\qquad\quad 0
\end{array}
$$

(c) The answer is $x^3-4x^2+3x+1-\dfrac{4x}{x^2+x+1}$, because

$$
\begin{array}{rrrrrl}
(x^5 & -3x^4 & & & +1) & \div (x^2+x+1) = x^3-4x^2+3x+1 \\
x^5 & +x^4 & +x^3 & & & \\
\hline
 & -4x^4 & -x^3 & & +1 & \\
 & -4x^4 & -4x^3 & -4x^2 & & \\
\hline
 & & 3x^3 & +4x^2 & +1 & \\
 & & 3x^3 & +3x^2 & +3x & \\
\hline
 & & & x^2 & -3x+1 & \\
 & & & x^2 & +x+1 & \\
\hline
 & & & & -4x & \text{remainder}
\end{array}
$$

(d) The answer is

$$
3x^5+6x^3-3x^2+12x-12+\frac{28x^2-36x+13}{x^3-2x+1},
$$

because

$$
\begin{array}{rrrrrrrl}
(3x^8 & & & & & +x^2 & +1) & \div (x^3-2x+1) = 3x^5+6x^3-3x^2+12x-12 \\
3x^8 & -6x^6 & +3x^5 & & & & & \\
\hline
 & 6x^6 & -3x^5 & & & +x^2 & +1 & \\
 & 6x^6 & & -12x^4 & +6x^3 & & & \\
\hline
 & & -3x^5 & +12x^4 & +6x^3 & +x^2 & +1 & \\
 & & -3x^5 & & +6x^3 & -3x^2 & & \\
\hline
 & & & 12x^4 & -12x^3 & +4x^2 & +1 & \\
 & & & 12x^4 & & -24x^2 & +12x & \\
\hline
 & & & & -12x^3 & +28x^2 & -12x+1 & \\
 & & & & -12x^3 & & +24x-12 & \\
\hline
 & & & & & 28x^2 & -36x+13 & \text{remainder}
\end{array}
$$

4. (a) Since the graph intersects the x-axis at the two points $x=-1$ and $x=3$, we try the quadratic function $f(x)=a(x+1)(x-3)$, for some constant $a>0$. But the graph passes through the point $(1,-2)$, so we need $f(1)=-2$. Since $f(1)=-4a$ for our chosen function, $a=\frac{1}{2}$. This leads to the formula $y=\frac{1}{2}(x+1)(x-3)$.

(b) Because the equation $f(x) = 0$ must have roots $x = -3, 1, 2$, we try the cubic function $f(x) = b(x+3)(x-1)(x-2)$. Then $f(0) = 6b$. According to the graph, $f(0) = -12$. So $b = -2$, and hence $y = -2(x+3)(x-1)(x-2)$.

(c) Here we try a cubic polynomial of the form $y = c(x+3)(x-2)^2$, with $x = 2$ as a double root. Then $f(0) = 12c$. From the graph we see that $f(0) = 6$, and so $c = \frac{1}{2}$. This leads to the formula $y = \frac{1}{2}(x+3)(x-2)^2$.

8. Polynomial division gives

$$
\begin{array}{lll}
(x^2 - & \gamma x &) \div (x+\beta) = x - (\beta+\gamma) \\
\;x^2 + & \beta x & \\
\hline
& -(\beta+\gamma)x & \\
& -(\beta+\gamma)x - \beta(\beta+\gamma) & \\
\hline
& \qquad\qquad \beta(\beta+\gamma) & \text{remainder}
\end{array}
$$

and so

$$E = \alpha\left[x - (\beta+\gamma) + \frac{\beta(\beta+\gamma)}{x+\beta}\right] = \alpha x - \alpha(\beta+\gamma) + \frac{\alpha\beta(\beta+\gamma)}{x+\beta}.$$

4.8 Power functions

4. (a) C. The graph is a parabola and since the coefficient in front of x^2 is positive, it has a minimum point.

(b) D. The function is defined for $x \le 2$ and crosses the y-axis at $y = 2\sqrt{2} \approx 2.8$.

(c) E. The graph is a parabola and since the coefficient in front of x^2 is negative, it has a maximum point.

(d) B. When x increases, y decreases, and y becomes close to -2 when x is large.

(e) A. The function is defined for $x \ge 2$ and increases as x increases.

(f) F. Let $y = 2 - (\frac{1}{2})^x$. Then y increases as x increases. For large values of x, one has y close to 2.

4.10 Logarithmic functions

3. (a) $3^x 4^{x+2} = 8$ when $3^x 4^x 4^2 = 8$ or $(12)^x 4^2 = 8$, and so $12^x = 1/2$. Taking the natural log of each side gives $x \ln 12 = -\ln 2$, so $x = -\ln 2/\ln 12$.

(b) Since $\ln x^2 = 2\ln x$, the equation reduces to $7\ln x = 6$, so $\ln x = 6/7$, and thus $x = e^{6/7}$.

(c) One possible way to solve the equation is to rewrite it as $4^x(1 - 4^{-1}) = 3^x(3-1)$, or $4^x \cdot (3/4) = 3^x \cdot 2$, so $(4/3)^x = 8/3$, implying that $x = \ln(8/3)/\ln(4/3)$. Alternatively, start by dividing both sides by 3^x to obtain $(4/3)^x(1-1/4) = 3-1 = 2$, so $(4/3)^x = 8/3$ as before.

For parts (d)–(f) below, we use the definition $a^{\log_a x} = x$ for all $a, x > 0$.

(d) $\log_2 x = 2$ implies that $2^{\log_2 x} = 2^2$ or $x = 4$.

(e) $\log_x e^2 = 2$ implies that $x^{\log_x e^2} = x^2$ or $e^2 = x^2$. Hence $x = e$.

(f) $\log_3 x = -3$ implies that $3^{\log_3 x} = 3^{-3}$ or $x = 1/27$.

4. Directly, the equation $Ae^{rt} = Be^{st}$ implies that $e^{rt}/e^{st} = B/A$, so $e^{(r-s)t} = B/A$. Taking the ln of each side gives $(r-s)t = \ln(B/A)$, so $t = \dfrac{1}{r-s} \ln \frac{B}{A}$. Alternatively, apply ln to each side of the original equation, leading to $\ln A + rt = \ln B + st$. This gives the same solution.

5. Let t denote the number of years after 1990. Assuming continuous exponential growth, when the GDP of the two nations is the same, one must have $1.2 \cdot 10^{12} \cdot e^{0.09t} = 5.6 \cdot 10^{12} \cdot e^{0.02t}$. Applying the answer found in part (a), we obtain

$$t = \frac{1}{0.09 - 0.02} \ln \frac{5.6 \cdot 10^{12}}{1.2 \cdot 10^{12}} = \frac{1}{0.07} \ln \frac{14}{3} \approx 22$$

According to this, the two countries would have the same GDP approximately 22 years after 1990, so in 2012.

Review exercises for Chapter 4

14. (a) $p(x) = x(x^2 + x - 12) = x(x-3)(x+4)$, because $x^2 + x - 12 = 0$ for $x = 3$ and $x = -4$.

(b) ± 1, ± 2, ± 4, ± 8 are the only possible integer zeros. By trial and error we find that $q(2) = q(-4) = 0$, so $2(x-2)(x+4) = 2x^2 + 4x - 16$ is a factor for $q(x)$. By polynomial division we find that $q(x) \div (2x^2 + 4x - 16) = x - 1/2$, so $q(x) = 2(x-2)(x+4)(x-1/2)$.

17. Check by direct calculation that $p(2) = \frac{1}{4}2^3 - 2^2 - \frac{11}{4}2 + \frac{15}{2} = 2 - 4 - \frac{11}{2} + \frac{15}{2} = 0$, so $x - 2$ must be a factor of $p(x)$. By direct division, we find that $p(x) \div (x-2) = \frac{1}{4}(x^2 - 2x - 15)$, which factors as $\frac{1}{4}(x+3)(x-5)$, so $x = -3$ and $x = 5$ are the two other zeros. (Alternatively, $p(x)$ has the same zeros as $q(x) = 4p(x) = x^3 - 4x^2 - 11x + 30$. Any integer zero of $q(x)$ must be a factor of 30, so the only possibilities are ± 1, ± 2, ± 3, ± 5, ± 10, ± 15, and ± 30. But it is tedious to check all 14 possible values of x in order to find the zeros in this way, and it does not find any noninteger roots.)

19. For the left-hand graph, note that for $x \neq 0$, one has $y = f(x) = \dfrac{a + b/x}{1 + c/x}$, so that y tends to a as x becomes large positive or negative. The graph shows that $a > 0$. There is a break point at $x = -c$, and $-c > 0$, so $c < 0$. Also $f(0) = b/c > 0$, so $b < 0$. The right-hand graph of the quadratic function g is a parabola which is convex, so $p > 0$. Moreover $r = g(0) < 0$. Finally, $g(x)$ has its minimum at $x = x^* = -q/2p$. Since $x^* > 0$ and $p > 0$, we conclude that $q < 0$.

23. (a) $\ln(x/e^2) = \ln x - \ln e^2 = \ln x - 2$ for $x > 0$.

(b) $\ln(xz/y) = \ln(xz) - \ln y = \ln x + \ln z - \ln y$ for $x, y, z > 0$

(c) $\ln(e^3 x^2) = \ln e^3 + \ln x^2 = 3 + 2\ln x$ for $x > 0$. (In general, $\ln x^2 = 2\ln|x|$.)

(d) When $x > 0$, note that $\ln(1/x) = -\ln x$ and so

$$\tfrac{1}{2}\ln x - \tfrac{3}{2}\ln(1/x) - \ln(x+1) = 2\ln x - \ln(x+1) = \ln x^2 - \ln(x+1) = \ln[x^2/(x+1)]$$

5 Properties of Functions

5.3 Inverse functions

4. (a) The function f does have an inverse since it is one-to-one. This is shown in the table by the fact that all the numbers in the second row, the domain of f^{-1}, are different. The inverse assigns to each number in the second row, the corresponding number in the first row. In particular, $f^{-1}(2) = -1$.

   (b) Since $f(x)$ increases by 2 for each unit increase in x, one has $f(x) = 2x + a$ for a suitable constant a. But then $a = f(0) = 4$, so $f(x) = 2x + 4$. Solving $y = 2x + 4$ for x yields $x = \frac{1}{2}y - 2$, so interchanging x and y gives $y = f^{-1}(x) = \frac{1}{2}x - 2$.

9. (a) $(x^3 - 1)^{1/3} = y \iff x^3 - 1 = y^3 \iff x = (y^3 + 1)^{1/3}$. If we use x as the independent variable, $f^{-1}(x) = (x^3 + 1)^{1/3}$. $\mathbb{R}$ is the domain and range for both f and f^{-1}.

   (b) The domain consists of all $x \neq 2$. For all such x one has

   $$\frac{x+1}{x-2} = y \iff x+1 = y(x-2) \iff (1-y)x = -2y-1 \iff x = \frac{-2y-1}{1-y} = \frac{2y+1}{y-1}.$$

   Using x as the independent variable, $f^{-1}(x) = (2x+1)/(x-1)$. The domain of the inverse is all $x \neq 1$.

   (c) Here one has $y = (1 - x^3)^{1/5} + 2 \iff y - 2 = (1 - x^3)^{1/5} \iff (y-2)^5 = 1 - x^3 \iff x^3 = 1 - (y-2)^5$, which is equivalent to $x = [1 - (y-2)^5]^{1/3}$. With x as the free variable, $f^{-1}(x) = [1 - (x-2)^5]^{1/3}$. For both f and f^{-1}, the whole of the real line $\mathbb{R}$ is both the domain and the range.

10. (a) The domain is $\mathbb{R}$ and the range is $(0, \infty)$, so the inverse is defined on $(0, \infty)$. From $y = e^{x+4}$ one has $\ln y = x + 4$, so $x = \ln y - 4$, $y > 0$.

    (b) The range is $\mathbb{R}$, which is the domain of the inverse. From $y = \ln x - 4$, one has $\ln x = y + 4$, and so $x = e^{y+4}$.

    (c) The domain is $\mathbb{R}$. On this domain the function is increasing, with $y \to \ln 2$ as $x \to -\infty$ and $y \to \infty$ as $x \to \infty$. So the range of the function is $(\ln 2, \infty)$. From $y = \ln\left(2 + e^{x-3}\right)$ one has $e^y = 2 + e^{x-3}$, so $e^{x-3} = e^y - 2$. Hence, $x = 3 + \ln(e^y - 2)$ for $y > \ln 2$.

5.4 Graphs of equations

1. (a) The curve intersects the axes $x = 0$ and $y = 0$ at the points $(0, \pm\sqrt{3})$ and $(\pm\sqrt{6}, 0)$ respectively. It is also entirely bounded by the rectangle whose four corners are $(\pm\sqrt{6}, \pm\sqrt{3})$. Moreover, it is symmetric about both axes, since all its points take the form $(\pm\sqrt{\xi}, \pm\sqrt{\eta})$, where ξ, η are any pair of positive real numbers satisfying $\xi^2 + 2\eta^2 = 6$. Putting $x = y$ yields the four points $(\pm\sqrt{2}, \pm\sqrt{2})$ on the curve. More points can be found by fixing any x satisfying $-6 < x < 6$, then solving for y.[11]

   (b) The same argument as in (a) shows that the curve intersects only the axis $x = 0$, at $(0, \pm 1)$. There are no points on the graph where $y^2 < 1$. As in (a), it is symmetric about

[11] The curve is called an ellipse. See the next section.

both axes. It comes in two separate parts: below $y = -1$; above $y = 1$. Putting $x^2 = 1$ and then $x^2 = 9$ yields the additional points $(\pm 1, \pm\sqrt{2})$ and $(\pm 3, \pm\sqrt{10})$.[12]

5.5 Distance in the plane

8. The method of completing the square used in problem 5 shows that

$$
\begin{aligned}
x^2 + y^2 + Ax + By + C = 0 &\iff x^2 + Ax + y^2 + By + C = 0 \\
&\iff x^2 + Ax + \left(\tfrac{1}{2}A\right)^2 + y^2 + By + \left(\tfrac{1}{2}B\right)^2 = \frac{1}{4}(A^2 + B^2 - 4C) \\
&\iff \left(x + \tfrac{1}{2}A\right)^2 + \left(y + \tfrac{1}{2}B\right)^2 = \frac{1}{4}(A^2 + B^2 - 4C).
\end{aligned}
$$

Provided that $A^2 + B^2 > 4C$, the last equation is that of a circle centred at $(-\frac{1}{2}A, -\frac{1}{2}B)$ with radius $\frac{1}{2}\sqrt{A^2 + B^2 - 4C}$. If $A^2 + B^2 = 4C$, the graph consists only of the point $(-\frac{1}{2}A, -\frac{1}{2}B)$. For $A^2 + B^2 < 4C$, the solution set is empty.

5.6 General functions

1. In each case, except (c), the rule defines a function because it associates with each member of the original set a unique member in the target set. For instance, in (d), if the volume V of a sphere is given, the formula $V = \frac{4}{3}\pi r^3$ in the appendix implies that the radius is $r = (3V/4\pi)^{1/3}$. But then the formula $S = 4\pi r^2$ in the appendix gives the surface area. Substituting for r in this formula gives $S = 4\pi(3V/4\pi)^{2/3} = (36\pi)^{1/3}V^{3/2}$ that expresses the surface area of a sphere as a function of its volume.

Review exercises for Chapter 5

7. (a) The function f is defined and strictly increasing for $e^x > 2$, i.e. $x > \ln 2$. Its range is $\mathbb{R}$ because $f(x) \to -\infty$ as $x \to \ln 2^+$, and $f(x) \to \infty$ as $x \to \infty$. From $y = 3 + \ln(e^x - 2)$, we get $\ln(e^x - 2) = y - 3$, and so $e^x - 2 = e^{y-3}$, or $e^x = 2 + e^{y-3}$, so $x = \ln(2 + e^{y-3})$. Hence $f^{-1}(x) = \ln(2 + e^{x-3})$, $x \in \mathbb{R}$.

   (b) Note that f is strictly increasing. Moreover, $e^{-\lambda x} \to \infty$ as $x \to -\infty$, and $e^{-\lambda x} \to 0$ as $x \to \infty$. Therefore, $f(x) \to 0$ as $x \to -\infty$, and $f(x) \to 1$ as $x \to \infty$. So the range of f, and therefore the domain of f^{-1}, is $(0, 1)$. Because $y = \dfrac{a}{e^{-\lambda x} + a}$ we get $e^{-\lambda x} + a = a/y$, so $e^{-\lambda x} = a(1/y - 1)$, or $-\lambda x = \ln a + \ln(1/y - 1)$. Thus $x = -(1/\lambda)\ln a - (1/\lambda)\ln(1/y - 1)$, and therefore the inverse is $f^{-1}(x) = -(1/\lambda)\ln a - (1/\lambda)\ln(1/x - 1)$, with $x \in (0, 1)$.

6 Differentiation

6.2 Tangents and derivatives

6. For parts (a)–(c) we set out the explicit steps of the recipe in given in the book; for parts (d)–(f) we still follow the recipe, but express the steps more concisely.

[12] The graph is a hyperbola. See the next section.

(a) (A) $f(a+h) = f(0+h) = 3h+2$; (B) $f(a+h) - f(a) = f(h) - f(0) = 3h+2-2 = 3h$; (C)–(D) $[f(h) - f(0)]/h = 3$; (E) $[f(h) - f(0)]/h = 3 \to 3$ as $h \to 0$, so $f'(0) = 3$. The slope of the tangent at $(0,2)$ is 3.

(b) (A) $f(a+h) = f(1+h) = (1+h)^2 - 1 = 1 + 2h + h^2 - 1 = 2h + h^2$;
(B) $f(1+h) - f(1) = 2h + h^2$; (C)–(D) $[f(1+h) - f(1)]/h = 2 + h$;
(E) $[f(1+h) - f(1)]/h = 2 + h \to 2$ as $h \to 0$, so $f'(1) = 2$.

(c) (A) $f(3+h) = 2 + 3/(3+h)$; (B) $f(3+h) - f(3) = 2 + 3/(3+h) - 3 = -h/(3+h)$;
(C)–(D) $[f(3+h) - f(3)]/h = -1/(3+h)$;
(E) $[f(3+h) - f(3)]/h = -1/(3+h) \to -1/3$ as $h \to 0$, so $f'(3) = -1/3$.

(d) $[f(h) - f(0)]/h = (h^3 - 2h)/h = h^2 - 2 \to -2$ as $h \to 0$, so $f'(0) = -2$.

(e) $\frac{1}{h}[f(-1+h) - f(-1)] = \frac{1}{h}\left[-1 + h + \frac{1}{-1+h} + 2\right]$, which simplifies to $\frac{h^2-1+1}{h(h-1)} = \frac{h}{h-1}$. This tends to 0 as $h \to 0$, so $f'(0) = 0$.

(f) $\frac{1}{h}[f(1+h) - f(1)] = \frac{1}{h}[(1+h)^4 - 1] = \frac{1}{h}[(h^4 + 4h^3 + 6h^2 + 4h + 1) - 1] = h^3 + 4h^2 + 6h + 4 \to 4$ as $h \to 0$, so $f'(1) = 4$.

9. (a) Applying the formula $(a-b)(a+b) = a^2 - b^2$ with $a = \sqrt{x+h}$ and $b = \sqrt{x}$ gives $(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x}) = (x+h) - x = h$.

(b) By direct computation,

$$\frac{f(x+h) - f(x)}{h} = \frac{(\sqrt{x+h} - \sqrt{x}\,)(\sqrt{x+h} + \sqrt{x}\,)}{h(\sqrt{x+h} + \sqrt{x}\,)} = \frac{1}{\sqrt{x+h} + \sqrt{x}}.$$

(c) From part (b), as $h \to 0$ one has $\frac{f(x+h) - f(x)}{h} \to \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$.

6.5 A dash of limits

5. (a) $\frac{1/3 - 2/3h}{h-2} = \frac{3h\,(1/3 - 2/3h)}{3h(h-2)} = \frac{h-2}{3h(h-2)} = \frac{1}{3h} \to \frac{1}{6}$ as $h \to 2$.

(b) When $x \to 0$, then $(x^2 - 1)/x^2 = 1 - 1/x^2 \to -\infty$ as $x \to 0$.

(c) $\frac{32t - 96}{t^2 - 2t - 3} = \frac{32(t-3)}{(t-3)(t+1)} = \frac{32}{t+1} \to 8$, as $t \to 3$, so $\sqrt[3]{\frac{32t-96}{t^2-2t-3}} \to \sqrt[3]{8} = 2$ as $t \to 3$.

(d) $\frac{\sqrt{h+3} - \sqrt{3}}{h} = \frac{(\sqrt{h+3} - \sqrt{3}\,)(\sqrt{h+3} + \sqrt{3}\,)}{h(\sqrt{h+3} + \sqrt{3}\,)} = \frac{1}{\sqrt{h+3} + \sqrt{3}} \to \frac{1}{2\sqrt{3}}$ as $h \to 0$.

(e) $\frac{t^2 - 4}{t^2 + 10t + 16} = \frac{(t+2)(t-2)}{(t+2)(t+8)} = \frac{t-2}{t+8} \to -\frac{2}{3}$ as $t \to -2$.

(f) Observe that $4 - x = (2 + \sqrt{x})(2 - \sqrt{x})$, so $\lim_{x \to 4} \frac{2 - \sqrt{x}}{4 - x} = \lim_{x \to 4} \frac{1}{2 + \sqrt{x}} = \frac{1}{4}$.

6. (a) $\frac{f(x) - f(1)}{x - 1} = \frac{x^2 + 2x - 3}{x - 1} = \frac{(x-1)(x+3)}{x-1} = x + 3 \to 4$ as $x \to 1$

(b) $\frac{f(x) - f(1)}{x - 1} = x + 3 \to 5$ as $x \to 2$.

(c) $\dfrac{f(2+h)-f(2)}{h} = \dfrac{(2+h)^2+2(2+h)-8}{h} = \dfrac{h^2+6h}{h} = h+6 \to 6$ as $h \to 0$.

(d) $\dfrac{f(x)-f(a)}{x-a} = \dfrac{x^2+2x-a^2-2a}{x-a} = \dfrac{x^2-a^2+2(x-a)}{x-a} = \dfrac{(x-a)(x+a)+2(x-a)}{x-a}$, which equals $x+a+2$ and tends to $2a+2$ as $x \to a$.

(e) Same answer as part (d), putting $x-a=h$.

(f) $\dfrac{f(a+h)-f(a-h)}{h} = \dfrac{(a+h)^2+2a+2h-(a-h)^2-2a+2h}{h} = 4a+4$, which tends to $4a+4$ as a limit as $h \to 0$.

6.7 Sums, products and quotients

3. (a) $y = \frac{1}{x^6} = x^{-6} \Rightarrow y' = -6x^{-7}$, using the power rule (6.6.4).

   (b) $y = x^{-1}(x^2+1)\sqrt{x} = x^{-1}x^2x^{1/2} + x^{-1}x^{1/2} = x^{3/2} + x^{-1/2} \Rightarrow y' = \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-3/2}$.

   (c) $y = x^{-3/2} \Rightarrow y' = -\frac{3}{2}x^{-5/2}$.

   (d) $y = \dfrac{x+1}{x-1} \Rightarrow y' = \dfrac{1\cdot(x-1)-(x+1)\cdot 1}{(x-1)^2} = \dfrac{-2}{(x-1)^2}$.

   (e) $y = \dfrac{x}{x^5} + \dfrac{1}{x^5} = x^{-4} + x^{-5} \Rightarrow y' = -\dfrac{4}{x^5} - \dfrac{5}{x^6}$.

   (f) $y = \dfrac{3x-5}{2x+8} \Rightarrow y' = \dfrac{3(2x+8)-2(3x-5)}{(2x+8)^2} = \dfrac{34}{(2x+8)^2}$

   (g) $y = 3x^{-11} \Rightarrow y' = -33x^{-12}$.

   (h) $y = \dfrac{3x-1}{x^2+x+1} \Rightarrow y' = \dfrac{3(x^2+x+1)-(3x-1)(2x+1)}{(x^2+x+1)^2} = \dfrac{-3x^2+2x+4}{(x^2+x+1)^2}$

6. (a) $y' = 6x-12 = 6(x-2) \geq 0 \iff x \geq 2$, so y is increasing in $[2,\infty)$.

   (b) $y' = x^3-3x = x(x^2-3) = x(x-\sqrt{3})(x+\sqrt{3})$, so, using a sign diagram, one sees that y is increasing in $[-\sqrt{3},0]$ and in $[\sqrt{3},\infty)$.

   (c) y is increasing in $[-\sqrt{2},\sqrt{2}]$, since

   $$y' = \frac{2(2+x^2)-(2x)(2x)}{(2+x^2)^2} = \frac{2(2-x^2)}{(x^2+2)^2} = \frac{2(\sqrt{2}-x)(\sqrt{2}+x)}{(x^2+2)^2}.$$

   (d) y is increasing in $(-\infty,x_1]$ and in $[0,x_2]$, since

   $$y' = \frac{(2x-3x^2)(x+1)-(x^2-x^3)}{2(x+1)^2} = \frac{-2x^3-2x^2+2x}{2(x+1)^2} = \frac{-x(x-x_1)(x-x_2)}{(x+1)^2},$$

   where $x_{1,2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$.

7. (a) $y' = -1-2x = -3$ when $x=1$, so the slope of the tangent is -3. Since $y=1$ when $x=1$, the point–slope formula gives $y-1 = -3(x-1)$, or $y = -3x+4$.

   (b) $y = 1-2(x^2+1)^{-1}$, so $y' = 4x/(x^2+1)^2 = 1$ and $y=0$ when $x=1$. The tangent is therefore $y = x-1$.

   (c) $y = x^2-x^{-2}$, so $y' = 2x+2x^{-3} = 17/4$. Also $y = 15/4$ when $x=2$, so the point–slope formula gives $y-15/4 = (17/4)(x-2)$, or $y = (17/4)x-19/4$.

(d) Here $y' = \dfrac{4x^3(x^3+3x^2+x+3)-(x^4+1)(3x^2+6x+1)}{[(x^2+1)(x+3)]^2} = -\dfrac{1}{9}$. Also $y = \frac{1}{3}$ when $x = 0$, so the tangent is $y = \frac{1}{3} - \frac{1}{9}x = -(x-3)/9$.

9. (a) By the quotient rule, $y = \dfrac{at+b}{ct+d} \Rightarrow y' = \dfrac{a(ct+d)-(at+b)c}{(ct+d)^2} = \dfrac{ad-bc}{(ct+d)^2}$.

(b) $y = t^n\left(a\sqrt{t}+b\right) = at^{n+1/2} + bt^n \Rightarrow y' = (n+1/2)at^{n-1/2} + nbt^{n-1}$.

(c) $y = \dfrac{1}{at^2+bt+c} \Rightarrow y' = \dfrac{0\cdot(at^2+bt+c) - 1\cdot(2at+b)}{(at^2+bt+c)^2} = \dfrac{-2at-b}{(at^2+bt+c)^2}$.

6.8 The chain rule

3. (a) Write $y = \dfrac{1}{(x^2+x+1)^5} = (x^2+x+1)^{-5} = u^{-5}$, where $u = x^2+x+1$. The chain rule gives $y' = (-5)u^{-6}u' = -5(2x+1)(x^2+x+1)^{-6}$.

(b) With $u = x + \sqrt{x+\sqrt{x}}$, $y = \sqrt{u} = u^{1/2}$, so $y' = \frac{1}{2}u^{-1/2}u'$. Now, $u = x + v^{1/2}$, with $v = x + x^{1/2}$. Then $u' = 1 + \frac{1}{2}v^{-1/2}v'$, where $v' = 1 + \frac{1}{2}x^{-1/2}$. Thus, in the end,

$$y' = \tfrac{1}{2}u^{-1/2}u' = \tfrac{1}{2}\left[x + (x+x^{1/2})^{1/2}\right]^{-1/2}\left[1 + (\tfrac{1}{2}(x+x^{1/2})^{-1/2}(1+\tfrac{1}{2}x^{-1/2})\right].$$

(c) See the answer in the book.

6.10 Exponential functions

4. (a) $y' = 3x^2 + 2e^{2x}$ is obviously positive everywhere, so y is increasing in $(-\infty,\infty)$.

(b) $y' = 10xe^{-4x} + 5x^2(-4)e^{-4x} = 10x(1-2x)e^{-4x}$. A sign diagram shows that y is increasing in $[0,1/2]$.

(c) $y' = 2xe^{-x^2} + x^2(-2x)e^{-x^2} = 2x(1-x)(1+x)e^{-x^2}$. A sign diagram shows that y is increasing in $(-\infty,-1]$ and in $[0,1]$.

6.11 Logarithmic functions

3. For most of these problems we need the chain rule. That is important in itself! But it implies in particular that if $u = f(x)$ is a differentiable function of x that satisfies $f(x) > 0$, then $\dfrac{\mathrm{d}}{\mathrm{d}x}\ln u = \dfrac{u'}{u}$.

(a) $y = \ln(\ln x) = \ln u$ with $u = \ln x$ implies that $y' = \dfrac{1}{u}u' = \dfrac{1}{\ln x}\dfrac{1}{x} = \dfrac{1}{x\ln x}$.

(b) $y = \ln\sqrt{1-x^2} = \ln u$ with $u = \sqrt{1-x^2}$ implies that

$$y' = \frac{1}{u}u' = \frac{1}{\sqrt{1-x^2}}\frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{1-x^2}$$

Alternatively, $\sqrt{1-x^2} = (1-x^2)^{1/2} \Rightarrow y = \frac{1}{2}\ln(1-x^2)$, and so on.

(c) $y = e^x\ln x \Rightarrow y' = e^x\ln x + e^x\dfrac{1}{x} = e^x\left(\ln x + \dfrac{1}{x}\right)$.

(d) $y = e^{x^3}\ln x^2 \Rightarrow y' = 3x^2 e^{x^3}\ln x^2 + e^{x^3}\dfrac{1}{x^2}2x = e^{x^3}\left(3x^2\ln x^2 + \dfrac{2}{x}\right)$.

(e) $y = \ln(e^x+1) \Rightarrow y' = \dfrac{e^x}{e^x+1}$.

(f) $y = \ln(x^2+3x-1) \Rightarrow y' = \dfrac{2x+3}{x^2+3x-1}$.

(g) $y = 2(e^x-1)^{-1} \Rightarrow y' = -2e^x(e^x-1)^{-2}$.

(h) $y = e^{2x^2-x} \Rightarrow y' = (4x-1)e^{2x^2-x}$.

5. (a) One must have $x^2 > 1$, i.e. $x > 1$ or $x < -1$.

   (b) $\ln(\ln x)$ is defined when $\ln x$ is defined and positive, that is, for $x > 1$.

   (c) The fraction $\dfrac{1}{\ln(\ln x)-1}$ is defined when $\ln(\ln x)$ is defined and different from 1. From (b), $\ln(\ln x)$ is defined when $x > 1$. Further, $\ln(\ln x) = 1 \iff \ln x = e \iff x = e^e$. We conclude that $\dfrac{1}{\ln(\ln x)-1}$ is defined if and only if $x > 1$ with $x \neq e^e$.

6. (a) The function is defined for $4 - x^2 > 0$, that is in $(-2,2)$. $f'(x) = -2x/(4-x^2) \geq 0$ in $(-2,0]$, so this is where y is increasing.

   (b) The function is defined for $x > 0$. $f'(x) = x^2(3\ln x + 1) \geq 0$ for $\ln x \geq -1/3$, or $x \geq e^{-1/3}$, so y is increasing in $[e^{-1/3},\infty)$.

   (c) The function is defined for $x > 0$, and

$$y' = \frac{2(1-\ln x)(-1/x)2x - 2(1-\ln x)^2}{4x^2} = \frac{(1-\ln x)(\ln x - 3)}{2x^2}.$$

   A sign diagram reveals that y is increasing in x when $1 \leq \ln x \leq 3$ and so for x in $[e, e^3]$.

9. In these problems we can use logarithmic differentiation. Alternatively we can write the functions in the form $f(x) = e^{g(x)}$ and then use the fact that $f'(x) = e^{g(x)}g'(x) = f(x)g'(x)$.

   (a) Let $f(x) = (2x)^x$. Then $\ln f(x) = x\ln(2x)$, so $\frac{f'(x)}{f(x)} = 1\cdot\ln(2x) + x\cdot\frac{1}{2x}\cdot 2 = \ln(2x)+1$. Hence, $f'(x) = f(x)(\ln(2x)+1) = (2x)^x(\ln x + \ln 2 + 1)$.

   (b) $f(x) = x^{\sqrt{x}} = \left(e^{\ln x}\right)^{\sqrt{x}} = e^{\sqrt{x}\ln x}$, so

$$f'(x) = e^{\sqrt{x}\ln x}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(\sqrt{x}\ln x) = x^{\sqrt{x}}\left(\frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x}\right) = x^{\sqrt{x}-\frac{1}{2}}(\tfrac{1}{2}\ln x + 1).$$

   (c) Here $\ln f(x) = x\ln\sqrt{x} = \frac{1}{2}x\ln x$, so $f'(x)/f(x) = \frac{1}{2}(\ln x + 1)$, which implies that $f'(x) = \frac{1}{2}(\sqrt{x})^x(\ln x + 1)$.

11. (a) See the answer in the book.

   (b) Let $f(x) = \ln(1+x) - \frac{1}{2}x$. Then $f(0) = 0$ and moreover $f'(x) = 1/(x+1) - \frac{1}{2} = (1-x)/2(x+1)$. This is positive in $(0,1)$, so $f(x) > 0$ in $(0,1)$, thus establishing the left-hand inequality. To prove the other inequality, put $g(x) = x - \ln(1+x)$. Then $g(0) = 0$ and $g'(x) = 1 - 1/(x+1) = x/(x+1) > 0$ in $(0,1)$, so the conclusion follows.

   (c) Let $f(x) = 2(\sqrt{x}-1) - \ln x$. Then $f(1) = 0$ and $f'(x) = (1/\sqrt{x}) - 1/x = (\sqrt{x}-1)/x$, which is positive for $x > 1$. The conclusion follows.

Review exercises for Chapter 6

15. (a) $y' = \frac{2}{x}\ln x \geq 0$ if $x \geq 1$.

(b) $y' = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} \geq 0 \iff e^x \geq e^{-x} \iff e^{2x} \geq 1 \iff x \geq 0$

(c) $y' = 1 - \dfrac{3x}{x^2+2} = \dfrac{(x-1)(x-2)}{x^2+2} \geq 0 \iff x \leq 1$ or $x \geq 2$, as a sign diagram will show.

7 Derivatives in Use

7.1 Implicit differentiation

3. (a) Implicit differentiation w.r.t. x yields $(*)$ $1 - y' + 3y + 3xy' = 0$. Solving for y' yields $y' = (1+3y)/(1-3x)$. The definition of the function implies that $y = (x-2)/(1-3x)$. Substituting this in the expression for y' gives $y' = -5/(1-3x)^2$. Differentiating $(*)$ w.r.t. x gives $-y'' + 3y' + 3y' + 3xy'' = 0$. Inserting $y' = (1+3y)/(1-3x)$ and solving for y'' gives $y'' = 6y'/(1-3x) = -30/(1-3x)^3$.

(b) Implicit differentiation w.r.t. x yields $(*)$ $5y^4y' = 6x^5$, so $y' = 6x^5/5y^4 = (6/5)x^{1/5}$. Differentiating $(*)$ w.r.t. x gives $20y^3(y')^2 + 5y^4y'' = 30x^4$. Inserting $y' = 6x^5/5y^4$ and solving for y'' yields $y'' = 6x^4y^{-4} - 4y^{-1}(y'')^2 = 6x^4y^{-4} - (144/25)x^{10}y^{-9} = (6/25)x^{-4/5}$.

8. (a) $y + xy' = g'(x) + 3y^2y'$, and solve for y'.

(b) $g'(x+y)(1+y') = 2x + 2yy'$, and solve for y'.

(c) First, in order to differentiate $g(x^2y)$ w.r.t. x, put $z = g(u)$ where $u = x^2y$. It follows that $z' = g'(u)u'$ where $u' = 2xy + x^2y'$. So one has $2(xy+1)(y+xy') = g'(x^2y)(2xy + x^2y')$. Now solve for y'.

10. (a) Differentiate w.r.t. x while keeping in mind that y depends on x. This yields the equation $2(x^2+y^2)(2x+2yy') = a^2(2x-2yy')$. Then solve for y'.

(b) Note that $x = 0$ would imply that $y = 0$. Excluding this possibility, we see that $y' = 0$ when $x^2 + y^2 = a^2/2$, or $y^2 = \frac{1}{2}a^2 - x^2$. Inserting this into the given equation yields $x = \pm\frac{1}{4}a\sqrt{6}$ and so $y = \pm\frac{1}{2}a\sqrt{2}$. This yields the four points on the graph at which the tangent is horizontal.

7.2 Economic examples

4. (a) Using (ii) and (iii) to substitute for C and Y respectively in equation (i), one has $Y = f(Y) + I + \bar{X} - g(Y)$.

(b) Differentiating w.r.t. I yields

$$\mathrm{d}Y/\mathrm{d}I = f'(Y)(\mathrm{d}Y/\mathrm{d}I) + 1 - g'(Y)(\mathrm{d}Y/\mathrm{d}I) = (f'(Y) - g'(Y))(\mathrm{d}Y/\mathrm{d}I) + 1 \qquad (*)$$

Thus, $\mathrm{d}Y/\mathrm{d}I = 1/[1 - f'(Y) + g'(Y)]$. Imports should increase when income increases, so $g'(Y) > 0$. It follows that $\mathrm{d}Y/\mathrm{d}I > 0$.

(c) Differentiating $(*)$ w.r.t. I yields, in simplified notation, $\mathrm{d}^2Y/\mathrm{d}I^2 = (f'' - g'')(\mathrm{d}Y/\mathrm{d}I) + (f' - g')(\mathrm{d}^2Y/\mathrm{d}I^2)$, so $\mathrm{d}^2Y/\mathrm{d}I^2 = (f'' - g'')(\mathrm{d}Y/\mathrm{d}I)/(1 - f' + g')^2 = (f'' - g'')/(1 - f' + g')^3$.

7.3 Differentiating the inverse

5. (a) $\mathrm{d}y/\mathrm{d}x = -e^{-x-5}$, so $\mathrm{d}x/\mathrm{d}y = 1/(\mathrm{d}y/\mathrm{d}x) = 1/(-e^{-x-5}) = -e^{x+5}$.

   (b) $\mathrm{d}y/\mathrm{d}x = -e^{-x}/(e^{-x}+3)$, so $\mathrm{d}x/\mathrm{d}y = -(e^{-x}+3)/e^{-x} = -1-3e^{x}$

   (c) Implicit differentiation w.r.t. x yields $y^3 + x(3y^2)(\mathrm{d}y/\mathrm{d}x) - 3x^2y - x^3(\mathrm{d}y/\mathrm{d}x) = 2$. Solve for $\mathrm{d}y/\mathrm{d}x$, and then invert.

7.4 Linear approximations

3. (a) $f(0) = 1$ and $f'(x) = -(1+x)^{-2}$, so $f'(0) = -1$. Then $f(x) \approx f(0) + f'(0)x = 1 - x$.

   (b) $f(0) = 1$ and $f'(x) = 5(1+x)^4$, so $f'(0) = 5$. Then $f(x) \approx f(0) + f'(0)x = 1 + 5x$.

   (c) $f(0) = 1$ and $f'(x) = -\frac{1}{4}(1-x)^{-3/4}$, so $f'(0) = -\frac{1}{4}$. Then $f(x) \approx f(0)+f'(0)x = 1-\frac{1}{4}x$.

8. (a) We must differentiate $3xe^{xy^2}$. To do so, first use the product rule in order to get $\frac{\mathrm{d}}{\mathrm{d}x}3xe^{xy^2} = 3e^{xy^2} + 3x\frac{\mathrm{d}}{\mathrm{d}x}e^{xy^2}$. Then the chain rule gives $\frac{\mathrm{d}}{\mathrm{d}x}e^{xy^2} = e^{xy^2}(y^2 + x2yy')$. Overall, therefore, implicit differentiation yields $3e^{xy^2} + 3xe^{xy^2}(y^2 + x2yy') - 2y' = 6x + 2yy'$. Putting $x = 1$ and $y = 0$ gives $3 - 2y' = 6$, so $y' = -3/2$.

   (b) $y(x) \approx y(1) + y'(1)(x-1) = -\frac{3}{2}(x-1)$

7.5 Polynomial approximations

2. $f'(x) = (1+x)^{-1}$, $f''(x) = -(1+x)^{-2}$, $f'''(x) = 2(1+x)^{-3}$, $f^{(4)}(x) = -6(1+x)^{-4}$, $f^{(5)}(x) = 24(1+x)^{-5}$. Then $f(0) = 0$, $f'(0) = 1$, $f''(0) = -1$, $f'''(0) = 2$, $f^{(4)}(0) = -6$, $f^{(5)}(0) = 24$, and so

$$\begin{aligned} f(x) &\approx f(0) + \frac{1}{1!}f'(0)x + \frac{1}{2!}f''(0)x + \frac{1}{3!}f'''(0)x^3 + \frac{1}{4!}f^{(4)}(0)x^4 + \frac{1}{5!}f^{(5)}(0)x^5 \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5. \end{aligned}$$

3. With $f(x) = 5(\ln(1+x) - \sqrt{1+x}) = 5\ln(1+x) - 5(1+x)^{1/2}$ we get

$$f'(x) = 5(1+x)^{-1} - \frac{5}{2}(1+x)^{-1/2} \text{ and } f''(x) = -5(1+x)^{-2} + \frac{5}{4}(1+x)^{-3/2}$$

   So $f(0) = -5$, $f'(0) = \frac{5}{2}$, and $f''(0) = -\frac{15}{4}$. Hence, the Taylor polynomial of order 2 about $x = 0$ is $f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = -5 + \frac{5}{2}x - \frac{15}{8}x^2$.

7.6 Taylor's formula

4. (a) We use Taylor's formula (7.6.3) with $g(x) = (1+x)^{1/3}$ and $n = 2$. Then $g'(x) = \frac{1}{3}(1+x)^{-2/3}$, $g''(x) = -\frac{2}{9}(1+x)^{-5/3}$, and $g'''(x) = \frac{10}{27}(1+x)^{-8/3}$, so $g(0) = 1$, $g'(0) = \frac{1}{3}$, $g''(0) = -\frac{2}{9}$, $g'''(c) = \frac{10}{27}(1+c)^{-8/3}$. It follows that $g(x) = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + R_3(x)$, where $R_3(x) = \frac{1}{6}\frac{10}{27}(1+c)^{-8/3}x^3 = \frac{5}{81}(1+c)^{-8/3}x^3$.

   (b) $c \in (0, x)$ and $x \geq 0$, so $(1+c)^{-8/3} \leq 1$, and the inequality follows.

(c) Note that $\sqrt[3]{1003} = 10(1 + 3 \cdot 10^{-3})^{1/3}$. Using the approximation in part (a) gives

$$(1 + 3 \cdot 10^{-3})^{1/3} \approx 1 + \frac{1}{3} 3 \cdot 10^{-3} - \frac{1}{9}(3 \cdot 10^{-3})^2 = 1 + 10^{-3} - 10^{-6} = 1.000999$$

So $\sqrt[3]{1003} \approx 10.00999$. By part (b), the error $R_3(x)$ in the approximation $(1+3 \cdot 10^{-3})^{1/3} \approx 1.000999$ satisfies $|R_3(x)| \leq \frac{5}{81}(3 \cdot 10^{-3})^3 = \frac{5}{3} 10^{-9}$. Hence the error in the approximation $\sqrt[3]{1003} \approx 10.00999$ is $10|R_3(x)| \leq \frac{50}{3} 10^{-9} < 2 \cdot 10^{-8}$, implying that the answer is correct to 7 decimal places.

7.7 Elasticities

9. (a) $\mathrm{El}_x A = \dfrac{x}{A} \dfrac{\mathrm{d}A}{\mathrm{d}x} = 0$.

(b) $\mathrm{El}_x(fg) = \dfrac{x}{fg}(fg)' = \dfrac{x}{fg}(f'g + fg') = \dfrac{xf'}{f} + \dfrac{xg'}{g} = \mathrm{El}_x f + \mathrm{El}_x g$.

(c) $\mathrm{El}_x \dfrac{f}{g} = \dfrac{x}{(f/g)} \left(\dfrac{f}{g}\right)' = \dfrac{xg}{f}\left(\dfrac{gf' - fg'}{g^2}\right) = \dfrac{xf'}{f} - \dfrac{xg'}{g} = \mathrm{El}_x f - \mathrm{El}_x g$.

(d) See the answer in the book.

(e) This case is like (d), but with $+g$ replaced by $-g$, and $+g'$ by $-g'$.

(f) $z = f(u)$ and $u = g(x)$ imply that

$$\mathrm{El}_x z = \frac{x}{z} \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{x}{u} \frac{u}{z} \frac{\mathrm{d}z}{\mathrm{d}u} \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{u}{z} \frac{\mathrm{d}z}{\mathrm{d}u} \frac{x}{u} \frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{El}_u f(u) \, \mathrm{El}_x u.$$

7.8 Continuity

3. By the properties of continuous functions stated in the book, all the functions are continuous wherever they are defined. So (a) and (d) are defined everywhere. In (b) we must exclude $x = 1$; in (c) the function is defined for $x < 2$. Next, in (e) we must exclude values of x that make the denominator 0. These values satisfy $x^2 + 2x - 2 = 0$, or $(x + 1)^2 = 3$, so they are $x = \pm\sqrt{3} - 1$. Finally, in (f), the first fraction requires $x > 0$, and then the other fraction is also defined.

7.9 More on limits

2. (a) $\lim_{x \to 0^+}(x^2 + 3x - 4) = 0^2 + 3 \cdot 0 - 4 = -4$.

(b) $|x| = -x$ for $x < 0$. Hence, $\displaystyle\lim_{x \to 0^-} \frac{x + |x|}{x} = \lim_{x \to 0^-} \frac{x - x}{x} = \lim_{x \to 0^-} 0 = 0$.

(c) $|x| = x$ for $x > 0$. Hence, $\displaystyle\lim_{x \to 0^+} \frac{x + |x|}{x} = \lim_{x \to 0^+} \frac{x + x}{x} = \lim_{x \to 0^+} 2 = 2$.

(d) As $x \to 0^+$ one has $\sqrt{x} \to 0$ and so $-1/\sqrt{x} \to -\infty$.

(e) As $x \to 3^+$ one has $x - 3 \to 0^+$ and so $x/(x - 3) \to \infty$.

(f) As $x \to 3^-$ one has $x - 3 \to 0^-$, and so $x/(x - 3) \to -\infty$.

5. (a) Vertical asymptote, $x = -1$. Moreover, $x^2 \div (x + 1) = x - 1 + 1/(x + 1)$, so $y = x - 1$ is an asymptote as $x \to \pm\infty$.

(b) No vertical asymptote. Moreover. $(2x^3-3x^2+3x-6)\div(x^2+1) = 2x-3+(x-3)/(x^2+1)$, so $y = 2x-3$ is an asymptote as $x \to \pm\infty$.

(c) Vertical asymptote, $x = 1$. Moreover, $(3x^2+2x) \div (x-1) = 3x+5+5/(x-1)$, so $y = 3x+5$ is an asymptote as $x \to \pm\infty$.

(d) Vertical asymptote, $x = 1$. Moreover,

$$(5x^4 - 3x^2 + 1) \div (x^3 - 1) = 5x + (-3x^2 + 5x + 1)/(x^3 - 1)$$

So $y = 5x$ is an asymptote as $x \to \pm\infty$.

9. This is rather tricky because the denominator is 0 at $x_{1,2} = 2 \pm \sqrt{3}$. A sign diagram shows that $f(x) > 0$ only in $(-\infty, 0)$ and in (x_1, x_2). The text explains where f increases. See also Fig. SM7.9.9.

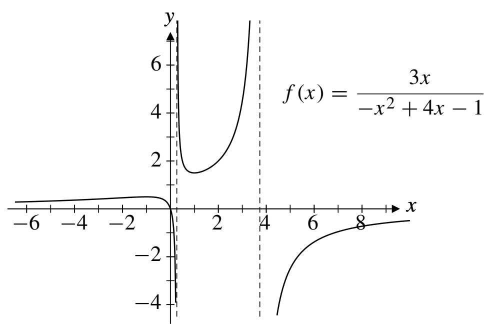

Figure SM7.9.9

7.10 The intermediate value theorem and Newton's method

4. Recall from (4.7.6) that any integer root of the equation $f(x) = x^4 + 3x^3 - 3x^2 - 8x + 3 = 0$ must be a factor of the constant term 3. The way to see this directly is to notice that we must have

$$3 = -x^4 - 3x^3 + 3x^2 + 8x = x(-x^3 - 3x^2 + 3x + 8)$$

and if x is an integer then the bracketed expression is also an integer. Thus, the only possible integer solutions are ± 1 and ± 3. Trying each of these possibilities, we find that only -3 is an integer solution.

There are three other real roots, with approximate values $x_0 = -1.9$, $y_0 = 0.4$, and $z_0 = 1.5$. If we use Newton's method once for each of these roots we get the more accurate approximations

$$x_1 = -1.9 - \frac{f(-1.9)}{f'(-1.9)} = -1.9 - \frac{-0.1749}{8.454} \approx -1.9 + 0.021 = -1.879$$
$$y_1 = 0.4 - \frac{f(0.4)}{f'(0.4)} = 0.4 - \frac{-0.4624}{-8.704} \approx 0.4 - 0.053 = 0.347$$
$$z_1 = 1.5 - \frac{f(1.5)}{f'(1.5)} = 1.5 - \frac{-0.5625}{16.75} \approx 1.5 + 0.034 = 1.534$$

7.12 L'Hôpital's rule

3. (a) $\lim_{x\to 1} \dfrac{x-1}{x^2-1} = \text{“}0/0\text{”} = \lim_{x\to 1} \dfrac{1}{2x} = \frac{1}{2}$. Alternatively, use $x^2 - 1 = (x+1)(x-1)$.

   (b) $\lim_{x\to -2} \dfrac{x^3+3x^2-4}{x^3+5x^2+8x+4} = \text{“}0/0\text{”} = \lim_{x\to -2} \dfrac{3x^2+6x}{3x^2+10x+8} = \text{“}0/0\text{”} = \lim_{x\to -2} \dfrac{6x+6}{6x+10} = 3$.

   (c) $\lim_{x\to 2} \dfrac{x^4-4x^3+6x^2-8x+8}{x^3-3x^2+4} = \text{“}0/0\text{”} = \lim_{x\to 2} \dfrac{4x^3-12x^2+12x-8}{3x^2-6x} = \text{“}0/0\text{”} =$
   $\lim_{x\to 2} \dfrac{12x^2-24x+12}{6x-6}$, which equals 2.

   (d) $\lim_{x\to 1} \dfrac{\ln x - x + 1}{(x-1)^2} = \text{“}0/0\text{”} = \lim_{x\to 1} \dfrac{(1/x)-1}{2(x-1)} = \text{“}0/0\text{”} = \lim_{x\to 1} \dfrac{(-1/x^2)}{2} = -\dfrac{1}{2}$.

   (e) $\lim_{x\to 1} \dfrac{1}{x-1} \ln\left(\dfrac{7x+1}{4x+4}\right) = \lim_{x\to 1} \dfrac{\ln(7x+1) - \ln(4x+4)}{x-1} = \text{“}0/0\text{”} = \lim_{x\to 1} \dfrac{\frac{7}{7x+1} - \frac{4}{4x+4}}{1} = \dfrac{3}{8}$.

   (f) $\lim_{x\to 1} \dfrac{x^x - x}{1 - x + \ln x} = \text{“}0/0\text{”} = \lim_{x\to 1} \dfrac{x^x(\ln x + 1) - 1}{-1 + 1/x} = \text{“}0/0\text{”} = \lim_{x\to 1} \dfrac{x^x(\ln x + 1)^2 + x^x(1/x)}{-1/x^2}$,
   which equals -2, where we have used Example 6.11.4 to differentiate x^x.

9. Here

$$L = \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{1/g(x)}{1/f(x)} = \text{“}0/0\text{”} = \lim_{x\to a} \frac{-1/(g(x))^2}{-1/(f(x))^2} \cdot \frac{g'(x)}{f'(x)}$$
$$= \lim_{x\to a} \frac{(f(x))^2}{(g(x))^2} \cdot \frac{g'(x)}{f'(x)} = L^2 \lim_{x\to a} \frac{g'(x)}{f'(x)} = L^2 \lim_{x\to a} \frac{1}{f'(x)/g'(x)}$$

   The conclusion follows. (This argument ignores problems with “division by 0”, when either $f'(x)$ or $g'(x)$ tends to 0 as x tends to a.)

Review exercises for Chapter 7

10. (a) We must have $\dfrac{1+x}{1-x} > 0$, so the domain of f is the interval $-1 < x < 1$. As $x \to 1^-$, one has $f(x) \to \infty$; as $x \to -1^-$, one has $f(x) \to -\infty$. Since $f'(x) = 1/(1-x^2) > 0$ when $-1 < x < 1$, f is strictly increasing and the range of f is $\mathbb{R}$.

    (b) From $y = \dfrac{1}{2} \ln \dfrac{1+x}{1-x}$ one has $\ln \dfrac{1+x}{1-x} = 2y$, so $\dfrac{1+x}{1-x} = e^{2y}$. Then solve for x.

12. (a) $f'(x) = 2/(2x+4) = (x+2)^{-1}$ and $f''(x) = -(x+2)^{-2}$. We get $f(0) = \ln 4$, $f'(0) = 1/2$, and $f''(0) = -1/4$, so $f(x) \approx f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = \ln 4 + x/2 - x^2/8$.

    (b) $g'(x) = -(1/2)(1+x)^{-3/2}$ and $g''(x) = (3/4)(1+x)^{-5/2}$. We get $g(0) = 1$, $g'(0) = -1/2$, and $g''(0) = 3/4$, so $g(x) \approx 1 - x/2 + 3x^2/8$.

    (c) $h'(x) = e^{2x} + 2xe^{2x}$ and $h''(x) = 4e^{2x} + 4xe^{2x}$. We get $h(0) = 0$, $h'(0) = 1$, and $h''(0) = 4$, so $h(x) \approx x + 2x^2$.

15. With $x = \frac{1}{2}$ and $n = 5$, formula (7.6.6) yields

$$e^{\frac{1}{2}} = 1 + \frac{\frac{1}{2}}{1!} + \frac{(\frac{1}{2})^2}{2!} + \frac{(\frac{1}{2})^3}{3!} + \frac{(\frac{1}{2})^4}{4!} + \frac{(\frac{1}{2})^5}{5!} + \frac{(\frac{1}{2})^6}{6!} e^c,$$

where c is some number between 0 and $\frac{1}{2}$. Now,

$$R_6(\frac{1}{2}) = \frac{(\frac{1}{2})^6}{6!}e^c < \frac{(\frac{1}{2})^6}{6!}2 = \frac{1}{23040} \approx 0.00004340,$$

where we used the fact that $c < \frac{1}{2}$ implies $e^c < e^{\frac{1}{2}} < 2$. Thus, it follows that

$$e^{\frac{1}{2}} \approx 1 + \frac{\frac{1}{2}}{1!} + \frac{(\frac{1}{2})^2}{2!} + \frac{(\frac{1}{2})^3}{3!} + \frac{(\frac{1}{2})^4}{4!} + \frac{(\frac{1}{2})^5}{5!} = 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \frac{1}{384} + \frac{1}{3840} \approx 1.6486979.$$

Because the error is less than 0.000043, the approximation $e^{\frac{1}{2}} \approx 1.649$ is correct to 3 decimal places.

23. (a) $\lim_{x\to 3^-}(x^2 - 3x + 2) = 9 - 9 + 2 = 2$.

(b) Tends to $+\infty$.

(c) $\dfrac{3-\sqrt{x+17}}{x+1}$ tends to $+\infty$ as $x \to -1^-$, but to $-\infty$ as $x \to -1^+$, so there is no limit as $x \to -1$.

(d) Here

$$\begin{aligned}\lim_{x\to 0}\frac{(2-x)e^x - x - 2}{x^3} &= \text{“}0/0\text{”} = \lim_{x\to 0}\frac{-e^x + (2-x)e^x - 1}{3x^2} = \text{“}0/0\text{”}\\ &= \lim_{x\to 0}\frac{-e^x - e^x + (2-x)e^x}{6x} = \lim_{x\to 0}\frac{-xe^x}{6x} = \lim_{x\to 0}\frac{-e^x}{6} = -\frac{1}{6}\end{aligned}$$

Note that by cancelling x at the penultimate step, we have avoided using l'Hôpital's rule a third time.

(e) Here

$$\begin{aligned}\lim_{x\to 3}\left(\frac{1}{x-3} - \frac{5}{x^2-x-6}\right) &= \lim_{x\to 3}\frac{x^2-6x+9}{x^3-4x^2-3x+18} = \text{“}0/0\text{”}\\ &= \lim_{x\to 3}\frac{2x-6}{3x^2-8x-3} = \text{“}0/0\text{”} = \lim_{x\to 3}\frac{2}{6x-8} = \frac{1}{5}\end{aligned}$$

(f) $\displaystyle\lim_{x\to 4}\frac{x-4}{2x^2-32} = \text{“}0/0\text{”} = \lim_{x\to 4}\frac{1}{4x} = \frac{1}{16}$. Alternatively, note that $2x^2 - 32 = 2(x^2-16) = 2(x-4)(x+4)$, and then cancel $x - 4$.

(g) If $x \neq 2$, then $\dfrac{x^2-3x+2}{x-2} = \dfrac{(x-2)(x-1)}{x-2} = x - 1$, which tends to 1 as $x \to 2$.

(h) If $x \neq -1$, then

$$\frac{4-\sqrt{x+17}}{2x+2} = \frac{(4-\sqrt{x+17})(4+\sqrt{x+17})}{(2x+2)(4+\sqrt{x+17})} = \frac{16-x-17}{(2x+2)(4+\sqrt{x+17})} = \frac{-1}{2(4+\sqrt{x+17})}$$

which tends to $-\frac{1}{16}$ as $x \to -1$.

(i) $\displaystyle\lim_{x\to\infty}\frac{(\ln x)^2}{3x^2} = \frac{1}{3}\lim_{x\to\infty}\left(\frac{\ln x}{x}\right)^2 = 0$, because of (7.12.3).

24. When $x \to 0$, the numerator tends to $\sqrt{b} - \sqrt{d}$ and the denominator to 0, so the limit does not exist when $d \neq b$. If $d = b$, however, then

$$\lim_{x\to 0}\frac{\sqrt{ax+b} - \sqrt{cx+b}}{x} = \text{“}0/0\text{”} = \lim_{x\to 0}\frac{\left[\frac{1}{2}a(ax+b)^{-1/2} - \frac{1}{2}c(cx+b)^{-1/2}\right]}{1} = \frac{a-c}{2\sqrt{b}}.$$

8 Single-variable Optimization

8.2 Simple tests for extreme points

2. Note that
$$h'(x) = \frac{8(3x^2+4)-(8x)(6x)}{(3x^2+4)^2} = \frac{8(2-\sqrt{3}x)(2+\sqrt{3}x)}{(3x^2+4)^2},$$
so h has critical points at $x_1 = -2\sqrt{3}/3$ and $x_2 = 2\sqrt{3}/3$. A sign diagram shows that $h'(x) < 0$ in $(-\infty, x_1)$ and in (x_2, ∞), whereas $h'(x) > 0$ in (x_1, x_2). Therefore h is strictly decreasing in $(-\infty, x_1]$, strictly increasing in $[x_1, x_2]$, and strictly decreasing again in $[x_2, \infty)$. Then, because $h(x) \to 0$ as $x \to \pm\infty$, it follows that the maximum of h occurs at $x_2 = 2\sqrt{3}/3$ and the minimum at $x_1 = -2\sqrt{3}/3$.

8. (a) $y' = e^x - 2e^{-2x}$ and $y'' = e^x + 4e^{-2x}$. Hence $y' = 0$ when $e^x = 2e^{-2x}$, or $e^{3x} = 2$, i.e. $x = \frac{1}{3}\ln 2$. Since $y'' > 0$ everywhere, the function is convex and this is a minimum point.
   (b) $y' = -2(x-a) - 4(x-b) = 0$ when $x = \frac{1}{3}(a+2b)$. This is a maximum point since $y'' = -6$.
   (c) $y' = 1/x - 5 = 0$ when $x = \frac{1}{5}$. This is a maximum point since $y'' = -1/x^2 < 0$ for all $x > 0$.

10. (a) $f'(x) = k - A\alpha e^{-\alpha x} = 0$ when $x = x_0 = (1/\alpha)\ln(A\alpha/k)$. Note that $x_0 > 0$ if and only if $A\alpha > k$. Moreover $f''(x) = A\alpha^2 e^{-\alpha x} > 0$ for all $x \geq 0$, so x_0 solves the minimization problem.
    (b) Substituting for A in the answer to (a) gives the expression for the optimal height x_0. Its value increases as p_0 (probability of flooding) or V (cost of flooding) increases, but decreases as δ (interest rate) or k (marginal construction cost) increases. The signs of these responses are obviously what an economist would expect.

8.3 Economic examples

2. (a) $\pi(Q) = Q(a-Q) - kQ = (a-k)Q - Q^2$ so $\pi'(Q) = (a-k) - 2Q = 0$ for $Q = Q^* = \frac{1}{2}(a-k)$. This maximizes π because $\pi''(Q) < 0$. The monopoly profit is
$$\pi(Q^*) = -[\frac{1}{2}(a-k)]^2 + (a-k)\frac{1}{2}(a-k) = \frac{1}{4}(a-k)^2.$$
   (b) $\mathrm{d}\pi(Q^*)/\mathrm{d}k = -\frac{1}{2}(a-k) = -Q^*$, as in Example 8.3.3.
   (c) The new profit function is $\hat{\pi}(Q) = \pi(Q) + sQ = (a-k)Q - Q^2 + sQ$. Then $\hat{\pi}'(Q) = a - k - 2Q + s = 0$ when $\hat{Q} = \frac{1}{2}(a-k+s)$. Evidently $\hat{Q} = \frac{1}{2}(a-k+s) = a-k$ provided $s = a-k$, which is the subsidy required to induce the monopolist to produce $a-k$ units.

8.4 The extreme value theorem

2. In all cases the maximum and minimum exist by the extreme value theorem. Follow the recipe described in Example 8.4.1:

(a) $f'(x) = -2$ for all x in $[0, 3]$, so the recipe tells us that both the maximum and minimum points are at the ends of the interval $[0, 3]$. Since $f(0) = -1$ and $f(3) = -7$, the maximum is at $x = 0$, the minimum at $x = 3$.[13]

(b) $f(-1) = f(2) = 10$ and $f'(x) = 3x^2 - 3 = 0$ at $x = \pm 1$. The only critical point in the interval $[-1, 2]$ is $x = 1$, where $f(1) = 6$. There are two maxima at the endpoints, and a minimum at $x = 1$.

(c) $f(x) = x + 1/x$, so $f(1/2) = f(2) = 5/2$ at the endpoints. Also, $f'(x) = 1 - 1/x^2 = 0$ at $x = \pm 1$. The only critical point in the interval $[\frac{1}{2}, 2]$ is $x = 1$, where $f(1) = 2$. There are two maxima at the endpoints, and a minimum at $x = 1$.

(d) At the endpoints one has $f(-1) = 4$ and $f(\sqrt{5}) = 0$. Because $f'(x) = 5x^2(x^2 - 3)$, there are two critical points in the interval $[-1, \sqrt{5}]$ at $x = 0$ and $x = \sqrt{3}$. The values at these critical points are $f(0) = 0$ and $f(\sqrt{3}) = -6\sqrt{3}$. The maximum is at $x = -1$ and the minimum is at $x = \sqrt{3}$.

(e) $f'(x) = 3x^2 - 9000x + 6 \cdot 10^6 = 3(x - 1000)(x - 2000) = 0$ when $x = 1000$ and $x = 2000$. At these critical points $f(1000) = 2.5 \cdot 10^9$ and $f(2000) = 2 \cdot 10^9$. There is a minimum at the endpoint $x = 0$ and a maximum at $x = 3000$.

6. (a) $(f(2) - f(1))/(2 - 1) = (4 - 1)/1 = 3$ and $f'(x) = 2x$, so $f'(x) = 3$ when $x = x^* = 3/2$.

(b) $(f(1) - f(0))/1 = (0 - 1)/1 = -1$ and $f'(x) = -x/\sqrt{1 - x^2} = -1$ when $x = \sqrt{1 - x^2}$. Squaring each side of the last equation gives $x^2 = 1 - x^2$ and so $x^2 = \frac{1}{2}$. This has two solutions $x = \pm\frac{1}{2}\sqrt{2}$, of which only the positive solution satisfies $x = \sqrt{1 - x^2}$. So we require $x = x^* = \frac{1}{2}\sqrt{2}$.

(c) $(f(6) - f(2))/4 = -1/6$ and $f'(x) = -2/x^2 = -1/6$ when $-12/x^2 = -1$ or $x^2 = 12$, and so $x = \pm\sqrt{12}$. The required solution in $[2, 6]$ is $x = x^* = \sqrt{12} = 2\sqrt{3}$.

(d) $(f(4) - f(0))/4 = 1/4 = (\sqrt{25} - \sqrt{9})/4 = (5 - 3)/4 = 1/2$ and $f'(x) = \frac{1}{2}2x/\sqrt{9 + x^2} = x/\sqrt{9 + x^2} = 1/2$ when $2x = \sqrt{9 + x^2}$. Squaring each side of the last equation gives $4x^2 = 9 + x^2$ and so $3x^2 = 9$. This has two solutions $x = \pm\sqrt{3}$, of which only the positive solution satisfies $x/\sqrt{9 + x^2} = 1/2$. So we require $x = x^* = \sqrt{3}$.

8.5 Further economic examples

4. (a) $\pi(Q) = 1840Q - (2Q^2 + 40Q + 5000) = 1800Q - 2Q^2 - 5000$. Since $\pi'(Q) = 1800 - 4Q = 0$ for $Q = 450$, and $\pi''(Q) = -4 < 0$, it follows that $Q = 450$ maximizes profits.

(b) $\pi(Q) = 2200Q - 2Q^2 - 5000$. Since $\pi'(Q) = 2200 - 4Q = 0$ for $Q = 550$, and $\pi''(Q) = -4 < 0$, it follows that $Q = 550$ maximizes profits.

(c) $\pi(Q) = -2Q^2 - 100Q - 5000$. Here $\pi'(Q) = -4Q - 100 < 0$ for all $Q \geq 0$, so the endpoint $Q = 0$ maximizes profits.

8.6 Local extreme points

2. (a) This function is strictly decreasing, so it has no extreme points.

[13] Actually the sign of $f'(x)$ alone implies that the maximum is at the lower end of the interval, and the minimum at the upper end.

(b) $f'(x) = 3x^2 - 3 = 0$ for $x = \pm 1$. Because $f''(x) = 6x$, we have $f''(-1) = -6$ and $f''(1) = 6$, so $x = -1$ is a local maximum point, and $x = 1$ is a local minimum point.

(c) $f'(x) = 1 - 1/x^2 = 0$ for $x = \pm 1$. With $f''(x) = 2/x^3$, we have $f''(-1) = -2$ and $f''(1) = 2$, so $x = -1$ is a local maximum point, and $x = 1$ is a local minimum point.

(d) $f'(x) = 5x^4 - 15x^2 = 5x^2(x^2 - 3)$ and $f''(x) = 20x^3 - 30x$. There are three critical points at $x = 0$ and $x = \pm\sqrt{3}$. Because $f''(0) = 0$, whereas $f''(-\sqrt{3}) = -20 \cdot 3\sqrt{3} + 30\sqrt{3} = -30\sqrt{3} < 0$ and $f''(\sqrt{3}) = 30\sqrt{3} > 0$, there is a local maximum at $x = -\sqrt{3}$ and a local minimum at $x = \sqrt{3}$.

(e) This parabola has a local (and global) minimum at $x = 3$.

(f) $f'(x) = 3x^2 + 6x = 3x(x + 2)$ and $f''(x) = 6x + 6$. There are two critical points at $x = 0$ and $x = -2$. Because $f''(0) = 6$ and $f''(-2) = -6$, there is a local maximum at $x = -2$ and a local minimum at $x = 0$.

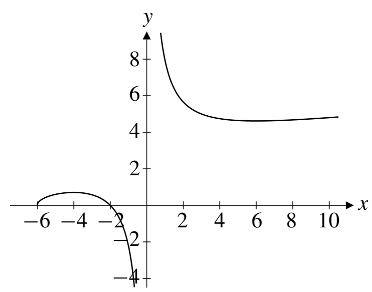

Figure SM8.6.3

3. See the graph in Fig. SM8.6.3.

(a) The function $f(x)$ is defined if and only if $x \neq 0$ and $x \geq -6$. We have $f(x) = 0$ at $x = -6$ and at $x = -2$. At any other point x in the domain, $f(x)$ has the same sign as $(x + 2)/x$, so $f(x) > 0$ if $x \in (-6, -2)$ or $x \in (0, \infty)$.

(b) We first find the derivative of f:

$$f'(x) = -\frac{2}{x^2}\sqrt{x+6} + \frac{x+2}{x}\frac{1}{2\sqrt{x+6}} = \frac{-4x - 24 + x^2 + 2x}{2x^2\sqrt{x+6}} = \frac{(x+4)(x-6)}{2x^2\sqrt{x+6}}$$

By means of a sign diagram we see that $f'(x) > 0$ if $-6 < x < -4$, $f'(x) < 0$ if $-4 < x < 0$, $f'(x) < 0$ if $0 < x < 6$, $f'(x) > 0$ if $6 < x$. Hence, f is strictly decreasing in $[-4, 0)$ and in $(0, 6]$, strictly increasing in $[-6, -4]$ and in $[6, \infty)$. It follows from the first-derivative test (Thm. 8.6.1) that the two points $x = -4$ and $x = 6$ are respectively a local maximum and a local minimum, with $f(-4) = \frac{1}{2}\sqrt{2}$ and $f(6) = \frac{4}{3}\sqrt{8} = 8\sqrt{2}/3$. Also, according to the definition (8.6.1), the point $x = -6$ is another local minimum point.

(c) Since $\lim_{x\to 0}\sqrt{x+6} = 6 > 0$, while $\lim_{x\to 0^-}(1 + 2/x) = -\infty$ and $\lim_{x\to 0^+}(1 + 2/x) = \infty$, we see that $\lim_{x\to 0^-} f(x) = -\infty$ and $\lim_{x\to 0^+} f(x) = \infty$. Furthermore, as $x \to \infty$,

$$f'(x) = \left(\frac{x^2 - 2x - 24}{2x^2} \cdot \frac{1}{\sqrt{x+6}}\right) = \left[\left(\frac{1}{2} - \frac{1}{x} - \frac{12}{x^2}\right) \cdot \frac{1}{\sqrt{x+6}}\right] \to \frac{1}{2} \cdot 0 = 0.$$

7. $f(x) = x^3 + ax + b \to \infty$ as $x \to \infty$, and $f(x) \to -\infty$ as $x \to -\infty$. By the intermediate value theorem, the continuous function f has at least one real root. We have $f'(x) = 3x^2 + a$. We consider two cases.

   First, in case $a \geq 0$, one has $f'(x) > 0$ for all $x \neq 0$, so f is strictly increasing, and there is only one real root. Note that $4a^3 + 27b^2 \geq 0$ in this case.

   Second, in case $a < 0$, one has $f'(x) = 0$ for $x = \pm\sqrt{-a/3} = \pm\sqrt{p}$, where $p = -a/3$. Because $f''(x) = 3x$, the function f has a local maximum at $x = -\sqrt{p}$, where $y = b + 2p\sqrt{p}$, and a local minimum at $x = \sqrt{p}$, where $y = b - 2p\sqrt{p}$. If either of these local extreme values is 0, the equation has a double root, which is the case if and only if $4p^3 = b^2$, that is, if and only if $4a^3 + 27b^2 = 0$. Otherwise, the equation has three real roots if and only if the local maximum value is positive and the local minimum value is negative. This occurs if and only if both $b > -2p\sqrt{p}$ and $b < 2p\sqrt{p}$. Equivalent conditions are $|b| < 2p\sqrt{p} \iff b^2 < 4p^3 \iff 4a^3 + 27b^2 < 0$.

8.7 Inflection points

3. The answers given in the text can be found in a straightforward way by considering the signs of the following derivatives:

   (a) $y' = -e^{-x}(1 + x)$, $y'' = xe^{-x}$.

   (b) $y' = (x - 1)/x^2$, $y'' = (2 - x)/x^3$.

   (c) $y' = x^2e^{-x}(3 - x)$, $y'' = xe^{-x}(x^2 - 6x + 6)$.

   (d) $y' = \frac{1-2\ln x}{x^3}$, $y'' = \frac{6\ln x - 5}{x^4}$.

   (e) $y' = 2e^x(e^x - 1)$, $y'' = 2e^x(2e^x - 1)$.

   (f) $y' = e^{-x}(2 - x^2)$, $y'' = e^{-x}(x^2 - 2x - 2)$.

Review exercises for Chapter 8

8. (a) The answer given in the text is easily found from the derivative $h'(x) = \dfrac{e^x(2 - e^{2x})}{(2 + e^{2x})^2}$.

   (b) The function h is strictly increasing in $(-\infty, \frac{1}{2}\ln 2]$, which includes $(-\infty, 0]$. Also $h(x) \to 0$ as $x \to -\infty$, and $h(0) = 1/3$. Thus, h defined on $(-\infty, 0]$ has an inverse defined on $(0, 1/3]$ with values in $(-\infty, 0]$. To find the inverse, note that $\dfrac{e^x}{2 + e^{2x}} = y$ if and only if $y(e^x)^2 - e^x + 2y = 0$. This quadratic equation in e^x has the roots $e^x = (1 \pm \sqrt{1 - 8y^2}])/2y$. We require the solution to satisfy $x \leq 0$ and so $e^x \leq 1$ when $0 < y < 1/3$. Now, taking the positive square root would give $e^x > 1/2y > 6$ when $0 < y < 1/3$. So we must have $e^x = (1 - \sqrt{1 - 8y^2})/2y$, implying that $x = \ln(1 - \sqrt{1 - 8y^2}) - \ln(2y)$. Using x as the free variable gives the inverse function $h^{-1}(x) = \ln(1 - \sqrt{1 - 8x^2}) - \ln(2x)$. The function and its inverse are graphed in Fig. SM8.R.8.

10. (a) Because $\sqrt[3]{u}$ is defined for all real u, the only points x not in the domain are the ones for which $x^2 - a = 0$, or $x = \pm\sqrt{a}$. So the domain of f consists of all $x \neq \pm\sqrt{a}$. Since $\sqrt[3]{u} > 0$ if and only if $u > 0$, the denominator in the expression for $f(x)$, $\sqrt[3]{x^2 - a}$, is

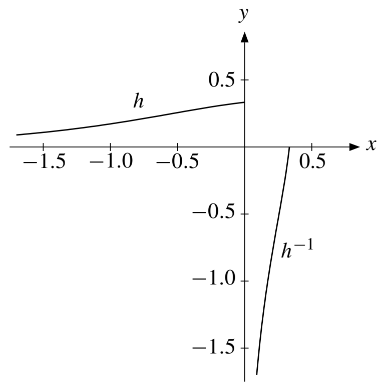

Figure SM8.R.8

positive if and only if $x^2 > a$, i.e. if and only if $x < -\sqrt{a}$ or $x > \sqrt{a}$. The numerator in the expression for $f(x)$ is x, and a sign diagram then reveals that $f(x)$ is positive in $(-\sqrt{a}, 0)$ and in $(\sqrt{a}, \infty)$. he The graph of f is symmetric about the origin because

$$f(-x) = \frac{-x}{\sqrt[3]{(-x)^2 - a}} = -\frac{x}{\sqrt[3]{x^2 - a}} = -f(x),$$

(b) Writing $\sqrt[3]{x^2 - a}$ as $(x^2 - a)^{1/3}$ and then differentiating yields

$$f'(x) = \frac{1 \cdot (x^2 - a)^{1/3} - x \cdot \frac{1}{3}(x^2 - a)^{-2/3} \cdot 2x}{(x^2 - a)^{2/3}} = \frac{x^2 - a - x \cdot \frac{1}{3} 2x}{(x^2 - a)^{4/3}} = \frac{\frac{1}{3}(x^2 - 3a)}{(x^2 - a)^{4/3}},$$

Here the second equality was obtained by multiplying both denominator and numerator by $(x^2 - a)^{2/3}$. Of course, $f'(x)$ is not defined at $\pm\sqrt{a}$. Except at these points, the denominator is always positive (since $(x^2 - a)^{4/3} = ((x^2 - a)^{1/3})^4$). The numerator, $\frac{1}{3}(x^2 - 3a) = \frac{1}{3}(x + \sqrt{3a})(x - \sqrt{3a})$, is 0 at $x = \pm\sqrt{3a}$, nonnegative in $(-\infty, -\sqrt{3a}]$ and in $[\sqrt{3a}, \infty)$. Since f and f' are not defined at $\pm\sqrt{a}$, we find that $f(x)$ is increasing in $(-\infty, -\sqrt{3a}]$ and in $[\sqrt{3a}, \infty)$, decreasing in $[-\sqrt{3a}, -\sqrt{a})$, in $(-\sqrt{a}, \sqrt{a})$, and in $(\sqrt{a}, \sqrt{3a}]$. It follows that $x = -\sqrt{3a}$ is a local maximum point and $x = \sqrt{3a}$ is a local minimum point.

(c) Differentiating once more, we find that

$$f''(x) = \frac{\frac{2}{3}x(x^2 - a)^{4/3} - \frac{1}{3}(x^2 - 3a) \cdot \frac{4}{3}(x^2 - a)^{1/3} \cdot 2x}{(x^2 - a)^{8/3}} = \frac{\frac{2}{9}x(9a - x^2)}{(x^2 - a)^{7/3}}.$$

Here the second equality was obtained by dividing each term in the denominator and the numerator by $(x^2 - a)^{1/3}$, then simplifying the numerator. The resulting expression for $f''(x)$ shows that there are inflection points where x equals $-3\sqrt{a}$, 0, and $3\sqrt{a}$.[14]

11. Note first that $f(x) \to 0$ as $x \to \pm\infty$, as can be shown by dividing both numerator and denominator by x^3, then taking limits. Next, differentiation yields

$$f'(x) = \frac{18x^2(x^4 + x^2 + 2) - 6x^3(4x^3 + 2x)}{(x^4 + x^2 + 2)^2} = \frac{-6x^2(x^4 - x^2 - 6)}{(x^4 + x^2 + 2)^2} = \frac{-6x^2(x^2 - 3)(x^2 + 2)}{(x^4 + x^2 + 2)^2},$$

[14] $f''(x)$ is 0 at these points, and changes sign around each of them.

so f has critical points at $x = 0$ and at $x = \pm\sqrt{3}$. Moreover, f' changes sign from negative to positive as x increases through $-\sqrt{3}$, then it switches back to negative as x increases through $\sqrt{3}$. It follows that $x = \sqrt{3}$ is a local (and global) maximum point, that $x = -\sqrt{3}$ is a local (and global) minimum point, and $x = 0$ is neither. (It is an inflection point.) Note moreover that $f(-x) = -f(x)$ for all x, so the graph is symmetric about the origin. The graph of f is shown in Fig. A8.R.11 in the book.

9 Integration

9.1 Indefinite integrals

3. (a) $\int (t^3 + 2t - 3)\,\mathrm{d}t = \int t^3\,\mathrm{d}t + \int 2t\,\mathrm{d}t - \int 3\,\mathrm{d}t = \frac{1}{4}t^4 + t^2 - 3t + C$.

   (b) $\int (x-1)^2\,\mathrm{d}x = \int (x^2 - 2x + 1)\,\mathrm{d}x = \frac{1}{3}x^3 - x^2 + x + C$. Alternatively, since $\frac{\mathrm{d}}{\mathrm{d}x}(x-1)^3 = 3(x-1)^2$, we have $\int (x-1)^2\,\mathrm{d}x = \frac{1}{3}(x-1)^3 + C_1$. This agrees with the first answer, with $C_1 = C + 1/3$.

   (c) $\int (x-1)(x+2)\,\mathrm{d}x = \int (x^2 + x - 2)\,\mathrm{d}x = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + C$.

   (d) First evaluate $(x+2)^3 = x^3 + 6x^2 + 12x + 8$, then integrate term by term to get $\int (x+2)^3\,\mathrm{d}x = \frac{1}{4}x^4 + 2x^3 + 6x^2 + 8x + C$. Or more directly, check that $\int (x+2)^3 = \frac{1}{4}(x+2)^4 + C_1$.

   (e) $\int (e^{3x} - e^{2x} + e^x)\,\mathrm{d}x = \frac{1}{3}e^{3x} - \frac{1}{2}e^{2x} + e^x + C$.

   (f) $\displaystyle\int \frac{x^3 - 3x + 4}{x}\,\mathrm{d}x = \int \left(x^2 - 3 + \frac{4}{x}\right)\mathrm{d}x = \frac{1}{3}x^3 - 3x + 4\ln|x| + C$

4. (a) First simplify the integrand to obtain $\dfrac{(y-2)^2}{\sqrt{y}} = \dfrac{y^2 - 4y + 4}{\sqrt{y}} = y^{3/2} - 4y^{1/2} + 4y^{-1/2}$. Next, integrate term by term to get

$$\int \frac{(y-2)^2}{\sqrt{y}}\mathrm{d}y = \int (y^{3/2} - 4y^{1/2} + 4y^{-1/2})\,\mathrm{d}y = \frac{2}{5}y^{5/2} - \frac{8}{3}y^{3/2} + 8y^{1/2} + C.$$

   (b) By polynomial division, the integrand is $\dfrac{x^3}{x+1} = x^2 - x + 1 - \dfrac{1}{x+1}$. It follows that

$$\int \frac{x^3}{x+1}\,\mathrm{d}x = \frac{x^3}{3} - \frac{x^2}{2} + x - \ln|x+1| + C.$$

   (c) $\dfrac{\mathrm{d}}{\mathrm{d}x}(1+x^2)^{16} = 16(1+x^2)^{15} \cdot 2x = 32x(1+x^2)^{15}$, so $\int x(1+x^2)^{15}\,\mathrm{d}x = \frac{1}{32}(1+x^2)^{16} + C$.

11. $f'(x) = \int (x^{-2} + x^3 + 2)\,\mathrm{d}x = -x^{-1} + \frac{1}{4}x^4 + 2x + C$. With $f'(1) = \frac{1}{4}$ we have $\frac{1}{4} = -1 + \frac{1}{4} + 2 + C$, so $C = -1$. Now integration yields

$$f(x) = \int \left(-x^{-1} + \frac{x^4}{4} + 2x - 1\right)\mathrm{d}x = -\ln x + \frac{x^5}{20} + x^2 - x + D.$$

With $f(1) = 0$ we have $0 = -\ln 1 + \frac{1}{20} + 1 - 1 + D$, so $D = -\frac{1}{20}$.

9.2 Area and definite integrals

5. We consider only parts (c) and (f).

(c) $\displaystyle\int_{-2}^{3}\left(\frac{1}{2}x^2-\frac{1}{3}x^3\right)\mathrm{d}x = \Big|_{-2}^{3}\big(\frac{1}{6}x^3-\frac{1}{12}x^4\big) = \Big|_{-2}^{3}\frac{1}{12}x^3(2-x) = -\frac{27}{12}+\frac{32}{12}=\frac{5}{12}.$

(f) $\displaystyle\int_{2}^{3}\left(\frac{1}{t-1}+t\right)\mathrm{d}t = \Big|_{2}^{3}\left[\ln(t-1)+\frac{1}{2}t^2\right] = \ln 2+\frac{9}{2}-\frac{4}{2}=\ln 2+\frac{5}{2}.$

6. (a) $f(x)=x^3-3x^2+2x$ and so $f'(x)=3x^2-6x+2=0$ for $x_0=1-\sqrt{3}/3$ and $x_1=1+\sqrt{3}/3$. We see that $f'(x)>0 \iff x<x_0$ or $x>x_1$. Also, $f'(x)<0 \iff x_0<x<x_1$. So f is (strictly) increasing in $(-\infty,x_0]$ and in $[x_1,\infty)$, but (strictly) decreasing in $[x_0,x_1]$.

(b) See the graph in the text. The integral is

$$\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 (x^3-3x^2+2x)\,\mathrm{d}x = \Big|_0^1\big(\tfrac{1}{4}x^4-x^3+x^2\big) = \tfrac{1}{4}-0=\tfrac{1}{4}$$

9.3 Properties of indefinite integrals

4. Directly, one has $\displaystyle\int_0^1 (x^{p+q}+x^{p+r})\,\mathrm{d}x = \Big|_0^1 \frac{x^{p+q+1}}{p+q+1}+\frac{x^{p+r+1}}{p+r+1} = \frac{1}{p+q+1}+\frac{1}{p+r+1}$

5. Equality (a) implies $a+b=6$. Also, $f''(x)=2ax+b$, so equality (b) implies $2a+b=18$. It follows that $a=12$ and $b=6$, so $f'(x)=12x^2-6x$. But then $f(x)=\int(12x^2-6x)\,\mathrm{d}x = 4x^3-3x^2+C$, and since we want $\int_0^2(4x^3-3x^2+C)=18$, we must have $16-8+2C=18$, hence $C=5$.

6. (a) See the answer given in the book.

(b) $\displaystyle\int_0^1 (x^2+2)^2\,\mathrm{d}x = \int_0^1 (x^4+4x^2+4)\,\mathrm{d}x = \Big|_0^1\big(\frac{1}{5}x^5+\frac{4}{3}x^3+4x\big) = \frac{83}{15}.$

(c) Note that $\displaystyle\frac{x^2+x+\sqrt{x+1}}{x+1} = \frac{x(x+1)+(x+1)^{1/2}}{x+1} = x+(x+1)^{-1/2}$, and so

$$\int_0^1 \frac{x^2+x+\sqrt{x+1}}{x+1}\,\mathrm{d}x = \Big|_0^1\big(\tfrac{1}{2}x^2+2(x+1)^{1/2}\big) = \tfrac{1}{2}+2\sqrt{2}-2 = 2\sqrt{2}-\tfrac{3}{2}.$$

(d) Clearing fractions, $\displaystyle A\frac{x+b}{x+c}+\frac{d}{x} = A\frac{x+c+b-c}{x+c}+\frac{d}{x} = A+\frac{A(b-c)}{x+c}+\frac{d}{x}$. Now integrate.

11. $W(T)=K(1-e^{-\varrho T})/\varrho T$. Here $W(T)\to 0$ as $T\to\infty$, and by l'Hôpital's rule, $W(T)\to K$ as $T\to 0^+$. For $T>0$, we find $W'(T)=Ke^{-\varrho T}(1+\varrho T-e^{\varrho T})/\varrho T^2<0$ because $e^{\varrho T}>1+\varrho T$ (see Problem 6.11.11). We conclude that $W(T)$ is strictly decreasing and that $W(T)\in(0,K)$.

12. (a) $\displaystyle f'(x) = \frac{2}{\sqrt{x+4}\,(\sqrt{x+4}-2)} > 0$ for all $x>0$. Also, $f(x)\to-\infty$ as $x\to 0$, whereas $f(x)\to\infty$ as $x\to\infty$. It follows that f is strictly increasing on $(0,\infty)$, with range equal to $\mathbb{R}$. Hence f has an inverse defined on $\mathbb{R}$. To find the inverse, note that $y = 4\ln(\sqrt{x+4}-2) \iff \ln(\sqrt{x+4}-2)=y/4 \iff \sqrt{x+4}=e^{y/4}+2 \iff x+4=(e^{y/4}+2)^2 \iff x=e^{y/2}+4e^{y/4}$. It follows that the inverse is $g(x)=e^{x/2}+4e^{x/4}$.

(b) See Fig. A9.3.12.

(c) In Fig. A9.3.12 the graphs of f and g are symmetric about the line $y = x$, so the areas of A and B are equal. But the area of B is the difference between: (i) the area of a rectangle with base a and height 10; (ii) the area below the graph of g over the interval $[0, a]$. Therefore,

$$A = B = 10a - \int_0^a (e^{x/2} + 4e^{x/4})\,\mathrm{d}x = 10a - 2e^{a/2} - 16e^{a/4} + 2 + 16$$

Because $a = f(10) = 4\ln(\sqrt{14}-2)$, we have $e^{a/2} = (\sqrt{14}-2)^2 = 14-4\sqrt{14}+4 = 18-4\sqrt{14}$ and also $e^{a/4} = \sqrt{14} - 2$. Hence,

$$A = B = 10a - 2(18 - 4\sqrt{14}) - 16(\sqrt{14} - 2) + 18 = 40\ln(\sqrt{14} - 2) + 14 - 8\sqrt{14} \approx 6.26$$

9.4 Economic applications

2. (a) Let n be the total number of individuals. The number of individuals with income in the interval $[b, 2b]$ is then $N = n\int_b^{2b} Br^{-2}\,\mathrm{d}r = n\Big|_b^{2b} -Br^{-1} = \frac{nB}{2b}$.
Their total income is $M = n\int_b^{2b} Br^{-2}r\,\mathrm{d}r = n\int_b^{2b} Br^{-1}\,\mathrm{d}r = n\Big|_b^{2b} B\ln r = nB\ln 2$.
Hence the mean income is $m = M/N = 2b\ln 2$.

(b) Total demand is $x(p) = \int_b^{2b} nD(p,r)f(r)\,\mathrm{d}r = \int_b^{2b} nAp^\gamma r^\delta Br^{-2}\,\mathrm{d}r = nABp^\gamma\int_b^{2b} r^{\delta-2}\,\mathrm{d}r$
Evaluating the integral gives

$$x(p) = nABp^\gamma \int_b^{2b} r^{\delta-2}\,\mathrm{d}r = nABp^\gamma \Big|_b^{2b} \frac{r^{\delta-1}}{\delta-1} = nABp^\gamma b^{\delta-1}\frac{2^{\delta-1}-1}{\delta-1}$$

9.5 Integration by parts

1. (a) See the answer given in the book.

(b) $\int 3xe^{4x}\,\mathrm{d}x = 3x\cdot\frac{1}{4}e^{4x} - \int 3\cdot\frac{1}{4}e^{4x}\,\mathrm{d}x = \frac{3}{4}xe^{4x} - \frac{3}{16}e^{4x} + C$.

(c) $\int(1+x^2)e^{-x}\,\mathrm{d}x = (1+x^2)(-e^{-x}) - \int 2x(-e^{-x})\,\mathrm{d}x = -(1+x^2)e^{-x} + 2\int xe^{-x}\,\mathrm{d}x$. Using the answer to (a) to evaluate the last integral, we get

$$\int(1+x^2)e^{-x}\mathrm{d}x = -(1+x^2)e^{-x} - 2xe^{-x} - 2e^{-x} + C = -(x^2+2x+3)e^{-x} + C$$

(d) $\int x\ln x\,\mathrm{d}x = \frac{1}{2}x^2\ln x - \int\frac{1}{2}x^2\frac{1}{x}\,\mathrm{d}x = \frac{1}{2}x^2\ln x - \int\frac{1}{2}x\,\mathrm{d}x = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$.

2. (a) See the answer given in the book.

(b) Recall that $\dfrac{\mathrm{d}}{\mathrm{d}x}2^x = 2^x\ln 2$, so $2^x/\ln 2$ is the indefinite integral of 2^x. It follows that

$$\int_0^2 x2^x\,\mathrm{d}x = \Big|_0^2 x\frac{2^x}{\ln 2} - \int_0^2\frac{2^x}{\ln 2}\,\mathrm{d}x = \frac{8}{\ln 2} - \Big|_0^2\frac{2^x}{(\ln 2)^2} = \frac{8}{\ln 2} - \left(\frac{4}{(\ln 2)^2} - \frac{1}{(\ln 2)^2}\right) = \frac{8}{\ln 2} - \frac{3}{(\ln 2)^2}.$$

(c) First use integration by parts on the indefinite integral. By Eq. (9.5.1) with $f(x) = x^2$ and $g(x) = e^x$,

$$\int x^2e^x\,\mathrm{d}x = x^2e^x - \int 2xe^x\,\mathrm{d}x. \qquad (*)$$

To evaluate the last integral we must use integration by parts once more. With $f(x) = 2x$ and $g(x) = e^x$, we get $\int 2xe^x\,dx = 2xe^x - \int 2e^x\,dx = 2xe^x - (2e^x + C)$. Inserted into $(*)$, this gives $\int x^2e^x\,dx = x^2e^x - 2xe^x + 2e^x + C$. So

$$\int_0^1 x^2e^x\,dx = \Big|_0^1 (x^2e^x - 2xe^x + 2e^x) = (e - 2e + 2e) - (0 - 0 + 2) = e - 2$$

Alternatively and more directly, use formula (9.5.2) to obtain

$$\int_0^1 x^2e^x\,dx = \Big|_0^1 x^2e^x - 2\int_0^1 xe^x\,dx = e - 2\left(\Big|_0^1 xe^x - \int_0^1 e^x\,dx\right) = e - 2\left(e - \Big|_0^1 e^x\right) = e - 2.$$

(d) We must write the integrand in the form $f(x)g'(x)$. If we let $f(x) = x$ and $g'(x) = \sqrt{1+x} = (1+x)^{1/2}$, then what is g? A certain amount of reflection should suggest that it is worth trying $g(x) = \frac{2}{3}(1+x)^{3/2}$. Using (9.5.2) then gives

$$\int_0^3 x\sqrt{1+x}\,dx = \Big|_0^3 x\cdot\frac{2}{3}(1+x)^{3/2} - \int_0^3 1\cdot\frac{2}{3}(1+x)^{3/2}dx \;= 3\cdot\frac{2}{3}\cdot 4^{3/2} - \frac{2}{3}\Big|_0^3 \frac{2}{5}(1+x)^{5/2}$$

It follows that $\int_0^3 x\sqrt{1+x}\,dx = 16 - \frac{4}{15}(4^{5/2} - 1) = 16 - \frac{4}{15}\cdot 31 = 7\frac{11}{15}$. Alternatively, we could have found the indefinite integral of $x\sqrt{1+x}$ first, and then evaluated the definite integral by using definition (9.2.3) of the definite integral.

Figure SM9.5.2(d) shows the area under the graph of $y = x\sqrt{1+x}$ over the interval $[0, 3]$. You should ask yourself whether $7\frac{11}{15}$ is a reasonable estimate of the area of A.

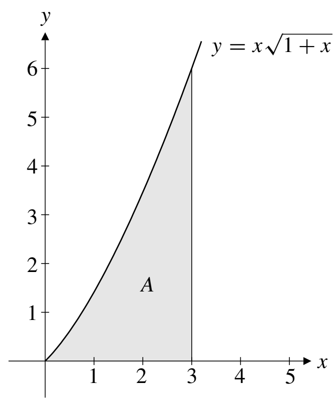

Figure SM9.5.2(d)

6. (a) By formula (9.5.2), $\displaystyle\int_0^T te^{-rt}\,dt = \Big|_0^T t\frac{-1}{r}e^{-rt} - \int_0^T \frac{-1}{r}e^{-rt}\,dt = \frac{-T}{r}e^{-rT} + \frac{1}{r}\int_0^T e^{-rt}\,dt$ $\displaystyle = \frac{-T}{r}e^{-rT} + \frac{1}{r}\Big|_0^T \frac{-1}{r}e^{-rt} = \frac{1}{r^2}(1 - (1+rT)e^{-rT})$. Multiply this expression by b.

(b) $\int_0^T (a + bt)e^{-rt}\,dt = a\int_0^T e^{-rt}\,dt + b\int_0^T te^{-rt}\,dt$ and so on using the result of part (a).

(c) $\int_0^T (a - bt + ct^2)e^{-rt}\,dt = a\int_0^T e^{-rt}\,dt - b\int_0^T te^{-rt}\,dt + c\int_0^T t^2e^{-rt}\,dt$. Now use the results of parts (a) and (b) together with

$$\int_0^T t^2e^{-rt}\,dt = \Big|_0^T t^2\left(-\frac{1}{r}\right)e^{-rt} - \int_0^T 2t\left(-\frac{1}{r}\right)e^{-rt}\,dt = -\frac{1}{r}T^2e^{-rT} + \frac{2}{r}\int_0^T te^{-rt}\,dt.$$

9.6 Integration by substitution

2. (a) See the answer given in the book.

   (b) With $u = x^3+2$ we get $\mathrm{d}u = 3x^2\,\mathrm{d}x$ and $\int x^2 e^{x^3+2}\,\mathrm{d}x = \int \frac{1}{3}e^u\,\mathrm{d}u = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3+2} + C$.

   (c) A first try might be $u = x+2$, which gives $\mathrm{d}u = \mathrm{d}x$ and

$$\int \frac{\ln(x+2)}{2x+4}\,\mathrm{d}x = \int \frac{\ln u}{2u}\,\mathrm{d}u.$$

   This, however, looks no simpler than the original integral.

   A better idea is to substitute $u = \ln(x+2)$. Then $\mathrm{d}u = \dfrac{1}{x+2}\mathrm{d}x$ and

$$\int \frac{\ln(x+2)}{2x+4}\,\mathrm{d}x = \int \frac{1}{2}u\,\mathrm{d}u = \frac{1}{4}u^2 + C = \frac{1}{4}\big[\ln(x+2)\big]^2 + C.$$

   (d) A first possible substitution is $u = 1+x$, which gives $\mathrm{d}u = \mathrm{d}x$ and then

$$\int x\sqrt{1+x}\,\mathrm{d}x = \int (u-1)\sqrt{u}\,\mathrm{d}u = \int (u^{3/2} - u^{1/2})\,\mathrm{d}u$$

   This can be integrated to give

$$\int x\sqrt{1+x}\,\mathrm{d}x = \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C = \frac{2}{5}(1+x)^{5/2} - \frac{2}{3}(1+x)^{3/2} + C$$

   A second possible substitution is $u = \sqrt{1+x}$. Now $u^2 = 1+x$ and $2u\,\mathrm{d}u = \mathrm{d}x$. The integral becomes

$$\int x\sqrt{1+x}\,\mathrm{d}x = \int (u^2-1)u\cdot 2u\,\mathrm{d}u = \int (2u^4 - 2u^3)\,\mathrm{d}u$$

   and so on. Check that you get the same answer.[15]

   (e) With $u = 1+x^2$ one has $x^2 = u-1$, and $\mathrm{d}u = 2x\,\mathrm{d}x$, so

$$\int \frac{x^3}{(1+x^2)^3}\,\mathrm{d}x = \int \frac{x^2\cdot x}{(1+x^2)^3}\,\mathrm{d}x = \tfrac{1}{2}\int \frac{u-1}{u^3}\,\mathrm{d}u = \tfrac{1}{2}\int (u^{-2} - u^{-3})\,\mathrm{d}u$$
$$= -\tfrac{1}{2}u^{-1} + \tfrac{1}{4}u^{-2} + C = -\frac{1}{2(1+x^2)} + \frac{1}{(1+x^2)^2} + C$$

   (f) With $u = \sqrt{4-x^3}$ one has $u^2 = 4-x^3$ and $2u\,\mathrm{d}u = -3x^2\,\mathrm{d}x$, so

$$\int x^5\sqrt{4-x^3}\,\mathrm{d}x = \int x^3\sqrt{4-x^3}x^2\,\mathrm{d}x = \int (4-u^2)u\left(-\frac{2}{3}\right)u\,\mathrm{d}u$$
$$= \int \left(-\frac{8}{3}u^2 + \frac{2}{3}u^4\right)\mathrm{d}u = -\frac{8}{9}u^3 + \frac{2}{15}u^5 + C = -\frac{8}{9}(4-x^3)^{3/2} + \frac{2}{15}(4-x^3)^{5/2} + C$$

6. (a) $I = \int_0^1 (x^4 - x^9)(x^5-1)^{12}\,\mathrm{d}x = \int_0^1 -x^4(x^5-1)^{13}\,\mathrm{d}x$. Introduce $u = x^5 - 1$. Then $\mathrm{d}u = 5x^4\mathrm{d}x$. Now we use (9.6.2) along with the facts that $u = -1$ when $x = 0$ and $u = 0$ when $x = 1$. The integral becomes $I = -\int_{-1}^0 \frac{1}{5}u^{13}\,\mathrm{d}u = -\Big|_{-1}^0 \frac{1}{70}u^{14} = \frac{1}{70}$.

[15] Actually, even integration by parts works in this case. See Problem 9.5.2(d).

(b) With $u = \sqrt{x}$ one has $u^2 = x$ and $2u\,\mathrm{d}u = \mathrm{d}x$. Then,[16]

$$\int \frac{\ln x}{\sqrt{x}}\,\mathrm{d}x = 2\int \ln u^2\,\mathrm{d}u = 4\int \ln u\,\mathrm{d}u = 4(u\ln u - u) + C$$
$$= 4\sqrt{x}\ln\sqrt{x} - 4\sqrt{x} + C = 2\sqrt{x}\ln x - 4\sqrt{x} + C$$

(c) With $u = 1 + \sqrt{x}$ one has $(u-1)^2 = x$, so $2(u-1)\,\mathrm{d}u = \mathrm{d}x$. Again we use (9.6.2) along with the facts that $u = 1$ when $x = 0$ and $u = 3$ when $x = 4$. The specified integral becomes[17]

$$\int_1^3 \frac{2(u-1)}{\sqrt{u}}\,\mathrm{d}u = 2\int_1^3 (u^{1/2} - u^{-1/2})\,\mathrm{d}u = 2\Big|_1^3 \left(\tfrac{2}{3}u^{3/2} - 2u^{1/2}\right) = \frac{8}{3}.$$

7. (a) With $u = 1 + e^{\sqrt{x}}$ one has $\mathrm{d}u = \frac{1}{2\sqrt{x}} \cdot e^{\sqrt{x}}\,\mathrm{d}x$. Now $x = 1$ gives $u = 1 + e$ and $x = 4$ gives $u = 1 + e^2$. Thus,

$$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}\,(1 + e^{\sqrt{x}})}\,\mathrm{d}x = \int_{1+e}^{1+e^2} \frac{2}{u}\,\mathrm{d}u = 2\Big|_{1+e}^{1+e^2} \ln u = 2\ln(1+e^2) - 2\ln(1+e).$$

(b) A natural substitution is $u = e^x + 1$ leading to $\mathrm{d}u = e^x\,\mathrm{d}x$ and so $\mathrm{d}x = \mathrm{d}u/e^x = \mathrm{d}u/(u-1)$. When $x = 0$ one has $u = 2$, and when $x = 1/3$ one has $u = e^{1/3} + 1$. Thus,

$$\int_0^{1/3} \frac{1}{e^x + 1}\,\mathrm{d}x = \int_2^{e^{1/3}+1} \frac{1}{u(u-1)}\,\mathrm{d}u = \int_2^{e^{1/3}+1} \left(\frac{1}{u-1} - \frac{1}{u}\right)\mathrm{d}u$$
$$= \Big|_2^{e^{1/3}+1} (\ln|u-1| - \ln|u|) = \tfrac{1}{3} - \ln(e^{1/3}+1) + \ln 2 = \ln 2 - \ln(e^{-1/3}+1)$$

because $\frac{1}{3} - \ln(e^{1/3}+1) = \ln[e^{1/3}/(e^{1/3}+1)] = -\ln(e^{-1/3}+1)$.[18]

(c) With $z^4 = 2x - 1$ one has $4z^3\,\mathrm{d}z = 2\,\mathrm{d}x$. Also $x = 8.5$ gives $z = 2$ and $x = 41$ gives $z = 3$. The integral becomes

$$\int_2^3 \frac{2z^3}{z^2 - z}\,\mathrm{d}z = 2\int_2^3 \frac{z^2}{z-1}\,\mathrm{d}z = 2\int_2^3 \left(z + 1 + \frac{1}{z-1}\right)\mathrm{d}z$$
$$= 2\Big|_2^3 \left[\tfrac{1}{2}z^2 + z + \ln(z-1)\right] = 7 + 2\ln 2$$

9.7 Infinite intervals of integration

3. (a) See the answer given in the book.

[16] Integration by parts also works in this case, with $f(x) = \ln x$ and $g'(x) = 1/\sqrt{x}$.

[17] The substitution $u = \sqrt{1 + \sqrt{x}}$ also works.

[18] Rewriting the integrand as $\dfrac{e^{-x}}{1 + e^{-x}}$, the suggested substitution $t = e^{-x}$, with $\mathrm{d}t = -e^{-x}\,\mathrm{d}x$ works as well. Verify that you get the same answer.

(b) Using a simplified notation and the result in Example 1(a), we have

$$\int_0^{\infty}\left(x-\frac{1}{\lambda}\right)^2\lambda e^{-\lambda x}\,\mathrm{d}x = -\Big|_0^{\infty}\left(x-\frac{1}{\lambda}\right)^2 e^{-\lambda x}+\int_0^{\infty}2\left(x-\frac{1}{\lambda}\right)e^{-\lambda x}\,\mathrm{d}x$$
$$=\frac{1}{\lambda^2}+2\int_0^{\infty}xe^{-\lambda x}\,\mathrm{d}x-\frac{2}{\lambda}\int_0^{\infty}e^{-\lambda x}\,\mathrm{d}x=\frac{1}{\lambda^2}+\frac{2}{\lambda^2}-\frac{2}{\lambda^2}=\frac{1}{\lambda^2}$$

where we have used both the result of Example 1(a) and part (a) in order to derive the penultimate equality.

(c) Using the result of part (b) in order to derive the penultimate equality, we get

$$\int_0^{\infty}\left(x-\frac{1}{\lambda}\right)^3\lambda e^{-\lambda x}\,\mathrm{d}x = -\Big|_0^{\infty}\left(x-\frac{1}{\lambda}\right)^3 e^{-\lambda x}+\int_0^{\infty}3\left(x-\frac{1}{\lambda}\right)^2 e^{-\lambda x}\,\mathrm{d}x$$
$$=-\frac{1}{\lambda^3}+\frac{3}{\lambda}\int_0^{\infty}\left(x-\frac{1}{\lambda}\right)^2\lambda e^{-\lambda x}\,\mathrm{d}x=-\frac{1}{\lambda^3}+\frac{3}{\lambda}\cdot\frac{1}{\lambda^2}=\frac{2}{\lambda^3}$$

5. (a) $f'(x)=(1-3\ln x)/x^4=0$ at $x=e^{1/3}$, with $f'(x)>0$ for $x<e^{1/3}$ and $f'(x)<0$ for $x>e^{1/3}$. Hence f has a maximum at $(e^{1/3},1/3e)$. Since $f(x)\to-\infty$ as $x\to 0^+$, there is no minimum. Use l'Hôpital's rule to show that $f(x)\to 0$ as $x\to\infty$.

(b) Here, $\int_a^b x^{-3}\ln x\,\mathrm{d}x=-\big|_a^b\,\frac{1}{2}x^{-2}\ln x+\int_a^b\frac{1}{2}x^{-3}\,\mathrm{d}x=\big|_a^b\left(-\frac{1}{2}x^{-2}\ln x-\frac{1}{4}x^{-2}\right)$.
This diverges when $b=1$ and $a\to 0$. But $\int_1^{\infty}x^{-3}\ln x\,\mathrm{d}x=1/4$.

7. Provided that both limits exist, the integral is the sum of

$$I_1=\lim_{\varepsilon\to 0^+}\int_{-2+\varepsilon}^{3}(1/\sqrt{x+2}\,)\,\mathrm{d}x\quad\text{and}\quad I_2=\lim_{\varepsilon\to 0^+}\int_{-2}^{3-\varepsilon}(1/\sqrt{3-x}\,)\,\mathrm{d}x.$$

Here, $I_1=\lim_{\varepsilon\to 0^+}\big|_{-2+\varepsilon}^{3}(2\sqrt{x+2}\,)=\lim_{\varepsilon\to 0^+}(2\sqrt{5}-2\sqrt{\varepsilon}\,)=2\sqrt{5}$,
and $I_2=\lim_{\varepsilon\to 0^+}\big|_{-2}^{3-\varepsilon}(-2\sqrt{3-x}\,)=\lim_{\varepsilon\to 0^+}(-2\sqrt{\varepsilon}+2\sqrt{5}\,)=2\sqrt{5}$.

12. All three parts (a)–(c) use the substitution $u=\dfrac{x-\mu}{\sqrt{2}\sigma}$, which gives $\mathrm{d}u=\dfrac{1}{\sigma\sqrt{2}}\mathrm{d}x$, and so $\mathrm{d}x=\sigma\sqrt{2}\,\mathrm{d}u$.

(a) $\displaystyle\int_{-\infty}^{+\infty}f(x)\,\mathrm{d}x=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-u^2}\,\mathrm{d}u=1$, by Eq. (9.7.8).

(b) $\displaystyle\int_{-\infty}^{+\infty}xf(x)\,\mathrm{d}x=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}(\mu+\sqrt{2}\sigma u)\,e^{-u^2}\,\mathrm{d}u=\mu$, using part (a) and Example 9.7.3.

(c) First, $\displaystyle\int_{-\infty}^{+\infty}(x-\mu)^2f(x)\,\mathrm{d}x=\int_{-\infty}^{+\infty}2\sigma^2u^2\frac{1}{\sigma\sqrt{2\pi}}e^{-u^2}\sigma\sqrt{2}\,\mathrm{d}u=\sigma^2\frac{2}{\sqrt{\pi}}\int_{-\infty}^{+\infty}u^2e^{-u^2}\,\mathrm{d}u$.

Now integrating by parts yields $\int_a^b u^2e^{-u^2}\,\mathrm{d}u=-\frac{1}{2}\big|_a^b ue^{-u^2}+\int_a^b\frac{1}{2}e^{-u^2}\,\mathrm{d}u$ for all real a,b. Because $\big|_a^b ue^{-u^2}=be^{-b^2}-ae^{-a^2}\to 0$ as $a\to-\infty$ and $b\to\infty$, it follows that $\int_{-\infty}^{+\infty}u^2e^{-u^2}\,\mathrm{d}u=\int_{-\infty}^{+\infty}\frac{1}{2}e^{-u^2}\,\mathrm{d}u=\frac{1}{2}\sqrt{\pi}$ by part (a). Hence $\int_{-\infty}^{+\infty}(x-\mu)^2f(x)\,\mathrm{d}x=\sigma^2$.

9.8 A glimpse at differential equations

10. (a) If $f \neq r$, the equation can be rewritten as $\dot{x} = (r-f)x\left[1 - \dfrac{x}{(1-f/r)K}\right]$.
There are two constant solutions $x \equiv 0$ and $x \equiv (1-f/r)K$, though the latter is negative, so biologically meaningless, unless $f \leq r$. With $\bar{r} = r - f$ and $\bar{K} = (1-f/r)K$, the equation is $x = \bar{r}x(1 - x/\bar{K})$. Using (9.8.7), the solution is

$$x(t) = \frac{\bar{K}}{1 + \dfrac{\bar{K} - x_0}{x_0}e^{-\bar{r}t}} = \frac{(1-f/r)K}{1 + \dfrac{(1-f/r)K - x_0}{x_0}e^{-(r-f)t}}$$

In the special case when $f = r$, the equation reduces to $\dot{x} = -rx^2/K$. Separating the variables gives $-\mathrm{d}x/x^2 = (r/K)\,\mathrm{d}t$, whereupon integration gives $1/x = (r/K)t + C$. If $x(0) = x_0$, we get $C = 1/x_0$, so the solution is $x(t) = 1/[(r/K)t + 1/x_0] \to 0$ as $t \to \infty$.

(b) When $f > r$, the solution to the differential equation given in part (a) is still valid even though both $\bar{K}$ and $\bar{r}$ are negative. Because the fish rate f exceeds the replenishment rate r, however, the fish stock steadily declines. Indeed, as $t \to \infty$ one has $e^{-(r-f)t} \to \infty$ and in fact the solution of the equation satisfies $x(t) \to 0$. That is, the fish stock tends to extinction.

9.9 Separable and linear differential equations

2. (a) $\mathrm{d}x/\mathrm{d}t = e^{2t}/x^2$. Separate: $\int x^2\,\mathrm{d}x = \int e^{2t}\,\mathrm{d}t$. Integrate: $\frac{1}{3}x^3 = \frac{1}{2}e^{2t} + C_1$. Solve for x: $x^3 = \frac{3}{2}e^{2t} + 3C_1 = \frac{3}{2}e^{2t} + C$, with $C = 3C_1$. Hence, $x = \sqrt[3]{\frac{3}{2}e^{2t} + C}$.[19]

(b) $\mathrm{d}x/\mathrm{d}t = e^{-t}e^{x}$, so $\int e^{-x}\,\mathrm{d}x = \int e^{-t}\,\mathrm{d}t$. Integrate: $-e^{-x} = -e^{-t} + C_1$. Solve for x: $e^{-x} = e^{-t} + C$, with $C = -C_1$. Hence, $-x = \ln(e^{-t} + C)$, so $x = -\ln(e^{-t} + C)$.

(c) This follows directly from (9.9.3).

(d) Similar to (a).

(e) By Eq. (9.9.5), $x = Ce^{2t} + e^{2t}\int(-t)e^{-2t}\,\mathrm{d}t = Ce^{2t} - e^{2t}\int te^{-2t}\,\mathrm{d}t$. Here

$$\int te^{-2t}\,\mathrm{d}t = t\left(-\frac{1}{2}\right)e^{-2t} + \frac{1}{2}\int e^{-2t}\mathrm{d}t = \left(-\frac{1}{2}t - \frac{1}{4}\right)e^{-2t}$$

and thus

$$x = Ce^{2t} - e^{2t}\left(-\frac{1}{2}t - \frac{1}{4}\right)e^{-2t} = Ce^{2t} + \frac{1}{2}t + \frac{1}{4}.$$

(f) By (9.9.5), $x = Ce^{-3t} + e^{-3t}\int e^{3t}te^{t^2-3t}\mathrm{d}t = Ce^{-3t} + e^{-3t}\int te^{t^2}\mathrm{d}t = Ce^{-3t} + \frac{1}{2}e^{t^2-3t}$.

5. From (iii), $L = L_0e^{\beta t}$, so $\dot{K} = \gamma K^{\alpha}L_0e^{\beta t}$, a separable equation. Writing the equation as $K^{-\alpha}\dot{K} = \gamma L_0e^{\beta t}$, we get

$$\int K^{-\alpha}\mathrm{d}K = \int \gamma L_0e^{\beta t}\mathrm{d}t, \quad \text{so} \quad \frac{1}{1-\alpha}K^{1-\alpha} = \frac{\gamma L_0}{\beta}e^{\beta t} + C_1.$$

[19] It is important to include the constant at the integration step. Adding it only at the end would lead to an error: $\frac{1}{3}x^3 = \frac{1}{2}e^{2t}$, $x^3 = \frac{3}{2}e^{2t}$, $x = \sqrt[3]{\frac{3}{2}e^{2t}} + C$. This is a solution only if $C = 0$, and is not the general solution.

Hence, $K^{1-\alpha} = \dfrac{(1-\alpha)\gamma L_0}{\beta} e^{\beta t} + (1-\alpha)C_1$, and at $t=0$, $K_0^{1-\alpha} = \dfrac{(1-\alpha)\gamma L_0}{\beta} + (1-\alpha)C_1$. It follows that $K^{1-\alpha} = \dfrac{(1-\alpha)\gamma L_0}{\beta}(e^{\beta t}-1) + K_0^{1-\alpha}$, from which we find K.

Review exercises for Chapter 9

4. (a) $5/4$. (See Example 9.7.2.)
   (b) $\Big|_0^1 \frac{1}{20}(1+x^4)^5 = 31/20$.
   (c) $\Big|_0^\infty 5te^{-t} - \int_0^\infty 5e^{-t}\,\mathrm{d}t = 5\Big|_0^\infty e^{-t} = -5$.
   (d) $\int_1^e (\ln x)^2\,\mathrm{d}x = \Big|_1^e x(\ln x)^2 - 2\int_1^e \ln x\,\mathrm{d}x = e - 2\Big|_1^e(x\ln x - x) = e-2$.
   (e) $\Big|_0^2 \frac{2}{9}(x^3+1)^{3/2} = \frac{2}{9}(9^{3/2}-1) = 52/9$.
   (f) $\Big|_{-\infty}^0 \frac{1}{3}\ln(e^{3z}+5) = \frac{1}{3}(\ln 6 - \ln 5) = \frac{1}{3}\ln(6/5)$.
   (g) $\frac{1}{4}\Big|_{1/2}^{e/2} x^4\ln(2x) - \frac{1}{4}\int_{1/2}^{e/2} x^3\,\mathrm{d}x = \frac{1}{4}(e/2)^4 - \frac{1}{16}[(e/2)^4 - (1/2)^4] = (1/256)(3e^4+1)$.
   (h) Introduce $u = \sqrt{x}$. Then $u^2 = x$, so $2u\,\mathrm{d}u = \mathrm{d}x$, and

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x = \int_1^\infty \frac{e^{-u}\,2u}{u}\,\mathrm{d}u = 2\int_1^\infty e^{-u}\,\mathrm{d}u = 2e^{-1}.$$

5. (a) With $u = 9+\sqrt{x}$ one has $x = (u-9)^2$ and so $\mathrm{d}x = 2(u-9)\,\mathrm{d}u$. Also, $u=9$ when $x=0$ and $u=14$ when $x=25$. Thus,

$$\int_0^{25} \frac{1}{9+\sqrt{x}}\,\mathrm{d}x = \int_9^{14} \frac{2(u-9)}{u}\,\mathrm{d}u = \int_9^{14}\left(2 - \frac{18}{u}\right)\mathrm{d}u = 10 - 18\ln\frac{14}{9}.$$

   (b) With $u = \sqrt{t+2}$ one has $t = u^2-2$ and so $\mathrm{d}t = 2u\,\mathrm{d}u$. Also, $u=2$ when $t=2$ and $u=3$ when $t=7$. Hence,

$$\int_2^7 t\sqrt{t+2}\,\mathrm{d}t = \int_2^3 (u^2-2)u\cdot 2u\,\mathrm{d}u = 2\int_2^3 (u^4-2u^2)\,\mathrm{d}u = 2\Big|_2^3\left(\frac{1}{5}u^5 - \frac{2}{3}u^3\right)$$
$$= 2\left[\left(\frac{243}{5} - \frac{54}{3}\right) - \left(\frac{32}{5} - \frac{16}{3}\right)\right] = \frac{422}{5} - \frac{76}{3} = \frac{886}{15}.$$

   (c) With $u = \sqrt[3]{19x^3+8}$ one has $u^3 = 19x^3+8$ and so $3u^2\mathrm{d}u = 57x^2\mathrm{d}x$. Also, $x=0$ gives $u=2$ and $x=1$ gives $u=3$. Therefore

$$\int_0^1 57x^2\sqrt[3]{19x^3+8}\,\mathrm{d}x = \int_2^3 3u^3\,\mathrm{d}u = \Big|_2^3 \frac{3}{4}u^4 = 195/4$$

10. (a) As in Example 9.4.3, first we need to find P^* and Q^*. From the equilibrium condition $f(Q^*) = 100 - 0.05Q^* = g(Q^*) = 0.1Q^* + 10$, we obtain $0.15Q^* = 90$, and so $Q^* = 600$. Then $P^* = g(Q^*) = 0.1Q^* + 10 = 70$. Moreover,

$$\text{CS} = \int_0^{600}(f(Q) - P^*)\,\mathrm{d}Q = \int_0^{600}(30 - 0.05Q)\,\mathrm{d}Q = \Big|_0^{600}\left(30Q - \tfrac{1}{2}0.05Q^2\right) = 9000$$
$$\text{PS} = \int_0^{600}(P^* - g(Q))\,\mathrm{d}Q = \int_0^{600}(60 - 0.1Q)\,\mathrm{d}Q = \Big|_0^{600}\left(60Q - \tfrac{1}{2}0.1Q^2\right) = 18000$$

(b) Equilibrium occurs when $50/(Q^*+5) = 4.5 + 0.1Q^*$. Clearing fractions and then simplifying, we obtain $(Q^*)^2 + 50Q^* - 275 = 0$. The only positive solution is $Q^* = 5$, and then $P^* = 5$.

$$\text{CS} = \int_0^5 \left[\frac{50}{Q+5} - 5\right] \mathrm{d}Q = \Big|_0^5 [50\ln(Q+5) - 5Q] = 50\ln 2 - 25$$

$$\text{PS} = \int_0^5 (5 - 4.5 - 0.1Q)\,\mathrm{d}Q = \Big|_0^5 (0.5Q - 0.05Q^2) = 2.5 - 1.25 = 1.25$$

11. (a) $f'(t) = 4[2\ln t \cdot (1/t) \cdot t - (\ln t)^2 \cdot 1]/t^2 = 4(2 - \ln t)\ln t/t^2$, and

$$f''(t) = 4\frac{[2\cdot(1/t) - 2\ln t \cdot (1/t)]\,t^2 - [2\ln t - (\ln t)^2]\,2t}{t^4} = 8\frac{(\ln t)^2 - 3\ln t + 1}{t^3}$$

(b) Here, $f'(t) = 0 \iff \ln t(2 - \ln t) = 0 \iff \ln t = 2 \text{ or } \ln t = 0 \iff t = e^2 \text{ or } t = 1$. But $f''(1) = 8 > 0$ and $f''(e^2) = -8e^{-6}$, so $t = 1$ is a local minimum point and $t = e^2 \approx 7.4$ is a local maximum point. We find $f(1) = 0$ and $f(e^2) = 16e^{-2} \approx 2.2$.

(c) The function is positive on $[1, e^2]$, so the area is $A = 4\int_1^{e^2} \frac{1}{t}(\ln t)^2\,\mathrm{d}t$. With $u = \ln t$ as a new variable, $\mathrm{d}u = \frac{1}{t}\mathrm{d}t$. When $t = 1$ then $u = 0$, and $t = e^2$ implies $u = 2$.

Hence, $A = 4\int_0^2 u^2\,\mathrm{d}u = 4\Big|_0^2 \frac{1}{3}u^3 = \frac{32}{3}$.

13. (a) Separable. $\int x^{-2}\,\mathrm{d}x = \int t\,\mathrm{d}t$, and so $-1/x = \frac{1}{2}t^2 + C_1$, or $x = 1/(C - \frac{1}{2}t^2)$ (with $C = -C_1$).

(b) Direct use of (9.9.3).

(c) Direct use of (9.9.3).

(d) Using (9.9.5), $x = Ce^{-5t} + 10e^{-5t}\int te^{5t}\,\mathrm{d}t$.
Here $\int te^{5t}\,\mathrm{d}t = t\frac{1}{5}e^{5t} - \frac{1}{5}\int e^{5t}\,\mathrm{d}t = \frac{1}{5}te^{5t} - \frac{1}{25}e^{5t}$.
Thus $x = Ce^{-5t} + 10e^{-5t}(\frac{1}{5}te^{5t} - \frac{1}{25}e^{5t}) = Ce^{-5t} + 2t - \frac{2}{5}$.

(e) $x = Ce^{-t/2} + e^{-t/2}\int e^{t/2}e^t\,\mathrm{d}t = Ce^{-t/2} + e^{-t/2}\int e^{3t/2}\,\mathrm{d}t = Ce^{-t/2} + e^{-t/2}\frac{2}{3}e^{3t/2}$
$= Ce^{-t/2} + \frac{2}{3}e^t$.

(f) $x = Ce^{-3t} + e^{-3t}\int t^2e^{3t}\,\mathrm{d}t = Ce^{-3t} + e^{-3t}\left(\frac{1}{3}t^2e^{3t} - \frac{2}{3}\int te^{3t}\,\mathrm{d}t\right)$
$= Ce^{-3t} + \frac{1}{3}t^2 - \frac{2}{3}e^{-3t}\left(\frac{1}{3}te^{3t} - \frac{1}{3}\int e^{3t}\,\mathrm{d}t\right) = Ce^{-3t} + \frac{1}{3}t^2 - \frac{2}{9}t + \frac{2}{27}$.

10 Topics in Financial Mathematics

10.2 Continuous compounding

6. With $g(x) = (1 + r/x)^x$ for all $x > 0$ one has $\ln g(x) = x\ln(1 + r/x)$. Differentiating gives

$$g'(x)/g(x) = \ln(1 + r/x) + x(-r/x^2)/(1 + r/x) = \ln(1 + r/x) - (r/x)/(1 + r/x),$$

as claimed in the problem. Putting $h(u) = \ln(1+u) - u/(1+u)$, one has $h'(u) = u/(1+u)^2 > 0$ for $u > 0$, so $h(u) > 0$ for all $u > 0$, implying that $g'(x)/g(x) = h(r/x) > 0$ for all $x > 0$. So $g(x)$ is strictly increasing for $x > 0$. Because $g(x) \to e^r$ as $x \to \infty$, it follows that $g(x) < e^r$ for all $x > 0$. Continuous compounding of interest is best for the lender.

10.4 Geometric series

6. Let x denote the number of years beyond 1971 that the extractable resources of iron will last. We need to solve the equation $794 + 794 \cdot 1.05 + \cdots + 794 \cdot (1.05)^x = 249 \cdot 10^3$. Using Eq. (10.4.3) to sum the left hand side, the equation is $794[(1.05)^{x+1} - 1]/(1.05 - 1) = 249 \cdot 10^3$ or $(1.05)^{x+1} = 249 \cdot 10^3 \cdot 0.05/794 + 1 = 12450/794 + 1 \approx 16.68$. Using a calculator, we find $x \approx (\ln 16.68/\ln 1.05) - 1 \approx 56.68$, so the resources will be exhausted part way through the year 2028.

8. (a) The quotient of this infinite series is e^{-rt}, so the sum is $f(t) = \dfrac{P(t)e^{-rt}}{1 - e^{-rt}} = \dfrac{P(t)}{e^{rt} - 1}$.

   (b) By direct computation, $f'(t) = \dfrac{P'(t)(e^{rt} - 1) - P(t)re^{rt}}{(e^{rt} - 1)^2}$. Now $t^* > 0$ can maximize $f(t)$ only if $f'(t^*) = 0$, or $P'(t^*)(e^{rt^*} - 1) = rP(t^*)e^{rt^*}$, so $P'(t^*)/P(t^*) = r/(1 - e^{-rt^*})$.

   (c) $\displaystyle\lim_{r\to 0} \frac{r}{1 - e^{-rt^*}} = \text{“}0/0\text{”} = \lim_{r\to 0} \frac{1}{t^* e^{-rt^*}} = \frac{1}{t^*}$

11. See Fig. SM10.4.11. If $p > 1$, then $\sum_{n=1}^{\infty}(1/n^p) = 1 + \sum_{n=2}^{\infty}(1/n^p)$ is finite because $\sum_{n=2}^{\infty}(1/n^p)$ is the sum of the shaded rectangles, and this sum is certainly less than the area under the curve $y = 1/x^p$ over $[1, \infty)$, which is equal to $1/(p-1)$.
    If $p \le 1$, the sum $\sum_{n=1}^{\infty}(1/n^p)$ is the sum of the larger rectangles in the figure, and this sum is larger than the area under the curve $y = 1/x^p$ over $[1, \infty)$, which is unbounded when $p \le 1$. Hence, $\sum_{n=1}^{\infty}(1/n^p)$ diverges in this case.

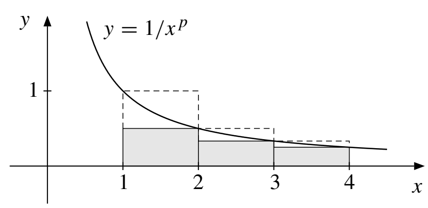

Figure SM10.4.11

10.7 Internal rate of return

5. According to (10.7.1), the internal rate must satisfy

$$-100\,000 + \frac{10\,000}{1+r} + \frac{10\,000}{(1+r)^2} + \cdots + \frac{10\,000}{(1+r)^{20}} = 0$$

After dividing all the terms by 10 000, and putting $s = 1/(1+r)$, we have to show that the equation $f(s) = s^{20} + s^{19} + \cdots + s^2 + s - 10 = 0$ has a unique positive solution. Since $f(0) = -10$ and $f(1) = 10$, the intermediate value theorem (Theorem 7.10.1) implies that there exists a number s^* between 0 and 1 such that $f(s^*) = 0$. This s^* is the unique positive root because $f'(s) > 0$ for all $s > 0$. In fact, from (10.4.3), $f(s) = -10 + (s - s^{21})/(1-s)$, and $f(s^*) = 0 \iff (s^*)^{21} - 11s^* + 10 = 0$. Exercise 7.R.26 asks for an approximation to the unique root of this equation in the interval $(0, 1)$. An answer is $s^* \approx 0.928$, so $r^* = 1/s^* - 1 \approx 0.0775$, which means that the internal rate of return is about $7\frac{3}{4}\%$.

Review exercises for Chapter 10

9. We use formula (10.5.3) on the future value of an annuity: $(5000/0.04)[(1.04)^4-1] = 21\,232.32$.

10. The last of seven payments will be on 1st January 2016, when the initial balance of 10 000 will have earned interest for 10 years. So K must solve $10\,000 \cdot (1.04)^{10} + K[(1.04)^8 - 1]/0.04 = 70\,000$. We find that $K \approx 5990.49$.

13. (a) $f'(t) = 100e^{\sqrt{t}/2}e^{-rt}\left(\frac{1}{4\sqrt{t}} - r\right)$. We see that $f'(t) = 0$ for $t = t^* = 1/16r^2$. Since $f'(t) > 0$ for $t < t^*$ and $f'(t) < 0$ for $t > t^*$, it follows that t^* maximizes $f(t)$.

    (b) $f'(t) = 200e^{-1/t}e^{-rt}\left(\frac{1}{t^2} - r\right)$. We see that $f'(t) = 0$ for $t = t^* = 1/\sqrt{r}$. Since $f'(t) > 0$ for $t < t^*$ and $f'(t) < 0$ for $t > t^*$, it follows that t^* maximizes $f(t)$.

11 Functions of Many Variables

11.2 Partial derivatives with two variables

5. Parts (a)–(c) are easy. As for (d), note that $z = x^y = (e^{\ln x})^y = e^{y \ln x} = e^u$ with $u = y \ln x$. Then $z'_x = e^u u'_x = x^y(y/x) = yx^{y-1}$. Similarly $z'_y = e^u u'_y = x^y \ln x$. Moreover, $z''_{xx} = (\partial/\partial x)(yx^{y-1}) = y(y-1)x^{y-2}$.[20] Similarly $z''_{yy} = (\partial/\partial y)(x^y \ln x) = x^y(\ln x)^2$ and $z''_{xy} = (\partial/\partial y)(yx^{y-1}) = x^{y-1} + yx^{y-1}\ln x$.

11.3 Geometric representation

9. (a) It helps to regard the figure as a contour map of a mountain, whose level curves join points at the same height above mean sea level. Near P the terrain is rising in the direction of the positive x-axis, so $f'_x(P) > 0$, and it is also sloping down in the direction of the positive y-axis so $f'_y(P) < 0$. Near Q, the terrain slopes in the opposite directions. Hence $f'_x(Q) < 0$ and $f'_y(Q) > 0$.

    (b) (i) The line $x = 1$ has no point in common with any of the given level curves.
    (ii) The line $y = 2$ intersects the level curve $z = 2$ at $x = 2$ and $x = 6$ (approximately).

    (c) If you start at the point $(6, 0)$ and move up along the line $2x + 3y = 12$, the first marked level curve you meet is $z = f(x, y) = 1$. Moving further you meet level curves with higher z-values. The level curve with the highest z-value you meet is $z = 3$, which is the level curve that just touches the straight line.

10. The inequalities on the partial derivatives imply that: $F(1,0) - F(0,0) = \int_0^1 F'_1(x,0)\,\mathrm{d}x \geq \int_0^1 2\,\mathrm{d}x = 2$; $F(2,0) - F(1,0) = \int_1^2 F'_1(x,0)\,\mathrm{d}x \geq 2$; $F(0,1) - F(0,0) = \int_0^1 F'_2(0,y)\,\mathrm{d}y \leq 1$; $F(1,1) - F(0,1) = \int_0^1 F'_1(x,1)\,\mathrm{d}x \geq 2$; and $F(1,1) - F(1,0) = \int_0^1 F'_2(1,y)\,\mathrm{d}y \leq 1$.

11.5 Functions of more variables

3. (a) In the first week it buys $120/50 = 2.4$ million shares, followed successively by $120/60 = 2$ million shares, $120/45 = 2.667$ million, $120/40 = 3$ million, $120/75 = 1.6$ million, and finally $120/80 = 1.5$ million in the sixth week. The total is 13.167 million shares.

[20] When differentiating x^{y-1} partially w.r.t. x, one treats y as a constant, so the rule $\mathrm{d}x^a/\mathrm{d}x = ax^{a-1}$ applies.

(b) The arithmetic mean price in dollars is $350/6 = 58.33$. But the total cost at that price of the 13.167 million shares which the fund has acquired would be $13.167 \times 58.33 = 768.031$ million dollars. So using the arithmetic mean price would overstate the fund's cost by 48.031 million dollars. A more accurate statement of the mean price is $720 \ / \ 13.167 = 54.68$ dollars per share. Routine arithmetic shows that this is the harmonic mean of the six prices defined by formula (c) of Example 2.

11.6 Partial derivatives with more variables

2. Parts (a)–(d) are routine. As for (e) and (f):

(e) $f(x, y, z) = (x^2 + y^3 + z^4)^6 = u^6$, with $u = x^2 + y^3 + z^4$. Then:

$$\begin{aligned} f_1' &= 6u^5 u_1' = 6(x^2 + y^3 + z^4)^5 \, 2x = 12x(x^2 + y^3 + z^4)^5 \\ f_2' &= 6u^5 u_2' = 6(x^2 + y^3 + z^4)^5 \, 3y^2 = 18y^2(x^2 + y^3 + z^4)^5 \\ f_3' &= 6u^5 u_3' = 6(x^2 + y^3 + z^4)^5 \, 4z^3 = 24z^3(x^2 + y^3 + z^4)^5 \end{aligned}$$

(f) $f(x, y, z) = e^{xyz} = e^u$, with $u = xyz$, gives $f_1' = e^u u_1' = e^{xyz} yz$.
Similarly, $f_2' = e^u u_2' = e^{xyz} xz$, and $f_3' = e^u u_3' = e^{xyz} xy$.

10. From $f = x^{y^z}$ we get $\ln f = y^z \ln x$.
Differentiating w.r.t. x yields $f_x'/f = y^z/x$, and so $f_x' = f y^z/x = x^{y^z} y^z/x = y^z x^{y^z - 1}$.
Differentiating w.r.t. y yields $f_y'/f = z y^{z-1} \ln x$, and so $f_y' = z y^{z-1} (\ln x) x^{y^z}$.
Differentiating w.r.t. z yields $f_z'/f = y^z (\ln y)(\ln x)$, and so $f_z' = y^z (\ln x)(\ln y) x^{y^z}$.

11. For $(x, y) \neq (0, 0)$ one has

$$f_1'(x, y) = y(x^4 + 4x^2y^2 - y^4)(x^2 + y^2)^{-2} \quad \text{and} \quad f_2'(x, y) = x(x^4 - 4x^2y^2 - y^4)(x^2 + y^2)^{-2}$$

Thus, for $y \neq 0$, $f_1'(0, y) = -y$. Because $f_1'(0, 0) = \lim_{h \to 0}[f(h, 0) - f(0, 0)]/h = 0$, this is also correct for $y = 0$. Similarly, $f_2'(x, 0) = x$ for all x.
It follows that $f_{12}''(0, y) = (\partial/\partial y) f_1'(0, y) = -1$ for all y. In particular, $f_{12}''(0, 0) = -1$.
Similarly, $f_{21}''(x, 0) = (\partial/\partial x) f_2'(x, 0) = 1$ for all x, and so $f_{21}''(0, 0) = 1$.

Straightforward differentiation shows that, for $(x, y) \neq (0, 0)$,

$$f_{12}''(x, y) = f_{21}''(x, y) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2 + y^2)^3}. \tag{$*$}$$

Thus, *away from the origin*, the two cross partial derivatives are equal, in accordance with Young's theorem. At the origin, however, we have seen that $f_{12}''(0, 0) = -1$ and $f_{21}''(0, 0) = 1$. Therefore, at least one of f_{12}'' and f_{21}'' must be discontinuous there. Indeed, it follows from ($*$) that $f_{12}''(x, 0) = 1$ for all $x \neq 0$ and $f_{12}''(0, y) = -1$ for all $y \neq 0$. Thus, as close to $(0, 0)$ as we want, we can find points where f_{12}'' equals 1 and also points where f_{12}'' equals -1. Therefore f_{12}'' cannot be continuous at $(0, 0)$. The same argument shows that f_{21}'' cannot be continuous at $(0, 0)$.

11.7 Economic applications

2. (a) $Y'_K = aAK^{a-1}$ and $Y'_K = aBL^{a-1}$, so $KY'_K + LY'_L = aAK^a + aBL^a = a(AK^a + BL^a) = aY$.

   (b) $KY'_K + LY'_L = KaAK^{a-1}L^b + LAK^a bL^{b-1} = aAK^a L^b + bAK^a L^b = (a+b)Y$.

   (c) $Y'_K = \dfrac{2aKL^5 - bK^4L^2}{(aL^3 + bK^3)^2}$ and $Y'_L = \dfrac{2bK^5L - aK^2L^4}{(aL^3 + bK^3)^2}$, so $KY'_K + LY'_L$ is

$$\frac{2aK^2L^5 - bK^5L^2 + 2bK^5L^2 - aK^2L^5}{(aL^3 + bK^3)^2} = \frac{K^2L^2(aL^3 + bK^3)}{(aL^3 + bK^3)^2} = \frac{K^2L^2}{aL^3 + bK^3} = Y.$$

   Note that according to Section 12.6 the functions in (a), (b), and (c) are homogeneous of degrees a, $a+b$, and 1, respectively, so these results are immediate implications of Euler's Theorem, (12.6.2).

7. Here

$$\begin{aligned} Y'_K &= (-\mu/\varrho)a(-\varrho)K^{-\varrho-1}Ae^{\lambda t}\left[aK^{-\varrho} + bL^{-\varrho}\right]^{-(\mu/\varrho)-1} \\ &= \mu aK^{-\varrho-1}Ae^{\lambda t}\left[aK^{-\varrho} + bL^{-\varrho}\right]^{-(\mu/\varrho)-1}, \\ Y'_L &= (-\mu/\varrho)b(-\varrho)L^{-\varrho-1}Ae^{\lambda t}\left[aK^{-\varrho} + bL^{-\varrho}\right]^{-(\mu/\varrho)-1} \\ &= \mu bL^{-\varrho-1}Ae^{\lambda t}\left[aK^{-\varrho} + bL^{-\varrho}\right]^{-(\mu/\varrho)-1} \end{aligned}$$

   Thus, $KY'_K + LY'_L = \mu(aK^{-\varrho} + bL^{-\varrho})Ae^{\lambda t}\left[aK^{-\varrho} + bL^{-\varrho}\right]^{-(\mu/\varrho)-1} = \mu Y$.
   Once again, this function is homogeneous of degree μ, so the result is an immediate implication of Euler's Theorem, (12.6.2).

11.8 Partial elasticities

4. Note that $\dfrac{\partial}{\partial m}\left(\dfrac{pD}{m}\right) = p\dfrac{mD'_m - D}{m^2} = \dfrac{p}{m^2}(mD'_m - D) = \dfrac{pD}{m^2}(\mathrm{El}_m D - 1) > 0$ if and only if $\mathrm{El}_m D > 1$. So pD/m increases with m if $\mathrm{El}_m D > 1$.[21]

Review exercises for Chapter 11

12. (a) See the answer given in the book.

    (b) We want to find all (x, y) that satisfy: (i) $4x^3 - 8xy = 0$ and (ii) $4y - 4x^2 + 4 = 0$. From equation (i), $4x(x^2 - 2y) = 0$, which implies that $x = 0$ or $x^2 = 2y$.
    For $x = 0$, equation (ii) yields $y = -1$, so $(x, y) = (0, -1)$ is one solution.
    For $x^2 = 2y$, (ii) reduces to $4y - 8y + 4 = 0$, or $y = 1$. But then $x^2 = 2$, so $x = \pm\sqrt{2}$. Hence, two additional solutions are $(x, y) = (\pm\sqrt{2}, 1)$.

12 Tools for Comparative Statics

12.1 A simple chain rule

6. (a) If $z = F(x, y) = x + y$ with $x = f(t)$ and $y = g(t)$, then $F'_1 = F'_2 = 1$, so the chain rule, formula (12.1.1), gives $\mathrm{d}z/\mathrm{d}t = 1 \cdot f'(t) + 1 \cdot g'(t) = f'(t) + g'(t)$.

[21] Using the formulas in Exercise 7.7.9, the result also follows from the fact that $\mathrm{El}_m(pD/m) = \mathrm{El}_m\, p + \mathrm{El}_m D - \mathrm{El}_m\, m = \mathrm{El}_m D - 1$.

(b) This case is like (a), except that $F_2' = -1$ so the chain rule gives $\mathrm{d}z/\mathrm{d}t = f'(t) - g'(t)$.

(c) If $z = F(x, y) = xy$ with $x = f(t)$ and $y = g(t)$, then $F_1'(x, y) = y$, $F_2'(x, y) = x$, $\mathrm{d}x/\mathrm{d}t = f'(t)$, and $\mathrm{d}y/\mathrm{d}t = g'(t)$, so formula (12.1.1) gives

$$\frac{\mathrm{d}z}{\mathrm{d}t} = F_1'(x, y)\frac{\mathrm{d}x}{\mathrm{d}t} + F_2'(x, y)\frac{\mathrm{d}y}{\mathrm{d}t} = yf'(t) + xg'(t) = f'(t)g(t) + f(t)g'(t).$$

(d) If $z = F(x, y) = \dfrac{x}{y}$ with $x = f(t)$ and $y = g(t)$, then $F_1'(x, y) = \dfrac{1}{y}$, $F_2'(x, y) = -\dfrac{x}{y^2}$, $\dfrac{\mathrm{d}x}{\mathrm{d}t} = f'(t)$, and $\dfrac{\mathrm{d}y}{\mathrm{d}t} = g'(t)$. So formula (12.1.1) gives

$$\frac{\mathrm{d}z}{\mathrm{d}t} = F_1'(x, y)\frac{\mathrm{d}x}{\mathrm{d}t} + F_2'(x, y)\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{y}f'(t) - \frac{x}{y^2}g'(t) = \frac{yf'(t) - xg'(t)}{y^2} = \frac{f'(t)g(t) - f(t)g'(t)}{(g(t))^2}.$$

(e) If $z = F(x, y) = G(x)$, independent of y, with $x = f(t)$, then $F_1' = G'$ and $F_2' = 0$, so (12.1.1) gives $\mathrm{d}z/\mathrm{d}t = G'(x) \cdot f'(t)$, which is the chain rule for one variable.

7. Let $U(x) = u(x, h(x))$. Then

$$U'(x) = u_1' + u_2'h'(x) = \frac{\alpha x^{\alpha-1}}{x^\alpha + z^\alpha} + \left(\frac{\alpha z^{\alpha-1}}{x^\alpha + z^\alpha} - \frac{\alpha}{z}\right)\frac{4a}{3}x^3(ax^4 + b)^{-2/3}$$

Because the term in large parentheses equals $-\alpha x^\alpha / z(x^\alpha + z^\alpha)$, simplifying gives

$$U'(x) = \frac{\alpha x^{\alpha-1}}{x^\alpha + z^\alpha} - \frac{\alpha x^\alpha}{z(x^\alpha + z^\alpha)}\frac{4ax^3}{3z^2} = \frac{\alpha x^{\alpha-1}(3z^3 - 4ax^4)}{3(x^\alpha + z^\alpha)z^3}$$

But $z^3 = ax^4 + b$ so $3z^3 - 4ax^4 = 3b - ax^4$. It follows that $U'(x) = 0$ when $x = x^* = \sqrt[4]{3b/a}$, whereas $U'(x) > 0$ for $x < x^*$ and $U'(x) < 0$ for $x > x^*$. Hence x^* maximizes U.

8. Differentiating (12.1.1) w.r.t. t yields $\mathrm{d}^2z/\mathrm{d}t^2 = (\mathrm{d}/\mathrm{d}t)[F_1'(x, y)\,\mathrm{d}x/\mathrm{d}t] + (\mathrm{d}/\mathrm{d}t)[F_2'(x, y)\,\mathrm{d}y/\mathrm{d}t]$. Here, $\dfrac{\mathrm{d}}{\mathrm{d}t}\left[F_1'(x, y)\dfrac{\mathrm{d}x}{\mathrm{d}t}\right] = \left[F_{11}''(x, y)\dfrac{\mathrm{d}x}{\mathrm{d}t} + F_{12}''(x, y)\dfrac{\mathrm{d}y}{\mathrm{d}t}\right]\dfrac{\mathrm{d}x}{\mathrm{d}t} + F_1'(x, y)\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}$,
while $\dfrac{\mathrm{d}}{\mathrm{d}t}\left[F_2'(x, y)\dfrac{\mathrm{d}y}{\mathrm{d}t}\right] = \left[F_{21}''(x, y)\dfrac{\mathrm{d}x}{\mathrm{d}t} + F_{22}''(x, y)\dfrac{\mathrm{d}y}{\mathrm{d}t}\right]\dfrac{\mathrm{d}y}{\mathrm{d}t} + F_2'(x, y)\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2}$.
The conclusion follows from summing these two while assuming that $F_{12}'' = F_{21}''$.

12.2 Chain rules for many variables

2. (a) Let $z = F(x, y) = xy^2$ with $x = t + s^2$ and $y = t^2s$. Then $F_1'(x, y) = y^2$, $F_2'(x, y) = 2xy$, $\partial x/\partial t = 1$, and $\partial y/\partial t = 2ts$. Now (12.2.1) gives

$$\frac{\partial z}{\partial t} = F_1'(x, y)\frac{\partial x}{\partial t} + F_2'(x, y)\frac{\partial y}{\partial t} = y^2 + 2xy2ts = (t^2s)^2 + 2(t + s^2)t^2s2ts = t^3s^2(5t + 4s^2).$$

The other partial derivative $\partial z/\partial s$ is found in the same way.

(b) $\dfrac{\partial z}{\partial t} = F_1'(x, y)\dfrac{\partial x}{\partial t} + F_2'(x, y)\dfrac{\partial y}{\partial t} = \dfrac{2y}{(x + y)^2}e^{t+s} + \dfrac{-2sx}{(x + y)^2}e^{ts}$, etc.

12. (a) Let $v = x^3+y^3+z^3-3xyz$, so that $u = \ln v$. Then $\partial u/\partial x = (1/v)(\partial v/\partial x) = (3x^2-3yz)/v$. Similarly, $\partial u/\partial y = (3y^2-3xz)/v$, and $\partial u/\partial z = (3z^2-3xy)/v$. Hence,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{1}{v}(3x^3-3xyz) + \frac{1}{v}(3y^3-3xyz) + \frac{1}{v}(3z^3-3xyz) = \frac{3v}{v} = 3$$

which proves (i). Equation (ii) is then proved by elementary algebra.

(b) Note that f is here a function of *one* variable. With $z = f(u)$ where $u = x^2y$, we get $\partial z/\partial x = f'(u)u'_x = 2xyf'(x^2y)$. Likewise, $\partial z/\partial y = x^2 f'(x^2y)$, so $x\,\partial z/\partial x = 2x^2yf'(x^2y) = 2y\,\partial z/\partial y$.

12.3 Implicit differentiation along a level curve

2. (a) See the answer given in the book.

(b) Let $F(x,y) = x - y + 3xy$. Then $F'_1 = 1+3y$, $F'_2 = -1+3x$, $F''_{11} = 0$, $F''_{12} = 3$, and $F''_{22} = 0$. So $y' = -F'_1/F'_2 = -(1+3y)/(-1+3x)$. Moreover, using Eq. (12.3.4),

$$y'' = -\frac{1}{(F'_2)^3}\left[F''_{11}(F'_2)^2 - 2F''_{12}F'_1F'_2 + F''_{22}(F'_1)^2\right] = \frac{6(1+3y)(-1+3x)}{(-1+3x)^3} = \frac{6(1+3y)}{(-1+3x)^2}.$$

(c) Let $F(x,y) = y^5 - x^6$. Then $F'_1 = -6x^5$, $F'_2 = 5y^4$, $F''_{11} = -30x^4$, $F''_{12} = 0$, $F''_{22} = 20y^3$, so $y' = -F'_1/F'_2 = -(-6x^5/5y^4) = 6x^5/5y^4$. Moreover, using Eq. (12.3.4),

$$y'' = -\frac{1}{(5y^4)^3}\left[(-30x^4)(5y^4)^2 + 20y^3(-6x^5)^2\right] = \frac{6x^4}{y^4} - \frac{144x^{10}}{25y^9}.$$

3. (a) With $F(x,y) = 2x^2+xy+y^2$, one has $y' = -F'_1/F'_2 = -(4x+y)/(x+2y) = -4$ at $(2,0)$. Moreover, using (12.3.4) gives $y'' = -(28x^2+14y^2+14xy)/(x+2y)^3$. At $(2,0)$ this gives $y'' = -14$. The point–slope formula for the tangent gives $y = -4x+8$.

(b) $y' = 0$ requires $y = -4x$. Inserting this into the original equation gives a quadratic equation $14x^2 - 8 = 0$ for x. The roots of this equation are $x = \pm 2\sqrt{7}/7$. Along with the corresponding values of y, this gives the two points indicated in the text.

12.4 More general cases

3. (a) Here, Eq. $(*)$ is $P/2\sqrt{L^*} = w$. Solve for L^*. The rest is routine.

(b) The first-order condition is now

$$Pf'(L^*) - C'_L(L^*,w) = 0 \tag{$*$}$$

To find the partial derivatives of L^*, we will differentiate $(*)$ partially w.r.t. P and w. First, we find the partial derivative of $Pf'(L^*)$ w.r.t. P using the product rule. The result is $f'(L^*) + Pf''(L^*)(\partial L^*/\partial P)$. Then the partial derivative of $C'_L(L^*,w)$ w.r.t. P is $C''_{LL}(L^*,w)(\partial L^*/\partial P)$. So differentiating $(*)$ w.r.t. P yields

$$f'(L^*) + Pf''(L^*)\frac{\partial L^*}{\partial P} - C''_{LL}(L^*,w)\frac{\partial L^*}{\partial P} = 0$$

Solving for $\partial L^*/\partial P$ gives the answer.

Second, differentiating $(*)$ w.r.t. w yields

$$[Pf''(L^*) - C''_{LL}(L^*, w)]\frac{\partial L^*}{\partial w} - C''_{Lw}(L^*, w) = 0.$$

Solving for $\partial L^*/\partial w$ gives the answer.

6. $F'_1(x, y) = e^{y-3} + y^2$ and $F'_2(x, y) = xe^{y-3} + 2xy - 2$. Hence, the slope of the tangent to the level curve $F(x, y) = 4$ at the point $(1, 3)$ is $y' = -F'_1(1, 3)/F'_2(1, 3) = -10/5 = -2$. Then use the point–slope formula.

7. Taking the logarithm of both sides, we get $(1 + c \ln y) \ln y = \ln A + \alpha \ln K + \beta \ln L$.
   Differentiating partially with respect to K gives $\dfrac{c}{y}\dfrac{\partial y}{\partial K} \ln y + (1 + c \ln y)\dfrac{1}{y}\dfrac{\partial y}{\partial K} = \dfrac{\alpha}{K}$.
   Solving for $\partial y/\partial K$ yields the given answer. Then $\partial y/\partial L$ is found in the same way.

12.5 Elasticity of substitution

3. When $F(K, L) = AK^aL^b$, the first- and second-order partial derivatives satisfy $F'_K = aF/K$, $F'_L = bF/L$, $F''_{KK} = a(a - 1)F/K^2$, $F''_{KL} = abF/KL$, and $F''_{LL} = b(b - 1)F/L^2$. But then the numerator of the expression for σ_{yx} is

$$-F'_K F'_L(KF'_K + LF'_L) = -(aF/K)(bF/L)(a + b)F = -ab(a + b)F^3/KL,$$

   whereas the denominator is

$$\begin{aligned} KL &\left[(F'_L)^2F''_{KK} - 2F'_K F'_L F''_{KL} + (F'_K)^2F''_{LL}\right] \\ &= KLF^3[b^2a(a - 1) - 2a^2b^2 + a^2b(b - 1)]/K^2L^2 = -ab(a + b)F^3/KL. \end{aligned}$$

   It follows that $\sigma_{KL} = 1$.

12.6 Homogeneous functions of two variables

3. That f is homogeneous of degree 3 is straightforward. As for the properties,

   (12.6.2): $xf'_1(x, y) + yf'_2(x, y) = x(y^2 + 3x^2) + y(2xy) = 3(x^3 + xy^2) = 3f(x, y)$;

   (12.6.3): It is easy to see that $f'_1(x, y) = y^2 + 3x^2$ and $f'_2(x, y) = 2xy$ are homogeneous of degree 2;

   (12.6.4): $f(x, y) = x^3 + xy^2 = x^3[1 + (y/x)^2] = x^3f(1, y/x) = y^3[(x/y)^3 + x/y] = y^3f(x/y, 1)$;

   (12.6.5): $x^2f''_{11} + 2xyf''_{12} + y^2f''_{22} = x^2(6x) + 2xy(2y) + y^2(2x) = 6x^3 + 6xy^2 = 3 \cdot 2f(x, y)$.

9. Let C and D denote the the numerator and the denominator in the expression for σ_{yx} in Exercise 12.5.3. Because F is homogeneous of degree one, Euler's theorem implies that $C = -F'_1F'_2F$, and (12.6.6) implies that $xF''_{11} = -yF''_{12}$ and $yF''_{22} = -xF''_{21} = -xF''_{12}$. Hence,

$$\begin{aligned} D = xy\left[(F'_2)^2F''_{11} - 2F'_1F'_2F''_{12} + (F'_1)^2F''_{22}\right] &= -F''_{12}\left[y^2(F'_2)^2 + 2xyF'_1F'_2 + x^2(F'_1)^2\right] \\ &= -F''_{12}(xF'_1 + yF'_2)^2 = -F''_{12}F^2, \end{aligned}$$

   using Euler's theorem again. Hence $\sigma_{xy} = C/D = (-F'_1F'_2F)/(-F''_{12}F^2) = F'_1F'_2/FF''_{12}$.

12.7 Homogeneous and homothetic functions

1. Parts (a) and (f) are easy. Here are solutions of the other parts:

   (b) Note that $xg'_x + yg'_y + zg'_z = g(x,y,z) + 2$. This does not equal $kg(x,y,z)$ for any k, so Euler's theorem implies that g is not homogeneous of any degree.

   (c) $h(tx,ty,tz) = \dfrac{\sqrt{tx}+\sqrt{ty}+\sqrt{tz}}{tx+ty+tz} = \dfrac{\sqrt{t}\,(\sqrt{x}+\sqrt{y}+\sqrt{z})}{t(x+y+z)} = t^{-1/2}h(x,y,z)$ for all $t>0$, so h is homogeneous of degree $-1/2$.

   (d) $G(tx,ty) = \sqrt{txty}\ln\dfrac{(tx)^2+(ty)^2}{txty} = t\sqrt{xy}\ln\dfrac{t^2(x^2+y^2)}{t^2xy} = t\,G(x,y)$ for all $t>0$, so G is homogeneous of degree 1.

   (e) $xH'_x + yH'_y = x(1/x)+y(1/y) = 2$. Since 2 does not equal $k(\ln x + \ln y)$ for any constant k, Euler's theorem implies that H is not homogeneous of any degree.

2. (a) We find that

$$
\begin{aligned}
f(tx_1,tx_2,tx_3) &= \frac{(tx_1tx_2tx_3)^2}{(tx_1)^4+(tx_2)^4+(tx_3)^4}\left(\frac{1}{tx_1}+\frac{1}{tx_2}+\frac{1}{tx_3}\right)\\
&= \frac{t^6(x_1x_2x_3)^2}{t^4(x_1^4+x_2^4+x_3^4)}\cdot\frac{1}{t}\cdot\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}\right) = tf(x_1,x_2,x_3),
\end{aligned}
$$

   so f is homogeneous of degree 1.

   (b) We find that x is homogeneous of degree μ because

$$
\begin{aligned}
x(tv_1,tv_2,\dots,tv_n) &= A\left[\delta_1(tv_1)^{-\varrho}+\delta_2(tv_2)^{-\varrho}+\cdots+\delta_n(tv_n)^{-\varrho}\right]^{-\mu/\varrho}\\
&= A\left[t^{-\varrho}(\delta_1v_1^{-\varrho}+\delta_2v_2^{-\varrho}+\cdots+\delta_nv_n^{-\varrho})\right]^{-\mu/\varrho}\\
&= (t^{-\varrho})^{-\mu/\varrho}A\left[\delta_1v_1^{-\varrho}+\delta_2v_2^{-\varrho}+\cdots+\delta_nv_n^{-\varrho}\right]^{-\mu/\varrho}\\
&= t^{\mu}A\left[\delta_1v_1^{-\varrho}+\delta_2v_2^{-\varrho}+\cdots+\delta_nv_n^{-\varrho}\right]^{-\mu/\varrho} = t^{\mu}x(x_1,x_2,x_3)
\end{aligned}
$$

5. (a) We use definition (12.7.6). Suppose $(x_1y_1)^2+1 = (x_2y_2)^2+1$. Then $(x_1y_1)^2 = (x_2y_2)^2$. If $t>0$, then $(tx_1ty_1)^2+1 = (tx_2ty_2)^2+1 \iff t^4(x_1y_1)^2+1 = t^4(x_2y_2)^2+1$. This holds if and only if $t^4(x_1y_1)^2 = t^4(x_2y_2)^2 \iff (x_1y_1)^2 = (x_2y_2)^2$, so f is homothetic.

   (b) From $\dfrac{2(x_1y_1)^2}{(x_1y_1)^2+1} = \dfrac{2(x_2y_2)^2}{(x_2y_2)^2+1}$ we get $2(x_1y_1)^2[(x_2y_2)^2+1] = 2(x_2y_2)^2[(x_1y_1)^2+1]$ and so $(x_1y_1)^2 = (x_2y_2)^2$. For all $t>0$ one has

$$
\begin{aligned}
\frac{2(tx_1ty_1)^2}{(tx_1ty_1)^2+1} = \frac{2(tx_2ty_2)^2}{(tx_2ty_2)^2+1} &\iff \frac{2t^4(x_1y_1)^2}{t^4(x_1y_1)^2+1} = \frac{2t^4(x_2y_2)^2}{t^4(x_2y_2)^2+1}\\
&\iff \frac{(x_1y_1)^2}{t^4(x_1y_1)^2+1} = \frac{(x_2y_2)^2}{t^4(x_2y_2)^2+1}
\end{aligned}
$$

   Now for all $a,b>0$ notice that $\dfrac{a}{t^4a+1} = \dfrac{b}{t^4b+1} \iff abt^4+a = abt^4+b \iff a=b$.

So we have proved that for all $t > 0$ one has

$$\frac{2(tx_1ty_1)^2}{(tx_1ty_1)^2+1} = \frac{2(tx_2ty_2)^2}{(tx_2ty_2)^2+1} \iff (x_1y_1)^2 = (x_2y_2)^2.$$

It follows that f is homothetic.

(c) $f(1,0) = 1 = f(0,1)$, but $f(2,0) = 4 \neq 8 = f(0,2)$. This is enough to show that f is not homothetic.

(d) $g(x,y) = x^2y$ is homogeneous of degree 3 and $u \to e^u$ is strictly increasing, so f is homothetic according to (12.7.7).

7. Define $\Delta = \ln C(t\mathbf{w}, y) - \ln C(\mathbf{w}, y)$. It suffices to prove that $\Delta = \ln t$, because then $C(t\mathbf{w}, y)/C(\mathbf{w}, y) = e^{\Delta} = t$. We find that

$$\Delta = \sum_{i=1}^{n} a_i[\ln(tw_i) - \ln w_i] + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}[\ln(tw_i)\ln(tw_j) - \ln w_i \ln w_j] + \ln y \sum_{i=1}^{n} b_i[\ln(tw_i) - \ln w_i].$$

Since $\ln(tw_i) - \ln w_i = \ln t + \ln w_i - \ln w_i = \ln t$
and $\ln(tw_i)\ln(tw_j) - \ln w_i \ln w_j = (\ln t)^2 + \ln t \ln w_i + \ln t \ln w_j$, this reduces to

$$\Delta = \ln t \sum_{i=1}^{n} a_i + \frac{1}{2}(\ln t)^2 \sum_{i,j=1}^{n} a_{ij} + \frac{1}{2}\ln t \sum_{j=1}^{n} \ln w_i \sum_{i=1}^{n} a_{ij} + \frac{1}{2}\ln t \sum_{i=1}^{n} \ln w_j \sum_{j=1}^{n} a_{ij} + \ln y \ln t \sum_{i=1}^{n} b_i.$$

The restrictions on the parameters a_i, a_{ij}, and b_i imply that $\Delta = \ln t + 0 + 0 + 0 + 0 = \ln t$.

12.8 Linear approximations

7. We use formula (12.8.3).

(a) Here, $\partial z/\partial x = 2x$ and $\partial z/\partial y = 2y$. At the point $(1,2,5)$ we get $\partial z/\partial y = 2$ and $\partial z/\partial x = 4$, so the tangent plane has the equation $z - 5 = 2(x-1) + 4(y-2)$ or $z = 2x + 4y - 5$.

(b) From $z = (y - x^2)(y - 2x^2) = y^2 - 3x^2y + 2x^4$ we get $\partial z/\partial x = -6xy + 8x^3$ and $\partial z/\partial y = 2y - 3x^2$. Thus, at the point $(1,3,2)$ we have $\partial z/\partial x = -10$ and $\partial z/\partial y = 3$. The tangent plane is given by the equation $z - 2 = -10(x-1) + 3(y-3)$ or $z = -10x + 3y + 3$.

8. $g(0) = f(\mathbf{x}^0)$, $g(1) = f(\mathbf{x})$. Using formula (12.2.3), one derives

$$g'(t) = f_1'(\mathbf{x}^0 + t(\mathbf{x} - \mathbf{x}^0))(x_1 - x_1^0) + \cdots + f_n'(\mathbf{x}^0 + t(\mathbf{x} - \mathbf{x}^0))(x_n - x_n^0)$$

Putting $t = 0$ gives $g'(0) = f_1'(\mathbf{x}^0)(x_1 - x_1^0) + \cdots + f_n'(\mathbf{x}^0)(x_n - x_n^0)$, and the conclusion follows.

12.9 Differentials

4. $T(x,y,z) = [x^2 + y^2 + z^2]^{1/2} = u^{1/2}$, where $u = x^2 + y^2 + z^2$. Then $\mathrm{d}T = \frac{1}{2}u^{-1/2}\,\mathrm{d}u = u^{-1/2}(x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z)$. For $x = 2$, $y = 3$, and $z = 6$, we have $u = 49$, $T = 7$ and $\mathrm{d}T = \frac{1}{7}(x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z) = \frac{1}{7}(2\,\mathrm{d}x + 3\,\mathrm{d}y + 6\,\mathrm{d}z)$. Thus,

$$T(2 + 0.01, 3 - 0.01, 6 + 0.02) \approx T(2,3,6) + \frac{1}{7}[2 \cdot 0.01 + 3(-0.01) + 6 \cdot 0.02]$$

$$= 7 + \frac{1}{7} \cdot 0.11 \approx 7.015714$$

A calculator gives the better approximation $T(2.01, 2.99, 6.02) \approx \sqrt{49.2206} \approx 7.015739$.

12.11 Differentiating systems of equations

3. Since we are asked to find the partials of y_1 and y_2 w.r.t. x_1 only, we might as well differentiate the system partially w.r.t. x_1 only to obtain the two simultaneous equations:

$$3 - \frac{\partial y_1}{\partial x_1} - 9y_2^2 \frac{\partial y_2}{\partial x_1} = 0 \quad \text{and} \quad 3x_1^2 + 6y_1^2 \frac{\partial y_1}{\partial x_1} - \frac{\partial y_2}{\partial x_1} = 0$$

Solving these for the partials gives the answers in the book.

An alternative method, in particular if one needs all the partials, is to use total differentiation:

$$3\,\mathrm{d}x_1 + 2x_2\,\mathrm{d}x_2 - \mathrm{d}y_1 - 9y_2^2\,\mathrm{d}y_2 = 0 \quad \text{and} \quad 3x_1^2\,\mathrm{d}x_1 - 2\,\mathrm{d}x_2 + 6y_1^2\,\mathrm{d}y_1 - \mathrm{d}y_2 = 0$$

Letting $\mathrm{d}x_2 = 0$ and solving for $\mathrm{d}y_1$ and $\mathrm{d}y_2$ leads to $\mathrm{d}y_1 = A\,\mathrm{d}x_1$ and $\mathrm{d}y_2 = B\,\mathrm{d}x_1$, where $A = \partial y_1/\partial x_1$ and $B = \partial y_2/\partial x_1$.

4. Differentiating with respect to M, while remembering that Y and r are functions of the independent variables a and M, one obtains $I'(r)r'_M = S'(Y)Y'_M$ and $aY'_M + L'(r)r'_M = 1$. Writing this as a linear equation system in standard form, we get

$$-S'(Y)\,Y'_M + I'(r)\,r'_M = 0 \quad \text{and} \quad a\,Y'_M + L'(r)\,r'_M = 1$$

Using either ordinary elimination or formula (2.4.2) gives the solution

$$Y'_M = \frac{I'(r)}{S'(Y)L'(r) + aI'(r)} \quad \text{and} \quad r'_M = \frac{S'(Y)}{S'(Y)L'(r) + aI'(r)}$$

Review exercises for Chapter 12

4. $X = Ng(u)$ where $u = \varphi(N)/N$. Then $\mathrm{d}u/\mathrm{d}N = [\varphi'(N)N - \varphi(N)]/N^2 = (1/N)(\varphi'(N) - u)$. Using the product rule and the chain rule for differentiation, one obtains

$$\frac{\mathrm{d}X}{\mathrm{d}N} = g(u) + Ng'(u)\frac{\mathrm{d}u}{\mathrm{d}N} = g(u) + g'(u)(\varphi'(N) - u).$$

Differentiating the right-hand side w.r.t. N gives

$$\begin{aligned}\frac{\mathrm{d}^2X}{\mathrm{d}N^2} &= g'(u)\frac{\mathrm{d}u}{\mathrm{d}N} + g''(u)\frac{\mathrm{d}u}{\mathrm{d}N}\left[\varphi'(N) - u\right] + g'(u)\left[\varphi''(N) - \frac{\mathrm{d}u}{\mathrm{d}N}\right] \\ &= \frac{1}{N}g''\left(\frac{\varphi(N)}{N}\right)\left[\varphi'(N) - \frac{\varphi(N)}{N}\right]^2 + g'\left(\frac{\varphi(N)}{N}\right)\varphi''(N)\end{aligned}$$

11. Taking the elasticity of each side of the equation gives $\mathrm{El}_x(y^2e^xe^{1/y}) = \mathrm{El}_x\,y^2 + \mathrm{El}_x\,e^x + \mathrm{El}_x\,e^{1/y} = 0$. Here $\mathrm{El}_x\,y^2 = 2\,\mathrm{El}_x\,y$ and $\mathrm{El}_x\,e^x = x$. Moreover, $\mathrm{El}_x\,e^{1/y} = \mathrm{El}_x\,e^u$, where $u = 1/y$, so $\mathrm{El}_x\,e^u = u\,\mathrm{El}_x(1/y) = (1/y)(\mathrm{El}_x\,1 - \mathrm{El}_x\,y) = -(1/y)\,\mathrm{El}_x\,y$. All in all, $2\,\mathrm{El}_x\,y + x - (1/y)\,\mathrm{El}_x\,y = 0$, so $\mathrm{El}_x\,y = xy/(1-2y)$.[22]

[22] We used the rules for elasticities in Exercise 7.7.9. If you are not comfortable with these rules, you can find y' by implicit differentiation and then use the formula $\mathrm{El}_x\,y = (x/y)y'$.

16. (a) Differentiating, then gathering all terms in $\mathrm{d}p$ and $\mathrm{d}L$ on the left-hand side, one has

$$\text{(i) } F'(L)\,\mathrm{d}p + pF''(L)\,\mathrm{d}L = \mathrm{d}w \qquad \text{(ii) } F(L)\,\mathrm{d}p + (pF'(L) - w)\,\mathrm{d}L = L\,\mathrm{d}w + \mathrm{d}B$$

Since we know that $pF'(L) = w$, (ii) implies that $\mathrm{d}p = (L\mathrm{d}w + \mathrm{d}B)/F(L)$. Substituting into (i) and solving for $\mathrm{d}L$, we obtain $\mathrm{d}L = [(F(L) - LF'(L))\mathrm{d}w - F'(L)\mathrm{d}B]/pF(L)F''(L)$. It follows that

$$\frac{\partial p}{\partial w} = \frac{L}{F(L)}, \quad \frac{\partial p}{\partial B} = \frac{1}{F(L)}, \quad \frac{\partial L}{\partial w} = \frac{F(L) - LF'(L)}{pF(L)F''(L)}, \quad \frac{\partial L}{\partial B} = -\frac{F'(L)}{pF(L)F''(L)}$$

(b) Because all variables including p are positive, whereas $F'(L) > 0$ and $F''(L) < 0$, it is clear that $\partial p/\partial w$, $\partial p/\partial B$, and $\partial L/\partial B$ are all positive.

The sign of $\partial L/\partial w$ is the opposite of the sign of $F(L) - LF'(L)$. From the equations in the model, we get $F'(L) = w/p$ and $F(L) = (wL + B)/p$, so $F(L) - LF'(L) = B/p > 0$. Therefore $\partial L/\partial w < 0$.

19. (a) The first-order condition for a maximum is $P'(t) = V'(t)e^{-rt} - rV(t)e^{-rt} - me^{-rt} = 0$. Cancelling e^{-rt}, we see that t^* can only maximize the present value provided $(*)$ is satisfied. The equation says that the marginal increase $V'(t^*)$ in market value per unit of time from keeping the car a little longer must equal the sum of the interest cost $rV(t^*)$ per unit time from waiting to receive the sales revenue, plus the maintenance cost m per unit of time.

(b) Differentiating $P(t)$ again gives

$$P''(t) = V''(t)e^{-rt} - rV'(t)e^{-rt} - rV'(t)e^{-rt} + r^2V(t)e^{-rt} + rme^{-rt}.$$

Gathering terms, we have

$$P''(t) = [V''(t) - rV'(t)]e^{-rt} + [V'(t) - rV(t) - m](-re^{-rt}).$$

At the critical point t^* the last square bracket is 0, so the condition $P''(t^*) < 0$ reduces to $D = V''(t^*) - rV'(t^*) < 0$.

(c) Taking the differential of $(*)$ yields $V''(t^*)\,\mathrm{d}t^* = dr\,V(t^*) + r\,V'(t^*)\,\mathrm{d}t^* = dm$. Hence

$$\frac{\partial t^*}{\partial r} = \frac{V(t^*)}{V''(t^*) - rV'(t^*)} = \frac{V(t^*)}{D} \quad \text{and} \quad \frac{\partial t^*}{\partial m} = \frac{1}{V''(t^*) - rV'(t^*)} = \frac{1}{D}$$

Assuming that $V(t^*) > 0$ (otherwise it would be better to scrap the car immediately), both partial derivatives are negative. A small increase in either the interest rate or the maintenance cost makes the owner want to sell the car a bit sooner.

13 Multivariable Optimization

13.2 Two variables: sufficient conditions

3. Solving the budget equation to express x as a function of y and z yields $x = 108 - 3y - 4z$. Then utility as a function of y and z is $U = (108 - 3y - 4z)yz$. Necessary first-order conditions

for a maximum are $U'_y = 108z - 6yz - 4z^2 = 0$ and $U'_z = 108y - 3y^2 - 8yz = 0$. Because y and z are assumed to be positive, these two equations reduce to $6y + 4z = 108$ and $3y + 8z = 108$, with solution $y = 12$ and $z = 9$.[23]

13.3 Local extreme points

3. (a) The first-order partial derivatives of f are $f'_1 = (2x - ay)e^y$, $f'_2 = x(x - ay - a)e^y$. The second-order partial derivatives are

$$f''_{11} = 2e^y, \quad f''_{12} = (2x - ay - a)e^y, \quad f''_{22} = x(x - ay - 2a)e^y.$$

The critical points where $f'_1 = 0$ and $f'_2 = 0$ are the solutions of the two-equation system:

$$\text{(i) } 2x - ay = 0; \quad \text{(ii) } x(x - ay - a) = 0$$

If $x = 0$, then (i) gives $y = 0$, because $a \neq 0$.
If $x \neq 0$, then (ii) gives $x = ay + a$, whereas (i) gives $x = \frac{1}{2}ay$. Hence $x = -a$ and $y = -2$.
So there are two critical points, $(0, 0)$ and $(-a, -2)$.
To determine the nature of each critical point (x_0, y_0), we use the second-derivative test, with $A = f''_{11}(x_0, y_0)$, $B = f''_{12}(x_0, y_0)$, and $C = f''_{22}(x_0, y_0)$. The test gives:

Point	A	B	C	$AC - B^2$	Result
$(0, 0)$	2	$-a$	0	$-a^2$	Saddle point
$(-a, -2)$	$2e^{-2}$	$-ae^{-2}$	a^2e^{-2}	a^2e^{-4}	Local minimum

(b) The critical point with $x^* \neq 0$ is $(x^*, y^*) = (-a, -2)$.
At this point one has $f^*(a) = f(-a, -2) = -a^2e^{-2}$ and $\mathrm{d}f^*(a)/\mathrm{d}a = -2ae^{-2}$.
On the other hand, for the function $\hat{f}(x, y, a) = (x^2 - axy)e^y$,
one has $\hat{f}'_3(x, y, a) = -xye^y$ and $\hat{f}'_3(x^*, y^*, a) = -x^*y^*e^{y^*} = -2ae^{-2}$.
Thus the equation $\hat{f}'_3(x^*, y^*, a) = \mathrm{d}f^*(a)/\mathrm{d}a$ is true.
Of course, this is also what the envelope theorem tells us — see Eq. (13.7.2).

4. (a) $V'_t(t, x) = f'_t(t, x)e^{-rt} - rf(t, x)e^{-rt} = 0$ and $V'_x(t, x) = f'_x(t, x)e^{-rt} - 1 = 0$, so at the optimum one has $f'_t(t^*, x^*) = rf(t^*, x^*)$ and $f'_x(t^*, x^*) = e^{rt^*}$.

(b) See the answer given in the book.

(c) $V(t, x) = g(t)h(x)e^{-rt} - x$, so $V'_t = h(x)(g'(t) - rg(t))e^{-rt}$, $V'_x = g(t)h'(x)e^{-rt} - 1$. Moreover, $V''_{tt} = h(x)(g''(t) - 2rg'(t) + r^2g(t))e^{-rt}$, $V''_{tx} = h'(x)(g'(t) - rg(t))e^{-rt}$, and $V''_{xx} = g(t)h''(x)e^{-rt}$. Because the first-order condition $g'(t^*) = rg(t^*)$ is satisfied at (t^*, x^*), there one has $V''_{tx} = 0$, as well as $V''_{xx} < 0$ provided that $h''(x^*) < 0$, and $V''_{tt} = h(x^*)[g''(t^*) - r^2g(t^*)]e^{-rt^*} < 0$ provided that $g''(t^*) < r^2g(t^*)$. When both stated conditions are satisfied, one also has $V''_{xx}V''_{tt} - (V''_{xt})^2 > 0$. These inequalities are sufficient to ensure that (t^*, x^*) is a local maximum point.

(d) The first-order conditions in (b) reduce to $e^{\sqrt{t^*}}/2\sqrt{t^*} = re^{\sqrt{t^*}}$, so $t^* = 1/4r^2$, and also $1/(x^* + 1) = e^{1/4r}/e^{1/2r}$, or $x^* = e^{1/4r} - 1$. We check that the two conditions in (c) are

[23] Theorem 13.2.1 cannot be used directly to prove optimality. However, it can be applied to the equivalent problem of maximizing $\ln U$. See Theorem 13.6.3.

satisfied. Obviously, $h''(x^*) = -(1+x^*)^{-2} < 0$. Moreover,

$$g''(t^*) = \frac{1}{4t^*\sqrt{t^*}} e^{\sqrt{t^*}}(\sqrt{t^*} - 1) = r^2(1-2r)e^{1/2r},$$

whereas $r^2 g(t^*) = r^2 e^{\sqrt{t^*}} = r^2 e^{1/2r}$. Hence $g''(t^*) < r^2 g(t^*)$ provided that $r^2(1-2r) < r^2$, which is true for all $r > 0$.

6. (a) We need to have $1 + x^2 y > 0$. When $x = 0$, one has $f(0, y) = 0$. For $x \neq 0$, one has $1 + x^2 y > 0 \iff y > -1/x^2$. The figure in the book shows part of the graph of f. Note that $f = 0$ on the x-axis and on the y-axis.

   (b) See the answer given in the book.

   (c) Here, $f''_{11}(x, y) = \dfrac{2y - 2x^2y^2}{(1+x^2y)^2}$, $f''_{12}(x, y) = \dfrac{2x}{(1+x^2y)^2}$, and $f''_{22}(x, y) = \dfrac{-x^4}{(1+x^2y)^2}$.
   The second-order derivatives at all points of the form $(0, b)$ are $f''_{11}(0, b) = 2b$, $f''_{12}(0, b) = 0$, and $f''_{22}(0, b) = 0$. Hence $AC - B^2 = 0$ at all the critical points, so the second-derivative test tells us nothing about these points.

   (d) See the answer given in the book.

13.4 Linear models with quadratic objectives

2. (a) See the answer given in the book.

   (b) The new profit function is $\hat{\pi} = -bp^2 - dp^2 + (a + \beta b)p + (c + \beta d)p - \alpha - \beta(a + c)$.
   The price that maximizes profits is easily seen to be $\hat{p} = [a + c + \beta(b + d)]/2(b + d)$.

   (c) In the case $\beta = 0$, the answers in part (a) simplify to $p^* = \frac{1}{2}a/b$ and $q^* = \frac{1}{2}c/d$, with maximized profit $\pi(p^*, q^*) = \frac{1}{4}a^2/b + \frac{1}{4}c^2/d - \alpha$. But when price discrimination is prohibited, the answer in part (b) becomes $\hat{p} = (a + c)/2(b + d)$, with maximized profit $\hat{\pi}(\hat{p}) = (a + c)^2/4(b + d) - \alpha$. The firm's loss of profit is $\pi(p^*, q^*) - \hat{\pi}(\hat{p}) = \dfrac{(ad - bc)^2}{4bd(b + d)} \geq 0$.
   Note that this loss is 0 if and only if $ad = bc$, in which case $p^* = q^*$, so the firm wants to charge the same price in each market anyway.

4. (a) The four data points are $(x_0, y_0) = (0, 11.29)$, $(x_1, y_1) = (1, 11.40)$, $(x_2, y_2) = (2, 11.49)$, and $(x_3, y_3) = (3, 11.61)$, where x_0 corresponds to 1970, etc.[24] Using the method of least squares set out in Example 4, we find that $\mu_x = \frac{1}{4}(0 + 1 + 2 + 3) = 1.5$, $\mu_y = \frac{1}{4}(11.29 + 11.40 + 11.49 + 11.61) = 11.45$, and $\sigma_{xx} = \frac{1}{4}[(0-1.5)^2 + (1-1.5)^2 + (2-1.5)^2 + (3-1.5)^2] = 1.25$. Moreover, $\sigma_{xy} = \frac{1}{4}[(-1.5)(11.29 - 11.45) + (-0.5)(11.40 - 11.45) + (0.5)(11.49 - 11.45) + (1.5)(11.61 - 11.45)]$, which is equal to 0.13125, so formula (**) implies that $\hat{a} = \sigma_{xy}/\sigma_{xx} = 0.105$ and $\hat{b} = \mu_y - \hat{a}\mu_x \approx 11.45 - 0.105 \cdot 1.5 = 11.29$.

   (b) With $z_0 = \ln 274$, $z_1 = \ln 307$, $z_2 = \ln 436$, and $z_3 = \ln 524$, the four data points are $(x_0, z_0) = (0, 5.61)$, $(x_1, z_1) = (1, 5.73)$, $(x_2, z_2) = (2, 6.08)$, and $(x_3, z_3) = (3, 6.26)$. As before, $\mu_x = 1.5$ and $\sigma_{xx} = 1.25$. Moreover, $\mu_z = \frac{1}{4}(5.61 + 5.73 + 6.08 + 6.26) = 5.92$ and $\sigma_{xz} \approx \frac{1}{4}[(-1.5)(5.61-5.92) + (-0.5)(5.73-5.92) + (0.5)(6.08-5.92) + (1.5)(6.26-5.92)] = 0.2875$. Hence $\hat{c} = \sigma_{xz}/\sigma_{xx} = 0.23$, $\hat{d} = \mu_z - \hat{c}\mu_x = 5.92 - 0.23 \cdot 1.5 = 5.575$.

[24] The numbers y_t are approximate, as are most subsequent results.

(c) If the time trends $\ln(\text{GDP}) = ax + b$ and $\ln(\text{FA}) = cx + d$ had continued, then FA would have grown to equal 1% of GNP by the time x that solves $\ln(FA/GNP) = \ln 0.01$ or $(c - a)x + d - b = \ln 0.01$. Hence $x = (b - d + \ln 0.01)/(c - a)$. Inserting the numerical estimates found in parts (a) and (b) gives

$$x \approx (11.29 - 5.575 - 4.605)/(0.23 - 0.105) = 1.11/0.125 = 8.88.$$

The goal would be reached in late 1978.

5. (a) The two firms' combined profit is $px + qy - (5 + x) - (3 + 2y)$, or substituting for x and y, $(p - 1)(29 - 5p + 4q) + (q - 2)(16 + 4p - 6q) - 8$, which simplifies to $26p + 24q - 5p^2 - 6q^2 + 8pq - 69$. This is a concave function of p and q. The first-order conditions are the two equations $26 - 10p + 8q = 0$ and $24 - 12q + 8p = 0$. The unique solution is $p = 9$, $q = 8$, which gives a maximum. The corresponding production levels are $x = 16$ and $y = 4$. Firm A's profit is 123, whereas B's is 21.

(b) Firm A's profit is now $\pi_A(p) = (p - 1)(29 - 5p + 4q) - 5 = 34p - 5p^2 + 4pq - 4q - 34$, with q fixed. This quadratic polynomial is maximized at $p = p_A(q) = \frac{1}{5}(2q + 17)$. Likewise, firm B's profit is now $\pi_B(q) = qy - 3 - 2y = 28q - 6q^2 + 4pq - 8p - 35$, with p fixed. This quadratic polynomial is maximized at $q = q_B(p) = \frac{1}{3}(p + 7)$.

(c) Equilibrium occurs where the price pair (p, q) satisfies the two equations $p = p_A(q) = \frac{1}{5}(2q+17)$ and $q = q_B(p) = \frac{1}{3}(p+7)$ simultaneously. Substituting from the second equation into the first yields $p = \frac{1}{5}\left(2\frac{1}{3}(p + 7) + 17\right)$ or, after clearing fractions, $15p = 2p + 14 + 51$. Hence prices are $p = 5$ and $q = 4$, whereas production levels are $x = 20$, $y = 12$, and profits are 75 for A and 21 for B, respectively.

(d) Starting at $(9, 8)$, first firm A moves to $p_A(8) = 33/5 = 6.6$, then firm B answers by moving to $q_A(6.6) = 13.6/3 \approx 4.53$, then firm A responds by moving to near $p_A(4.53) = 26.06/5 = 5.212$, and so on. After the first horizontal move away from $(9, 8)$, the process keeps switching between moves vertically down from the curve $p = p_A(q)$, and moves horizontally across to the curve $q = q_B(p)$, as shown in Fig. SM13.4.5. These moves never cross either curve, and in the limit the process converges to the equilibrium $(5, 4)$ found in part (c).

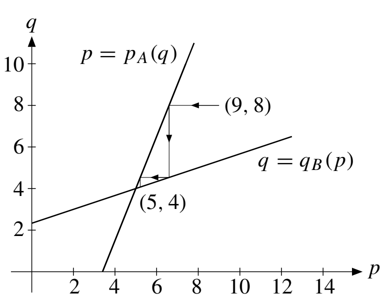

Figure SM13.4.5

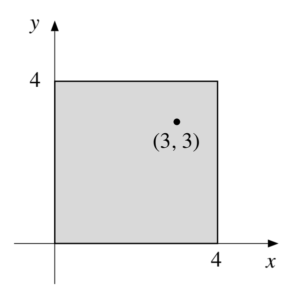

Figure SM13.5.2

13.5 The Extreme Value Theorem

2. (a) The continuous function f is defined on a closed, bounded set S, as seen in Fig. SM13.5.2, so the extreme value theorem ensures that f attains both a maximum and a minimum

over S.

Critical points are where: (i) $f_1'(x,y) = 3x^2 - 9y = 0$, and (ii) $f_2'(x,y) = 3y^2 - 9x = 0$. From (i), $y = \frac{1}{3}x^2$, which inserted into (ii) yields $\frac{1}{3}x(x^3 - 27) = 0$. The only solutions are $x = 0$ and $x = 3$. Thus the only critical point in the interior of S is $(x,y) = (3,3)$. We proceed by examining the behaviour of $f(x,y)$ along the boundary of S — i.e., along the four edges of S.

(i) $y = 0$, $x \in [0,4]$. Then $f(x,0) = x^3 + 27$, which has minimum at $x = 0$, and maximum at $x = 4$.

(ii) $x = 4$, $y \in [0,4]$. Then $f(4,y) = y^3 - 36y + 91$. The function $g(y) = y^3 - 36y + 91$, $y \in [0,4]$ has $g'(y) = 3y^2 - 36 = 0$ at $y = \sqrt{12}$. Possible extreme points along (II) are $(4,0)$, $(4,\sqrt{12})$, and $(4,4)$.

(iii) $y = 4$, $x \in [0,4]$. Then $f(x,4) = x^3 - 36x + 91$, and as in (II) we see that possible extreme points are $(0,4)$, $(\sqrt{12},4)$, and $(4,4)$.

(iv) $x = 0$, $y \in [0,4]$. As in case (I) there are possible extreme points at $(0,0)$ and $(0,4)$.

This results in six candidates, where the function values are $f(3,3) = 0$, $f(0,0) = 27$, $f(4,0) = f(0,4) = 91$, $f(4,\sqrt{12}) = f(\sqrt{12},4) = 91 - 24\sqrt{12} \approx 7.86$, $f(0,0) = 27$. The conclusion follows.

(b) The constraint set $S = \{(x,y) : x^2 + y^2 \le 1\}$ consists of points that lie on or inside a circle around the origin with radius 1. This is a closed and bounded set, and $f(x,y) = x^2 + 2y^2 - x$ is continuous. Therefore the extreme value theorem ensures that f attains both a maximum and a minimum over S.

Critical points for f occur where $f_x'(x,y) = 2x - 1 = 0$ and $f_y'(x,y) = 4y = 0$. So the only critical point for f is $(x_1,y_1) = (1/2,0)$, which is an interior point of S.

An extreme point that does not lie in the interior of S must lie on the boundary of S, that is, on the circle $x^2 + y^2 = 1$. *Along this circle* we have $y^2 = 1 - x^2$, and therefore

$$f(x,y) = x^2 + 2y^2 - x = x^2 + 2(1 - x^2) - x = 2 - x - x^2$$

where x runs through the interval $[-1,1]$.[25] The function $g(x) = 2 - x - x^2$ has one critical point in the interior of $[-1,1]$, namely $x = -1/2$, so any extreme values of $g(x)$ must occur either for this value of x or at one the endpoints ± 1 of the interval $[-1,1]$. Any extreme points for $f(x,y)$ on the boundary of S must therefore be among the points

$$(x_2,y_2) = (-\frac{1}{2}, \frac{1}{2}\sqrt{3}),\ \ (x_3,y_3) = (-\frac{1}{2}, -\frac{1}{2}\sqrt{3}),\ \ (x_4,y_4) = (1,0),\ \ (x_5,y_5) = (-1,0)$$

Now, $f(\frac{1}{2},0) = -\frac{1}{4}$, $f(-\frac{1}{2}, \pm\frac{1}{2}\sqrt{3}) = \frac{9}{4}$, $f(1,0) = 0$, and $f(-1,0) = 2$. The conclusion in the text follows.

3. The set S is shown in Fig. A13.5.3 in the book. It is clearly closed and bounded, so the continuous function f has a maximum in S. The critical points are where $\partial f/\partial x = 9 - 12(x+y) = 0$ and $\partial f/\partial y = 8 - 12(x+y) = 0$. But $12(x+y) = 9$ and $12(x+y) = 8$ give a contradiction. Hence, there are no critical points at all. The maximum value of f must therefore occur on the boundary, which consists of five parts. Either the maximum value

[25] It is a common error to overlook this restriction.

occurs at one of the five corners or "extreme points" of the boundary, or else at an interior point of one of five straight "edges." The function values at the five corners are $f(0,0) = 0$, $f(5,0) = -105$, $f(5,3) = -315$, $f(4,3) = -234$, and $f(0,1) = 2$.

We proceed to examine the behaviour of f at interior points along each of the five edges.

(i) $y = 0$, $x \in (0,5)$. The behaviour of f is determined by the function $g_1(x) = f(x,0) = 9x - 6x^2$ for $x \in (0,5)$. If this function of one variable has a maximum in $(0,5)$, it must occur at a critical point where $g_1'(x) = 9 - 12x = 0$, and so at $x = 3/4$. We find that $g_1(3/4) = f(3/4,0) = 27/8$.

(ii) $x = 5$, $y \in (0,3)$. Define $g_2(y) = f(5,y) = 45 + 8y - 6(5+y)^2$ for $y \in (0,3)$. Here $g_2'(y) = -52 - 12y$, which is negative throughout $(0,3)$, so there are no critical points on this edge.

(iii) $y = 3$, $x \in (4,5)$. Define $g_3(x) = f(x,3) = 9x + 24 - 6(x+3)^2$ for $x \in (4,5)$. Here $g_3'(x) = -27 - 12x$, which is negative throughout $(4,5)$, so there are no critical points on this edge either.

(iv) $-x+2y = 2$, or $y = x/2+1$, with $x \in (0,4)$. Define the function $g_4(x) = f(x, x/2+1) = -27x^2/2 - 5x + 2$ for $x \in (0,4)$. Here $g_4'(x) = -27x - 5$, which is negative in $(0,4)$, so there are no critical points here.

(v) $x = 0$, $y \in (0,1)$. Define $g_5(y) = f(0,y) = 8y - 6y^2$. Then $g_5'(y) = 8 - 12y = 0$ at $y = 2/3$, with $g_5(2/3) = f(0,2/3) = 8/3$.

After comparing the values of f at the five corners of the boundary and at the two points found on the edges labelled (I) and (V) respectively, we conclude that the maximum value of f is $27/8$, which is achieved at $(3/4, 0)$.

5. (a) First, $f_1'(x,y) = e^{-x}(1-x)(y-4)y = 0$ when $x = 1$, or $y = 0$, or $y = 4$.
Second, $f_2'(x,y) = 2xe^{-x}(y-2) = 0$ when $x = 0$ or $y = 2$.
It follows that the critical points are $(1,2)$, $(0,0)$ and $(0,4)$.
Moreover, $f_{11}''(x,y) = e^{-x}(x-2)(y^2-4y)$, $f_{12}''(x,y) = e^{-x}(1-x)(2y-4)$, $f_{22}'' = 2xe^{-x}$.
The critical points can be classified as follows:

(x,y)	A	B	C	$AC - B^2$	Type of point
(1, 2)	$4e^{-1}$	0	$2e^{-1}$	$8e^{-2}$	Local minimum
(0, 0)	0	-4	0	-16	Saddle point
(0, 4)	0	4	0	-16	Saddle point

(b) We show that the range of f is unbounded both above and below. Indeed, there is no global maximum because $f(1,y) = e^{-1}(y^2 - 4y) \to \infty$ as $y \to \infty$. Nor is there any global minimum because $f(-1,y) = -e(y^2 - 4y) \to -\infty$ as $y \to \infty$.

(c) The set S is obviously bounded. The boundary of S consists of the four edges of the rectangle, and all points on these line segments belong to S. Hence S is closed. Since f is continuous, the extreme value theorem tells us that f has global maximum and minimum points in S. These global extreme points must be either critical points if f in the interior of S, or points on the boundary of S. The only critical point of f in the interior of S is $(1,2)$. The function value at this point is $f(1,2) = -4e^{-1} \approx -1.4715$.
The four edges are most easily investigated separately:

(i) Along (I), $y = 0$ and $f(x, y) = f(x, 0)$ is identically 0.

(ii) Along (II), $x = 5$ and $f(x, y) = 5e^{-5}(y^2 - 4y)$ for $y \in [0, 4]$. This has its least value for $y = 2$ and its greatest value for $y = 0$ and for $y = 4$. The function values are $f(5, 2) = -20e^{-5} \approx -0.1348$ and $f(5, 0) = f(5, 4) = 0$.

(iii) Along edge (III), $y = 4$ and $f(x, y) = f(x, 4) = 0$.

(iv) Finally, along (IV), $x = 0$ and $f(x, y) = f(0, y) = 0$.

Collecting all these results, we see that f attains its least value (on S) at the point $(1, 2)$ and its greatest value (namely 0) at all points of the line segments (I), (III) and (IV).

(d) $y' = -\dfrac{f_1'(x, y)}{f_2'(x, y)} = -\dfrac{e^{-x}(1 - x)(y - 4)y}{2xe^{-x}(y - 2)} = \dfrac{(x - 1)(y - 4)y}{2x(y - 2)} = 0$ when $x = 1$.

13.6 The general case

4. To calculate f_x' is routine. The derivative of $\int_y^z e^{t^2}\, dt$ w.r.t. y, keeping z constant, can be found using (9.3.7): it is $-e^{y^2}$. The derivative of $\int_y^z e^{t^2}\, dt$ w.r.t. z, keeping y constant, can be found using (9.3.6): it is e^{z^2}. Thus $f_y' = 2 - e^{y^2}$ and $f_z' = -3 + e^{z^2}$. Since each of the three partials depends only on one variable and is 0 for two different values of that variable, there are eight critical points (as indicated in the book).

13.7 Comparative statics and the Envelope Theorem

3. (a) With profits π as given in the problem, first-order conditions for a maximum are

$$\pi_K' = \frac{2}{3}pK^{-1/3} - r = 0, \quad \pi_L' = \frac{1}{2}pL^{-1/2} - w = 0, \quad \pi_T' = \frac{1}{3}pT^{-2/3} - q = 0.$$

Thus, $K^{-1/3} = 3r/2p$, $L^{-1/2} = 2w/p$, and $T^{-2/3} = 3q/p$. Raising each side of the equation $K^{-1/3} = 3r/2p$ to the power of -3 yields $K = (3r/2p)^{-3} = (2p/3r)^3 = (8/27)p^3r^{-3}$. A similar method can be used to find L and T.

(b) Routine algebra: See the answer given in the book.

5. (a) Differentiating $pF_K'(K^*, L^*) = r$ using the product rule gives

$$F_K'(K^*, L^*)\mathrm{d}p + p\,\mathrm{d}[F_K'(K^*, L^*)] = \mathrm{d}r.$$

Moreover, $\mathrm{d}[F_K'(K^*, L^*)] = F_{KK}''(K^*, L^*)\,\mathrm{d}K^* + F_{KL}''(K^*, L^*)\,\mathrm{d}L^*$.[26]
This explains the first displayed equation, replacing $\mathrm{d}K$ by $\mathrm{d}K^*$ and $\mathrm{d}L$ by $\mathrm{d}L^*$.
The second is derived in the same way.

(b) Rearrange the equation system by moving the differentials of the exogenous prices p, r, and w to the right-hand side, while suppressing the notation indicating that the partials are evaluated at (K^*, L^*):

$$\begin{aligned} pF_{KK}''\,\mathrm{d}K^* + pF_{KL}''\,\mathrm{d}L^* &= \mathrm{d}r - F_K'\,\mathrm{d}p \\ pF_{LK}''\,\mathrm{d}K^* + pF_{LL}''\,\mathrm{d}L^* &= dw - F_L'\,\mathrm{d}p \end{aligned}$$

[26] To see why, note that $\mathrm{d}g(K^*, L^*) = g_K'(K^*, L^*)\,\mathrm{d}K^* + g_L'(K^*, L^*)\,\mathrm{d}L^*$. Then let $g = F_K'$.

Define $\Delta = F''_{KK}F''_{LL} - F''_{KL}F''_{LK} = F''_{KK}F''_{LL} - (F''_{KL})^2$. Using (2.4.2) and cancelling p, we get $\mathrm{d}K^* = \dfrac{-F'_K F''_{LL} + F'_L F''_{KL}}{p\Delta}\,\mathrm{d}p + \dfrac{F''_{LL}}{p\Delta}\,\mathrm{d}r + \dfrac{-F''_{KL}}{p\Delta}\,\mathrm{d}w$.

In the same way $\mathrm{d}L^* = \dfrac{-F'_L F''_{KK} + F'_K F''_{LK}}{p\Delta}\,\mathrm{d}p + \dfrac{-F''_{LK}}{p\Delta}\,\mathrm{d}r + \dfrac{F''_{KK}}{p\Delta}\,\mathrm{d}w$.

We can now read off the required partials.

(c) See the answer given in the book.[27]

6. (a) (i) $R'_1(x_1^*, x_2^*) + \sigma = C'_1(x_1^*, x_2^*)$: marginal revenue plus subsidy equals marginal cost.

(ii) $R'_2(x_1^*, x_2^*) = C'_2(x_1^*, x_2^*) + \tau = 0$: marginal revenue equals marginal cost plus tax.

(b) See the answer given in the book, which introduces the notation

$$D = (R''_{11} - C''_{11})(R''_{22} - C''_{22}) - (R''_{12} - C''_{12})^2.$$

(c) Taking the total differentials of (i) and (ii) yields

$$(R''_{11} - C''_{11})\mathrm{d}x_1^* + (R''_{12} - C''_{12})\mathrm{d}x_2^* = -\mathrm{d}\sigma, \quad (R''_{21} - C''_{21})\mathrm{d}x_1^* + (R''_{22} - C''_{22})\mathrm{d}x_2^* = \mathrm{d}\tau$$

Solving for $\mathrm{d}x_1^*$ and $\mathrm{d}x_2^*$ yields, after rearranging,

$$\mathrm{d}x_1^* = \frac{-(R''_{22} - C''_{22})\mathrm{d}\sigma - (R''_{12} - C''_{12})\mathrm{d}\tau}{D}, \quad \mathrm{d}x_2^* = \frac{(R''_{21} - C''_{21})\mathrm{d}\sigma + (R''_{11} - C''_{11})\mathrm{d}\tau}{D}$$

From the assumptions in the problem and the fact that $D > 0$ from (b), we find that the partial derivatives are

$$\frac{\partial x_1^*}{\partial \sigma} = \frac{-R''_{22} + C''_{22}}{D} > 0, \qquad \frac{\partial x_1^*}{\partial \tau} = \frac{-R''_{12} + C''_{12}}{D} > 0,$$

$$\frac{\partial x_2^*}{\partial \sigma} = \frac{R''_{21} - C''_{21}}{D} < 0, \qquad \frac{\partial x_2^*}{\partial \tau} = \frac{R''_{11} - C''_{11}}{D} < 0,$$

Note that the signs of these partial derivatives accord with economic intuition. For example, if the tax on good 2 increases, then the production of good 1 increases, while the production of good 2 decreases.

(d) Follows from the expressions in (c) because $R''_{12} = R''_{21}$ and $C''_{12} = C''_{21}$.

Review exercises for Chapter 13

2. (a) The profit function is $\pi(Q_1, Q_2) = 120Q_1 + 90Q_2 - 0.1Q_1^2 - 0.1Q_1Q_2 - 0.1Q_2^2$.
First-order conditions for maximal profit are
$\pi'_1(Q_1, Q_2) = 120 - 0.2Q_1 - 0.1Q_2 = 0$ and $\pi'_2(Q_1, Q_2) = 90 - 0.1Q_1 - 0.2Q_2 = 0$.
We find $(Q_1, Q_2) = (500, 200)$.
Moreover, $\pi''_{11}(Q_1, Q_2) = -0.2 \le 0$, $\pi''_{12}(Q_1, Q_2) = -0.1$, and $\pi''_{22}(Q_1, Q_2) = -0.2 \le 0$.
Since also $\pi''_{11}\pi''_{22} - (\pi''_{12})^2 = 0.03 \ge 0$, $(500, 200)$ maximizes profits.

[27] Recall that $F''_{LL} < 0$ follows from $(**)$ in Example 13.3.3.

(b) The profit function is now $\hat{\pi}(Q_1, Q_2) = P_1Q_1 + 90Q_2 - 0.1Q_1^2 - 0.1Q_1Q_2 - 0.1Q_2^2$. First-order conditions for maximal profit become
$\hat{\pi}_1' = P_1 - 0.2Q_1 - 0.1Q_2 = 0$ and $\hat{\pi}_2' = 90 - 0.1Q_1 - 0.2Q_2 = 0$.
In order to induce the choice $Q_1 = 400$,
the first-order conditions imply that $P_1 - 80 - 0.1Q_2 = 0$ and $90 - 40 - 0.2Q_2 = 0$.
It follows that $Q_2 = 250$ and $P_1 = 105$.

4. (a) Critical points are given by:
(i) $f_1'(x, y) = 3x^2 - 2xy = x(3x - 2y) = 0$; and (ii) $f_2'(x, y) = -x^2 + 2y = 0$.
From (i), $x = 0$ or $3x = 2y$. If $x = 0$, then (ii) gives $y = 0$.
If $3x = 2y$, then (ii) gives $3x = x^2$, and so $x = 0$ or $x = 3$.
If $x = 3$, then (ii) gives $y = x^2/2 = 9/2$. So the critical points are $(0, 0)$ and $(3, 9/2)$.

(b) Critical points are given by: (i) $f_1'(x, y) = ye^{4x^2-5xy+y^2}(8x^2 - 5xy + 1) = 0$;
and (ii) $f_2'(x, y) = xe^{4x^2-5xy+y^2}(2y^2 - 5xy + 1) = 0$.
If $y = 0$, then (i) is satisfied and (ii) holds only when $x = 0$.
If $x = 0$, then (ii) is satisfied and (i) holds only if $y = 0$.
Thus, in addition to $(0, 0)$, any other critical point must satisfy both $8x^2 - 5xy + 1 = 0$ and $2y^2 - 5xy + 1 = 0$.
Subtracting the second of these equations from the first yields $8x^2 = 2y^2$, or $y = \pm 2x$.
Inserting $y = -2x$ into $8x^2 - 5xy + 1 = 0$ yields $18x^2 + 1 = 0$, which has no solutions.
Inserting $y = 2x$ into $8x^2 - 5xy + 1 = 0$ yields $-2x^2 + 1 = 0$, so $x = \pm\frac{1}{2}\sqrt{2}$.
We conclude that the critical points are: $(0, 0)$ and $(\frac{1}{2}\sqrt{2}, \sqrt{2}\,)$, $(-\frac{1}{2}\sqrt{2}, -\sqrt{2}\,)$.

(c) Critical points occur where: (i) $f_1'(x, y) = 24xy - 48x = 24x(y - 2) = 0$;
and (ii) $f_2'(x, y) = 12y^2 + 12x^2 - 48y = 12(x^2 + y^2 - 4y) = 0$.
From (i) $x = 0$ or $y = 2$.
If $x = 0$, then (ii) gives $y(y - 4) = 0$, so $y = 0$ or $y = 4$. So $(0, 0)$ and $(0, 4)$ are critical points.
If $y = 2$, then (ii) gives $x^2 - 4 = 0$, so $x = \pm 2$. Hence $(2, 2)$ and $(-2, 2)$ are also critical points.

6. (a) With $\pi = p(K^a + L^b + T^c) - rK - wL - qT$, the first-order conditions for (K^*, L^*, T^*) to maximize π are

$$\pi_K' = pa(K^*)^{a-1} - r = 0, \quad \pi_L' = pb(L^*)^{a-1} - w = 0, \quad \pi_T' = pc(T^*)^{a-1} - q = 0$$

Hence, $K^* = (ap/r)^{1/(1-a)}$, $L^* = (bp/w)^{1/(1-a)}$, $T^* = (cp/q)^{1/(1-a)}$.

(b) $\pi^* = \Gamma +$ terms that do not depend on r, where $\Gamma = p(ap/r)^{a/(1-a)} - r(ap/r)^{1/(1-a)}$. Some algebraic manipulations yield

$$\begin{aligned}\Gamma &= \left(\frac{a}{r}\right)^{a/(1-a)} p^{1/(1-a)} - (ap)^{1/(1-a)} r^{-a/(1-a)} \\ &= \left(a^{a/(1-a)} - a^{1/(1-a)}\right) p^{1/(1-a)} r^{-a/(1-a)} \\ &= (1 - a)a^{a/(1-a)} p^{1/(1-a)} r^{-a/(1-a)}.\end{aligned}$$

Then,

$$\frac{\partial \pi^*}{\partial r} = \frac{\partial \Gamma}{\partial r} = -aa^{a/(1-a)} p^{1/(1-a)} r^{-a/(1-a)} = -a^{1/(1-a)} p^{1/(1-a)} r^{\frac{-1}{1-a}} = -\left(\frac{ap}{r}\right)^{1/(1-a)}.$$

(c) We apply (13.7.2) to this case, where $\pi(K, L, T, p, r, w, q) = pQ - rK - wL - qT$ with $Q = K^a + L^b + T^c$, and $\pi^*(p, r, w, q) = pQ^* - rK^* - wL^* - qT^*$. With the partial derivatives of π evaluated at $(K^*, L^*, T^*, p, r, w, q)$ where output is Q^*, one should have $\partial\pi^*/\partial p = \pi'_p = Q^*$, $\partial\pi^*/\partial r = \pi'_r = -K^*$, $\partial\pi^*/\partial w = \pi'_w = -L^*$, and $\partial\pi^*/\partial w = \pi'_w = -T^*$. From (b), we have the second property. The other three equations can be verified by rather tedious algebra in a similar way.

8. (a) $f'_1(x, y) = 2x - y - 3x^2$, $f'_2(x, y) = -2y - x$, $f''_{11}(x, y) = 2 - 6x$, $f''_{12}(x, y) = -1$, $f''_{22}(x, y) = -2$. Critical points occur where $2x - y - 3x^2 = 0$ and $-2y - x = 0$. The last equation yields $y = -x/2$, which inserted into the first equation gives $\frac{5}{2}x - 3x^2 = 0$. It follows that there are two critical points, $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (5/6, -5/12)$. These points are classified in the following table:

(x, y)	A	B	C	$AC - B^2$	Type of point
$(0, 0)$	2	-1	-2	-5	Saddle point
$(\frac{5}{6}, -\frac{5}{12})$	-3	-1	-2	5	Local maximum

(b) f is concave where $f''_{11} \leq 0$, $f''_{22} \leq 0$, and $f''_{11}f''_{22} - (f''_{12})^2 \geq 0$ — i.e., where $2 - 6x \leq 0$, $-2 \leq 0$, and $(2 - 6x)(-2) - (-1)^2 \geq 0$. These conditions are equivalent to $x \geq 1/3$ and $x \geq 5/12$. Since $5/12 > 1/3$, the function f is concave in the set S consisting of all (x, y) where $x \geq 5/12$.

(c) The critical point $(x_2, y_2) = (5/6, -5/12)$ found in (a) belongs to S. Since f is concave in S, this is a (global) maximum point for f in S, and $f_{\max} = \frac{25}{36} - \frac{25}{144} + \frac{25}{72} - \frac{125}{216} = \frac{125}{432}$.

9. (a) Critical points are where $x - 1 = -ay$ and $a(x - 1) = y^2 - 2a^2y$. Multiplying the first equation by a gives $-a^2y = a(x - 1) = y^2 - 2a^2y$, by the second equation. Hence $a^2y = y^2$, implying that $y = 0$ or $y = a^2$. Since $x = 1 - ay$, the critical points are $(1, 0)$ and $(1 - a^3, a^2)$.[28]

(b) The function value at the critical point in (a) is

$$\tfrac{1}{2}(1 - a^3)^2 - (1 - a^3) + a^3(-a^3) - \tfrac{1}{3}a^6 + a^2 \cdot a^4 = -\tfrac{1}{2} + \tfrac{1}{6}a^6,$$

whose derivative w.r.t. a is a^5. On the other hand, the partial derivative of f w.r.t. a, keeping x and y constant, is $\partial f/\partial a = y(x - 1) + 2ay^2$. Evaluated at $x = 1 - a^3$, $y = a^2$, this partial derivative is also a^5, thus confirming the envelope theorem.

(c) $f''_{11} = 1$, $f''_{22} = -2y + 2a^2$, $f''_{12} = a$, and $f''_{11}f''_{22} - (f''_{12})^2 = a^2 - 2y$. Thus f is convex if and only if $-2y + 2a^2 \geq 0$ and $-2y + a^2 \geq 0$, which is equivalent to $a^2 \geq y$ and $a^2 \geq 2y$. It follows that $f(x, y)$ is convex in that part of the xy-plane where $y \leq \frac{1}{2}a^2$.

14 Constrained Optimization

14.1 The Lagrange Multiplier method

4. (a) With $\mathcal{L}(x, y) = x^2 + y^2 - \lambda(x + 2y - 4)$, the first-order conditions are $\mathcal{L}'_1 = 2x - \lambda = 0$ and $\mathcal{L}'_2 = 2y - 2\lambda = 0$. From these equations we get $2x = y$, which inserted into the constraint gives $x + 4x = 4$. So $x = 4/5$ and $y = 2x = 8/5$, with $\lambda = 2x = 8/5$.

[28] Since you were asked only to show that $(1 - a^3, a^2)$ is a critical point, it would suffice to verify that it makes both partials equal to 0.

(b) The same method as in (a) gives $2x - \lambda = 0$ and $4y - \lambda = 0$, so $x = 2y$. From the constraint we get $x = 8$ and $y = 4$, with $\lambda = 16$.

(c) The first-order conditions imply that $2x + 3y = \lambda = 3x + 2y$, which gives $x = y$. So the solution is $(x, y) = (50, 50)$ with $\lambda = 250$.

9. (a) With $\mathcal{L} = x^a + y - \lambda(px + y - m)$, the first-order conditions for (x^*, y^*) to solve the problem are

$$\mathcal{L}'_x = a(x^*)^{a-1} - \lambda p = 0, \qquad \mathcal{L}'_y = 1 - \lambda = 0.$$

Thus $\lambda = 1$, and $x^* = x^*(p, m) = kp^{-1/(1-a)}$ where $k = a^{1/(1-a)}$.
Then $y^* = y^*(p, m) = m - kp^{-a/(1-a)}$.

(b) $\partial x^*/\partial p = -x^*/p(1-a) < 0$, $\partial x^*/\partial m = 0$, $\partial y^*/\partial p = ax^*/(1-a) > 0$, and $\partial y^*/\partial m = 1$.

(c) The optimal expenditure on good x is $px^*(p, m) = kp^{-a/(1-a)}$, so $\mathrm{El}_p\, px^*(p, m) = -a/(1-a) < 0$. In particular, the expenditure on good x will decrease as its price increases.

(d) We see that $x^* = (1/2p)^2$, $y^* = m - 1/4p$, so $U^*(p, m) = \sqrt{x^*} + y^* = (1/2p) + m - 1/4p = m + 1/4p$, and the required identity is obvious.

10. (a) With $\mathcal{L}(x, y) = 100 - e^{-x} - e^{-y} - \lambda(px + qy - m)$, the first-order conditions $\mathcal{L}'_x = \mathcal{L}'_y = 0$ imply that $e^{-x} = \lambda p$ and $e^{-y} = \lambda q$. Hence, $x = -\ln(\lambda p) = -\ln \lambda - \ln p$ and $y = -\ln \lambda - \ln q$. Inserting these expressions for x and y into the constraint yields $-p(\ln \lambda + \ln p) - q(\ln \lambda + \ln q) = m$ and so $\ln \lambda = -(m + p \ln p + q \ln q)/(p + q)$. Therefore

$$x(p, q, m) = \frac{m + q\ln(q/p)}{p + q} \quad \text{and} \quad y(p, q, m) = \frac{m + p\ln(p/q)}{p + q}.$$

(b) $x(tp, tq, tm) = [tm + tq\ln(tq/tp)]/(tp + tq) = x(p, q, m)$, so x is homogeneous of degree 0. In the same way we see that $y(p, q, m)$ is homogeneous of degree 0.

14.2 Interpreting the Lagrange Multiplier

4. (a) With $\mathcal{L}(x, y) = \sqrt{x} + y - \lambda(x + 4y - 100)$, the first-order conditions for (x^*, y^*) to solve the problem are:

$$\text{(i)}\ \partial\mathcal{L}/\partial x = 1/2\sqrt{x^*} - \lambda = 0, \qquad \text{(ii)}\ \partial\mathcal{L}/\partial y = 1 - 4\lambda = 0.$$

From (ii), $\lambda = 1/4$, which inserted into (i) yields $\sqrt{x^*} = 2$, so $x^* = 4$. Then $y^* = 25 - \frac{1}{4}4 = 24$, and maximal utility is $U^* = \sqrt{x^*} + y^* = 26$.

(b) Denote the new optimal values of x and y by $\hat{x}$ and $\hat{y}$. If 100 is changed to 101, still $\lambda = 1/4$ and $\hat{x} = 4$. The constraint now gives $4 + 4\hat{y} = 101$, so that $\hat{y} = 97/4 = 24.25$, with $\hat{U} = \sqrt{\hat{x}} + \hat{y} = 26.25$. The increase in maximum utility is therefore $\hat{U} - U^* = 0.25 = \lambda$.[29]

(c) The necessary conditions for optimality are now $\partial\mathcal{L}/\partial x = 1/2\sqrt{x^*} - \lambda p = 0$, $\partial\mathcal{L}/\partial y = 1 - \lambda q = 0$. Proceeding in the same way as in (a), we find $\lambda = 1/q$, $\sqrt{x^*} = q/2p$, and so $x^* = q^2/4p^2$, with $y^* = m/q - q/4p$.[30]

[29] In general, the increase in utility is *approximately* equal to the value of the Lagrange multiplier.

[30] Note that $y^* > 0 \iff m > q^2/4p$. Also, note that if we solve the constraint for y, the utility function is $u(x) = \sqrt{x} + (m - px)/q$. We see that $u'(x) = 1/2\sqrt{x} - p/q = 0$ for $x^* = q^2/4p^2$ and $u''(x) = -(1/4)x^{-3/2} < 0$ when $x > 0$. So we have found the maximum.

5. (a) The first-order conditions given in the main text imply that $px^* = pa + \alpha/\lambda$ and $qy^* = qb + \beta/\lambda$. Substituting these into the budget constraint gives $m = px^* + qy^* = pa + qb + (\alpha+\beta)/\lambda = pa + qb + 1/\lambda$, so $1/\lambda = m - (pa+qb)$. The expressions given in $(**)$ are now easily established.[31]

   (b) With U^* as given in the answer provided in the text, differentiating partially while remembering that $\alpha + \beta = 1$ gives

$$\frac{\partial U^*}{\partial m} = \frac{\alpha}{m-(pa+qb)} + \frac{\beta}{m-(pa+qb)} = \frac{1}{m-(pa+qb)} = \lambda > 0.$$

   Moreover, $\dfrac{\partial U^*}{\partial p} = \dfrac{-\alpha a}{m-(pa+qb)} - \dfrac{\alpha}{p} + \dfrac{-\beta a}{m-(pa+qb)} = \dfrac{-a}{m-(pa+qb)} - \dfrac{\alpha}{p} = -a\lambda - \dfrac{\alpha}{p}$,
   whereas $-\dfrac{\partial U^*}{\partial m}x^* = -\lambda\left(a + \dfrac{\alpha}{\lambda p}\right) = -a\lambda - \dfrac{\alpha}{p}$, so $\partial U^*/\partial p = -\partial U^*/\partial m \cdot x^*$.
   The last equality is shown in the same way.

6. A formula for $f(x,T)$ is

$$x\int_0^T \left[-t^3 + (\alpha T^2 + T - 1)t^2 + (T - \alpha T^3)t\right] dt = x\Big|_0^T \left[-\tfrac{1}{4}t^4 + (\alpha T^2 + T - 1)\tfrac{1}{3}t^3 + (T - \alpha T^3)\tfrac{1}{2}t^2\right]$$
$$= -\frac{1}{6}\alpha x T^5 + \frac{1}{12}xT^4 + \frac{1}{6}xT^3 = \frac{1}{12}xT^3(2 + T - 2\alpha T^2)$$

   A similar, but easier, calculation shows that $g(x,T) = \int_0^T (xtT - xt^2)\,dt = \frac{1}{6}xT^3$. The Lagrangian for the producer's problem is

$$\mathcal{L} = \frac{1}{12}xT^3(2 + T - 2\alpha T^2) - \lambda\left(\frac{1}{6}xT^3 - M\right).$$

   The two first-order conditions are

$$\frac{1}{12}T^3(2 + T - 2\alpha T^2) - \frac{1}{6}\lambda T^3 = 0 \quad \text{and} \quad \frac{1}{12}xT^2(6 + 4T - 10\alpha T^2) - \frac{1}{2}\lambda x T^2 = 0.$$

   These equations imply that $\lambda = \frac{1}{2}(2 + T - 2\alpha T^2) = \frac{1}{6}(6 + 4T - 10\alpha T^2)$. It follows that $4\alpha T^2 = T$. One solution is $T = 0$, but this is inconsistent with the constraint $\frac{1}{6}xT^3 = M$. Hence, the solution we are interested is $T = 1/4\alpha$, implying that $\lambda = 1 + 1/16\alpha$. Substituting into the constraint $g(x,T) = M$ determines $x = 6MT^{-3} = 384M\alpha^3$. Because $xT^3 = 6M$, the maximum profit is $f^*(M) = M + M/8\alpha - \alpha M/16\alpha^2 = M + M/16\alpha$, whose derivative w.r.t. M is indeed λ.[32]

14.3 Multiple solution candidates

1. (a) With $\mathcal{L}(x,y) = 3xy - \lambda(x^2 + y^2 - 8)$, the first-order conditions are $\mathcal{L}_1' = 3y - 2\lambda x = 0$ and $\mathcal{L}_2' = 3x - 2\lambda y = 0$. These can be rewritten as: (i) $3y = 2\lambda x$; (ii) $3x = 2\lambda y$.

[31] We can interpret a and b as minimum subsistence quantities of the two goods, in which case the assumption $pa + qb < m$ means that the consumer can afford to buy (a,b).

[32] We note that the maximum by itself is much easier to find if one substitutes the constraint $M = \frac{1}{6}xT^3$ into the objective function $f(x,T)$, which then becomes the function $-\alpha MT^2 + \frac{1}{2}MT + M$ of T alone, with α and M as parameters. The first-order condition for T to maximize this expression is $-2\alpha MT + \frac{1}{2}M = 0$, implying that $T = 1/4\alpha$. However, this method does not find λ.

If $x = 0$, then (i) gives $y = 0$; conversely, if $y = 0$, then (ii) gives $x = 0$. But $(x, y) = (0, 0)$ does not satisfy the constraint. Hence $x \neq 0$ and $y \neq 0$. Equating the ratio of the left-hand sides of (i) and (ii) to the ratio of their right-hand sides, we get $y/x = x/y$ or $x^2 = y^2$. Finally, using the constraint gives $x^2 = y^2 = 4$.

The four solution candidates are therefore $(2, 2)$ and $(-2, -2)$ with $\lambda = 3/2$, as well as $(2, -2)$ and $(-2, 2)$ with $\lambda = -3/2$. The corresponding function values are $f(2, 2) = f(-2, -2) = 12$ and $f(2, -2) = f(-2, 2) = -12$.

Because f is continuous and the constraint set is a circle, which is closed and bounded, the extreme value theorem implies that a maximum and a minimum do exist. The function values tell us that $(2, 2)$ and $(-2, -2)$ solve the maximization problem, whereas $(-2, 2)$ and $(2, -2)$ solve the minimization problem.

(b) With $\mathcal{L} = x + y - \lambda(x^2 + 3xy + 3y^2 - 3)$, the first-order conditions are

$$1 - 2\lambda x - 3\lambda y = 0 \quad \text{and} \quad 1 - 3\lambda x - 6\lambda y = 0.$$

These equations give us $1 = 2\lambda x + 3\lambda y = 3\lambda x + 6\lambda y$. In particular, $\lambda(3y + x) = 0$. Here $\lambda = 0$ is impossible, so $x = -3y$. Inserting this into the constraint reduces it to $3y^2 = 3$, with solutions $y = \pm 1$. So the first-order conditions give two solution candidates:

$$(x, y, \lambda) = (3, -1, \tfrac{1}{3}) \quad \text{and} \quad (x, y, \lambda) = (-3, 1, -\tfrac{1}{3}).$$

The objective function is continuous and the constraint curve is closed and bounded — actually, an ellipse — see (5.5.5). So the extreme value theorem ensures that both a maximum and minimum exist. The two function values $f(3, -1) = 2$ and $f(-3, 1) = -2$ tell us that the maximum is at $(3, -1)$, the minimum at $(-3, 1)$.

2. (a) With $\mathcal{L} = x^2 + y^2 - 2x + 1 - \lambda(x^2 + 4y^2 - 16)$, the first-order conditions are

$$\text{(i) } 2x - 2 - 2\lambda x = 0; \qquad \text{(ii) } 2y - 8\lambda y = 0.$$

Equation (i) implies that $x \neq 0$ and then $\lambda = 1 - 1/x$, whereas equation (ii) shows that $y = 0$ or $\lambda = 1/4$. If $y = 0$, then $x^2 = 16 - 4y^2 = 16$, so $x = \pm 4$, which then gives $\lambda = 1 \mp 1/4$. If $y \neq 0$, then $\lambda = 1/4$ and (i) gives $2x - 2 - x/2 = 0$, so $x = 4/3$. The constraint $x^2 + 4y^2 = 16$ now yields $4y^2 = 16 - 16/9 = 128/9$, so $y = \pm\sqrt{32/9} = \pm 4\sqrt{2}/3$. Thus, there are four solution candidates:

$$\begin{array}{ll} \text{(a) } (x, y, \lambda) = (4, 0, 3/4); & \text{(b) } (x, y, \lambda) = (4/3, 4\sqrt{2}/3, 1/4); \\ \text{(c) } (x, y, \lambda) = (-4, 0, 5/4); & \text{(d) } (x, y, \lambda) = (4/3, -4\sqrt{2}/3, 1/4). \end{array}$$

Of these four, checking function values shows that (a) and (b) both give a maximum, whereas (c) and (d) both give a minimum.

(b) The Lagrangian is $\mathcal{L} = \ln(2 + x^2) + y^2 - \lambda(x^2 + 2y - 2)$. Hence, the necessary first-order conditions for (x, y) to be a minimum point are

$$\text{(i) } \frac{\partial}{\partial x}\mathcal{L} = 2x/(2 + x^2) - 2\lambda x = 0; \quad \text{(ii) } \frac{\partial}{\partial y}\mathcal{L} = 2y - 2\lambda = 0; \quad \text{(iii) } x^2 + 2y = 2.$$

From (i) we get $x\big(1/(2 + x^2) - \lambda\big) = 0$, so $x = 0$ or $\lambda = 1/(2 + x^2)$.

Now if $x = 0$, then (iii) gives $y = 1$, so $(x_1, y_1) = (0, 1)$ is a first solution candidate. Alternatively, if $x \neq 0$, then (ii) and then (i) imply that $y = \lambda = 1/(2 + x^2)$. In this case, inserting $y = 1/(2 + x^2)$ into (iii) gives

$$x^2 + 2/(2 + x^2) = 2 \iff 2x^2 + x^4 + 2 = 4 + 2x^2 \iff x^4 = 2 \iff x = \pm\sqrt[4]{2}.$$

From (iii), $y = 1 - \frac{1}{2}x^2 = 1 - \frac{1}{2}\sqrt{2}$. This gives us the two other solution candidates

$$(x_2, y_2) = (\sqrt[4]{2}, 1 - \tfrac{1}{2}\sqrt{2}); \qquad (x_3, y_3) = (-\sqrt[4]{2}, 1 - \tfrac{1}{2}\sqrt{2}).$$

Comparing function values, we see that

$$f(x_1, y_1) = f(0, 1) = \ln 2 + 1 \approx 1.69$$
$$f(x_2, y_2) = f(x_3, y_3) = \ln(2 + \sqrt{2}) + (1 - \tfrac{1}{2}\sqrt{2})^2 = \ln(2 + \sqrt{2}) + \tfrac{3}{2} - \sqrt{2} \approx 1.31.$$

Hence, the minimum points for $f(x, y)$ subject to the constraint are (x_2, y_2) and (x_3, y_3).

4. (a) With $\mathcal{L} = 24x - x^2 + 16y - 2y^2 - \lambda(x^2 + 2y^2 - 44)$, the two first-order conditions are:

$$\text{(i) } \mathcal{L}_1' = 24 - 2x - 2\lambda x = 0; \qquad \text{(ii) } \mathcal{L}_2' = 16 - 4y - 4\lambda y = 0.$$

From (i) $x(1 + \lambda) = 12$ and from (ii) $y(1 + \lambda) = 4$. Eliminating λ from (i) and (ii) we get $x = 3y = 12/(1 + \lambda)$, with $\lambda \neq -1$. Inserted into the constraint, $11y^2 = 44$, so $y = \pm 2$, and then $x = \pm 6$. So there are two solution candidates, $(x, y) = (6, 2)$ and $(-6, -2)$, both with $\lambda = 1$. Computing the objective function at these two points, the only possible maximum is at $(x, y) = (6, 2)$. Since the objective function is continuous and the constraint curve is closed and bounded (an ellipse), the extreme value theorem assures us that there is indeed a maximum at this point.

(b) According to (14.2.3) the change is approximately $\lambda \cdot 1 = 1$.

14.4 Why the Lagrange method works

4. Before trying to find the minimum, consider the graph of the curve $g(x, y) = 0$, as shown in Fig. SM14.4.4. It consists of three pieces:

   (i) the continuous curve $y = \sqrt{x}(x + 1)$ in the positive quadrant;

   (ii) the continuous curve $y = -\sqrt{x}(x + 1)$, which is the reflection of curve (i) about the x-axis;

   (iii) the isolated point $(-1, 0)$. The problem is to minimize the square of the distance d from the point $(-2, 0)$ to a point on this graph. The minimum of f obviously occurs at the isolated point $(-1, 0)$, with $f(-1, 0) = 1$.

   With the Lagrangian $\mathcal{L} = (x + 2)^2 + y^2 - \lambda[y^2 - x(x + 1)^2]$, we have

$$\mathcal{L}_1' = 2(x + 2) + \lambda[(x + 1)^2 + 2x(x + 1)]$$

   and $\mathcal{L}_2' = 2y(1 - \lambda)$. Note that $\mathcal{L}_2' = 2y(1 - \lambda) = 0$ only if $\lambda = 1$ or $y = 0$. For $\lambda = 1$, we find that $\mathcal{L}_1' = 3(x + 1)^2 + 2 > 0$ for all x. For $y = 0$, the constraint gives $x = 0$ or $x = -1$. At $x = 0$, we have $\mathcal{L}_1' = 4 + \lambda = 0$ whereas $\mathcal{L}_1' = 2$ at $x = -1$.

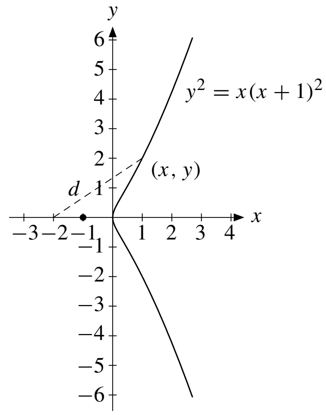

Figure SM14.4.4

Thus the Lagrange multiplier method produces a unique solution candidate $(x, y) = (0, 0)$ with $\lambda = -4$, which does correspond to a *local* minimum. The *global* minimum at $(-1, 0)$, however, fails to satisfy the first-order conditions $\mathcal{L}'_1 = \mathcal{L}'_2 = 0$ for any value of λ, because $\mathcal{L}'_1 = 2$ at this point. So the Lagrange multiplier method cannot locate this minimum. Note that at $(-1, 0)$ both $g'_1(-1, 0)$ and $g'_2(-1, 0)$ are 0.

14.5 Sufficient conditions

4. With $\mathcal{L} = x^a + y^a - \lambda(px + qy - m)$, the first-order conditions are:

$$\mathcal{L}'_1 = ax^{a-1} - \lambda p = 0 \quad \text{and} \quad \mathcal{L}'_2 = ay^{a-1} - \lambda q = 0.$$

If $\lambda = 0$, then $(x, y) = (0, 0)$, which does not satisfy the constraint. It follows that $\lambda \neq 0$ and then $x = (\lambda p/a)^{1/(a-1)}$, $y = (\lambda q/a)^{1/(a-1)}$. Inserting these values of x and y into the budget constraint gives $(\lambda/a)^{1/(a-1)}(p^{a/(a-1)} + q^{a/(a-1)}) = m$. To reduce notation, define $R = p^{a/(a-1)} + q^{a/(a-1)}$ as in the answer in the book. Then we have $(\lambda/a)^{1/(1-a)} = m/R$. Therefore $x = mp^{1/(a-1)}/R$ and $y = mq^{1/(a-1)}/R$.

14.6 Additional variables and constraints

7. The Lagrangian is $\mathcal{L} = x + y - \lambda(x^2 + 2y^2 + z^2 - 1) - \mu(x + y + z - 1)$, which has critical points when

$$\text{(i)}\ \mathcal{L}'_x = 1 - 2\lambda x - \mu = 0; \qquad \text{(ii)}\ \mathcal{L}'_y = 1 - 4\lambda y - \mu = 0; \qquad \text{(iii)}\ \mathcal{L}'_z = -2\lambda z - \mu = 0.$$

From (ii) and (iii) we get $1 = \lambda(4y - 2z)$, and in particular $\lambda \neq 0$. From (i) and (ii), $\lambda(x - 2y) = 0$ and so $x = 2y$. Substituting this value for x into the constraints gives $6y^2 + z^2 = 1$ and $3y + z = 1$. Thus $z = 1 - 3y$, implying that $1 = 6y^2 + (1 - 3y)^2 = 15y^2 - 6y + 1$. Hence $y = 0$ or $y = 2/5$, implying that $x = 0$ or $4/5$, and that $z = 1$ or $-1/5$. The only two solution candidates are $(x, y, z) = (0, 0, 1)$ with $\lambda = -1/2$, $\mu = 1$, and $(x, y, z) = (4/5, 2/5, -1/5)$

with $\lambda = 1/2$, $\mu = 1/5$. Because $x+y$ is 0 at $(0,0,1)$ and 6/5 at $(4/5, 2/5, -1/5)$, these are respectively the minimum and the maximum.[33]

8. (a) For the given Cobb–Douglas utility function, one has $U_j'(\mathbf{x}) = \alpha_j U(\mathbf{x})/x_j$. So (14.6.6) with $k=1$ implies that $p_j/p_1 = U_j'(\mathbf{x})/U_1'(\mathbf{x}) = \alpha_j x_1/\alpha_1 x_j$. Thus $p_j x_j = (a_j/a_1)p_1 x_1$. Inserting this into the budget constraint for $j = 2, \dots, n$, gives $p_1x_1 + (a_2/a_1)p_1x_1 + \cdots + (a_n/a_1)p_1x_1 = m$, which implies that $p_1x_1 = a_1m/(a_1 + \cdots + a_n)$. Similarly, $p_jx_j = a_jm/(a_1 + \cdots + a_n)$ for $k = 1, \dots, n$.

   (b) From (14.6.6) with $k=1$, we get $x_j^{a-1}/x_1^{a-1} = p_j/p_1$ and so $x_j/x_1 = (p_j/p_1)^{-1/(1-a)}$, or $p_jx_j/p_1x_1 = (p_j/p_1)^{1-1/(1-a)} = (p_j/p_1)^{-a/(1-a)}$. Inserting this into the budget constraint for $j = 2, \dots, n$, gives

$$p_1x_1\left[1 + \left(\frac{p_2}{p_1}\right)^{-a/(1-a)} + \cdots + \left(\frac{p_n}{p_1}\right)^{-a/(1-a)}\right] = m$$

   so $p_1x_1 = \dfrac{mp_1^{-a/(1-a)}}{\sum_{i=1}^n p_i^{-a/(1-a)}}$. Arguing similarly for each j, we get

$$p_jx_j = \frac{mp_j^{-a/(1-a)}}{\sum_{i=1}^n p_i^{-a/(1-a)}} \quad \text{for } j = 1, \dots, n.$$

14.7 Comparative statics

2. Here, $\mathcal{L} = x + 4y + 3z - \lambda(x^2 + 2y^2 + \frac{1}{3}z^2 - b)$, so necessary first-order conditions are:

$$\text{(i) } \mathcal{L}_1' = 1 - 2\lambda x = 0 \qquad \text{(ii) } \mathcal{L}_2' = 4 - 4\lambda y = 0 \qquad \text{(iii) } \mathcal{L}_3' = 3 - \frac{2}{3}\lambda z = 0$$

   It follows that $\lambda \neq 0$, and so $x = 1/2\lambda$, $y = 1/\lambda$, $z = 9/2\lambda$. Inserting these values into the constraint yields $[(1/4) + 2 + (27/4)]\lambda^{-2} = b$ and so $\lambda^2 = 9/b$, implying that $\lambda = \pm 3/\sqrt{b}$. The value of the objective function is $x + 4y + 3z = 18/\lambda$, so $\lambda = -3/\sqrt{b}$ determines the minimum point. This is $(x, y, z) = (a, 2a, 9a)$, where $a = -\sqrt{b}/6$. For the last verification, see the answer in the book.

4. (a) With $\mathcal{L} = x^2 + y^2 + z - \lambda(x^2 + 2y^2 + 4z^2 - 1)$, necessary conditions are:

$$\text{(i) } \partial\mathcal{L}/\partial x = 2x - 2\lambda x = 0 \qquad \text{(ii) } \partial\mathcal{L}/\partial y = 2y - 4\lambda y = 0 \qquad \text{(iii) } \partial\mathcal{L}/\partial z = 1 - 8\lambda z = 0$$

   From (i), $2x(1-\lambda) = 0$, so there are two possibilities: $x = 0$ or $\lambda = 1$.

   (A) Suppose $x = 0$. From (ii), $2y(1-2\lambda) = 0$, so $y = 0$ or $\lambda = 1/2$.

      (A.1) If $y = 0$, then the constraint gives $4z^2 = 1$, so $z^2 = 1/4$, or $z = \pm 1/2$. Equation (iii) gives $\lambda = 1/8z$, so we have two solution candidates: $P_1 = (0, 0, 1/2)$ with $\lambda = 1/4$; and $P_2 = (0, 0, -1/2)$ with $\lambda = -1/4$.

[33] The two constraints determine the curve in three dimensions which is the intersection of an ellipsoid (see Fig. 11.4.2) and a plane. Because an ellipsoid is a closed bounded set, so is this curve. By the extreme value theorem, the continuous function $x + y$ does attain a maximum and a minimum over this closed bounded set.

(A.2) If $\lambda = 1/2$, then (iii) gives $z = 1/8\lambda = 1/4$. It follows from the constraint that $2y^2 = 3/4$ (recall that we assumed $x = 0$), and hence $y = \pm\sqrt{3/8} = \pm\sqrt{6}/4$. So new candidates are: $P_3 = (0, \sqrt{6}/4, 1/4)$ with $\lambda = 1/2$; and $P_4 = (0, -\sqrt{6}/4, 1/4)$ with $\lambda = 1/2$.

(B) Suppose $\lambda = 1$. Equation (iii) yields $z = 1/8$, and (ii) gives $y = 0$. From the constraint, $x^2 = 15/16$, so $x = \pm\sqrt{15}/4$. This gives us two more solution andidates: $P_5 = (\sqrt{15}/4, 0, 1/8)$ with $\lambda = 1$; and $P_6 = (-\sqrt{15}/4, 0, 1/8)$ with $\lambda = 1$.

For $k = 1, 2, \ldots, 6$, let f_k denote the value of the criterion function f at the candidate point P_k. Routine calculation shows that $f_1 = 1/2$, $f_2 = -1/2$, $f_3 = f_4 = 5/8$, and $f_5 = f_6 = 17/16$. It follows that both P_5 and P_6 solve the maximization problem, whereas P_2 solves the minimization problem.

(b) See the answer given in the book.

5. The Lagrangian is $\mathcal{L} = rK + wL - \lambda(K^{1/2}L^{1/4} - Q)$, so first-order necessary conditions for (K^*, L^*) to solve the problem are:

$$\text{(i)}\ \mathcal{L}'_K = r - \tfrac{1}{2}\lambda(K^*)^{-1/2}(L^*)^{1/4} = 0 \qquad \text{and} \qquad \text{(ii)}\ \mathcal{L}'_L = w - \tfrac{1}{4}\lambda(K^*)^{1/2}(L^*)^{-3/4} = 0$$

together with the constraint: (iii) $(K^*)^{1/2}(L^*)^{1/4} = Q$.

Together (i) and (ii) imply that $r/w = 2L^*/K^*$ and so $L^* = rK^*/2w$. Inserting the latter into (iii) gives $Q = (K^*)^{1/2}(rK^*/2w)^{1/4} = (K^*)^{3/4}2^{-1/4}r^{1/4}w^{-1/4}$. Solving for K^* gives the answer in the text. The answers for L^* and $C^* = rK^* + wL^*$ follow if we observe that $2^{1/3} = 2 \cdot 2^{-2/3}$. Verifying $(*)$ in Example 14.7.3 is easy.

14.8 Nonlinear programming: a simple case

5. (a) With the Lagrangian $\mathcal{L}(x, y) = 2 - (x-1)^2 - e^{y^2} - \lambda(x^2 + y^2 - a)$, the Kuhn–Tucker conditions are:

$$\text{(i)}\ -2(x-1) - 2\lambda x = 0; \quad \text{(ii)}\ -2ye^{y^2} - 2\lambda y = 0; \quad \text{(iii)}\ \lambda \ge 0, \text{ with } \lambda = 0 \text{ if } x^2 + y^2 < a.$$

From (i), $x = (1+\lambda)^{-1}$. Moreover, (ii) reduces to $y(e^{y^2} + \lambda) = 0$, and so $y = 0$, because $e^{y^2} + \lambda$ is always positive.

(I) *Assume that* $\lambda = 0$. Then equation (i) gives $x = 1$. In this case we must have $a \ge x^2 + y^2 = 1$.

(II) *Assume that* $\lambda > 0$. Then (iii) gives $x^2 + y^2 = a$, and so $x = \pm\sqrt{a}$ (remember that $y = 0$). Because $x = 1/(1+\lambda)$ and $\lambda > 0$ we must have $0 < x < 1$, so $x = \sqrt{a}$ and $a = x^2 < 1$. It remains to find the value of λ and check that it is > 0. From equation (i) we get $\lambda = 1/x - 1 = 1/\sqrt{a} - 1 > 0$.

Conclusion: The only point that satisfies the Kuhn–Tucker conditions is $(x, y) = (1, 0)$ if $a \ge 1$ and $(\sqrt{a}, 0)$ if $0 < a < 1$. The corresponding value of λ is 0 or $1/\sqrt{a} - 1$, respectively.

(b) Since $\mathcal{L}''_{11} = -2 - 2\lambda < 0$, $\mathcal{L}''_{22} = -e^{y^2}(2 + 4y^2) - 2\lambda < 0$, and $\mathcal{L}''_{12} = 0$, the Lagrangian $\mathcal{L}(x, y)$ is concave, so we have found the solution in both cases(I) and (ii) in part (a)..

(c) If $a \in (0,1)$ we have $f^*(a) = f(\sqrt{a}, 0) = 2 - (\sqrt{a} - 1)^2 - 1 = 2\sqrt{a} - a$, and for $a \geq 1$ we get $f^*(a) = f(1,0) = 1$. The derivative of f^* is as given in the book, but note that in order to find the derivative $\mathrm{d}f^*(a)/\mathrm{d}a$ when $a = 1$, we need to show that the right and left derivatives (see Section 7.9 in the book)

$$(f^*)'(1^+) = \lim_{h \to 0^+} \frac{f^*(1+h) - f^*(1)}{h} \qquad \text{and} \qquad (f^*)'(1^-) = \lim_{h \to 0^-} \frac{f^*(1+h) - f^*(1)}{h}$$

exist and are equal. But the left and right derivatives are respectively equal to the derivatives of the differentiable functions $g_-(a) = 2\sqrt{a} - a$ and $g_+(a) = 1$ at $a = 1$, which are both 0. Hence $(f^*)'(1)$ exists and equals 0.

14.9 Multiple inequality constraints

2. The Lagrangian is $\mathcal{L} = \alpha \ln x + (1-\alpha)\ln y - \lambda(px + qy - m) - \mu(x - \bar{x})$, and the Kuhn–Tucker conditions for (x^*, y^*) to solve the problem are:

$$\text{(i)}\ \mathcal{L}'_1 = \frac{\alpha}{x^*} - \lambda p - \mu = 0 \qquad \text{(iii)}\ \lambda \geq 0,\ \text{with } \lambda = 0 \text{ if } px^* + qy^* < m$$

$$\text{(ii)}\ \mathcal{L}'_2 = \frac{1-\alpha}{y^*} - \lambda q = 0 \qquad \text{(iv)}\ \mu \geq 0,\ \text{with } \mu = 0 \text{ if } x^* < \bar{x}.$$

Since $\alpha \in (0,1)$, (ii) implies that $\lambda > 0$ and so (iii) entails $px^* + qy^* = m$.

Suppose $\mu = 0$. Then (i) and (ii) imply that $\alpha/px^* = (1-\alpha)/qy^*$, so $qy^* = (1-\alpha)px^*/\alpha$. Then the budget constraint implies that $px^* + (1-\alpha)px^*/\alpha = m$, so $x^* = m\alpha/p$ and then $y^* = (1-\alpha)m/q$, with $\lambda = 1/m$. This solution is valid as long as $x^* \leq \bar{x}$, that is $m \leq p\bar{x}/\alpha$.

Suppose $\mu > 0$. Then $x^* = \bar{x}$ and $y^* = m/q - p\bar{x}/q = (m - p\bar{x})/q$, with $\lambda = (1-\alpha)/(m - p\bar{x})$ and $\mu = \alpha/\bar{x} - \lambda p = (\alpha m - p\bar{x})/\bar{x}(m - p\bar{x})$. Note that if $m > p\bar{x}/\alpha$, then $m > p\bar{x}$ since $\alpha < 1$. We conclude that if $m > p\bar{x}/\alpha$, then λ and μ are both positive and so (i)–(iv) are all satisfied.

Since $\mathcal{L}'_{11} = -\alpha/x^2 < 0$, $\mathcal{L}'_{22} = -\alpha/y^2 < 0$, and $\mathcal{L}'_{12} = 0$, the Lagrangian $\mathcal{L}(x,y)$ is concave, so we have found the solution in both cases.

3. (a) See the answer given in the book.

   (b) With the constraints $g_1(x,y) = -x - y \leq -4$, $g_2(x,y) = -x \leq 1$, and $g_3(x,y) = -y \leq -1$, the Lagrangian is $\mathcal{L} = x + y - e^x - e^{x+y} - \lambda_1(-x - y + 4) - \lambda_2(-x - 1) - \lambda_3(-y + 1)$. Formulating the complementary slackness conditions as in (14.8.5), the Kuhn–Tucker conditions are that there exist nonnegative numbers λ_1, λ_2, and λ_3 such that:

$$\text{(i)}\ \mathcal{L}'_x = 1 - e^x - e^{x+y} + \lambda_1 + \lambda_2 = 0 \qquad \text{(iii)}\ \lambda_1(-x - y + 4) = 0$$

$$\text{(ii)}\ \mathcal{L}'_y = 1 - e^{x+y} + \lambda_1 + \lambda_3 = 0 \qquad \text{(iv)}\ \lambda_2(-x - 1) = 0$$

$$\text{(v)}\ \lambda_3(-y + 1) = 0$$

From (ii), $e^{x+y} = 1 + \lambda_1 + \lambda_3$. Inserting this into (i) yields $\lambda_2 = e^x + \lambda_3 \geq e^x > 0$. Because $\lambda_2 > 0$, (iv) implies that $x = -1$. So any solution must lie on the line (II) in Fig. A14.9.3 in the book. This implies that the third constraint $y \geq 1$ must be slack.[34] So from (v) we get $\lambda_3 = 0$, and then (ii) gives $\lambda_1 = e^{x+y} - 1 \geq e^4 - 1 > 0$. Thus from (iii), the first constraint is active, so $y = 4 - x$. We already showed that $x = -1$, so $y = 5$.

[34] Algebraically, because $x + y \geq 4$ and $x = -1$, we have $y \geq 4 - x = 5 > 1$.

(c) Hence the only possible solution is $(x^*, y^*) = (-1, 5)$. Because $\mathcal{L}(x, y)$ is concave, as the sum of concave functions, we have found the optimal point.

4. (a) The feasible set is shown in Fig. A14.9.4 in the book. The function to be maximized is $f(x, y) = x + ay$. The level curves of this function are straight lines with slope $-1/a$ if $a \neq 0$, and vertical lines if $a = 0$. The dashed line in the figure is such a level curve (for $a \approx -0.25$). The maximum point for f is that point in the feasible region that we shall find if we make a parallel displacement of this line as far to the right as possible without losing contact with the shaded region.[35]

   To apply the recipe of Section 14.9, we write the second constraint as $-x - y \leq 0$, and consider the Lagrangian $\mathcal{L}(x, y) = x + ay - \lambda_1(x^2 + y^2 - 1) + \lambda_2(x + y)$. The Kuhn–Tucker conditions are:

   (i) $\mathcal{L}_1'(x, y) = 1 - 2\lambda_1 x + \lambda_2 = 0$, (iii) $\lambda_1 \geq 0$, with $\lambda_1 = 0$ if $x^2 + y^2 < 1$,

   (ii) $\mathcal{L}_2'(x, y) = a - 2\lambda_1 y + \lambda_2 = 0$, (iv) $\lambda_2 \geq 0$, with $\lambda_2 = 0$ if $x + y > 0$.

   (b) From (i), $2\lambda_1 x = 1 + \lambda_2 \geq 1$. But (iii) implies that $\lambda_1 \geq 0$, so in fact $\lambda_1 > 0$ and $x > 0$. Because $\lambda_1 > 0$, (iii) also implies that any maximum point must lie on the semi-circle where $x > 0$ and $x^2 + y^2 = 1$.

   First consider the case $x + y = 0$. Because $x^2 + y^2 = 1$ and $x > 0$, the only possible solution is $x = \frac{1}{2}\sqrt{2}$ and $y = -\frac{1}{2}\sqrt{2}$.

   Now, adding equations (i) and (ii) gives $0 = 1 + a - 2\lambda_1(x + y) + 2\lambda_2 = 1 + a + 2\lambda_2$ since $x + y = 0$. But $\lambda_2 \geq 0$, so $a = 1 - 2\lambda_2 \leq -1$ in this case. Then equation (i) along with $\lambda_2 = \frac{1}{2}(1 - a)$ and $x = \frac{1}{2}\sqrt{2}$ gives $\lambda_1 = (1 + \lambda_2)/2x = (1 - a)/4x = \sqrt{2}(1 - a)/4$.

   Second, consider the alternative case $x + y > 0$. Then (iv) implies that $\lambda_2 = 0$, so (i) and (ii) reduce to $1 - 2\lambda_1 x = 0$ and $a - 2\lambda_1 y = 0$, and so $x = 1/(2\lambda_1)$ and $y = a/(2\lambda_1)$.

   Inserting these into $x^2 + y^2 = 1$ yields $(1/4\lambda_1)^2(1 + a^2) = 1$, and so $\lambda_1 = \frac{1}{2}\sqrt{1 + a^2}$. Hence $x = (1 + a^2)^{-1/2}$ and $y = a(1 + a^2)^{-1/2}$. Because $x + y = (1 + a)(1 + a^2)^{-1/2} > 0$ in this case, it must occur when $a > -1$.

   Conclusion: The only points satisfying the Kuhn–Tucker conditions are those given in the text. Since the feasible set is closed and bounded and f is continuous, it follows from the extreme value theorem that extreme points exists.

5. The Lagrangian is $\mathcal{L} = y - x^2 + \lambda y + \mu(y - x + 2) - \nu(y^2 - x)$. The Kuhn—Tucker conditions are:

   (i) $\mathcal{L}_x' = -2x - \mu + \nu = 0$; (iii) $\lambda \geq 0$, with $\lambda = 0$ if $y > 0$;

   (ii) $\mathcal{L}_y' = 1 + \lambda + \mu - 2\nu y = 0$; (iv) $\mu \geq 0$, with $\mu = 0$ if $y - x > -2$;

   (v) $\nu \geq 0$, with $\nu = 0$ if $y^2 < x$.

   From (ii), $2\nu y = 1 + \lambda + \mu > 0$, so $y > 0$ and $\nu > 0$, implying from (v) that $y^2 = x$. Also (iii) implies $\lambda = 0$, and $2\nu y = 1 + \mu$. Also, (i) implies that $x = \frac{1}{2}(\nu - \mu)$.

[35] Why to the right?

Suppose $\mu > 0$. Then $y - x + 2 = y - y^2 + 2 = 0$ with roots $y = -1$ and $y = 2$. Only $y = 2$ is feasible. Then $x = y^2 = 4$. Because $\lambda = 0$, conditions (i) and (ii) become $-\mu + \nu = 8$ and $\mu - 4\nu = -1$, so $\nu = -7/3$. This contradicts $\nu \geq 0$, so $(x, y) = (4, 2)$ is not a candidate.

Therefore $\mu = 0$ after all. Thus $x = \frac{1}{2}\nu = y^2$ and, by (ii), $1 = 2\nu y = 4y^3$. Hence $y = 4^{-1/3}$, implying $x = 4^{-2/3}$. This is the only remaining candidate. It is the solution with $\lambda = 0$, $\mu = 0$, and $\nu = 1/2y = 4^{-1/6}$.

6. (a) See Fig. A14.9.6 in the book. Note that for (x, y) to be admissible, one must have $e^{-x} \leq y \leq 2/3$, and so $e^x \geq 3/2$. This implies, in particular, that $x > 0$.

   (b) The Lagrangian is $\mathcal{L} = -(x + \frac{1}{2})^2 - \frac{1}{2}y^2 - \lambda_1(e^{-x} - y) - \lambda_2(y - \frac{2}{3})$, so the Kuhn–Tucker conditions are:

   (i) $\mathcal{L}_1'(x, y) = -(2x + 1) + \lambda_1 e^{-x} = 0$; (iii) $\lambda_1 \geq 0$, with $\lambda_1 = 0$ if $e^{-x} < y$;

   (ii) $\mathcal{L}_2'(x, y) = -y + \lambda_1 - \lambda_2 = 0$; (iv) $\lambda_2 \geq 0$, with $\lambda_2 = 0$ if $y < 2/3$.

   From (i), $\lambda_1 = (2x + 1)e^x \geq 3/2$, because of part (a), implying that $y = e^{-x}$. From (ii), $\lambda_2 = \lambda_1 - y \geq 3/2 - 2/3 > 0$, so $y = 2/3$ because of (iv). This gives the solution candidate $(x^*, y^*) = (\ln(3/2), 2/3)$, with $\lambda_1 = 3[\ln(3/2) + 1/2]$ and $\lambda_2 = 3\ln(3/2) + 5/6$. The Lagrangian is easily seen to be concave as a function of (x, y) when $\lambda_1 \geq 0$, so this is indeed the solution.

   Here is an alternative argument: Suppose $\lambda_1 = 0$. Then from (ii), $y = -\lambda_2 \leq 0$, contradicting $y \geq e^{-x}$. So $\lambda_1 > 0$, and (iii) gives $y = e^{-x}$.
   Suppose $\lambda_2 = 0$. Then from (ii), $\lambda_1 = y = e^{-x}$ and (i) gives $e^{-2x} = 2x + 1$. Define $g(x) = 2x + 1 - e^{-2x}$. Then $g(0) = 0$ and $g'(x) = 2 + 2e^{-2x} > 0$. So the equation $e^{-2x} = 2x + 1$ has no solution except $x = 0$. Thus $\lambda_2 > 0$, etc.

14.10 Nonnegativity constraints

2. The Lagrangian, without multipliers for the non-negativity constraints, is $\mathcal{L} = xe^{y-x} - 2ey - \lambda(y - 1 - x/2)$, so the first-order conditions (14.10.3) and (14.10.4) are

   (i) $\mathcal{L}_x' = e^{y-x} - xe^{y-x} + \frac{1}{2}\lambda \leq 0$, and $\mathcal{L}_x' = 0$ if $x > 0$;

   (ii) $\mathcal{L}_y' = xe^{y-x} - 2e - \lambda \leq 0$, and $\mathcal{L}_y' = 0$ if $y > 0$;

   (iii) $\lambda \geq 0$, with $\lambda = 0$ if $y < 1 + \frac{1}{2}x$.

   If $x = 0$, then (i) implies $e^y + \frac{1}{2}\lambda \leq 0$, which is impossible, so $x > 0$. Then from (i) we get (iv) $xe^{y-x} = e^{y-x} + \frac{1}{2}\lambda$.

   Suppose first that $\lambda > 0$. Then (iii) and $y \leq 1 + x/2$ imply (v) $y = 1 + \frac{1}{2}x$. Thus $y > 0$ and from (ii) we have $xe^{y-x} = 2e + \lambda$. Using (iv) and (v), we get $\lambda = 2e^{y-x} - 4e = 2e(e^{-\frac{1}{2}x} - 2)$. But then $\lambda > 0$ implies that $e^{-\frac{1}{2}x} > 2$, which contradicts $x \geq 0$.

   This leaves $\lambda = 0$ as the only possibility. Then (iv) gives $x = 1$. If $y > 0$, then (ii) yields $e^{y-1} = 2e$, and so $y - 1 = \ln(2e) = \ln 2 + 1$. With $x = 1$ this contradicts the constraint

$y \leq 1 + \frac{1}{2}x$. Hence $y = 0$, so we see that $(x, y) = (1, 0)$ is the only point satisfying all the first-order conditions, with $\lambda = 0$.[36]

3. The Lagrangian is $\mathcal{L} = x_1 + 3x_2 - x_1^2 - x_2^2 - k^2 - \lambda(x_1 - k) - \mu(x_2 - k)$. A feasible triple (x_1^*, x_2^*, k^*) solves the problem if and only if there exist numbers λ and μ such that

$$\begin{aligned}
&\text{(i)} \ \partial\mathcal{L}/\partial x_1 = 1 - 2x_1^* - \lambda \leq 0, \text{ and } 1 - 2x_1^* - \lambda = 0 \text{ if } x_1^* > 0;\\
&\text{(ii)} \ \partial\mathcal{L}/\partial x_2 = 3 - 2x_2^* - \mu \leq 0, \text{ and } 3 - 2x_2^* - \mu = 0 \text{ if } x_2^* > 0;\\
&\text{(iii)} \ \partial\mathcal{L}/\partial k = -2k^* + \lambda + \mu \leq 0, \text{ and } -2k^* + \lambda + \mu = 0 \text{ if } k^* > 0;\\
&\text{(iv)} \ \lambda \geq 0, \quad \text{with } \lambda = 0 \text{ if } x_1^* < k^*;\\
&\text{(v)} \ \mu \geq 0, \text{ with } \mu = 0 \text{ if } x_2^* < k^*.
\end{aligned}$$

If $k^* = 0$, then feasibility requires $x_1^* = 0$ and $x_2^* = 0$, and so (i) and (ii) imply that $\lambda \geq 1$ and $\mu \geq 3$, which contradicts (iii). Thus, $k^* > 0$. Next, if $\mu = 0$, then (ii) and (iii) imply that $x_2^* \geq 3/2$ and $\lambda = 2k^* > 0$. So $x_1^* = k^* = 1/4$, contradicting $x_2^* \leq k^*$. So $\mu > 0$, which implies that $x_2^* = k^*$. Now, if $x_1^* = 0 < k^*$, then $\lambda = 0$, which contradicts (i).

So $0 < x_1^* = \frac{1}{2}(1 - \lambda)$. Next, if $\lambda > 0$, then $x_1^* = k^* = x_2^* = \frac{1}{2}(1 - \lambda) = \frac{1}{2}(3 - \mu) = \frac{1}{2}(\lambda + \mu)$ by (i), (ii), and (iii) respectively. But the last two equalities are only satisfied when $\lambda = -1/3$ and $\mu = 5/3$, which contradicts $\lambda \geq 0$. So $\lambda = 0$ after all, with $x_2^* = k^* > 0$, $\mu > 0$, $x_1^* = \frac{1}{2}(1 - \lambda) = \frac{1}{2}$. Now, from (iii) it follows that $\mu = 2k^*$ and so, from (ii), that $3 = 2x_2^* + \mu = 4k^*$. The only possible solution is, therefore, $(x_1^*, x_2^*, k^*) = (1/2, 3/4, 3/4)$, with $\lambda = 0$ and $\mu = 3/2$.

Finally, with $\lambda = 0$ and $\mu = \frac{3}{2}$, the Lagrangian $x_1 + 3x_2 - x_1^2 - x_2^2 - k^2 - \frac{3}{2}(x_2 - k)$ is a quadratic function of (x_1, x_2, k), which has a maximum at the critical point (x_1^*, x_2^*, k^*). As stated at the end of the recipe in Section 14.9, this is sufficient for the same (x_1^*, x_2^*, k^*) to solve the problem.

Review exercises for Chapter 14

3. (a) If sales x of the first commodity are increased, the increase in net profit per unit increase in x is the sum of three terms:
   (i) $p(x^*)$, which is the gain in revenue due to the extra output;
   (ii) $-p'(x^*)x^*$, which is the loss in revenue from selling x^* units due to the reduced price;
   (iii) $-C_1'(x^*, y^*)$, which is minus the marginal cost of the additional output.

   In fact $p(x^*) + p'(x^*)x^*$ is the derivative of the revenue function $R(x) = p(x)x$ at $x = x^*$, usually called the marginal revenue. The first necessary condition therefore states that the marginal revenue from increasing x must equal the (partial) marginal cost. The argument when considering any variation in the sales y of the second commodity is just the same.

[36] The extreme value theorem cannot be applied here because the feasible set (the set of all points satisfying the constraints) is unbounded — it includes the point $(x, 0)$ for arbitrarily large x. However, you were told to assume that the problem has a solution.

(b) With the restriction $x + y \leq m$, we have to add the condition $\lambda \geq 0$, with $\lambda = 0$ if $\hat{x} + \hat{y} < m$.

5. (a) The Lagrangian is $\mathcal{L} = x^2 + y^2 - 2x + 1 - \lambda(\frac{1}{4}x^2 + y^2 - b)$; the first-order conditions are:

$$\mathcal{L}_1' = 2x - 2 - \frac{1}{2}\lambda x = 0; \qquad \mathcal{L}_2' = 2y - 2\lambda y = 0; \qquad \frac{1}{4}x^2 + y^2 = b.$$

From (ii), one has $(1 - \lambda)y = 0$, and thus $\lambda = 1$ or $y = 0$.

(I) Suppose first that $\lambda = 1$. Then (i) gives $x = \frac{4}{3}$, and from (iii) we have $y^2 = b - \frac{1}{4}x^2 = b - \frac{4}{9}$, which gives $y = \pm\sqrt{b - \frac{4}{9}}$. This gives two candidates: $(x_1, y_1) = (4/3, \sqrt{b - \frac{4}{9}})$ and $(x_2, y_2) = (4/3, -\sqrt{b - \frac{4}{9}})$.

(II) If $y = 0$, then from (iii), $x^2 = 4b$, i.e. $x = \pm 2\sqrt{b}$. This gives two further candidates: $(x_3, y_3) = (2\sqrt{b}, 0)$ and $(x_4, y_4) = (-2\sqrt{b}, 0)$.

Evaluating the objective function at these different solutions candidates gives:

$$f(x_1, y_1) = f(x_2, y_2) = b - 1/3;$$
$$f(x_3, y_3) = (2\sqrt{b} - 1)^2 = 4b - 4\sqrt{b} + 1;$$
$$f(x_4, y_4) = (-2\sqrt{b} - 1)^2 = 4b + 4\sqrt{b} + 1.$$

Clearly, (x_4, y_4) is the maximum point. To decide which of the points (x_3, y_3), (x_1, y_1), or (x_2, y_2) give the minimum, we have to decide which of the two values $4b - 4\sqrt{b} + 1$ and $b - \frac{1}{3}$ is smaller. The difference between these two is

$$4b - 4\sqrt{b} + 1 - \left(b - \tfrac{1}{3}\right) = 3\left(b - \tfrac{4}{3}\sqrt{b} + \tfrac{4}{9}\right) = 3\left(\sqrt{b} - \tfrac{2}{3}\right)^2 > 0$$

since $b > \frac{4}{9}$. Thus the minimum occurs at (x_1, y_1) and (x_2, y_2).

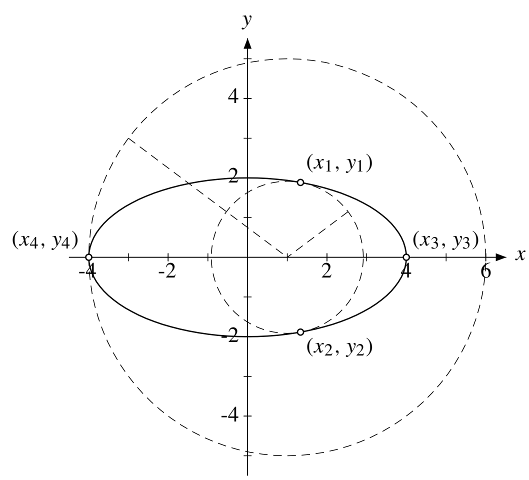

Figure SM14.R.5

The constraint $x^2/4 + y^2 = b$ describes the ellipse which is indicated by the solid curve in Fig. SM14.R.5. The objective function $f(x, y) = (x - 1)^2 + y^2$ is the square of the distance between (x, y) and the point $(1, 0)$. The level curves for f are therefore circles centred at $(1, 0)$, and in the figure we see those two that pass through the maximum and minimum points.

7. (a) With $\mathcal{L} = x^2 - 2x + 1 + y^2 - 2y - \lambda[(x+y)\sqrt{x+y+b} - 2\sqrt{a}]$, the first-order conditions are:

   (i) $\mathcal{L}'_1 = 2x - 2 - \lambda[\sqrt{x+y+b} + (x+y)/2\sqrt{x+y+b}] = 0$,

   (ii) $\mathcal{L}'_2 = 2y - 2 - \lambda[\sqrt{x+y+b} + (x+y)/2\sqrt{x+y+b}] = 0$.

   From these equations it follows immediately that $2x-2 = 2y-2$, so $x = y$. The constraint gives $2x\sqrt{2x+b} = 2\sqrt{a}$. Cancelling 2 and then squaring each side, one obtains the second equation in $(*)$ of the main text.

   (b) Differentiating yields:

$$\mathrm{d}x = \mathrm{d}y; \qquad 6x^2\,\mathrm{d}x + x^2\,\mathrm{d}b + 2bx\,\mathrm{d}x = \mathrm{d}a.$$

   From these equations we easily read off the first-order partials of x and y w.r.t. a and b. Further,

$$\frac{\partial^2 x}{\partial a^2} = \frac{\partial}{\partial a}\left(\frac{\partial x}{\partial a}\right) = \frac{\partial}{\partial a}\frac{1}{6x^2+2bx} = -\frac{12x+2b}{(6x^2+2bx)^2}\frac{\partial x}{\partial a} = -\frac{12x+2b}{(6x^2+2bx)^3} = -\frac{6x+b}{4x^3(3x+b)^3}.$$

8. With $\mathcal{L} = 10 - (x-2)^2 - (y-1)^2 - \lambda(x^2+y^2-a)$, the Kuhn–Tucker conditions are:

   (i) $\mathcal{L}'_x = -2(x-2) - 2\lambda x = 0$;

   (ii) $\mathcal{L}'_y = -2(y-1) - 2\lambda y = 0$.

   (iii) $\lambda \geq 0$, with $\lambda = 0$ if $x^2 + y^2 < a$;

   Since the Lagrangian is concave when $\lambda \geq 0$, these conditions are sufficient for a maximum. One case occurs when $\lambda = 0$, implying that $(x, y) = (2, 1)$. This is valid when $a \geq x^2 + y^2 = 5$. The other case is when $\lambda > 0$. Then (i) implies that $x = 2/(1+\lambda)$ and (ii) implies that $y = 1/(1+\lambda)$. Because (iii) implies that $x^2 + y^2 = a$, we have $5/(1+\lambda)^2 = a$ and so $\lambda = \sqrt{5/a} - 1$, which is positive when $a < 5$. The solution then is $(x, y) = (2\sqrt{a/5}, \sqrt{a/5})$.

9. (a) See the answer given in the book.

   (b) The numbers (i)–(vi) in the following refer to the answer to (a) in the book. From (ii) and (vi) we see that $\lambda_1 = 0$ is impossible. Thus $\lambda_1 > 0$, and from (iii) and (v), we see that:

$$(x^*)^2 + r(y^*)^2 = m. \tag{vii}$$

   Now,

   (I) Assume $\lambda_2 = 0$. Then from (i) and (ii), $y^* = 2\lambda_1 x^*$ and $x^* = 2\lambda_1 r y^*$, so $x^* = 4\lambda_1^2 r x^*$. But (vi) implies that $x^* \neq 0$. Hence $\lambda_1^2 = 1/4r$ and thus $\lambda_1 = 1/2\sqrt{r}$. Then $y^* = x^*/\sqrt{r}$, which inserted into (vii) and solved for x^* yields $x^* = \sqrt{m/2}$ and then $y^* = \sqrt{m/2r}$. Note that $x^* \geq 1 \iff \sqrt{m/2} \geq 1 \iff m \geq 2$. Thus, for $m \geq 2$, a solution candidate is $x^* = \sqrt{m/2}$ and $y^* = \sqrt{m/2r}$, with $\lambda_1 = 1/2\sqrt{r}$ and $\lambda_2 = 0$.

(II) Assume $\lambda_2 > 0$. Then $x^* = 1$ and from (vii) we have $r(y^*)^2 = m - 1$. By (ii), one has $y^* \geq 0$, so $y^* = \sqrt{(m-1)/r}$. Inserting these values into (i) and (ii), then solving for λ_1 and λ_2, one obtains $\lambda_1 = 1/2\sqrt{r(m-1)}$ and furthermore, $\lambda_2 = (2-m)/\sqrt{r(m-1)}$. Note that $\lambda_2 > 0 \iff 1 < m < 2$. Thus, for $1 < m < 2$, the only solution candidate is $x^* = 1$, $y^* = \sqrt{(m-1)/r}$, with $\lambda_1 = 1/2\sqrt{r(m-1)}$ and $\lambda_2 = (2-m)/\sqrt{r(m-1)}$.

The objective function is continuous and the constraint set is obviously closed and bounded, so by the extreme value theorem there has to be a maximum. The solution candidates we have found are therefore optimal.[37]

(c) For $m \geq 2$, one has $V(r, m) = m/2\sqrt{r}$, so $V'_m = 1/2\sqrt{r} = \lambda_1$, and $V'_r = -m/4\sqrt{r^3}$, whereas $\mathcal{L}'_r = -\lambda_1(y^*)^2 = -(1/2\sqrt{r})m/2r = -m/4\sqrt{r^3}$.
For $1 < m < 2$, one has $V(r, m) = \sqrt{(m-1)/r}$, so $V'_m = 1/2\sqrt{r(m-1)} = \lambda_1$, and $V'_r = -(1/2)\sqrt{(m-1)/r^3}$, whereas
$\mathcal{L}'_r = -\lambda_1(y^*)^2 = -[1/2\sqrt{r(m-1)}](m-1)/r = -(1/2)\sqrt{(m-1)/r^3}$.

15 Matrix and Vector Algebra

15.1 Systems of linear equations

6. The equation system is:

$$\begin{cases} \quad 0.712y - c = -95.05 \\ 0.158x - s + 0.158c = 34.30 \\ x - y - s + c = 0 \\ x = 93.53 \end{cases}$$

Solve the first equation for y as a function of c. Insert this expression for y and $x = 93.53$ into the third equation. Solve it to get s as a function of c. Insert the results into the second equation and solve for c, and then solve for y and s in turn. The answer is given in the main text.

15.3 Matrix multiplication

6. We know that $\mathbf{A}$ is an $m \times n$ matrix. Let $\mathbf{B}$ be a $p \times q$ matrix. The matrix product $\mathbf{AB}$ is defined if and only if $n = p$, and $\mathbf{BA}$ is defined if and only if $q = m$. So for both $\mathbf{AB}$ and $\mathbf{BA}$ to be defined, it is necessary and sufficient that $\mathbf{B}$ is an $n \times m$ matrix.

7. We know from the previous exercise that $\mathbf{B}$ must be a 2×2 matrix. So let $\mathbf{B} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$.

Then $\mathbf{BA} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} x + 2y & 2x + 3y \\ z + 2w & 2z + 3w \end{pmatrix}$

[37] Alternatively, $\mathcal{L}''_{11} = -2\lambda_1 \leq 0$, $\mathcal{L}''_{22} = -2r\lambda_1 \leq 0$, and $\Delta = \mathcal{L}''_{11}\mathcal{L}''_{22} - (\mathcal{L}''_{12})^2 = 4r\lambda_1^2 - 1$. In the case $m \geq 2$, $\Delta = 0$, and in the case $1 < m < 2$, $\Delta = 1/(m-1) > 0$. Thus in both cases, $\mathcal{L}(x, y)$ is concave. So the Kuhn–Tucker conditions are sufficient for a maximum.

while $\mathbf{AB} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x+2z & y+2w \\ 2x+3z & 2y+3w \end{pmatrix}$.
Hence, $\mathbf{BA} = \mathbf{AB}$ if and only if:
(i) $x+2y = x+2z$; (ii) $2x+3y = y+2w$; (iii) $z+2w = 2x+3z$; and (iv) $2z+3w = 2y+3w$. Equations (i) and (iv) are both true if and only if $y = z$; then equations (ii) and (iii) are also both true if and only if, in addition, $x = w - y$. To summarize, all four equations hold if and only if $y = z$ and $x = w - y$. Hence, the matrices $\mathbf{B}$ that commute with $\mathbf{A}$ are precisely those of the form $\mathbf{B} = \begin{pmatrix} w-y & y \\ y & w \end{pmatrix} = w \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + y \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ where y and w can be any real numbers.

15.4 Rules for matrix multiplication

2. We start by performing the multiplication

$$\begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax+dy+ez \\ dx+by+fz \\ ex+fy+cz \end{pmatrix}.$$

Next,

$$(x,\ y,\ z) \begin{pmatrix} ax+dy+ez \\ dx+by+fz \\ ex+fy+cz \end{pmatrix} = ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz,$$

which is a 1×1 matrix.

8. (a) Direct verification yields (i) because

$$\mathbf{A}^2 = (a+d)\mathbf{A} - (ad-bc)\mathbf{I}_2 = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}$$

(b) For the matrix $\mathbf{A}$ in part (a), one has $\mathbf{A}^2 = \mathbf{0}$ if $a+d = 0$ and $ad = bc$, so one example with $\mathbf{A}^2 = \mathbf{0} \neq \mathbf{A}$ is

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

(c) By part (a), $\mathbf{A}^3 = (a+d)\mathbf{A}^2 - (ad-bc)\mathbf{A}$. So $\mathbf{A}^3 = \mathbf{0}$ implies that $(a+d)\mathbf{A}^2 = (ad-bc)\mathbf{A}$ and then, multiplying each side by $\mathbf{A}$ once more, that $(a+d)\mathbf{A}^3 = (ad-bc)\mathbf{A}^2$. If $\mathbf{A}^3 = \mathbf{0}$, therefore, one has two cases:

$$\text{(i)}\ \ \mathbf{A}^2 = \mathbf{0}; \qquad\qquad \text{(ii)}\ \ ad - bc = 0.$$

But in case (ii), one has $(a+d)\mathbf{A}^2 = \mathbf{0}$, which gives rise to two subcases:

$$\text{(ii)(a)}\ \ \mathbf{A}^2 = \mathbf{0}; \qquad\qquad \text{(ii)(b)}\ \ ad - bc = 0 \text{ and } a + d = 0.$$

Now, even in case (ii)(b), the result of part (a) implies that $\mathbf{A}^2 = \mathbf{0}$, which is therefore true in every case.

15.5 The transpose

6. In general, for any natural number $n > 3$, one has

$$[(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_{n-1})\mathbf{A}_n]' = \mathbf{A}_n'[\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_{n-1}]'.$$

As the induction hypothesis, suppose the result is true for $n-1$. Then the last expression becomes $\mathbf{A}_n'\mathbf{A}_{n-1}'\cdots,\mathbf{A}_2'\mathbf{A}_1'$, so the result is true for n.

8. (a) $\mathbf{TS} = \mathbf{S}$ is shown in the book. A similar argument shows that $\mathbf{T}^2 = \frac{1}{2}\mathbf{T} + \frac{1}{2}\mathbf{S}$. To prove the last equality, we do not have to consider the individual elements. Instead, we premultiply the last equation by $\mathbf{T}$ and then use $\mathbf{TS} = \mathbf{S}$ to obtain

$$\mathbf{T}^3 = \mathbf{TT}^2 = \mathbf{T}\left(\tfrac{1}{2}\mathbf{T} + \tfrac{1}{2}\mathbf{S}\right) = \tfrac{1}{2}\mathbf{T}^2 + \tfrac{1}{2}\mathbf{TS} = \tfrac{1}{2}\left(\tfrac{1}{2}\mathbf{T} + \tfrac{1}{2}\mathbf{S}\right) + \tfrac{1}{2}\mathbf{S} = \tfrac{1}{4}\mathbf{T} + \tfrac{3}{4}\mathbf{S}.$$

(b) We prove by induction that the appropriate formula is

$$\mathbf{T}^n = 2^{1-n}\mathbf{T} + (1 - 2^{1-n})\mathbf{S}. \tag{$*$}$$

This formula is correct for $n = 1$.[38] Suppose $(*)$ is true for $n = k$. Then premultiplying by $\mathbf{T}$ and using the two first equalities in (a), one obtains

$$\begin{aligned}\mathbf{T}^{k+1} &= \mathbf{TT}^k = \mathbf{T}(2^{1-k}\mathbf{T} + (1-2^{1-k})\mathbf{S}) = 2^{1-k}\mathbf{T}^2 + (1-2^{1-k})\mathbf{TS} = \\ &2^{1-k}(\tfrac{1}{2}\mathbf{T} + \tfrac{1}{2}\mathbf{S}) + (1-2^{1-k})\mathbf{S} = 2^{-k}\mathbf{T} + 2^{-k}\mathbf{S} + \mathbf{S} - 2\cdot 2^{-k}\mathbf{S} = 2^{-k}\mathbf{T} + (1-2^{-k})\mathbf{S}\end{aligned}$$

which is formula $(*)$ for $n = k+1$.

15.6 Gaussian elimination

3. Apply the following elementary operations successively:

   (i) subtract the third equation from the first;

   (ii) subtract the new first equation from the two others;

   (iii) interchange the second and the third equation;

   (iv) multiply the second equation by -3 and add it to the third equation.

The result is

$$\begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{pmatrix} 2 & 1 & 4 & 3 & 1 \\ 1 & 3 & 2 & -1 & 3c \\ 1 & 1 & 2 & 1 & c^2 \end{pmatrix} \end{array} \sim \begin{pmatrix} 1 & 0 & 2 & 2 & 1-c^2 \\ 0 & 1 & 0 & -1 & 2c^2-1 \\ 0 & 0 & 0 & 0 & -5c^2+3c+2 \end{pmatrix}$$

The last row of the last matrix indicates that the system has solutions if and only if $-5c^2 + 3c + 2 = 0$, that is, if and only if $c = 1$ or $c = -2/5$. For these particular values of c we get the solutions in the book.[39]

[38] And, by part (a), also for $n = 2, 3$.

[39] The final answer can take many equivalent forms depending on how you arrange the elementary operations.

4. After moving the first row down to row number three, then applying the indicated elementary row operations, we obtain the successive augmented matrices

$$\begin{pmatrix} 1 & 2 & 1 & b_2 \\ 3 & 4 & 7 & b_3 \\ a & 1 & a+1 & b_1 \end{pmatrix} \begin{array}{c} -3 \quad -a \\ \\ \\ \end{array} \sim \begin{pmatrix} 1 & 2 & 1 & b_2 \\ 0 & -2 & 4 & b_3 - 3b_2 \\ 0 & 1-2a & 1 & b_1 - ab_2 \end{pmatrix} \begin{array}{c} \\ \times -\frac{1}{2} \\ \\ \end{array}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & b_2 \\ 0 & 1 & -2 & \frac{3}{2}b_2 - \frac{1}{2}b_3 \\ 0 & 1-2a & 1 & b_1 - ab_2 \end{pmatrix} \begin{array}{c} \\ 2a-1 \\ \\ \end{array}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & b_2 \\ 0 & 1 & -2 & \frac{3}{2}b_2 - \frac{1}{2}b_3 \\ 0 & 0 & 3-4a & b_1 + (2a - \frac{3}{2})b_2 + (\frac{1}{2} - a)b_3 \end{pmatrix}$$

Looking at the element in row 3 and column 3 of the final augmented matrix makes it obvious that there is a unique solution if and only if $3 - 4a \neq 0$ or equivalently $a \neq 3/4$.

5. Put $a = 3/4$ in Exercise 15.6.4. Then the last row in the last augmented matrix in the above solution becomes $(0, 0, 0, b_1 - \frac{1}{4}b_3)$. It follows that if $b_1 \neq \frac{1}{4}b_3$ there is no solution. If $b_1 = \frac{1}{4}b_3$ there is an infinite set of solutions. For an arbitrary real t, there is a unique solution with $z = t$. Then the second equation gives $y = \frac{3}{2}b_2 - \frac{1}{2}b_3 + 2t$, and finally the first equation gives $x = -2b_2 + b_3 - 5t$.

15.8 Geometric interpretation of vectors

2. (a) See the answer given in the book.

   (b) See Fig. A15.8.2 in the book. According to the point–point formula, the line through $(3, 1)$ and $(-1, 2)$ has the equation $x_2 = -\frac{1}{4}x_1 + \frac{7}{4}$ or $x_1 + 4x_2 = 7$. Call this line L. Any point (x_1, x_2) on L satisfies $x_1 + 4x_2 = 7$ and equals $(-1 + 4\lambda, 2 - \lambda)$ for $\lambda = \frac{1}{4}(x_1 + 1) = 2 - x_2$. So there is a one-to-one correspondence between points:
   (i) that lie on the line segment joining $\mathbf{a} = (3, 1)$ and $\mathbf{b} = (-1, 2)$;
   (ii) whose coordinates can be written as $(-1 + 4\lambda, 2 - \lambda)$ for some λ in $[0, 1]$.

   (c) This is, precisely, line L: as we let $\lambda \in \mathbb{R}$, the previous argument gives us a one-to-one correspondence between points:
   (i) that lie on the line L going through $\mathbf{a} = (3, 1)$ and $\mathbf{b} = (-1, 2)$;
   (ii) whose coordinates can be written as $(-1 + 4\lambda, 2 - \lambda)$ for some λ in $\mathbb{R}$.

15.9 Lines and planes

3. First, note that $(5, 2, 1) - (1, 0, 2) = (4, 2, -1)$ and $(2, -1, 4) - (1, 0, 2) = (1, -1, 2)$ are two vectors in the plane. The normal (p_1, p_2, p_3) to the plane must be orthogonal to both these vectors, so $(4, 2, -1) \cdot (p_1, p_2, p_3) = 4p_1 + 2p_2 - p_3 = 0$ and $(1, -1, 2) \cdot (p_1, p_2, p_3) = p_1 - p_2 + 2p_3 = 0$. One solution of these two equations is $(p_1, p_2, p_3) = (1, -3, -2)$. Then using formula (4) with $(a_1, a_2, a_3) = (2, -1, 4)$ yields $(1, -3, -2) \cdot (x_1 - 2, x_2 + 1, x_3 - 4) = 0$, or $x_1 - 3x_2 - 2x_3 = -3$.[40]

[40] A more pedestrian approach is to assume that the equation is $ax + by + cz = d$ and require the three points to satisfy the equation: $a + 2c = d$, $5a + 2b + c = d$, $2a - b + 4c = d$. Solve for a, b, and c in terms of d, insert the results

7. (a) See the answer given in the book.

   (b) By elementary operations:

$$
\begin{pmatrix} 2 & 2 & -1 & 2 \\ 1 & -3 & 1 & 0 \\ 3 & 4 & -1 & 1 \end{pmatrix}
\sim
\begin{pmatrix} 1 & -3 & 1 & 0 \\ 2 & 2 & -1 & 2 \\ 3 & 4 & -1 & 1 \end{pmatrix}
\begin{matrix} -2 \quad -3 \\ \\ \\ \end{matrix}
\sim
\begin{pmatrix} 1 & -3 & 1 & 0 \\ 0 & 8 & -3 & 2 \\ 0 & 13 & -4 & 1 \end{pmatrix}
\begin{matrix} \\ 1/8 \\ \\ \end{matrix}
$$

$$
\sim
\begin{pmatrix} 1 & -3 & 1 & 0 \\ 0 & 1 & -3/8 & 1/4 \\ 0 & 13 & -4 & 1 \end{pmatrix}
\begin{matrix} \\ -13 \\ \\ \end{matrix}
\sim
\begin{pmatrix} 1 & -3 & 1 & 0 \\ 0 & 1 & -3/8 & 1/4 \\ 0 & 0 & 7/8 & -9/4 \end{pmatrix}
\begin{matrix} \\ \\ 8/7 \end{matrix}
$$

$$
\sim
\begin{pmatrix} 1 & -3 & 1 & 0 \\ 0 & 1 & -3/8 & 1/4 \\ 0 & 0 & 1 & -18/7 \end{pmatrix}
\begin{matrix} \\ 3 \\ \\ \end{matrix}
$$

$$
\sim
\begin{pmatrix} 1 & 0 & -1/8 & 3/4 \\ 0 & 1 & -3/8 & 1/4 \\ 0 & 0 & 1 & -18/7 \end{pmatrix}
\begin{matrix} \\ \\ 3/8 \quad 1/8 \end{matrix}
\sim
\begin{pmatrix} 1 & 0 & 0 & 3/7 \\ 0 & 1 & 0 & -5/7 \\ 0 & 0 & 1 & -18/7 \end{pmatrix}.
$$

   The solution is $x_1 = 3/7$, $x_2 = -5/7$, $x_3 = -18/7$.

   (c) We again use elementary operations:

$$
\begin{pmatrix} 1 & 3 & 4 & 0 \\ 5 & 1 & 1 & 0 \end{pmatrix}
\begin{matrix} -5 \\ \\ \end{matrix}
\sim
\begin{pmatrix} 1 & 3 & 4 & 0 \\ 0 & -14 & -19 & 0 \end{pmatrix}
\begin{matrix} \\ -1/14 \end{matrix}
$$

$$
\sim
\begin{pmatrix} 1 & 3 & 4 & 0 \\ 0 & 1 & 19/14 & 0 \end{pmatrix}
\begin{matrix} \\ -3 \end{matrix}
\sim
\begin{pmatrix} 1 & 0 & -1/14 & 0 \\ 0 & 1 & 19/14 & 0 \end{pmatrix}
$$

   The solution is $x_1 = (1/14)x_3$, $x_2 = -(19/14)x_3$, for arbitrary x_3. That is, there is one degree of freedom.

10. (a) See the answer given in the book.

    (b) In part (a) we saw that $\mathbf{a}$ can be produced even without throwing away outputs.
    For $\mathbf{b}$ to be possible if we are allowed to throw away output, there must exist a scalar λ in $[0, 1]$ such that $6\lambda + 2 \geq 7$, $-2\lambda + 6 \geq 5$, and $-6\lambda + 10 \geq 5$.
    These inequalities reduce to $\lambda \geq 5/6$, $\lambda \leq 1/2$, $\lambda \leq 5/6$, which are incompatible.

    (c) Revenue is $R(\lambda) = p_1x_1 + p_2x_2 + p_3x_3 = (6p_1 - 2p_2 - 6p_3)\lambda + 2p_1 + 6p_2 + 10p_3$.
    If the constant slope $6p_1 - 2p_2 - 6p_3$ is > 0, then $R(\lambda)$ is maximized at $\lambda = 1$;
    if $6p_1 - 2p_2 - 6p_3$ is < 0, then $R(\lambda)$ is maximized at $\lambda = 0$.
    Only in the special case where $6p_1 - 2p_2 - 6p_3 = 0$ can the two plants both remain in use.

11. (a) If $\mathbf{PQ} - \mathbf{QP} = \mathbf{P}$, then $\mathbf{PQ} = \mathbf{QP} + \mathbf{P}$, and so

$$\mathbf{P}^2\mathbf{Q} = \mathbf{P}(\mathbf{PQ}) = \mathbf{P}(\mathbf{QP} + \mathbf{P}) = (\mathbf{PQ})\mathbf{P} + \mathbf{P}^2 = (\mathbf{QP} + \mathbf{P})\mathbf{P} + \mathbf{P}^2 = \mathbf{QP}^2 + 2\mathbf{P}^2.$$

    Thus, $\mathbf{P}^2\mathbf{Q} - \mathbf{QP}^2 = 2\mathbf{P}^2$. Moreover,

$$\mathbf{P}^3\mathbf{Q} = \mathbf{P}(\mathbf{P}^2\mathbf{Q}) = \mathbf{P}(\mathbf{QP}^2 + 2\mathbf{P}^2) = (\mathbf{PQ})\mathbf{P}^2 + 2\mathbf{P}^3 = (\mathbf{QP} + \mathbf{P})\mathbf{P}^2 + 2\mathbf{P}^3 = \mathbf{QP}^3 + 3\mathbf{P}^3.$$

---

into the equation $ax + by + cz = d$ and cancel d.

Hence, $\mathbf{P}^3\mathbf{Q} - \mathbf{Q}\mathbf{P}^3 = 3\mathbf{P}^3$.

To prove the result for general k, suppose that $\mathbf{P}^n\mathbf{Q} - \mathbf{Q}\mathbf{P}^n = n\mathbf{P}^n$ as the induction hypothesis for $n = k$. Then, for $n = k + 1$, one has

$$\begin{aligned}\mathbf{P}^n\mathbf{Q} = \mathbf{P}(\mathbf{P}^k\mathbf{Q}) = \mathbf{P}(\mathbf{Q}\mathbf{P}^k + k\mathbf{P}^k) = (\mathbf{P}\mathbf{Q})\mathbf{P}^k + k\mathbf{P}^{k+1} \\ = (\mathbf{Q}\mathbf{P} + \mathbf{P})\mathbf{P}^k + k\mathbf{P}^{k+1} = \mathbf{Q}\mathbf{P}^{k+1} + (k+1)\mathbf{P}^{k+1},\end{aligned}$$

so the induction hypothesis is also true for $n = k + 1$.

16 Determinants and Inverse Matrices

16.1 Determinants of order 2

9. (a) See the answer given in the book.

   (b) The suggested substitutions produce the two equations

   $$Y_1 = c_1Y_1 + A_1 + m_2Y_2 - m_1Y_1; \qquad Y_2 = c_2Y_2 + A_2 + m_1Y_1 - m_2Y_2$$

   or

   $$(1 - c_1 - m_1)Y_1 - m_2Y_2 = A_1; \qquad -m_1Y_1 + (1 - c_2 - m_2)Y_2 = A_2$$

   This can be rewritten in the matrix form

   $$\begin{pmatrix} 1 - c_1 - m_1 & -m_2 \\ -m_1 & 1 - c_2 - m_2 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

   For these equations to be soluble, we need to assume that $D = (1 - c_1 - m_1)(1 - c_2 - m_2) - m_1m_2 \neq 0$. When $D \neq 0$, the answers in the text can be derived using Cramer's rule.

   (c) Y_2 depends linearly on A_1. Economists usually assume that D given in part (b) is positive, as it will be provided that the parameters c_1, c_2, m_1, m_2 are all sufficiently small. Then increasing A_1 by one unit changes Y_2 by the factor $m_1/D \geq 0$, so Y_2 increases when A_1 increases.

   Here is an economic explanation: An increase in A_1 increases nation 1's income, Y_1. This in turn increases nation 1's imports, M_1. However, nation 1's imports are nation 2's exports, so this causes nation 2's income, Y_2, to increase, and so on.

16.2 Determinants of order 3

1. (a) Sarrus's rule yields $\begin{vmatrix} 1 & -1 & 0 \\ 1 & 3 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 0 - 2 + 0 - 0 - 0 - 0 = -2.$

   (b) By Sarrus's rule $\begin{vmatrix} 1 & -1 & 0 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{vmatrix} = 3 - 2 - 0 - 0 - 4 - (-1) = -2.$

(c) Because $a_{21} = a_{31} = a_{32} = 0$, the only non-zero term in the expansion (16.2.2) is the product of the terms on the main diagonal. The determinant is therefore adf.[41]

(d) By Sarrus's rule, $\begin{vmatrix} a & 0 & b \\ 0 & e & 0 \\ c & 0 & d \end{vmatrix} = aed + 0 + 0 - bec - 0 - 0 = e(ad - bc)$.

3. (a) The determinant of the coefficient matrix is $|\mathbf{A}| = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & -1 \end{vmatrix} = -4$.

The numerators in (16.2.6) are

$$\begin{vmatrix} 2 & -1 & 1 \\ 0 & 1 & -1 \\ -6 & -1 & -1 \end{vmatrix} = -4\,, \qquad \begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ -1 & -6 & -1 \end{vmatrix} = -8\,, \qquad \begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \\ -1 & -1 & -6 \end{vmatrix} = -12$$

Hence, (16.2.6) yields the solution $x_1 = 1$, $x_2 = 2$, and $x_3 = 3$. Inserting this into the original system of equations confirms that this is a correct answer.

(b) The determinant of the coefficient matrix is equal to -2, and the numerators in (16.2.6) are all 0, so the unique solution is $x_1 = x_2 = x_3 = 0$.

(c) Follow the pattern in part (a) to get the answer in the book.

8. (a) Substituting $T = d + tY$ into the expression for C gives $C = a - bd + b(1-t)Y$. Substituting for C in the expression for Y then yields $Y = a + b(Y - d - tY) + A_0$. Then solve for Y, T, and C in turn to derive the answers given in (b) below.

(b) We write the system as $\begin{pmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{pmatrix} \begin{pmatrix} Y \\ C \\ T \end{pmatrix} = \begin{pmatrix} A_0 \\ a \\ d \end{pmatrix}$.

With $D = \begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix} = 1 + bt - b$, Cramer's rule yields[42]

$$Y = \frac{1}{D} \begin{vmatrix} A_0 & -1 & 0 \\ a & 1 & b \\ d & 0 & 1 \end{vmatrix} = \frac{a - bd + A_0}{1 - b(1-t)}, \qquad C = \frac{1}{D} \begin{vmatrix} 1 & A_0 & 0 \\ -b & a & b \\ -t & d & 1 \end{vmatrix} = \frac{a - bd + A_0 b(1-t)}{1 - b(1-t)},$$

and $T = \dfrac{1}{D} \begin{vmatrix} 1 & -1 & A_0 \\ -b & 1 & a \\ -t & 0 & d \end{vmatrix} = \dfrac{t(a + A_0) + (1-b)d}{1 - b(1-t)}$.

16.3 Determinants in general

1. (a) Each of the three determinants is a sum of $4! = 24$ terms. In (a) there is only one nonzero term. In fact, according to (16.3.4), the value of the determinant is 24.

[41] Alternatively, Sarrus's rule gives the same answer.

[42] This problem is meant to train you in using Cramer's rule. It is also a warning against its overuse, since solving the equations by systematic elimination is much more efficient.

(b) Only two terms in the sum are nonzero: the product of the elements on the main diagonal, which is $1 \cdot 1 \cdot 1 \cdot d$, with a plus sign; and the term

$$\begin{vmatrix} 1 & 0 & 0 & \boxed{1} \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ \boxed{a} & b & c & d \end{vmatrix}$$

Since there are 5 rising lines between the pairs, the sign of the product $1 \cdot 1 \cdot 1 \cdot a$ must be minus. So the value of the determinant is $d - a$. (c) 4 terms are nonzero. See the answer given in the book.

16.4 Basic rules for determinants

14. The answer in the text amounts to the following successive transformations

$$D_n = \begin{vmatrix} a+b & a & \cdots & a \\ a & a+b & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & a \end{vmatrix} = \begin{vmatrix} na+b & na+b & \cdots & na+b \\ a & a+b & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & a+b \end{vmatrix}$$

$$= (na+b) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & a+b & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & a+b \end{vmatrix} = (na+b) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & b & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b \end{vmatrix}$$

According to (16.3.4), the last determinant is b^{n-1}. Thus $D_n = (na+b)b^{n-1}$.

16.5 Expansion by cofactors

1. (a) See the answer given in the book.

(b) One possibility is to expand by the second row or the third column, because they both have two zero entries. But it is easier first to use elementary operations to get a row or a column with at most one non-zero element. For example:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 0 & 11 \\ 2 & -1 & 0 & 3 \\ -2 & 0 & -1 & 3 \end{vmatrix} \begin{array}{cc} -2 & 2 \\ & \\ \leftarrow & \\ & \leftarrow \end{array} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 0 & 11 \\ 0 & -5 & -6 & -5 \\ 0 & 4 & 5 & 11 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 11 \\ -5 & -6 & -5 \\ 4 & 5 & 11 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 & 11 \\ 0 & -6 & -60 \\ 0 & 5 & 55 \end{vmatrix} \begin{array}{cc} -5 & 4 \\ \leftarrow & \\ & \leftarrow \end{array} = -1 \begin{vmatrix} -6 & -60 \\ 5 & 55 \end{vmatrix} = -(-330 + 300) = 30$$

(c) See the book for the simple answer. When computing determinants one can use elementary column as well as row operations, but column operations become meaningless when solving linear equation systems using Gaussian elimination.

16.6 The inverse of a matrix

8. By direct computation, $\mathbf{B}^2 + \mathbf{B} = \begin{pmatrix} 3/2 & -5 \\ -1/4 & 3/2 \end{pmatrix} + \begin{pmatrix} -1/2 & 5 \\ 1/4 & -1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$.

   One can either verify by direct matrix multiplication that $\mathbf{B}^3 - 2\mathbf{B} + \mathbf{I} = \mathbf{0}$, or somewhat more easily, use the relation $\mathbf{B}^2 + \mathbf{B} = \mathbf{I}$ to argue that $\mathbf{B}^2 = \mathbf{I} - \mathbf{B}$ and so

   $$\mathbf{B}^3 - 2\mathbf{B} + \mathbf{I} = \mathbf{B}(\mathbf{I} - \mathbf{B}) - 2\mathbf{B} + \mathbf{I} = \mathbf{B} - \mathbf{B}^2 - 2\mathbf{B} + \mathbf{I} = -\mathbf{B}^2 - \mathbf{B} + \mathbf{I} = \mathbf{0}.$$

   Furthermore $\mathbf{B}^2 + \mathbf{B} = \mathbf{I}$ implies that $\mathbf{B}(\mathbf{B} + \mathbf{I}) = \mathbf{I}$. It follows from (16.6.4) that $\mathbf{B}^{-1} = \mathbf{B} + \mathbf{I}$.

16.7 A general formula for the inverse

1. (a) $|\mathbf{A}| = 10 - 12 = -2$, and the adjugate is $\begin{pmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix}$, so the inverse is

   $$\mathbf{A}^{-1} = -\frac{1}{2}\begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} -5/2 & 3/2 \\ 2 & -1 \end{pmatrix}$$

   (b) The adjugate of $\mathbf{B}$ is $\operatorname{adj}\mathbf{B} = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = \begin{pmatrix} 1 & 4 & 2 \\ 2 & -1 & 4 \\ 4 & -2 & -1 \end{pmatrix}$

   and $|\mathbf{B}| = b_{11}C_{11} + b_{21}C_{21} + b_{31}C_{31} = 1 \cdot 1 + 2 \cdot 4 + 0 \cdot 2 = 9$ by expansion along the first column. Hence,

   $$\mathbf{B}^{-1} = \frac{1}{9}(\operatorname{adj}\mathbf{B}) = \frac{1}{9}\begin{pmatrix} 1 & 4 & 2 \\ 2 & -1 & 4 \\ 4 & -2 & -1 \end{pmatrix}$$

   (c) Since the second column of $\mathbf{C}$ equals -2 times its third column, the determinant of $\mathbf{C}$ is zero, so there is no inverse.

3. The determinant of $\mathbf{I} - \mathbf{A}$ is $|\mathbf{I} - \mathbf{A}| = 0.496$, and the adjugate is

   $$\operatorname{adj}(\mathbf{I} - \mathbf{A}) = \begin{pmatrix} 0.72 & 0.64 & 0.40 \\ 0.08 & 0.76 & 0.32 \\ 0.16 & 0.28 & 0.64 \end{pmatrix}.$$

   Hence $(\mathbf{I} - \mathbf{A})^{-1} = \dfrac{1}{0.496}\operatorname{adj}(\mathbf{I} - \mathbf{A}) \approx \begin{pmatrix} 1.45161 & 1.29032 & 0.80645 \\ 0.16129 & 1.53226 & 0.64516 \\ 0.32258 & 0.56452 & 1.29032 \end{pmatrix}$,

   rounded to five decimal places. For an exact answer, note that $1000/496 = 125/62$

   and $\operatorname{adj}(\mathbf{I} - \mathbf{A}) = \begin{pmatrix} 0.72 & 0.64 & 0.40 \\ 0.08 & 0.76 & 0.32 \\ 0.16 & 0.28 & 0.64 \end{pmatrix} = \dfrac{1}{25}\begin{pmatrix} 18 & 16 & 10 \\ 2 & 19 & 8 \\ 4 & 7 & 16 \end{pmatrix}$,

   which gives $(\mathbf{I} - \mathbf{A})^{-1} = \dfrac{5}{62}\begin{pmatrix} 18 & 16 & 10 \\ 2 & 19 & 8 \\ 4 & 7 & 16 \end{pmatrix}$.

4. Let $\mathbf{B}$ denote the $n \times p$ matrix whose k^{th} column has the elements $b_{1k}, b_{2k}, \dots, b_{nk}$. The p systems of n equations in n unknowns can be expressed as $\mathbf{AX} = \mathbf{B}$, where $\mathbf{A}$ is $n \times n$ and $\mathbf{X}$ is $n \times p$. Following the method illustrated in Example 2, exactly the same row operations that transform the $n \times 2n$ matrix $(\mathbf{A} : \mathbf{I})$ into $(\mathbf{I} : \mathbf{A}^{-1})$ will also transform the $n \times (n+p)$ matrix $(\mathbf{A} : \mathbf{B})$ into $(\mathbf{I} : \mathbf{B}^*)$, where $\mathbf{B}^*$ is the matrix with elements b^*_{ij}.[43] When $k = r$, the solution to the system is $x_1 = b^*_{1r}$, $x_2 = b^*_{2r}$, … , $x_n = b^*_{nr}$.

5. (a) Using elementary operations,

$$\left(\begin{array}{cc|cc}1&2&1&0\\3&4&0&1\end{array}\right)\begin{array}{c}-3\\ \end{array} \sim \left(\begin{array}{cc|cc}1&2&1&0\\0&-2&-3&1\end{array}\right)\begin{array}{c}\\ -\frac{1}{2}\end{array}$$

$$\sim \left(\begin{array}{cc|cc}1&2&1&0\\0&1&\frac{3}{2}&-\frac{1}{2}\end{array}\right)\begin{array}{c}\\ -2\end{array} \sim \left(\begin{array}{cc|cc}1&0&-2&1\\0&1&\frac{3}{2}&-\frac{1}{2}\end{array}\right)$$

(b) In this case,

$$\left(\begin{array}{ccc|ccc}1&2&3&1&0&0\\2&4&5&0&1&0\\3&5&6&0&0&1\end{array}\right)\begin{array}{c}-2 \quad -3\\ \\ \end{array} \sim \left(\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&0&-1&-2&1&0\\0&-1&-3&-3&0&1\end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&-1&-3&-3&0&1\\0&0&-1&-2&1&0\end{array}\right)\begin{array}{c}\\ \times(-1)\\ \times(-1)\end{array} \sim \left(\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&1&3&3&0&-1\\0&0&1&2&-1&0\end{array}\right)\begin{array}{c}\\ -2\\ \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc}1&0&-3&-5&0&2\\0&1&3&3&0&-1\\0&0&1&2&-1&0\end{array}\right)\begin{array}{c}\\ \\ -3 \quad 3\end{array} \sim \left(\begin{array}{ccc|ccc}1&0&0&1&-3&2\\0&1&0&-3&3&-1\\0&0&1&2&-1&0\end{array}\right)$$

(c) We see that the third row equals the first row multiplied by -3, so the matrix has no inverse.

16.8 Cramer's Rule

1. (a) The determinant $|\mathbf{A}|$ of the coefficient matrix is

$$|\mathbf{A}| = \begin{vmatrix}1&2&-1\\2&-1&1\\1&-1&-3\end{vmatrix} = 19.$$

The determinants in (16.8.2) are

$$\begin{vmatrix}-5&2&-1\\6&-1&1\\-3&-1&-3\end{vmatrix} = 19, \quad \begin{vmatrix}1&-5&-1\\2&6&1\\1&-3&-3\end{vmatrix} = -38, \quad \begin{vmatrix}1&2&-5\\2&-1&6\\1&-1&-3\end{vmatrix} = 38$$

According to (16.8.4) the solution is

$$x = 19/19 = 1, \quad y = -38/19 = -2, \quad \text{and} \quad z = 38/19 = 2$$

Inserting this into the original system of equations confirms that this is the correct answer.

[43] In fact, because these row operations are together equivalent to premultiplication by $\mathbf{A}^{-1}$, it must be true that $\mathbf{B}^* = \mathbf{A}^{-1}\mathbf{B}$.

(b) The determinant $|\mathbf{A}|$ of the coefficient matrix is $\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{vmatrix}$. Subtracting the fourth column from the second leaves only one non-zero element in the second column, and so reduces the determinant to $-\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = -1$.

We ask you to check that the other determinants in (16.8.2) are

$$\begin{vmatrix} 3 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 6 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{vmatrix} = 3\,, \quad \begin{vmatrix} 1 & 3 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 6 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{vmatrix} = -6\,, \quad \begin{vmatrix} 1 & 1 & 3 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 6 & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix} = -5\,, \quad \begin{vmatrix} 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 6 \\ 0 & 1 & 0 & 1 \end{vmatrix} = 5.$$

According to (16.8.4) the solution is $x = -3$, $y = 6$, $z = 5$, and $u = -5$. Inserting this into the original system of equations confirms that this is the correct answer.[44]

3. According to Theorem 16.8.2, the system has nontrivial solutions if and only if the determinant of the coefficient *equal* to 0. Expansion along the first row gives

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a\begin{vmatrix} c & a \\ a & b \end{vmatrix} - b\begin{vmatrix} b & a \\ c & b \end{vmatrix} + c\begin{vmatrix} b & c \\ c & a \end{vmatrix} = a(bc - a^2) - b(b^2 - ac) + c(ab - c^2)$$
$$= 3abc - a^3 - b^3 - c^3.$$

Thus the system has nontrivial solutions if and only if $3abc - a^3 - b^3 - c^3 = 0$.

Review exercises for Chapter 16

5. One of many ways to evaluate the determinant is to expand along column 3, which gives

$$|\mathbf{A}| = \begin{vmatrix} q & -1 & q-2 \\ 1 & -p & 2-p \\ 2 & -1 & 0 \end{vmatrix} = (q-2)\begin{vmatrix} 1 & -p \\ 2 & -1 \end{vmatrix} - (2-p)\begin{vmatrix} q & -1 \\ 2 & -1 \end{vmatrix}$$
$$= (q-2)(-1+2p) - (2-p)(-q+2) = (q-2)(p+1),$$

Now, $|\mathbf{A} + \mathbf{E}| = \begin{vmatrix} q+1 & 0 & q-1 \\ 2 & 1-p & 3-p \\ 3 & 0 & 1 \end{vmatrix} = (1-p)\begin{vmatrix} q+1 & q-1 \\ 3 & 1 \end{vmatrix} = 2(p-1)(q-2)$.

For the rest, see the answer in the text.

[44] Of course, there is a much quicker way to solve these four equations. Subtracting the fourth from the third yields $z = 5$ immediately. Then the second equation gives $x = -3$; the first gives $y = 6$; and the last gives $u = -5$.

8. (a) This becomes easy after noting that

$$\mathbf{U}^2 = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} = \begin{pmatrix} n & n & \dots & n \\ n & n & \dots & n \\ \vdots & \vdots & \ddots & \vdots \\ n & n & \dots & n \end{pmatrix} = n\mathbf{U}$$

(b) The trick is to note that

$$\mathbf{A} = \begin{pmatrix} 4 & 3 & 3 \\ 3 & 4 & 3 \\ 3 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} = \mathbf{I}_3 + 3\mathbf{U}$$

From (a), $(\mathbf{I}_3 + 3\mathbf{U})(\mathbf{I}_3 + b\mathbf{U}) = \mathbf{I}_3 + (3 + b + 3 \cdot 3b\mathbf{U}) = \mathbf{I}_3 + (3 + 10b)\mathbf{U}$. This can be made equal to $\mathbf{I}_3$ by choosing $b = -3/10$. It follows that

$$\mathbf{A}^{-1} = (\mathbf{I}_3 + 3\mathbf{U})^{-1} = \mathbf{I}_3 - (3/10)\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{3}{10} & \frac{3}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{3}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{3}{10} & \frac{3}{10} \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 7 & -3 & -3 \\ -3 & 7 & -3 \\ -3 & -3 & 7 \end{pmatrix}$$

10. (a) Gaussian elimination with elementary row operations yields

$$\begin{pmatrix} a & 1 & 4 & 2 \\ 2 & 1 & a^2 & 2 \\ 1 & 0 & -3 & a \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3 & a \\ 2 & 1 & a^2 & 2 \\ a & 1 & 4 & 2 \end{pmatrix} \quad -2 \quad -a$$

$$\sim \begin{pmatrix} 1 & 0 & -3 & a \\ 0 & 1 & a^2+6 & -2a+2 \\ 0 & 1 & 3a+4 & -a^2+2 \end{pmatrix} \quad -1 \sim \begin{pmatrix} 1 & 0 & -3 & a \\ 0 & 1 & a^2+6 & -2a+2 \\ 0 & 0 & -a^2+3a-2 & -a^2+2a \end{pmatrix}$$

It follows that the system has a unique solution if and only if $-a^2 + 3a - 2 \neq 0$ — i.e., if and only if $a \neq 1$ and $a \neq 2$. If $a = 2$, the last row consists only of 0's so there are infinitely many solutions, whereas if $a = 1$, there are no solutions.

(b) If we perform the same elementary operations on the associated extended matrix as in part (a), then the fourth column is transformed as follows:

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \to \begin{pmatrix} b_3 \\ b_2 \\ b_1 \end{pmatrix} \to \begin{pmatrix} b_3 \\ b_2 - 2b_3 \\ b_1 - ab_3 \end{pmatrix} \to \begin{pmatrix} b_3 \\ b_2 - 2b_3 \\ b_1 - b_2 + (2-a)b_3 \end{pmatrix}.$$

Thus, the final extended matrix is $\begin{pmatrix} 1 & 0 & -3 & b_3 \\ 0 & 1 & a^2+6 & b_2 - 2b_3 \\ 0 & 0 & -a^2+3a-2 & b_1 - b_2 + (2-a)b_3 \end{pmatrix}$.

So there are infinitely many solutions if and only if all elements in the last row are 0, which is true if and only if: either (i) $a = 1$ and $b_1 - b_2 + b_3 = 0$; or (ii) $a = 2$ and $b_1 = b_2$.

15. (a) See the answer given in the book.

(b) The trick is to note that the cofactor expansions of $|\mathbf{A}|$, $|\mathbf{B}|$ and $|\mathbf{C}|$ along the r^{th} row take the respective forms $\sum_{j=1}^{n} a_{rj}C_{rj}$, $\sum_{j=1}^{n} b_{rj}C_{rj}$ and $\sum_{j=1}^{n}(a_{rj}+b_{rj})C_{rj}$ for exactly the same collection of cofactors C_{rj}, for $j = 1, 2, \ldots, n$. Then,

$$|\mathbf{C}| = \sum_{j=1}^{n}(a_{rj}+b_{rj})C_{rj} = \sum_{j=1}^{n} a_{rj}C_{rj} + \sum_{j=1}^{n} b_{rj}C_{rj} = |\mathbf{A}| + |\mathbf{B}|$$

16. It is a bad idea to use "brute force" here. Note instead that rows 1 and 3 in the determinant have "much in common" and so do rows 2 and 4. So begin by subtracting row 3 from row 1, and row 4 from row 2. According to Theorem 16.4.1(vi), this does not change the value of the determinant. This gives, if we thereafter use Theorem 16.4.1(iii),

$$\begin{vmatrix} 0 & a-b & 0 & b-a \\ b-a & 0 & a-b & 0 \\ x & b & x & a \\ a & x & b & x \end{vmatrix} = (a-b)^2 \begin{vmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ x & b & x & a \\ a & x & b & x \end{vmatrix} = (a-b)^2 \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ x & b & x & a+b \\ a & x & b & 2x \end{vmatrix}$$

The last equality is obtained by adding column 2 to column 4 in the middle determinant. If we expand the last determinant along row 1, we obtain successively

$$-(a-b)^2 \begin{vmatrix} -1 & 1 & 0 \\ x & x & a+b \\ a & b & 2x \end{vmatrix} = -(a-b)^2[-2x^2 + b(a+b) - 2x^2 + a(a+b)]$$
$$= (a-b)^2[4x^2 - (a+b)^2] = (a-b)^2[2x - (a+b)][2x + (a+b)].$$

The conclusion follows.

17 Linear Programming

17.1 A graphical approach

3. The set A corresponds to the shaded polygon in Fig. SM17.1.3. The following arguments explain the answers given in the book.

(a) The solution is obviously at the uppermost point P in the polygon because it has the largest x_2 coordinate among all points in A. Point P is where the two lines $-2x_1 + x_2 = 2$ and $x_1 + 2x_2 = 8$ intersect, and the solution of these two equations is $(x_1, x_2) = (4/5, 18/5)$.

(b) The point in A with the largest x_1 coordinate is obviously $Q = (8, 0)$.

(c) The line $3x_1 + 2x_2 = c$ for one typical value of c is the dashed line in Fig. SM17.1.3. As c increases, the line moves out farther and farther to the north-east. The line that has the largest value of c, and still has a point in common with A, is the one that passes through the point Q in the figure.

(d) The line $2x_1 - 2x_2 = c$ (or $x_2 = x_1 - c/2$) makes a 45° angle with the x_1 axis, and intersects the x_1 axis at $c/2$. As c decreases, the line moves up and to the left. The line in this family that has the smallest value of c, and still has a point in common with A, is the one that passes through the point P in the figure.

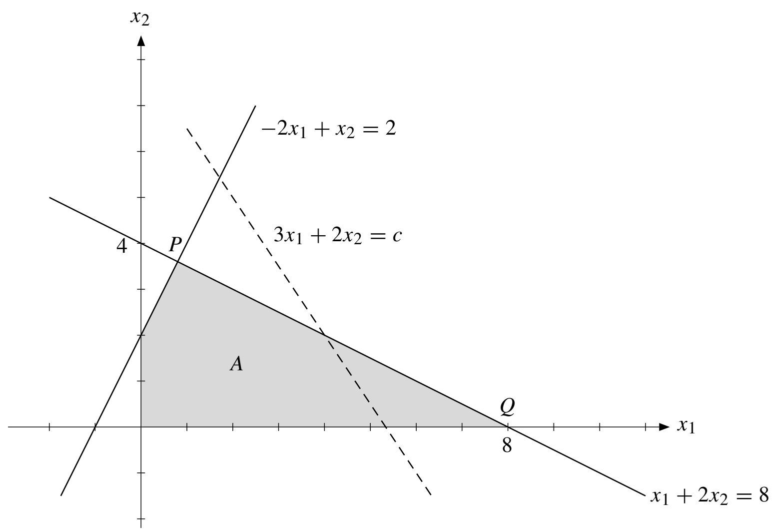

Figure SM17.1.3

(e) The line $2x_1 + 4x_2 = c$ is parallel to the line $x_1 + 2x_2 = 8$ in the figure. As c increases, the line moves out farther and farther to the north-east. The line with points in common with A that has the largest value of c is obviously the one that coincides with the line $x_1 + 2x_2 = 8$. So all points on the line segment between P and Q are solutions.

(f) The line $-3x_1 - 2x_2 = c$ is parallel to the dashed line in the figure, and intersects the x_1 axis at $-c/3$. As c decreases, the line moves out farther and farther to the north-east, so the solution is at $Q = (8, 0)$. (We could also argue like this: Minimizing $-3x_1 - 2x_2$ subject to $(x_1, x_2) \in A$ is obviously equivalent to maximizing $3x_1 + 2x_2$ subject to $(x_1, x_2) \in A$, so the solution is the same as the one in part (c).)

17.2 Introduction to Duality Theory

1. (a) See Fig. A17.1.1a in the answer section of the book. When $3x_1 + 2x_2 \leq 6$ is replaced by $3x_1 + 2x_2 \leq 7$ in Exercise 17.1.1, the feasible set expands because the steeper line through P is moved out to the right. The new optimal point is at the intersection of the lines $3x_1 + 2x_2 = 7$ and $x_1 + 4x_2 = 4$, and it follows that the solution is $(x_1, x_2) = (2, 1/2)$. The old maximum value of the objective function was 36/5. The new optimal value is $3 \cdot 2 + 4 \cdot \frac{1}{2} = 8 = 40/5$, and the difference in optimal value is $u_1^* = 4/5$.

   (b) When $x_1 + 4x_2 \leq 4$ is replaced by $x_1 + 4x_2 \leq 5$, the feasible set expands because the line $x_1 + 4x_2 = 4$ is moved up. The new optimal point is at the intersection of the lines $3x_1 + 2x_2 = 6$ and $x_1 + 4x_2 = 5$, and it follows that the solution is $(x_1, x_2) = (7/5, 9/10)$. The old maximum value of the objective function was 36/5. The new optimal value is 39/5, and the difference in optimal value is $u_2^* = 3/5$.

   (c) See the answer given in the book.

17.3 The Duality Theorem

1. (a) From Fig. A17.3.1a in the text it is clear that as c increases, the dashed line moves out farther and farther to the north-east. The line that has the largest value of c and still has a point in common with the feasible set, is the one that passes through the point P, which has coordinates $(x, y) = (0, 3)$, where the associated maximum value is $2 \cdot 0 + 3 \cdot 7 = 21$.

   (b) In Fig. A17.3.1b, as c decreases, the dashed line moves farther and farther to the south-west. The line that has the smallest value of c and still has a point in common with the feasible set, is the one that passes through the point P, which has coordinates $(u_1, u_2) = (0, 1)$. The associated minimum value is $20u_1 + 21u_2 = 21$.

   (c) Yes, because the maximum of the primal in (a) and the minimum of the dual in (b) both equal 21.

3. (a) See the answer given in the book.

   (b) The problem is illustrated graphically in Fig. SM17.3.3, which makes clear the answer given in the book.

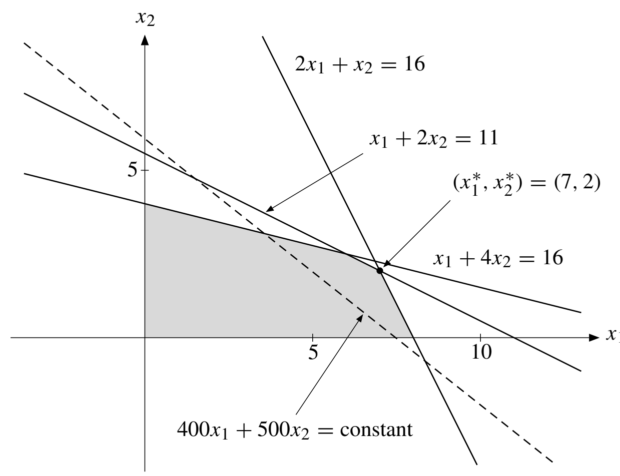

Figure SM17.3.3

   (c) Relaxing the first constraint to $2x_1 + x_2 \leq 17$ allows the solution to move out to the intersection of the two lines $2x_1 + x_2 = 17$ and $x_1 + 2x_2 = 11$. So the new solution is $x_1 = 23/3$, $x_2 = 5/3$, where the profit is 3900.

   Relaxing the second constraint to $x_1 + 4x_2 \leq 17$ makes no difference, because some capacity in division 2 remained unused anyway at $(7, 2)$.

   Relaxing the third constraint to $x_1 + 2x_2 \leq 12$, the solution is no longer where the lines $2x_1 + x_2 = 16$ and $x_1 + 2x_2 = 12$ intersect, namely at $(x_1, x_2) = (20/3, 8/3)$, because this would violate the second constraint $x_1 + 4x_2 \leq 16$. Instead, as a carefully drawn graph shows, the solution occurs where the first and second constraints both bind, at the intersection of the two lines $2x_1 + x_2 = 16$ and $x_1 + 4x_2 = 16$, namely at $(x_1, x_2) = (48/7, 16/7)$. The resulting profit is $27200/7 = 3885\frac{5}{7} < 3900$. So it is division 1 that should have its capacity increased. Indeed, if the capacity of division 3 is increased by 1 hour per day, some of that increase has to go to waste because of the limited capacities in divisions 1 and 2.

17.4 A general economic interpretation

2. (a) The problem is similar to Exercise 17.3.3. The linear program is set out in the book. As a carefully drawn graph will show, the solution occurs at the intersection of the two lines $6x_1 + 3x_2 = 54$ and $5x_1 + 5x_2 = 50$, where $(x_1, x_2) = (8, 2)$. Note that the maximum profit is $300 \cdot 8 + 200 \cdot 2 = 2800$.

   (b) The dual problem is

$$\min \ (54u_1 + 48u_2 + 50u_3) \quad \text{s.t.} \quad \begin{cases} 6u_1 + 4u_2 + 5u_3 \geq 300 \\ 3u_1 + 6u_2 + 5u_3 \geq 200 \\ \qquad u_1, u_2, u_3 \geq 0 \end{cases}$$

   The optimal solution of the primal is $x_1^* = 8$, $x_2^* = 2$. Since the optimal solution of the primal has both x_1^* and x_2^* positive, the first two constraints in the dual are satisfied with equality at the optimal triple (u_1^*, u_2^*, u_3^*). But the second constraint of the primal problem is satisfied with inequality, because $4x_1^* + 6x_2^* = 44 < 48$. Hence $u_2^* = 0$, with $6u_1^* + 5u_3^* = 300$ and $3u_1^* + 5u_3^* = 200$. It follows that $u_1^* = 100/3$, $u_2^* = 0$, and $u_3^* = 20$. Moreover, the optimum value $54u_1^* + 48u_2^* + 50u_3^* = 2800$ of the dual equals the optimum value of the primal.

   (c) See the answer given in the book.

17.5 Complementary slackness

3. (a) See the answer given in the book.

   (b) The dual is given in the text. It is illustrated graphically in Fig. SM17.5.3, where (1) labels the first, (2) the second, and (3) the third constraint. The parallel dashed lines are level curves of the objective $300x_1 + 500x_2$. We see from the figure that optimum occurs at the point where the first and the third constraint are satisfied with equality—i.e., where $10x_1^* + 20x_2^* = 10\,000$ and $20x_1^* + 20x_2^* = 11\,000$. The solution is $x_1^* = 100$ and $x_2^* = 450$. The maximum value of the objective function is $300 \cdot 100 + 500 \cdot 450 = 255\,000$.

   Since both variables in the dual problem are positive at the optimum, complementary slackness implies that the two constraints in the primal problem must be satisfied with equality at the optimum, that is,

$$10y_1^* + 20y_2^* + 20y_3^* = 300$$
$$20y_1^* + 10y_2^* + 20y_3^* = 500$$

   Moreover, the second constraint in the dual problem is satisfied with strict inequality at the optimum, since $20x_1^* + 10x_2^* = 6500 < 8000$. Therefore the corresponding variable in the primal problem must be zero at the optimum, so $y_2^* = 0$. Together with the two equations above this implies that $y_1^* = 20$, $y_3^* = 5$. The maximum value of the objective function is $10\,000 \cdot 20 + 8\,000 \cdot 0 + 11\,000 \cdot 5 = 255\,000$.

   (c) If the cost per hour in factory 1 increases by 100, this has no effect on the constraints in the primal, but does increase the right-hand side of the first constraint of the dual by 100. An approximate answer of $100 \cdot 20 = 2000$ is the increase in cost that would result

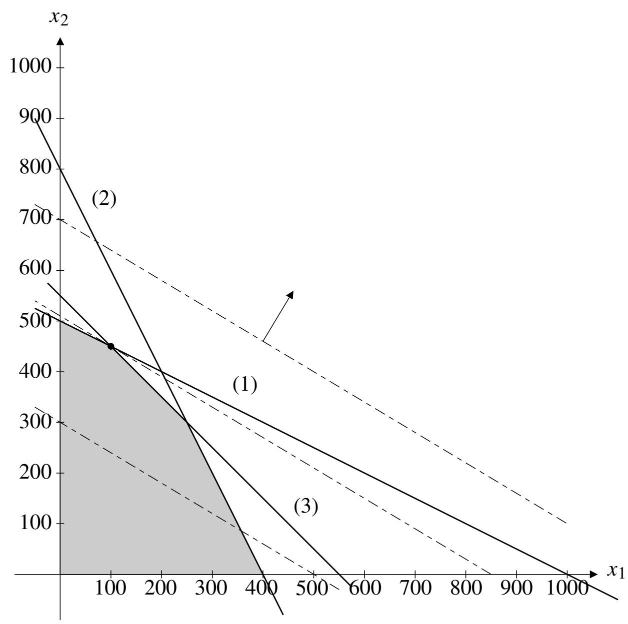

Figure SM17.5.3

from choosing the same feasible point $(y_1^*, y_2^*, y_3^*) = (20, 0, 5)$ in the primal. This may over-estimate the increased minimum cost, however, because it may be better to switch some production away from factory 1, now it has become more expensive.

To find the exact answer, we check whether any production is switched by considering the dual linear program again. It is unchanged except that the first constraint becomes $10x_1 + 20x_2 \le 10100$. Looking again at Fig. SM17.5.3, the solution to the dual still occurs at the intersection of the lines (1) and (3), even after (1) has been shifted out. In particular, therefore, the solution to the altered primal satisfies exactly the same constraints (including nonnegativity constraints) with equality, and is therefore the same point. So the earlier estimate of 2000 for the increased cost is indeed accurate.

Review exercises for Chapter 17

2. (a) Regard the given LP problem as the primal and denote it by (P). Its dual is shown in answer section of the book. Let us denote that by (D). If you draw the feasible set for (D) and a line $-x_1 + x_2 = c$, you see that as c increases, the line moves to the north-west. The line with the largest value of c which intersects the feasible set does so at the point $(0, 8)$, at the intersection of three lines $-x_1 + 2x_2 = 16$, $-2x_1 - x_2 = -8$, and $x_1 = 0$ that define three of the constraints, but the constraint $x_1 \ge 0$ is redundant.

   (b) We see that when $x_1 = 0$ and $x_2 = 8$, the second and fourth constraints in (D) are satisfied with strict inequality, so $y_2 = y_4 = 0$ at the optimum of (P). Also, since $x_2 = 8 > 0$, the second constraint in (P) must be satisfied with equality at the optimum — i.e., $2y_1 - y_3 = 1$. But then we see that the objective function in (P) can be reduced from $16y_1 + 6y_2 - 8y_3 - 15y_4$ to $16y_1 - 8y_3 = 8(2y_1 - y_3) = 8$. We conclude that any (y_1, y_2, y_3, y_4) of the form $(y_1, y_2, y_3, y_4) = (\frac{1}{2}(1 + b), 0, b, 0)$ must solve (P) provided its components are nonnegative and the first constraint in (P) is satisfied.[45] The first constraint reduces to $-\frac{1}{2}(1+b) - 2b \geq -1$, or $b \leq \frac{1}{5}$. We conclude that $(\frac{1}{2}(1+b), 0, b, 0)$ is optimal provided $0 \leq b \leq \frac{1}{5}$.

(c) The objective function in (D) changes to $kx_1 + x_2$, but the constraints remain the same. The solution $(0, 8)$ found in part (a) also remains unchanged provided that the slope $-k$ of the level curve $x_2 = 8 - kx_1$ through the point $(0, 8)$ remains positive and no less than the slope 1/2 of the line $-x_1 + 2x_2 = 16$. Hence $k \leq -\frac{1}{2}$.

4. (a) If $a = 0$, it is a linear programming problem, whose answer appears in the main text.

(b) When $a \geq 0$ we follow the techniques in Section 14.10 and consider the Lagrangian

$$\begin{aligned}\mathcal{L} = (500 - ax_1)x_1 + 250x_2 - \lambda_1(0.04x_1 + 0.03x_2 - 100) \\ - \lambda_2(0.025x_1 + 0.05x_2 - 100) - \lambda_3(0.05x_1 - 100) - \lambda_4(0.08x_2 - 100).\end{aligned}$$

Then the Kuhn–Tucker conditions, with nonnegativity constraints, are that there exist numbers λ_1, λ_2, λ_3, and λ_4, such that

(i) $\partial\mathcal{L}/\partial x_1 = 500 - 2ax_1 - 0.04\lambda_1 - 0.025\lambda_2 - 0.05\lambda_3 \leq 0$, with equality if $x_1 > 0$;

(ii) $\partial\mathcal{L}/\partial x_2 = 250 - 0.03\lambda_1 - 0.05\lambda_2 - 0.08\lambda_4 \leq 0$, with equality if $x_2 > 0$;

(iii) $\lambda_1 \geq 0$, and $\lambda_1 = 0$ if $0.04x_1 + 0.03x_2 < 100$;

(iv) $\lambda_2 \geq 0$, and $\lambda_2 = 0$ if $0.025x_1 + 0.05x_2 < 100$;

(v) $\lambda_3 \geq 0$, and $\lambda_3 = 0$ if $0.05x_1 < 100$;

(vi) $\lambda_4 \geq 0$, and $\lambda_4 = 0$ if $0.08x_2 < 100$.

(c) The Kuhn–Tucker conditions are sufficient for optimality since the Lagrangian is easily seen to be concave in (x_1, x_2) for $a \geq 0$. To find when $(x_1, x_2) = (2000, 2000/3)$ remains optimal, we must find Lagrange multipliers such that all the Kuhn–Tucker conditions are still satisfied. Note that (i) and (ii) must still be satisfied with equality. Moreover, the inequalities in (iv) and (vi) are strict, so $\lambda_2 = \lambda_4 = 0$. Then (ii) gives $\lambda_1 = 25000/3$. It remains to check for which values of a one can satisfy (i) for a λ_3 that also satisfies (v). From (i), $0.05\lambda_3 = 500 - 4000a - 0.04(25000/3) = 500/3 - 4000a \geq 0$ if and only if $a \leq 1/24$.

5. (a) See the answer in the main text.

(b) The admissible set is shown as the shaded infinite polygon in Fig. SM17.R.5. Lines I, II, and III show where the respective constraints are satisfied with equality. The dotted lines are level curves for the objective function $100y_1+100y_2$. We see that the objective function has its smallest value at P, the intersection of the two lines I and II. The coordinates (y_1^*, y_2^*) for P are given by the two equations $3y_1^* + 2y_2^* = 6$ and $y_1^* + 2y_2^* = 3$. Thus $P = (y_1^*, y_2^*) = (3/2, 3/4)$. The optimal value of the objective function is $100(y_1^* + y_2^*) = 225$.

(c) Since the dual has a solution, the duality theorem tells us that (a) also has an optimal solution, which we denote by (x_1^*, x_2^*, x_3^*). Since the third constraint in the dual is satisfied with strict inequality, we must have $x_3^* = 0$. Moreover, both constraints in the dual must be satisfied with equality at the optimum because both dual variables are positive in

[45] The second constraint we already know is satisfied with equality.

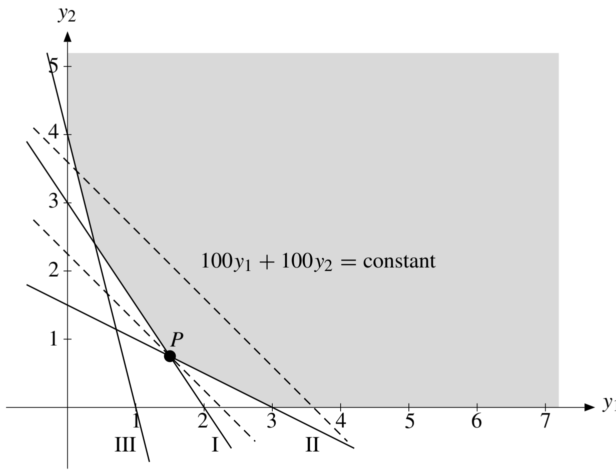

Figure SM17.R.5

optimum. Hence $3x_1^* + x_2^* = 100$ and $2x_1^* + 2x_2^* = 100$, which gives $x_1^* = x_2^* = 25$. The maximal profit is $6x_1^* + 3x_2^* + 4x_3^* = 225$, equal to the value of the dual, as expected.

(d) To a first-order approximation, profit increases by $y_1^* \Delta b_1 = 1.5$, so the new maximal profit is 226.5. For this approximation to be exact, the optimal point in the dual must not change when b_1 is increased from 100 to 101. This is obviously true, as one sees from Fig. SM17.R.5.

(e) The maximum value in the primal is equal to the minimum value in the dual. Given the same approximation as in part (d), this equals $b_1y_1^* + b_2y_2^*$, which is obviously homogeneous of degree 1 in b_1 and b_2. More generally, let $F(b_1, b_2)$ denote the minimum value of the dual set out in part (b).

Given any $\alpha > 0$, note that minimizing $\alpha b_1 y_1 + \alpha b_2 y_2$ over the constraint set of the dual gives the same solution (y_1^*, y_2^*) as minimizing $b_1y_1 + b_2y_2$ over the same constraint set. Hence $F(\alpha b_1, \alpha b_2) = \alpha F(b_1, b_2)$ for all $\alpha > 0$, so F is homogeneous of degree 1.