Mathematical Economics:

Exact Differential Equations and Second Order Differential Equations

Introduction

If a differential equation has variable coefficient and variable term, we have a non-homogeneous differential equation. The solution is not as easy as that of a differential equation with constant coefficient and constant term.

This module will try to solve a differential equation with variable coefficient and variable term.

Objectives:

The objectives of this module are:

1. *Define an exact differential equation*
2. *Solve an exact differential equation*
3. *Solve a Second Order Differential Equation*

Terminology

1. Exact equation: An equation that is independent of its path.
2. Inexact equation: an equation that is dependent on its path. It is a differential form that cannot be expressed as the differential of a function.
3. State function: A "state function" is used to describe a particular state. It is independent of the path. The value of the function is determined at a point.
4. Path Function: A "path function" is used to describe the path traversed from one point to another.

31.1. Exact equations:

Consider a function $R(x, y) = C$

$x \text{ and } y \text{ are variables and } C \text{ is a constant}.$

Using the concept of total derivative, we get,

$$dR(x,y) = \frac{\partial R}{\partial x}dx + \frac{\partial R}{\partial y}dy - - - - - - - - (31.1)$$

Since the function has two variables, we use partial derivatives.

In equation 31.1, if the differential is set equal to zero, or $dR = 0$, we get

$$\frac{\partial R}{\partial x}dx + \frac{\partial R}{\partial y}dy = 0 - - - - - - - - (31.2)$$

Equation 31.2 is called an exact equation.

This is because the $LHS \text{ is exactly the differential of the function } R(x, y)$

Therefore, an equation is an exact equation, if $dR = 0$ (total derivative=0), for some function $R(x, y) = C$

In general, a differential equation of the form

$$M\,dx + Ndy = 0 - - - - - - - - (31.3)$$

will be exact if and only if there exists a function $R(x, y)$ such that $M = \frac{\partial R}{\partial x} \text{ and } N = \frac{\partial R}{\partial y}$.

A. Test for exact equation:

To test if an equation is exact or not, we use Young's Theorem, which states that

$$\frac{\partial^2 R}{\partial y \partial x} = \frac{\partial^2 R}{\partial x \partial y}$$

$$or, \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} - - - - - - - - - (31.4)$$

Since equation 31.2 involve the use of partial differentials, it is also termed as **exact differential equation.**

Example:

Consider the function $(y + 2x)dx + (x + 3y^2)dy = 0$

This equation is similar to equation 31.3.

Let $(y + 2x) = M \text{ and } (x + 3y^2) = N$

To test if this equation is an exact equation, we find $\frac{\partial M}{\partial y} \text{and } \frac{\partial N}{\partial x}$

Therefore,

$$\frac{\partial M}{\partial y} = \frac{\partial (y + 2x)}{\partial y} = 1$$

$$\frac{\partial N}{\partial x} = \frac{\partial (x + 3y^2)}{\partial x} = 1$$

Since

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 1$$

It can be said that the equation is an exact equation.

B. Difference between exact and inexact equation:

Inexact differential equation is an equation that is dependent on its path. It is a differential form that cannot be expressed as the differential of a function.

Example:

Consider the equation

$$4x\, dy + \frac{4}{5} y\, dx = 0 - - - - - (31.5)$$

If we take $4x = N \ and \ \frac{4}{5}y = M$

Then,

$$\frac{\partial M}{\partial y} = \frac{4}{5} \ \ and \ \frac{\partial N}{\partial x} = 4$$

Since

$$\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial y}$$

Equation 31.5 is not an exact equation or is an inexact equation.

An inexact differential equation can be made exact by multiplying every term of the equation by a common factor, called “integrating factor”. Interested learners may go through the references at the end of this module for more details.)

Note:

Exact equation may be linear or non-linear. This module will use only linear exact differential. equation.

C. State equation and Path equation:

Exact differential equation is independent of the path. These equations are therefore significant to test if a system is independent of its path. Exact equations are widely used in the field of physics and engineering.

At this point, it will be necessary to give a hint about a "state function" and "a path function".

A "state function" is used to describe a particular state. It is independent of the path. The value of the function is determined at a point.

For example: density, temperature, pressure, energy are examples of state function.

A "path function" is used to describe the path traversed from one point to another.

For example: heat and work are examples of path functions.

In Physics, thermodynamic systems exist in a particular state. A system passes through different states and returns to its initial state and in this process, it may perform work on its surroundings.

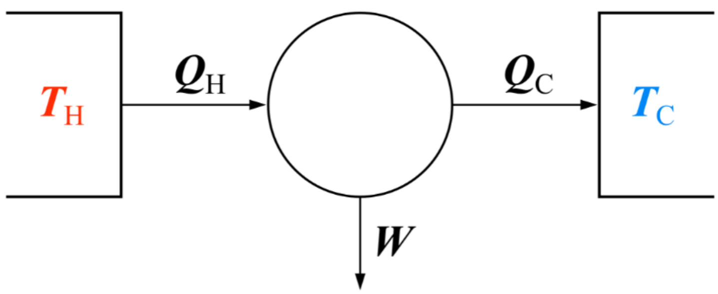

Image 1: Carnot Engine Diagram

[Source: https://en.wikipedia.org/wiki/File:Carnot_heat_engine_2.svg]

In the Carnot Engine, heat (Q_H) flows from a furnace (T_H) through the substance (circle) and remaining heat (Q_c) flows to the cold sink (T_c), forcing the substance to do mechanical work (W) on the surroundings.

These concepts are also used in Economics. Price is an example of a state function. Growth is an example of a path function. The example of path function from economics is explained in Module 32.

31.2: Solving an Exact Differential Equation:

Consider equation 31.3 in section 31.1 of this module

$$M\,dx + N dy = 0$$

This may be considered as the general form of an exact differential equation. Note that

$$M = \frac{\partial R}{\partial x} \; and \; N = \frac{\partial R}{\partial y}$$

In order to solve an exact differential equation, let us take the previous example discussed in section 31.1

$$(y + 2x)dx + (x + 3y^2)dy = 0$$

Let $(y + 2x) = M \; and \; (x + 3y^2) = N$

Now,

$$\frac{\partial M}{\partial y} = \frac{\partial(y + 2x)}{\partial y} = 1$$

$$\frac{\partial N}{\partial x} = \frac{\partial(x + 3y^2)}{\partial x} = 1$$

Hence it is an exact equation.

In order to solve this exact equation, we follow the underlying step.

Step 1:

Integrate M. Since M is a partial differential equation, $M = \frac{\partial R}{\partial x}$, this means that the function R contains integral of M with respect to x. In differentiating $R(x, y)$ partially with respect to x, any additive term containing only the variable y or a constant term would be zero, therefore, we must take care to replace such terms in the integration process. In this example, $\varphi(y)$ have been used, which acts like a constant but is not exactly the same as a constant.

Therefore, we have a preliminary solution

$$R\,(x, y) = \int M dx \; + \; \varphi(y) = \int (y + 2x)dx \; + \; \varphi(y) = xy + x^2 + \varphi(y)$$

Step 2:

Differentiate the result obtained in step 1 with respect to y.

Therefore,

$$\frac{\partial R}{\partial y} = x + \varphi'(y) ------(31.6)$$

Step 3:

Equating equation 31.6 with $N = (x + 3y^2)$, we get

$$\varphi'(y) = (3y^2) - - - - (31.7)$$

Step 4:

Integrate equation 31.7

Therefore,

$$\varphi(y) = \int \varphi'(y)dy = \int 3y^2\, dy = y^3$$

Step 5:

Combine results of Step 1 and Step 4, gives

$$R(x, y) = xy + x^2 + y^3$$

31.3. Solving Second Order Differential Equations

If a differential equation has a derivative of order 2 that is, $\frac{d^2y}{dt^2}$, it is termed as a second order differential equation. A derivative of second order means a derivative of a derivative. To keep it simple, we will consider only the linear second order differential equation with constant coefficient and constant term.

The general form of a second order differential equation with constant coefficient and constant term may be represented as

$$\frac{d^2y}{dt^2} + \alpha_1 \frac{dy}{dt} + \alpha_2 y = \beta - - - - - (31.8)$$

α *and* β *are constants*

If $\beta = 0$, we have a homogeneous equation and if $\beta \neq 0$, we have a non-homogeneous equation. The solution of a second order differential equation consists of two parts- complementary solution (y_c) and particular solution (y_p). The complete solution or general solution is the sum of the complementary solution and particular solution, that is, $y_c + y_p$

A. Complementary solution:

To find the complementary solution we consider the homogeneous part of equation 31.8, that is RHS =0

Thus, we have

$$\frac{d^2y}{dt^2} + \alpha_1 \frac{dy}{dt} + \alpha_2 y = 0 - - - - - - (31.9)$$

Here, we may take the help of the first order differential equation with constant term and constant coefficient. Using equation 30.8, 30.9 and 30.14 in Module 30, section 30.3, it is clear that the complementary solution consists of an exponential part Ae^{mt}.

If we have a function $y = Ae^{mt}$, then

$$\frac{dy}{dt} = mAe^{mt} \text{ and } \frac{d^2y}{dt^2} = m^2Ae^{mt}$$

Therefore, equation 31.8 may be transformed into

$$m^2Ae^{mt} + \alpha_1 mAe^{mt} + \alpha_2 Ae^{mt} = 0$$

$$or, \qquad Ae^{mt}(m^2 + \alpha_1 m + \alpha_2) = 0$$

Note that the expression in bracket is of quadratic form, and the roots is given by using the quadratic equation formula. Thus, the roots of the quadratic equation will be

$$m_1, m_2 = \frac{-\alpha_1 \pm \sqrt{\alpha_1^2 - 4\alpha_2}}{2}$$

There is an interesting relationship between the two roots

i) The sum of the roots is always equal to $-\alpha_1$

$$m_1 + m_2 = \frac{-\alpha_1 + \sqrt{\alpha_1^2 - 4\alpha_2}}{2} + \frac{-\alpha_1 - \sqrt{\alpha_1^2 - 4\alpha_2}}{2} = -\alpha_1$$

ii) The product of the roots is always equal to α_2

$$m_1 \times m_2 = \frac{-\alpha_1 + \sqrt{\alpha_1^2 - 4\alpha_2}}{2} \times \frac{-\alpha_1 - \sqrt{\alpha_1^2 - 4\alpha_2}}{2} = \alpha_2$$

Therefore, there will be two solutions of $y = Ae^{mt}$

$$y_1 = A_1e^{m_1t} \text{ and } y_2 = A_2e^{m_2t}$$

But we have only one complementary solution. So, we either take only one solution or combine both the solution. But the first alternative gives only one arbitrary constant and in obtaining a second order derivative we lose two constants, so to revert from a second order derivative, we consider the second alternative.

Thus,

$$y_c = y_1 + y_2$$

Note that the solutions depend on the roots of the equation and there may be three cases:

Case 1: $\alpha_1^2 > 4\alpha_2$

In this case, we have two real roots and the complementary solution is given by

$$y_1 + y_2 = A_1 e^{m_1 t} + A_2 e^{m_2 t}$$

Case 2: $\alpha_1^2 = 4\alpha_2$

In this case, both the roots are equal and the complementary solution is given by

$y_1 + y_2 = A_1 e^{m_1 t} + A_2 e^{m_2 t} = (A_1 + A_2)e^{mt}$ ($\because\ m_1 = m_2 = m$)

If $A_1 + A_2 = A_3$, then the complementary solution is given by

$$y_c = A_3 e^{mt}$$

But we have only one arbitrary constant, hence we cannot obtain the original function from the second order differential equation. Therefore, we need another term that is independent of $A_3 e^{mt}$ and that may be $A_4 t e^{mt}$

Therefore, complementary solution in this case may be given as

$$y_c = A_3 e^{mt} + A_4 t e^{mt}$$

Case 3: $\alpha_1^2 < 4\alpha_2$

In this case, we have complex roots. Since this course is limited to only real numbers, we will not discuss this case in this course. Interested learners may refer to "learn more" section for advanced learning.

B. Particular Solution:

Finding the particular solution is easy. We consider the simplest solution of y as a constant. Then,

$$\frac{dy}{dt} = 0 \ and\ \frac{d^2y}{dt^2} = 0$$

And equation 31.8 becomes

$$\alpha_2 y = \beta$$

$$or, \qquad y_p = \frac{\beta}{\alpha_2}\ (\alpha_2 \neq 0)$$

Therefore, the complete solution for the second order differential equation is given as

Case 1:

$$y(t) = A_1 e^{m_1 t} + A_2 e^{m_2 t} + \frac{\beta}{\alpha_2}$$

Case 2:

$$y(t) = A_3 e^{mt} + A_4 t e^{mt} + \frac{\beta}{\alpha_2}$$