Mathematical Economics: Differentiation- An Introduction

20.1 Introduction:

You must have observed that price of goods in the market keep changing. Unknowingly, you change your demand according to the increase or decrease in the price. Though this is a psychological process, this change in the behavior of the consumer actually helps to define the character of a good, that is if a good is a necessary good or a luxury good.

This module will introduce the technique to measure a change and how the change in independent variables affects the dependent variable.

Objectives

The objectives of this module are:

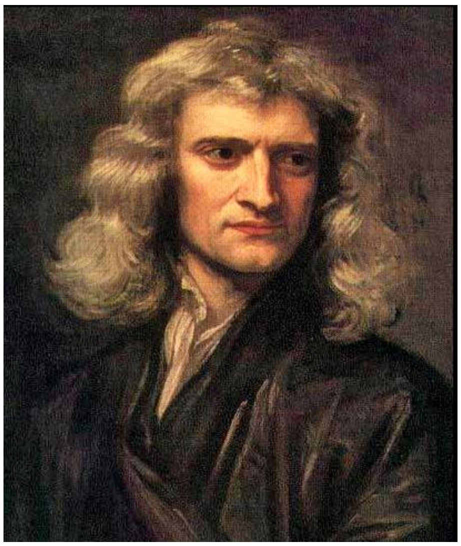
1. *Define* the concept of derivative
2. *Explore* the applications of differentiation in the real world

Terminology

1. Calculus: a branch of mathematics that deals with change and motion, technically called derivatives and integrals
2. Derivative: the rate of change of one variable with respect to another variable and used particularly for infinitesimal changes
3. Differentiation: a process of finding the derivative
4. Partial differentiation: a process of finding the derivative for more than one independent variable; a process to study the effect of one independent variable on the dependent variable, keeping the other independent variables constant
5. Total differentiation: a process to find the effect on the dependent variable when all the independent variables undergo a change simultaneously
6. Differential: actual change of a function, actual because it deals with infinitesimal changes
7. Jacobian Matrix: a matrix of all first-order partial derivatives of a vector-valued function
8. Hessian Matrix: a matrix of second-order partial derivatives of a scalar-valued function
9. Economic cycle: a natural fluctuation of an economy between growth and depression

20.2. Origin of Calculus

Introductory economics involve the use of basic mathematics. But a detailed study of economics involves the use of more advanced mathematics including calculus. The introductory part of calculus has been already discussed in the earlier part of this course where you were introduced to functions, the backbone of calculus. The discovery of calculus is generally attributed to Isaac Newton and Gottfried Wilhelm Leibniz. However, there was a controversy over who invented calculus first.



Issac Newton

[Source: Godfrey Kneller - http://www.newton.cam.ac.uk/art/portrait.html **]**

Gottfried Wilhelm Leibniz

[Source: https://en.wikipedia.org/wiki/File:Christoph_Bernhard_Francke_-_Bildnis_des_Philosophen_Leibniz_(ca._1695).jpg **]**

Nevertheless, the invention of calculus has helped to create mathematical economic models to arrive at optimum solutions. The two types of calculus that is used are differential calculus and integral calculus. This module will focus on differential calculus.

Differential calculus is the branch of mathematics that studies the rates at which quantities change. The introductory concept in differential calculus is the "derivative". The process of finding the derivative is called differentiation.

20.3. Derivative in Mathematics:

The basic concept in understanding the concept of a derivative is the concept of a function. A function is a relationship between a dependent variable and an independent variable, expressed as $y = f(x)$, where y is the dependent variable and x is the independent variable.

In order to find the change of the dependent variable as the independent variable changes, the ratio $\frac{\Delta y}{\Delta x}$ is used. This ratio is called the rate of change or the slope and Δ is a symbol used to denote “change”.

Geometrically, the slope between two points on a curve may be shown as follows:

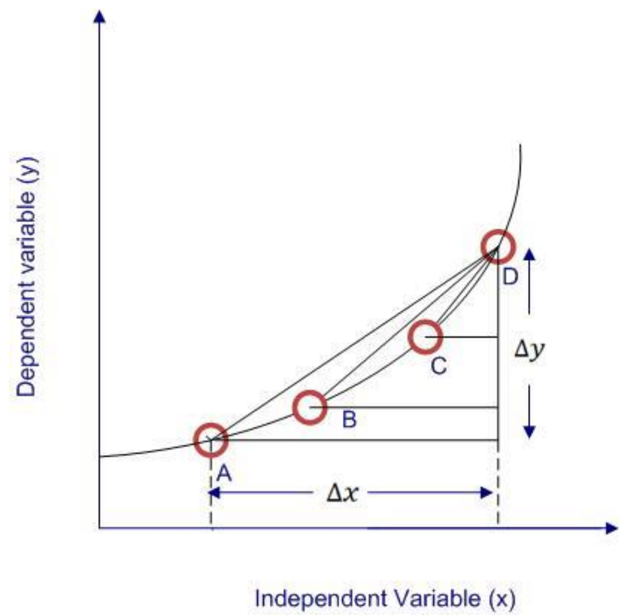


Fig 20.1: Derivative of a function

The difference between point A and point D on the curve or the change in x is given by Δx . As A moves closer to D, Δx becomes infinitesimally small, or $\Delta x \rightarrow 0$. As the distance between A and D reduces, $\frac{\Delta y}{\Delta x}$ tends to a number and this limiting value is called the derivative of the function $y = f(x)$. It can be seen from the figure that as Δx decreases, the slope of the straight line becomes steeper; AD is flatter than BD that is flatter than CD. Conversely, CD is steeper than BD that is steeper than AD. Moving even closer, a point will be reached when the straight line will be tangent at the point D. Therefore, at point D, the slope of the curve will be equal to the slope of the straight line tangent at that point on the curve.

Mathematically, the derivative of a function $y = f(x)$ is expressed as $\frac{dy}{dx}$ or $f'(x)$

Thus, $$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \text{ ------------------- (20.1)}$$

Therefore, the concept of a derivative depends on the concept of limit.

A. Different notations used for derivative of a function:

a) Equation (1) was invented by Leibniz. It says that $\frac{\Delta y}{\Delta x}$ approaches $\frac{dy}{dx}$ and is used for a first order derivative. Higher order derivatives are represented as $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$, for second order and third order respectively and so on.

b) Joseph-Louis Lagrange used the notation f' for derivative of a function. Higher order derivatives are represented a f'', f''', for second order and third order respectively and so on.

c) Newton used the notation $\dot{y}$ and $\ddot{y}$ for first order and second order derivative of y respectively.

d) Leonhard Euler used a differential operation D. Therefore, the first derivative is represented as Df and higher order derivatives are represented as D^2f, D^3f for second and third derivative respectively and so on.

B. Higher order derivative:

Second order derivative is the differentiation of the first order derivative. Third order derivative is the derivative of the second order derivative and so on. For example, the second order derivative of the function $y = f(x)$ is given as $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$ and third order derivative is given as $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ and so on.

Note: 1. Δx cannot be equal to zero, otherwise, $\frac{\Delta y}{\Delta x} = \infty$, which is undefined.

2. Δ is used for large change and $'d'$ is used for infinitesimal change.

3. First derivative tells about slope of the function and second derivative tells about the curvature.

20.4 Basic Rules of Differentiation:

There are several rules that are followed in differentiation. Some basic rules may be classified under two categories.

A. Rules of differentiation with one explanatory variable
B. Rules of differentiation with two or more functions of the same explanatory variable.

A. One explanatory variable	B. Two or more functions of same explanatory variable
Constant Rule: $\boldsymbol{If\ y = f(x)\ some\ constant,}$ $\boldsymbol{then\ \frac{dy}{dx} = 0}$	**Addition and Subtraction Rule**: If $y = f(x) = m(x) \pm n\,(x)$, $then\ \frac{dy}{dx} = m'(x) \pm n'(x)$
Power Rule: If $\mathbf{y} = \boldsymbol{f(x) = x^n, then\ \frac{dy}{dx} = nx^{n-1}}$	**Product Rule**: If $y = f(x) = m(x) \times n\,(x)$, $then\ \frac{dy}{dx} = m(x) \times n'(x) + m'(x) \times n(x)$
Logarithmic Rule: **If** $\boldsymbol{y = f(x) = \log x\,,\ then\ \frac{dy}{dx} = \frac{1}{x}}$	**Quotient Rule**: If $y = f(x) = \frac{m(x)}{n}(x)$, $then\ \frac{dy}{dx} = \frac{\{n(x)m'(x) - m(x)n'(x)\}}{[n(x)]^2}$
Exponential Rule: 1. **If** $\boldsymbol{y = f(x) = e^x, then\ \frac{dy}{dx} = e^x}$ 2. If $\boldsymbol{y = f(x) = e^{ax},\ then\ \frac{dy}{dx} = ae^{ax}}$	

20.5. Partial Differentiation and Total Differentiation.

In section 7.1.2 and section 7.1.3, a derivative was defined for a function with only one explanatory variable. If there are more than one explanatory variables in a function, such that, $y = f(x_{1,}x_2)$, then the differentiation of the function may be defined in two ways:

a) **Partial derivative**:

This derivative study the impact of one explanatory variable on the dependent variable, keeping all the other explanatory variables unchanged. Again, the process of finding the partial derivative is known as partial differentiation.

Using the definition of derivative from section 7.1.2, we may write the partial derivative of a function with two explanatory variables $y = f(x_{1,}x_2)$ as,

$$\frac{\partial y}{\partial x_1} = f_1 = \lim_{\Delta x_1 \to 0} \frac{\Delta y}{\Delta x_1} ----(20.2)$$

and

$$\frac{\partial y}{\partial x_2} = f_2 = \lim_{\Delta x_2 \to 0} \frac{\Delta y}{\Delta x_2} ----(20.3)$$

In equation (2), the variable x_2 is kept unchanged or assumed to be a constant and in equation (3), the variable x_3 is kept unchanged or assumed to be a constant.

Higher order partial derivatives may be represented as

$$\boldsymbol{f_{11}} = \frac{\partial^2 y}{\partial x_1^2} = \frac{\partial}{\partial x_1}(\frac{\partial y}{\partial x_1}) \ ;$$

$$\boldsymbol{f_{22}} = \frac{\partial^2 y}{\partial x_2^2} = \frac{\partial}{\partial x_2}(\frac{\partial y}{\partial x_2});$$

$$\boldsymbol{f_{12}} = \frac{\partial^2 y}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_2}(\frac{\partial y}{\partial x_1})$$ and so on.

b) **Differential and Total differentiation:**

The concept of differential and total differentiation is used for functions with more than one explanatory (independent) variable. Before understanding total differentiation, the concept of a 'differential' must be clear. There is a very thin line of distinction between a 'differential' and a 'derivative'. A derivative is the rate of change of one variable with respect to another variable, while a differential is the actual change of a function. Hence, for a function $y = f(x_{1,}x_2)$, the total change in y, due to a simultaneous infinitesimal change in x_1 and x_2 is called total differential of the function and the process of finding the total differential is called total differentiation.

In other words, total differential of y is the rate of change of y with respect to x_1 multiplied by the change in x_1 plus the rate of change of y with respect to x_2 multiplied by the change in x_2.

Therefore, $dy = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2$

The rules for partial differentiation and total differentiation are the same as the rules of differentiation discussed in section 7.1.3.

20.6. Jacobian Matrix and Jacobian Determinant:

Jacobian Matrix is the matrix of all first-order partial derivatives of a vector-valued function. The concept of Jacobian matrix and Jacobian determinant are named after a German mathematician, Carl Gustav Jacob Jacobi.

Carl Gustav Jacob Jacobi

[Source: http://www.sil.si.edu/digitalcollections/hst/scientific-identity/explore.htm]

Suppose we have a set of n functions in n variables

$$y_1 = f(x_1, x_2, \dots \dots x_n)$$

$$y_2 = f(x_1, x_2, \dots \dots x_n)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$y_n = f(x_1, x_2, \dots \dots x_n)$$

There will be $n \times n$ partial derivatives that may be represented in an $n \times n$ matrix. This matrix of partial derivatives is called Jacobian matrix, denoted by J.

Therefore,

$$J_n = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}$$

The corresponding Jacobian determinant is denoted by

$$|J_n| = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

More specifically,

$$|J_2| = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} for\ two\ functions\ with\ two\ variables\ and\ so\ on$$

20.7. Hessian Matrix and Hessian Determinant:

Hessian matrix is a square matrix of second-order partial derivatives of a scalar-valued function. This matrix was developed by Ludwig Otto Hesse and is used in large-scale optimization problems.

Ludwig Otto Hesse

[Source: https://en.wikipedia.org/wiki/File:Ludwig_Otto_Hesse.jpg]

In other words, the Jacobian of the derivatives $\frac{\partial y_1}{\partial x_1}, \frac{\partial y_1}{\partial x_2}, \ldots\ldots \frac{\partial y_1}{\partial x_n}$, of a function,

$y = f(x_{1,}x_2\ldots\ldots\ldots x_n)$ with respect to $x_{1,}x_2\ldots\ldots\ldots x_n$ is called the Hessian matrix and the corresponding determinant is called the Hessian determinant.

Thus, Hessian matrix is denoted by,

$$H_n = \begin{bmatrix} \frac{\partial^2 y_1}{\partial x_1^2} & \frac{\partial^2 y_1}{\partial x_1 \partial x_2} & \cdots & \frac{\partial y_1}{\partial x_1 \partial x_n} \\ \frac{\partial^2 y_2}{\partial x_2 \partial x_1} & \frac{\partial^2 y_2}{\partial x_2^2} & \cdots & \frac{\partial y_2}{\partial x_2 \partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial^2 y_n}{\partial x_n \partial x_1} & \frac{\partial^2 y_n}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 y_n}{\partial x_n^2} \end{bmatrix}$$

And the corresponding Hessian determinant is denoted by

$$|H_n| = \begin{vmatrix} \frac{\partial^2 y_1}{\partial x_1^2} & \frac{\partial^2 y_1}{\partial x_1 \partial x_2} & \cdots & \frac{\partial y_1}{\partial x_1 \partial x_n} \\ \frac{\partial^2 y_2}{\partial x_2 \partial x_1} & \frac{\partial^2 y_2}{\partial x_2^2} & \cdots & \frac{\partial y_2}{\partial x_2 \partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial^2 y_n}{\partial x_n \partial x_1} & \frac{\partial^2 y_n}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 y_n}{\partial x_n^2} \end{vmatrix}$$

More specifically,

$$|H_1| = \frac{\partial^2 y}{\partial x_1^2}\ , |H_2| = \begin{vmatrix} \dfrac{\partial^2 y}{\partial x_1^2} & \dfrac{\partial^2 y}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 y}{\partial x_2 \partial x_1} & \dfrac{\partial^2 y}{\partial x_2^2} \end{vmatrix} \text{and so on}$$

20.8 Bordered Hessian Determinant:

This determinant is used in solving certain optimization problems with constraints.

If we have a function, $y = f(x_{1,}x_2 \ldots\ldots\ldots x_n)$, which is subject to a constraint $h(x_{1,}x_2 \ldots\ldots\ldots x_n) = c$

Here, $y = f(x_{1,}x_2 \ldots\ldots\ldots x_n)$ is called the objective function and $h(x_{1,}x_2 \ldots\ldots\ldots x_n) = c$ is called the constraint.

To find the optimum solution to such problems, we construct the Lagrange function

$$L = f(x_{1,}x_2 \ldots\ldots\ldots x_n) + \lambda[c - h(x_{1,}x_2 \ldots\ldots\ldots x_n)]$$

$$\text{where } \lambda \text{ is known as the Lagrange multiplier}$$

The Lagrange function is a technique of combining the objective function and the constraint into one equation so that the first order condition for maxima or minima can still be applied. In such cases, we use the concept of bordered Hessian determinant.

The bordered Hessian determinant for a function with n variables and n constraints is given as

$$|\overline{H}| = \begin{vmatrix} 0 & h_1 & h_2 \cdots & h_n \\ h_1 & L_{11} & L_{12} \ \ldots & L_{1n} \\ \ldots.. & \ldots \ldots & & \ldots.. \\ h_n & L_{n1} & L_{n2} \cdots & L_{nn} \end{vmatrix}$$

Here, $h_1 = \frac{\partial h}{\partial x_1}, h_2 = \frac{\partial h}{\partial x_2}$, and so on, and $L_{11} = \frac{\partial}{\partial x_1}\left(\frac{\partial L}{\partial x_1}\right),\quad L_{12} = \frac{\partial}{\partial x_1}(\frac{\partial L}{\partial x_2})$, and so on.

More specifically,

$$|\overline{H_2}| = \begin{vmatrix} 0 & h_1 & h_2 \\ h_1 & L_{11} & L_{12} \\ h_2 & L_{21} & L_{22} \end{vmatrix}, |\overline{H_3}| = \begin{vmatrix} 0 & h_1 & h_2 & h_3 \\ h_1 & L_{11} & L_{12} & L_{13} \\ h_2 & L_{21} & L_{22} & L_{23} \\ h_3 & L_{31} & L_{32} & L_{33} \end{vmatrix}, \text{and so on.}$$

20.9. Applications of Derivative and Differentiation:

Differentiation is extensively used in day to day life. Below are few of the examples where the application of derivative/differentiation may be seen.

1) **Physics:** In Physics, the concept of derivative is used in studying the velocity of an object. Velocity is the rate of change of position of an object with respect to time. Derivative is also used to study acceleration, which is the rate of change of velocity with respect to time. Potential energy and kinetic energy also use the concept of derivative.

Image 1: Velocity

[Source https://en.wikipedia.org/wiki/File:ALMS_Prototypes.jpg]

In Image 1: Velocity, when the racing cars turn on the curved track, there is a change in direction and their velocity is not constant. This change in the speed or velocity uses the concept of derivative.

2) **Astronomy:** Astronomers use the concept of derivative to study the planetary motions. They use derivative in space travel, to find how much fuel a rocket or a satellite will need for acceleration so that it can cross the stratosphere.

Luna 1 was launched in 1959 and was the first known man-made object to achieve escape velocity from the Earth. Escape velocity is the speed at which a body must be travelling in order to escape the gravitational attraction of a particular planet or other object.

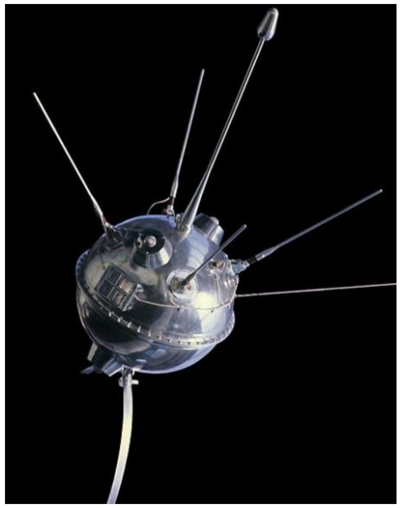


Image 2: Spacecraft Luna 1 (Mechta)

[Source: https://en.wikipedia.org/wiki/File:RIAN_archive_510848_Interplanetary_station_Luna_1_-_blacked.jpg]

3) **Sports:** Derivative is used in different kind of sports such as running, basketball, baseball, etc. The concept of optimization helps to improve the performance of the players. For example, in basketball, the shape of the ball, according to Newton, has a reaction depending on the force applied to it.

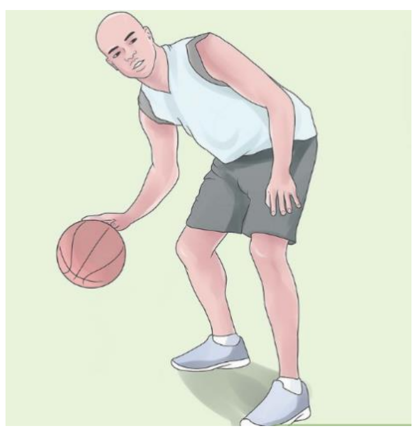


Image 3: Bouncing of Basketball

[Source: https://www.wikihow.com/Apply-Math-and-Geometry-in-Basketball#/Image:Apply-Math-and-Geometry-in-Basketball-Step-3.jpg]

A relative amount of force will have to be applied to the ball to keep it on the right track and at the same time maintaining the speed of the player.

4) **Biology:** Biologists also use derivative in their research projects. In biochemistry, Michaelis–Menten kinetics is one of the best known models of enzyme kinetics. For many enzymes, the rate of catalysis, which is defined as the number of moles of product formed per second, varies with the substrate concentration as shown in the figure. This model takes the form of an equation that shows the rate of enzymatic reactions using derivatives.

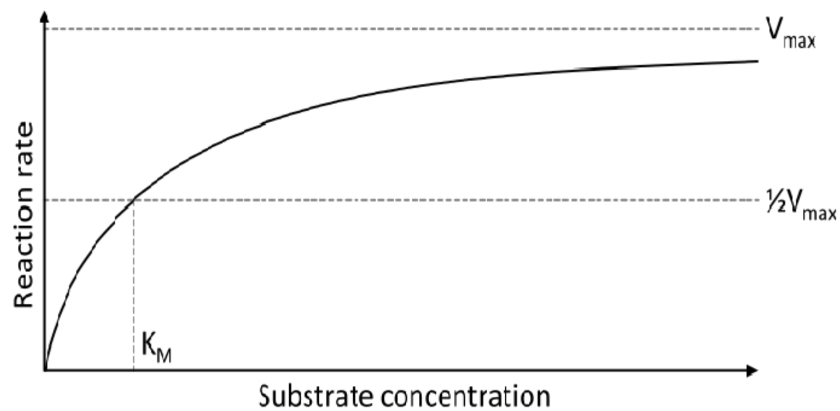


Image 4: Michaelis–Menten saturation curve for an enzyme reaction
[Source: https://en.wikipedia.org/wiki/File:Michaelis_Menten_curve_2.svg**]**

Michaelis–Menten saturation curve for an enzyme reaction shows the relation between the substrate concentration and reaction rate.

5. **Economics and Business:** In Economics, the concept of derivative plays a very important role in decision making. The marginal functions used in economics that is defined as the change of one variable with respect to another variable uses the concept of derivative. Many economic models are constructed in continuous time and use the time derivative. For example, the economic cycle or business cycle or trade cycle represents a wave of the Gross Domestic Product (GDP). The fluctuations mean a change, and the rate of change is measured using the concept of derivative.

20.10 Application of Differentiation in Economic Theories

Differentiation is extensively used in economic research to study functional relationships between economic variables. Below are few examples:

1. **Microeconomic Theories** such as the Demand Theory uses derivative to measure the price elasticity of demand, income elasticity of demand and cross elasticity of demand.

$$e_p = \frac{dQ}{dP} \times \frac{P}{Q}$$

   Here,
   e_p is the price elasticity of demand
   P is the price
   Q is the quantity demanded
   $\frac{dQ}{dP}$ is the change in quantity demanded with respect to change in price

2. **Macroeconomic Theories** such as the Theory of Consumption uses derivative to study the marginal propensity to consume and the marginal propensity to save. Marginal Propensity to Consume (MPC) is defined as the change in consumption when income changes and Marginal Propensity to Save is the change in Savings when income changes.

Therefore, in the consumption function,

$$C = a + m\,(Y - T)$$

C is induced consumption, Y is income, T is tax, m is the MPC defined as $\frac{dC}{dY}$ and 'a' is autonomous consumption.

3. **Theories of Growth** such as the Harrod Domar Growth Model use the dot notation of derivative. According to this model, the output growth rate is equal to the savings rate times the marginal product of capital minus the depreciation rate. Dot notation ($\dot{Y}$) is used for the derivative of a variable with respect to time.

Mathematically,

$$\frac{\dot{Y}}{Y} = sc - \delta$$

Here,

$\frac{\dot{Y}}{Y}$ is the output growth rate; s is savings rate, c is marginal product of capital and δ is the depreciation rate.

4. **Theory of Distribution** uses the Euler's Theorem. If the total output (Q) is a function of two inputs, Labour (L) and Capital (K), then the production function may be represented as $Q = f(L, K)$. If $Q = f(L, K)$ is linearly homogenous, then

$$K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L} \equiv Q$$

The economic interpretation of this theorem is that, under constant returns to scale, if each input is paid an amount equal to its marginal product, $\frac{\partial Q}{\partial K}$, the marginal product of capital and $\frac{\partial Q}{\partial L}$, the marginal product of labour, the total product (Q) will be exactly exhausted.

