Matrix Algebra and Linear Models

Suppose you purchase the following items from the market. The price per unit of each item is given

Items	Price per unit (₹)
10 Chocolates	10
2 packets of balloon	5
5 brownies	20
5 birthday caps	15
1 packet glitter candles	50

What is your total expenditure?

Simple mathematics says that $Expenditure = Price \times Number\ of\ quantities$

Therefore, $Total\ expenditure = 10 \times 10 + 2 \times 5 + 5 \times 20 + 5 \times 15 + 1 \times 50 =$₹$335$

Will you be surprised if you knew that you actually did a matrix multiplication? Do you want to know how?

This module will show how matrix algebra can help to make basic economic calculations.

Objectives

The objectives of the module are:

1. *Explain* the basic structure of a matrix
2. *C*onvert linear models to matrix form
3. *Explore* the applications of matrix algebra

Terminology

1. Scalar: a quantity with magnitude only
2. Vector: a quantity with magnitude and direction
3. Matrix: a collection of vectors
4. Circular flow matrix: a matrix that shows how resources, products and money flow in an economy
5. Input-Output Matrix: a matrix that shows the interdependence between different sectors of an economy
6. Matrix organization: an organization with more than one boss
7. Matrix Game: a game where two players with opposite interests play within a given set of rules and strategies.

11.1. Vector and Scalar

The word "vector" originates from a Latin word that means "carrier" or "transporter". The term "vector was introduced by William Rowan Hamilton and later Joshiah Williard Gibbs and Oliver Heaviside developed vector analysis.

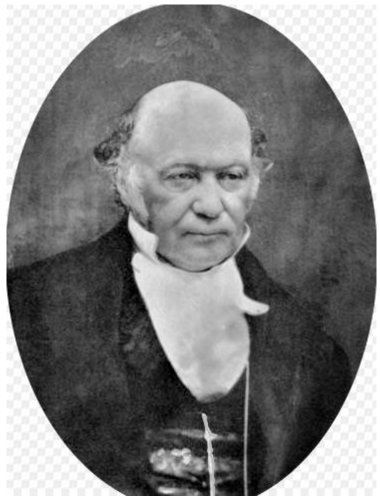

Image 11.1 William Rowan Hamilton[1]

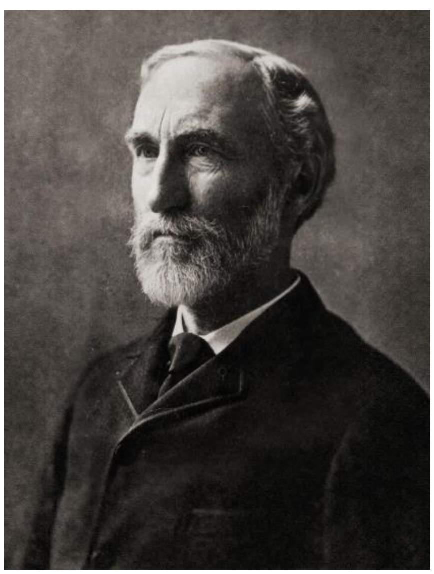

Image 11.2 Joshiah Williard Gibbs[2]

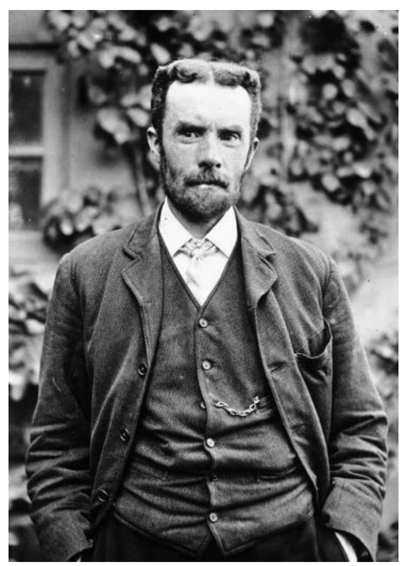

Image 11.3 Oliver Heaviside[3]

[1]https://en.wikipedia.org/wiki/William_Rowan_Hamilton#/media/File:William_Rowan_Hamilton_portrait_oval_combined.png

[2]https://en.wikipedia.org/wiki/Josiah_Willard_Gibbs#/media/File:Josiah_Willard_Gibbs_-from_MMS-.jpg
[3]https://en.wikipedia.org/wiki/Oliver_Heaviside#/media/File:Oheaviside.jpg

Examples of vector:

Acceleration or velocity studied in Physics, are vectors. In biology, a vector is an insect that carries a disease from one person to another. In Mathematics, it is used to denote a quantity that has magnitude (size or how much) as well as a direction.

A vector is different from a scalar. Scalars have a magnitude only. In other words, a scalar is a number but a vector is a number with direction. (Both scalars and vectors are part of "tensors").

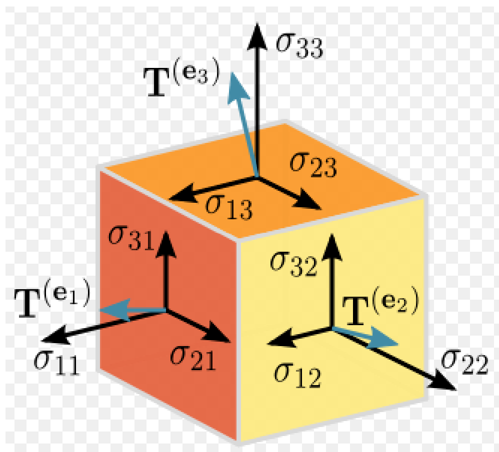

Image 11.4: Second-order Cauchy stress tensor

[Source: https://en.wikipedia.org/wiki/Tensor#/media/File:Components_stress_tensor.svg **]**

Example, Speed has only magnitude but no direction and hence is a scalar. On the other hand, displacement has a magnitude as well as a direction, and hence is a vector.

Example of scalar and vector in Economics:

In Economics, demand for a good, may be called a vector because it has a magnitude, quantity demanded and since demand is inversely related to price, it also has a direction, stating that quantity demanded moves in the opposite direction with price.

Further, price elasticity of demand, is the percentage change in quantity demanded with respect to the change in the price.

Thus,

$$\textit{Price Elasticity of Demand } (e_p) = \left| \frac{\%\ change\ in\ quantity\ demanded}{\%\ change\ in\ price} \right|$$

It is an absolute value and tells by how much quantity demanded changes when the price changes. But elasticity has no direction, and hence it is a scalar.

However, if a negative sign is attached to the price elasticity of demand, it says that quantity demanded changes inversely with price.

Elasticity may be of different types:

a) Elastic
b) Inelastic
c) Perfectly elastic
d) Perfectly inelastic
e) Unitary elastic

In Mathematical language, a vector is a special type of matrix that has only one row or one column. If a matrix has one row, it is referred to as a "row vector". If a matrix has one column, it is referred to as a "column vector".

Thus

R = $\begin{bmatrix} 100 & 200 & 300 \end{bmatrix}$ is a row vector with a collection of numbers 100, 200 and 300 in one row.

C = $\begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$ is a column vector with a collection of numbers 10, 20 and 30

Each of the elements in R and C are vectors.

Diagrammatically, a vector may be represented by an arrow as follows:

$$\text{A (Tail)} \xrightarrow{\quad\quad V \quad\quad} \text{(Head) B}$$

The distance between A and B has a magnitude. A magnitude is a number that may be used for any variable such as height, weight, length, income, price, and so on, and is a scalar quantity. The arrow represents that the line moves from A to B by some amount and also has a displacement in a particular direction. Thus, V may be called a vector.

Vectors may be represented in a co-ordinate system with the x-axis and the y-axis and it generally starts from the origin. They are 2D (Two-dimensional) vectors. Consider the following diagram:

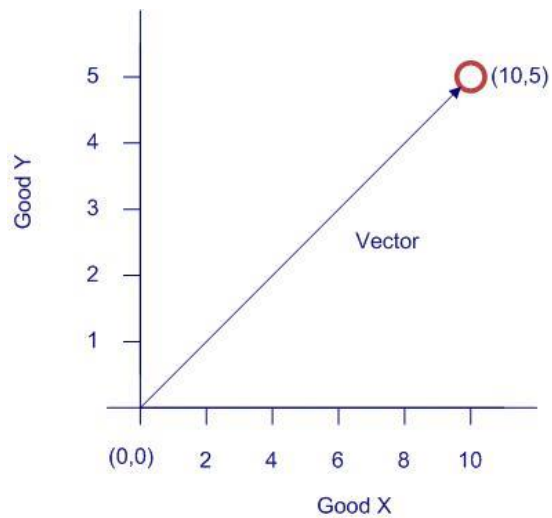

Fig 11.1: Vector

Fig. 11.1 shows a commodity space. The co-ordinate (10, 5) says that the units consumed of good X are 10 and the units of Good Y are 5. Drawing a line from the origin to the co-ordinates (10, 5), gives the vector and represents the movement from the origin. In other words, it is a displacement from zero to 10 units of Good X and zero to 5 units of Good Y. This may be represented with an arrow as shown in the figure.

Note: Zero vector means that the magnitude is zero and it does not have any length

11..2. Difference between a vector and matrix

A vector is a quantity having a magnitude and a direction. A matrix is a collection of vectors. All the elements in a matrix are movements or connections from one point in space to another.

For example,

The economic variable, Price, may be written as a column vector as:

$$P = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

P is a 2 x 1 column vector and may be called a price vector.

On the other hand, a matrix is an array of rows and columns written as:

$$Q = \begin{bmatrix} 10 & 5 \\ 5 & 10 \end{bmatrix}$$

Q is a 2 x 2 matrix and is a collection of vectors.

For example, the matrix Q may show a connection between the quantities sold of two goods in two different places. The columns may represent the quantity sold of two goods, Say Good X and Good Y, by a seller and the rows may represent two different markets, market A and market B.

Thus, Q is a collection of vectors that may diagrammatically be shown as follows:

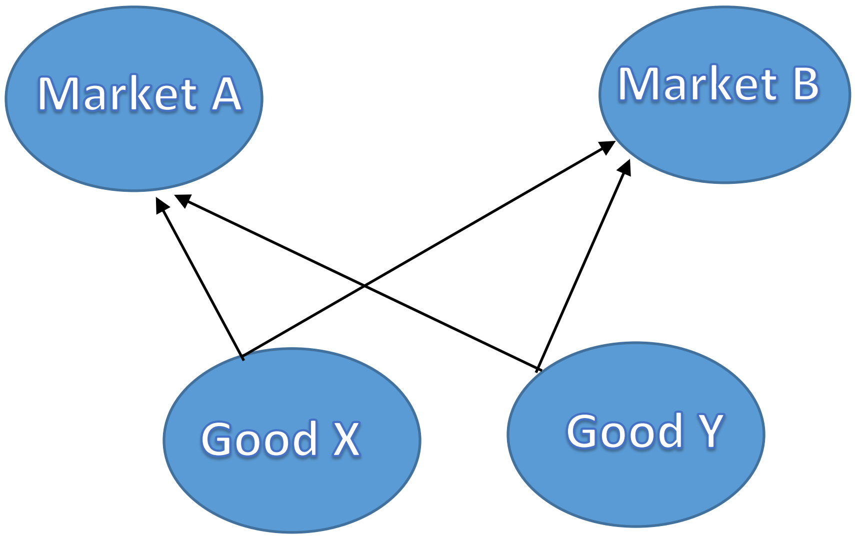

Fig 11.2: Matrix of two goods sold in two markets

Fact: A matrix is a storehouse of information. The higher the dimension, greater is the information stored.

11.3. Converting linear models to matrix form

Matrix algebra is used to solve linear equations. Finding a solution to a single linear equation is not difficult. Even two linear equations may be solved by using the elimination method. But as the number of linear equations in a system of equation exceeds two, solution by the usual elimination method becomes difficult. Matrix algebra may be used in such cases to find the solution to a simultaneous equation system.

A simultaneous system of linear equations may be expressed into matrix form by re-arranging the equations in rows and columns.

Example1:

To understand the method of conversion, let us take a simple example with only two equations. The same procedure may be applied for higher number of equations.

Given two linear equations,

$$\begin{aligned} x + y &= 24 \\ 2x \quad &= 12 \end{aligned}$$

The solution gives, $x = 6 \; and \; y = 18$

This problem may be solved by using matrix algebra as follows:

Step 1: Write down the co-efficient of the variables by arranging them in rows and columns within a pair of brackets and give any name by using letters from the alphabets. This matrix is known as the co-efficient matrix.

Step 2: Write a column matrix of the variables in the system. This may be called the variable matrix.

Step 3: Write a column matrix of the constants.

Step 3: Multiply the co-efficient matrix with the variable matrix and equate it with the constant matrix.

Note: If the variables in the equations are not arranged, it has to be arranged first and the order (position) has to be followed in the rest of the equations as well. The co-efficient of a missing variable is zero.

For the above example, the variables are already arranged. So, we can move to the first step of constructing matrices

1) Co-efficient matrix is given by C = $\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$. The order of the matrix is 2 x 2
2) Column vector of the variables is given by V = $\begin{bmatrix} x \\ y \end{bmatrix}$. The order of the vector is 2 x 1
3) Column vector of constants is given by F = $\begin{bmatrix} 24 \\ 12 \end{bmatrix}$. The order of the vector is 2 x 1

Thus a simultaneous equation system with two equations has been transformed into matrix form as

$$\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 24 \\ 12 \end{bmatrix}$$

And may be written in short hand as CV = F

Note: Matrix method of solution may not necessarily be used to find solutions for small system of equations, say, a two equations' system. This method becomes useful as the number of equations and variables increases.

Example 2:

Given the following simultaneous equation system with three equations:

$$4x + 2y - z = 40$$

$$2x + 3y \quad = 43$$

$$x \quad + z = 38$$

In the above system, z is a missing variable in the second equation and y is a missing variable in the third equation. The co-efficient for z and y will be zero.

Note: The position of the variables is important. If x is placed in the first place, y in the second and z in the third place in the first equations, the same positions must be chosen for the rest of the equations.

Following the steps of conversion, the simultaneous equation system may be written in matrix form as:

$$\begin{bmatrix} 4 & 2 & -1 \\ 2 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 40 \\ 43 \\ 38 \end{bmatrix}$$

Let the co-efficient matrix be called E. Thus, $E = \begin{bmatrix} 4 & 2 & -1 \\ 2 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

The variable matrix be called R. Thus, $R = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

And the constant matrix be called C. Thus, $C = \begin{bmatrix} 40 \\ 43 \\ 38 \end{bmatrix}$

In short hand, the three equations have been converted into a single equation, ER = C. E is a matrix of order 3 x 3, R is a column vector of order 3 x 1 and C is a column vector of order 3 x 1.

Similar conversions may be made for higher number of equations.

Note: While multiplying the matrices, the rule for multiplication must be kept in mind. The rule says that, two matrices can be multiplied if the column of the first matrix is equal to the row of the second matrix. Therefore, R cannot be pre-multiplied with E, because the number of columns of the first matrix that is R is 1 and it is not equal to the number of rows in the second matrix that is 3.

11.4. Use of matrix in Economics:

Matrix may be used to solve economic problems. Below are some popular applications of matrix in economics and business

1. **Circular Flow matrix:** This is a matrix of the Circular Flow Economy that explains how resources, products and money flow in an economy.

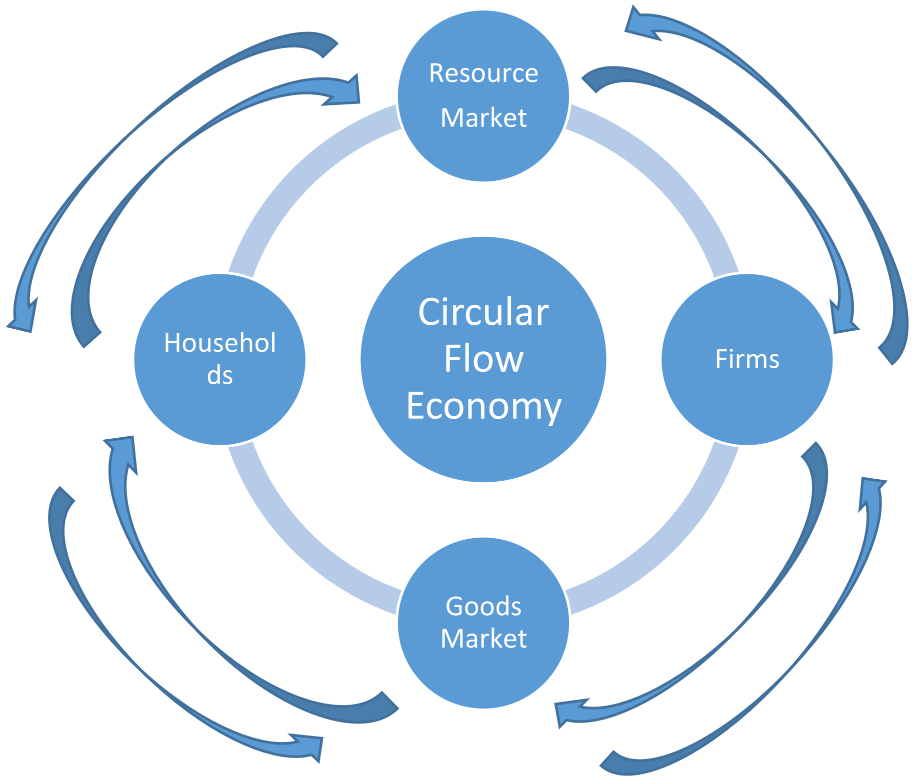

Fig 11.3: Matrix of Circular Flow Economy

The arrows in the inner circle move clockwise and show the flow of goods and services. Households provide labour to the resource market and that labour is distributed to the firms. The firms utilize the services of the labourers to produce goods that go to the goods' market. The goods are then distributed to the households.

The arrows in the outer circle move anti-clockwise and show the flow of money in the economy. In return for the services offered by the labourers, they are paid wages, salaries or bonus, etc. And in return for the goods received by the households, revenue is collected by the firms.

Thus a circular connection is established in the economy between the households and the firms and this is expressed as the Circular Flow Matrix.

2. **Input-Output matrix:** The concept of an input-output matrix was given by the Nobel laureate Wassily Leontief. This matrix was used to model and analyze the relationships

between different sectors in an economy or to analyze the distribution of intermediate inputs.

Suppose there are three industries. The production of each industry is required by the own industry as well as distributed among the other two industries as well.

Table 11.1: Input-Output Matrix (Relationship between industries)

Industry	Industry A	Industry B	Industry C	Total
Industry A	50	30	20	100
Industry B	30	20	50	100
Industry C	20	50	30	100
Total	100	100	100	300

The total production of Industry A, B and C is 100 units each. Industry A keeps 50 units for itself and distributes 30 units and 20 units to industry B and C respectively. Industry B keeps 20 units for itself and distributes 30 units and 50 units to A and C respectively. Similarly, Industry C keeps 30 units for itself and distributes 20 units and 50 units to Industry A and B respectively.

This relationship established between the three industries is termed as the input-output matrix. Besides, matrix applications are also seen in the following:

3. **Matrix Organization:** An organizational structure refers to how the people in an organization are grouped and to whom they report. A matrix organizational structure is one where there is more than one boss. This is a complex structure with horizontal and vertical production lines.

Suppose an organization has three projects to be completed A, B and C. Each project has a project manager. There are four different functional divisions of each project -research division, production division, sales division and finance division. The project managers move horizontally intersecting with the and the functional managers move vertically down. This may be arranged as shown in Fig. 11.4.

	Research Department	Production Department	Sales Department	Finance Department
Project A				
Project B				
Project C				

Fig. 11.4: Matrix Organization Structure

Though there are several challenges, some of the major advantages of a matrix organization structure are:

1. Optimum and Effective use of resources: Since projects run parallelly, there is optimum and effective use of both physical and human resources and no resources goes as waste or remain idle.
2. Skill development: There is an opportunity for employees to learn and develop new skills. For example, an employee from the research division gets to lean about sales or finance.

4. Efficiency: Different projects work independently and parallelly with their respective experts. This leads to high returns at a low cost.

Matrix Game: This term is used in Game Theory. Game theory is not limited to economics alone but has variety of applications. A matrix game is a term used for a finite "Two-person zero sum game." Two players with opposite interests play a game within a given set of rules and strategies.

Fig. 11.5 shows a payoff matrix of Firm I. Firm I has four strategies, X_1, X_2, X_3 and X_4 and Firm II has five strategies, Y_1, Y_2, Y_3, Y_4 and Y_5

		Firms II's strategies				
		Y_1	Y_2	Y_3	Y_4	Y_5
Firm I's Strategies	X_1	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}
	X_2	P_{21}	P_{22}	P_{23}	P_{24}	P_{25}
	X_3	P_{31}	P_{32}	P_{33}	P_{34}	P_{35}
	X_4	P_{41}	P_{42}	P_{43}	P_{44}	P_{45}

Fig. 11.5: Payoff matrix of Firm I in Game Theory

