Mathematical Economics:

Applications of Differential Equations in Economics

Introduction:

Economic Researchers use differential equations to study the functional relationships between variables. There are a wide range of applications of differential equations in economics and finance. Some of the examples that uses differential equations are presented in this module.

Objectives:

1. *Explore* the applications of differential equations in economics
2. *Explain* the economic interpretation of some mathematical functions

Terminology:

1. Gross Domestic Product (GDP): a monetary measure of the sum of all final goods and services produced in an economy during a particular period of time.
2. Lorenz curve: a curve that represents distribution of income or wealth or consumption expenditure
3. Growth rate: the rate at which economy grows and defined as the ratio of marginal function to total function
4. Productivity: a measure of efficiency of a person (labour), machine (capital), land and so on. Productivity can be average or marginal and measured in converting inputs to outputs.
5. e: also called Eulers number. In Economics, it is interpreted as the result of a special process of interest compounding.
6. Instantaneous growth: growth at a particular point.

32.1 Some examples of the use of Differential equations in Economics:

Differential equations are widely used in economic research. Some of the examples that uses the technique of differential equations are given below:

1. **To calculate the rate of change of Gross Domestic Product (GDP) with time:**

Gross Domestic product (GDP) is a monetary measure of the sum of all final goods and services produced in an economy in a particular period of time. In order to measure the rate of change of GDP with respect to time, economists have used the concept of differential calculus.

The rate of change of GDP is said to be proportional to the current GDP.

If $Y(t)$ is the current state of GDP, then $\frac{dY}{dt}$ is the rate of change of current state with respect to time and it is proportional to the current GDP.

Thus,

$$\frac{dY}{dt} = gY(t), \qquad g \text{ is the growth rate}$$

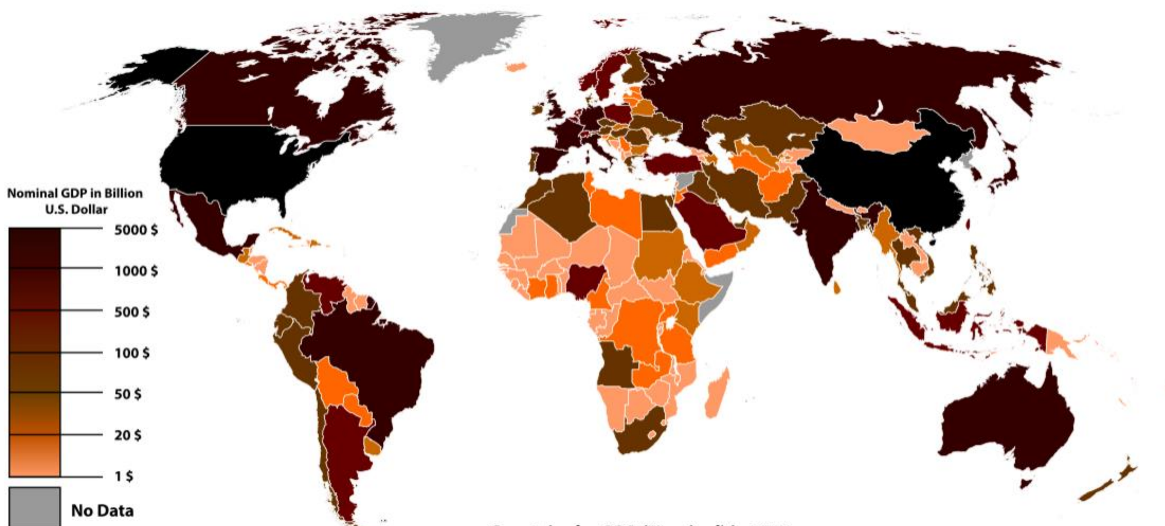

Image 1: Map of world economies by size of GDP (nominal) in USD

[Source: https://en.wikipedia.org/wiki/File:Countries_by_GDP_(Nominal)_in_2014.svg]

2. **To compute average income of population and minimum income levels:**

Lorenz curve is a popular curve in Economics that represents the distribution of income or wealth. It is a graphical representation that shows the relation between cumulative percentage of population and the cumulative percentage of income.

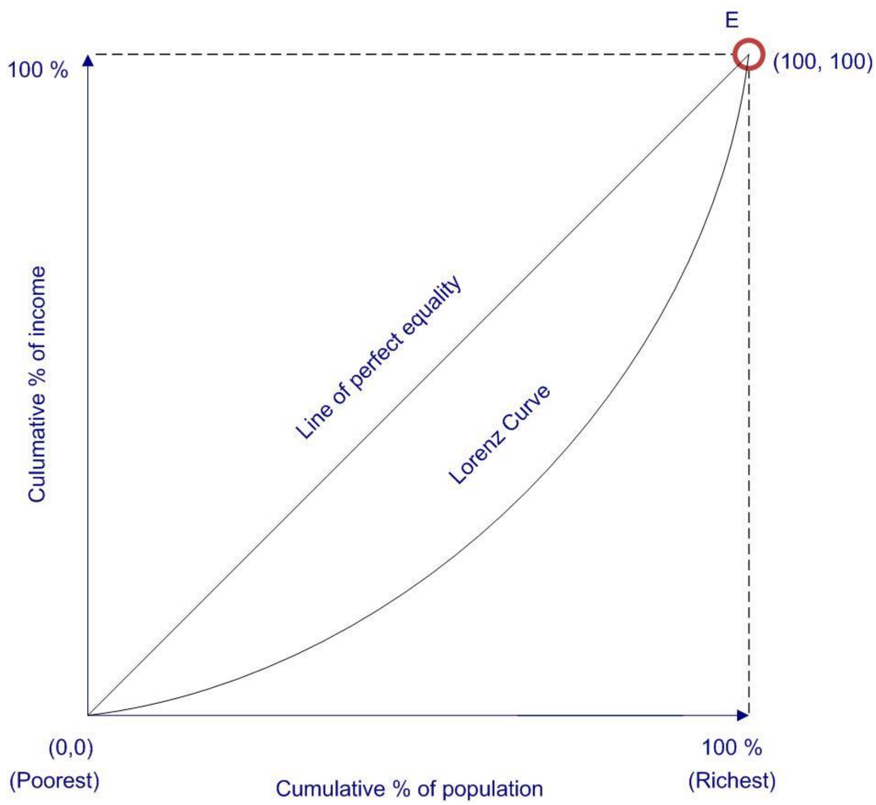

Figure 32.1: A Lorenz Curve

The x- axis measures the cumulative percentage of population from the poorest to the richest and the y axis measures the cumulative percentage of income from the lowest to the highest. The point E shows that 100 percent of the population receive 100 percent of the income. The line of equality is a straight line from the origin and is a 45-degree line. This line shows that there is equal distribution of income in the economy. In other words, lowest 10 % of population receive 10 % of income, 50% of population receive 50% of income and highest 100 % of population receive 100 % of income.

But this scenario generally does not exist in an economy and the Lorenz curve lie below the 45-degree line showing a situation of inequality. The further away from the 45-degree line, the higher is the inequality.

The cumulative income is related to individual incomes by the derivative of the Lorenz curve.

Suppose a Lorenz curve is represented by $F(x)$, x is the percent of population; then an individual's contribution to Lorenz curve is given by the derivative $F'(x)$.

"Comparing individual incomes to average incomes leads to a differential equation." Says K¨ampke, T., Pestel, R., Radermacher, F.J.

If a segment of population x receive a proportion $F(x)$ of income. Assuming the total income of an economy as 1, the remaining income $1 - F(x)$ is distributed among the remaining fraction of population $1 - x$. This remaining fraction is the richer section and the average income of all richer individuals will be represented by $\frac{1-F(x)}{1-x}$. An individual with income x is supposed to have a constant fraction, δ, where $\delta < 1$. Further, if an individual's contribution to Lorenz curve is the derivative of the curve, then

$$F'(x) = \delta \frac{1 - F(x)}{1 - x}$$

Is a linear non-homogeneous differential equation.

3. **To explain an economy's growth rate:**

Growth models are constructed to explain an economy's growth rate. Different economists have constructed different growth models. To illustrate the use of differential equations in growth models, below are two growth models:

a) Harrod-Domar Model:

This model was developed independently by Roy. F. Harrod in 1939 and Evsey Domar in 1946. According to this model, an economy's growth rate is explained in terms of the level of saving and productivity of capital. The output growth rate is determined with calculus using dot notation, for the derivative of a variable with respect to time.

Mathematically, the output growth rate is given by:

$$\frac{\dot{Y}}{Y} = sc - \delta$$

Here $\frac{\dot{Y}}{Y}$ is the output growth rate

s is savings rate

c is marginal product of capital

and δ is the rate of depreciation of capital stock

b) Solow-Swan Model:

This model was developed independently by Robert Solow and Trevor Swan in 1956. This is a non-linear system consisting of a single ordinary differential equation and attempts to explain long-run economic growth by considering capital accumulation, population growth and technological progress. The key equation of Solow-Swan model is of Cobb-Douglas type Production function given as

$$Y = K^{\alpha}(AL)^{1-\alpha} \ \ (0 < \alpha < 1)$$

Y is the output

L and K are labour and capital used

α is the elasticity of output with respect capital.

A is the labour-augmenting technology and AL is the effective labour

The main interest of this model is the capital intensity k, the capital stock per unit of effective labour.

The marginal product of capital is given as

$$MP_K = \frac{\partial Y}{\partial K} = \frac{\alpha A^{1-\alpha}}{(K/L)^{1-\alpha}}$$

This equation is non-linear equation consisting of a single ordinary differential equation and attempts to explain long-run economic growth.

In addition, differential equations have a wide range of applications such as to study trade cycles, economic chaos, dynamic stability conditions of equilibrium, to examine dynamic behavior of economic models such as Cournot Duopoly Model with constant marginal costs and increasing marginal costs, Walrasian-Marshallian adjustments, Cagan Monetary Model and to examine the dynamics of unemployment within the framework of growth theory.

Discussion of all the applications is beyond the scope of this course. Interested learners may refer to “learn more” section for detailed study.

32.2. Economic interpretation of exponential ‘e’:

‘e’ is a mathematical constant and used as the base on natural logarithm. It is also called “Euler’s number” and is named after the Swiss mathematician, Leonhard Euler.

In the previous modules, while explaining the solution of first order and second order differential equations, $'e'$ appeared as a term in the general solution. The mathematical interpretation of $'e'$ may be explained as follows:

If we have a function $f\ (x) = (1 + \frac{1}{x})^x$

As $x\ increases$, $f(x)$ will also become larger and larger.

If $x = 1, f\ (1) = (1 + \frac{1}{1})^1 = 2$

If $x = 2, f\ (2) = (1 + \frac{1}{2})^2 = 2.25$

If $x = 3, f\ (3) = (1 + \frac{1}{3})^3 = 2.3691$

If $x = 4, f\ (4) = (1 + \frac{1}{4})^4 = 2.4414$

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If $x = 100, f\ (100) = (1 + \frac{1}{100})^{100} = 2.7048$

If x increases indefinitely, $f(x)$ will converge to the number $'e = 2.71828\ \ldots.'$

Mathematically,

$$e \equiv \lim_{x\to\infty} f(x) = \lim_{x\to\infty}(1 + \frac{1}{x})^x$$

In Economics, this number can be interpreted as the result of a special process of interest compounding. The expressions $e, e^t, Ae^t\ and\ Ae^{mt}$ are economically interpreted as continuous interest compounding. $'m'$denotes the nominal interest rate. However, this is not an exclusive interpretation. It can be applied to the growth of population, wealth or real capital.

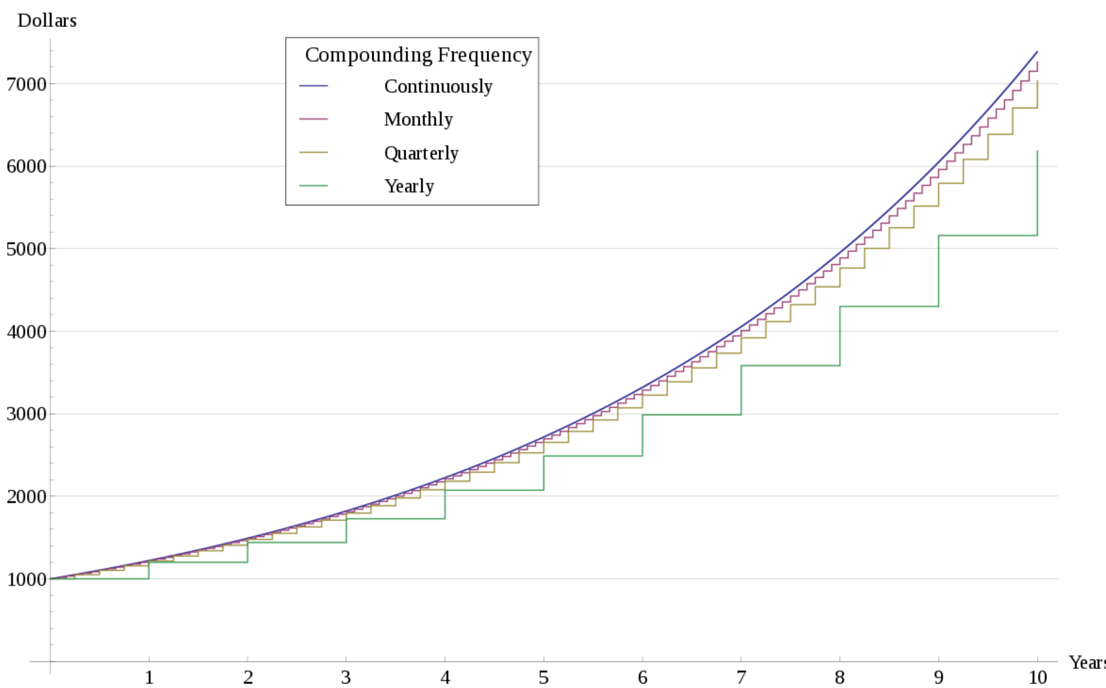

Image 2: An example of compound interest

[Source: https://en.wikipedia.org/wiki/File:Compound_Interest_with_Varying_Frequencies.svg]

Image 2 shows the effect of 20 percent annual interest on an initial amount of $1000 investment at various compounding frequencies.

32.3. Instantaneous Rate of Growth:

In the expression, Ae^{mt}, 'm' is the instantaneous rate of growth.

If a variable x is a function of time (t), $x = f(t)$, the instantaneous rate of growth is defined as

$$m = \frac{dx/dt}{x} = \frac{f'(t)}{f(t)}$$

Further, if $f(t)$ is the total function the economic interpretation of first derivative $f'(t)$ is the marginal function.

Therefore,

Rate of growth may be defined as

$$m = \frac{marginal\ function}{total\ function}$$

Suppose we have a function

$$S = Ae^{mt} - - - - - - - - (32.1)$$

$S\ may\ denote\ a\ sum\ of\ money\ or\ population$

The equation 32.1 is a stock or is a function at a given point of time and S takes unique values at one point in time.

Differentiating equation 32.1 with respect to t, gives

$$\frac{dS}{dt} = mAe^{mt} = mS - - - - - (32.2)$$

Now, by definition, the rate of growth of S is expressed as a ratio of the rate of change of S to itself.

Therefore, for any given point,

$$\text{Rate of growth of S} = \frac{dS/dt}{S} = \frac{mAe^{mt}}{Ae^{mt}} = m$$

The change of S, represents a flow. This gives the values of S during a period of time.

Note:

a) Since $\frac{dS}{dt} = mAe^{mt} = mS$ takes different values at different points of time t and the ratio $\frac{dS/dt}{S}$ has reference to an instant time, it is known as the instantaneous rate of growth. However, by definition, growth occurs over a period of time, therefore, instantaneous growth rate means if the rate m at a particular point continues undisturbed for one unit of time (example, one year)
b) Instantaneous rate of growth may not be a constant for all growth situations.
c) For an exponential function, $S = Ae^{mt}$, the percentage rate of growth is constant at all points but the absolute value increases as time moves ahead.