Mathematical Economics

First-Order Linear Differential Equations

Introduction:

The fundamental idea of solving differential equation is by integration. But the economy is complex. The dynamics of market equilibrium and growth models may be easily solved by using some other advanced methods. This module will discuss these methods.

Objective

The objectives of this module are:

1. *Identify* the different types of differential equations
2. *Explain* the concept of linear differential equation
3. *Solve* the first-order linear differential equation

Terminology

1. Ordinary differential equation: a differential equation containing one or more functions of an independent variable and the derivatives of those functions.
2. Partial differential equation: a differential equation containing more than one independent variable and their derivatives.
3. Linear differential equation: a differential equation having linear terms
4. Non-linear differential equation: a differential equation with non-linear terms
5. Homogeneous differential equation: a differential equation in which the RHS is zero
6. Non-homogeneous differential equation: a differential equation where RHS is non-zero
7. Taylor series: expansion of a function $y = f(x)$ at any point of x
8. Maclaurin series: a special case of Taylor series that shows the expansion of a series at $x = 0$
9. Final solution of a differential equation: the sum of complementary solution and particular solution
10. Complementary solution: the solution of a homogeneous differential equation
11. Particular solution: the solution of a non-homogeneous differential equation
12. Complete solution: sum of complementary solution and particular solution

30.1 Different types of differential equations:

Differential equations may be of several types based on the type of function, order or homogeneity.

a) **Ordinary differential equation:** If a differential equation contains one independent variable and the derivative of that variable, it is termed as an ordinary differential equation.

For example: $\frac{dy}{dx} = x$ is an ordinary differential equation

b) **Partial differential equation:** If a differential equation contains more than one independent variables and their derivatives, it is termed as a partial differential equation. It gets the name partial because it contains a partial derivative.

For example: If $u = f(x, y)$, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = x^3 - y^3$ is a partial differential equation. $\partial\ is\ used\ for\ partial\ differentiation$

c) **Linear differential equation:** If a differential equation has dependent variable with order 1 and the derivatives also have degree 1, it is termed as a linear differential equation.

For example: $\frac{dy}{dx} + 5y = 10$ is a linear differential equation as the power of y and the derivative is 1.

d) **Non-linear differential equation:** A differential equation of the form

$$\frac{dy}{dx} = y^2$$

it is termed as non-linear differential equation. Here the unknown function has a quadratic form apart from the derivative.

Riccati equation is a first-order ordinary differential equation that is quadratic in the unknown function. It is expressed as

$$\frac{dy}{dx} = y^2 + x$$

Abel equation is a first-order ordinary differential equation that is cubic in the unknown function. It is expressed as

$$\frac{dy}{dx} = y^3 + x$$

e) **Homogeneous differential equation:** In a differential equation, if the constant term on the right-hand side is zero, it is termed as a homogeneous differential equation.

For example: $\frac{dy}{dx} + 5y = 0$ is a homogenous differential equation.

f) **Non-homogeneous differential equation:** In a differential equation, if the constant term on the right-hand side is non-zero, it is termed as a non- homogeneous differential equation.

For example: $\frac{dy}{dx} + 5y = 10$ is a non- homogenous differential equation.

g) **Differential equation by order:** A differential equation may also be categorized by the order of the derivative in the equation. If differential equation has a derivative with order 1, it is termed as first order differential equation. Similarly, if a differential equation has 2 as the highest derivative, it is termed as a second order differential equation.

For example: $\frac{d^3u}{dx^3} + 2\,\frac{du}{dx} + u = 5x$ is an example of a third order differential equation as the highest power (order) of the derivative is 3.

30.2 Concept of Linearization of a funtion

As was defined in the previous section, a linear differential equation consists of linear terms but linear differential equations may be approximations to non-linear equations. In dynamics, linearization is a method for measuring the stability of an equilibrium point of a system of non-linear differential equations. It is used in the field of physics, ecology, engineering and economics. In economics, the use of the method of linearization can be seen in the utility maximization problem and cost optimization.

A. Taylor Series and Maclaurin Series:

The linear approximation of a function is the first order Taylor expansion. Taylor series was introduced by Brook Taylor. Maclaurin series is a special case of Taylor series and was named after Colin Maclaurin.

Image 30.1: Brook Taylor

[Source:https://en.wikipedia.org/wiki/Brook_Taylor#/media/File:BTaylor.jpg **]**

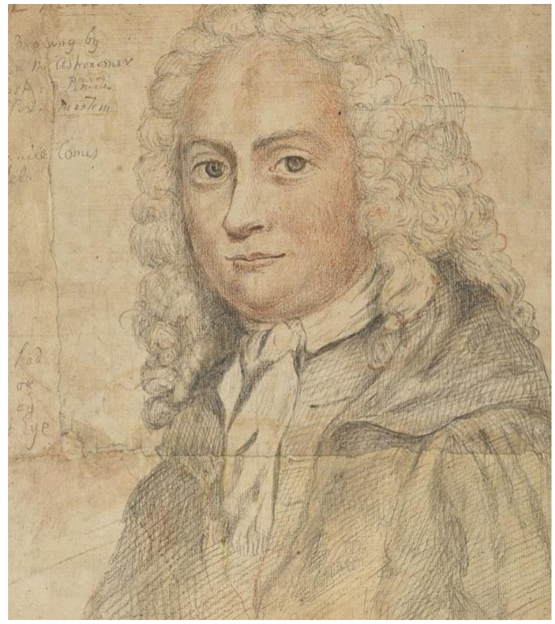

Image 30.2: Colin Maclaurin

[Source:https://en.wikipedia.org/wiki/Colin_Maclaurin#/media/File:Colin_Maclaurin_(Buchan-Ferguson).jpg **]**

Before understanding the Taylor series and Maclaurin series, it is important to understand what "expansion" of a function mean. If $y = f(x)$ is a function, the expansion of the function means to transform the function such that the coefficients are expressed as derivatives, at the point of expansion. Therefore, expansion of the function $y = f(x)$ at $x = x_0$ is known as a Taylor Series. The expansion of the function at $x = 0$ is a special case of Taylor expansion and is known as Maclaurin Series.

You will agree that higher the degree of a polynomial, the number of turning points of the graph increases. In order to get a good approximation of higher degree polynomials, the Taylor series and the Maclaurin series may be used.

B. Use of Taylor series in Economics:

i) Decisions involving individual risks

ii) Measuring the effect of changes in the price of bond due to changes in yield.

iii) Approximate loss distribution of portfolio

Note: This module will concentrate in solving the linear differential equation.

Interested learners may refer to research papers mentioned in the "learn more" section.

30.3 First Order Linear Differential Equation:

If $y = f(t)$ is a function, then

$$\frac{dy}{dt} = f'(t) \text{ ----------------------- } (30.1)$$

is defined as a differential equation. Since the order of the derivative in equation 30.1 is 1, it is termed as a first order linear differential equation. There is however no restriction on the independent variable t. Another characteristic of a linear differential equation is that the no products of y or any of its derivative is present.

A. General form of a first order differential equation:

The first order linear differential equation may take the general form as

$$\frac{dy}{dt} + u(t)y = v(t) - - - - - (30.2)$$

$u \text{ and } v \text{ are functions of } t.$

B. First order linear differential equation with constant term and constant coefficient:

In Equation 30.2, $u(t)$ and $v(t)$ are functions. If $u(t) and\ v(t)$ take constant values, then we have a linear differential equation with constant coefficient and constant term.

For example:

$$\frac{dy}{dt} + my = n - - - - - - (30.3)$$

$m \text{ and } n \text{ are constants}$

The solution of equation 30.3 is divided into two parts:

a) Complementary solution:

In order to obtain the complementary solution, we consider the homogeneous part of equation 30.3. Thus,

$$\frac{dy}{dt} + my = 0 - - - - - - (30.4)$$

is a homogeneous differential equation in which the right-hand side is equal to zero.

Re-writing equation 30.4, we get

$$\frac{dy}{dt} = -my$$

$$or, \qquad \frac{1}{y}\frac{dy}{dt} = -m ------(30.5)$$

Now, solving equation 30.5 means to find the value of y. By definition, integration is the reverse of differentiation.

Therefore, integrating equation 30.5 gives,

$$\int \frac{1}{y}\frac{dy}{dt}dt = \int -mdt$$

By substitution rule and log rule, we get

$$LHS = \int \frac{dy}{y} = \ln|y| + c_1 \; ------(30.6)$$

And

$$RHS = \int -mdt = -mt + c_2 ----- (30.7)$$

Equating 30.6 and 30.7, gives

$$\ln|y| + c_1 = -mt + c_2$$

$$\text{or,} \quad \ln|y| = -mt + c \qquad (c = c_1 + c_2)$$

To obtain ln|y|, we take antilog, that gives

$$e^{\ln|y|} = e^{-mt+c}$$

$$or, \qquad |y| = e^{-mt}e^{c}$$

$$or, \qquad |y| = Ae^{-mt} \qquad (A = e^{c})$$

Thus,

$$y(t) = Ae^{-mt} -------- (30.8)$$

$$A\ is\ any\ arbitrary\ constant$$

To find the value of A, if we set $t = 0$, then $y(0) = Ae^0 = A$

Therefore,

$$y(t) = y(0)e^{-mt} ------(30.9)$$

Equation 30.8 is the general solution and equation 30.9 is the definite solution.

It is to be noted that the general solution or the definite solution do not have any derivative but is a function of t.

Let us call the complementary solution as $y_c = Ae^{-mt} - - - (30.10)$

b) Particular solution:

In order to find the particular solution, we consider the simplest solution, $y = k, where\ k\ is\ a\ constant$. Then equation 30.3 becomes

$$my = n - - - - - - (30.11)$$

$\because diferentiation\ of\ a\ constant\ or\ \frac{dk}{dt} = 0$

$$or, \qquad y = \frac{n}{m}\ (m \neq 0)$$

Let us call the particular solution as $y_p = \frac{n}{m}\ (m \neq 0) - - - - - (30.12)$

Now, the final solution is given by the sum of the complementary solution and the particular solution.

Therefore,

$$y(t) = y_c + y_p = Ae^{-mt} + \frac{n}{m}$$

Taking an initial solution $at\ t = 0$, we get

$$y(0) = A + \frac{n}{m}$$

$$or, \qquad A = y(0) - \frac{n}{m} - - - - - - - - - (30.13)$$

$$\therefore\ y(t) = \left[y(0) - \frac{n}{m}\right] e^{-mt} + \frac{n}{m} - - - - (30.14)$$

Example:

Solve the differential equation

$$\frac{dy}{dt} + 4y = 16$$

With an initial condition $y(0) = 10$

Solution: Here, $m = 4\ and\ n = 16$

The complementary solution is given by $y_c = Ae^{-mt}$ (equation 30.10)

$$or, \qquad y_c = Ae^{-4t}$$

Now, the initial condition is given as $y(0) = 10$, and we have

$$A = y(0) - \frac{n}{m} \qquad (from\ equation\ 30.13)$$

$$or, \qquad A = 10 - \frac{16}{4} = 6$$

$$\therefore y_c = 6e^{-4t}$$

The particular solution is given by

$$y_p = \frac{n}{m} = \frac{16}{4} = 4$$

Therefore, the final solution is given by

$$y(t) = y_c + y_p = 6e^{-4t} + 4$$

30.4. First Order Differential Equation with variable coefficient and variable term:

Consider equation 30.2 in section 30.3 (A),

$$\frac{dy}{dt} + u(t)y = v(t)$$

If the RHS of this equation is zero, we have a homogeneous differential equation with variable coefficient and constant term.

Therefore,

$$\frac{dy}{dt} + u(t)y = 0$$

$$or, \qquad \frac{1}{y}\frac{dy}{dt} = -u(t) - - - - - (30.15)$$

Using the same procedure as in section 30.3, we arrive at the general solution as

$$y(t) = Ae^{-\int u(t)dt} - - - - - - - - (30.16)$$

The difference between equation 30.8 and equation 30.16 is that, the solution has the integration of a function as the power of the exponential on the RHS.

Equation 30.16 is the general solution of the differential equation of homogeneous form with variable coefficient.

Note: What will be the solution of a differential equation with variable coefficient and variable term? *(Hint:* Use the concept of Exact Differential Equation explained in Module 31.)

