Mathematical Economics

Duality in Linear Programming

25.1. Introduction:

Linear Programming problems deal with optimizing an objective function subject to certain constraints. The objective function may be maximized or minimized. For example, a production firm producing two products using three resources in limited quantity wants to find the amount of the two products that must be produced with the given resources such that the total profit from both the products is maximised. This problem may be solved by constructing the linear programming problem.

Given the same data of the production firm, if the firm wants to determine the amount of resources that must be utilised for each unit of a product such that the cost is minimised, how will the firm achieve this objective?

This module will try to answer this question.

A. Objectives:

The objectives of the module are:

1. *Explain the concept of duality*.
2. *Differentiate* the mathematical formulation of a dual program from a primal program
3. *Determine the optimal solution of a primal program via its dual*

B. Terminology:

1. Duality: a situation of being dual
2. Dual: double or two parts, not necessarily opposites
3. Primal program: original program
4. Dummy variable: a slack or surplus variable that is not part of the original model but may enter the objective function with zero coefficient
5. Slack variable: a variable that is added to an inequality constraint to transform it into an equality
6. Surplus variable: a variable that is subtracted from an inequality constraint to transform it into an equality

25.2. Meaning and application of Duality

'Duality' is a situation of being 'dual'. The term 'dual' generally means double or "two parts". It does not necessarily mean that one part is the "opposite" of the other; rather may be considered complementary. In other words, one true statement may be obtained by another statement simply by interchanging words.

Joseph Diez Gergonne, a French mathematician and logician was the first to explain this concept in projective geometry in 1825. (Compared to elementary geometry, Projective geometry is not based on the concept of distance)

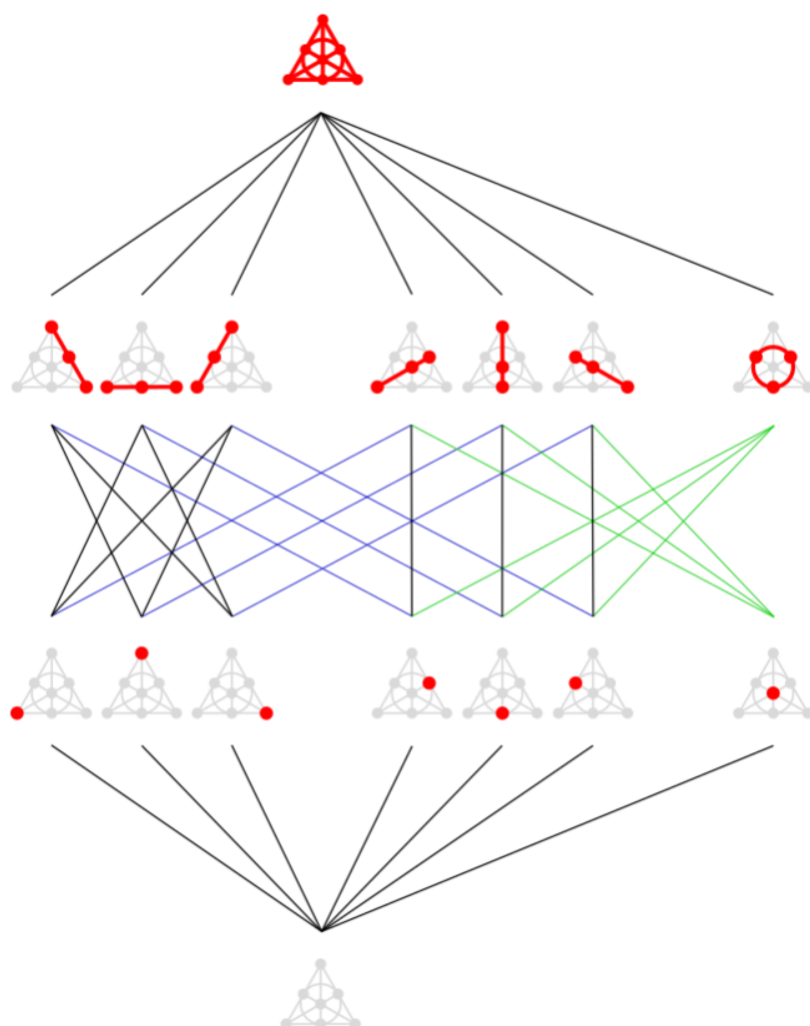

Image25.1: Duality in the Fano Plane

[Source: https://en.wikipedia.org/wiki/File:Fano_plane_Hasse_diagram.svg]

The Fano Plane is the finite projective plane with the smallest possible number of points and lines: 7 points and 7 lines, with 3 points on every line and 3 lines through every point. (the seventh line is actually the circle with 3 points)
The concept of duality is used in different fields of study such as mathematics, philosophy, logic, psychology, physics, music and economics.

Examples of duality:

1. In projective geometry, the statements “two points determine a line” and “two lines determine a point” are dual statements.
2. In set theory, the statements “contained in” and “contains” are dual statements, that can be interchanged by union and intersection of sets.
3. In logic, “implies” and “is implied by” are dual statements
4. In physics, “wave- particle duality” in the theory of quantum mechanics, says that light and matter exhibit properties of both wave and particle.
5. In philosophy, a “coin” and “two sides of a coin” are dual statements.
6. In psychology, humans are considered to have a mind (non-physical) and body (physical). According to Descartes- Cartesian dualism, duality proposes that the mind controls the body, but that the body can also influence the rational mind, such as when people act out of passion.
7. In music, apparently music is a double toroidal system at its most fundamental level.
8. In economics, “maximizing profit” and “minimizing cost” are dual statements.

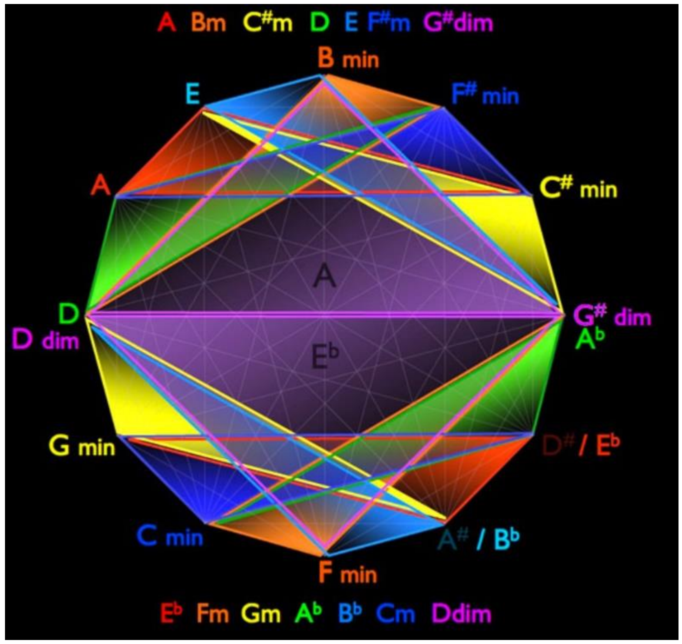

Image 25.2: Balanced pattern showing how it takes two opposite keys to create the whole system of music.[2]

[Source: http://www.cosmometry.net/tri-tone-duality-of-music]

25.3. Mathematical Formulation of Dual Linear Program

The mathematical formulation of a dual linear program is obtained from the primal linear program. The original linear programming problem is known as the primal and the transformed problem is known as the dual. However, the dual of a dual program is again the primal program.

Consider the general formulation of a primal linear programming problem in $'n'$ variables and $'m'$ constraints

$$\left.\begin{array}{l}
\textit{Maximize} \\
\qquad Y = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \\
\textit{subject to} \\
\qquad a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq c_1 \\
\qquad a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \leq c_2 \\
\qquad \text{---} \\
\qquad \text{---} \\
\qquad a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq c_m
\end{array}\right\} \qquad (25.1)$$

$$\textit{and} \quad x_1 \geq 0, x_2 \geq 0, \ldots \ldots x_n \geq 0$$

In order to obtain the dual of the linear program (25.1), there are certain rules of transformation. These rules are as follows:

1) The path of optimization is reversed. If the original (primal) linear program is a maximization problem, the dual program becomes a minimization problem.
2) The 'performance variable' changes. If the 'performance variable' of the primal program is Y, it will change and may be represented as Y^*
3) The number of constraints in the primal program becomes the number of choice variables in the dual program. Thus, if the primal program has $'n'$ choice variables and $'m'$ constraints then the dual program will have $'m'$ choice variables and $'n'$ constraints. (there may be cases where the number of choice variables may be equal to the number of constraints)
4) If primal choice variables are represented by x, dual choice variables may be represented by y.
5) The co-efficient of the choice variables in the primal program becomes the constants of the constraints.
6) The constants of the constraints in the primal program becomes the co-efficient of the choice variables in the dual.
7) The co-efficient of the choice variables in the constraints are transposed. That is, the co-efficient of a row becomes the co-efficient of a column and vice versa.
8) The inequality sign changes. If the primal program has $\geq$ sign, it changes to $\leq$ sign.

Following the above rules, the dual program of equation (25.1) may be written as:

$$\left.\begin{array}{l}
Minimize \\
\qquad Y^* = c_1\, y_1 + c_2 y_2 + \cdots + \; c_m y_m \\
subject\ to \\
\qquad a_{11}\, y_1 + a_{21} y_2 + \cdots + \; a_{m1} y_m \geq \alpha_1 \\
\qquad a_{12}\, y_1 + a_{22} y_2 + \cdots + \; a_{m2} y_m \geq \alpha_2 \\
\qquad \text{---} \\
\qquad \text{---} \\
\qquad a_{1n}\, y_1 + a_{2n} y_2 + \cdots + \; a_{mn} y_m \geq \alpha_n
\end{array}\right\} \quad (25.2)$$

$$and \qquad y_1 \geq 0, y_2 \geq 0, \dots \dots y_m \geq 0$$

Equation (25.2) is the dual program of the primal program equation (25.1)

25.4. Matrix representation of Primal and Dual Program

The primal program and dual program obtained in section 25.3 may be represented in matrix notation as follows:

Primal Program		**Dual Program**	
$Maximize$		$Minimize$	
	$Y = \alpha' x$		$Y^* = c' y$
$subject\ to$		$subject\ to$	
	$Ax \leq c$		$A' y \geq \alpha$
and	$x \geq 0$	and	$y \geq 0$

25.5. Solving a Primal Program via its Dual Program

The solution of a primal program may be obtained via its dual in three steps:

Step 1: Obtain the dual program of the primal program using the rules of transformation

Step 2: Find the optimum solution for the dual program

Step 3: Express the dual choice variables in terms of primal choice variables by using the following Duality Theorems.

A. **Duality Theorems:**

a) If an optimal feasible solution exists, the optimal values obtained for the primal program is equal to that of the dual program. Thus, $Y = Y^*$

b) If a choice variable is a linear program is non-zero, the corresponding slack or surplus variable in the counter program will be zero. Therefore, the dual choice variables must be ultimately expressed in terms of primal choice variables by using the constraints of the primal program that includes the slack and surplus variables.

B. **Slack and surplus variables:**

A slack variable is a variable that is generally added to $\leq$ inequality constraint to transform the constraint into a strict equation.

A surplus variable is a variable that is generally subtracted from a $\geq$ inequality constraint to transform into a strict equation.

If we have an inequality, $x_1 + x_2 \leq 4$, in order to transform it into a strict equation, we may write

$x_1 + x_2 + S_1 = 4$ -------------------------- (25.3)

If we have an inequality, $x_1 + x_2 \geq 4$, in order to transform it into a strict equation, we may write

$x_1 + x_2 - S_2 = 4$ ---------------------------(25.4)

In equation (25.3), S_1 is a slack variable and in equation (25.4), S_2 is a surplus variable. Slack variables and surplus variables are included in the objective function of a linear programming problem by giving a co-efficient zero.

Example:

Consider a linear programming problem

$$\begin{aligned}
&Maximize \\
&\qquad R = 4x + 3y \\
&subject\ to \\
&\qquad x + y \leq 4 \\
&\qquad 2x + y \leq 6 \\
&and \qquad x \geq 0, y \geq 0
\end{aligned}$$

R is the performance variable

$x\ and\ y$ are the primal choice variables

In this example, there are two choice variables and two constraints. The graphical analysis of the primal program gave the optimum solution, $x = 2\ and\ y = 2$

In order to solve the linear programming problem via its dual, two steps are to be followed as mentioned in section 25.5.

Step 1: Obtain the dual program of the primal program using the rules of transformation

Step 2: Find the optimum solution for the dual program

Step 3: Express the dual choice variables in terms of primal choice variables by using the Duality Theorems.

Step 1: The dual program of the above primal program may be written as:

$$\begin{aligned}
&\textit{Minimize} \\
&\qquad R^* = 4x_1 + 6y_1 \\
&\textit{subject to} \\
&\qquad x_1 + 2y_1 \geq 4 \\
&\qquad x_1 + y_1 \geq 3 \\
&\textit{and} \qquad x_1 \geq 0, y_1 \geq 0
\end{aligned}$$

R^* is the performance variable

x_1 *and* y_1 are the dual choice variables

Step 2: The optimum solution for the dual program may be found by using the usual steps as was explained in Module 24.

Therefore,

Expressing the inequality constraints as strict equations, we get,

$$x_1 + 2y_1 = 4$$

$$x_1 + y_1 = 3$$

The graph of the two equations are traced and the feasible region is identified.

Therefore,

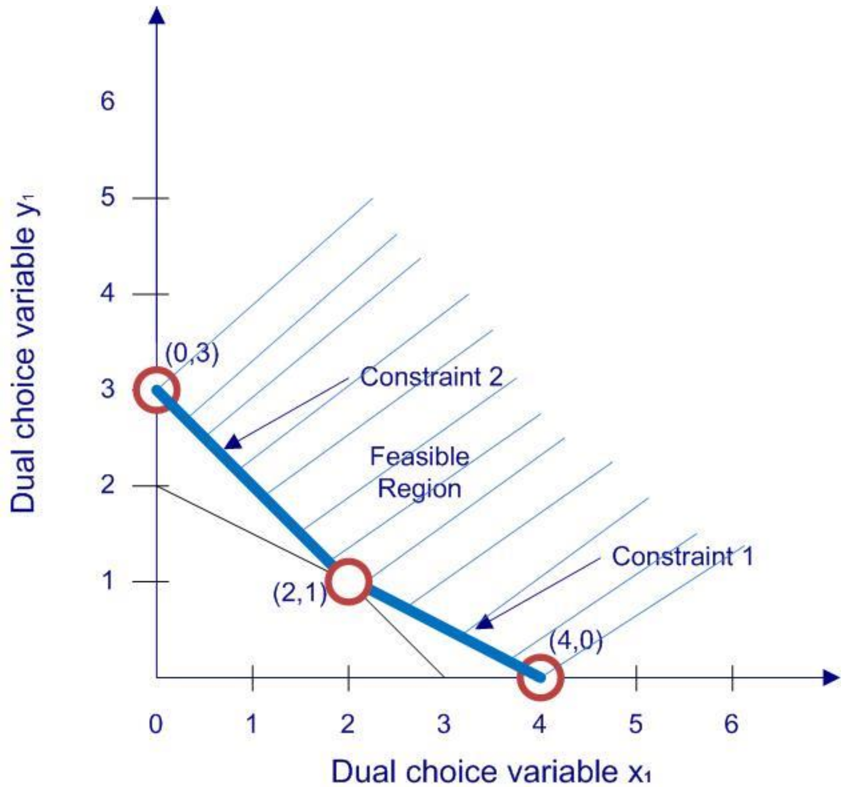

Fig. 25.1: Graph of constraints of dual program

The basic feasible solutions are the corner points of the feasible region. The co-ordinates of three corner points are (0,3), (2,1) and (4,0). The co-ordinates (2,1) in the co-ordinate plane are found by solving the intersecting equations.

Therefore, solving

$$x_1 + 2y_1 = 4$$
$$x_1 + y_1 = 3$$

gives $y_1 = 1 \; and \; x_1 = 2$

The basic feasible solutions are obtained are: (0,3), (2,1) and (4,0).

Next, by tracing out the graph of the objective function we arrive at the optimum solution. The slope of the objective function is $-\frac{2}{3}$

Therefore,

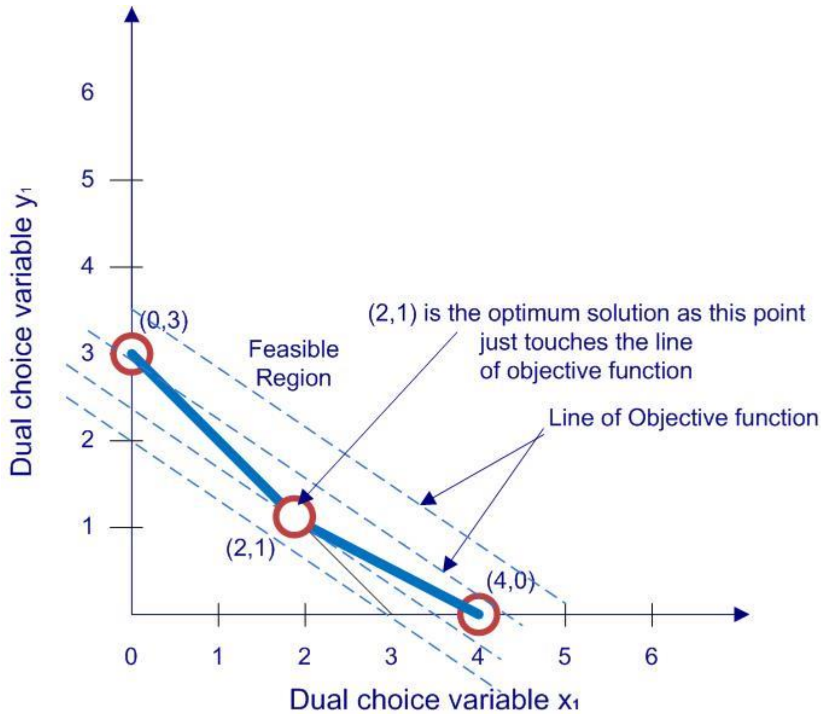

Fig. 25.2: Optimum solution of Dual Program

The optimum solution of the dual program is therefore $x_1 = 2\ and\ y_1 = 1$

Step 3: The final step in solving the primal program via the dual program is to express the dual optimum solution in terms of primal choice variables by using the Duality Theorems. Duality theorem A, part b) says that:

If a dual choice variable in a linear program is non-zero, the corresponding slack or surplus variable in the counter program will be zero. Therefore, the dual choice variables must be ultimately expressed in terms of primal choice variables by using the constraints of the primal program that includes the slack and surplus variables.

Therefore, the constraints of the primal program may be expressed as strict equations by using slack variables as follows:

$$x + y + s_1 = 4$$

$$2x + y + s_2 = 6$$

Putting $s_1 = 0\ and\ s_2 = 0$, we get

$$x + y = 4$$

$$2x + y = 6$$

Solving we get, $x = 2\ and\ y = 2$

