Solved Problems: Differentiation and Optimization

22.1 A television set costs ₹25,000 in the market. A particular shop gets a demand of 10 television sets a month at this price. After few months, the price was lowered to ₹20,000 and the demand increased to 15 television sets in a month. After few more months the price dropped again to ₹15,000 and the demand increased further to 20 television sets.

a) Can you explain the above market situation graphically?

b) According to you, what are the observations on the relationship between quantity demanded and the price? Can you think of a way to explain the relationship logically and mathematically?

c) Can you identify the type of commodity from the information you have with you?

Solution:

a) Since the price and quantity demanded are changing, they are the economic variables.

Given,

Initial price = ₹25,000 Quantity demanded at ₹25,000 is 10

New price = ₹20,000 New Quantity demanded at ₹20,000 is 15

Next New price = ₹15,000 New Quantity demanded at ₹15,000 is 20

The graph for the above situation is shown in Fig 22.1

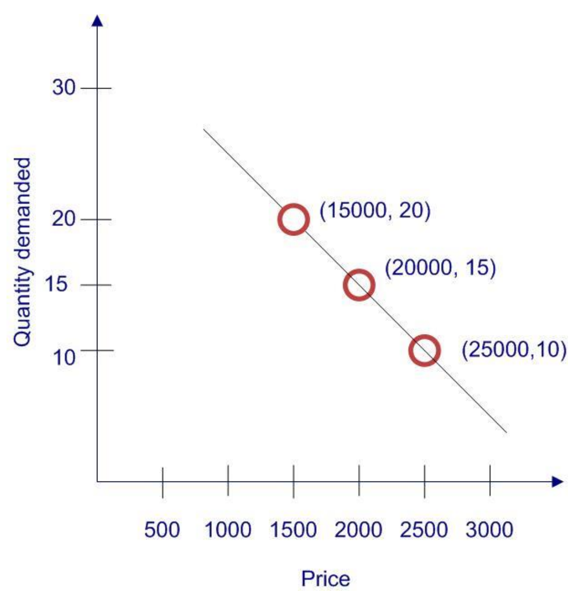

Fig. 22.1: Graph for quantity demanded of Television set

b) Observation:

Since quantity demanded changed when price changed, price may be considered as independent variable and quantity demanded as the dependent variable. When the price was reduced, quantity demanded increased. Therefore, the graph is downward sloping as there is an inverse relationship between price and quantity demanded.

In Economics, the response of the change in quantity demanded to a change in the price is measured by the concept called 'Elasticity'. You will generally come across three types of elasticity of demand

i) price elasticity

ii) income elasticity and

iii) cross elasticity.

The three types are explained below:

i) **Price elasticity of demand**: measures the responsive of quantity demanded to a change in the price. It is sometimes also referred as elasticity of demand. Price elasticity of demand may be measured by two methods:

Method 1: Point elasticity of demand method

This method is used to find the elasticity at a point. In other words, for infinitesimal or very small changes. (see concept of a derivative). Mathematically, it may be given as

$$e_p = (-)\frac{percentage\ change\ in\ quantity\ demanded}{percentage\ change\ in\ price}$$

$e_p\ is\ the\ point\ elasticity\ of\ demand$

Symbolically,

$$e_p = (-)\frac{\frac{dq}{q} \times 100}{\frac{dp}{p} \times 100}\ , q\ is\ the\ quantity\ demanded\ and\ p\ is\ the\ price$$

Or, $$e_p = (-)\frac{dq}{dp} \times \frac{p}{q}$$

The negative sign is used in the formula to represent the inverse relationship between price and quantity demanded.

Note: $'d'$ is used for infinitesimal change.

Method 1: Arc elasticity of demand

This method is used for large changes and is given by the formula:

$$e_a = (-)\frac{\dfrac{Change\ in\ quantity\ demanded}{Average\ of\ the\ two\ quantities}}{\dfrac{Change\ in\ price}{Average\ of\ the\ two\ prices}}$$

e_a *is the arc elasticity of demand*

Symbolically,

$$e_a = (-)\frac{\dfrac{\Delta q}{\frac{q_1 + q_2}{2}} \times 100}{\dfrac{\Delta p}{\frac{p_1 + p_2}{2}} \times 100}$$

$$Or, \quad e_a = (-)\frac{\dfrac{\Delta q}{q_1 + q_2}}{\dfrac{\Delta p}{p_1 + p_2}}$$

$$Or, \quad e_a = (-)\frac{\Delta q}{\Delta p} \times \frac{p_1 + p_2}{q_1 + q_2}$$

p_1 *is the initial price,* p_2 *is the new price,* q_1 *is the inital quantity and* q_2 *is the new quantity*

Observations:

- Point elasticity is obtained by differentiation and derivative of an inverse relationship will yield a negative number. A minus sign before the formula makes price elasticity a positive number.
- Price elasticity may be greater than one. $e_p > 1$(elastic). If it is greater than one, it means that the percentage change in quantity demanded is greater than the percentage change in the price.
- Price elasticity may be less than one. $e_p < 1$ (inelastic). If the price elasticity is less than one, it means that if the price changes by a certain percentage, the quantity demanded changes by a lesser percentage.

- Price elasticity may be equal to one. $e_p = 1$. In this case, the percentage change in quantity demanded is equal to the percentage change in price.
- Elasticity is different at different points on a demand curve. There is a simple geometric formula to find the elasticity at different points on a linear demand curve y. It is given by

$$e_p = \frac{Lower\ segment\ of\ demand\ curve}{Upper\ segment\ of\ demand\ curve}$$

The lower segment and upper segment are measured as the segment below the mid-point and the segment above the mid-point of the demand curve respectively.

Graphically, this is shown in Fig 22.2.

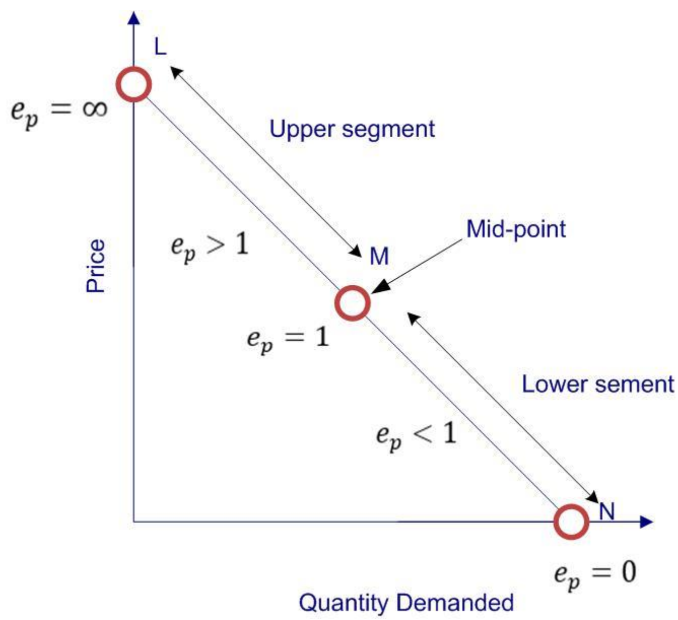

Fig. 22.2: Elasticity is different at different points on a demand curve

- At the mid-point M, lower segment is equal to upper segment and $e_p = 1$
- Above the mid-point, lower segment is greater than the upper segment and $e_p > 1$
- Below the mid-point, lower segment is less than the upper segment and $e_p < 1$
- At point L, upper segment is equal to zero and $e_p = \infty$
- At point N, lower segment is equal to zero and $e_p = 0$

Note: Both price and quantity demanded are interdependent and the curve is not affected even when we measure quantity dependent on the x axis and price on the y-axis. Economists generally measure quantity demanded on the x-axis and price on the y-axis to explain situations where producers and consumers come together on the same plane.

When $e_p = 0$: Perfect Inelastic Demand

This means that for a percentage change in price, there is no change in quantity demanded. This is a case of perfect inelasticity. This situation is graphically shown in Fig. 22.3.

When $e_p = \infty$: Perfect Elastic Demand

This means that quantity demanded changes even if there is no change in price. This is a case of perfect elasticity. This situation is graphically shown in Fig. 22.4.

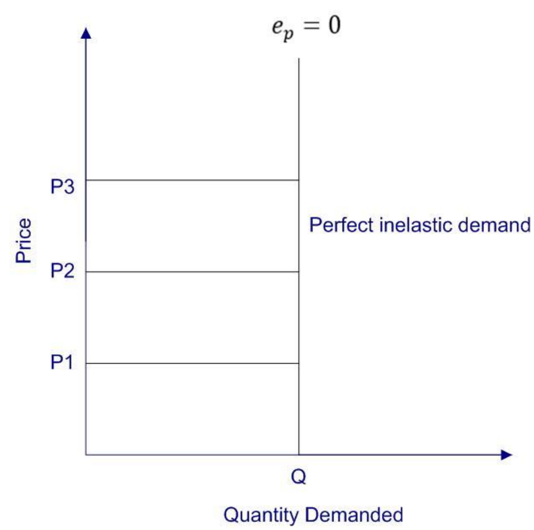

Fig. 22.3: Inelastic demand curve

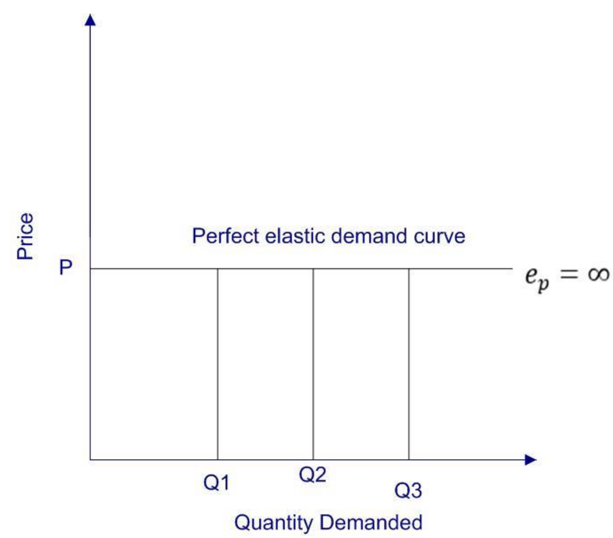

Fig. 22.4: Elastic demand curve

ii) **Income Elasticity of Demand:** measures the responsiveness of quantity demanded to a change in the income of a consumer. It is represented by the following formula

$$e_i = \frac{percentage\ change\ in\ quantity\ demanded}{percentage\ change\ in\ income}$$

$$Or, \qquad e_i = \frac{\frac{\Delta q}{q} \times 100}{\frac{\Delta i}{i} \times 100}\ , i\ is\ the\ income$$

$$Or, \qquad e_i = \frac{\Delta q}{\Delta i} \times \frac{i}{q}$$

Observations:

- Income elasticity may be positive. This means that when income increase, quantity demanded also increases and when income decreases, quantity demanded falls. That is, there is a positive relationship between income and quantity demanded.
- Income elasticity may be negative. This means that when income increase, quantity demanded decreases and when income decreases, quantity demanded increases. That is, there is a negative relationship between income and quantity demanded.
- Income elasticity may be greater than one. $e_i > 1$. This means that the percentage change of quantity demanded is greater than the percentage change in income.
- Income elasticity may be less than one. $e_i < 1$. This means that the percentage change in quantity demanded is less than the percentage change in income.

iii) **Cross elasticity of demand:** measures the responsiveness of demand of one commodity to the change in the price of some related commodities. It is represented by the formula:

$$e_c = \frac{percentage\ change\ in\ quantity\ demanded\ of\ good\ y}{percentage\ change\ in\ price\ of\ good\ x}$$

Symbolically,

$$e_c = \frac{\frac{\Delta q_y}{q_y} \times 100}{\frac{\Delta p_x}{p_x} \times 100}\ , i\ is\ the\ income$$

$$Or, \qquad e_c = \frac{\Delta q_y}{\Delta p_x} \times \frac{p_x}{q_y}$$

Observations:

- Cross price elasticity may be positive. $e_c > 0$. This means that when the price of good x changes, the quantity demanded of good y changes in the same direction. That is, if price of good x increases, quantity demanded of good y increases and if price of good x decreases, quantity demanded of good y decreases.
- Cross price elasticity may be negative. $e_c < 0$. This means that when the price of good x changes, the quantity demanded of good y changes in the opposite direction. That is, if price of good x increases, quantity demanded of good y decreases and if price of good x decreases, quantity demanded of good y increases.

c) Based on the price elasticity of demand, goods may be categorized as:

- Necessary goods ($e_p < 1 \; or \; e_p = 0$)
- Luxury goods ($e_p > 1 \; or \; e_p = \infty$)

Based on income elasticity of demand, goods may be categorized as:

- Normal goods (positive e_i)
- Inferior goods (negative e_i)
- Luxury goods or superior goods ($e_i > 1$)

Based on cross elasticity of demand, goods may be classified as:

- Substitute goods ($e_c > 0$)
- Complementary goods ($e_c < 0$)

22.2. For a linear demand function, $D = m - nP$, where D is the quantity demanded, P is the price, m is intercept and n is the slope, show that n is a component of the elasticity of demand.

Solution:

The linear demand function is given as $D = m - nP$ and the price elasticity of demand is given by the formula

$$e_p = (-)\frac{dD}{dP} \times \frac{P}{Q} \text{ -------------------- (22.1)}$$

If we differentiate the equation with respect to P, we get

$$\frac{dD}{dP} = \frac{d}{dP}(m - nP)$$

$$or\ ,\ \frac{dD}{dP} = -n \quad \text{----------------------------(22.2)}$$

Now, $\frac{dD}{dP}$ is a component of the elasticity formula given in equation (1). Therefore, it is shown from equation (2) that the slope n is a component of the elasticity of demand.

22.3. Given the simple national income model,

$$\boldsymbol{Y = C + I,} \quad \text{--------------------}\textbf{\textit{(22.3)}}$$

$$\boldsymbol{C = \alpha + \beta Y} \quad \text{------------------}\textbf{\textit{(22.4)}}$$

$\boldsymbol{Y\ is\ national\ income, C\ is\ consumption\ and\ I\ is\ investment, \alpha\ and\ \beta\ are\ parameters.}$

- ***a) How will the consumption change when income changes?***
- ***b) Is there any effect on income as the investment changes?***
- ***c) If there is an investment of ₹1000 and the rate of consumption with respect to income is 0.5, what is the equilibrium income? (assume there is no autonomous consumption) If income is to be increased to 2500, what should be the target for investment?***

Solution:

a) To find the change in consumption, when income changes, differentiate equation (2) with respect to Y.

Therefore,

$$\frac{dC}{dY} = \frac{d}{dY}(\alpha + \beta Y)$$

$$or, \quad \frac{dC}{dY} = \beta$$

In Economics, this rate of change of consumption with respect to Income is termed as Marginal Propensity to Consume or in short as MPC and α is termed as the autonomous consumption. MPC always lie between 0 and 1.

b) In order to find the effect of change in investment on the national income, differentiate equation (1) with respect to I

Substituting equation (2) in equation (1) gives

$$Y = \alpha + \beta Y + I$$

$$or, (1-\beta)Y = \alpha + I$$

$$or, Y = \frac{\alpha}{1-\beta} + (\frac{1}{1-\beta})I$$

Therefore,

$$\frac{dY}{dI} = \frac{d}{I}[\frac{\alpha}{1-\beta} + (\frac{1}{1-\beta})I] \text{------------------------(22.5)}$$

$$or, \frac{dY}{dI} = \frac{1}{1-\beta}$$

$$or,\ dY = \frac{1}{1-\beta}dI$$

If we represent $k = \frac{1}{1-\beta}$ then, $dY = kdI,$ $k\ is\ termed\ as\ the\ multiplier$

A multiplier is therefore the rate at which income changes when investment changes.

c) Given,

Investment = ₹1000

Rate of consumption with respect to income, that is MPC or β = 0.5

Equilibrium Income is given by

$$Y = C + I,$$

$$or\ Y = \beta Y + I, assuming\ \alpha = 0$$

$$or\ (1-\beta)Y = I$$

$$or, Y = \frac{1}{(1-\beta)}I$$

$$or, Y = \frac{1}{1-0.5} \times 1000$$

$$or\ Y = 2000$$

Now, if income is to be increased to 2500, what should be the target for investment?

We know that

$$dY = kdI, \qquad k = \frac{1}{1-\beta}$$

Since income increases by 500, and k =2,

$$500 = 2dI$$

$$or, dI \ = \frac{500}{2} = 250$$

That is, investment should be increased by 250 or ₹1000 +250 = ₹1250

22.4. Suppose you are the producer of a firm that has a total revenue function

$$\boldsymbol{TR = 250q - 2q^2}$$

Where TR is total revenue and q is the output produced. How much output should you produce such that the total revenue is maximized?

Solution:

Given, $TR = 250q - 2q^2$

We are to find the value of q that will maximize TR.

From condition of maximization, there are two conditions for maximization or minimization

First condition says that the slope of the curve must be zero or first derivative must be equal to zero. That is, $\frac{dTR}{dq} = 0$

Second condition says that the curve must be inverted U, or the second derivative must be negative. That is, $\frac{d^2TR}{dq^2} < 0$

Therefore,

$$\frac{dTR}{dq} = \frac{d}{dq}(250q - 2q^2) = 0$$

$$or, \; 250 - 2q = 0 \text{ ----------------(22.6)}$$

$$or, \qquad q = 125$$

Now,

For $q = 125$ to be the maximum output, $\frac{d^2TR}{dq^2} < 0$

From equation 22.6, $\frac{d^2TR}{dq^2} = \frac{d}{dq}\left(\frac{dTR}{dq}\right) = \frac{d}{dq}(\, 250 - 2q) = -2 < 0$

Therefore, at $q = 125$, the total revenue is maximized.

22.5. ***A monopolist discriminates prices between two market. The price equations are given as***

$$\boldsymbol{P_1 = 60 - 4Q_1}$$

$$\boldsymbol{P_2 = 42 - 3Q_2}$$

$\boldsymbol{Q_1 \text{ is the output in Market } 1 \text{ and } Q_2 \text{ is the outputin market } 2.}$ ***The total cost function is given as*** $\boldsymbol{TC = 50 + 12Q}$ ***and*** $\boldsymbol{Q = Q_1 + Q_2}$

- ***a) What amount of output should be produced so that the profit of the firm is maximized?***
- ***b) What is the maximum profit?***
- ***c) What is the profit maximizing prices in the market?***

Solution:

a) Profit is defined as the difference between revenue and cost. It is generally denoted as

$$\pi = TR - TC$$

$TR \text{ is total revenue and } TC \text{ is total cost}$

The total revenue is defined as the product price and output. The total revenue in the two markets is given as

$$TR_1 = P_1 \times Q_1 = (60 - 4Q_1)Q_1 = 60Q_1 - 4Q_1^2$$

$$TR_2 = P_2 \times Q_2 = (42 - 3Q_2)Q_2 = 42Q_2 - 3Q_2^2$$

The cost function of the monopolist is given as $TC = 50 + 12Q$

Therefore, the profit function is given as

$$\pi = TR_1 + TR_2 - TC$$

Substituting the TR functions and $Q = Q_1 + Q_2$, gives

$$\pi = (60Q_1 - 4Q_1^2) + (42Q_2 - 3Q_2^2) - \{50 + 12(Q_1 + Q_2)\}$$

The first order condition for maximization requires $\frac{\partial \pi}{\partial Q_1} = 0 \; and \; \frac{\partial \pi}{\partial Q_2} = 0$

Now,

$\frac{\partial \pi}{\partial Q_1} = 0$ implies $\frac{\partial}{\partial Q_1}(60Q_1 - 4Q_1^2 + 42Q_2 - 3Q_2^2 - 50 - 12Q_1 - 12Q_2) = 0$

$$or, \; 60 - 8Q_1 - 12 = 0$$

$$or \; \boldsymbol{Q_1 = 6}$$

$\frac{\partial \pi}{\partial Q_2} = 0$ implies $\frac{\partial}{\partial Q_2}(60Q_1 - 4Q_1^2 + 42Q_2 - 3Q_2^2 - 50 - 12Q_1 - 12Q_2) = 0$

$$or, \; 42 - 6Q_2 - 12 = 0$$

$$or \; \boldsymbol{Q_2 = 5}$$

The second order condition for maximization requires the Hessian Determinant $|H_1| < 0$ and $|H_2| > 0$

Now, $|H_1| = \frac{\partial^2 \pi}{\partial Q_1^2} = \frac{\partial \pi}{\partial Q_1}\left(60 - 8Q_1 - 12\right) = -8 < 0$

And $|H_2| = \begin{vmatrix} \frac{\partial^2 \pi}{\partial Q_1^2} & \frac{\partial^2 \pi}{\partial Q_1 \partial Q_2} \\ \frac{\partial^2 \pi}{\partial Q_2 \partial Q_1} & \frac{\partial^2 \pi}{\partial Q_2^2} \end{vmatrix} = \begin{vmatrix} -8 & 0 \\ 0 & -6 \end{vmatrix} = 48 > 0$

Since both the conditions are satisfied, at $Q_1 = 6 \; and \; or \; Q_2 = 5$, the profit is maximized.

b) The maximum profit

$$\pi = (60 \times 6 - 4 \times 6^2) + (42 \times 5 - 3 \times 5^2) - \{50 + 12(6+5)\} = \mathbf{169}$$

c) The profit maximizing prices are

$$P_1 = 60 - 4 \times 6 = 36 \qquad \because Q_1 = 6$$

$$P_2 = 42 - 3 \times 5 = 27 \qquad \because Q_2 = 5$$

22.6. The consumer purchases two goods x and y. His utility function is given by $u = 18xy + 9y$. The purchase of the goods is limited by a budget whose equation is given as $6x + 3y = 15$. What amount of the two goods should the consumer purchase within the limited budget so that her utility is maximized?

Solution:

Maximizing a utility function requires us to construct the Lagrange function. The Lagrange function is given as

$$L = 18xy + 9y + \lambda(15 - 6x - 3y)$$

The first order condition requires

$$\frac{\partial L}{\partial x} = 0, \frac{\partial L}{\partial y} = 0 \ \ and \ \ \frac{\partial L}{\partial \lambda} = 0$$

Therefore,

$$\frac{\partial L}{\partial x} = 0 => \frac{\partial}{\partial x}[18xy + 9y + \lambda(15 - 6x - 3y)] = 0$$

$$or, \qquad 18y - 6\lambda = 0$$

$$or, \ \lambda = 3y - - - - - - - - (22.7)$$

$$\frac{\partial L}{\partial y} = 0 => \frac{\partial}{\partial y}[18xy + 9y + \lambda(15 - 6x - 3y)] = 0$$

$$or, \qquad 18x + 9 - 3\lambda = 0$$

$$or, \ \lambda = 6x + 3 - - - - - - - - (22.8)$$

$$\frac{\partial L}{\partial \lambda} = 0 => \frac{\partial}{\partial \lambda}[18xy + 9y + \lambda(15 - 6x - 3y)] = 0$$

$$or, \quad 15 - 6x - 3y = 0$$

$$or, \; 6x + 3y = 15 - - - - - - - - - - (22.9)$$

Equating equation (22.7) and (22.8) gives

$$3y = 6x + 3$$

$$or, \quad 6x - 3y = -3 - - - - - - - - - - (22.10)$$

Now, solving equation (22.9) and (22.10) gives

$$x = 1 \; and \; y = 3$$

In order to prove that the values of x and y maximize the utility, the second order condition requires the Bordered Hessian determinant, $|\overline{H_2}| > 0$

$$|\overline{H_2}| = \begin{vmatrix} 0 & h_1 & h_2 \\ h_1 & L_{11} & L_{12} \\ h_2 & L_{21} & L_{22} \end{vmatrix} \qquad h \; is \; the \; constraint \; function \; and \; L \; is \; the \; Lagrange \; function$$

Here, $h_1 = \frac{\partial h}{\partial x}$, $h_2 = \frac{\partial h}{\partial y}$, $L_{11} = \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial x}\right)$, $L_{12} = \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial y}\right)$, $L_{21} = \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial x}\right)$, $L_{22} = \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial y}\right)$

(Refer Module 21, Section 21.8)

$$|\overline{H_2}| = \begin{vmatrix} 0 & 6 & 3 \\ 6 & 0 & 18 \\ 3 & 18 & 0 \end{vmatrix} > 0$$

Therefore, the consumer will have to purchase $1 \; unit \; of \; x \; and \; 3 \; units \; of \; y$ so that her utility is maximized.

