Matrix- An Introduction

Suppose you have to disseminate some information or data that is scattered. What would you do? Can you think of a way to organize the data so that the receiver gets a clear picture and so that is becomes easy to store for future use?

Objectives

The objectives of this module are:

1. Discover the concept of a matrix
2. Interpret the structure of a matrix

Terminology

1. Matrix: a system or environment where something originates and develops.
2. Square matrix: a matrix where number of rows and columns are equal
3. Diagonal matrix: a square matrix where the diagonal elements are non-zero and all other elements are zero
4. Scalar matrix: a diagonal matrix where all elements in the diagonal are equal
5. Identity matrix: a diagonal matrix where all diagonal elements are equal to 1
6. Triangular matrix: a matrix where all elements above or below the diagonal are zero
7. Upper triangular matrix: a matrix where all elements above the diagonal elements are non-zero and elements below the diagonal are zero
8. Lower triangular matrix: a matrix where all elements below the diagonal elements are non-zero and elements above the diagonal elements are zero
9. Transpose of a matrix: a matrix obtained by interchanging the rows and columns of the original matrix
10. Symmetric matrix: if the transpose of a matrix is equal to the original matrix
11. Null matrix or zero matrix: all elements in a matrix are zero

10.1. Origin and meaning of term Matrix:

Research shows that the term matrix dates back to the second century BC, though traces were also seen from the fourth century BC. It is a Latin word used for “mother” and was later realized that it was used to represent the “womb”.

James Joseph Sylsevter, an English mathematician and Arthur Cayley, a British Mathematician were the early contributors of the matrix theory. In 1848, James Joseph Sylsevter introduced the term *matrix* which is a Latin word for *womb*. Later in 1856, Arthur Cayley introduced matric multiplication and inverse matrix.

Image 10.1 Joseph Sylvester [1]

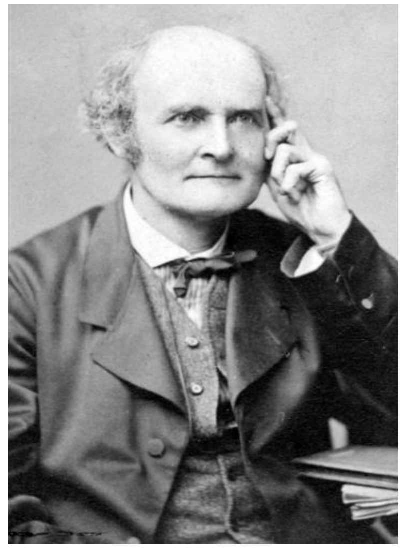

Image 10.2 Arthur Cayley [2]

[1]https://en.wikipedia.org/wiki/James_Joseph_Sylvester#/media/File:James_Joseph_Sylvester.jpg
[2]https://en.wikipedia.org/wiki/Arthur_Cayley#/media/File:Arthur_Cayley.jpg

The word matrix is widely used in many different fields of study. A matrix is a system or an environment in which something originates and develops. It is an arrangement or an ordered series of information. Matrix is used as a noun and the plural term is called "matrices".

10.2. Uses of matrix:

The word "matrix" is widely used. It is used to represent large systems in a compact form. It is used as a form of short-hand. Below are few examples where matrix may be used in the world around us.

a) Biology: In biology, it is used to represent the tissue between cells in which more specialized structures are embedded.
b) Geology: In geology, this word may be used to explain masses of rocks. For example, matrix of sedimentary rocks means a collection of finely grained sedimentary materials.
c) Geography: In Geography, matrices may be used in remote sensing or to study seismic waves.
d) Physics: Matrix may be used in every branch of Physics including optics, electromagnetism, and quantum mechanics to study physical phenomena.
e) Computer Science: In the world of computing, matrix may be used in computer graphics.

f) Literature: In literature, it is used to represent a system of lines and roads that cross each other and form a series of different shapes in between.

g) Economics: In Economics, matrix may be used in a variety of ways, such as, study of a market, study the interaction between industries in an economy, study the connection between inputs and outputs of a production system.

10.3. Writing a matrix in Mathematics

A. Structure:

In Mathematics, matrix is an array of numbers, symbols or parameters that are arranged in rows and columns, generally written within a pair of second brackets, "[ ]", parentheses "( )" or sometimes double lines, "||". It is a system where information is stored. Each member of a matrix is known as an element of the matric and the elements are separated by spaces. There may or may not be a relationship between the elements. Generally, the element '0' means there is no relationship and '1' or any other number that means there is a relationship.

Matrices may be rectangular or square in shape. A rectangular matrix has different number of rows and columns. A square matrix has the same number of rows and columns.

A matrix is generally denoted by a letter of the alphabet. Thus, a matrix Z may be written as:

$$Z=\begin{pmatrix} 2 & 4 & 6 \\ 7 & 1 & 0 \\ 0 & 1 & 8 \end{pmatrix}$$

In short hand, the elements of a matric are represented as $Z_{i\,j}$, where i represents the element in the i-th row and j is the element in the j-th column.

Thus, Z_{21} is the element in the 2 nd row and the first column and that is the number 7. Similarly, Z_{32} is the element in the third row and second column that is number 1 and so on.

Z_{43} is the element in the fourth row and third column. Since there is no fourth row in the matrix Z, hence, this is undefined.

Figure 4.1 is an example of an m x n (read as m by n) matrix; that is, there are m horizontal rows and n vertical columns.

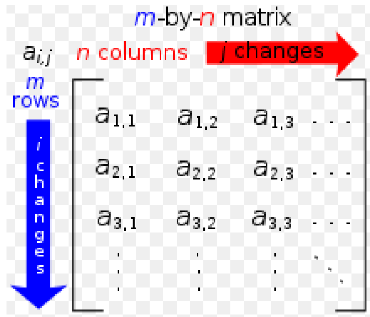

Image 10.3 : m x n matrix

[Source: https://en.wikipedia.org/wiki/Matrix_(mathematics)#/media/File:Matrix.svg]

The elements of a matrix are usually denoted by a letter with two subscripts. In the above example, the elements are denoted by a_{11}, a_{12}, a_{21}, a_{22} and so on. a_{11} represents the element in the first row and first column; a_{21} represents the element in the second row and first column and so on.

B. Order or dimension of a matrix:

The total number of elements in a matrix is the order or the dimension of the matric. The order is represented by the product of the rows and columns and is written on the bottom right hand corner of the matrix.

Note: There is a rule for writing the order of a matrix, and that is, the number of rows is multiplied by the number or columns. Thus, if "r" is the number of rows and "c" is the number of columns, then the order of the matrix with "r" rows and "c" columns may be written as

"r x c".

Thus,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2 \times 2}$$

is a matrix with two rows and two columns and may be written as 2 x 2 matrix. There are

(2 x 2) = 4 elements in this matrix. Since the number of rows and columns are equal, this is an example of a square matrix.

Similarly,

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix}_{2 \times 3}$$

is a matrix with two rows and three columns and may be written as 2 x 3 matrix. There are

(2 x 3) = 6 elements in this matrix. Since the number of rows and columns are not equal, this is an example of a rectangular matrix.

Let us name the rows as D and E and the columns as A B C.

$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} D \\ E \end{array} & \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix} \end{array}$$

The possible connections that may be derived from the above matrix are DA, DB, DC, EA, EB and EC. The element 1 means that there is some connection between the first row D and the first column A, element 2 means that there is some connection between D and B and so on. Thus there are six different connections.

10.4. Types of matrices used in Mathematics:

There may be different types of matrices depending upon the field of study. In Mathematics, matrices are broadly categorized as rectangular or square matrix, as discussed above. Further, matrices may also be of the following types:

a) **Diagonal Matrix**: A diagonal matrix is formed only in square matrices, where the number of rows and columns are equal. If all the elements that form the diagonal of a matrix are non-zero and all the other elements are zero, it is termed as a diagonal matrix.

Example,

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

The diagonal elements of the above matrix are 10, 14 and 6, that are non-zero and all the other elements are zero.

The interdependence between industries in an economy may be represented by using a diagonal matrix. If the output of one industry is not used as input in some other industry, the element is zero. For example, sugarcane is used only in the sugarcane industry but is not used in the groundnut industry.

b) Scalar Matrix: If all the elements in the diagonal of a diagonal matrix are equal, it is a scalar matrix.

Example,

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

In the above diagonal matrix, all the elements in the diagonal are equal, that is four. Such matrices are termed as scalar matrix.

c) Identity Matrix: if all the elements in the diagonal of a diagonal matrix are equal to one, it is an Identity Matrix.

Example,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In the above matrix, all the elements in the diagonal are equal and equal to one. Thus, it is a scalar matrix and is also an identity matrix.

An industry with a single line of production may form an identity matrix. For example, the water transport industry has only a single line of production. Hence the product mix may be represented by unity or one.

Note: A scalar matrix may not be an identity matrix, but an identity matrix is always a scalar matrix. Similarly, a diagonal matrix may be a scalar matrix or an identity matrix, but a scalar matrix or an identity matrix is always a diagonal matrix.

d) Triangular Matrix: A matrix may be a lower triangular matrix or an upper triangular matrix. If all the elements above the diagonal elements of the matrix are zero and the elements below the diagonal matrix are non-zero, then it is referred as a lower triangular matrix. On the other hand, if all the elements below the diagonal elements are zero and the elements above the diagonal elements are non-zero, it is termed as an upper triangular matrix.

Example,

$$\begin{bmatrix} 1 & 3 & 8 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Upper Triangular Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 4 & 0 \\ 6 & 9 & 1 \end{bmatrix}$$

Lower Triangular Matrix

e) **Transpose of a matrix:** If the rows and columns of a matrix are interchanged, the new matrix formed is called the transpose of the original matrix. It is generally denoted by A^T or A'.

Example,

If Y = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a matrix of order 2 x 2 then the transpose of Y or Y' = $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Here, the elements of the first row, that is a and b become the elements of the first column. Similarly, elements of the second row, i.e. c and d become the elements of the second column.

f) Symmetric matrix: If the transpose of a matrix is equal to the original matrix, then the original matrix is called a symmetric matrix. Symmetry occurs only in square matrices.

Example,

If M = $\begin{bmatrix} 1 & 3 & 5 \\ 3 & 7 & 9 \\ 5 & 9 & 2 \end{bmatrix}$ is 3 x 3 matrix. Then interchanging the rows and columns, gives the transpose of M or M' = $\begin{bmatrix} 1 & 3 & 5 \\ 3 & 7 & 9 \\ 5 & 9 & 2 \end{bmatrix}$. Thus, M = M'. Then M is known as a symmetric matrix.

g) Null or Zero Matrix: If all the elements of a matrix are zero, then it is known as a zero matrix or a null matrix.

Example,

Z = $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a zero matrix or a null matrix.

10.5 Basic Matrix Operations

A. Addition and subtraction

Rule:

In order to add or subtract two or more matrices, the simple rule is that the order of the matrices must be the same. Thus, a 2 x 3 matrix cannot be added with or subtracted from a 3 x 2 matrix, because the number of rows and columns of the matrices are different.

The resultant matrix after matrix addition or subtraction is of the same order as the original matrices.

Example:

Suppose there are three products, Product 1, Product 2 and Product 3 that are sold in two different markets, Market A and Market B.

The monthly sales of the three products sold in Market A and Market B for January, February and March is given as follows:

Table 10.1: Monthly Sales of three products

Products	Market A			Market B		
	Jan	Feb	Mar	Jan	Feb	Mar
Product 1	10	20	30	5	15	10
Product 2	5	10	10	10	20	30
Product 3	20	5	10	10	5	20

The above information may be arranged in matrix form as follows:

$$M = \begin{bmatrix} 10 & 20 & 30 \\ 5 & 10 & 10 \\ 20 & 5 & 10 \end{bmatrix} \text{ and } N = \begin{bmatrix} 5 & 15 & 10 \\ 10 & 20 & 30 \\ 10 & 5 & 20 \end{bmatrix}$$

M and N are the two markets.

The first row in both the matrices represents Product 1, the second row represents product 2 and the third row represents product 3. The first column represents month of January, second column represents month of February and third column represents month of March.

Now, by using matrix addition,

$$M + N = \begin{bmatrix} 10+5 & 20+15 & 30+10 \\ 5+10 & 10+20 & 10+30 \\ 20+10 & 5+5 & 10+20 \end{bmatrix}$$

Or, $(M + N) = R \text{ (say)} = \begin{bmatrix} 15 & 35 & 40 \\ 15 & 30 & 40 \\ 30 & 10 & 30 \end{bmatrix}$

The elements in the above matrix give the total sales of the three products for January, February and March.

Thus, the element 15 in the first row and first column is the total sales of product 1 from market A and B in the month of January.

B. Matrix Multiplication

Rule:

Before multiplying two matrices, it should be kept in mind that the number of columns of the first matrix must be equal to the number of rows in the second matrix.

Number of Columns of 1st Matrix = Number of Rows in 2nd Matrix

In the example of the monthly sales of three products, suppose the prices of the products are fixed at ₹5, ₹10 and ₹2.

This may be written as a column vector as:

$$P = \begin{bmatrix} 5 \\ 10 \\ 2 \end{bmatrix}_{3x1}$$

Given the matrix of total sales:

$$R = \begin{bmatrix} 15 & 35 & 40 \\ 15 & 30 & 40 \\ 30 & 10 & 30 \end{bmatrix}_{3x3}$$

Matrix multiplication is carried out as follows:

Therefore,

$$RP = \begin{bmatrix} 15 & 35 & 40 \\ 15 & 30 & 40 \\ 30 & 10 & 30 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \\ 2 \end{bmatrix}$$

$$RP = \begin{bmatrix} 15x5 + 35x10 + 40x2 \\ 15x5 + 30x10 + 40x2 \\ 30x5 + 10x10 + 30x2 \end{bmatrix}$$

$$RP = \begin{bmatrix} 75 + 350 + 80 \\ 75 + 300 + 80 \\ 150 + 100 + 60 \end{bmatrix}$$

$$RP = \begin{bmatrix} 75 + 350 + 80 \\ 75 + 300 + 80 \\ 150 + 100 + 60 \end{bmatrix}$$

$$RP = \begin{bmatrix} 505 \\ 455 \\ 310 \end{bmatrix}$$

Note: Matrix algebra is applicable only to linear-equation systems

