Mathematical Economics:

Solved Problems: Differential Equations

33.1: Solve the following Differential equation

$$\frac{dy}{dt} + 5y = 8 \text{ given the initial } y(0) = 3$$

Solution:

The differential equation

$$\frac{dy}{dt} + 5y = 8 --------(33.1)$$

is a differential equation with constant coefficient and constant term.

For a differential equation of the form

$$\frac{dy}{dt} + my = n$$

The final solution is given as,

$$y(t) = \left[y(0) - \frac{n}{m}\right] e^{-mt} + \frac{n}{m} ------(33.2)$$

Comparing equation 33.1 with the general form, we have

$$m = 5 \; and \; n = 8$$

Therefore, substituting the values in equation 33.2, gives

$$y(t) = \left[3 - \frac{8}{5}\right] e^{-5t} + \frac{8}{5}$$

$$or, \qquad y(t) = \left[\frac{7}{5}\right] e^{-5t} + \frac{8}{5}$$

33.2 Predicting Population

The annual growth rate of population in India is 1.1 percent. (World Bank, 2017). If the current population of India is 1.3 billion, what will be the population after 10 years?

Solution:

It is observed that economic growth follows an exponential function. Suppose the growth function is represented as

$$Y = Ae^{mt} ------(33.3)$$

A is an arbitrary constant or some initial value,

m is the rate of growth, and

t is the time.

Given

Annual growth rate = 1.1 percent or $m = 0.011$

Current population = 1.3 billion or $A = 1.3$

Time period = 10 years or $t = 10$

Substituting the values in equation 33.3, gives

$$Y = 1.3e^{0.011(10)} = 1.3e^{0.11} = 1.3 \times 1.11627 = 1.45115 \ (value\ of\ e = 2.71824)$$

Therefore, the population after 10 years is predicted to be 1.45115 billion.

33.3 Making Investments

Suppose you are planning for your child's education after 10 years that requires an amount of ₹ 3,00,000. If you plan to invest from today and if your investment grows at an annual rate of 5 percent, how much should you invest today so that you have of ₹ 3,00,000 after 10 years?

Solution:

This is again an example of compounding interest. The rate of growth is given by

$$Y = Ae^{mt}$$

Here, $Y = 3,00,000, m = 5\ and\ t = 10$

We have to find the amount to be invested today, that is the value of A

Substituting the values in the equation $Y = Ae^{mt}$, gives

$$300000 = Ae^{0.05(10)}$$

$$or, \qquad A = 300000e^{-0.05(10)}$$

$$or, \qquad A = 300000 \times e^{-0.5} = 181961$$

Therefore, in order to have an amount of ₹ 3,00,000 after 10 years at 5 percent rate of interest, you need to invest an amount of approximately ₹ 1,81,961

33.4. Dynamic Stability of Market Price

Given the demand and supply functions of a partial equilibrium market model for a single commodity

$$Demand\ function\colon\ Q_D = \alpha - \beta P \qquad \alpha, \beta > 0$$

$$Supply\ function\colon\ Q_s = -\gamma + \delta P \qquad \gamma, \delta > 0$$

Suppose, with time, t, if the price adjusts to the excess demand $(Q_D - Q_S)$ by a rate of m.

a) Obtain the time path of price

b) Obtain the condition for dynamic stability of the market price

Solution:

a) It is given that Price adjusts to the excess demand by a rate $m.$ This may be mathematically expressed as follows:

$$\frac{dP}{dt} = m(Q_D - Q_S) - - - - - - - - (33.4)$$

Substituting demand and supply equations in equation 33.2, gives

$$\frac{dP}{dt} = m[(\alpha - \beta P) - (-\gamma + \delta P)]$$

$$or, \quad \frac{dP}{dt} = m[(\alpha + \gamma - \beta P - \delta P)]$$

$$or, \quad \frac{dP}{dt} = -m(\beta + \delta)P + m(\alpha + \gamma\ )$$

$$or, \quad \frac{dP}{dt} + m(\beta + \delta)P = m(\alpha + \gamma\ ) - - - - - - - - (33.5)$$

Equation 33.5, is a first order linear differential equation with constant term and constant coefficient.

In order to obtain the time path of price, we have to solve the differential equation and find the value of P in equation 33.5. Since equation 33.5 is a first order linear differential equation with constant term and constant coefficient, the solution is given as the sum of the complementary solution and particular solution. ***(Module 30, Section 30.3, B)***

Following the general form of a first order differential equation with constant coefficient and constant term

$$\frac{dy}{dt} + my = n$$

The final solution is given as,

$$y(t) = \left[y(0) - \frac{n}{m}\right] e^{-mt} + \frac{n}{m}$$

Comparing the general form with equation 33.3,

$y = P$

$m = m(\beta + \delta)$

$n = m(\alpha + \gamma)$

Therefore, the time path of P is given by using the final solution as

$$P(t) = \left[P(0) - \frac{m(\alpha + \gamma)}{m(\beta + \delta)}\right] e^{-m(\beta+\delta)t} + \frac{m(\alpha + \gamma)}{m(\beta + \delta)}$$

$$or, \qquad P(t) = \left[P(0) - \frac{(\alpha + \gamma)}{(\beta + \delta)}\right] e^{-m(\beta+\delta)t} + \frac{(\alpha + \gamma)}{(\beta + \delta)} - - - - - (33.6)$$

$P(0)$ is the initial price.

Now, the equilibrium price of the market model is given by

$$\bar{P} = \frac{(\alpha + \gamma)}{(\beta + \delta)}$$

And since, $m, \beta, and\ \delta\ are\ constants\ and\ all\ are > 0,$ let $m(\beta + \delta) = \lambda$, a constant

Substituting the equilibrium Price and λ, in equation 33.4, gives

$$P(t) = [P(0) - \bar{P}]e^{-\lambda t} + \bar{P} - - - - - - - - (33.7)$$

Equation 33.5 gives the time path of price.

b) An equilibrium is said to be dynamically stable when the time path of the relevant variable converges to its equilibrium value.

In our example, equilibrium will be dynamically stable when the time path of price $P(t)$ converges to the equilibrium price, $\bar{P}$, that is, $P(t) \to \bar{P}$

From equation 33.7, it may be seen that price will be stable, that is, $P(t) = \bar{P}$, if

$$[P(0) - \bar{P}]e^{-\lambda t} = 0 - - - - - - (33.8)$$

Now, as $t \to \infty$, $e^{-\lambda t} \to 0$ and $P(t)$ will move towards the equilibrium price $\bar{P}$ thus showing the dynamic stability of market equilibrium.

Therefore, the condition for dynamic stability requires

$$\lambda > 0\ or\ m(\beta + \delta) > 0$$

$m \text{ is the adjustment coefficient},$

$\beta \text{ is the slope of the demand curve and}$

$\delta \text{ is the slope of the supply curve}.$

33.4 Harrod-Domar Growth Model:

Obtain the time path of growth in income given a simple national income model

$$Y = C + I + A \text{------}(33.9)$$

$$C = mY \text{---------------------------}(33.10)$$

$$I = \alpha \frac{dY}{dt} \text{---------------------}(33.11)$$

$Y \text{ is Natinal Income}$

$C \text{ is consumption}$

$I \text{ is induced investment}$

$A \text{ is autonomous investment}$

$m \text{ is the MPC and } \alpha \text{ is the accelerator}$

Solution:

Substituting equation 33.10 and 33.11 in equation 33.9, gives

$$Y = mY + \alpha \frac{dY}{dt} + A \text{------}(33.12)$$

$$or,\ \alpha \frac{dY}{dt} = (1 - m)Y - A$$

$$or,\ \frac{dY}{dt} = \frac{(1-m)}{\alpha} Y - \frac{A}{\alpha}$$

$$or,\ \frac{dY}{dt} = \frac{s}{\alpha} Y - \frac{A}{\alpha} \qquad (\because 1 - m = s)$$

$$or,\ \frac{dY}{dt} = \frac{s}{\alpha}\left(Y - \frac{A}{s}\right)$$

$$or,\ \frac{dY}{dt} = \lambda\left(Y - \frac{A}{s}\right) \text{----------------------}(33.13) \quad \left(\lambda = \frac{s}{\alpha}\right)$$

Equation 33.13 is of the form of a first order linear differential equation.

The solution of equation 33.13 will give the time path of growth in income.

There may be two cases:

Case 1: When the autonomous investment (A) is fixed

$$y = \left(Y - \frac{A}{s}\right)$$

then

$$\frac{dy}{dt} = \frac{dY}{dt}$$

Substituting this value in equation 33.13 gives

$$\frac{dy}{dt} = \lambda y$$

The solution to this differential equation gives

$$y(t) = y(0)e^{\lambda t} \text{ --------------------(33.14)}$$

Equation 33.14 is an exponential function and it may be said that there is exponential growth of income at the rate of λ

Case 2: When autonomous investment is progressive or $\boldsymbol{A = A_0 e^{mt}}$

When the autonomous investment is progressive, equation 33.13 may be written as

$$\frac{dY}{dt} = \lambda\left(Y - \frac{A_0 e^{mt}}{s}\right) - - - - - (33.15)$$

If we consider the progressive growth in income equal to progressive growth in investment, That is $Y(t) = \bar{Y}(0)e^{mt}$, then

$$\frac{dY}{dt} = m\bar{Y}(0)e^{mt} - - - - - - - - (33.16)$$

Using equation 33.16 in equation 33.15, gives

$$mY(0)e^{mt} = \lambda\left(\bar{Y}(0)e^{mt} - \frac{A_0 e^{mt}}{s}\right) - - - - - - - - - (33.17)$$

$or,\qquad (m-\lambda)\bar{Y}(0)e^{mt} = -\frac{A_0 e^{mt}}{\alpha}\qquad$ (why?)

$or,\qquad (\lambda - m)\bar{Y}(0) = \frac{A_0}{\alpha}$

$or,\qquad \bar{Y}(0) = \frac{A_0}{\alpha(\lambda - m)} - - - - - - - - (33.18)$

In order to trace the time path,

Let $\qquad y(t) = Y(t) - \bar{Y}(0)e^{mt} - - - - - - (33.19)$

Then, at $t = 0$,

$$y(0) = Y(0) - \bar{Y}(0) - - - - - (33.20)$$

Subtracting equation 33.17 from equation 33.15, gives

$$\frac{dY}{dt} - m\bar{Y}(0)e^{mt} = \left(\lambda Y - \frac{A_0 e^{mt}}{s}\right) - \left[\lambda\bar{Y}(0)e^{mt} - \frac{A_0 e^{mt}}{s}\right]$$

$$or,\ \frac{dY}{dt} - m\bar{Y}(0)e^{mt} = \lambda[Y - \bar{Y}(0)e^{mt}]$$

$$or,\ \frac{dy}{dt} = \lambda y\text{-----------------}(33.21)$$

Since from equation 33.19,

$$\frac{dy}{dt} = \frac{dY}{dt} - m\bar{Y}(0)e^{mt}$$

Therefore, the final solution from equation 33.19 is given as

$$Y(t) = y(t) + \bar{Y}(0)e^{mt}$$

$$or,\ Y(t) = y(0)e^{\lambda t} + \bar{Y}(0)e^{mt} \qquad (from\ 33.14)$$

Using equation 33.20, we get

$$Y(t) = [Y(0) - \bar{Y}(0)]e^{\lambda t} + \bar{Y}(0)e^{mt} - - - - - - (33.19)$$

Where $\bar{Y}(0) = \frac{A_0}{\alpha(\lambda - m)}$ (from equation 33.18)

Note:

i) If $Y(0) = \bar{Y}(0)$, then the time path of income is equal to the equilibrium rate of growth of income at the same rate as autonomous investment. $(\boldsymbol{A_0 e^{mt}})$

ii) If $Y(0) \neq \bar{Y}(0)$, then the time path of income will progress away from the equilibrium path by a rate of $\lambda = \frac{s}{\alpha}$

