Mathematical Economics

Solved Problems: Input-Output Analysis

19.1. Complete the following Input-Output tables using proper functions:

A

Input	***Output***
1	***5***
2	***10***
3	***15***
4	***20***

B

Input	***Output***
6	***36***
7	***49***
8	***64***
9	***81***

Solution:

A. In Input-Output table A, as the value of the input increase by one, the value of the output increases by 5.

Therefore, the function that may be used for table A is

$$Y = 5X$$

B. In Input-Output table B, as the value of the input increase by one, the value of the output increases by square of the input.

Therefore, the function that may be used for table B is

$$Y = X^2$$

19.2. A farmer sells potatoes and tomatoes in a market. During the day, he gets 3 customers. The first customer demands 2 kg of potatoes and 1 kg of tomatoes. The second customer demands 1 kg of potatoes and 2 kg of tomatoes. The third customer demands 5 kg of potatoes and 2 kg of tomatoes. The price of potatoes per kg is ₹30 and the price of tomatoes per kg is ₹20.

a) Draw the input-output table for the given situation and find the revenue earned by the farmer from each customer and the total revenue.

b) Formulate a mathematical expression to show the relationship between the price of potatoes and tomatoes, the quantities sold and the revenue.

c) If a fourth customer visits his shop and demands 3 kg of potatoes and 2 kg of tomatoes, calculate the total revenue earned by the farmer using the mathematical equation formulated in question b.

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Solution:

a) The situation may be represented in an Input-Output table as follows:

Customer	**Input 1** **Potatoes sold (in kg)**	**Input 2** **Tomatoes sold (in kg)**	**Output** **Revenue (in ₹)** **(Price of Potatoes per kg = ₹30** **Price of Tomatoes per kg = ₹20)**
1	2	1	80
2	1	2	70
3	5	2	190
Total	**8**	**5**	**340**

From the table:

The revenue earned from first customer = (₹ 30×2)+ (₹ 20×1)= ₹80

The revenue earned from second customer = (₹ 30×1)+ (₹ 20×2) = ₹70

The revenue earned from third customer = (₹ 30×5)+ (₹ 20×2) = ₹190

Total revenue earned by the farmer = ₹(80+70+190) = ₹340

b) Let Y be the revenue, X_1 be the quantity demanded of potatoes and X_2 be the quantity demanded of tomatoes. Also let P_x be the price of potatoes per unit and P_y be the price of tomatoes per unit.

The mathematical expression may be written as:

$$P_x X_1 + P_y X_2 = Y$$

c) Using the mathematical expression, the total revenue for 3 kg of potatoes and 2 kg of tomatoes will be

$$P_x X_1 + P_y X_2 = Y$$
$$30 \times 3 + 20 \times 2 = 130$$

Therefore, the total revenue for 3 kg of potatoes and 2 kg of tomatoes is ₹130

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19.3. Suppose an economy consists of two industries- steel and automobiles. In order to produce automobiles, the economy requires steel and automobiles. Similarly, in order to produce steel, the economy requires automobiles and steel. To produce one-rupee worth of steel, the steel industry requires 0.2 paisa worth of steel and 0.7 paisa worth of automobiles. To produce one-rupee worth of automobile, the automobile industry requires 0.5 paisa worth of steel and 0.1 paisa worth of automobiles. Also suppose that the economy has to export ₹15000 worth of steel and ₹5000 worth of automobiles.

a) Express the above problem as an input-output model.

b) How much of worth of steel and automobiles should be produced to meet the total demand?

Solution:

a) Let X be the total steel production and Y be the total automobile production. Let us construct an input-output table to understand the problem more clearly.

	Steel	**Automobile**	**Export (in thousands)**
Steel	0.2	0.5	15
Automobile	0.7	0.1	5

Table 19.1: Input -Output table

Mathematically, we may express total production of steel and automobiles as follows:

$$X = 0.2X + 0.5Y + 15 - - - - - (1)$$

$$Y = 0.7X + 0.1Y + 5 - - - - - - (2)$$

Equation (1) says that total steel is produced by using steel, automobiles and a part of the production goes for export. Similarly, equation (2) says that total automobile is produced by using steel, automobile and a part goes for export. Export may be considered as the final demand vector.

b) Using matrix algebra, equation (1) and (2) may be expressed as:

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0.2 & 0.5 \\ 0.7 & 0.1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} 15 \\ 5 \end{bmatrix}$$

Or, $$Z = AZ + E$$

Or, $$Z = (I - A)^{-1}E - - - - - - (3)$$

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Here, $Z = 2 \times 1\ column$ vector

$A = 2 \times 2\ matrix\ of\ co-efficients\ or\ technology\ matrix$

$E = 2 \times 1\ column\ vector\ of\ exports$

Equation (3) may be solved using matrix inversion.

Therefore,

$$(I-A)^{-1} = \frac{Adj\ (I-A)}{|I-A|}$$

$$(I-A) = \begin{bmatrix} 1-0.2 & 0-0.5 \\ 0-0.7 & 1-0.1 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.5 \\ -0.7 & 0.9 \end{bmatrix}$$

Or, $Adj\ (I-A) = \begin{bmatrix} 0.9 & 0.5 \\ 0.7 & 0.8 \end{bmatrix}$

Further, $|I-A| = \begin{vmatrix} 0.8 & -0.5 \\ -0.7 & 0.9 \end{vmatrix} = (0.8 \times 0.9) - (0.7 \times 0.5) = 0.37$

$$Now, \quad Z = (I-A)^{-1}E = \frac{Adj\ (I-A)}{|I-A|}E = \frac{1}{0.37}\begin{bmatrix} 0.9 & 0.5 \\ 0.7 & 0.8 \end{bmatrix}\begin{bmatrix} 15 \\ 5 \end{bmatrix} = \begin{bmatrix} 43.235 \\ 39.189 \end{bmatrix}$$

$$Or, \quad \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 43.235 \\ 39.189 \end{bmatrix}$$

Since the values are in thousand rupees, the amount of steel that needs to be produced is worth ₹43,235 and amount of automobile that needs to be produced is worth ₹39,189

19.4 ***Given is an input-output table of an economy offering services worth lakhs of rupees***.

	Banking	**Insurance**	**Education and Research**	**Total Output**
Banking	80000	8000	30000	**122000**
Insurance	21000	6000	3000	**30000**
Education and Research	0	0	10000	**110000**

Table 19.2: Input-Output Table for Services (In lakhs of rupees)

Convert the input-output table into a matrix.

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Solution:

Converting the given input-output table into a matrix requires us to convert the large numbers into one unit. For example, how much worth of rupees of banking is needed to provide banking services worth one rupee. Similarly, how much worth of banking is needed to produce insurance worth one rupee and so on.

This matrix is called the input- coefficient matrix. Recall the mathematics behind finding the input coefficient.

In general, $\frac{y_{ij}}{Y_j}$ will give the amount of output of the i-th commodity used as intermediate input to produce one unit of output of the j-th commodity.

Let $\frac{y_{ij}}{Y_j} = b_{ij}$ ---------------------- (1)

Then, $y_{ij} = b_{ij}Y_j$ ---------------------- (2)

b_{ij} is known as the input-coefficient or technical co-efficient.

Using the concept of input coefficient, gives us the input-coefficient matrix for the current example as shown on Table 19.3

	Banking	**Insurance**	**Education and Research**
Banking	$\frac{80000}{122000} = 0.65$	$\frac{8000}{30000} = 0.26$	$\frac{30000}{110000}$ =0.27
Insurance	$\frac{21000}{122000} = 0.17$	$\frac{6000}{30000} = 0.2$	$\frac{3000}{110000} = 0.02$
Education and Research	$\frac{0}{122000} = 0$	$\frac{0}{30000} = 0$	$\frac{10000}{30000} = 0.33$

Table: 19.3 Input-coefficient matrix

Interpretation:

The banking sector requires 0.27 lakh rupees to provide education and research services worth 1 lakh rupee.

Similarly, Education and Research requires 0.33 lakhs rupees in its own sector to provide education and research worth 1 lakh rupee.

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19.5 ***Given is the input-co-efficient matrix (A) obtained from an input-output table of three industries $\boldsymbol{L, M\ and\ N}$.***

$$C = \begin{bmatrix} \mathbf{0.3} & \mathbf{0.2} & \mathbf{0.4} \\ \mathbf{0} & \mathbf{0.2} & \mathbf{0.1} \\ \mathbf{0.1} & \mathbf{0.2} & \mathbf{0.2} \end{bmatrix}$$

Suppose there is a demand for the products to use for final consumption and it is given as 300, 400 and 500 for industry $\boldsymbol{L, M\ and\ N}$ respectively.

a) ***Formulate a mathematical expression for the total output from the given information.***
b) ***Obtain the matrix form of the model constructed.***
c) ***How much output should each of the three industries produce to meet the total demand? (Hint: Using Cramer's rule.)***

Solution:

a) The vector of demand for final consumption from industry $L, M\ and\ N$ may be written as

$$F = \begin{bmatrix} 300 \\ 500 \\ 400 \end{bmatrix}$$

Let $L, M\ and\ N$ be the output that needs to be produced to meet the intermediate demand and the final demand.

Given the input-coefficient matrix, the total output may be mathematically expressed as:

$$\left.\begin{aligned} L &= 0.3L + 0.2M + 0.4N + 300 \\ M &= 0.2M + 0.1N + 500 \\ N &= 0.1L + 0.2M + 0.2N + 400 \end{aligned}\right\rbrack \quad (19.1)$$

We therefore have a system of simultaneous equation.

b) This system of equations in equation (19.1) may be transformed into a single equation by using the matrix transformation method.

Simplifying the equations gives

$$L - 0.3L - 0.2M - 0.4N = 300$$

$$M - 0.2M - 0.1N = 500$$

$$-0.1L - 0.2M + N - 0.2N = 400$$

Now, expressing the equations in matrix form gives

$$\begin{bmatrix} 0.7 & -0.2 & -0.4 \\ 0 & 0.8 & -0.1 \\ -0.1 & -0.2 & 0.8 \end{bmatrix} \begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} 300 \\ 500 \\ 400 \end{bmatrix}$$

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c) In order to find the values of L, M and N we use the Cramer's Rule

Note that the co-efficient matrix is actually the result of subtracting the given input-coefficient matrix from an identity matrix, that is,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.3 & 0.2 & 0.4 \\ 0 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.7 & -0.2 & -0.4 \\ 0 & 0.8 & -0.1 \\ -0.1 & -0.2 & 0.8 \end{bmatrix}$$

Let us name this resultant matrix as $I - C$, where $I \text{ is the identity matrix}$

Now, using Cramer's rule ***(refer module 12, section 12.5)***, gives

$$L = \frac{|I - C_L|}{|I - C|} = \frac{\begin{vmatrix} 300 & -0.2 & -0.4 \\ 500 & 0.8 & -0.1 \\ 400 & -0.2 & 0.8 \end{vmatrix}}{\begin{vmatrix} 0.7 & -0.2 & -0.4 \\ 0 & 0.8 & -0.1 \\ -0.1 & -0.2 & 0.8 \end{vmatrix}} = \frac{442}{0.4} = 1105$$

$$M = \frac{|I - C_M|}{|I - C|} = \frac{\begin{vmatrix} 0.7 & 300 & -0.4 \\ 0 & 500 & -0.1 \\ -0.1 & 400 & 0.8 \end{vmatrix}}{\begin{vmatrix} 0.7 & -0.2 & -0.4 \\ 0 & 0.8 & -0.1 \\ -0.1 & -0.2 & 0.8 \end{vmatrix}} = \frac{291}{0.4} = 727.5$$

$$N = \frac{|I - C_N|}{|I - C|} = \frac{\begin{vmatrix} 0.7 & -0.2 & 300 \\ 0 & 0.8 & 500 \\ -0.1 & -0.2 & 400 \end{vmatrix}}{\begin{vmatrix} 0.7 & -0.2 & -0.4 \\ 0 & 0.8 & -0.1 \\ -0.1 & -0.2 & 0.8 \end{vmatrix}} = \frac{328}{0.4} = 820$$

Therefore, the output that should be produced by the three industries to meet the total demand is $1105, 727.5 \text{ and } 820$ respectively for industry $L, M \text{ and } N$

Note:

You can verify the answer by putting the values of $L, M \text{ and } N$ in the system of equation (19.1)

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