Matrix and Market Model

Introduction:

The market in any economy is a complex entity with a large number of forces that determine the demand, supply and price of goods. Single equations that were discussed in the earlier modules of this course have certain limitations and fail to show the real scenario.

So, can mathematics help to understand a complex market situation? How to solve problem where a large number of variables interact and influence each other?

This module will try to explore the method of solving complex market models.

Objectives

The objectives of this module are:

1. *Distinguish* between single equation and simultaneous equations
2. *Interpret* the structure of simultaneous equations
3. *Transform* a system of simultaneous equations into a single equation in matrix form
4. *Solve* the market model using matrix algebra

Terminology

1. Identity: an equality equation where function on the LHS is the same as that on the RHS and the equation is true for all values of the variables.
2. Conditional equation: an equation that is true only for particular values of the variables.
3. Simultaneous equation: a system of equations that describe joint dependence of variables
4. Endogenous variable: a variable that may be behave as dependent and independent and whose value is determined from the model
5. Exogenous variable: a variable that is purely an independent variable and whose value is given from outside the model

14.1 Equations, Single equation and Simultaneous Equations:

The term 'equation' is a mathematical expression of equality. It contains two or more variables. An equation has two expressions connected by an "=" sign. The expression to the left of "=" sign is called Left Hand Side (in short, LHS) and the expression to the right of "=" sign is called Right Hand Side (in short, RHS). The "=" sign was invented by Robert Recorde, a physician and mathematician.

Robert Recorde

[Source: https://en.wikipedia.org/wiki/Robert_Recorde#/media/File:Robert_recorde.jpg]

Solving an equation means to find the values of the variables that makes the equation true. An equation may be an identity or it may be conditional. An identity is true for all the values of the variables, while a conditional equation is true for only particular values of the variables.

Example:

The equation $2x + 3 = 7$ is a conditional equation because this equation is true only for $x = 2$. On the other hand, an equation $2\,(5 + 2x) = 10 + 4x$ is an identity because it is true for all values of x.

14.2. Equations and Models used in Economics:

The use of mathematics in solving economic problems dates back to the 17th century. However the importance of mathematics in economic analysis has gained ground since the 20th century. Also, in recent years, researchers have started using theories and methods originally developed by physicists to solve economic problems.

Below is a list of few of the equations and models that are used in economics:

Table 5.1: List of few equations and models used in Economics

S. No.	Name of the equation	Name of the scientist	Application
1	Bernoulli equation/ Differential equation	Daniel Bernoulli/ Jacob Bernoulli	Change of Gross Domestic Product (GDP) over time
2.	Black-Scholes Equation	Fisher Black and Myron Scholes	Mathematical Finance
3.	Euler-Lagrange Equation	Leonhard Euler and Joseph Louis Lagrange	Optimization problems
4.	Fisher Equation	Irving Fisher	Financial mathematics
5.	Kolmogorov Equations	Andrey Kolmogorov	Markov stochastic processes
6.	Slutsky Equation	Eugen Slutsky	Consumer Theory
7.	Gravity Model of International Trade	Walter Isard	Predicts bilateral trade flows
8.	Kinetic Exchange Model	Meghnad Saha, B.N. Shrivastava and Bernoit Mandelbrot	Distribution of income and wealth in an economy
9.	Chaos Theory	Edward Lorenz	Inequality and chaotic behaviour
10.	Game Theory	Dov Monderer and Lloyd Shapley	Potential game

There are a large number of other equations and models that are used in Economics. The above list was prepared to give the reader an idea of the use of mathematics and physics in solving economic problems.

14.3. Simultaneous equations in Economics:

Two or more equations form a system of simultaneous equations. It is a system that describes the joint dependence of variables. In single equations, there is a one-way cause and effect relationship. The independent variables are the cause and dependent variable is the effect. In a simultaneous equation system, there is a two-way cause and effect relationship. The independent variable may act as a cause in one equation and may act as an effect in another. Similarly, the dependent variable may act as the effect in one equation and may act as the cause in another.

Example:

In the simple economic model for a consumer, the demand equation is expressed as function of price. Mathematically, $D = f(P)$, where D is quantity demanded (effect) of a good and P is the price (cause) of the good. This mathematical expression tells that Quantity Demanded depends on the Price, all other determinants remaining constant. However, there may be instances, where the price of the good is determined by the quantity demanded. In such a case, price behaves as the dependent variable (effect) and quantity demanded (cause) behaves as the independent variable. The mathematical expression may then be written as $P = f(D)$.

Therefore, we have two equations in the above situation, or a simultaneous system where one equation shows that quantity demanded is dependent and another shows that quantity demanded is independent.

A. Basic terms in simultaneous equations:

i) **Endogenous variable:** an endogenous variable is a variable whose value is determined from the model. In a simultaneous equation system, an endogenous variable may be dependent or independent.

ii) **Exogenous variable:** an exogenous variable is a variable whose value is given. It is determined from external sources. It behaves as a purely independent variable in the equations of the system of simultaneous equations.

Example:

$$\left.\begin{aligned} y_1 &= 3y_2 - 2x_1 + x_2 - - - - - (14.1) \\ y_2 &= y_3 + x_3 - - - - - - - - - - (14.2) \\ y_3 &= y_1 - y_2 - 2x_3 - - - - - - - - (14.3) \end{aligned}\right\} \text{(A)}$$

(A) is a system of simultaneous equation with 3 equations and 6 variables. Out of the variables, y_1, y_2 *and* y_3 are endogenous variables, since they behave as both dependent and independent variables and x_1, x_2 *and* x_3 are exogenous variables, since they are purely independent.

B. General Structure

The general structure of a simultaneous equation with "m" equations and "m" endogenous variables may be written as follows:

$$\left.\begin{aligned} Y_{1t} &= \beta_{12}Y_{2t} + \beta_{13}Y_{3t} + \ldots\ldots\ldots + \beta_{1m}Y_{mt} + \gamma_{11}X_{1t} + \gamma_{12}X_{2t} + \ldots\ldots + \gamma_{1k}X_{kt} \\ Y_{2t} &= \beta_{21}Y_{1t} + \beta_{23}Y_{3t} + \ldots\ldots\ldots + \beta_{2m}Y_{mt} + \gamma_{21}X_{1t} + \gamma_{22}X_{2t} + \ldots\ldots + \gamma_{2k}X_{kt} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ Y_{mt} &= \beta_{m1}Y_{1t} + \beta_{m2}Y_{2t} + \ldots\ldots + \beta_{m-1}Y_{m-1t} + \gamma_{m1}X_{1t} + \gamma_{m2}X_{2t} + \ldots\ldots + \gamma_{mk}X_{kt} \end{aligned}\right\} \text{(B)}$$

(B) is the general structure of a system of simultaneous equations with "m" equations and "m" endogenous variables.

$Y_{1t}, Y_{2t} \ldots\ldots Y_{mt}$ are the endogenous variables whose values are to be determined from the model.

$X_{1t}, X_{2t} \ldots\ldots X_{kt}$ are the exogenous variables that is given from external sources.

C. Condition for solving a system of simultaneous equations:

Number of Equations = Number of Endogenous Variables

D. Application of simultaneous equations:

The real world is complex. Therefore, a single equation model is generally considered to be unrealistic. Simultaneous equations help to understand the complex economy in a much better way. Below are few examples where simultaneous equations may be used:

a) Market Model
b) National Income Model
c) Input-Output Analysis
d) Optimization problems (with equality constraint and inequality constraint)

14.4. Transforming a system of simultaneous equation to matrix form

A simultaneous equation is a system of two or more equations. Solving a system of simultaneous equations by the elimination and substitution method becomes a tedious job. In such cases, matrix algebra may be used to transform a system of simultaneous equations into a single equation to obtain unique solutions.

Example:

Given a system of simultaneous equations

$$a_1x_1 + b_1x_2 + c_1x_3 = z_1 \quad \text{--------------------} (1)$$

$$a_2x_1 + b_2x_2 + c_2x_3 = z_2 \quad \text{--------------------} (2)$$

$$a_3x_1 + b_3x_2 + c_3x_3 = z_3 \quad \text{--------------------} (3)$$

Here, x_1, x_2 and x_3 are variables

$a, b\ and\ c$ are parameters

$z_1,\ z_2$ and z_3 are constants

The system of three equations may be transformed into the matrix form as follows:

Step 1: Construct a matrix of all the co-efficient in the system

Arrange all the co-efficient in the system of equations in a particular order, Name the matrix as A.

Thus,

$$A = \begin{bmatrix} a_1 & b_1 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

A is a matrix of order 3 x 3

Step 2: Construct a column vector of the variables.

Name the column vector of variables as X. Thus,

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$ X is a column vector of order 3 x 1

Note: *A column vector is constructed so that it can fulfill the condition required for matrix multiplication. The condition says that the number of columns in the first matrix must be equal to the number of rows in the second matrix.*

Step 3: Multiply the co-efficient matrix and the column vector of variables.

Thus,

$$AX = \begin{bmatrix} a_1 & b_1 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Step 4: Construct a column vector of constants

Name the constant vector as Z. Thus

$$Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix},$$ Z is a column vector of order 3 x 1

Step 5: Equate the product to the constant vector

Thus AX =Z ----------(4)

$$\begin{bmatrix} a_1 & b_1 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

Step 6: Multiply both sides of equation (4) by A^{-1} (Inverse of A)

Therefore,

$$AA^{-1}X = A^{-1}Z$$

We know by property of inverse f a matrix, $AA^{-1} = I$, where I is an Identity matrix

Therefore, $X = A^{-1}Z$ ------------(5)

Thus it is seen that the system of three equations has been transformed into a single equation.

14.4. Solving the Market Model using matrix algebra

Solving a system of simultaneous equations means finding the value of the variables (unknowns) that will satisfy all the equations simultaneously (at the same time).

Sometimes, when there are a large number of equations, solving the system of equations by using the most common substitution and elimination method may become very difficult. In such cases, matrix algebra can be used to solve a system of simultaneous equations.

The procedure for solving a simultaneous equation system is explained for the market model.

The market model comprises of three equations

$$Demand\ Equations: D = a - bP$$

$$Supply\ Equation: \quad S = -c + dP$$

$$Equilibrium: \qquad D = S$$

Where D is the quantity demanded,

S is the quantity supplied

P is the price

a,b,c and d are the parameters

The main task of solving the model is to find the values of the equilibrium price and quantity, i.e the values of D, S and P.

In order to solve this model by using the matrix method, it has to be first transformed into the matrix form.

Thus, re-arranging the model by putting the variables on the left hand side and the constants on the right hand side, we get,

$$D + bP = a$$

$$S - dP = -c$$

$$D - S = 0$$

Further, to construct the matrix of co-efficients, we write,

$$D + 0 \times S + bP = a$$

$$0 \times D + S - dP = -c$$

$$D - S + 0 \times P = 0$$

Now, transforming the equations into matrix form we get,

$$\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} D \\ S \\ P \end{bmatrix} = \begin{bmatrix} a \\ -c \\ 0 \end{bmatrix}$$

Naming the co-efficient matrix as A, variable matrix as X and constant matrix as Z, we get

$$AX = Z, or\ X = A^{-1}Z$$

Finding the Inverse of matrix A

The inverse of matrix A is given by

$$\boxed{A^{-1} = \frac{Adj\ A}{|A|}}$$

Adj A is the Adjoint of A and |A| is the determinant of A

For a solution to exist, $|A| \neq 0$

In the example of the market model,

$|A| = \begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{vmatrix}$ = 1 (-d) -0 +b(-1) = -d-b = -(b+d)

Adj A is calculated by finding out the transpose of the co-factor matrix.

Co-factor of an element is obtained by the following formula,

$$\boxed{\boldsymbol{C_{ij} = (-1)^{i+j} M_{ij}}}$$

C_{ij} is the co-factor of the element in the i-th row and j-th column
M_{ij} is the minor of the element in the i-th row and j-th column

Minor of an element is found out by striking of the row and column in which the element is placed and taking the determinant of the remaining elements.

Thus, in the above example of the market model,

Minor of element 1 in the first row and first column is given by,

$M_{11} = \begin{vmatrix} 1 & -d \\ -1 & 0 \end{vmatrix}$ after striking out the first row and first column.

Thus $M_{11} = -d$

Now, co-factor of the element 1, $C_{11} = (-1)^{1+1} M_{11}$, or $C_{11} = -d$

Similarly, co-factors of the remaining elements of matrix A is obtained by using the same formula.

Thus Co-factor of element 0 in first row and second column is given by

$C_{12} = (-1)^{1+2} M_{12} = (-1)^3 \begin{vmatrix} 0 & -d \\ 1 & 0 \end{vmatrix}$ or $C_{12} = -d$

$C_{13} = (-1)^{1+3} M_{13} = (-1)^4 \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}$ or $C_{13} = -1$

$C_{21} = (-1)^{2+1} M_{21} = (-1)^3 \begin{vmatrix} 0 & b \\ -1 & 0 \end{vmatrix}$ or $C_{21} = -b$

$C_{22} = (-1)^{2+2} M_{22} = (-1)^4 \begin{vmatrix} 0 & b \\ 1 & 0 \end{vmatrix}$ or $C_{22} = -b$

$C_{23} = (-1)^{2+3} M_{23} = (-1)^5 \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix}$ or $C_{23} = 1$

$C_{31} = (-1)^{3+1}M_{31} = (-1)^4 \begin{vmatrix} 0 & b \\ 1 & -d \end{vmatrix}$ or $C_{31} = b$

$C_{32} = (-1)^{3+2}M_{32} = (-1)^5 \begin{vmatrix} 1 & b \\ 0 & -d \end{vmatrix}$ or $C_{32} = d$

$C_{33} = (-1)^{3+3}M_{33} = (-1)^4 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ or $C_{33} = 1$

The co-factor matrix is given by

$$\begin{bmatrix} -d & -d & -1 \\ -b & -b & 1 \\ b & d & 1 \end{bmatrix}$$

And $Adj\ A = \begin{bmatrix} -d & -b & b \\ -d & -b & d \\ -1 & 1 & 1 \end{bmatrix}$

Therefore,

$$A^{-1} = \frac{1}{-(b+d)} \begin{bmatrix} -d & -b & b \\ -d & -b & d \\ -1 & 1 & 1 \end{bmatrix}$$

Further,

$$X = A^{-1}Z = \frac{1}{-(b+d)} \begin{bmatrix} -d & -b & b \\ -d & -b & d \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ -c \\ 0 \end{bmatrix} = \frac{1}{-(b+d)} \begin{bmatrix} -ad + bc \\ -ad + bc \\ -a - c \end{bmatrix}$$

Or $X = \begin{bmatrix} D \\ S \\ P \end{bmatrix} = \frac{1}{-(b+d)} \begin{bmatrix} -ad + bc \\ -ad + bc \\ -a - c \end{bmatrix}$

Thus,

Equilibrium quantity $D = \frac{ad-bc}{b+d}$

Equilibrium supplied $S = \frac{ad-bc}{b+d}$ and

Equilibrium Price $P = \frac{a+c}{b+d}$

