Core Algebra Only

This note provides a concise, self-contained algebraic foundation for the toy model, without any physical interpretation. Only algebraic structures, constraints, and their formal consequences are included.

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1. Algebraic Framework

We consider a hierarchy of real algebras constructed via the Cayley–Dickson process:

$\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O} \to \mathbb{S} \to \dots$

where: - $\mathbb{C}$: complex numbers (2D) - $\mathbb{H}$: quaternions (4D) - $\mathbb{O}$: octonions (8D) - $\mathbb{S}$: sedenions (16D)

Each algebra is equipped with: - a multiplication operation, - an involution (conjugation), - a quadratic form (norm) defined as $|x|^2 = x * \bar{x}$.

Only $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are **division algebras** with **multiplicative norms**.

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2. Norm Structure

For any element written in real coordinates:

- $\mathbb{C}$**:** $x = a + bi \quad \to \quad |x|^2 = a^2 + b^2$
- $\mathbb{H}$**:** $x = a + b\,i + c\,j + d\,k \quad \to \quad |x|^2 = a^2 + b^2 + c^2 + d^2$
- $\mathbb{O}$**:** $x = a + \Sigma\, b_k e_k \quad (k = 1\dots7) \quad \to \quad |x|^2 = a^2 + \Sigma\, b_k^2$
- $\mathbb{S}$**:** $x = a + \Sigma\, b_k e_k \quad (k = 1\dots15) \quad \to$ $|x|^2$ still defined but **not multiplicative**

A **unit-norm element** satisfies:

$|x| = 1 \quad \Leftrightarrow \quad$ sum of squares of all coordinates = 1.

This single constraint provides: - boundedness of all coordinates ($|\text{component}| \leq 1$), - quadratic admissibility conditions determining which coordinate sets are allowed.

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3. Hierarchical Embedding

Each algebra embeds naturally into the next: - $\mathbb{C} \subset \mathbb{H}$ via identification of i, - $\mathbb{H} \subset \mathbb{O}$ via preservation of quaternionic triples, - $\mathbb{O} \subset \mathbb{S}$ via Cayley–Dickson doubling.

For any x in a lower algebra treated as an element of a higher algebra: - its norm is preserved, - its coordinates occupy a subspace of the larger coordinate system, - additional imaginary directions are simply appended.

This produces an algebraic hierarchy where each stage strictly expands the coordinate space while retaining the previous one as a subalgebra.

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4. Multiplicativity and Its Breakdown

4.1. Division Algebras

In $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ the norm is **multiplicative**:

|xy| = |x|\,|y|

This implies: - absence of zero divisors, - invertibility of every nonzero element, - stability of norm under algebraic operations.

4.2. Post-Octonionic Algebras

In $\mathbb{S}$ (sedenions) and higher Cayley–Dickson algebras: - the norm is **not** multiplicative, - **zero divisors exist**, i.e., $x \neq 0$, $y \neq 0$ but $xy = 0$, - multiplicative structure loses alternativity and other stabilizing properties.

Consequences: - norm no longer determines invertibility, - elements with $|x| = 1$ may still produce degenerate products, - polynomial equations may develop nonunique (or infinitely many) roots.

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5. Unit-Norm Constraint as a Filter

Given $|x| = 1$: - all coordinates satisfy $\text{coordinate}^2 \leq 1$, - coordinate combinations violating the sphere constraint are excluded, - higher-dimensional algebras impose stricter admissibility because more coordinates must jointly satisfy the unit sphere.

Thus each algebraic level acts as a **filter**: - $\mathbb{C}$ filters only (a, b), - $\mathbb{H}$ filters (a, b, c, d), - $\mathbb{O}$ filters $(a, b_1 \ldots b_7)$, etc.

The constraint strengthens as dimensionality grows.

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6. Coordinate Renormalization (Perspective Maps)

For any nonzero coordinate w_i of $x = (w_0, w_1, \ldots, w_n)$, define a rescaling map:

$s_i = 1 / w_i \qquad v_j{}^{()}_i = w_j \cdot s_i$

This produces n+1 distinct normalized representations (“perspectives”), each placing one coordinate at 1.

Properties: - algebraically valid for any $w_i \neq 0$, - preserves ratios between components, - produces alternative coordinate systems within the same algebraic element.

These maps are purely algebraic and do not enforce any physical interpretation.

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7. Root Structure and Admissibility

Given a polynomial or quadratic constraint on components (e.g., norm or other algebraically defined expressions), admissible solutions are those whose coordinates satisfy: - boundedness within the unit sphere, - absence of contradictions with algebraic relations (e.g., multiplication rules), - compatibility with the signature arising from the chosen algebra.

In division algebras: - roots tend to be isolated and unique.

In non-division algebras: - zero divisors cause degenerate or nonunique root sets.

This reflects a purely algebraic transition from structured to pathological solution spaces.

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8. Summary of Core Algebraic Facts

1. Only $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ are division algebras with multiplicative norms.
2. Norm = 1 defines a unit hypersphere that all coordinates must satisfy.
3. Each algebra strictly extends the previous one as a subspace.
4. Higher algebras add new imaginary directions without affecting existing ones.
5. Sedenions and higher algebras contain zero divisors and lose multiplicativity.
6. Norm-preserving constraints act as algebraic filters for allowable coordinate sets.
7. Rescaling by any nonzero coordinate yields alternative algebraic representations.
8. Polynomial equations have stable roots only up to the octonionic level; root structure becomes degenerate in sedenions and above.

This completes the algebra-only core needed for the toy model, free of any physical assumptions or interpretations.