MATH2004 Tutorial 6

1. Use Green's Theorem to evaluate $\oint_C 2xydx + x^2y^3dy$ where C is the rectangle $D = [0,2]\times[0,3]$ with positive orientation.

   Solution:
$$\oint_C 2xydx + x^2y^3dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$
$$= \iint_D (2xy^3 - 2x)dxdy = \int_0^3 \int_0^2 (2xy^3 - 2x)dxdy = \int_0^3 \int_0^2 (4y^3 - 4)dy = 69.$$

2. Use Green's Theorem to evaluate $\oint_C 3xydx + 6x^2y^3dy$ where C is the triangle with vertices $(0,0),(2,0),(2,4)$, with positive orientation.

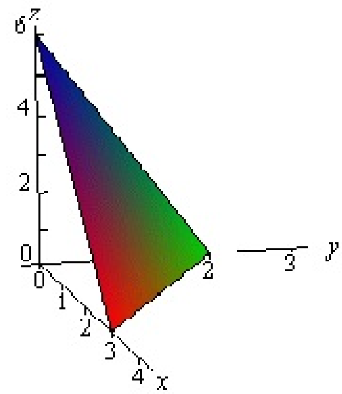
   Solution: D is the triangle with vertices $(0,0),(2,0),(2,4)$. Then $D = R_{yx} : 0 \le y \le 2x, 0 \le x \le 2$. Let $P = 3xy$, $Q = 6x^2y^3$. Then $Q_x - P_y = 12xy^3 - 3x$.
$$\oint_C 3xydx + 6x^2y^3dy = \iint_D (12xy^3 - 3x)dxdy = \int_0^2 (\int_0^{2x} (12xy^3 - 3x)dy)dx$$
$$= \int_0^2 (48x^5 - 6x^2)dx = 496.$$

3. Evaluate $\oint_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = 3z\vec{i} + 3x^2\vec{j} + 6y\vec{k}$, C is the curve of intersection of the plane $2x + 2y + z = 6$ and the cylinder $x^2 + y^2 = 1$.

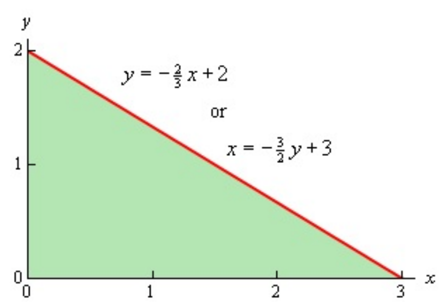
   Solution:
$$\vec{n} = \frac{2\vec{i} + 2\vec{j} + \vec{k}}{\sqrt{2^2 + 2^2 + 1^2}} = \frac{1}{3}(2\vec{i} + 2\vec{j} + \vec{k}).$$
$$curl\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} = 6\vec{i} + 3\vec{j} + 6x\vec{k}.$$
   Thus
$$curl\vec{F}\cdot\vec{n} = 6 + 2x.$$
   By Stokes' Theorem,
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S curl\vec{F}\cdot\vec{n}dS = \iint_S (6 + 2x)dS = \iint_D (6 + 2x)(3dA)$$
$$= 3\int_0^1 \int_0^{2\pi} (6 + 2r\cos\theta)rdrd\theta = 18\pi.$$

4. Find the triple integral $\iiint_E 2x\ dV$, where E is the under the plane $2x + 3y + z = 6$ that lies in the first octant.

Solution: This region can be illustrated as follows:



The limits of z are: $0 \le z \le 6 - 2x - 3y$. The region D in the xy-plane is the triangle with vertices at (0,0), (3,0) (0,2):



The triple integral is evaluated by the iterated integral:

$$\iiint_E 2x dV = \iint_D \int_0^{6-2x-3y} (2x) dz dA$$

$$= \int_0^3 \int_0^{-\frac{2}{3}x+2} [2x(6-2x-3y)] dy dx = \int_0^3 [\frac{4}{3}x^3 - 8x^2 + 12x] dx = 9.$$

5. Find the integral $\iiint_E z dV$, where E is within $x^2 + y^2 = 4$, below $z = \sqrt{4 + x^2 + y^2}$, above $z = \sqrt{4 - x^2 - y^2}$.

Solution: By cylindrical coordinates,

$$E = \{0 \le r \le 2, 0 \le \theta \le 2\pi, \sqrt{4 - r^2} \le z \le \sqrt{4 + r^2}\}.$$

$$\iiint_E z dV = \int_0^{2\pi} \int_0^2 \int_{\sqrt{4-r^2}}^{\sqrt{4+r^2}} z dz\ dr\ d\theta$$

$$= \int_0^{2\pi} \int_0^2 \left(\frac{1}{2} z^2\right) |_{z=\sqrt{4-r^2}}^{\sqrt{4+r^2}}\ r\ dr\ d\theta$$

$$= \int_0^{2\pi} \int_0^2 r^3\ dr\ d\theta$$

$$= 8\pi.$$

6. Find the integral $\iiint_E \sqrt{x^2+y^2} dV$, where E is the upper half of the region between the sphere $x^2+y^2+z^2=1$, and the sphere $x^2+y^2+z^2=4$.

Solution: We use the Spherical Coordinates:

$$x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta, z = \rho \cos\phi, dV = \rho^2 \sin\phi d\rho d\phi d\theta,$$

where $J(\rho, \phi, \theta) = \rho^2 \sin\phi$.

Since $1 \le x^2+y^2+z^2 = \rho^2 \le 4$, we see that $1 \le \rho \le 2, 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{2}$. Thus

$$\iiint_E \sqrt{x^2+y^2} dV = \iiint_E \rho \sin\phi\ \rho^2 \sin\phi d\rho d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \int_1^2 \rho^3 \sin^2\phi d\rho d\phi d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right) \left(\int_1^2 \rho^3 d\rho\right) \left(\int_0^{\pi/2} \frac{1}{2}(1 - \cos 2\phi) d\phi\right) = \frac{15\pi^2}{8}.$$