$Lecture3$

$rrison$

2 叉乘的李代数性质

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in V$

1. 封闭性

$[X, Y] = \begin{bmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{bmatrix} \in V$

2. 双线性

$[aX + bY, Z]$

$$= \begin{bmatrix} ax_2z_3 + by_2z_3 - ax_3z_2 - by_3z_2 \\ ax_3z_1 + by_3z_1 - ax_1z_3 - by_1z_3 \\ ax_1z_2 + by_1z_2 - ax_2z_1 - by_2z_1 \end{bmatrix} = \begin{bmatrix} ax_2z_3 - ax_3z_2 \\ ax_3z_1 - ax_1z_3 \\ ax_1z_2 - ax_2z_1 \end{bmatrix} + \begin{bmatrix} by_2z_3 - by_3z_2 \\ by_3z_1 - by_1z_3 \\ by_1z_2 - by_2z_1 \end{bmatrix}$$

$$= a\begin{bmatrix} x_2z_3 - x_3z_2 \\ x_3z_1 - x_1z_3 \\ x_1z_2 - x_2z_1 \end{bmatrix} + b\begin{bmatrix} y_2z_3 - y_3z_2 \\ y_3z_1 - y_1z_3 \\ y_1z_2 - y_2z_1 \end{bmatrix} = a[X, Z] + b[Y, Z]$$

$[Z, aX + bY]$

$$= \begin{bmatrix} ax_3z_2 + by_3z_2 - ax_2z_3 - by_2z_3 \\ ax_1z_3 + bx_1z_3 - ax_3z_1 - by_3z_1 \\ ax_2z_1 + by_2z_1 - ax_1z_2 - by_1z_2 \end{bmatrix} = \begin{bmatrix} ax_3z_2 - ax_2z_3 \\ ax_1z_3 - ax_3z_1 \\ ax_2z_1 - ax_1z_2 \end{bmatrix} + \begin{bmatrix} by_3z_2 - by_2z_3 \\ bx_1z_3 - by_3z_1 \\ by_2z_1 - by_1z_2 \end{bmatrix}$$

$$= a\begin{bmatrix} x_3z_2 - x_2z_3 \\ x_1z_3 - x_3z_1 \\ x_2z_1 - x_1z_2 \end{bmatrix} + b\begin{bmatrix} y_3z_2 - y_2z_3 \\ x_1z_3 - y_3z_1 \\ y_2z_1 - y_1z_2 \end{bmatrix} = a[Z, X] + b[Z, Y]$$

3. 自反性

$[X, X] = \begin{bmatrix} x_2x_3 - x_3x_2 \\ x_3x_1 - x_1x_3 \\ x_1x_2 - x_2x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

4. 雅可比等价

$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]]$

$$= \begin{bmatrix} x_2(y_1z_2 - y_2z_1) - x_3(y_3z_1 - y_1z_3) \\ x_3(y_2z_3 - y_3z_2) - x_1(y_1z_2 - y_2z_1) \\ x_1(y_3z_1 - y_1z_3) - x_2(y_2z_3 - y_3z_2) \end{bmatrix} + \begin{bmatrix} z_2(x_1y_2 - x_2y_1) - z_3(x_3y_1 - x_1y_3) \\ z_3(x_2y_3 - x_3y_2) - z_1(x_1y_2 - x_2y_1) \\ z_1(x_3y_1 - x_1y_3) - z_2(x_2y_3 - x_3y_2) \end{bmatrix} + \begin{bmatrix} y_2(z_1x_2 - z_2x_1) - y_3(z_3x_1 - z_1z_3) \\ y_3(z_2x_3 - z_3x_2) - y_1(z_1x_2 - z_2x_1) \\ y_1(z_3x_1 - z_1x_3) - y_2(z_2x_3 - z_3x_2) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

3 推导SE(3)指数映射

$$e^{\xi^\wedge} = I + \begin{bmatrix} \phi^\wedge & \rho \\ 0 & 0 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} \phi^\wedge & \rho \\ 0 & 0 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} (\phi^\wedge)^2 & \phi^\wedge\rho \\ 0 & 0 \end{bmatrix} + \cdots = \begin{bmatrix} \sum_{n=0}^{\infty}\frac{1}{n!}(\phi^\wedge)^n & \sum_{n=0}^{\infty}\frac{1}{(n+1)!}(\phi^\wedge)^n\rho \\ 0^T & 1 \end{bmatrix}$$

已知

$$R = e^{\phi^\wedge} = \sum_{n=0}^{\infty}\frac{1}{n!}(\phi^\wedge)^n = cos\theta \cdot I + (1-cos\theta)aa^T + sin\theta \cdot a^\wedge \tag{1}$$

令$J = \sum_{n=0}^{\infty}\frac{1}{(n+1)!}(\phi^\wedge)^n$,

$J = I + \frac{1}{2!}\phi^\wedge + \frac{1}{3!}(\phi^\wedge)^2 + \frac{1}{4!}(\phi^\wedge)^3 + \frac{1}{5!}(\phi^\wedge)^4 + \ldots$

$= aa^T - a^\wedge a^\wedge + \frac{1}{2!}\theta a^\wedge + \frac{1}{3!}\theta^2 a^\wedge a^\wedge - \frac{1}{4!}\theta^3 a^\wedge - \frac{1}{5!}\theta^4 a^\wedge a^\wedge + \ldots$

$= aa^T - a^\wedge a^\wedge(1 - \frac{1}{3!}\theta^2 + \frac{1}{5!}\theta^4 + \ldots) + a^\wedge(\frac{1}{2!}\theta - \frac{1}{4!}\theta^3 + \ldots)$

$= aa^T - \frac{1}{\theta}a^\wedge a^\wedge(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \ldots) + \frac{1}{\theta}a^\wedge(\frac{1}{2!}\theta^2 - \frac{1}{4!}\theta^4 + \ldots)$

$= aa^T - \frac{sin\theta}{\theta}a^\wedge a^\wedge + \frac{1-cos\theta}{\theta}a^\wedge$

$= aa^T - \frac{sin\theta}{\theta}(aa^T - I) + \frac{1-cos\theta}{\theta}a^\wedge$

$= \frac{sin\theta}{\theta}I + (1 - \frac{sin\theta}{\theta})aa^T + \frac{1-cos\theta}{\theta}a^\wedge$

then $e^{\xi^\wedge} = \begin{bmatrix} \sum_{n=0}^{\infty}\frac{1}{n!}(\phi^\wedge)^n & \sum_{n=0}^{\infty}\frac{1}{(n+1)!}(\phi^\wedge)^n\rho \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} R & J\rho \\ 0^T & 0 \end{bmatrix}$,
where $R = cos\theta \cdot I + (1-cos\theta)aa^T + sin\theta \cdot a^\wedge, J = \frac{sin\theta}{\theta}I + (1 - \frac{sin\theta}{\theta})aa^T + \frac{1-cos\theta}{\theta}a^\wedge$

4 伴随

证明$Ra^\wedge R^T = (Ra)^\wedge$
$Ra^\wedge R^T = R(a \times I)R^T = R(a \times (R^T R))R^T = (Ra) \times (RR^T R)R^T = (Ra)^\wedge RR^T RR^T = (Ra)^\wedge$

已知$((Ra)^\wedge)^n = Ra^\wedge R^T \cdot Ra^\wedge R^T \cdot \cdots \cdot Ra^\wedge R^T = R(a^\wedge)^n R^T$
证明$SO(3)$伴随性质：
$Re^{p^\wedge}R^T = \sum_{n=0}^{\infty} R\frac{1}{n!}(p^\wedge)^n R^T = \sum_{n=0}^{\infty}\frac{1}{n!}((Rp)^\wedge)^n = e^{((Rp)^\wedge)^n}$

5 轨迹的描绘

见code/draw_trajectory.cpp

6 轨迹的误差

见code/draw_trajectory.cpp里面的CalcRMSE().