*Lecture*3

rrison

2 叉乘的李代数性质

$X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}, Y=\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}, Z=\begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix}\in V$

1. 封闭性

$[X,Y]=\begin{bmatrix}x_2y_3-x_3y_2\\x_3y_1-x_1y_3\\x_1y_2-x_2y_1\end{bmatrix}\in V$

2. 双线性

$[aX+bY,Z]$

$$=\begin{bmatrix}ax_2z_3+by_2z_3-ax_3z_2-by_3z_2\\ax_3z_1+by_3z_1-ax_1z_3-by_1z_3\\ax_1z_2+by_1z_2-ax_2z_1-by_2z_1\end{bmatrix}=\begin{bmatrix}ax_2z_3-ax_3z_2\\ax_3z_1-ax_1z_3\\ax_1z_2-ax_2z_1\end{bmatrix}+\begin{bmatrix}by_2z_3-by_3z_2\\by_3z_1-by_1z_3\\by_1z_2-by_2z_1\end{bmatrix}$$

$$=a\begin{bmatrix}x_2z_3-x_3z_2\\x_3z_1-x_1z_3\\x_1z_2-x_2z_1\end{bmatrix}+b\begin{bmatrix}y_2z_3-y_3z_2\\y_3z_1-y_1z_3\\y_1z_2-y_2z_1\end{bmatrix}=a[X,Z]+b[Y,Z]$$

$[Z,aX+bY]$

$$=\begin{bmatrix}ax_3z_2+by_3z_2-ax_2z_3-by_2z_3\\ax_1z_3+bx_1z_3-ax_3z_1-by_3z_1\\ax_2z_1+by_2z_1-ax_1z_2-by_1z_2\end{bmatrix}=\begin{bmatrix}ax_3z_2-ax_2z_3\\ax_1z_3-ax_3z_1\\ax_2z_1-ax_1z_2\end{bmatrix}+\begin{bmatrix}by_3z_2-by_2z_3\\bx_1z_3-by_3z_1\\by_2z_1-by_1z_2\end{bmatrix}$$

$$=a\begin{bmatrix}x_3z_2-x_2z_3\\x_1z_3-x_3z_1\\x_2z_1-x_1z_2\end{bmatrix}+b\begin{bmatrix}y_3z_2-y_2z_3\\x_1z_3-y_3z_1\\y_2z_1-y_1z_2\end{bmatrix}=a[Z,X]+b[Z,Y]$$

3. 自反性

$[X,X]=\begin{bmatrix}x_2x_3-x_3x_2\\x_3x_1-x_1x_3\\x_1x_2-x_2x_1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

4. 雅可比等价

$[X,[Y,Z]]+[Z,[X,Y]]+[Y,[Z,X]]$

$$=\begin{bmatrix}x_2(y_1z_2-y_2z_1)-x_3(y_3z_1-y_1z_3)\\x_3(y_2z_3-y_3z_2)-x_1(y_1z_2-y_2z_1)\\x_1(y_3z_1-y_1z_3)-x_2(y_2z_3-y_3z_2)\end{bmatrix}+\begin{bmatrix}z_2(x_1y_2-x_2y_1)-z_3(x_3y_1-x_1y_3)\\z_3(x_2y_3-x_3y_2)-z_1(x_1y_2-x_2y_1)\\z_1(x_3y_1-x_1y_3)-z_2(x_2y_3-x_3y_2)\end{bmatrix}+\begin{bmatrix}y_2(z_1x_2-z_2x_1)-y_3(z_3x_1-z_1z_3)\\y_3(z_2x_3-z_3x_2)-y_1(z_1x_2-z_2x_1)\\y_1(z_3x_1-z_1x_3)-y_2(z_2x_3-z_3x_2)\end{bmatrix}$$

$$=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

3 推导SE(3)指数映射

$$e^{\xi^\wedge}=I+\begin{bmatrix}\phi^\wedge & \rho\\0 & 0\end{bmatrix}+\frac{1}{2!}\begin{bmatrix}\phi^\wedge & \rho\\0 & 0\end{bmatrix}+\frac{1}{3!}\begin{bmatrix}(\phi^\wedge)^2 & \phi^\wedge\rho\\0 & 0\end{bmatrix}+\cdots=\begin{bmatrix}\sum_{n=0}^{\infty}\frac{1}{n!}(\phi^\wedge)^n & \sum_{n=0}^{\infty}\frac{1}{(n+1)!}(\phi^\wedge)^n\rho\\0^T & 1\end{bmatrix}$$

已知

$$R=e^{\phi^\wedge}=\sum_{n=0}^{\infty}\frac{1}{n!}(\phi^\wedge)^n=cos\theta\cdot I+(1-cos\theta)aa^T+sin\theta\cdot a^\wedge \tag{1}$$

令$J=\sum_{n=0}^{\infty}\frac{1}{(n+1)!}(\phi^\wedge)^n$,

$J=I+\frac{1}{2!}\phi^\wedge+\frac{1}{3!}(\phi^\wedge)^2+\frac{1}{4!}(\phi^\wedge)^3+\frac{1}{5!}(\phi^\wedge)^4+\ldots$

$=aa^T-a^\wedge a^\wedge+\frac{1}{2!}\theta a^\wedge+\frac{1}{3!}\theta^2a^\wedge a^\wedge-\frac{1}{4!}\theta^3a^\wedge-\frac{1}{5!}\theta^4a^\wedge a^\wedge+\ldots$

$=aa^T-a^\wedge a^\wedge(1-\frac{1}{3!}\theta^2+\frac{1}{5!}\theta^4+\ldots)+a^\wedge(\frac{1}{2!}\theta-\frac{1}{4!}\theta^3+\ldots)$

$=aa^T-\frac{1}{\theta}a^\wedge a^\wedge(\theta-\frac{1}{3!}\theta^3+\frac{1}{5!}\theta^5+\ldots)+\frac{1}{\theta}a^\wedge(\frac{1}{2!}\theta^2-\frac{1}{4!}\theta^4+\ldots)$

$=aa^T-\frac{sin\theta}{\theta}a^\wedge a^\wedge+\frac{1-cos\theta}{\theta}a^\wedge$

$=aa^T-\frac{sin\theta}{\theta}(aa^T-I)+\frac{1-cos\theta}{\theta}a^\wedge$

$=\frac{sin\theta}{\theta}I+(1-\frac{sin\theta}{\theta})aa^T+\frac{1-cos\theta}{\theta}a^\wedge$

then $e^{\xi^\wedge}=\begin{bmatrix}\sum_{n=0}^{\infty}\frac{1}{n!}(\phi^\wedge)^n & \sum_{n=0}^{\infty}\frac{1}{(n+1)!}(\phi^\wedge)^n\rho\\0^T & 1\end{bmatrix}=\begin{bmatrix}R & J\rho\\0^T & 0\end{bmatrix}$,
where $R=cos\theta\cdot I+(1-cos\theta)aa^T+sin\theta\cdot a^\wedge, J=\frac{sin\theta}{\theta}I+(1-\frac{sin\theta}{\theta})aa^T+\frac{1-cos\theta}{\theta}a^\wedge$

4 伴随

证明$Ra^\wedge R^T=(Ra)^\wedge$

$(Ra)^\wedge R=(Ra)\times R=(Ra)\times(RR^TR)=R(a\times(R^TR))=R(a\times I)=Ra^\wedge$

$(Ra)^\wedge R=Ra^\wedge$

$\rightarrow Ra^\wedge R^T=(Ra)^\wedge$

已知$((Ra)^\wedge)^n=Ra^\wedge R^T\cdot Ra^\wedge R^T\cdot\cdots\cdot Ra^\wedge R^T=R(a^\wedge)^nR^T$
证明$SO(3)$伴随性质：
$Re^{p^\wedge}R^T=\sum_{n=0}^{\infty}R\frac{1}{n!}(p^\wedge)^nR^T=\sum_{n=0}^{\infty}\frac{1}{n!}((Rp)^\wedge)^n=e^{((Rp)^\wedge)^n}$

5 轨迹的描绘

见code/draw_trajectory.cpp

6 轨迹的误差

见code/draw_trajectory.cpp里面的CalcRMSE().