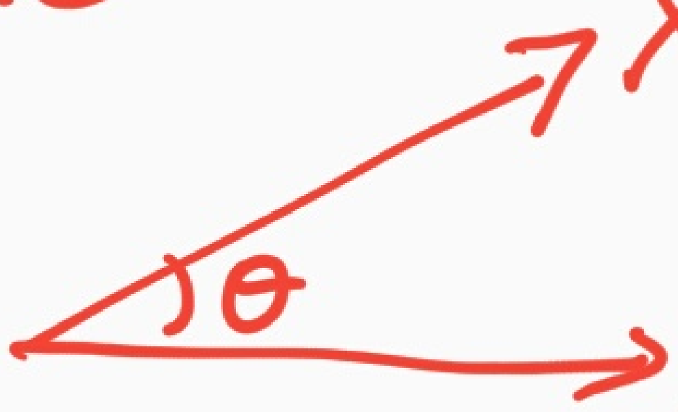
26.3.2019

Quiz:



$\bar{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\quad \bar{y} = \begin{bmatrix} -2 \\ 1 \\ \alpha \end{bmatrix}$

a) $\|\bar{x}\| = ?$

b) $\bar{x}$ yönündeki birim vektör nedir ($\hat{x} = ?$)

c) $\theta = 90^\circ$ için $\alpha = ?$

Çözüm:

a) $\|\bar{x}\| = \sqrt{\bar{x}^T \cdot \bar{x}} = \sqrt{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} = \sqrt{14}$

b) $\hat{x} = \dfrac{\bar{x}}{\|\bar{x}\|} = \dfrac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

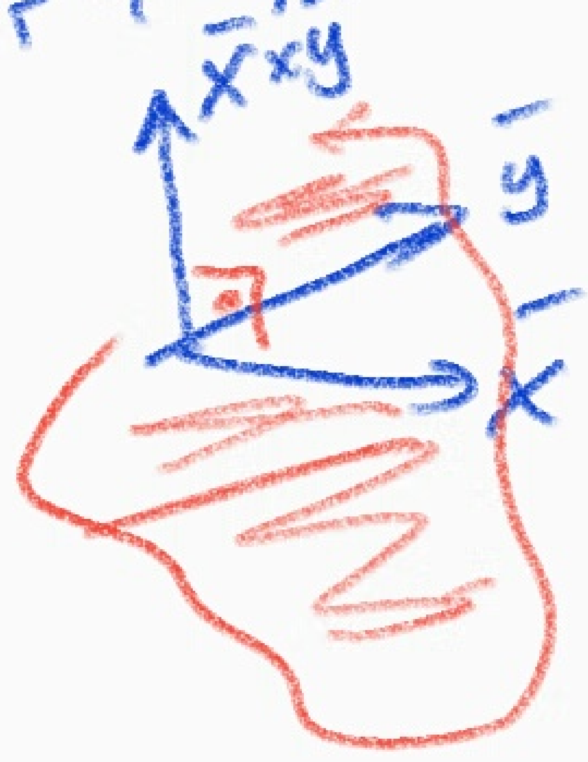
c) $\theta = 0^\circ \Rightarrow \langle \bar{x}, \bar{y} \rangle = \bar{x}^T \cdot \bar{y} = 0 \Rightarrow \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ \alpha \end{bmatrix} = 0$

$1 \cdot (-2) + 2 \cdot 1 + 3 \cdot \alpha = 0$

$\Rightarrow \alpha = 0$

iç çarpım (inner product): $\langle \bar{x}, \bar{y} \rangle = \bar{x}^T \cdot \bar{y} \rightarrow$ skaler

dış çarpım (outer product) $\bar{x} \times \bar{y} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$



$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \rightarrow \text{vektör}$$

$$\|\bar{x}+\bar{y}\|^2 = ? \qquad \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \bar{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad y_i \in C$$

$$\bar{z} = \bar{x}+\bar{y} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix}$$

$$\to = \|\bar{z}\|^2 = \bar{z}^H \cdot \bar{z} = [\bar{x}+\bar{y}]^H [\bar{x}+\bar{y}]$$

$$= [\bar{x}^H+\bar{y}^H][\bar{x}+\bar{y}]$$

$$= \bar{x}^H \cdot \bar{x} + \bar{x}^H \cdot y + \bar{y}^H \cdot \bar{x} + \bar{y}^H \cdot \bar{y}$$

$$= \|\bar{x}\|^2 + 2\langle \bar{x}, \bar{y} \rangle + \|\bar{y}\|^2$$

$$(\|\bar{x}\| + \|\bar{y}\|)^2 = \|\bar{x}\|^2 + 2\|\bar{x}\|\|\bar{y}\| + \|\bar{y}\|^2$$

$$\Rightarrow \boxed{\|\bar{x}+\bar{y}\|^2 \leq (\|\bar{x}\| + \|\bar{y}\|)^2}$$

Cauchy-Schwarz inequality

(triangle ″)

Orthogonal vektörler

$\langle \bar{x}, \bar{y} \rangle = 0$ ise $\bar{x}$ ve $\bar{y}$ vektörü orthogonal (birbirine dik, $\theta = 90°$) vektörlerdir.

$\bar{x} \perp \bar{y}$

↳ orthogonal

Orthonormal vektörler :

$\{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m\}$

$\|\bar{x}_i\| = 1$ ve $\bar{x}_i \perp \bar{x}_j$ $\quad i \neq j$, $\quad i,j = 1,2,\dots,m$

ise verilen vektörler orthonormal vektörlerdir.

$$\langle \bar{x}_i, \bar{x}_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} = \delta_{ij}$$

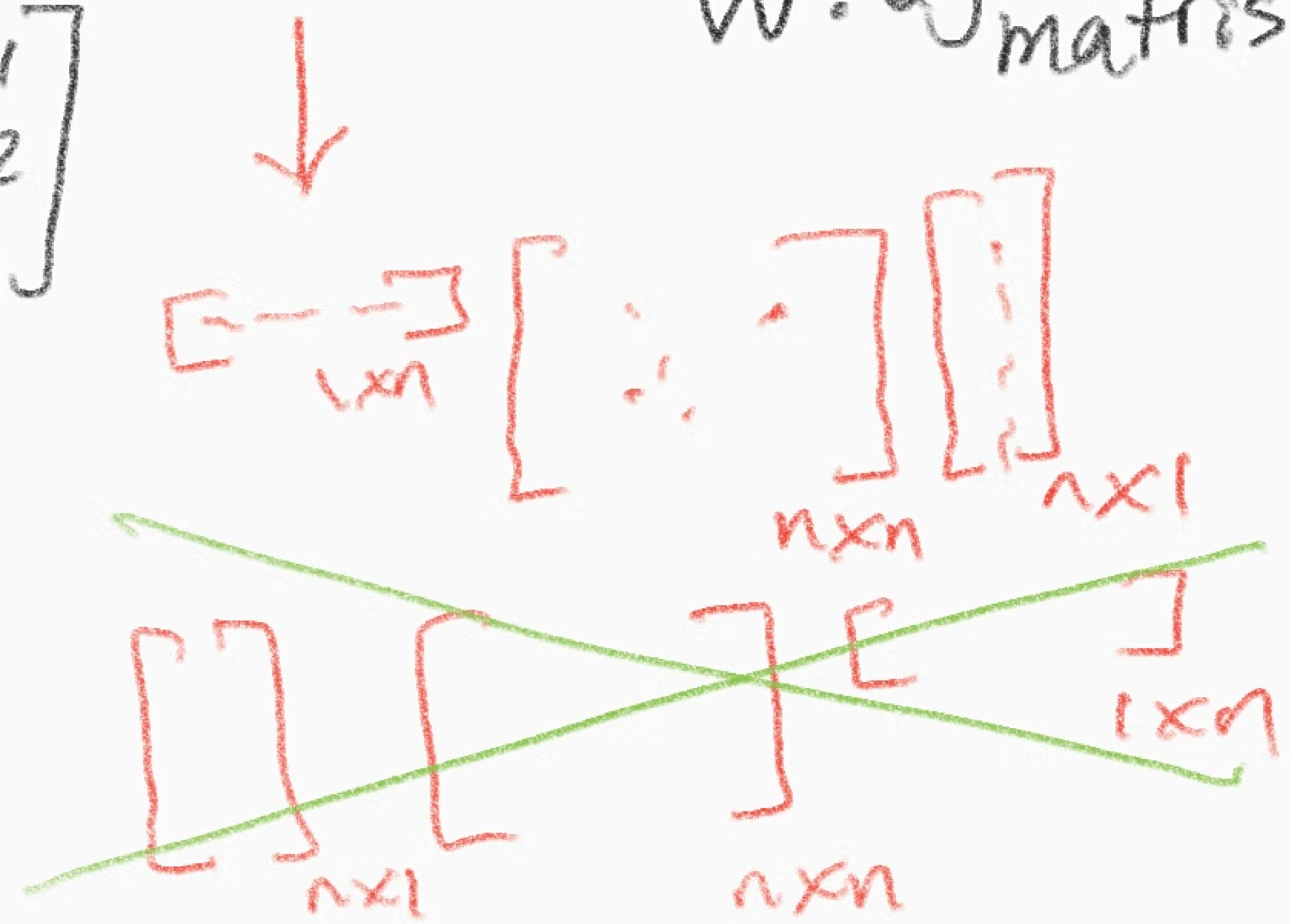
↓ dirac

Ağırlıklı İç Çarpım :

$$\langle \bar{x}, \bar{y} \rangle_W = \bar{x}^H_{1\times n} \, W_{n\times n} \, \bar{y}_{n\times 1} = \bar{y}^H W \bar{x}$$

W: ağırlık matrisi

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \bar{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$



örnek:

$$\bar{x} = \begin{bmatrix}1\\1\end{bmatrix} \qquad \bar{y} = \begin{bmatrix}2\\1\end{bmatrix}$$

$$\langle \bar{x}, \bar{y} \rangle = [1 \ \ 1]\begin{bmatrix}2\\1\end{bmatrix} = 3 \qquad \bar{x} \cancel{\perp} \bar{y}$$

$\bar{x}$, $\bar{y}$'ye dik değil.

problem tanımı:
öyle bir W matrisi bulalım ki

$$\langle \bar{x}, \bar{y} \rangle_W = 0 \qquad W = ?$$

$$\bar{x}^T \cdot W \cdot \bar{y} = 0$$

$$[1 \ \ 1]\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = 0$$

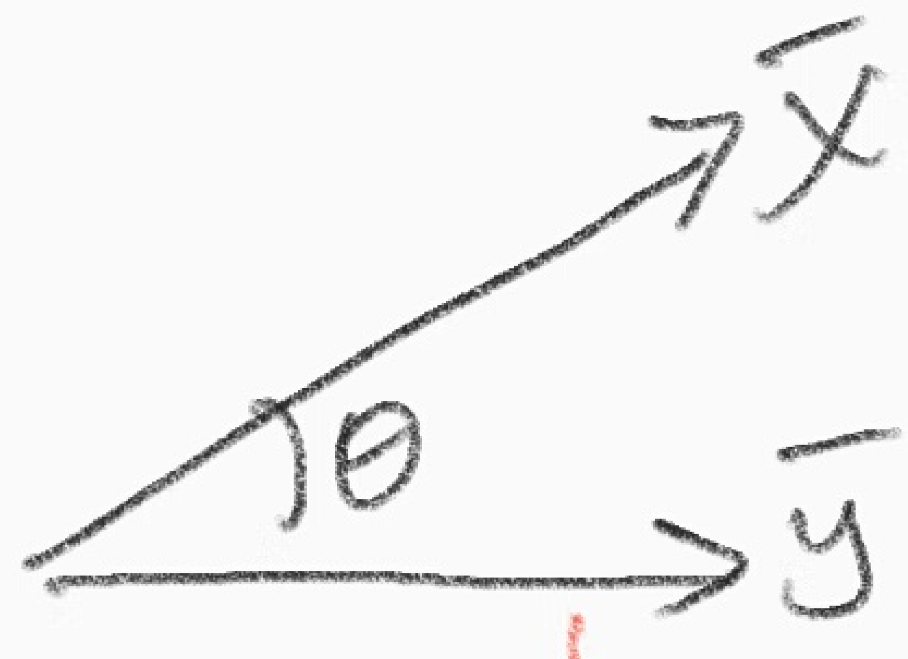
Matlab exercise:
Bu problemdeki W matrisi hesaplayınız?

$$[1 \ \ 1]\begin{bmatrix}2a+b\\2c+d\end{bmatrix} = 1.(2a+b) + 1.(2c+d) = 0$$

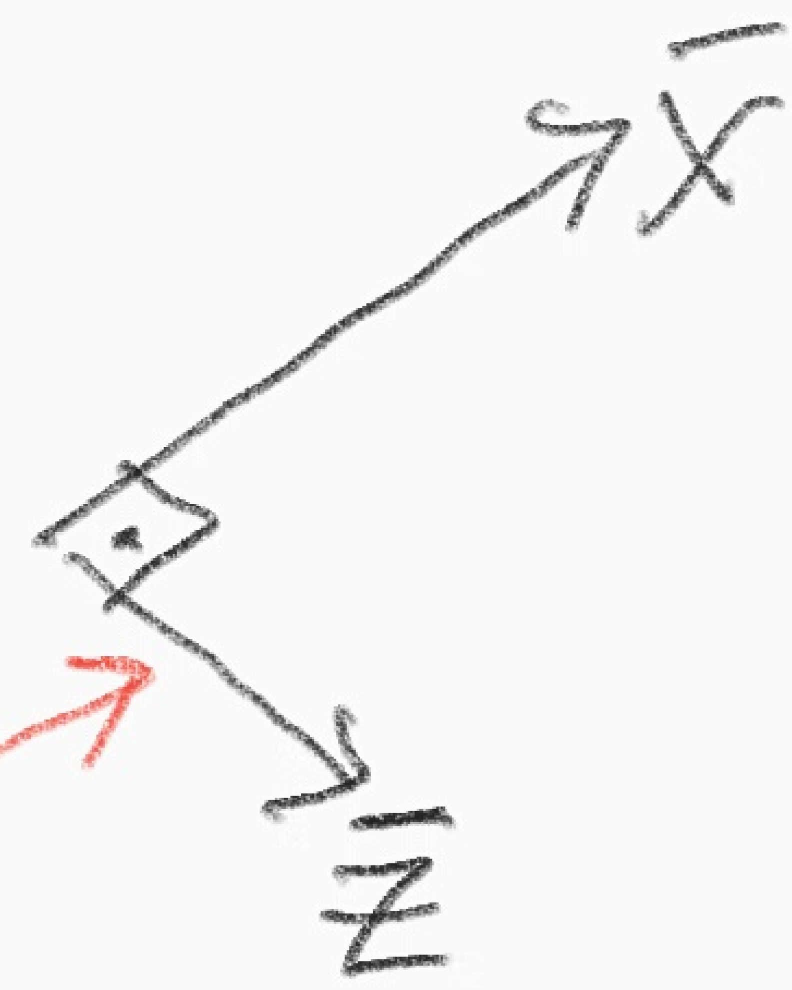
= - - - - - -

Bulmak zor!

$$W = \begin{bmatrix}\alpha & -\alpha\\ -\alpha & \alpha\end{bmatrix}_{2\times2} \qquad \alpha \in R$$



$\langle \bar{x}, \bar{y} \rangle \neq 0$



$\langle \bar{x}, \bar{z} \rangle = 0$

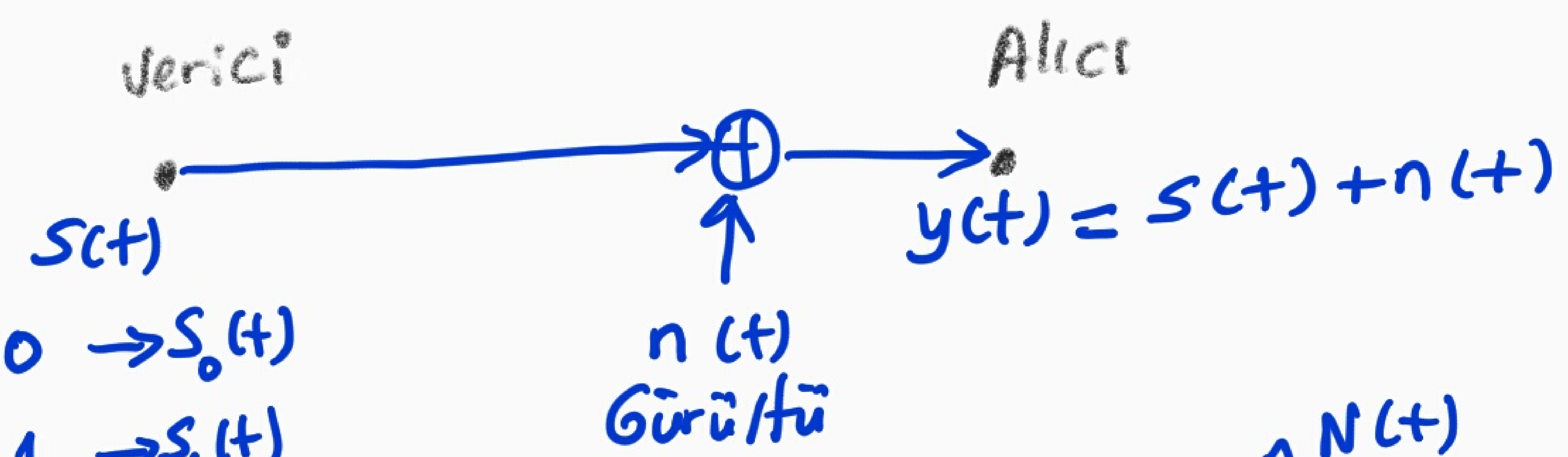
$$\bar{z} = \bar{\mathbf{W}}\bar{y}$$

↓ matris (transformasyon)

$$\bar{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

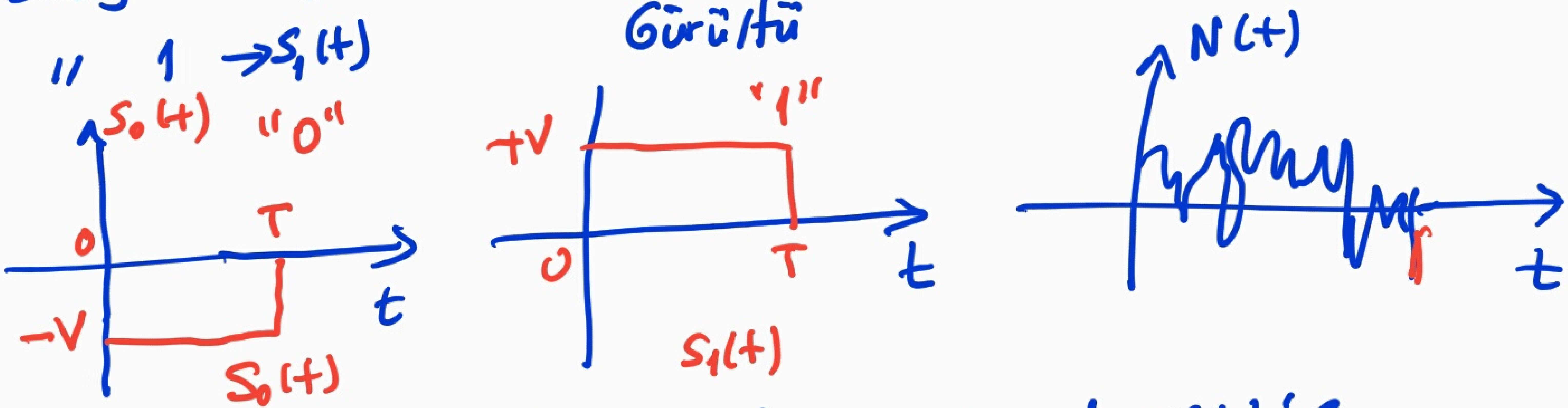
$$\bar{z} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Örnek : Sayısal Haberleşme



Binary 0 $\rightarrow s_0(t)$

" 1 $\rightarrow s_1(t)$



problem tanımı: alıcıda alınan sinyal y(t)'e bakarak gönderilen sinyali tahmin etmek.
"Maximum Olabilirlik Alıcısı"

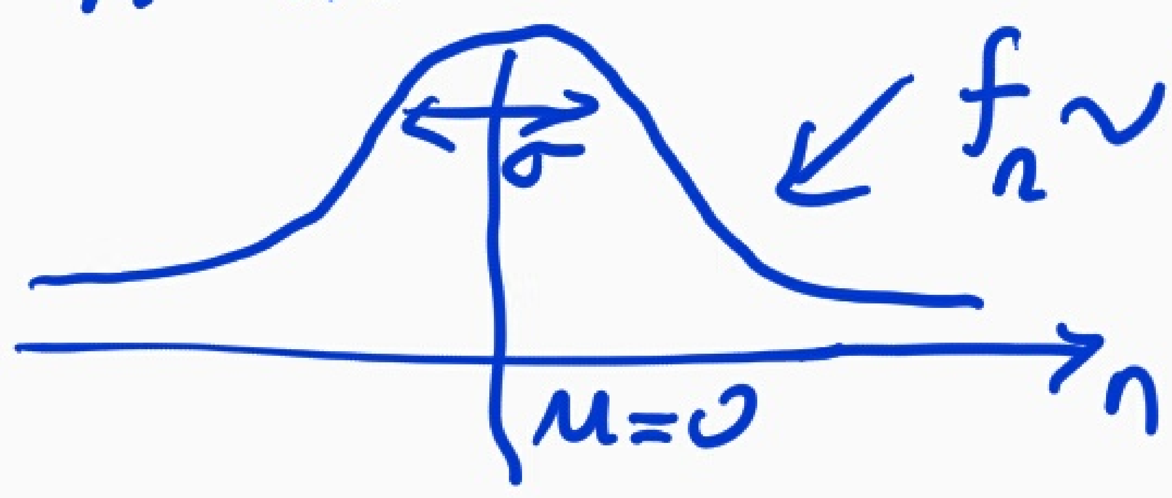
Çözüm : sinyalleri vektörel olarak gösterdiğimizi varsayalım
binary 0 $\Rightarrow \bar{s}_0$, binary 1 $\Rightarrow \bar{s}_1$ olsun.

Bu iki sinyalin gönderilme olasılıkları $P(s_0) = P(s_1) = \frac{1}{2}$

$$\bar{y} = \bar{s}_i + \bar{n} \qquad i = 0, 1$$

$\bar{n}$: Beyaz Gauss Gürültüsü
(white Gaussian Noise)

Gauss : $\bar{n}$ normal olasılık dağılımına sahip



$f_n \sim N(0,1)$ $\quad \underline{\sigma = 1}$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{n^2}{2\sigma^2}}$$

y sinyalinin olabilirlik fonksiyonu $\Rightarrow$

$$f(\bar{y}\,|\,\bar{s}) = \frac{1}{(2\pi)^{\frac{n}{2}}\,|R|^{\frac{1}{2}}}\, e^{\left\{-\frac{1}{2}\underbrace{(\bar{y}-\bar{s})^T}_{1\times n}\underbrace{R^{-1}}_{n\times n}\underbrace{(\bar{y}-\bar{s})}_{n\times 1}\right\}}$$

$\underbrace{\qquad}_{1\times 1}$

koşullu olasılık dağılımı

s verildiğinde y'nin olasılığı

$y_i = s_i + n_i$

$$f(y_i|s_i) = \frac{1}{\sqrt{2\pi\sigma_i^2}}\, e^{-\frac{(y_i-s_i)^2}{2\sigma_i^2}}$$

(skaler)

$i = 0, 1$

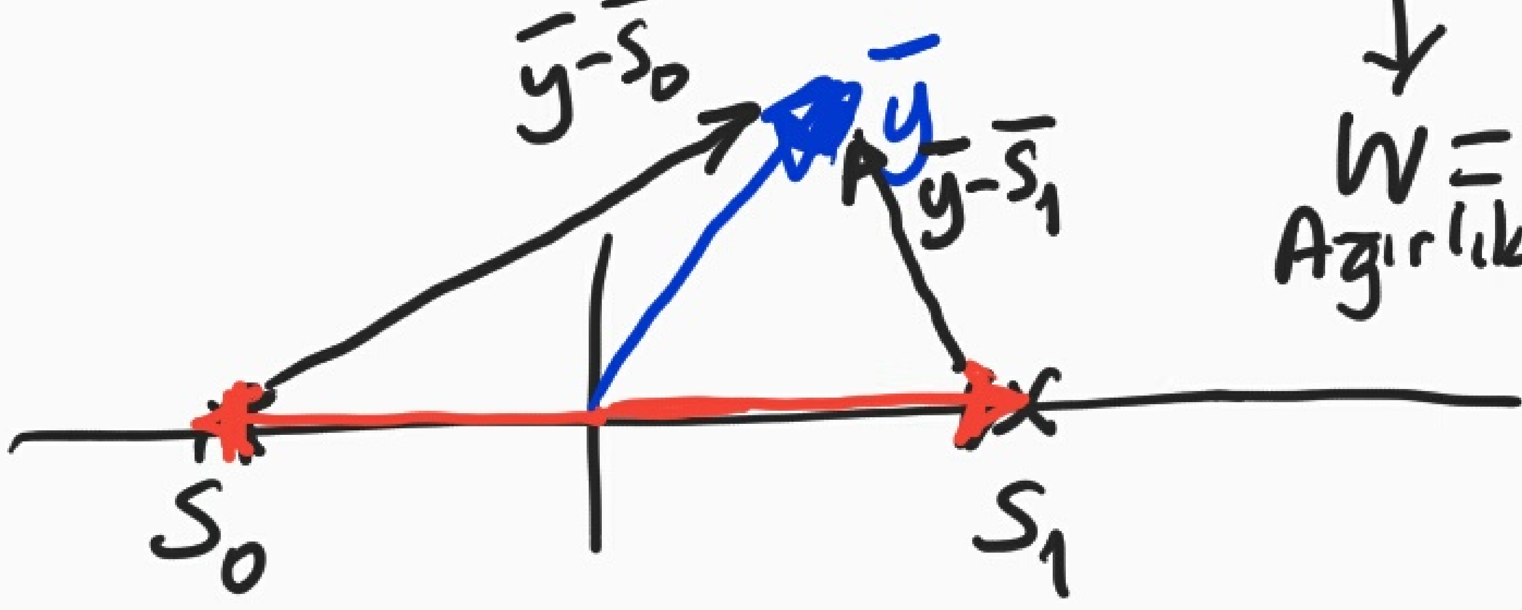
Kural:

Kural 1 $\Rightarrow L_0 = (\bar{y}-\bar{s}_0)^T R^{-1} (\bar{y}-\bar{s}_0)$

Kural 2 $\Rightarrow L_1 = (\bar{y}-\bar{s}_1)^T R^{-1} (\bar{y}-\bar{s}_1)$

$W = R^{-1}$

Ağırlıklı iç çarpım



$L_1 > L_0$

$\Rightarrow$ Gönderilen sinyal s_0

$L_1 < L_0$

$\Rightarrow$ Gönderilen sinyal s_1

$$\|\bar{y}-\bar{s}_i\|^2 = (\bar{y}-\bar{s}_i)^T(\bar{y}-\bar{s}_i) \quad i=0,1$$

"Maximum olabilirlik (benzerlik)

$\Rightarrow$ Minimum uzaklık