Problem set 1:

Problem 1:

How many ways can 10 people be seated in a row if
a. there are no restrictions on the seating arrangement?
b. persons A and B must sit next to each other?
c. there are 5 adults and 5 children, and no two adults nor two children can sit next to each other?
d. there are 5 married couples and each couple must sit together?

Solution:

a. $10 * 9 * ... * 1 = 10!$, since there is no restriction and each person is distinct.
b. We first treat the person A and B to be just one person then just like part a we take factorial of the amount of people which is $9!$ then multiply it with 2 since A and B can switch seats with each other. So the answer is $9! * 2$.
c. There is two scenarios for this, first one is that in the first seat a child sits and in the other scenario the adult sits. In either of the scenarios the probability is same. Since each scenario has to follow the pattern A,B,A,... the answer is: $5! * 5! * 2$.
d. Just like part b we consider each couple as one person then, $5!$ but since the couples can switch seats between themselves the answer is: $5! * 2^5$.

Problem 2:

At the local zoo, a new exhibit consisting of 3 different species of birds and 3 different species of reptiles is to be formed from a pool of 8 bird species and 6 reptile species. How many exhibits are possible if
a. there are no additional restrictions on which species can be selected?
b. 2 particular bird species cannot be placed together (e.g., they have a predator-prey relationship)?
c. 1 particular bird species and 1 particular reptile species cannot be placed together?

Solution:

a. $\binom{8}{5} * \binom{6}{3}$, since there is no restriction and each animal is distinct.
b. $\binom{6}{1} * \binom{6}{3}$, is the possible amounts of exhibits when we are taking the 2 particular bird species that can't be placed together. From here we just subtract the amount from the total amount of possible exhibits: $[\binom{8}{5} * \binom{6}{3}] - [\binom{6}{1} * \binom{6}{3}]$.
c. Just like part b: $[\binom{8}{5} * \binom{6}{3}] - [\binom{7}{2} * \binom{5}{2}]$.

Problem 3:

Given all the start-up activity going on in high-tech, you realize that applying combinatorics to investment strategies might be an interesting idea to pursue. Say you have $20 million that must be invested among 4 possible companies. Each investment must be in integral units of $1 million. How many different investment strategies are available if

a. an investment must be made in each company, and we must invest all $20 million? Assume we have a minimal investment requirement per company, where we must invest a minimal investment of $1, $2, $3, and $4 million dollars for company 1, 2, 3, and 4, respectively.

b. investments must be made in at least 3 of the 4 companies, and we must invest all $20 million? Assume we have the same minimal investment as in part (a), where should we choose to invest in company n (for n = 1, . . ., 4), we must invest a minimal of $n million.

c. we must invest less than or equal to $k million dollars total among the 4 companies, where k is an integer such that 10 ≤ k ≤ 20? Note that you can think of k as a constant that can be used in your answer. Assume in this part that we do not have a minimal investment.

Solution:

a. We have

$10 millionremainingtoinvestaftertheminimalinvestmentrequirementpercompany. Sincethisadi$

, since each divider and each million is indistinct we need to remove the ones we over counted.

b. For this part we have two scenarios, one in which we choose to invest in only three companies and in the other where we invest in all four companies. For the first, we choose 3 companies and place the minimal investment and then find the amount like in part a:

1. minimal investment = 6, possible events = $\binom{16}{2}$
2. minimal investment = 7, possible events = $\binom{15}{2}$
3. minimal investment = 8, possible events = $\binom{14}{2}$
4. minimal investment = 9, possible events = $\binom{13}{2}$

So the possible events for the first scenario is $\binom{16}{2}+\binom{15}{2}+\binom{14}{2}+\binom{13}{2}$. As for the second scenario it is the same as part a so: $\binom{13}{3}$. For the answer we just add them together: $\binom{16}{2}+\binom{15}{2}+\binom{14}{2}+\binom{13}{2}+\binom{13}{3}$.

c. For this we consider t, for every $0 \le t \le k$ as the amount of identical items, the amount of possible events is $\binom{t+3}{3}$ for $t = k$. But since we are asked to find the possible investments for $t \le k$ we need to add all the possible cases: $\sum_{n=0}^{k}\binom{t+3}{3}$ using the binomial identity we get the answer, $\binom{k+4}{4}$.

Problem 4:

Say a university is offering 3 programming classes: one in Java, one in C++, and one in Python. The classes are open to any of the 100 students at the university. There are:

- a total of 27 students in the Java class;
- a total of 28 students in the C++ class;
- a total of 20 students in the Python class;
- 12 students in both the Java and C++ classes (note: these students are also counted as being in each class in the numbers above);
- 5 students in both the Java and Python classes;
- 8 students in both the C++ and Python classes; and
- 2 students in all three classes (note: these students are also counted as being in each pair of classes in the numbers above).
  a. If a student is chosen randomly at the university, what is the probability that the student is not in any of the 3 programming classes?
  b. If a student is chosen randomly at the university, what is the probability that the student is taking exactly one of the three programming classes?
  c. If two different students are chosen randomly at the university, what is the probability that at least one of the chosen students is taking at least one of the programming classes?

Solution:

Total amount of students taking any of the 3 classes: $27+28+20-12-5-8+2=52$.
Total amount of students taking only one of the 3 classes: $12+10+9=31$
a. $\frac{100-52}{100}=0.48$.
b. $\frac{31}{100}=0.31$.
c. $0.48*0.48=0.2304 \rightarrow$ probability of both of the students not going to any of the programming classes. $1-0.2304=0.7696 \rightarrow$ at least one of them going to one programming class.

Problem 5:

If we assume that all possible poker hands (comprised of 5 cards from a standard 52 card deck) are equally likely, what is the probability of being dealt:
a. a flush? (A hand is said to be a flush if all 5 cards are of the same suit. Note that this definition means that straight flushes (five cards of the same suit in numeric sequence) are also considered flushes.)
b. two pairs? (This occurs when the cards have numeric values a, a, b, b, c, where a, b and c are all distinct.)
c. four of a kind? (This occurs when the cards have numeric values a, a, a, a, b, where a and b are distinct.)

Solution:

a. Amount of flushes is: $4 * \frac{\binom{13}{5}}{\binom{52}{5}}$.

b. $\frac{\binom{13}{2}*\binom{4}{2}*\binom{4}{2}*\binom{11}{1}*\binom{4}{1}}{\binom{52}{5}}$.

c. $\frac{\binom{13}{1}*\binom{4}{4}*\binom{12}{1}*\binom{4}{1}}{\binom{52}{5}}$.

Problem 6:

Say we roll a six-sided die six times. What is the probability that

a. we will roll two different numbers thrice (three times) each?

b. we will roll exactly one number exactly three times? Hint: Be careful of over counting.

Solution:

a. $\frac{\binom{6}{2}*\binom{6}{3}}{6^6}$.

b. $\frac{\binom{6}{1}*\binom{6}{3}*\frac{5!}{2!}}{6^6}$.

Problem 7:

Say we send out a total of 20 distinguishable emails to 12 distinct users, where each email we send is equally likely to go to any of the 12 users (note that it is possible that some users may not actually receive any email from us). What is the probability that the 20 emails are distributed such that there are 4 users who receive exactly 2 emails each from us and 3 users who receive exactly 4 emails each from us?

Solution:

$$\frac{\binom{12}{4}*\binom{8}{4}*\binom{20}{2}*\binom{18}{2}*\binom{16}{2}*\binom{14}{2}*\binom{12}{4}*\binom{8}{4}*\binom{4}{4}}{12^{20}}.$$

Problem 8:

Say a hacker has a list of n distinct password candidates, only one of which will successfully log her into a secure system.

a. If she tries passwords from the list at random, deleting those passwords that do not work, what is the probability that her first successful login will be (exactly) on her k-th try?

b. Now say the hacker tries passwords from the list at random, but does not delete previously tried passwords from the list. She stops after her first successful login attempt. What is the probability that her first successful login will be (exactly) on her k-th try?

Solution:

a. $\frac{n-1}{n}$, is the probability of choosing an incorrect password in the first try; $\frac{n-2}{n-1}$, is the probability of choosing an incorrect password in the second try and so on. Choosing the wrong password until the k-th time is: $\Pi_{t=0}^{k-2} \frac{n-(t+1)}{n-(t)} = \frac{n-(k-1)}{n}$ and choosing the correct password on the k-th time is: $\frac{1}{n-(k-1)}$. So the answer is $\frac{1}{n-(k-1)} * \frac{n-(k-1)}{n} = \frac{1}{n}$.
b. $\frac{n-1}{n}$, is the probability of choosing an incorrect password in the first try; $\frac{n-1}{n}$, is the probability of choosing an incorrect password in the second try and so on. Choosing the wrong password until the k-th time is: $\Pi_{t=0}^{k-2} \frac{n-1}{n} = \frac{(n-1)^{k-1}}{n^{k-1}}$ and choosing the correct password on the k-th time is: $\frac{1}{n}$. So the answer is $\frac{(n-1)^{k-1}}{n^{k-1}} * \frac{1}{n} = \frac{(n-1)^{k-1}}{n^k}$.

Problem 9:

Suppose that m strings are hashed (randomly) into N buckets, assuming that all N^m arrangements are equally likely. Find the probability that exactly k strings are hashed to the first bucket.

Solution:

$$\frac{\binom{m}{k} * (N-1)^{m-k}}{N^m}.$$

Problem 10:

To get good performance when working binary search trees (BST), we must consider the probability of producing completely degenerate BSTs (where each node in the BST has at most one child). See Lecture Notes # 2, Example 3, for more details on binary search trees.
a. If the integers 1 through n are inserted in arbitrary order into a BST (where each possible order is equally likely), what is the probability (as an expression in terms of n) that the resulting BST will have completely degenerate structure?
b. Using your expression from part (a), determine the smallest value of n for which the probability of forming a completely degenerate BST is less than 0.001 (i.e., 0.1%).

Solution:

a. There is two possible ways to make degenerate binary search trees: first one it goes in the ascending order, and the second one where it goes in the descending order. So the answer is: $\frac{2}{n!}$.

b. We want $\frac{2}{n!} \leq 0.001$, so: $2000 \leq n!, \quad \text{smallest: } n = 7$.