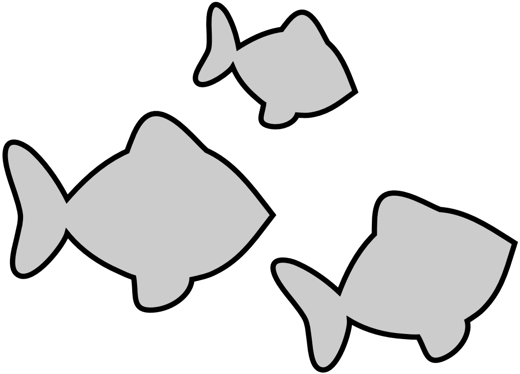
Introduction to Shape Manifolds

Geometry of Data

September 24, 2020

What is Shape?

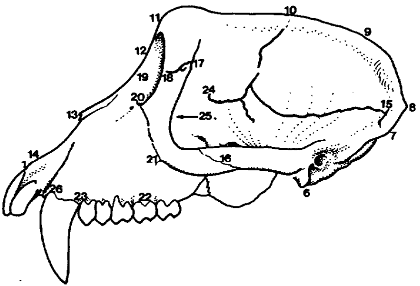


Shape is the geometry of an object modulo position, orientation, and size.

Geometry Representations

- Landmarks (key identifiable points)
- Boundary models (points, curves, surfaces, level sets)
- Interior models (medial, solid mesh)
- Transformation models (splines, diffeomorphisms)

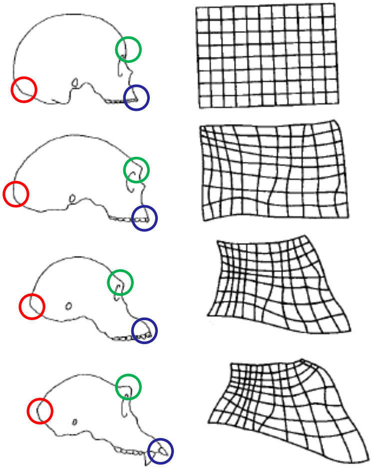
Landmarks



From Dryden & Mardia, 1998

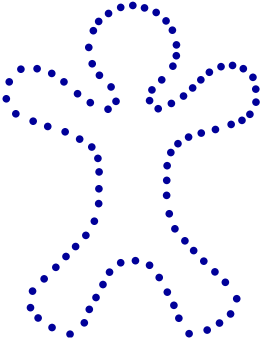
- A **landmark** is an identifiable point on an object that corresponds to matching points on similar objects.
- This may be chosen based on the application (e.g., by anatomy) or mathematically (e.g., by curvature).

Landmark Correspondence

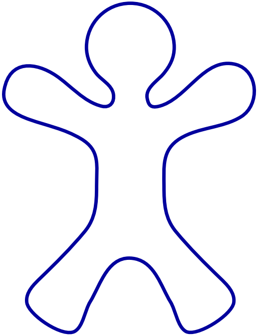


From C. Small, *The Statistical Theory of Shape*

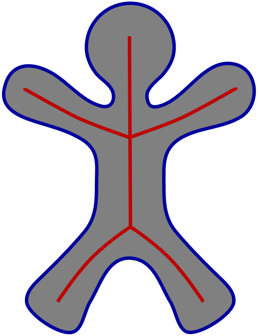
More Geometry Representations



Dense Boundary
Points

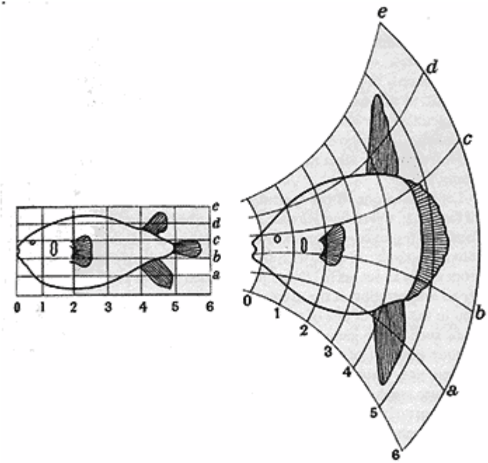


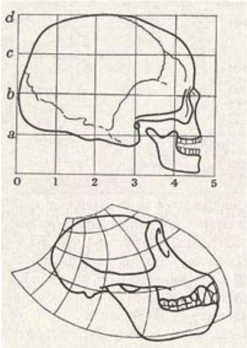
Continuous Boundary
(Fourier, splines)



Medial Axis
(solid interior)

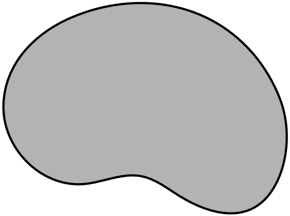
Transformation Models





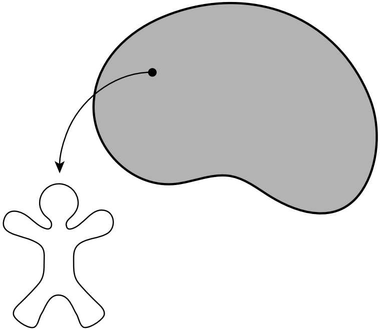
From D'Arcy Thompson, *On Growth and Form*, 1917.

Shape Spaces



A shape is a point in a high-dimensional, nonlinear manifold, called a **shape space**.

Shape Spaces



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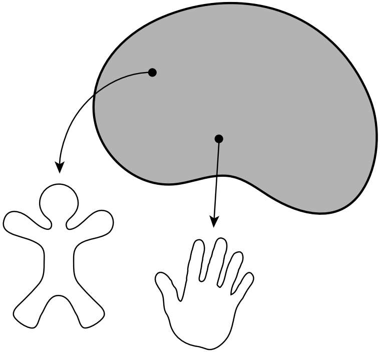

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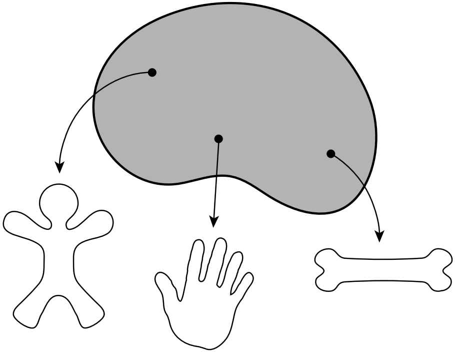

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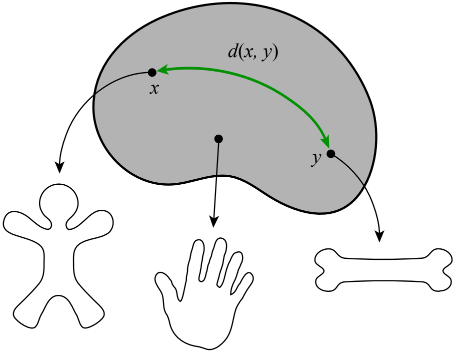
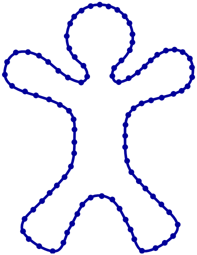


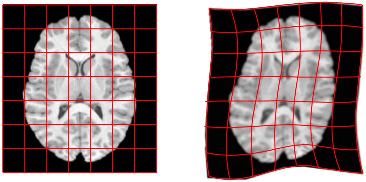
A metric space structure provides a comparison between two shapes.

Examples: Shape Spaces

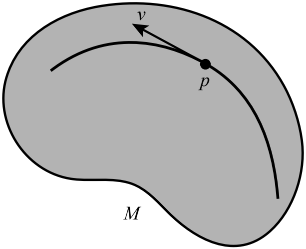
Kendall's Shape Space



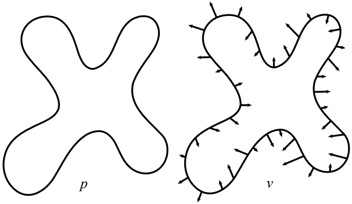
Space of Diffeomorphisms



Tangent Spaces



Infinitesimal change in shape:



A **tangent vector** is the velocity of a curve on M.

Shape Equivalences

Two geometry representations, x_1, x_2, are **equivalent** if they are just a translation, rotation, scaling of each other:

$$x_2 = \lambda R \cdot x_1 + v,$$

where λ is a scaling, R is a rotation, and v is a translation.

In notation: $x_1 \sim x_2$

Equivalence Classes

The relationship $x_1 \sim x_2$ is an **equivalence relationship**:

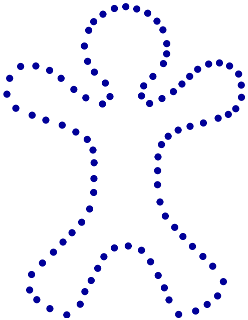
- Reflexive: $x_1 \sim x_1$
- Symmetric: $x_1 \sim x_2$ implies $x_2 \sim x_1$
- Transitive: $x_1 \sim x_2$ and $x_2 \sim x_3$ imply $x_1 \sim x_3$

We call the set of all equivalent geometries to x the **equivalence class** of x:

$$[x] = \{y \,:\, y \sim x\}$$

he set of all equivalence classes is our **shape space**.

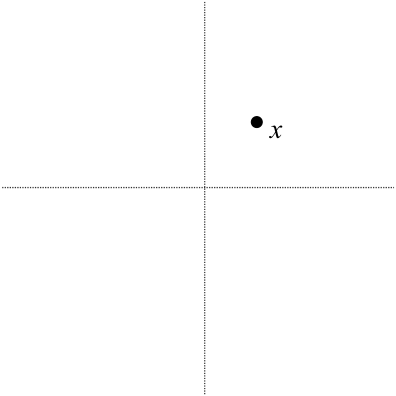
Kendall's Shape Space



- Define object with k points.
- Represent as a vector in $\mathbb{R}^{2k}$.
- Remove translation, rotation, and scale.
- End up with complex projective space, $\mathbb{CP}^{k-2}$.

Quotient Spaces

What do we get when we "remove" scaling from $\mathbb{R}^2$?



Notation: $[x] \in \mathbb{R}^2/\mathbb{R}^+$

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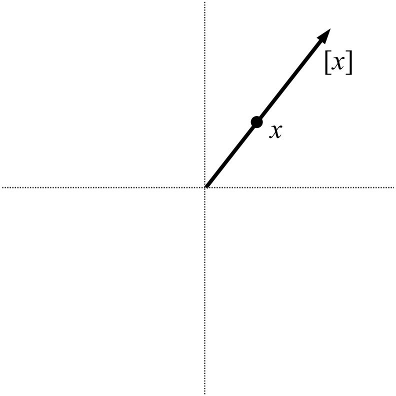

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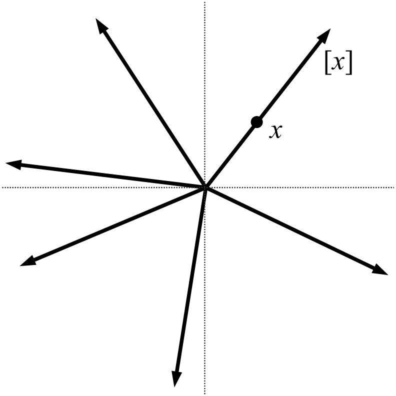

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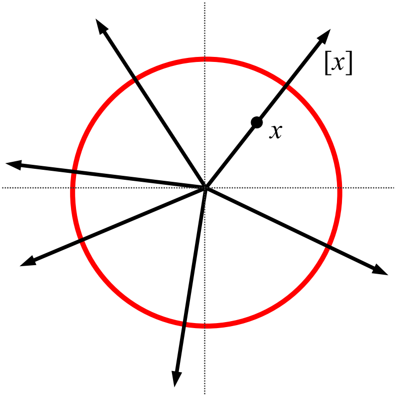

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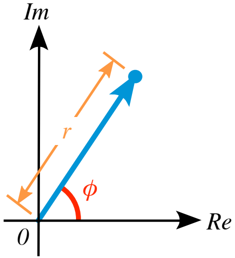
Constructing Kendall's Shape Space

- Consider planar landmarks to be points in the complex plane.
- An object is then a point $(z_1, z_2, \ldots, z_k) \in \mathbb{C}^k$.
- Removing **translation** leaves us with $\mathbb{C}^{k-1}$.
- How to remove **scaling** and **rotation**?

Scaling and Rotation in the Complex Plane



Recall a complex number can be written as $z = re^{i\phi}$, with modulus r and argument ϕ.

Complex Multiplication:

$$se^{i\theta} * re^{i\phi} = (sr)e^{i(\theta+\phi)}$$

Multiplication by a complex number $se^{i\theta}$ is equivalent to scaling by s and rotation by θ.

Removing Scale and Rotation

Multiplying a centered point set, $\mathbf{z} = (z_1, z_2, \ldots, z_{k-1})$, by a constant $w \in \mathbb{C}$, just rotates and scales it.

Thus the shape of $\mathbf{z}$ is an equivalence class:

$$[\mathbf{z}] = \{(wz_1, wz_2, \ldots, wz_{k-1}) : \forall w \in \mathbb{C}\}$$

This gives complex projective space $\mathbb{CP}^{k-2}$ – much like the sphere comes from equivalence classes of scalar multiplication in $\mathbb{R}^n$.

Alternative: Shape Matrices

Represent an object as a real $d \times k$ matrix.

Preshape process:

- Remove translation: subtract the row means from each row (i.e., translate shape centroid to 0).
- Remove scale: divide by the Frobenius norm.

Orthogonal Procrustes Analysis

Problem:

Find the rotation R^* that minimizes distance between two $d \times k$ matrices A, B:

$$R^* = \arg\min_{R \in \mathrm{SO}(d)} \|RA - B\|^2$$

Solution:

Let $U\Sigma V^T$ be the SVD of BA^T, then

$$R^* = UV^T$$

Geodesics in 2D Kendall Shape Space

Let A and B be $2 \times k$ shape matrices

1. Remove centroids from A and B
2. Project onto sphere: $A \leftarrow A/\|A\|$, $B \leftarrow B/\|B\|$
3. Align rotation of B to A with OPA
4. Now a geodesic is simply that of the sphere, S^{2k-1}

Where to Learn More

Books

- Dryden and Mardia, *Statistical Shape Analysis*, Wiley, 1998.
- Small, *The Statistical Theory of Shape*, Springer-Verlag, 1996.
- Kendall, Barden and Carne, *Shape and Shape Theory*, Wiley, 1999.
- Krim and Yezzi, *Statistics and Analysis of Shapes*, Birkhauser, 2006.