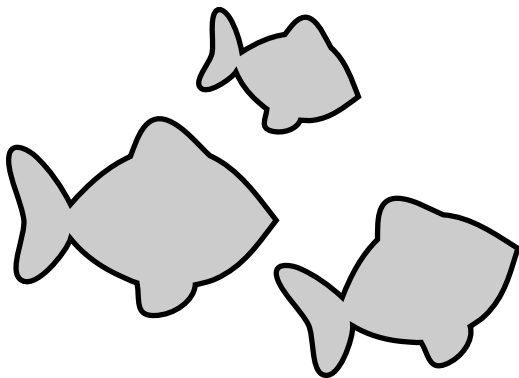


# Introduction to Shape Manifolds

Geometry of Data

September 28, 2021

# What is Shape?

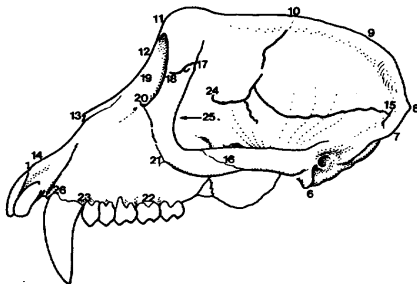


Shape is the geometry of an object modulo position, orientation, and size.

# Geometry Representations

- ▶ Landmarks (key identifiable points)
- ▶ Boundary models (points, curves, surfaces, level sets)
- ▶ Interior models (medial, solid mesh)
- ▶ Transformation models (splines, diffeomorphisms)

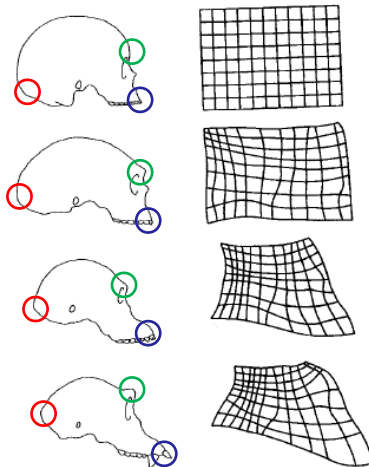
# Landmarks



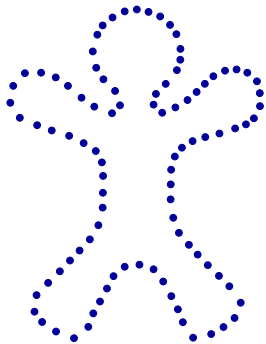
From Dryden & Mardia, 1998

- ▶ A **landmark** is an identifiable point on an object that corresponds to matching points on similar objects.
- ▶ This may be chosen based on the application (e.g., by anatomy) or mathematically (e.g., by curvature).

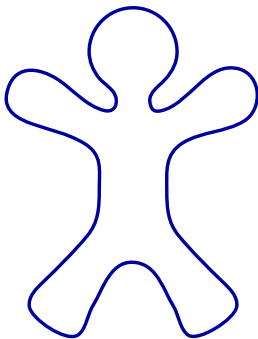
# Landmark Correspondence



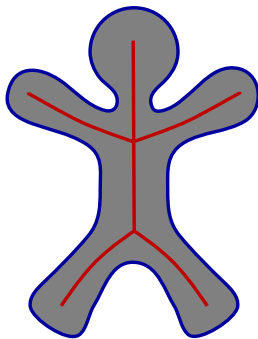
# More Geometry Representations



Dense Boundary  
Points

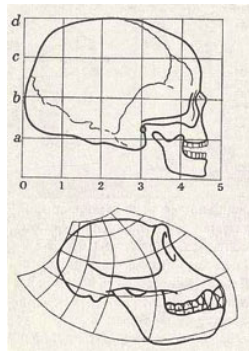
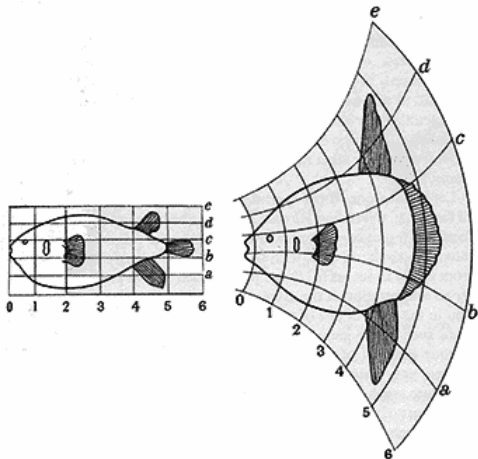


Continuous Boundary  
(Fourier, splines)



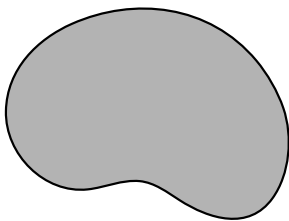
Medial Axis  
(solid interior)

# Transformation Models



From D'Arcy Thompson, *On Growth and Form*, 1917.

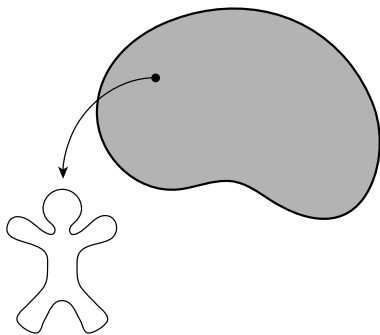
# Shape Spaces



A shape is a point in a high-dimensional, nonlinear manifold, called a **shape space**.

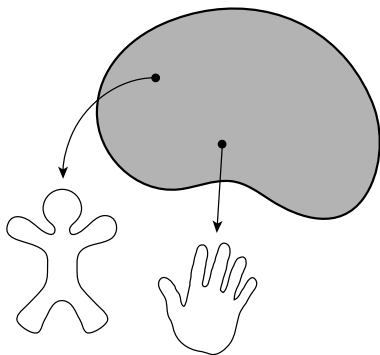


# Shape Spaces



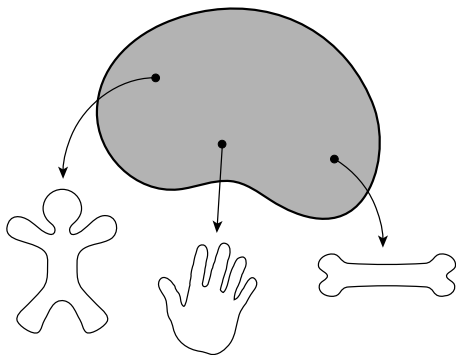
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# Shape Spaces



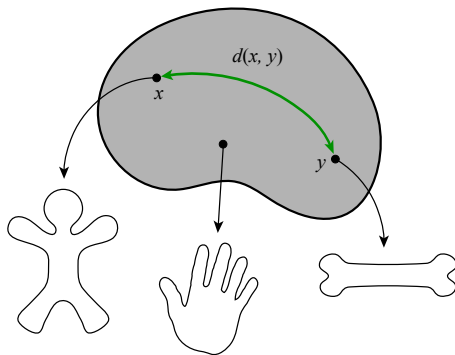
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# Shape Spaces



A shape is a point in a high-dimensional, nonlinear manifold, called a **shape space**.

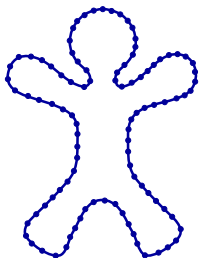
# Shape Spaces



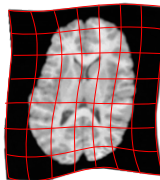
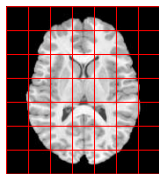
A metric space structure provides a comparison between two shapes.

# Examples: Shape Spaces

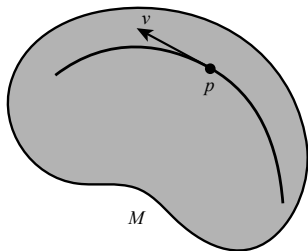
**Kendall's Shape Space**



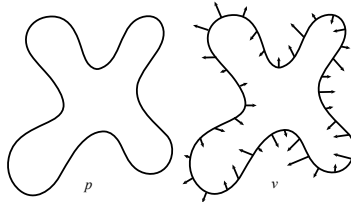
**Space of  
Diffeomorphisms**



# Tangent Spaces



Infinitesimal change in shape:



A **tangent vector** is the velocity of a curve on  $M$ .

# Shape Equivalences

Two geometry representations,  $x_1, x_2$ , are **equivalent** if they are just a translation, rotation, scaling of each other:

$$x_2 = \lambda R \cdot x_1 + v,$$

where  $\lambda$  is a scaling,  $R$  is a rotation, and  $v$  is a translation.

In notation:  $x_1 \sim x_2$

# Equivalence Classes

The relationship  $x_1 \sim x_2$  is an **equivalence relationship**:

- ▶ Reflexive:  $x_1 \sim x_1$
- ▶ Symmetric:  $x_1 \sim x_2$  implies  $x_2 \sim x_1$
- ▶ Transitive:  $x_1 \sim x_2$  and  $x_2 \sim x_3$  imply  $x_1 \sim x_3$

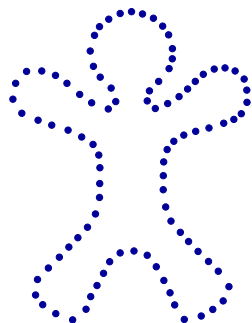
We call the set of all equivalent geometries to  $x$  the **equivalence class** of  $x$ :

$$[x] = \{y : y \sim x\}$$

The set of all equivalence classes is our **shape space**.



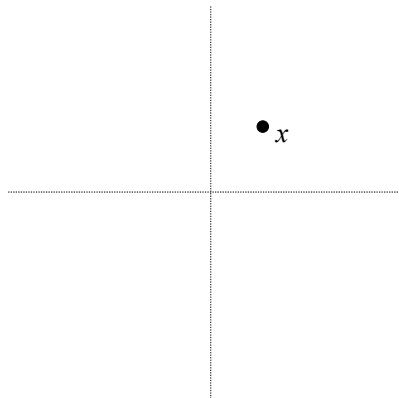
# Kendall's Shape Space



- ▶ Define object with  $k$  points.
- ▶ Represent as a vector in  $\mathbb{R}^{2k}$ .
- ▶ Remove translation, rotation, and scale.
- ▶ End up with complex projective space,  $\mathbb{CP}^{k-2}$ .

# Quotient Spaces

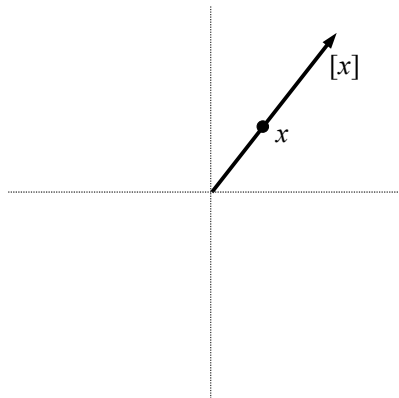
What do we get when we “remove” scaling from  $\mathbb{R}^2$ ?



Notation:  $[x] \in \mathbb{R}^2 / \mathbb{R}^+$

# Quotient Spaces

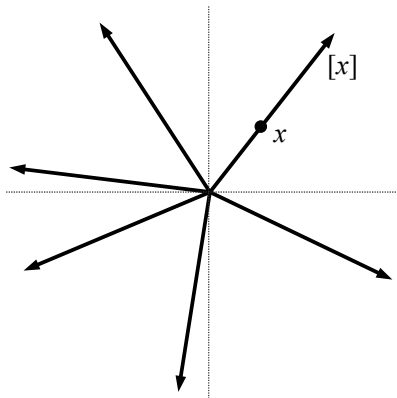
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# Quotient Spaces

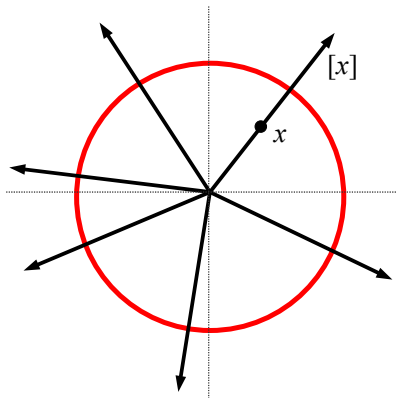
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# Quotient Spaces

What do we get when we “remove” scaling from  $\mathbb{R}^2$ ?

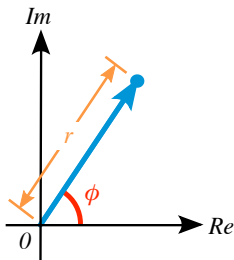


Notation:  $[x] \in \mathbb{R}^2 / \mathbb{R}^+$

# Constructing Kendall's Shape Space

- ▶ Consider planar landmarks to be points in the complex plane.
- ▶ An object is then a point  $(z_1, z_2, \dots, z_k) \in \mathbb{C}^k$ .
- ▶ Removing **translation** leaves us with  $\mathbb{C}^{k-1}$ .
- ▶ How to remove **scaling** and **rotation**?

# Scaling and Rotation in the Complex Plane



Recall a complex number can be written as  $z = re^{i\phi}$ , with modulus  $r$  and argument  $\phi$ .

Complex Multiplication:

$$se^{i\theta} * re^{i\phi} = (sr)e^{i(\theta+\phi)}$$

Multiplication by a complex number  $se^{i\theta}$  is equivalent to scaling by  $s$  and rotation by  $\theta$ .

# Removing Scale and Rotation

Multiplying a centered point set,  $\mathbf{z} = (z_1, z_2, \dots, z_{k-1})$ , by a constant  $w \in \mathbb{C}$ , just rotates and scales it.

Thus the shape of  $\mathbf{z}$  is an equivalence class:

$$[\mathbf{z}] = \{(wz_1, wz_2, \dots, wz_{k-1}) : \forall w \in \mathbb{C}\}$$

This gives complex projective space  $\mathbb{CP}^{k-2}$  – much like the sphere comes from equivalence classes of scalar multiplication in  $\mathbb{R}^n$ .



# Alternative: Shape Matrices

Represent an object as a real  $d \times k$  matrix.

## **Preshape process:**

- ▶ Remove translation: subtract the row means from each row (i.e., translate shape centroid to 0).
- ▶ Remove scale: divide by the Frobenius norm.

# Orthogonal Procrustes Analysis

## Problem:

Find the rotation  $R^*$  that minimizes distance between two  $d \times k$  matrices  $A, B$ :

$$R^* = \arg \min_{R \in \text{SO}(d)} \|RA - B\|^2$$

## Solution:

Let  $U\Sigma V^T$  be the SVD of  $BA^T$ , then

$$R^* = UV^T$$

# Geodesics in 2D Kendall Shape Space

Let  $A$  and  $B$  be  $2 \times k$  shape matrices

1. Remove centroids from  $A$  and  $B$
2. Project onto sphere:  $A \leftarrow A/\|A\|$ ,  $B \leftarrow B/\|B\|$
3. Align rotation of  $B$  to  $A$  with OPA
4. Now a geodesic is simply that of the sphere,  $S^{2k-1}$

# Where to Learn More

## Books

- ▶ Dryden and Mardia, *Statistical Shape Analysis*, Wiley, 1998.
- ▶ Small, *The Statistical Theory of Shape*, Springer-Verlag, 1996.
- ▶ Kendall, Barden and Carne, *Shape and Shape Theory*, Wiley, 1999.
- ▶ Krim and Yezzi, *Statistics and Analysis of Shapes*, Birkhauser, 2006.