Variational Autoencoders

Geometry of Data

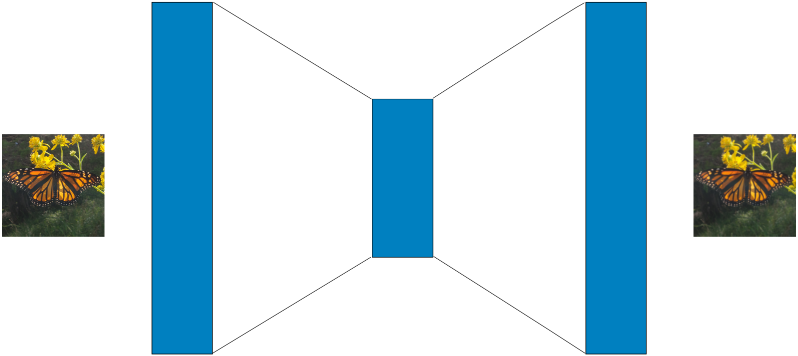
October 29, 2019

Talking about this paper:

Diederik Kingma and Max Welling, Auto-Encoding Variational Bayes, In *International Conference on Learning Representation (ICLR)*, 2014.

Autoencoders

Input Latent Space Output



$x \in \mathbb{R}^D$ $z \in \mathbb{R}^d$ $x' \in \mathbb{R}^D$

$$d << D$$

Autoencoders

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    1. Given data x, feedforward to x' output
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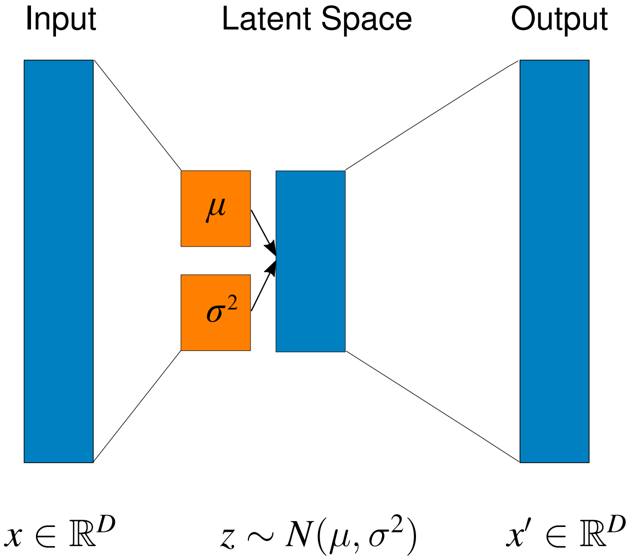
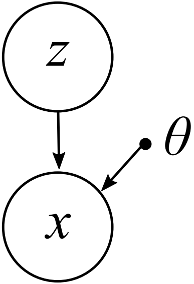
- **Not** a generative model!

Variational Autoencoders



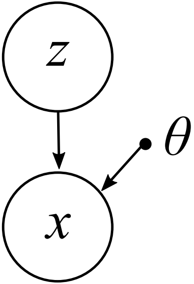
Generative Models



Sample a new x in two steps:

Prior:	$p(z)$
Generator:	$p_\theta(x \mid z)$

Generative Models



Sample a new x in two steps:

Prior: $p(z)$

Generator: $p_\theta(x \mid z)$

Now the analogy to the "encoder" is:

Posterior: $p(z \mid x)$

Bayesian Inference

Posterior via Bayes' Rule:

$$
\begin{aligned}
p(z \mid x) &= \frac{p_\theta(x \mid z)p(z)}{p(x)} \\
&= \frac{p_\theta(x \mid z)p(z)}{\int p_\theta(x \mid z)p(z)dz}
\end{aligned}
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Integral in denominator is (usually) intractable!

Could use Monte Carlo to approximate, but it's expensive

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Kullback-Leibler Divergence

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D_{\mathrm{KL}}(q \| p) &= -\int q(z) \log\left(\frac{p(z)}{q(z)}\right) dz \\
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The average *information gained* from moving from q to p

Variational Inference

Approximate intractable posterior $p(z \mid x)$ with a manageable distribution $q(z)$

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Minimize the KL divergence: $D_{\mathrm{KL}}(q(z) \| p(z \mid x))$

Evidence Lower Bound (ELBO)

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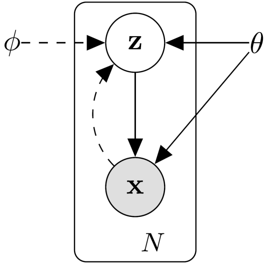
$$\log p(x) = D_{\mathrm{KL}}(q(z) \| p(z \mid x)) + L[q(z)]$$

ELBO: $L[q(z)] = E_q[\log p(z, x)] - E_q[\log q(z)]$

Variational Autoencoder

$q_\phi(z \mid x)$

Encoder Network



$p_\theta(x \mid z)$

Decoder Network

Maximize ELBO:

$$\mathcal{L}(\theta, \phi, x) = E_{q_\phi}[\log p_\theta(x, z) - \log q_\phi(z \mid x)]$$

VAE ELBO

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VAE ELBO

$$\begin{aligned}\mathcal{L}(\theta, \phi, x) &= E_{q_\phi}[\log p_\theta(x, z) - \log q_\phi(z \mid x)] \\ &= E_{q_\phi}[\log p_\theta(z) + \log p_\theta(x \mid z) - \log q_\phi(z \mid x)]\end{aligned}$$

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Use Monte Carlo approx., sampling $z^{(s)} \sim q_\phi(z \mid x)$:

$$\nabla_\phi E_{q_\phi}[\log p_\theta(x \mid z)] \approx \frac{1}{S} \sum_{s=1}^{S} \log p_\theta(x \mid z) \nabla_\phi \log q_\phi(z^{(s)} \mid x)$$

Reparameterization Trick

What about the other term?

$$-D_{\mathrm{KL}}(q_\phi(z \mid x) \| p_\theta(z))$$

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Instead of encoding z, encode parameters for a normal distribution, $N(\mu, \sigma^2)$

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$$q_\phi(z_j \mid x^{(i)}) = N(\mu_j^{(i)}, \sigma_j^{2(i)})$$
$$p_\theta(z) = N(0, I)$$

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KL divergence between these two is:

$$D_{\mathrm{KL}}(q_\phi(z \mid x^{(i)}) \| p_\theta(z)) = -\frac{1}{2} \sum_{j=1}^{d} \left(1 + \log(\sigma_j^{2(i)}) - (\mu_j^{(i)})^2 - \sigma_j^{2(i)}\right)$$

Results from Kingma & Welling