MATH230: Tutorial Two [Solutions]

Propositional Logic: Natural Deductions with Negation

Key ideas

- Write natural deduction proofs involving $\neg$ negation.
- Write natural deductions using Ex Falso i.e. $\perp$ elimination.

Relevant Topic: Propositional Logic
Relevant reading: L∃∀N Chapters 3,4 and Simon section 1.1

Hand in exercises: 1a, 1b, 1c, 1i, 1k
Due Friday @ 5pm to the submission box on Learn.

Discussion Questions

1. Show $A \vdash \neg\neg A$.

2. Show $(A \vee B) \wedge (A \vee C) \ \vdash \ A \vee (B \wedge C)$.

Hints

Read these hints and suggestions before reading the solutions below. It is important that you try to solve these problems first, before reading a solution. All hints from earlier labs still apply to this lab.

In this second lab we now use propositions of the form $\neg\alpha$. However, this requires no new rules to deal with the $\neg$ connective as we define $\neg\alpha := \alpha \to \bot$. Therefore, we need only the implication introduction and elimination rules to deal with negations.

This introduction of $\neg$ and $\bot$ can lead to inconsistent contexts i.e. $\Sigma \vdash \bot$ is possible for some Σ. Different logics deal with the introduction of $\bot$ in different ways. In this tutorial you will be asked to make use of the following $\bot$-elimination rule called *ex falso quodlibet*.

$$\dfrac{\begin{matrix}\Sigma \\ \vdots \\ \bot\end{matrix}}{A}\ \text{XF}$$

$\bot$ Elimination (Ex Falso)

More commonly called the *principle of explosion*, this rule of inference states that if a contradiction can be derived from Σ, then in fact any proposition at all follows.

Hints and Tips

> Always unpack $\neg\alpha$ for $\alpha \to \bot$ to help remember the rules that apply in eliminating and introducing $\neg$ negations.

Tutorial Exercises

1. **Minimal Logic.** Provide natural deduction proofs of the following sequents. These deductions require only the use of minimal logic; the introduction and elimination rules for $\wedge$ conjunction, $\vee$ disjunction,$\rightarrow$ implication, and the definition of $\neg$ negation as an implication.

   (a) $\neg A \;\vdash\; (C \rightarrow A) \rightarrow \neg C$

   Solution:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{C}^{2} \quad \overline{C \rightarrow A}^{1}}{A}\ \text{MP} \qquad \neg A}{\bot}\ \text{MP}}{\neg C}\ \rightarrow I, 2}{(C \rightarrow A) \rightarrow \neg C}\ \rightarrow I, 1$$

   (b) $A \wedge \neg B \;\vdash\; \neg(A \rightarrow B)$

   Solution:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \wedge \neg B}{A}\ \wedge E_l \quad \overline{A \rightarrow B}^{1}}{B}\ \text{MP} \quad \dfrac{A \wedge \neg B}{\neg B}\ \wedge E_r}{\bot}\ \text{MP}}{\neg(A \rightarrow B)}\ \rightarrow I, 1$$

   (c) $A \rightarrow C, \quad B \rightarrow D, \quad \neg C \vee \neg D \vdash \neg A \vee \neg B$

   Solution:

$$\dfrac{\neg C \vee \neg D \quad \dfrac{\dfrac{\dfrac{\dfrac{\overline{\neg C}^{2} \quad \dfrac{\overline{A}^{1} \quad A \rightarrow C}{C}\ \text{MP}}{\bot}\ \text{MP}}{\neg A}\ \rightarrow I, 1}{\neg A \vee \neg B}\ \vee I_r}{\neg C \rightarrow (\neg A \vee \neg B)}\ \rightarrow I, 2 \quad \dfrac{\dfrac{\dfrac{\dfrac{\overline{\neg D}^{3} \quad \dfrac{\overline{B}^{4} \quad B \rightarrow D}{D}\ \text{MP}}{\bot}\ \text{MP}}{\neg B}\ \rightarrow I, 4}{\neg A \vee \neg B}\ \vee I_l}{\neg D \rightarrow (\neg A \vee \neg B)}\ \rightarrow I, 3}{\neg A \vee \neg B}\ \vee E$$

   (d) $A, \quad \neg A \vdash \neg B$

   Solution:

$$\dfrac{\dfrac{A \quad \neg A}{\bot}\ \text{MP}}{\neg B}\ \rightarrow I$$

   (e) $A \rightarrow B,\ A \rightarrow \neg B \vdash \neg A$

   Solution:

$$\dfrac{\dfrac{\dfrac{\overline{A}^{1} \quad A \rightarrow B}{B}\ \text{MP} \quad \dfrac{\overline{A}^{1} \quad A \rightarrow \neg B}{\neg B}\ \text{MP}}{\bot}\ \text{MP}}{\neg A}\ \rightarrow I, 1$$

(f) $A \to \neg B \ \vdash \ B \to \neg A$

Solution:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A}^{2} \quad A \to \neg B}{\neg B}\,\mathsf{MP} \qquad \overline{B}^{1}}{\bot}\,\mathsf{MP}}{\neg A}\,\to I, 2}{B \to \neg A}\,\to I, 1$$

(g) $\neg(A \wedge B) \ \vdash \ A \to \neg B$

Solution:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A}^{1} \quad \overline{B}^{2}}{A \wedge B}\,\wedge I \qquad \neg(A \wedge B)}{\bot}\,\mathsf{MP}}{\neg B}\,\to I, 2}{A \to \neg B}\,\to I, 1$$

(h) $A \ \vdash \ \neg\neg A$

Solution:

$$\dfrac{\dfrac{A \quad \overline{\neg A}^{1}}{\bot}\,\mathsf{MP}}{\neg\neg A}\,\to I, 1$$

(i) $\neg\neg\neg A \ \vdash \ \neg A$

Solution:

$$\dfrac{\dfrac{\dfrac{\overline{A}^{1}}{\neg\neg A}\,\mathsf{THM} \qquad \neg\neg\neg A}{\bot}\,\mathsf{MP}}{\neg A}\,\to I, 1$$

(j) $\neg A \vee \neg B \vdash \neg(A \wedge B)$

Solution:

$$\dfrac{\dfrac{\neg A \vee \neg B \qquad \dfrac{\dfrac{\dfrac{\overline{A \wedge B}^{1}}{A}\,\wedge E_l \quad \overline{\neg A}^{2}}{\bot}\,\mathsf{MP}}{\neg A \to \bot}\,\to I, 2 \qquad \dfrac{\dfrac{\dfrac{\overline{A \wedge B}^{1}}{B}\,\wedge E_r \quad \overline{\neg B}^{3}}{\bot}\,\mathsf{MP}}{\neg B \to \bot}\,\to I, 3}{\bot}\,\vee E}{\neg(A \wedge B)}\,\to I, 1$$

(k) $\neg A \wedge \neg B \ \vdash \neg(A \vee B)$

Solution:

$$\dfrac{\dfrac{\overline{A \vee B}^{1} \qquad \dfrac{\dfrac{\dfrac{\neg A \wedge \neg B}{\neg A}\,\wedge E_l \quad \overline{A}^{2}}{\bot}\,\mathsf{MP}}{A \to \bot}\,\to I, 2 \qquad \dfrac{\dfrac{\dfrac{\neg A \wedge \neg B}{\neg B}\,\wedge E_r \quad \overline{B}^{3}}{\bot}\,\mathsf{MP}}{B \to \bot}\,\to I, 3}{\bot}\,\vee E}{\neg(A \vee B)}\,\to I, 1$$

(l) $\neg(A \lor B) \vdash \neg A \land \neg B$

Solution:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A}^{1}}{A \lor B}\lor I_r \quad \neg(A\lor B)}{\bot}\text{MP}}{\neg A}\to I,1 \qquad \dfrac{\dfrac{\dfrac{\overline{B}^{2}}{A\lor B}\lor I_l \quad \neg(A\lor B)}{\bot}\text{MP}}{\neg B}\to I,2}{\neg A \land \neg B}\land I$$

(m) $A \to \neg B \vdash \neg(A \land B)$

Solution:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A\land B}^{1}}{A}\land E_l \quad A\to\neg B}{\neg B}\text{MP} \quad \dfrac{\overline{A\land B}^{1}}{B}\land E_r}{\bot}\text{MP}}{\neg(A\land B)}\to I,1$$

(n) $\vdash \neg\neg(A \lor \neg A)$

Solution:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{A}^{2}}{A\lor\neg A}\lor I_r \quad \overline{\neg(A\lor\neg A)}^{1}}{\bot}\text{MP}}{\neg A}\to I,2}{A\lor\neg A}\lor I_l \quad \overline{\neg(A\lor\neg A)}^{1}}{\bot}\text{MP}}{\neg\neg(A\lor\neg A)}\to I,1$$

2. **Intuitionistic derivations.** Provide natural deduction proofs of the following. You do not need to use the *classical* $\bot$ rule for these questions, but may find that the *intuitionistic* $\bot$ rule is necessary.

(a) $A, \neg A \;\vdash\; B$

Solution:

$$\dfrac{\dfrac{A \quad \neg A}{\bot}\,\mathsf{MP}}{B}\,\mathsf{XF}$$

(b) $\neg A \;\vdash\; A \to B$

Solution:

$$\dfrac{\dfrac{\dfrac{\overline{A}^{1} \quad \neg A}{\bot}\,\mathsf{MP}}{B}\,\mathsf{XF}}{A \to B}\,\to I, 1$$

(c) $\neg A \lor B \;\vdash\; A \to B$

Solution:

$$\dfrac{\dfrac{\neg A \lor B \quad \dfrac{\dfrac{\dfrac{\overline{A}^{1} \quad \overline{\neg A}^{2}}{\bot}\,\mathsf{MP}}{B}\,\mathsf{XF}}{\neg A \to B}\,\to I, 2 \quad B \to B}{B}\,\lor E}{A \to B}\,\to I, 1$$

(d) $A \lor B,\ \neg A \;\vdash\; B$

Solution:

$$\dfrac{A \lor B \quad \dfrac{\dfrac{\dfrac{\overline{A}^{1} \quad \neg A}{\bot}\,\mathsf{MP}}{B}\,\mathsf{XF}}{A \to B}\,\to I, 1 \quad B \to B}{B}\,\lor E$$

(e) $\vdash\; \neg(B \to A) \to (A \to B)$

Solution:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{A}^{2}}{B \to A}\,\to I \quad \overline{\neg(B \to A)}^{1}}{\bot}\,\mathsf{MP}}{B}\,\mathsf{XF}}{A \to B}\,\to I, 2}{\neg(B \to A) \to (A \to B)}\,\to I, 1$$

(f) $A \to B, A \to \neg B \;\vdash\; A \to C$

Solution:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A}^{1} \quad A \to B}{B}\,\mathsf{MP} \quad \dfrac{\overline{A}^{1} \quad A \to \neg B}{\neg B}\,\mathsf{MP}}{\bot}\,\mathsf{MP}}{C}\,\mathsf{XF}}{A \to C}\,\to I, 1$$

(g) $A \lor B, \neg A, \neg B \ \vdash \ C$

Solution:

$$\dfrac{A \lor B \qquad \dfrac{\dfrac{\dfrac{\neg A \quad \overline{A}^{1}}{\bot}\,\mathsf{MP}}{C}\,\mathsf{XF}}{A \to C}\to I, 1 \qquad \dfrac{\dfrac{\dfrac{\neg B \quad \overline{B}^{1}}{\bot}\,\mathsf{MP}}{C}\,\mathsf{XF}}{B \to C}\to I, 2}{C}\lor E$$