MATH230: Tutorial Five

Peano Arithmetic

Key ideas

- Natural deduction practice,
- Proofs using the identity rules of inference,
- Prove first-order sentences in theories of arithmetic,
- Use the induction schema of Peano arithmetic, and
- Become exasperated enough to appreciate the help of proof assistants.

Relevant Topic: Peano Arithmetic
Relevant reading: Natural Number Game

Hand in exercises: 1c, 2a, 2b, 2c, and 5b
Due Friday @ 5pm to the submission box on Learn.

Discussion Questions

Proofs that make use of the axiom (schema) PA 7

$$[P(0) \land \forall x\ (P(x) \to P(s(x)))] \to \forall y(P(y))$$

will prove statements of the form $\forall y\ P(y)$ with the use of modus ponens. This requires proving the antecedent conjunction:

$$[P(0) \land \forall x\ (P(x) \to P(s(x)))]$$

This in turn requires proving each conjunct i.e. two proofs witnessing:

$$\text{PA} \vdash P(0) \qquad\qquad \text{PA} \ \vdash\ \forall x\ (P(x) \to P(s(x)))$$

If we piece these together, then we see that all proofs by induction have the form:

$$\dfrac{\dfrac{\dfrac{\vdots}{\mathcal{D}_{BC}}}{P(0)} \qquad \dfrac{\dfrac{\vdots}{\mathcal{D}_{IS}}}{\forall x\ (P(x) \to P(s(x)))}\ \forall I}{\dfrac{[P(0) \land \forall x\ (P(x) \to P(s(x)))]}{\forall y\ P(y)}\ \text{IND}}\ \land I$$

1. Identify the following steps involved in a proof by induction of the following sentence of Peano arithmetic:

$$\text{PA} \ \vdash\ \forall x\ (0 + x = x)$$

   (a) Identify the wff $P(x)$ to do induction on.
   (b) $\mathcal{D}_{BC}$: Write down the sequent PA $\vdash$ $P(0)$.
   (c) $\mathcal{D}_{IS}$: Write down the sequent $\text{PA}, P(n) \ \vdash\ P(s(n))$.

Tutorial Exercises

1. Give natural deductions of the following theorems of identity.

   (a) $\vdash\ \exists x\ (t = x)$

   (b) $\vdash\ \forall x \forall y\ x = y \rightarrow y = x$

   (c) $\vdash\ \forall x \forall y \forall z\ (x = y \wedge y = z) \rightarrow x = z$

   (d) $\vdash\ \forall x \forall y \forall z\ x \neq y \rightarrow (x \neq z \vee y \neq z)$ (RAA)

   Parts (b) and (c) together with proofs from lectures show that identity is reflexive, symmetric, and transitive. Thus behaving like an equivalence relation - as one would hope of the definition of equals!

2. In this question PA denotes the first-order theory of Peano arithmetic which has signature PA: $\{0, s, +, \times\}$ and axioms:

   PA1 $\forall x \neg(s(x) = 0)$

   PA2 $\forall x\ \forall y((s(x) = s(y)) \rightarrow (x = y))$

   PA3 $\forall x\ (x + 0 = x)$

   PA4 $\forall x\ \forall y\ (x + s(y) = s(x + y))$

   PA5 $\forall x\ (x \times 0 = 0)$

   PA6 $\forall x\ \forall y\ (x \times s(y) = (x \times y) + x)$

   PA7 $[P(0) \wedge \forall x\ (P(x) \rightarrow P(s(x)))] \rightarrow \forall y(P(y))$

   Provide deductions to prove the following sequents.

   (a) $\mathrm{PA} \vdash 1 + 1 = 2$

   (b) $\mathrm{PA} \vdash 3 \neq 1$

   (c) $\mathrm{PA} \vdash \forall x\ (x + 1 = s(x))$

   (d) $\mathrm{PA} \vdash \forall x\ (x \times 1 = x)$

3. The first-order language of Peano Arithmetic is often presented with an extra binary relation symbol $<$ where $x < y$ is given the usual interpretation: x is *strictly* less than y. In fact it is not necessary to add anything extra, for this relation can be defined using a sentence in PA as stated.

   Write down a wff in PA which defines the binary relation $<$ of being "strictly less than". Use this to write down formulae that represent: less than or equal to, strictly greater than, and greater than or equal to.

4. Write down well-formed formulae in the first-order language of PA corresponding to the following statements.

   (a) Each natural number is either equal to 0 or greater than 0.

   (b) If x is not less than y, then x equals y or y is less than x.

   (c) If x is less than or equal to y and y is less than or equal to x, then $x = y$.

5. The followings sequents all require the use of the induction axiom schema. Recall that all proofs using the induction schema have the following form:

$$
\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ \mathcal{D}_{BC}\end{matrix}}{P(0)} \qquad \dfrac{\begin{matrix}\vdots\\ \mathcal{D}_{IS}\end{matrix}}{\forall x\ (P(x) \to P(s(x)))}}{[P(0) \land \forall x\ (P(x) \to P(s(x)))]}\ \land I}{\forall y\ P(y)}\ IND
$$

For this reason, once the wff $P(x)$ is identified, it suffices to provide the base case deduction $\mathcal{D}_{BC}$ and induction step $\mathcal{D}_{IS}$. The sequents are stated in such a way as to mean induction on the variable x will be the easiest approach. Always do induction on the variable x.

(a) $\text{PA} \vdash \forall x\ (0 + x = x)$

(b) $\text{PA} \vdash \forall x\ (0 \times x = 0)$

(c) $\text{PA} \vdash \forall x\ (1 \times x = x)$

(d) $\text{PA} \vdash \forall x\ (x = 0 \lor \exists y (x = s(y)))$ (Challenge!)

(e) $\text{PA} \vdash \forall x\ \forall y\ [s(y) + x = s(y + x)]$ (Challenge!)

(f) $\text{PA} \vdash \forall x\ \forall y\ \forall z\ [(y + z) + x = y + (z + x)]$ (Challenge!)

(g) $\text{PA} \vdash \forall x\ \forall y\ [y + x = x + y]$ (Challenge!)

6. Visit the Natural Number Game to write computer checked proofs of these theorems of Peano arithmetic as well as theorems of propositional logic from previous tutorials.

7. Provide natural deductions of the following theorems of Peano Arithmetic. Beware each of these must be true, but some I have not provided natural deductions for. Some may require breaking down into further subgoals (lemma) to help. I recommend writing informal proofs, before formalising them with natural deductions.

(a) $\text{PA},\ 0 < a\ \vdash\ 0 < s(a)$

(b) $\text{PA}\ \vdash\ a < s(a)$

(c) $\text{PA},\ a < b\ \vdash\ s(a) < s(b)$

(d) $\text{PA},\ s(a) < s(b)\ \vdash\ a < b$

(e) $\text{PA},\ (a < b) \land (b < c)\ \vdash\ a < c$

(f) $\text{PA}\ \vdash\ \forall x\ [(x = 0) \lor (0 < x)]$ (Challenge!)

(g) $\text{PA}\ \vdash\ \forall x\ \forall y\ [\neg(x < y) \to ((x = y) \lor (y < x))]$ (Challenge!)

(h) $\text{PA}\ \vdash\ \forall x\ \forall y\ [(x \leq y) \land (y \leq x)] \to x = y$ (Challenge!)