**MATH230: Tutorial Three**

**Natural Deductions with Classical Logic**

Key ideas

- Write natural deductions using RAA.
- Prove LEM and DNE.
- Use LEM and DNE as derived rules of inference to avoid RAA directly.
- Witness the oddities of classical theorems.

Relevant Topic: Propositional Logic
Relevant reading: L∃∀N Chapter 5

Hand in exercises: 2a, 2b, 3a, 3b, 3e
**Due Friday @ 5pm to the submission box on Learn.**

## Discussion Questions

1. $\neg(A \wedge B) \vdash \neg A \vee \neg B$

**Tutorial Exercises**

1. **NOTE!** Make sure you have finished all of the minimal and intuitionistic natural deductions before doing this tutorial. It is more important that you understand those.

2. **LEM and DNE.** Prove each of the following fundamental theorems of classical logic making explicit use of the RAA mode of reasoning:

   (a) $\vdash A \vee \neg A$ [Challenge!]

   (b) $\neg\neg A \vdash A$

3. **Classical derivations.** Provide natural deduction proofs of the following. All rules *may* be required. Rather than making explicit use of RAA, it can be easier to appeal to LEM or DNE as *derived rules of inference*.

   (a) $\neg(A \wedge B) \vdash \neg A \vee \neg B$

   (b) $A \to B \vdash \neg A \vee B$

   (c) $\vdash (A \to B) \vee (B \to C)$

   (d) $\vdash A \to B \vee \neg B$

   (e) $\vdash (\neg A \to A) \to A$

   (f) $(A \to B) \vdash (A \to D) \vee (C \to B)$

   (g) $\neg(A \to B) \vdash A \wedge \neg B$

   (h) $\vdash ((A \to B) \to A) \to A$ [Challenge!]

**PTO**

4. In class we discussed how classical logic can be obtained from intuitionistic logic by adding the following rule of inference *reductio ad absurdum*: If $\overset{\Sigma}{\underset{\perp}{\mathcal{D}}}$ is a deduction of $\perp$ from $\Sigma$, then

$$\dfrac{\begin{array}{c}\overline{\cancel{\neg\alpha}}\\ \Sigma \\ \mathcal{D} \\ \perp\end{array}}{\alpha}\ \text{RAA}$$

is a derivation of $\alpha$ from the assumptions $\Sigma\backslash\{\neg\alpha\}$.

In this question we will explore this extension of logics in more detail. We will see that there are different methods for obtaining classical logic from minimal/intuitionistic logic. For the purposes of this question we introduce two more rules of inference:

**Double Negation Elimination**

$$\dfrac{\neg\neg\alpha \to \alpha \qquad \begin{array}{c}\Sigma \\ \mathcal{D} \\ \neg\neg\alpha\end{array}}{\alpha}\ \text{DNE}$$

**Law of Excluded Middle**

$$\dfrac{\alpha \vee \neg\alpha \qquad \begin{array}{c}\Sigma_1 \\ \mathcal{D}_1 \\ \alpha \to \gamma\end{array} \qquad \begin{array}{c}\Sigma_2 \\ \mathcal{D}_2 \\ \neg\alpha \to \gamma\end{array}}{\gamma}\ \text{LEM}$$

**Question:** Suppose you are given a proof witnessing the sequent $\Sigma, \neg\alpha \ \vdash \ \perp$.

(a) Using only minimal logic + DNE extend this proof to a proof witnessing the sequent $\Sigma \ \vdash \alpha$.

(b) Using only inituitionistic logic + LEM extend this proof to a proof witnessing the sequent $\Sigma \ \vdash \alpha$.

Use part (a) to argue that RAA = DNE in the presence of minimal logic. Where as part (b) shows RAA = LEM in the presence of intuitionistic logic. Finally, argue that all three are therefore equivalent modes of reasoning in the presence of intuitionistic logic: RAA = DNE = LEM.