MATH230: Tutorial Eight

Curry-Howard Correspondence

Key ideas

- Write context dependent typing derivations.
- Understand the connection between natural deductions and typing derivations.
- Write proof-terms witnessing theorems of minimal logic.

Relevant lectures: Typed Lambda Calculus Slides

Relevant reading: Type Theory and Functional Programming, Simon Thompson

Hand in exercises: 1a, 1d, 1e, 1k, 3

Due Friday @ 5pm to the submission box on Learn.

Discussion Questions

- Write a program of the specified type in the given context:

$$p : A \times (B \times C) \;\vdash\; (A \times B) \times C$$

$$\dfrac{\dfrac{\dfrac{p : A \times (B \times C)}{\mathsf{fst}\ \ p\ :\ A}\,\mathsf{fst} \qquad \dfrac{\dfrac{p : A \times (B \times C)}{\mathsf{snd}\ \ p\ :\ B \times C}\,\mathsf{snd}}{\mathsf{fst}\ \ (\mathsf{snd}\ \ p)\ :\ B}\,\mathsf{fst}}{(\mathsf{fst}\ \ p,\ \mathsf{fst}\ \ (\mathsf{snd}\ \ p))\ :\ A \times B}\,\times \qquad \dfrac{\dfrac{p\ :\ A \times (B \times C)}{\mathsf{snd}\ \ p\ :\ B \times C}\,\mathsf{snd}}{\mathsf{snd}\ \ (\mathsf{snd}\ \ p)\ :\ C}\,\mathsf{snd}}{((\mathsf{fst}\ \ p,\ \mathsf{fst}\ \ (\mathsf{snd}\ \ p)),\ \mathsf{snd}\ \ (\mathsf{snd}\ \ p))\ :\ (A \times B) \times C}\,\times$$

  This typing derivation shows the λ-term

$$((\mathsf{fst}\ \ p,\ \mathsf{fst}\ \ (\mathsf{snd}\ \ p)),\ \mathsf{snd}\ \ (\mathsf{snd}\ \ p))\ : (A \times B) \times C$$

  inhabits the stated type in the given context. Compare this to the natural deduction proof of the sequent

$$A \wedge (B \wedge C) \;\vdash\; (A \wedge B) \wedge C$$

- For a fixed typed A, prove that the type $(A \to A) \to A$ is uninhabited i.e. there is no term t of simple type theory that has this type.

  Solution:

  If this type were inhabited, then one would get a function for any choice of type A. If we choose $A = \bot$ to be the empty type, then we see that this is impossible. Indeed, there is a function $\bot \to \bot$. However, we can't get a function from the non-empty type $\bot \to \bot$ to the empty type $\bot$; as there is nothing to map to! Therefore, there can be no such function.

  The Curry-Howard correspondence gives us another avenue to prove this. If there were a program of this type, then there would be a natural deduction of $\vdash (A \to A) \to A$. By the soundness of the natural deduction calculus, we know we can only have proofs of logical tautologies. By writing a truth table for this proposition one can see this is not a tautology. This means there can be no derivation of the proposition. Via the Curry-Howard correspondence we see that there can be no function of this type.

Tutorial Exercises

1. For each $\Sigma \vdash \alpha$ provide a term of type α from the given Σ context.

   (a) $f : A \to (B \to C) \vdash B \to (A \to C)$

   Solution:

$$\cfrac{\cfrac{\cfrac{\overline{a : A}^{\,1} \quad f : A \to (B \to C)}{f\ a\ :\ B \to C}\,\mathsf{app} \qquad \overline{b : B}^{\,1}}{(f\ a)\ b\ :\ C}\,\mathsf{app}}{\lambda x.\ \lambda y.\ (f\ x)\ y\ : B \to (A \to C)}\,\lambda, 2, 1$$

   Compare this typing derivation to the natural derivation verifying the sequent

$$A \to (B \to C) \vdash B \to (A \to C)$$

$$\cfrac{\cfrac{\cfrac{\overline{A}^{\,1} \quad A \to (B \to C)}{B \to C}\,\mathsf{MP} \qquad \overline{B}^{\,1}}{C}\,\mathsf{MP}}{B \to (A \to C)}\,\to I, 2, 1$$

   We refer to the resulting λ-term as the proof object witnessing this theorem:

$$\lambda x.\ \lambda y.\ (f\ x)\ y\ : B \to (A \to C)$$

   (b) $t : A \times B \vdash B \times A$

   Solution:

$$\cfrac{\cfrac{t : A \times B}{\mathsf{snd}\ t : B}\,\mathsf{snd} \qquad \cfrac{t : A \times B}{\mathsf{fst}\ t : A}\,\mathsf{fst}}{(\mathsf{snd}\ t, \mathsf{fst}\ t) : B \times A}\,\times$$

   Compare this typing derivation to the natural deduction verifying the sequent

$$A \wedge B \vdash B \wedge A$$

$$\cfrac{\cfrac{A \wedge B}{B}\,\wedge E_R \qquad \cfrac{A \wedge B}{A}\,\wedge E_L}{B \wedge A}\,\wedge I$$

   We refer to the resulting λ-term as the proof object witnessing this theorem:

$$(\mathsf{snd}\ t, \mathsf{fst}\ t) : B \times A$$

(c) $t : A + B \;\vdash\; B + A$

Solution:

$$\dfrac{t : A + B \qquad \dfrac{\dfrac{\overline{a : A}\;1}{\mathsf{inr}\ a : B + A}\ \mathsf{inr}}{\lambda a.\ \mathsf{inr}\ a : A \to B + A}\ \lambda, 1 \qquad \dfrac{\dfrac{\overline{b : B}\;2}{\mathsf{inr}\ b : B + A}\ \mathsf{inl}}{\lambda b.\ \mathsf{inl}\ b : B \to B + A}\ \lambda, 2}{\mathsf{cases}\ t\ (\lambda a.\ \mathsf{inr}\ a)\ (\lambda b.\ \mathsf{inl}\ b)\ \ : B + A}\ \mathsf{cases}$$

Compare this typing derivation to the natural deduction verifying the sequent

$$A \vee B \;\vdash\; B \vee A$$

$$\dfrac{A \vee B \qquad \dfrac{\dfrac{\overline{A}\;1}{B \vee A}\ \vee I_R}{A \to B \vee A} \to I, 1 \qquad \dfrac{\dfrac{\overline{B}\;2}{B \vee A}\ \vee I_L}{B \to B \vee A} \to I, 2}{B \vee A}\ \vee E$$

We refer to the resulting λ-term as the proof object witnessing this theorem:

$$\mathsf{cases}\ t\ (\lambda a.\ \mathsf{inr}\ a)\ (\lambda b.\ \mathsf{inl}\ b)\ \ : B + A$$

(d) $f : (A \times B) \to C \vdash A \to (B \to C)$

Solution:

$$\dfrac{\dfrac{\dfrac{f \;:\; (A \times B) \to C \qquad \dfrac{\dfrac{}{a : A}\,1 \qquad \dfrac{}{b : B}\,2}{(a, b) : A \times B}\,\times}{f(a, b) \;:\; C}\,\mathsf{app}}{\lambda y.\ f(a, y) \;:\; B \to C}\,\lambda, 2}{\lambda x.\lambda y.\ f(x, y) \;:\; A \to (B \to C)}\,\lambda, 1$$

Compare this typing derivation to the natural deduction verifying the sequent:

$$(A \wedge B) \to C \;\vdash\; A \to (B \to C)$$

$$\dfrac{\dfrac{\dfrac{(A \wedge B) \to C \qquad \dfrac{\dfrac{}{A}\,1 \qquad \dfrac{}{B}\,2}{A \wedge B}\,\wedge I}{C}\,\mathsf{MP}}{B \to C}\,\to I, 2}{A \to (B \to C)}\,\to I, 1$$

The resulting λ-term (i.e. program) below

$$\lambda x.\lambda y.\ f(x, y) \;:\; A \to (B \to C)$$

is the proof-object witnessing the proof of the sequent.

(e) $f : A \to (B \to C) \vdash (A \times B) \to C$

Solution:

$$\dfrac{\dfrac{\dfrac{f : A \to (B \to C) \qquad \dfrac{\dfrac{}{p : A \times B}\,1}{\mathsf{fst}\ \ p \;:\; A}\,\mathsf{fst}}{f\ (\mathsf{fst}\ \ p) \;:\; B \to C}\,\mathsf{app} \qquad \dfrac{\dfrac{}{p : A \times B}\,1}{\mathsf{snd}\ \ p :\; B}\,\mathsf{snd}}{(f\ (\mathsf{fst}\ \ p))\ (\mathsf{snd}\ \ p) \;:\; C}\,\mathsf{app}}{\lambda x.\ (f\ (\mathsf{fst}\ \ x))\ (\mathsf{snd}\ \ x) \;:\; (A \times B) \to C}\,\lambda, 1$$

Compare this typing derivation to the natural deduction verifying the sequent:

$$A \to (B \to C) \;\vdash\; (A \times B) \to C$$

$$\dfrac{\dfrac{\dfrac{A \to (B \to C) \qquad \dfrac{\dfrac{}{A \wedge B}\,1}{A}\,\wedge E_l}{B \to C}\,\mathsf{MP} \qquad \dfrac{\dfrac{}{A \wedge B}\,1}{B}\,\wedge E_r}{C}\,\mathsf{MP}}{(A \wedge B) \to C}\,\to I, 1$$

The resulting λ-term (i.e. program) below

$$\lambda x.\ (f\ (\mathsf{fst}\ \ x))\ (\mathsf{snd}\ \ x) \;:\; (A \times B) \to C$$

is the proof-object witnessing the proof of the sequent.

(f) $f : A \to B \ \vdash A \to (B + C)$

Solution:

$$\dfrac{\dfrac{\dfrac{f : A \to B \quad \overline{a : A}^{\,1}}{f\ a : B}\,\mathsf{app}}{\mathsf{inl}\ (f\ a) : B + C}\,\mathsf{inl}}{\lambda x.\ \mathsf{inl}\ (f\ x) : A \to B + C}\,\lambda, 1$$

Compare this typing derivation to the natural deduction verifying the sequent:

$$A \to B \ \vdash A \to (B \vee C)$$

$$\dfrac{\dfrac{\dfrac{A \to B \quad \overline{A}^{\,1}}{B}\,\mathsf{MP}}{B \vee C}\,\vee I_L}{A \to B \vee C}\,\to I, 1$$

We refer to the resulting λ-term as the proof object witnessing this theorem:

$$\lambda x.\ \mathsf{inl}\ (f\ x) : A \to B + C$$

(g) $f : A \to B,\ g : B \to C \vdash A \to C$

Solution:

$$\dfrac{\dfrac{g\ :\ B \to C \qquad \dfrac{f : A \to B \quad \overline{a : A}^{\,1}}{f\ a\ :\ B}\,\mathsf{app}}{g(f\ a)\ :\ C}\,\mathsf{app}}{\lambda x.\ g(f\ x)\ :\ A \to C}\,\lambda, 1$$

Compare this typing derivation to the natural deduction verifying the sequent:

$$A \to B,\ B \to C \vdash\ A \to C$$

$$\dfrac{\dfrac{B \to C \qquad \dfrac{A \to B \quad \overline{A}^{\,1}}{B}\,\mathsf{MP}}{C}\,\mathsf{MP}}{A \to C}\,\to I, 1$$

The resulting λ-term (i.e. program) below

$$\lambda x.\ g(f\ x)\ :\ A \to C$$

is the proof-object witnessing the proof of the sequent.

(h) $t : A + B,\ f : A \to C,\ g : B \to D \vdash C + D$

Solution:

$$\dfrac{p : A+B \qquad \dfrac{\dfrac{\dfrac{f : A \to C \quad \overline{a : A}^{\,1}}{f\ a : C}\ \mathsf{app}}{\mathsf{inl}\ \ (f\ a) : C + D}\ \mathsf{inl}}{\lambda x.\ \mathsf{inl}\ \ (f\ x) : A \to C + D}\ \lambda, 1 \qquad \dfrac{\dfrac{\dfrac{g : B \to D \quad \overline{b : B}^{\,2}}{g\ b : D}\ \mathsf{app}}{\mathsf{inr}\ \ (g\ b) : C + D}\ \mathsf{inr}}{\lambda y.\ \mathsf{inl}\ \ (g\ y) : B \to C + D}\ \lambda, 2}{\mathsf{cases}\ \ p\ (\lambda x.\ \mathsf{inl}\ \ (f\ x))\ (\lambda y.\ \mathsf{inr}\ \ (g\ y))\ \ :\ \ C + D}\ \mathsf{cases}$$

Compare this typing derivation to the natural deduction verifying the sequent:

$$A \vee B,\ A \to C,\ B \to D \vdash\ C \vee D$$

$$\dfrac{A \vee B \qquad \dfrac{\dfrac{\dfrac{A \to C \quad \overline{A}^{\,1}}{C}\ \mathsf{MP}}{C \vee D}\ \vee I}{A \to C \vee D}\ \to I, 1 \qquad \dfrac{\dfrac{\dfrac{B \to D \quad \overline{B}^{\,2}}{D}\ \mathsf{MP}}{C \vee D}\ \vee I}{B \to C \vee D}\ \to I, 2}{C \vee D}\ \vee E$$

The resulting λ-term (i.e. program) below

$$\mathsf{cases}\ \ p\ (\lambda x.\ \mathsf{inl}\ \ (f\ x))\ (\lambda y.\ \mathsf{inr}\ \ (g\ y))\ \ :\ \ C + D$$

is the proof-object witnessing the proof of the sequent.

(i) $f : A \to B\ \vdash\ (C \to A) \to (C \to B)$

Solution:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{c : C}^{\,2} \quad \overline{g : C \to A}^{\,1}}{g\ c : A}\ \mathsf{app} \qquad f : A \to B}{f\ (g\ c) : B}\ \mathsf{app}}{\lambda y.\ f\ (g\ y)\ :\ C \to B}\ \lambda, 2}{\lambda x.\ \lambda y.\ f\ (x\ y)\ :\ (C \to B) \to (C \to B)}\ \lambda, 1$$

Compare this typing derivation to the natural deduction verifying the sequent:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{C}^{\,2} \quad \overline{C \to A}^{\,1}}{A}\ \mathsf{MP} \qquad A \to B}{B}\ \mathsf{MP}}{C \to B}\ \to I, 2}{(C \to B) \to (C \to B)}\ \to I, 1$$

The resulting λ-term (i.e. program) below

$$\lambda x.\ \lambda y.\ f\ (x\ y)\ :\ (C \to B) \to (C \to B)$$

is the proof-object witnessing the proof of the sequent.

(j) $t : (A \to B) \times (A \to C) \ \vdash \ A \to (B \times C)$

Solution:

$$\dfrac{\dfrac{\dfrac{\dfrac{t \ : \ (A \to B) \times (A \to C)}{\mathsf{fst} \ \ t : A \to B}\,\mathsf{fst} \qquad \dfrac{}{a : A}\,1}{(\mathsf{fst} \ \ t) \ a : B}\,\mathsf{app} \qquad \dfrac{\dfrac{t \ : \ (A \to B) \times (A \to C)}{\mathsf{snd} \ t : A \to C}\,\mathsf{snd} \qquad \dfrac{}{a : A}\,1}{(\mathsf{snd} \ t) \ a : C}\,\mathsf{app}}{((\mathsf{fst} \ \ t) \ a, (\mathsf{snd} \ \ t) \ a) : B \times C}\,\times}{\lambda x. \ ((\mathsf{fst} \ t) \ x, (\mathsf{snd} \ t) \ x) : A \to B \times C}\,\lambda, 1$$

Compare this typing derivation to the natural deduction verifying the sequent:

$$\dfrac{\dfrac{\dfrac{\dfrac{(A \to B) \wedge (A \to C)}{A \to B}\,\wedge E_l \qquad \dfrac{}{A}\,1}{B}\,\mathsf{MP} \qquad \dfrac{\dfrac{(A \to B) \times (A \to C)}{A \to C}\,\wedge E_r \qquad \dfrac{}{A}\,1}{C}\,\mathsf{MP}}{B \wedge C}\,\wedge I}{A \to B \wedge C}\,\to I, 1$$

The resulting λ-term (i.e. program) below

$$\lambda x. \ ((\mathsf{fst} \ \ t) \ x, (\mathsf{snd} \ \ t) \ x) : A \to B \wedge C$$

is the proof-object witnessing the proof of the sequent.

(k) $t : A \times (B + C) \ \vdash \ (A \times B) + (A \times C)$ [Handin exercise]

(l) $t : (A \times B) + (A \times C) \;\vdash\; A \times (B + C)$

(m) $t : A + (B \times C) \vdash (A + B) \times (A + C)$

(n) $t : (A + B) \times (A + C) \ \vdash \ A + (B \times C)$

Extras: For these extra problems consider $\perp$ to be type with no constructor or destructors. Furthermore, consider $\neg P$ to be shorthand for the function type: $\neg P := P \to \perp$.

(a) $f : \neg A \;\vdash\; (C \to A) \to \neg C$

(b) $t : A \times \neg B \;\vdash\; \neg(A \to B)$

(c) $f : A \to C,\ g : B \to D,\ t : \neg C + \neg D \vdash \neg A + \neg B$

(d) $t : A,\ f : \neg A \vdash \neg B$

(e) $f : A \to B,\ g : A \to \neg B \vdash \neg A$

(f) $f : A \to \neg B \;\vdash\; B \to \neg A$

(g) $f : \neg(A \times B) \;\vdash\; A \to \neg B$

(h) $t : A \;\vdash\; \neg\neg A$

(i) $f : \neg\neg\neg A \;\vdash\; \neg A$

(j) $t : \neg A + \neg B \vdash \neg(A \times B)$

(k) $f : \neg A \times \neg B \;\vdash \neg(A + B)$

(l) $f : \neg(A + B) \;\vdash \neg A \times \neg B$

(m) $f : A \to \neg B \;\vdash\; \neg(A \times B)$

(n) $\vdash\; \neg\neg(A + \neg A)$

2. Revisit Lab 1 and Lab 2. For each derivation in those labs, provide a proof-object witnessing a natural deduction of the sequent. You don't need to do any more derivations at this point!

3. This exercise shows you an example of a general observation first made by William Tait, relating the simplifications of proofs and the process of computation in the λ-calculus.

   Consider the following proof of the theorem

$$\vdash \; A \wedge B \to B$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \wedge B}^{\;1}}{B}\,\wedge E_R \qquad \dfrac{\overline{A \wedge B}^{\;1}}{A}\,\wedge_L}{B \wedge A}\,\wedge I}{B}\,\wedge E_L}{A \wedge B \to B}\,\to, 1$$

   (a) Determine the corresponding proof-object for this proof.

   (b) Why does the proof-object have a redex in it?

   (c) Perform the β-reduction on the proof object from (a).

   (d) What proof does the reduced proof-object correspond to?

   Solution:

   The natural deduction proof stated in the question corresponds to the following type construction:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{p \;:\; A \times B}^{\;1}}{B}\,\mathsf{snd} \qquad \dfrac{\overline{p \;:\; A \times B}^{\;1}}{A}\,\mathsf{fst}}{B \times A}\,\times}{B}\,\mathsf{fst}}{\lambda x : A \times B.\ \mathsf{fst}\ (\mathsf{snd}\ x, \mathsf{fst}\ x) \;:\; A \times B \to B}\,\lambda, 1$$

   The corresponding proof-object is

$$\lambda x : A \times B.\ \mathsf{fst}\ (\mathsf{snd}\ x, \mathsf{fst}\ x) \;:\; A \times B \to B$$

   Which can be β-reduced to

$$\lambda x : A \times B.\ \mathsf{snd}\ x \;:\; A \times B \to B$$

   This proof-object takes in a pair and returns the second of the pair.

   As a natural deduction this corresponds to the following:

$$\dfrac{\dfrac{\overline{A \wedge B}^{\;1}}{B}\,\wedge E_R}{A \wedge B \to B}\,\to, 1$$

   This simplified program corresponds to a shorter proof. In this sense β-reduction (i.e. computation!) is related to the simplification of proofs.

4. Prove that the type $A + B \to A$ is uninhabited i.e. there is no term t of simple type theory that has this type. Your proof should be an informal reason for why no such program can exist. You might refer to the corresponding minimal logic sequent to help your justification.

   Solution:

   Terms in $A+B$ are either of type A or type B. We can't, in general, get a function from $A + B$ to A because we have no function for processing Bs to As.

   The Curry-Howard correspondence gives us another avenue to prove this. If there were a program of this type, then there would be a natural deduction of $\vdash A \vee B \to A$. By the soundness of the natural deduction calculus, we know we can only have proofs of logical tautologies. By writing a truth table for this proposition one can see this is not a tautology. This means there can be no derivation of the proposition. Via the Curry-Howard correspondence we see that there can be no function of this type.