MATH230: Tutorial Four [Solutions]

Natural Deductions in First Order Logic

Key ideas

- Read and interpret formulae of first-order logic.
- Write formulae of first-order logic.
- Natural deductions with $\forall$ $\exists$ rules.

Relevant Topic: First-Order Logic.
Relevant reading: Logic and Proof Sections 7,8, and 9.

Hand in exercises: 1a, 1b, 1i, 1j, 1n
Due Friday @ 5pm to the submission box on Learn.

Discussion Questions

1. $\forall x \neg Fx \dashv\vdash \neg \exists x Fx$

Tutorial Exercises

1. Prove the following in the predicate calculus. You will need only the minimal logic rules of inference along with the introduction and elimination rules for the quantifiers.

   (a) $\forall x(Fx \to Gx) \vdash \forall x Fx \to \forall x Gx$

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x\ Fx \to Gx}{Ft \to Gt}\,\forall E \qquad \dfrac{\overline{\forall x\ Fx}^{\,1}}{Ft}\,\forall E}{Gt}\,\text{MP}}{\forall x\ Gx}\,\forall I}{\forall x\ Fx \to \forall x\ Gx}\,\to I, 1$$

   (b) $\forall x((Fx \vee Gx) \to Hx), \quad \forall x \neg Hx \vdash \forall x \neg Fx$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x\ (Fx \vee Gx \to Hx)}{Ft \vee Gt \to Ht}\,\forall E \qquad \dfrac{\overline{Ft}^{\,1}}{Ft \vee Gt}\,\vee I_R}{Ht}\,\text{MP} \qquad \dfrac{\forall x\ \neg Hx}{\neg Ht}\,\forall E}{\bot}\,\text{MP}}{\neg Ft}\,\to I, 1}{\forall x\ \neg Fx}\,\forall I$$

   (c) $\forall x(Fx \wedge Gx) \vdash \forall x Fx \wedge \forall x Gx$

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x\ (Fx \wedge Gx)}{Ft \wedge Gt}\,\forall E}{Ft}\,\wedge E_L}{\forall x\ Fx}\,\forall I \qquad \dfrac{\dfrac{\dfrac{\forall x\ (Fx \wedge Gx)}{Ft \wedge Gt}\,\forall E}{Gt}\,\wedge E_R}{\forall x\ Gx}\,\forall I}{(\forall x\ Fx) \wedge (\forall x\ Gx)}\,\wedge I$$

   (d) $\forall x Fx \wedge \forall x Gx \vdash \forall x(Fx \wedge Gx)$

$$\dfrac{\dfrac{\dfrac{\dfrac{(\forall x\ Fx) \wedge (\forall x\ Gx)}{\forall x\ Fx}\,\wedge E_L}{Ft}\,\forall E \qquad \dfrac{\dfrac{(\forall x\ Fx) \wedge (\forall x\ Gx)}{\forall x\ Gx}\,\wedge E_R}{Gt}\,\forall E}{Ft \wedge Gt}\,\wedge I}{\forall x\ (Fx \wedge Gx)}\,\forall I$$

(e) $\forall x(P \to Fx) \vdash P \to \forall x Fx$

(f) $P \to \forall x Fx \vdash \forall x(P \to Fx)$

(g) $\exists x(P \to Fx) \vdash P \to \exists x Fx$

(h) $\exists x \neg Fx \vdash \neg \forall x Fx$

$$\dfrac{\dfrac{\exists x\ Fx \qquad \dfrac{\dfrac{\overline{\neg Ft}^{1} \qquad \dfrac{\overline{\forall x\ Fx}^{2}}{Ft}\ \forall E}{\bot}\ \text{MP}}{\neg Ft \to \bot}\ \to I, 1}{\bot}\ \text{MP}}{\neg \forall x\ Fx}\ \to I, 2$$

(i) $\forall x \neg Fx \vdash \neg \exists x Fx$

$$\dfrac{\dfrac{\overline{\exists x\ Fx}^{2} \qquad \dfrac{\dfrac{\overline{Ft}^{1} \qquad \dfrac{\forall x\ \neg Fx}{\neg Ft}\ \forall E}{\bot}\ \text{MP}}{Ft \to \bot}\ \to I, 1}{\bot}\ \text{MP}}{\neg \exists x\ Fx}\ \to I, 2$$

(j) $\neg \exists x Fx \vdash \forall x \neg Fx$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{Ft}^{1}}{\exists x\ Fx}\ \exists I \qquad \neg \exists x\ Fx}{\bot}\ \text{MP}}{\neg Ft}\ \to I, 1}{\forall x\ \neg Fx}\ \forall I$$

(k) $\exists x Fx \to P \vdash \forall x(Fx \to P)$

(l) $\exists x(Fx \to Gx) \vdash \forall x Fx \to \exists x Gx$

$$\dfrac{\dfrac{\exists x\ (Fx \to Gx) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overline{\forall x\ Fx}^{1}}{Ft}\ \forall E \qquad \overline{Ft \to Gt}^{2}}{Gt}}{\exists x\ Gx}\ \exists I}{(Ft \to Gt) \to \exists x\ Gx}\ \to I, 2}{\exists x\ Gx}\ \exists E}{\forall x\ Fx \to \exists x\ Gx}\ \to I, 1$$

(m) $\forall x Fx \vdash \neg \exists x\ \neg Fx$

$$\dfrac{\dfrac{\overline{\exists x\ \neg Fx}^{1} \qquad \dfrac{\dfrac{\overline{\neg Ft}^{2} \qquad \dfrac{\forall x\ Fx}{Ft}\ \forall E}{\bot}\ \text{MP}}{\neg Ft \to \bot}\ \to I, 2}{\bot}\ \text{MP}}{\neg \exists x\ \neg Fx}\ \to I, 1$$

(n) $\exists x Fx \vdash \neg\forall x\ \neg Fx$

$$\dfrac{\dfrac{\exists x\ Fx \qquad \dfrac{\dfrac{\overline{Ft}^{2} \qquad \dfrac{\overline{\forall x\ \neg Fx}^{1}}{\neg Ft}\ \forall E}{\bot}\ \mathsf{MP}}{Ft \to \bot}\ \to I, 2}{\bot}\ \mathsf{MP}}{\neg\forall x\ \neg Fx}\ \to I, 1$$

(o) $\forall x\ (Fx \to \neg Gx)\ \vdash\ \neg\exists x\ (Fx \land Gx)$

$$\dfrac{\dfrac{\overline{\exists x\ (Fx \land Gx)}^{1} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\forall x\ (Fx \to \neg Gx)}{Ft \to \neg Gt}\ \forall E \qquad \dfrac{\overline{Ft \land Gt}^{2}}{Ft}\ \land E_L}{\neg Gt}\ \mathsf{MP} \qquad \dfrac{\overline{Ft \land Gt}^{2}}{Gt}\ \land E_R}{\bot}\ \mathsf{MP}}{Ft \land Gt \to \bot}\ \to I, 2}{\bot}\ \exists E}{\neg\exists x\ (Fx \land Gx)}\ \to I, 1$$

(p) $\vdash\ \exists x\ (Fx \lor Gx) \leftrightarrow \exists x\ Fx\ \lor\ \exists x\ Gx$

2. Prove the following in the predicate calculus. Ex falso or classical modes of reasoning (RAA, LEM, DNE) will be helpful for proving these theorems. Each of these are challenging!

   (a) $\forall x\ (Fx \lor Gx),\ \forall x\ \neg Gx\ \vdash\ \forall x\ Fx$

   This is the universally quantified version of the "disjunctive syllogism" from the Lab on intuitionistic logic. Once the quantifiers have been eliminated, then we can simply call on this theorem from propositional logic - that is what the rule DS is denoting.

$$
\begin{prooftree}
\AxiomC{$\forall x\ (Fx \lor Gx)$}
\RightLabel{$\forall E$}
\UnaryInfC{$Ft \lor Gt$}
\AxiomC{$\forall x\ \neg Gx$}
\RightLabel{$\forall E$}
\UnaryInfC{$\neg Gt$}
\RightLabel{DS}
\BinaryInfC{Ft}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x\ Fx$}
\end{prooftree}
$$

   (b) $\neg \forall x Fx \vdash \exists x \neg Fx$

   We will use RAA to prove this. Which means we can prove the following sequent instead, before using RAA to prove the original sequent:

$$\neg \forall x Fx, \neg \exists x \neg Fx \vdash \bot$$

$$
\begin{prooftree}
\AxiomC{$\overline{\neg Fa}^{1}$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x\ \neg Fx$}
\AxiomC{$\overline{\neg \exists x\ \neg Fx}^{2}$}
\RightLabel{MP}
\BinaryInfC{$\bot$}
\RightLabel{$\rightarrow I, 1$}
\UnaryInfC{$\neg\neg Fa$}
\RightLabel{DNE}
\UnaryInfC{Fa}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x\ Fx$}
\AxiomC{$\neg \forall x\ Fx$}
\RightLabel{MP}
\BinaryInfC{$\bot$}
\RightLabel{RAA, 2}
\UnaryInfC{$\exists x\ \neg Fx$}
\end{prooftree}
$$

   (c) $\neg \forall x\ \neg Fx \vdash \exists x Fx$

$$
\begin{prooftree}
\AxiomC{$\overline{Ft}^{1}$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x\ Fx$}
\AxiomC{$\overline{\neg \exists x\ Fx}^{2}$}
\RightLabel{MP}
\BinaryInfC{$\bot$}
\RightLabel{$\rightarrow I, 1$}
\UnaryInfC{$\neg Ft$}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x \neg Fx$}
\AxiomC{$\neg \forall x\ \neg Fx$}
\RightLabel{MP}
\BinaryInfC{$\bot$}
\RightLabel{RAA, 2}
\UnaryInfC{$\exists x\ Fx$}
\end{prooftree}
$$

   (d) $\neg \exists x\ \neg Fx \vdash \forall x Fx$

$$
\begin{prooftree}
\AxiomC{$\overline{\neg Ft}^{1}$}
\RightLabel{$\exists I$}
\UnaryInfC{$\exists x\ \neg Fx$}
\AxiomC{$\neg \exists x\ \neg Fx$}
\RightLabel{MP}
\BinaryInfC{$\bot$}
\RightLabel{RAA, 1}
\UnaryInfC{Ft}
\RightLabel{$\forall I$}
\UnaryInfC{$\forall x\ Fx$}
\end{prooftree}
$$

   The final $\forall I$ is valid as t has been discharged in the previous step.

Note: The exercises above show that in the presence of classical modes of reasoning it is sufficient to introduce only one of the quantifiers, as the other can be deduced from it. Sequents above show that we could define $\forall = \neg\exists\neg$ and $\exists = \neg\forall\neg$ in classical logic. Following the BHK, however, one needs to define both independently.