MATH230: Tutorial Five [Solutions]

Peano Arithmetic

Key ideas

- Natural deduction practice,
- Proofs using the identity rules of inference,
- Prove first-order sentences in theories of arithmetic,
- Use the induction schema of Peano arithmetic, and
- Become exasperated enough to appreciate the help of proof assistants.

Relevant Topic: Peano Arithmetic

Relevant reading: Natural Number Game

Hand in exercises: 1c, 2a, 2b, 2c, and 5b

Due Friday @ 5pm to the submission box on Learn.

Discussion Questions

Proofs that make use of the axiom (schema) PA 7

$$[P(0) \wedge \forall x\ (P(x) \to P(s(x)))] \to \forall y (P(y))$$

will prove statements of the form $\forall y\ P(y)$ with the use of modus ponens. This requires proving the antecedent conjunction:

$$[P(0) \wedge \forall x\ (P(x) \to P(s(x)))]$$

This in turn requires proving each conjunct i.e. two proofs witnessing:

$$\mathsf{PA} \vdash P(0) \qquad\qquad \mathsf{PA}\ \vdash\ \forall x\ (P(x) \to P(s(x)))$$

If we piece these together, then we see that all proofs by induction have the form:

$$\dfrac{\dfrac{\dfrac{\vdots}{\mathcal{D}_{BC}}}{P(0)} \qquad \dfrac{\dfrac{\vdots}{\mathcal{D}_{IS}}}{\forall x\ (P(x) \to P(s(x)))}\ \forall I}{\dfrac{[P(0) \wedge \forall x\ (P(x) \to P(s(x)))]}{\forall y\ P(y)}\ \mathsf{IND}}\ \wedge I$$

1. Identify the following steps involved in a proof by induction of the following sentence of Peano arithmetic:

$$\mathsf{PA}\ \vdash\ \forall x\ (0 + x = x)$$

   (a) Identify the wff $P(x)$ to do induction on.

   (b) $\mathcal{D}_{BC}$: Write down the sequent $\mathsf{PA}\ \vdash\ P(0)$.

   (c) $\mathcal{D}_{IS}$: Write down the sequent $\mathsf{PA}, P(n)\ \vdash\ P(s(n))$.

Tutorial Exercises

1. Give natural deductions of the following theorems of identity.

   (a) $\vdash\ \forall x \forall y\ x = y \rightarrow y = x$

$$\dfrac{\dfrac{\dfrac{\overline{a = a}^{\,=I} \qquad \overline{a = b}^{\,1}}{b = a}\ {=E}}{a = b \rightarrow b = a}\ {\rightarrow I, 1}}{\forall x\ \forall y\ (x = y \rightarrow y = x)}\ \forall I$$

   (b) $\vdash\ \forall x \forall y \forall z\ (x = y \wedge y = z) \rightarrow x = z$

   Solution: To introduce the $\forall$ we argue from a general case. Furthermore, we are to prove an implication, so we add a temporary hypothesis to discharge later. So the crux of the proof comes down to showing:

$$(a = b) \wedge (b = c)\ \vdash\ a = c$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\cancel{a = b \wedge b = c}}^{\,1}}{b = c}\ \wedge E_r \qquad \dfrac{\overline{\cancel{a = b \wedge b = c}}^{\,1}}{a = b}\ \wedge E_l}{a = c}\ {=E}}{(a = b \wedge b = c) \rightarrow a = c}\ {\rightarrow I, 1}}{\forall x\ \forall y\ \forall z\ (x = y \wedge y = z) \rightarrow x = z}\ \forall I$$

Parts (a) and (b) together with proofs from lectures show that identity is reflexive, symmetric, and transitive. Thus behaving like an equivalence relation - as one would hope of the definition of equals!

2. In this question PA denotes the first-order theory of Peano arithmetic which has signature PA: $\{0, s, +, \times\}$ and axioms:

   PA1 $\forall x \neg(s(x) = 0)$

   PA2 $\forall x\ \forall y((s(x) = s(y)) \to (x = y))$

   PA3 $\forall x\ (x + 0 = x)$

   PA4 $\forall x\ \forall y\ (x + s(y) = s(x + y))$

   PA5 $\forall x\ (x \times 0 = 0)$

   PA6 $\forall x\ \forall y\ (x \times s(y) = (x \times y) + x)$

   PA7 $[P(0) \wedge \forall x\ (P(x) \to P(s(x)))] \to \forall y(P(y))$

   Provide deductions to prove the following sequents.

   Comments: Axioms PA1 - PA6 are each of the form $\forall$... which means that $\forall E$ must be used to start these proofs. A lot of these proofs turn on the right choice of term to substitute in for x, y, z in the $\forall E$ step. Look at the statement you're trying to prove and find the correct substitution into an axiom to make it come out.

   Theorems involving the binary function $+$ will necessarily make use of PA3 and/or PA4 as those are the axioms defining the properties of addition.

   Theorems involving the binary function $\times$ will necessarily make use of PA5 and/or PA6 as those are the axioms defining the properties of multiplication.

   PA 1 is the only statement of the form "this is not equal to that". This means that showing any two things are not equal must ultimately boil down to showing that if they were equal, then (something like) 0=1 would follow.

   (a) PA $\vdash 1 + 1 = 2$

   Solution: First the sequent should be desugared as 1,2 are not terms in PA. So the sequent in PA is the following:

$$\text{PA} \ \vdash \ s(0) + s(0) = s(s(0))$$

$$\dfrac{\dfrac{\text{PA3}}{s(0) + 0 = s(0)} \qquad \dfrac{\text{PA4}}{s(0) + s(0) = s(s(0) + 0)}\ \forall E}{s(0) + s(0) = s(s(0))}\ {=}E$$

   (b) PA $\vdash 3 \neq 1$

   Solution: Desugaring into PA yields the following sequent:

$$\text{PA} \ \vdash \ \neg[s(s(s(0))) = s(0)]$$

   Recall that introducing a negation ultimately comes down to showing

$$\text{PA},\ s(s(s(0))) = s(0) \ \vdash \ \bot$$

   and then using $\to I$ to tidy up at the end.

$$
\dfrac{\dfrac{\dfrac{}{\cancel{s(s(s(0))) = s(0)}}\,1 \qquad \dfrac{\mathsf{PA2}}{[s(s(s(0))) = s(0)] \to [s(s(0)) = 0]}\,\forall E}{s(s(0)) = 0}\,\mathsf{MP} \qquad \dfrac{\mathsf{PA1}}{\neg[s(s(0)) = 0]}\,\forall E}{\dfrac{\bot}{\neg[s(s(s(0))) = s(0)]}\,\to I, 1}\,\mathsf{MP}
$$

(c) PA $\vdash \forall x\ (x + 1 = s(x))$

Solution: Desugaring into PA yields the following sequent:

$$
\mathsf{PA} \vdash \forall x\ [x + s(0) = s(x)]
$$

$$
\dfrac{\dfrac{\mathsf{PA3}}{a + 0 = a}\,\forall E \qquad \dfrac{\mathsf{PA4}}{a + s(0) = s(a + 0)}\,\forall E}{\dfrac{a + s(0) = s(a)}{\forall x\ [x + s(0) = s(x)]}\,\forall I}\,= E
$$

(d) PA $\vdash \forall x\ (x \times 1 = x)$

Solution: Desugaring into PA yields the following sequent:

$$
\mathsf{PA}\ \vdash\ \forall x\ [x \times s(0) = x]
$$

This proof will make use of Exercise 3a of this tutorial.

$$
\dfrac{\dfrac{\dfrac{\mathsf{PA6}}{s(a) \times s(0) = (s(a) \times 0) + s(a)}\,\forall E \qquad \dfrac{\mathsf{PA5}}{s(a) \times 0 = 0}\,\forall E}{s(a) \times s(0) = 0 + s(a)} \qquad \dfrac{\mathsf{PA}}{0 + s(a) = s(a)}\,\mathsf{THM}}{\dfrac{s(a) \times s(0) = s(a)}{\forall x\ [x \times s(0) = x]}\,\forall I}\,= E
$$

3. The first-order language of Peano Arithmetic is often presented with an extra binary relation symbol $<$ where $x < y$ is given the usual interpretation: x is *strictly* less than y. In fact it is not necessary to add anything extra, for this relation can be defined using a sentence in PA as stated.

   Write down a wff in PA which defines the binary relation $<$ of being "strictly less than". Use this to write down formulae that represent: less than or equal to, strictly greater than, and greater than or equal to.

4. Write down well-formed formulae in the first-order language of PA corresponding to the following statements.

   (a) Each natural number is either equal to 0 or greater than 0.

   (b) If x is not less than y, then x equals y or y is less than x.

   (c) If x is less than or equal to y and y is less than or equal to x, then $x = y$.

5. The followings sequents all require the use of the induction axiom schema. Recall that all proofs using the induction schema have the following form:

$$\dfrac{\dfrac{\dfrac{\vdots}{\mathcal{D}_{BC}}}{P(0)} \quad \dfrac{\dfrac{\vdots}{\mathcal{D}_{IS}}}{\forall x\ (P(x) \to P(s(x)))}}{\dfrac{[P(0) \land \forall x\ (P(x) \to P(s(x)))]}{\forall y\ P(y)}\ \text{IND}}\ \land I$$

For this reason, once the wff $P(x)$ is identified, it suffices to provide the base case deduction $\mathcal{D}_{BC}$ and induction step $\mathcal{D}_{IS}$. The sequents are stated in such a way as to mean induction on the variable x will be the easiest approach. Always do induction on the variable x.

(a) PA $\vdash \forall x\ (0 + x = x)$

Solution:

We prove this using induction on the wff

$$P(x) : \ (0 + x = x)$$

$\mathcal{D}_{BC}$: First state and prove the base case

$$\text{PA} \vdash (0 + 0 = 0)$$

$$\dfrac{\text{PA3}}{0 + 0 = 0}\ \forall E$$

$\mathcal{D}_{IS}$: Next state and prove the induction step

$$\text{PA},\ (0 + n = n)\ \vdash\ (0 + s(n) = s(n))$$

$$\dfrac{\dfrac{\dfrac{\text{PA4}}{0 + s(n) = s(0 + n)}\ \forall E \quad \dfrac{}{\cancel{0 + n = n}}\ \text{IH}}{0 + s(n) = s(n)}\ = E}{(0 + n = n) \to (0 + s(n) = s(n))}\ \to I, \text{IH}$$

(b) $\mathsf{PA} \vdash \forall x\ (0 \times x = 0)$

Solution:

This proof is by induction on the wff

$$P(x) : \ (0 \times x = 0)$$

$\mathcal{D}_{BC}$: First state and prove the base case

$$\mathsf{PA} \vdash 0 \times 0 = 0$$

$$\dfrac{\mathsf{PA5}}{0 \times 0 = 0}\ \forall E$$

$\mathcal{D}_{IS}$: Next state and prove the induction step

$$\mathsf{PA},\ 0 \times n = 0 \ \vdash \ 0 \times s(n) = 0$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{PA6}}{0 \times s(n) = 0 \times n + 0}\ \forall E \qquad \dfrac{\mathsf{PA3}}{0 \times n + 0 = 0 \times n}\ \forall E}{0 \times s(n) = 0 \times n}\ {=}E \qquad \dfrac{}{\cancel{0 \times n = 0}}\ \mathsf{IH}}{0 \times s(n) = 0}\ {=}E}{(0 \times n = 0) \to (0 \times s(n) = 0)}\ \to I, \mathsf{IH}$$

(c) $\text{PA} \vdash \forall x\ (1 \times x = x)$

Solution:

This proof is by induction on the wff

$$P(x):\ \ s(0) \times x = x$$

$\mathcal{D}_{BC}$: First state and prove the base case

$$\text{PA} \vdash s(0) \times 0 = 0$$

$$\dfrac{\text{PA5}}{s(0) \times 0 = 0}\ \forall E$$

$\mathcal{D}_{IS}$: Next state and prove the induction step

$$\text{PA},\ s(0) \times n = n\ \vdash\ s(0) \times s(n) = s(n)$$

The deduction below makes use of an exercise above.

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{PA6}}{s(0) \times s(n) = (s(0) \times n) + s(0)}\ \forall E \qquad \dfrac{}{\cancel{s(0) \times n = n}}\ \text{IH}}{s(0) \times s(n) = n + s(0)}\ {=}E \qquad \dfrac{\text{PA}}{n + s(0) = s(n)}\ \text{THM}}{s(0) \times s(n) = s(n)}\ {=}E}{(s(0) \times n = n) \to (s(0) \times s(n) = s(n))}\ \to I, \text{IH}$$

(d) PA $\vdash \forall x\ (x = 0 \lor \exists y(x = s(y)))$ (Challenge!)

Solution:

This proof is by induction on the wff

$$P(x): \ [x = 0 \lor \exists y(x = s(y))]$$

$\mathcal{D}_{BC}$: First state and prove the base case

$$\text{PA} \vdash [0 = 0 \lor \exists y(0 = s(y))]$$

$$\dfrac{\dfrac{}{0 = 0}\ {=}I}{0 = 0 \lor \exists y\ (0 = s(y))}\ \lor I$$

$\mathcal{D}_{IS}$: Next state and prove the induction step

$$\text{PA},\ [n = 0 \lor \exists y(n = s(y))]\ \vdash\ [s(n) = 0 \lor \exists y(s(n) = s(y))]$$

Since the induction hypothesis is a disjunction, the proof will finish with a disjunction elimination step. This requires proving the following sequents along the way:

$$\text{PA}\ \vdash\ (n = 0) \to [s(n) = 0 \lor \exists y(s(n) = s(y))]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{n = 0 \to s(n) = s(0)}\ \text{THM} \qquad \dfrac{}{\cancel{n = 0}}\ 1}{s(n) = s(0)}\ \text{MP}}{\exists y\ (s(n) = s(y))}\ \exists I}{(s(n) = 0) \lor \exists y\ (s(n) = s(y))}\ \lor I}{(n = 0) \to [(s(n) = 0) \lor \exists y\ (s(n) = s(y))]}\ \to I, 1$$

$$\text{PA}\ \vdash\ \exists y\ (n = s(y)) \to [s(n) = 0 \lor \exists y(s(n) = s(y))]$$

$$\dfrac{\dfrac{\dfrac{}{\cancel{\exists y\ (n = s(y))}}\ 2 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cancel{n = s(w)}}\ 3 \qquad \dfrac{}{(n = s(w)) \to (s(n) = s(s(w)))}\ \text{THM}}{s(n) = s(s(w))}\ \text{MP}}{\exists y\ (s(n) = s(y))}\ \exists I}{(s(n) = 0) \lor \exists y\ (s(n) = s(y))}\ \lor I}{n = s(w) \to [(s(n) = 0) \lor \exists y\ (s(n) = s(y))]}\ \to I, 3}{(s(n) = 0) \lor \exists y\ (s(n) = s(y))}\ \exists E}{\exists y\ (n = s(y)) \to [(s(n) = 0) \lor \exists y\ (s(n) = s(y))]}\ \to I, 2$$

See over the page for these steps combined with the disjunction elimination to complete the proof of the induction step.

$$\mathsf{PA},\ [n = 0 \vee \exists y(n = s(y))] \ \vdash \ [s(n) = 0 \vee \exists y(s(n) = s(y))]$$

$$\dfrac{\dfrac{}{n = 0 \vee \exists y\ (n = s(y))}\,\mathsf{IH} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{n = 0 \to s(n) = s(0)}\,\mathsf{THM} \qquad \dfrac{}{\cancel{n = 0}}\,1}{s(n) = s(0)}\,\mathsf{MP}}{\exists y\ (s(n) = s(y))}\,\exists I}{(s(n) = 0) \vee \exists y\ (s(n) = s(y))}\,\vee I}{(n = 0) \to [(s(n) = 0) \vee \exists y\ (s(n) = s(y))]}\,\to I, 1 \qquad \dfrac{\dfrac{\dfrac{}{\cancel{\exists y\ (n = s(y))}}\,2 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cancel{n = s(w)}}\,3 \qquad \dfrac{}{(n = s(w)) \to (s(n) = s(s(w)))}\,\mathsf{THM}}{s(n) = s(s(w))}\,\mathsf{MP}}{\exists y\ (s(n) = s(y))}\,\exists I}{(s(n) = 0) \vee \exists y\ (s(n) = s(y))}\,\vee I}{n = s(w) \to [(s(n) = 0) \vee \exists y\ (s(n) = s(y))]}\,\to I, 3}{(s(n) = 0) \vee \exists y\ (s(n) = s(y))}\,\exists E}{\exists y\ (n = s(y)) \to [(s(n) = 0) \vee \exists y\ (s(n) = s(y))]}\,\to I, 2}{[(s(n) = 0) \vee \exists y\ (s(n) = s(y))]}\,\vee E$$

(e) PA $\vdash \forall x\, \forall y\, [s(y) + x = s(y + x)]$ (Challenge!)

Solution:

This proof is by induction on the wff

$$P(x) :\ \forall y\, [s(y) + x = s(y + x)]$$

$\mathcal{D}_{BC}$: First state and prove the base case

$$\text{PA} \vdash \forall y\, [s(y) + 0 = s(y + 0)]$$

$$\dfrac{\dfrac{\dfrac{}{s(a) = s(a)}{=}I \quad \dfrac{\text{PA3}}{s(a) + 0 = s(a)}\forall E}{s(a) + 0 = s(a)}{=}E \quad \dfrac{\text{PA3}}{a = a + 0}\forall E}{s(a) + 0 = s(a + 0)}{=}E$$

$\mathcal{D}_{IS}$: Next state and prove the induction step

$$\text{PA},\ \forall y\, [s(y) + n = s(y + n)]\ \vdash\ \forall y\, [s(y) + s(n) = s(y + s(n))]$$

$$\dfrac{\dfrac{\dfrac{\text{PA4}}{s(a) + s(n) = s(s(a) + n)}\forall E \quad \dfrac{}{s(a) + n = s(a + n)}\text{IH}}{s(a) + s(n) = s(s(a + n))} \quad \dfrac{\text{PA4}}{a + s(n) = s(a + n)}\forall E}{s(a) + s(n) = s(a + s(n))}{=}E$$

These two proofs can be pieced together with the induction rule of inference to prove the original goal.

(f) PA $\vdash \forall x\ \forall y\ \forall z\ [(y+z)+x = y+(z+x)]$ (Challenge!)

This proof is by induction on the formula

$$P(x)\ :\ \forall y\ \forall z\ [(y+z)+x = y+(z+x)]$$

$\mathcal{D}_{BC}$: First state and prove the base case

$$\mathsf{PA}\ \vdash\ \forall y\ \forall z\ [(y+z)+0 = y+(z+0)]$$

$$\dfrac{\dfrac{\dfrac{\mathsf{PA3}}{(b+c)+0 = b+c}\ \forall E \qquad \dfrac{\mathsf{PA3}}{c+0 = c}\ \forall E}{(b+c)+0 = b+(c+0)}\ {=}E}{\forall y\ \forall z\ [(y+z)+0 = y+(z+0)]}\ \forall I$$

See the next page for induction step.

$\mathcal{D}_{IS}$: Next state and prove the induction step

$$\text{PA},\ \forall y\ \forall z\ [(y+z)+n = y+(z+n)]\ \vdash\ \forall y\ \forall z\ [(y+z)+s(n) = y+(z+s(n))]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{PA}4}{(b+c)+s(n)=s((b+c)+n)}\,\forall E \qquad \dfrac{}{(b+c)+n=b+(c+n)}\,\text{IH}}{(b+c)+s(n)=s(b+(c+n))}\,{=E} \qquad \dfrac{\text{PA}4}{b+s(c+n)=s(b+(c+n))}\,\forall E}{(b+c)+s(n)=b+s(c+n)}\,{=E} \qquad \dfrac{\text{PA}4}{c+s(n)=s(c+n)}\,\forall E}{(b+c)+s(n)=b+(c+s(n))}\,{=E}$$

(g) PA $\vdash \forall x\ \forall y\ [y + x = x + y]$ (Challenge!)

Solution:

This proof is by induction on the wff

$$P(x) :\ \forall y\ [y + x = x + y]$$

$\mathcal{D}_{BC}$: First state and prove the base case

$$\text{PA} \vdash \forall y\ (y + 0 = 0 + y)$$

$$\dfrac{\dfrac{\dfrac{}{a = a}\,{=I} \qquad \dfrac{\text{PA3}}{a + 0 = a}\,\forall E}{a + 0 = a}\,{=E} \qquad \dfrac{\text{PA}}{0 + a = a}\,\text{THM}}{a + 0 = 0 + a}\,{=E}$$

$\mathcal{D}_{IS}$: Next state and prove the induction step

$$\text{PA},\ \forall y\ (y + n = n + y)\ \vdash\ \forall y\ (y + s(n) = s(n) + y)$$

$$\dfrac{\dfrac{\dfrac{\text{PA4}}{a + s(n) = s(a + n)}\,\forall E \qquad \dfrac{}{a + n = n + a}\,\text{IH}}{a + s(n) = s(n + a)}\,{=E} \qquad \dfrac{\text{PA}}{s(n) + a = s(n + a)}\,\text{THM}}{a + s(n) = s(n) + a}\,{=E}$$

See the Logic section on the course webpage for a complete proof of the commutativity of addition, if you dare.

6. Provide natural deductions of the following theorems of Peano Arithmetic.

   (a) PA, $0 < a \ \vdash \ 0 < s(a)$

   Recall that $0 < a$ and $0 < s(a)$ are ($\exists$) existential claims:

$$0 < a := \ \exists x \ a = 0 + s(x)$$

$$0 < s(a) := \ \exists x \ s(a) = 0 + s(x)$$

   As such, we must use an instance of $\exists$ elimination to make use of such an hypothesis.

$$\dfrac{0 < a \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x \ x = 0 + x}\,\text{THM}}{s(s(t)) = 0 + s(s(t))}\,\forall E \qquad \dfrac{\dfrac{\dfrac{}{a = 0 + s(a)}^{1}}{s(a) = s(0 + s(t))}\,\text{CONG} \qquad \dfrac{\dfrac{}{\forall x \ x = 0 + x}\,\text{THM}}{s(t) = 0 + s(t)}\,\forall E}{s(a) = s(s(t))}\,= E}{s(a) = 0 + s(s(t))}\,= E}{0 < s(a)}\,\exists I}{a = 0 + s(a) \to 0 < s(a)}\,\to I, 1}{0 < s(a)}\,\exists E$$

   (b) PA, $a < b \ \vdash \ s(a) < s(b)$

$$a < b := \exists x \ b = a + s(x)$$

$$s(a) < s(b) := \exists x \ s(b) = s(a) + s(x)$$

$$\dfrac{a < b \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{b = a + s(t)}^{1}}{s(b) = s(a + s(t))}\,\text{CONG} \qquad \dfrac{\dfrac{}{\forall x \ \forall y \ s(x) + y = s(x + y)}\,\text{THM}}{s(a) + s(t) = s(a + s(t))}\,\forall E}{s(b) = s(a) + s(t)}\,= E}{s(a) < s(b)}\,\exists I}{b = a + s(t) \to s(a) < s(b)}\,\to I, 1}{s(a) < s(b)}\,\exists E$$

(c) PA, $(a < b) \wedge (b < c) \vdash a < c$

(d) PA $\vdash$ $\forall x \; [(x = 0) \vee (0 < x)]$ (Challenge!)

(e) PA $\vdash$ $\forall x\, \forall y\, [\neg(x < y) \to ((x = y) \vee (y < x))]$ (Challenge!)

(f) PA $\vdash \forall x \, \forall y \, [(x \leq y) \wedge (y \leq x)] \to x = y$ (Challenge!)