# Class February 23

1. Direct Products and Finite Abelian Groups

**Definition 0.1.** $G$ always denotes a group. G is the inner direct product of the subgroups $A, B \leq G$ if i) $A \lhd G$, $B \lhd G$ ii) $A \cap B = \{e\}$ iii) $G = AB$. The notation for direction products is $G = A \times B$.

**Lemma 0.2.** *Assume $G = A \times B$.*
*(a) $A$ and $B$ commute element-wise i.e. $ab = ba$ $\forall a \in A, b \in B$.*
*(b) if $A$ and $B$ are abelian, then so is $G$.*

*Proof.* (a) Consider the commutators $[a, b] := (aba^{-1})b^{-1} = a(ba^{-1}b^{-1}) \in A \cap B = \{e\}$
$\implies aba^{-1}b^{-1} = e \implies ab = ba\ \forall a \in A, b \in B$
(b) $g_1, g_2 \in G \implies \exists a_1, a_2 \in A, b_1, b_2 \in B$ s.t. $g_1 = a_1b_1$ and $g_2 = a_2b_2 \implies g_1g_2 = a_1b_1a_2b_2 = a_1a_2b_1b_2$ and because $A, B$ are abelian, this equals $a_2(a_1b_2)b_1 = a_2b_2a_1b_1 = g_2g_1$ $\square$

**Example 0.3.** (a) $V =< (12)(34) > \times < (13)(24) >\cong \mathbb{Z}_2 \times \mathbb{Z}_2$
(b) $U(8) = \{[1], [3], [5], [7]\} =< [3] > \times < [5] >\cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
(c) $\mathbb{Z}_6 =< [3] >\cong \mathbb{Z}_3\times < [2] >\cong \mathbb{Z}_2$
(d) $D_6 = \{b^i, ab^i \mid 0 \leq i \leq 5\}$ such that $a^2 = b^6 = e$, $aba^{-1} = aba = b^{-1}$. Therefore, $D_6 \cong< b^3 > \times\{e, b^2, b^4, a, ab^2, ab^4\} \implies D_6 \cong \mathbb{Z}_2 \times D_3$
(e) By contrast, neither $D_4$ nor $Q_8$ can be written as direct products of two proper subgroups (Exercise).
(f) Trivially, $\forall G, G = G \times \{e\}$

**Lemma 0.4.** *If $|G|= p^2$, $p$ is prime, then either $G$ is cyclic or $G = A \times B(\cong \mathbb{Z}_p \times \mathbb{Z}_p)$ with subgroups $A$ and $B$ of order $p$.*

# Class February 26