Some Ring Theory Class Notes

Class March 12

Conventions regarding 1 (multiplicative unity):

1. Every ring R has a multiplicatuve unity denoted by 1 or 1_R such that $1 * a = a * 1 \; \forall \; a \in R$. Note: $1 = 0$ in $R \Leftrightarrow R = \{0\}$ because $\forall a \in R$: $a = a * 1 = a * 0 = 0$.

2. Any subring S of R must contain 1_R. For subring, check

   (a) $1_R \in S$
   (b) $a \in S \implies -a \in S$
   (c) $a, b \in S \implies a + b \in S$
   (d) $a, b \in S \implies ab \in S$

   Note: An ideal I of R is a subring if and only if $I = R$ ($1 \in I \implies a = a * 1 \in I \; \forall \; a \in R$).

   Example 0.1. $R \times \{0\} = \{(a, 0) \mid a \in R\}$ is not a subring of $R \times R$ if $R \neq \{0\}$ since $(1, 1) \notin R \times \{0\}$. But $\{(a, a) \mid a \in R\}$ is a subring of $R \times R$.

3. For any ring homomorphism $\varphi : R \to S$ we require $\varphi(1_R) = 1_S$. Note that this is not a consequence of the other ring homomorphism properties:

   (a) $\varphi(a + b) = \varphi(a) + \varphi(b) \; \forall a, b \in R$
   (b) $\varphi(ab) = \varphi(a)\varphi(b) \; \forall \; a, b \in R$

   $\varphi(0) = 0$ is a consequence of (a): $\varphi(0) = \varphi(0+0) = \varphi(0)+\varphi(0) \implies 0 = \varphi(0)$. For multiplication, $\varphi(1) = \varphi(1 * 1) = \varphi(1) * \varphi(1)$ does not necessarily imply $1 = \varphi(1)$ since $\varphi(1)$ need not have a multiplicative inverse in S.

   Example 0.2. $\varphi : R \to R \times R$ which maps $a \to (a, 0)$ is NOT a ring homomorphism since $\varphi(1_R) = (1_R, 0) \neq 1_{R \times R}$ if $R \neq \{0\}$

   Example 0.3. $\psi : R \to R \times R$ which maps $a \to (a, a)$ is a ring homomorphism.

4. For an integral domain R (commutative without zero divisors) we also require $1 \neq 0 \Leftrightarrow R \neq \{0\}$ (neither integral domain nor a field)

   Example 0.4. (a) of fields: $\mathbb{R}, \mathbb{Z}_p$ (p prime), $\mathbb{Q}, \mathbb{C}$. $\mathbb{Q}(\sqrt{2}) := \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ subfield of $\mathbb{R}$. Check: $0 \neq x \in \mathbb{Q}(\sqrt{2}) \implies x^{-1} \in \mathbb{Q}(\sqrt{2})$ (need $\sqrt{2} \notin \mathbb{Q}$).

   (b) of integral domains which are no fields: $\mathbb{Z}$, when n is a prime $\implies \mathbb{Z}_n$ is an integral domain, but also a field. When n is not a prime $\implies \mathbb{Z}_n$ has zero divisors and isn't an integral domain. Specifically $\exists \; l, m \in \mathbb{N}$, $1 < l, m < n$ such that $n = lm \rightsquigarrow$ (modulo n). $[0] = [n] = [lm] = [l][m]$ in $\mathbb{Z}_n$ (such that $[l] \neq [0]$ and $[m] \neq [0]$.

   (c) $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ subring of $\mathbb{C}$; $\mathbb{Z}[\sqrt{2}]$ is a subring of $\mathbb{R}$.

   (d) commutative rings which are not integral domains. $\mathbb{Z}_n$, n is not prime. $\mathbb{Z} \times \mathbb{Z}$ has zero divisors e.g. $(1, 0) * (0, 1) = (0, 0)$.

   (e) of non-commutative rings:

      i. $M(n, R)$, $n \geq 2$ and R any ring $\neq \{0\}$. $\exists A, B \in M(n, R)$ such that $AB \neq BA$

      ii. Hamilton's quaternions $\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$ ($\cong \mathbb{R}^4$ as abelian group). Multiplication is induced by that $\mathbb{Q}$ and distributive laws $\rightsquigarrow$ example of skew field or division ring.