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1. Direct Products and Finite Abelian Groups

Definition 0.1. G always denotes a group. G is the inner direct product of the subgroups $A, B \leq G$ if i) $A \lhd G$, $B \lhd G$ ii) $A \cap B = \{e\}$ iii) $G = AB$. The notation for direction products is $G = A \times B$.

Lemma 0.2. *Assume $G = A \times B$.*
(a) A and B commute element-wise i.e. $ab = ba$ $\forall a \in A, b \in B$.
(b) if A and B are abelian, then so is G.

Proof. (a) Consider the commutators $[a, b] := (aba^{-1})b^{-1} = a(ba^{-1}b^{-1}) \in A \cap B = \{e\}$
$\implies aba^{-1}b^{-1} = e \implies ab = ba$ $\forall a \in A, b \in B$
(b) $g_1, g_2 \in G \implies \exists a_1, a_2 \in A, b_1, b_2 \in B$ s.t. $g_1 = a_1b_1$ and $g_2 = a_2b_2 \implies g_1g_2 = a_1b_1a_2b_2 = a_1a_2b_1b_2$ and because A, B are abelian, this equals $a_2(a_1b_2)b_1 = a_2b_2a_1b_1 = g_2g_1$ $\square$

Example 0.3. (a) $V =< (12)(34) > \times < (13)(24) > \cong \mathbb{Z}_2 \times \mathbb{Z}_2$
(b) $U(8) = \{[1], [3], [5], [7]\} =< [3] > \times < [5] > \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
(c) $\mathbb{Z}_6 =< [3] > \cong \mathbb{Z}_3 \times < [2] > \cong \mathbb{Z}_2$
(d) $D_6 = \{b^i, ab^i \mid 0 \leq i \leq 5\}$ such that $a^2 = b^6 = e$, $aba^{-1} = aba = b^{-1}$. Therefore, $D_6 \cong < b^3 > \times \{e, b^2, b^4, a, ab^2, ab^4\} \implies D_6 \cong \mathbb{Z}_2 \times D_3$
(e) By contrast, neither D_4 nor Q_8 can be written as direct products of two proper subgroups (Exercise).
(f) Trivially, $\forall G, G = G \times \{e\}$

Lemma 0.4. *If $|G| = p^2$, p is prime, then either G is cyclic or $G = A \times B (\cong \mathbb{Z}_p \times \mathbb{Z}_p)$ with subgroups A and B of order p.*

Proof. G is a p-group so the center $Z(G)$ is nontrivial. Assume $Z(G) = p \implies G/Z(G) \cong \mathbb{Z}_p \implies G/Z(G)$ is cyclic and therefore G is abelian. But this is a contradiction, because if G is abelian, $|G| = |Z(G)|$ by definition.Therefore, we know that G is abelian and $G = Z(G)$.
Assume G is not cyclic $\implies |g| = p$ $\forall g \in G \setminus \{e\}$. Pick any $a \in G \setminus \{e\}$ and set $A =< a > \leq G \implies |a| = |A| = p$. Therefore, $|G \setminus A| = p^2 - p > 0 \implies G \setminus A \neq \emptyset$. Pick $b \in G \setminus A$ and set $B :=< b > \leq G \implies |b| = |B| = p$.
Now, check (1) $A \lhd G, B \lhd G$ because G is abelian. (2) $A \cap B = \{e\}$. If $e \neq x \in A \cap B \implies |x| = p \implies A =< x >= B \implies b \in A$ which is a contradiction. (3) $G = AB$. $AB = \frac{|A||B|}{|A \cap B|} = \frac{p*p}{1} = p^2 = |G| \implies G = AB$ $\square$

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Lemma 0.5. *If $G = A \times B$ with subgroups $A, B \leq G$, then $|ab| = lcm(|a|, |b|)$ if $a \in A, b \in B$*

Remark 0.6. If $a, b \in G$ with $|a|, |b| < \infty$, then $|ab| \mid lcm(|a|, |b|)$ of $ab = ba$. If $ab \neq ba$, you cannot say anything about $|ab|$. If $ab = ba$, then $|ab| < lcm(|a|, |b|)$ is possible in general (e.g. $b = a^{-1}$). If $ab = ba$ and $gcd(|a|, |b|) = 1$, then $|ab| = |a||b| = lcm(|a|, |b|)$. This uses the fact that $< a > \cap < b >= \{e\}$.

Definition 0.7. The (outer) direct product of the groups A, B is defined as $A \times B = \{(a, b) | a \in A, b \in B\}$ as set with a binary operation $\implies (a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2)$ $\forall a_1, a_2 \in A$ and $\forall b_1, b_2 \in B$.

This yields a group since (i) $A \times B$ satisfies associativity since A and B do (ii) $e_{A\times B} = (e_A, e_B)$ and (iii) $(a,b)^{-1} = (a^{-1}, b^{-1})$ for $a \in A, b \in B$.
Define $\iota_A : A \to A \times B$ which maps $a \to (a, e)$ and $\iota_B : B \to A \times B$ which maps $b \to (e, b)$. Then, ι_A, ι_B are injective group homomorphisms.
$A \cong \iota_A(A) =: A' = \{(a,e) | a \in A\} \leq A \times B$ and $B \cong \iota_B(B) =: B' = \{(e,b) | b \in B\} leq A \times B$.

Remark 0.8. Properties of subgroups A', B' of $A \times B$:
(1) $A', B' \lhd A \times B$, e.g. $(\widetilde{a}, b)(a, e)(\widetilde{a}^{-1}, b^{-1}) = (\widetilde{a}a\widetilde{a}^{-1}, beb^{-1}) = (\widetilde{a}a\widetilde{a}^{-1}, e) \in A'$
(2) $A' \cap B' = \{(a,b) \in A \times B | b = e, a = e\} = \{(e,e)\}$.
(3) $G = A'B' \implies$ given $(a,b) \in A \times B$, then $(a,b) = (a,e)(e,b)$
The consequence is that the outer product equals the inner product such that $A \times B = A' \times B'$

Lemma 0.9. *Assume that $G = A \times B$ (inner; $A, B \leq G$) and that A', B' are groups with $A' \cong A$ and $B' \cong B$. Then, $G \cong A' \times B'$ (outer).*

An application of this is that if $|G| = A \times B$ with $|A| = |B| = p \implies G \cong \mathbb{Z}_p \times \mathbb{Z}_p$ (outer).

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