5. Hats in a box

Problem 5. Hats in a box

4/5 points (graded)

Each one of n persons, indexed by $1, 2, \ldots, n$, has a clean hat and throws it into a box. The persons then pick hats from the box, at random. Every assignment of the hats to the persons is equally likely. In an equivalent model, each person picks a hat, one at a time, in the order of their index, with each one of the remaining hats being equally likely to be picked. Find the probability of the following events.

(You need to answer all 5 questions before you can submit.)

- 1. Every person gets his or her own hat back.
 - $\frac{1}{n!}$
 - $\begin{array}{c} \bigcirc \quad \frac{1}{(n+1)!} \end{array}$
 - $\frac{1}{n}$
 - $\begin{array}{cc} \odot & \frac{1}{n+1} \end{array}$
- 2. Each one of persons $1, \ldots, m$ gets his or her own hat back, where $1 \leq m \leq n$.
 - $\frac{(n+m)!}{n!}$

$$\frac{n!}{(n+m)!}$$

 $\bigcirc \frac{m!}{n!}$

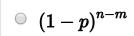
3. Each one of persons $1,\ldots,m$ gets back a hat belonging to one of the last m persons (persons $n-m+1,\ldots,n$), where $1\leq m\leq n$.

$$\begin{array}{c}
\bullet \\
\frac{1}{\binom{n}{m}}
\end{array}$$

- $igodots rac{m}{igodom{n}{m}}$
- $egin{array}{c} n m \\ n \\ m \end{array}$
- $igodots rac{n}{igodom{n}{m}}$

Now assume, in addition, that every hat thrown into the box has probability \boldsymbol{p} of getting dirty (independently of what happens to the other hats or who has dropped or picked it up). Find the probability that:

4. Persons $1, \ldots, m$ will pick up clean hats.



$$\bullet$$
 $m(1-p)^m \times$

$$\bigcirc \ (1-p)^m \checkmark$$

$$0 m(1-p)^{n-m}$$

5. Exactly m persons will pick up clean hats.

$$\stackrel{\textstyle \bigcirc}{=} \frac{\binom{n}{m}}{n!} (1-p)^m p^{n-m}$$

$$\bigcirc (1-p)^m p^{n-m}$$

$$\binom{n}{m}(1-p)^{n-m}p^m$$

$$igoplus \binom{n}{m}(1-p)^mp^{n-m} \checkmark$$

Solution:

- 1. Consider the sample space of all possible hat assignments. It has n! elements (n hat selections for the first person, after that n-1 for the second, etc.), with every assignment equally likely; hence each assignment has probability 1/n!. The event that everyone gets his or her own hat back corresponds to exactly one of these n! assignments. Therefore, the answer is 1/n!.
- 2. Consider the same sample space and probabilities as in the solution of part 1. The event of interest assigns the first m people to their own hats and allows for an arbitrary assignment of hats to the remaining n-m persons, so that there are (n-m)! possible assignments. The

- probability of an event with (n-m)! elements is (n-m)!/n!.
- 3. Consider the m hats belonging to the last m persons. There are m! ways to distribute these m hats among the first m persons. Then, there are (n-m)! ways to distribute the remaining n-m hats to everyone else. The probability of an event with m!(n-m)! elements is m!(n-m)!/n!, which is equal to $1/\binom{n}{m}$.
- 4. The probability of a given person picking up a clean hat is 1-p. By the independence assumption, the probability of m specific persons picking up clean hats is $(1-p)^m$.
- 5. Think of picking a clean hat as an independent Bernoulli trial with success probability 1-p. The probability of m successes out of n trials is $\binom{n}{m}(1-p)^mp^{n-m}$.

提交

You have used 2 of 2 attempts

Answers are displayed within the problem

讨论

Topic: Unit 3 / Problem Set / 5. Hats in a box

显示讨论

Learn About Verified Certificates

© All Rights Reserved