2. Introduction to Goodness of Fit Tests

Recap of Parametric Hypothesis Testing: The Uniform Statistical Model

1/1得分 (计入成绩)

Let X be a uniform random variable with the distribution $\textbf{Unif}\,[0, \theta^*]$.

We would like to test whether $H_0 : \theta^* = 2$ or $H_1 : \theta^* \neq 2$, with $\Theta = (0, \infty)$.

Let $X_1, \ldots, X_n$ be iid samples of X.

- Let $\overline{X_n}$ denote sample mean.
- Let $\widetilde{S}_n$ denote the unbiased sample variance of $X_1, \ldots, X_n$.
- Let $\widehat{\theta_n}^{\text{MLE}}$ denote the maximum likelihood estimator of θ.
- Let $\ell_n\left(\widehat{\theta_n}^{\text{MLE}}\right)$ denote the log-likelihood of n samples evaluated at the maximum likelihood estimator and $\ell_n(2)$ denote the log-likelihood of n samples under H_0.

Select from the following the tests that are **technically correct (that is, can be applied under this scenario) and that have the required level** $\alpha \in [0, 1]$.

Note: By asymptotic level of α, we require the probability of type-1 error under H_0 be at most α as $n \to \infty$. By non-asymptotic level of α, we require the probability of type-1 error under H_0 be at most α for every n.

- [ ] $\mathbf{1}\left\{\frac{|\overline{X_n}-1|}{\sqrt{\widetilde{S}_n/n}} > q_{\alpha/2}\right\}$ for non-asymptotic level α, where $q_{\alpha/2}$ is the $(1-\alpha/2)$-quantile of the Student's T distribution with $n-1$ degrees of freedom.
- [x] $\mathbf{1}\left\{\sqrt{n}\frac{|2\overline{X_n}-2|}{\sqrt{4/3}} > q_{\alpha/2}\right\}$ for asymptotic level α, where q_α is the $(1-\alpha)$-quantile of the standard normal random variable.
- [x] $\mathbf{1}\left\{\widehat{\theta_n}^{\text{MLE}} > 2 \text{ or } \widehat{\theta_n}^{\text{MLE}} \leq 1\right\}$ for asymptotic level α. 只要theta小于1，n接近正无穷了以后就会变成0。
- [ ] $\mathbf{1}\left\{2\left(\ell_n\left(\widehat{\theta_n}^{\text{MLE}}\right) - \ell_n(2)\right) > q_\alpha\right\}$ for asymptotic level α, where q_α is the $(1-\alpha)$-quantile of χ^2_1.

Solution:

Recall from Lecture 10 that the maximum likelihood estimator for the uniform statistical model is $\max_{i=1,\ldots,n} X_i$.

Technically incorrect: First, let us elaborate on the choices that are technically incorrect under this scenario: **Choices 1 and 4**. The first choice is attempting a Student's T test for non-asymptotic level α and this is technically incorrect because X is not a Gaussian random variable. Only if X is a Gaussian random variable will the test statistic (at least to our knowledge in this course) follow a Student's T distribution for a finite number of samples n.

Choice 4 is attempting a likelihood ratio test using the log-likelihoods evaluated at the maximum likelihood estimator and under H_0 and this choice is also technically incorrect because the MLE technical conditions are not satisfied for the uniform statistical model (recall the technical conditions for asymptotic normality of the MLE). Only if the MLE conditions are satsfied can this test be applied according to Wilk's theorem. regularity conditions

Technically correct and have an asymptotic level α: **Choices 2 and 3**. The third choice has an asymptotic level 0 because of the following:

$$
\begin{aligned}
P_{H_0}\left[\widehat{\theta_n}^{\text{MLE}} > 2 \text{ or } \widehat{\theta_n}^{\text{MLE}} \le 1\right] &= P_{H_0}\left[\widehat{\theta_n}^{\text{MLE}} > 2\right] + P_{H_0}\left[\widehat{\theta_n}^{\text{MLE}} \le 1\right] \\
&= P_{H_0}\left[\widehat{\theta_n}^{\text{MLE}} \le 1\right] \\
&= P_{H_0}\left[\max_{i=1,\ldots,n} X_i \le 1\right] \\
&= \left(\frac{1}{2}\right)^n \to 0.
\end{aligned}
$$

The second choice is a test that is both technically correct and has an asymptotic level α. This can be seen from the following:

- $2\overline{X_n}$ has an expectation equal to θ^*.
- The variance of $2\overline{X_n}$ under H_0 is

$$
\begin{aligned}
\text{Var}_{H_0}\left(2\overline{X_n}\right) &= \frac{4}{n}\text{Var}_{H_0}(X) \\
&= \frac{4}{n}\frac{4}{12} \\
&= \frac{4}{3n}.
\end{aligned}
$$

- An application of central limit theorem.

Remark: Wald's test cannot be written out because Fisher information does not exist for the uniform random variable.

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Answers are displayed within the problem

Goodness of Fit Tests: Motivation

that we may ask you we're looking at the t-test

where Gaussian assumption was actually, quite important.

So, we're going to want this.

And here, again, this looks like a parametric assumption, right?

It's a suppressant trick distribution.

But I'm really trying to testify in Galveston and implicitly

versus anything else.

Does x have a uniform distribution

with parameters 0 and 1 for the interval,

or am I anything else?

So, that's for the continuous cases.

And for the discrete case-- we might ask--

do I have this PMF or do I have any other PMF?

And actually turns out that the last one-- of course,

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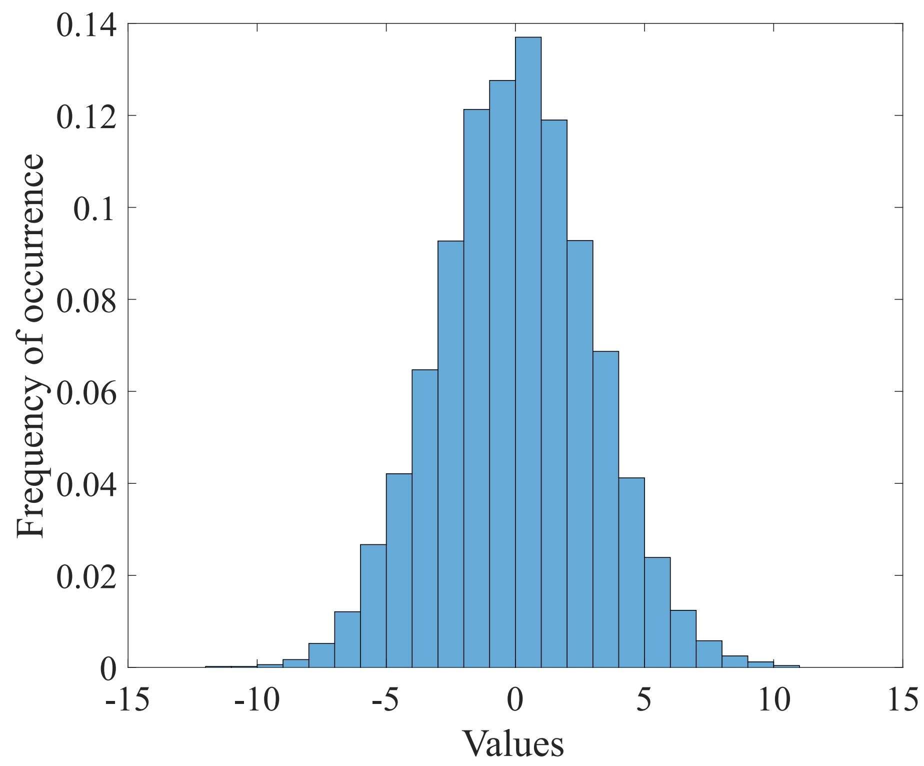
Intuition for Goodness of Fit Tests

1/1得分 (计入成绩)

In the topic goodness of fit testing, we want to decide whether our data can be modeled by a specific type of distribution (**e.g.**, uniform, Gaussian, Poisson). In practice, a useful tool for making such a decision is to use a **histogram** of the data set.

A histogram for a sample data set is shown below. The x-axis, which represents the sample space, is divided into the intervals $[i, i+1]$ for all $i \in \mathbb{Z}$. The bar over the interval $[i, i+1]$ represents **how many** data points took values in that interval.



Based on the histogram above, which of the following is a reasonable choice of distribution for our data?

- Bernoulli
- Uniform on the interval $[0, 1]$
- Gaussian
- χ^2 with some number of degrees of freedom

Solution:

First we examine the incorrect choices.

- The choices "Bernoulli" and "Uniform on the interval $[0, 1]$" are incorrect. We observe data points outside of the interval $[0, 1]$, so there is no way that our data could be distributed as either of these distributions.
- The choice "χ^2 with some number of degrees of freedom" is also incorrect. A χ^2 distributed random variable will not take negative values. Since our data set includes negative observations, it is impossible for the underlying distribution to be χ^2.

By process of elimination, the correct response is "Gaussian". The histogram seems to interpolate a bell-curved profile, so it is intuitive to conclude that our data is distributed as a Gaussian.

Remark: The histogram is a useful tool for gaining intuition as to what distribution the data follows. In the remainder of this unit, we will develop more rigorous methods to examine whether a particular data set is drawn from a given distribution.

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Concept Check: Terminology

2/3得分 (计入成绩)

Suppose you observe iid samples $X_1, \ldots, X_n \sim P$ from some **unknown** distribution $\mathbf{P}$. Let $\mathcal{F}$ denote a family of known distributions (for example, $\mathcal{F}$ could be the family of normal distributions $\{\mathcal{N}(\mu, \sigma^2)\}_{\mu \in \mathbb{R}, \sigma^2 > 0}$).

In the topic of **goodness of fit testing**, our goal is to answer the question "**Does $\mathbf{P}$ belong to the family $\mathcal{F}$?**".

In particular, parametric hypothesis testing is a particular case of goodness of fit testing (why?). However, we emphasize that in the context of parametric hypothesis testing, you must have a **parametric** statistical model for the data.

Categorize the following problems as examples of parameter estimation (as studied in Unit 2), parametric hypothesis testing (as studied in the previous two lectures), or goodness of fit testing (as introduced in the above video). (Choose all categories that apply.)

Problem 1: Estimate the bias of an unfair coin.

- [x] Parameter estimation
- [ ] Parametric hypothesis testing
- [ ] Goodness of fit testing

Problem 2: Decide if a 6-sided die is fair or not.

- [ ] Parameter estimation
- [x] Parametric hypothesis testing
- [ ] Goodness of fit testing

Problem 3: Decide if the heights of pine trees in Canada have a Gaussian distribution. Assume that the statistical model for this data is $\{\mathbb{R}, \Pi\}$ where Π denotes the set of **all** probability distributions with sample space $\mathbb{R}$. (In particular, this model is **non-parametric**.)

- [ ] Parameter estimation
- [ ] Parametric hypothesis testing
- [x] Goodness of fit testing

Solution:

We examine the problems in order.

- The first choice is an example of **parameter estimation**. The key-word **estimate** confirms that this is the correct response. We are not performing a hypothesis test on where the true parameter (the bias) lies in the parameter space. Rather, we want to come up with a good approximiation for the bias.

- The second choice is an example of **parametric hypothesis testing** and **goodness of fit testing**. A six-sided die has a statistical model $(\{1,2,3,4,5,6\}, \{\mathbf{P}_{\mathbf{p}}\}_{\mathbf{p}})$, where $\mathbf{p}$ could be any valid pmf. Testing if the die is fair is the same as testing if the true parameter is $(1/6, 1/6, 1/6, 1/6, 1/6, 1/6)$. Moreover, this is a goodness of fit test, because we want to figure out whether or not the **uniform distribution** on $\{1,2,3,4,5,6\}$ is a good fit for our data.

- The third choice is an example of **goodness of fit testing** only. For this example, the family of distributions is $\mathcal{F} = \{\mathcal{N}(\mu, \sigma^2)\}_{\mu \in \mathbb{R}, \sigma^2 > 0}$, and we want to figure out if the tree-height data can be modeled by some distribution in this family. Because our statistical model is non-parametric, this is not an example of parametric hypothesis testing or parameter estimation.

Remark: In general, goodness of fit testing is considered a topic in **non-parametric statistics**, in contrast to the material we have covered so far. You should keep in mind though that the topic of parametric hypothesis testing is a special case of goodness of fit testing. However, to handle non-parametric models we will need to develop new techniques.

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Answers are displayed within the problem