2. Independent normal random variables

Problem 1. Independent normal random variables

4/5 points (graded)

Let U, V, and W be independent standard normal random variables (that is, independent normal random variables, each with mean 0 and variance 1), and let $X = 5U + 12V$ and $Y = U - W$.

Standard Normal Table

Show

1. What is the probability that $X \geq 2.6$? (Give 3 decimal digits.)

   $\mathbf{P}(X \geq 2.6) =$ 0.4207 ✔ **Answer:** 0.421

2. $\mathbf{E}[XY] =$ 5 ✔ **Answer:** 5

3. $\text{Var}(X+Y) =$ 181 ✔ **Answer:** 181

4. Let H be a normal random variable with mean zero, and variance equal to 2. Let $a = \mathbf{E}[|H|]$. Find a.

   $a = \mathbf{E}[|H|] =$ 1/sqrt(pi) ✘ **Answer:** 2/sqrt(pi)

5. In this final part of the problem, we will find $\mathbf{E}[\max\{U, V\}]$, using the following argument. First, note that,

$$\max\{U,V\} - \min\{U,V\} = |U - V|,$$
$$\max\{U,V\} + \min\{U,V\} = U + V.$$

   Using your answer to previous part, and using the formulas above, obtain the answer, symbolically, as a function of the constant a defined in previous part.

$\mathbf{E}[\max\{U, V\}] =$ a/2 ✔ Answer: a/2

$\frac{a}{2}$

Solution:

1. Since X is a sum of independent normal random variables, X is also normal. Its mean and variance are, $\mathbf{E}[X] = \mathbf{E}[5U + 12V] = 5\mathbf{E}[U] + 12\mathbf{E}[V] = 0$, and $\mathsf{Var}(X) = \mathsf{Var}(5U + 12V) = 25 \cdot \mathsf{Var}(U) + 144 \cdot \mathsf{Var}(V) = 169$. Hence, letting N be a standard normal,

$$\begin{aligned} \mathbf{P}(X \geq 2.6) &= \mathbf{P}\left(\frac{X-0}{13} \geq \frac{2.6-0}{13}\right) \\ &= \mathbf{P}(N \geq \frac{2.6}{13}) \\ &= 1 - \Phi(0.2) \\ &\approx 1 - 0.579 \\ &= 0.421. \end{aligned}$$

2. Since U, V, and W are zero-mean and independent, we have,

$$\begin{aligned} \mathbf{E}[XY] &= \mathbf{E}[(5U + 12V)(U - W)] \\ &= \mathbf{E}[5U^2 - 5UW + 12UV - 12VW] \\ &= 5\mathbf{E}[U^2] - 5\mathbf{E}[U]\mathbf{E}[W] + 12\mathbf{E}[U]\mathbf{E}[V] - 12\mathbf{E}[V]\mathbf{E}[W] \\ &= 5. \end{aligned}$$

3. For this part, note that $X + Y = 6U + 12V + W$. Since U, V, and W are independent, we have,

$$\begin{aligned} \mathsf{Var}(X+Y) &= \mathsf{Var}(6U + 12V + W) \\ &= \mathsf{Var}(6U) + \mathsf{Var}(12V) + \mathsf{Var}(W) \\ &= 36 \cdot \mathsf{Var}(U) + 144 \cdot \mathsf{Var}(V) + \mathsf{Var}(W) \\ &= 181. \end{aligned}$$

4. For this part, we will integrate $|H|$ with respect to the density of H.

$$\begin{aligned} \mathbf{E} &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi}} e^{-h^2/4} |h| \, dh \\ &= 2\int_{0}^{\infty} \frac{1}{\sqrt{4\pi}} h e^{-h^2/4} \, dh \\ &= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} h e^{-h^2/4} \, dh. \end{aligned}$$

Using a change of variables, $y = h^2/2$, we have, $dy = h\ dh$, and the integral becomes,

$$\frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-y/2}\ dy = \frac{1}{\sqrt{\pi}} 2e^{-y/2}|_0^{\infty} = \frac{2}{\sqrt{\pi}}(1-0) = \frac{2}{\sqrt{\pi}}.$$

5. Finally, for this part, notice that $U - V$ is a normal distribution with mean 0, and variance 2. Hence, from the previous part, $\mathbf{E}[|U-V|] = 2/\sqrt{\pi} = a$. Then, using the given formulas, we have,

$$2\mathbf{E}[\max\{U,V\}] = \mathbf{E}[|U-V|] + \mathbf{E}[U+V] = a.$$

Therefore,

$$\mathbf{E}[\max\{U,V\}] = a/2.$$

提交 You have used 2 of 3 attempts

Answers are displayed within the problem

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