16. Exercise: Moments of the Beta distribution
Exercise: Moments of the Beta distribution

2/2 points (graded)
Suppose that Θ takes values in $[0, 1]$ and its PDF is of the form

$$f_\Theta(\theta) = a\theta(1 - \theta)^2, \quad \text{for } \theta \in [0, 1],$$

where a is a normalizing constant.

Use the formula

$$\int_0^1 \theta^\alpha (1 - \theta)^\beta \, d\theta = \frac{\alpha! \, \beta!}{(\alpha + \beta + 1)!}$$

to find the following:

a) $a =$ | 12 | ✔ **Answer:** 12

b) $\mathbf{E}[\Theta^2] =$ | 1/5 | ✔ **Answer:** 0.2

Solution:

a) Let $\mathrm{I}(\alpha, \beta)$ be the integral in the formula given in the problem statement. The normalizing constant must be equal to $1/\mathrm{I}(1, 2)$: this is needed for the PDF to integrate to 1. We have $\mathrm{I}(1, 2) = 2!/4! = 1/12$, so that $a = 12$.

b)

$$\mathbf{E}[\Theta^2] = \int_0^1 \theta^2 f_\Theta(\theta) \, d\theta = \int_0^1 a\theta^3 (1 - \theta)^2 \, d\theta = a \cdot \mathrm{I}(3, 2) = 12 \cdot \frac{3! \, 2!}{6!} = \frac{1}{5}.$$