Problem 3

Setup:

As on the previous page, let $X_1, \ldots, X_n$ be i.i.d. with pdf

$$f_\theta(x) = \theta x^{\theta-1} \mathbf{1}(0 \leq x \leq 1)$$

where $\theta > 0$.

(a)

2/2 points (graded)

Assume we do not actually get to observe $X_1, \ldots, X_n$. Instead let $Y_1, \ldots, Y_n$ be our observations where $Y_i = \mathbf{1}(X_i \leq 0.5)$. Our goal is to estimate θ based on this new data.

What distribution does Y_i follow?

First, choose the type of the distribution:

- (•) Bernoulli
- ( ) Poisson
- ( ) Normal
- ( ) Exponential

Second, enter the parameter of this distribution in terms of θ. Denote this parameter by m_θ. (If the distribution is normal, enter only 1 parameter, the mean).

$m_\theta =$ 0.5^theta **Answer:** 1/(2^theta)

0.5^θ

STANDARD NOTATION

Solution:

Note that Y is distributed over only two values and therefore is a distributed as a Bernoulli random variable. The parameter of the Bernoulli is

$$\mathbf{P}[Y=1] = \int_0^{1/2} \theta x^{\theta-1} dx = \frac{1}{2^\theta}$$

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Answers are displayed within the problem

(b)

1/1 point (graded)
Write down a statistical model associated to this experiment. Is the parameter θ identifiable?

- [x] Yes
- [ ] No

Solution:

Yes it is identifiable since $2^{-\theta}$, which is the parameter of the Bernoulli, is an injective function of θ.

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(c)

0.0/2.0 points (graded)
Compute the Fisher information $I(\theta)$.
(To answer this question correctly, your answer to part (a) needs to be correct.)

$I(\theta) =$ 1/((1/2)^theta*(1-(1/2)^theta)) **Answer:** (ln(2))^2/(2^theta-1)

STANDARD NOTATION 操这里我算错了

$$\mathrm{Ber}(m_\theta),\ m_\theta = (\tfrac{1}{2})^\theta = 2^{-\theta}$$

$$L(m_\theta) = m_\theta^{\,x} \cdot (1-m_\theta)^{1-x}$$

$$\ell(\theta) = \ln L(m_\theta) = -\theta \cdot x \cdot \ln 2 + (1-x)\cdot \ln(1-2^{-\theta})$$

$$\ell''(\theta) = \frac{2^\theta (x-1) \ln^2(2)}{(2^\theta - 1)^2}.$$

$$E(x-1) = (\tfrac{1}{2})^\theta - 1 = \frac{1-2^\theta}{2^\theta}$$

$$-E[\ell''(\theta)] = \frac{\ln^2(2)}{2^\theta - 1}$$

Solution:

The log likelihood of an observation Y is

$$\ell(Y;\theta) = \mathbf{1}_{Y=0} \ln(1-2^{-\theta}) + \mathbf{1}_{Y=1} \ln(2^{-\theta})$$

$$= \mathbf{1}_{Y=0} \ln(1-2^{-\theta}) - \theta \mathbf{1}_{Y=1} \ln(2).$$

Taking the second derivative one finds that

$$\ell''(Y;\theta) = \mathbf{1}_{Y=0} \frac{2^\theta (\ln(2))^2}{(2^\theta - 1)^2}$$

and therefore the Fisher Information is

$$I(\theta) = \mathbb{E}[-\ell''(Y;\theta)] = \frac{(\ln(2))^2}{2^\theta - 1}.$$

(Note that by definition, the Fisher Information does not depend on n.)

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Answers are displayed within the problem

(d)

2.0/2.0 points (graded)
Compute the maximum likelihood estimator $\hat{\theta}$ for θ in terms of $\overline{Y_n}$.

(Enter **barY_n** for $\overline{Y_n}$.)

$\hat{\theta} =$ ln(barY_n)/ln(1/2) **Answer:** -ln(barY_n)/ln(2)

STANDARD NOTATION

Solution:

Let n_0 and n_1 denote the number of 0's and 1's among $Y_1, \ldots, Y_n$. The log-likelihood of this observation is then

$$\ell(\theta) = n_0 \ln\left(1 - 2^{-\theta}\right) - n_1 \theta \ln(2).$$

The MLE $\hat{\theta}$ satisfies

$$\ell'\left(\hat{\theta}\right) = 0$$

which is equivalent to

$$\ell'\left(\hat{\theta}\right) = \frac{n_0 \ln(2)\, 2^{-\theta}}{1 - 2^{-\theta}} - n_1 \ln(2).$$

Rearranging and solving for $\hat{\theta}$ it follows that

$$\hat{\theta} = \frac{-\ln\left(\frac{n_1}{n_0+n_1}\right)}{\ln(2)} = \frac{-\ln \bar{Y_n}}{\ln(2)}.$$

Note that $\ell''(\theta) < 0$ so this is the unique maximum.

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Answers are displayed within the problem

(e)

2.0/2.0 points (graded)
Compute the method of moments estimator $\tilde{\theta}$ for θ.

(Enter **barY_n** for $\bar{Y_n}$.)

$\tilde{\theta} =$ ln(barY_n)/ln(1/2) **Answer:** -ln(barY_n)/ln(2)

STANDARD NOTATION

Solution:

Note trivially that

$$\mathbb{E}[Y_i] = 2^{-\theta}$$

and therefore $2^{-\tilde{\theta}} = \overline{Y_n}$. Thus

$$\tilde{\theta} = \frac{-\ln\left(\overline{Y_n}\right)}{\ln 2}.$$

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Answers are displayed within the problem

(f)

0/1 point (graded)
What is the asymptotic variance $V\left(\tilde{\theta}\right)$ of the method of moments estimator $\tilde{\theta}$?

$V\left(\tilde{\theta}\right) =$ (1/2)^theta*(1-(1/2)^theta) **Answer:** (2^theta-1)/(ln(2))^2

STANDARD NOTATION

Solution:

Note the the method of moments estimator and the MLE estimator are the same! Thus we can use the Theorem on MLE to determine that the asymptotic variance is

$$I(\theta)^{-1} = \frac{2^{\theta}-1}{\left(\ln\left(2\right)\right)^2}.$$

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Answers are displayed within the problem

(g)

1.5/3.0 points (graded)
Give a **formula** for the p-value for the test of

$$H_0 : \theta \leq 1 \quad \text{vs.} \quad H_1 : \theta > 1$$

based on the asymptotic distribution of $\hat{\theta}$.

To avoid double jeopardy, you may use V for the asymptotic variance $V\left(\theta_0\right)$, I for the Fisher information $I\left(\theta_0\right)$, **hattheta** for $\hat{\theta}$, or enter your answer directy without using V or I or **hattheta**.

(Enter **barY_n** for $\overline{Y}_n$, **hattheta** for $\hat{\theta}$. If applicable, enter **Phi(z)** for the cdf $\Phi\left(z\right)$ of a normal variable Z, **q(alpha)** for the quantile q_α for any numerical value α. Recall the convention in this course that $\mathbf{P}\left(Z \leq q_\alpha\right) = 1-\alpha$ for $Z \sim \mathcal{N}\left(0,1\right)$.)

p-value: sqrt(n/V)*(hattheta -1) **Answer:** 1-Phi(sqrt(n/V)*(hattheta-1))

Assume $n = 50$, and $\overline{Y_n} = 0.46$. Will you reject the null hypothesis at level $\alpha = 5\%$?

- Yes, reject the null hypothesis at level $\alpha = 5\%$.
- No, cannot reject the null hypothesis at level $\alpha = 5\%$.

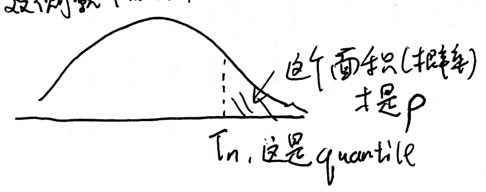
Correction Note: In an earlier version of this problem, the input instruction was: “ To avoid double jeopardy, you may use V for the appropriate estimator of the asymptotic variance $V(\hat{\theta})$ of the MLE $\hat{\theta}$, I for the Fisher information $I(\hat{\theta})$ evaluated at $\hat{\theta}$, **hattheta** for $\hat{\theta}$, or enter your answer directy without using V or I or **hattheta**."

CLT: $P\left(\sqrt{n}\frac{\hat{\theta}-E(\theta)}{\sigma}=q_a\right)=a$

代入E[θ]: $P\left(\sqrt{n}\frac{\hat{\theta}-1}{\sigma}=\sqrt{\frac{n}{V}}(\hat{\theta}-1)=q_a=T_n\right)=a$

单侧: $p=1-\Phi\left(\sqrt{\frac{n}{V}}(\hat{\theta}-1)\right)$

双侧就除以2,



STANDARD NOTATION

Solution:

Define the test statistic for this one-sided test as

$$T_n = \sqrt{\frac{n}{V}}\left(\hat{\theta}-1\right)$$

where $V=V(1)$ is the asymptotic variance evaluated at the boundary of the null hypothesis. Recall the p-value is the smallest level at which this test will reject H_0. Hence

$$p = 1-\Phi\left(\sqrt{\frac{n}{V(1)}}\left(\hat{\theta}-1\right)\right)$$

If $\bar{Y}_n = .46$ the MLE is

$$\hat{\theta} = \frac{-\ln(.46)}{\ln(2)} = 1.12.$$

The asymptotic variance of $\bar{\theta}$ given $\theta = 1$ is

$$V(1) = \frac{2^{\theta}-1}{(\ln(2))^2} = 2.08.$$

Therefore the desired p value is

$$p = \mathbf{P}\left(Z \geq \sqrt{n}(2.08)^{\frac{-1}{2}}\left(\bar{\theta}-1\right)\right)$$

$$= 1-\mathbf{P}\left(Z \geq .59\right)$$

$$= .2776$$

where Z is distributed as an $\mathcal{N}(0,1)$. Since $p > 0.05$, we fail to reject the null hypothesis at level $\alpha = 5\%$.

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Answers are displayed within the problem

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显示讨论

主题: Midterm Exam 1:Midterm Exam 1 / Problem 3