6. Bob and his coins

Problem 6. Bob and his coins

9.0/10.0 points (graded)

Bob has two coins, A and B, in front of him. The probability of Heads at each toss is $p = 0.5$ for coin A and $q = 0.9$ for coin B.

Bob chooses one of the two coins at random (both choices are equally likely).

He then continues with 5 tosses of the chosen coin; these tosses are conditionally independent given the choice of the coin.

Let:
H_i : the event that Bob's ith coin toss resulted in Heads;
N : the number of Heads in Bob's coin tosses.

1. For $i \in \{0, 1, \ldots, 5\}$, $p_N(i)$, the pmf of N, is of the form

$$\frac{1}{2}\binom{5}{a} b^5 + c\binom{5}{d} q^e (1-q)^f.$$

Find the coefficients $a, b, \ldots, f$. Your answer can be either a number or an expression involving i.

$a =$ i ✔ **Answer:** i

$b =$ 0.5 ✔ **Answer:** 0.5

0.5

$c =$ 1/2 ✔ **Answer:** 0.5

$\frac{1}{2}$

$d =$ i ✔ **Answer:** i

$e =$ i ✔ **Answer:** i

i

$f =$ 1-i ✘ **Answer:** 5-i

$1 - i$

2. Find $\mathbf{E}[N]$.

$\mathbf{E}[N] =$ 3.5 ✔ **Answer:** 3.5

3. Find the conditional variance of N, in a conditional model where we condition on having chosen coin A **and** the first two tosses resulting in Heads.

$\mathsf{Var}(N \mid A, H_1, H_2) =$ 3/4 ✔ **Answer:** 0.75

4. Are the events H_1 and $\{N = 5\}$ independent?

No ▼ ✔ **Answer:** No

5. Given that the 3rd toss resulted in Heads, what is the probability that coin A was chosen?

5/14 ✔ **Answer:** 5/14

STANDARD NOTATION

Solution:

1. Using the total probability theorem,

$$\begin{aligned}\mathbf{P}(N = i) &= \mathbf{P}(N = i \mid A)\mathbf{P}(A) + \mathbf{P}(N = i \mid B)\mathbf{P}(B) \\ &= \frac{1}{2}\binom{5}{i}p^5 + \frac{1}{2}\binom{5}{i}q^i(1-q)^{5-i}\end{aligned}$$

2. Using the total expectation theorem,

$$\mathbf{E}[N] = \frac{1}{2}(\mathbf{E}[N|A] + \mathbf{E}[N|B]) = \frac{1}{2}(5p + 5q) = \frac{7}{2}.$$

3. Conditioned on H_1, H_2, we have $N = 2 + M$ where M is the number of Heads in the last three tosses. Adding a constant does not change the variance. Thus,

$$\mathrm{Var}(N|A, H_1, H_2) = \mathrm{var}(M|A) = 3p(1-p) = 0.75.$$

4. No. It suffices to show $\{N=5\}$ and H_1^c are not independent. By part 1, we have

$$\mathbf{P}(N=5) = \frac{1}{2}p^5 + \frac{1}{2}q^5 \neq 0,$$

whereas

$$\mathbf{P}(N=5|H_1^c) = 0.$$

Since $\mathbf{P}(N=5) \neq \mathbf{P}(N=5|H_1^c)$, the events $N=5$ and H_1^c are not independent.

5. By Bayes' rule we have

$$\mathbf{P}(A|H_3) = \frac{\mathbf{P}(H_3|A)\mathbf{P}(A)}{\mathbf{P}(H_3)}.$$

We have $\mathbf{P}(A) = 0.5$ and $\mathbf{P}(H_3|A) = 0.5$. Also,

$$\mathbf{P}(H_3) = \mathbf{P}(H_3|A)\mathbf{P}(A) + \mathbf{P}(H_3|B)\mathbf{P}(B) = \frac{1}{2}(0.5 + 0.9) = 0.7.$$

Therefore,

$$\mathbf{P}(A|H_3) = \frac{0.5 * 0.5}{0.7} = \frac{5}{14} \sim 0.357.$$

提交 You have used 2 of 2 attempts

Answers are displayed within the problem

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