MITx Video | 6.041x | Spring 2014 | Video transcript | L24-2: Checkout counter example

So let us start with our example. Suppose that you go to a supermarket, and start observing customers arriving and leaving from a given checkout counter. Assume that there are two customers in the queue when you arrive.

For simplicity, assume also the customers come one at a time, that there is a single queue, and that the customer in front of the queue is the one getting served by the clerk. So what events of interest could happen then? A new customer could join the queue, which is an arrival. Or the customer currently being served is done, and leaves-- departure. Or both events could happen.

Now for making our model more precise, we need to specify the processes of the customer arrivals and departures. And for that, let's use some simple discrete time stochastic processes, which we have introduced before. So as usual, we first divide time into discrete time steps, say in seconds.

Here n equals 0 would correspond to the time when you arrived. For arrivals, let's assume that customers arrive according to a Bernoulli process with parameter p. And at that time, steps here, there is an arrival, perhaps this customer here. At 6, there is another customer.

So during each time interval, independently of what happened in the past, with probability p a new customer arrives. And with probability 1 minus p, no one comes. It may be useful to think about the following imaginary experiment. Imagine that during each time interval, nature independently flips a biased coin, which has a probability p of resulting in Heads, and 1 minus p in Tails. And whenever Heads is the result, then a new customer joins the queue.

So in our example here, here you obtain Tails, Heads, Tails, Tails, Tails, Heads, Tails, et cetera. From the lecture on Bernoulli process, remember that this implies that the time duration-- that means the number of time steps between two consecutive arrivals-- follows a geometric random variable with parameter p. So here in our example, that time duration of 4 is the result of a geometric random variable with parameter p.

So again, once in the queue, customers wait their turn until they start being served by the clerk. And typically, when a customer starts to check out, the number of times steps it takes to go through the entire process will depend on many factors, such as the total number of items selected, the speed or the mood of the clerk, and so on, so forth. We will model this variation by assuming that the service duration of any customer is the outcome of a random variable.

In particular, we will assume that the number of time steps it takes for any customer to check out is a geometric random variable with a constant parameter q. That is, the same q for each customer. So you might have a departure here. That's correspond to that customer here. And another departure at time step 6 correspond to that customer.

It may be useful, again, to think about another imaginary experiment to represent this service duration. Imagine the following. At the time a customer in the queue becomes the one to be served, that customer starts flipping a biased coin, which has a probability q of resulting in Heads. And it does so independently during each successive time steps, until Heads appear for the first time, which then indicates that the checkout service is done, and that the customer can leave.

So here in our example, you arrive at that time here. And this customer was being served. And during that time step, the customer flips a coin resulting in Tails, Tails, Tails, Heads. That customer now leaves.

The next customer start being served. At that time, flips a coin-- Tails, Heads-- then that customer leaves again. Finally, we will also assume that the processes modeling these arrivals and departures are independent of each other.

Now let us go back to our made up experiment, and assume that you have arrived at 6:45 PM. Consider the following question. What is the probability that you observe a customer leaving the checkout counter during the first time step?

In our example, since there were at least one customer in the queue, that probability is then simply q. However, if that queue had been empty when you arrive, then that probability would have been 0. Another question, would the queue be empty at 6:50 PM? That means 5 minutes later.

Well, it's hard to tell. If the initial length of the queue had been huge when you arrive at 6:45 PM, then the probability that it will be empty 5 minutes later would be very small, much smaller than the probability in that case, with two customers initially, or in that case with an initial empty queue. From these questions and answers, it looks like knowing the number of customers in the queue at any point in time not only provides a good description of the system at that time, but it does seem to capture the critical information we need in order to answer questions about the future evolution of the system.

So let us define the state of our system as the number of customers in the queue at each time step n, and see what we can do. So here, in our example, initially we had 2 customers. Then, time step 1, still 2 customers.

Time step 2, we have 1 arrival. So we have 3 customers. Time step 3 still 3. 4, 3 minus 1.

We have a departure, equals 2. So 5 will still be 2. Time step 6, we have an arrival and a departure. So we have 2. And so on and so forth.

Assume now that there is limited space in the supermarket, and that no more than 10 customers can be in the queue at any point in time. We can then give a graphical representation of all possible states for our system, as follows. A system can be 11 different states, from an empty queue with no customer, to a system at full capacity with 10 customers.

Let us now describe some possible transitions between these states, from one step to the next. Suppose first that the system is in state 2, and that one new customer arrives and no one leaves. So you will transition from 2 to 3. And this is what happened in this example, from time step 1 to time step 2.

Suppose now that you are in state 3, and that a customer leaves, and no one arrives. Then you will transition to state 2, like what happened between time step 3 and 4. What else? Well, you could also be in a given state at one time step, and stay in this same state at the next step.

How? It can happen in two ways. If there are no arrivals and no departure in the next step, and that was what happened between time step 4 to 5 here, 4 to 5. Or there is 1 arrival and 1 departure, like what happened between time step 5 and 6.

A graphical representation of all possible one-step transitions can be done with the help of arcs, such as here. In order to complete our model, we need to indicate the probabilities associated with these transitions. So again, assume that you're currently in state 2, with 2 customers in the queue. The probability of next going to state 3 here, with 1 more customer in the queue is simply a probability of having 1 arrival and no departures. On the other hand, the probability of being here, and going in transition next here, correspond to 1 departure and no arrival. Finally, the system can stay in state 2, like that, when there is 1 arrival and 1 departure, or no arrivals and no departures.

These transition probabilities would be similar if the current state were 1, 3, 9. For the two extreme states, the transition probabilities are a bit different. If you are in state 0, the queue is empty, and you can go to state 1 with 1 additional customer, with a probability p. Or there is no new customer coming, and you stay in state 0. And if the queue is at maximum capacity, either you stay at maximum capacity if there is no service, or you go down to 9 customers in a queue if you have a departure.

So one important fact. When you are in a given state, for example state 2, and you look at all possible transitions, could go to 3, could go to 1, could remain in 2. If you sum all the probabilities, p times 1 minus q plus q times 1 minus p plus this total probability here, pq plus 1 minus p times 1 minus q, you will get a total probability of 1.

Similarly, if you look at this probability here, they sum to 1. And these probabilities sum to 1. It's simply says that from one time step to the next, if you consider all possible transition probabilities, they all have to sum to 1. So in conclusion, this so-called transition probability graph, which is this representation here, provides a complete representation of a discrete time finite state Markov chain model of our simple supermarket checkout counter example.