

This is a rather theoretical exercise that has two purposes. One is to verify that the notion of convergence in probability is quite natural and that it has properties similar to the notion of convergence of sequences. And the second purpose is to get a little bit of practice with the formal definition of convergence in probability.

So what is the statement saying? It says that if we have a sequence of random variables that converges to a certain number,  $a$ , and this basically means that when  $n$  is large, the distribution is highly concentrated around  $a$ . And if we have another sequence of random variables that converges to a certain number,  $b$ , which means that the probability distribution of  $Y_n$  is heavily concentrated around  $b$ . In that case, then the probability distribution of the sum of the two random variables is heavily concentrated in the vicinity of  $a + b$ .

So what are we saying? If  $X_n$  is very close to  $a$  with high probability and  $Y_n$  is very close to  $b$  with high probability, then the sum will also be close to  $a + b$  with high probability. This is the intuitive content of the statement. Now we want to establish this formally.

Before establishing this statement, however, it will be a good practice to verify a property of this type for the ordinary convergence of sequences of numbers, not random variables. So let us do that. What we want to show is that if a sequence of numbers,  $a_n$ , converges to some number  $a$ , and another sequence converges to some number  $b$ , we want to show that in that case,  $a_n + b_n$  converges to the sum of  $a + b$ , and we want to do this formally.

So let us start with the definition of convergence. What does it mean that  $a_n$  converges to  $a$ ? It means that if I fix some positive  $\epsilon$ , then there exists some number or some time such that if we consider some  $n$  bigger than  $n_0$ , then  $a_n$  is close to  $a$  in the sense that this difference is less than  $\epsilon$ . Now this is true for any positive  $\epsilon$ , so if instead of  $\epsilon$ , I take  $\epsilon/2$ , this would also be true. Eventually, after some time, we will have the property that  $a_n - a$  is less than  $\epsilon/2$ .

Similarly, if  $b_n$  converges to  $b$ , then we will have the property that there exists some time-- let's call it  $n_0'$  prime-- such that if  $n$  is bigger than that particular time, then  $b_n - b$  is going to be less than  $\epsilon/2$ . So after time  $n_0$  and after time  $n_0'$ , these two inequalities will be true.

So if we wait long enough so that both of these inequalities are true, that is, if  $n$  is bigger than the maximum of  $n_0$  and  $n_0'$ , then we will have the following. We will have that  $a_n + b_n - a - b$  which, by an elementary inequality, is less than or equal to  $|a_n - a| + |b_n - b|$ .

Where is this inequality coming from? This is a general inequality about absolute values. If I give you two numbers, the absolute value of  $x + y$  is always less than or equal to the sum of the absolute values. So we're using this inequality where  $x$  is  $a_n - a$  and  $y$  is  $b_n - b$ . So we have this inequality, but when  $n$  is big enough,  $|a_n - a|$  is less than  $\epsilon/2$ .  $|b_n - b|$  is also less than  $\epsilon/2$ . And putting everything together, this is  $\epsilon$ .

So what have we shown? That if  $a_n$  converges to  $a$  and  $b_n$  converges to  $b$ , so that all these relations hold, then if  $n$  is large enough, then the difference between this number and that number is going to be less than  $\epsilon$ . And this is true for every positive  $\epsilon$ , but that's just the definition of convergence of this quantity to that quantity. And this is the proof of this elementary relation about convergence of numbers.

Now let us turn to convergence of random variables. We fix some  $\epsilon$  that's positive. In order to show convergence in probability, we want to look at this difference and look at the probability that this difference is bigger than  $\epsilon$  in magnitude. And we want to show that this quantity converges to 0. If it does, then we will have established convergence in probability because that's just the definition.

Now, this is the event that the sum of the random variables is close to  $a + b$ , and we want to use the fact that  $x_n$  is close to  $a$  and  $y_n$  is close to  $b$ . So this is the event that-- let's write it in a somewhat different way-- is the probability of the event that  $|x_n - a + y_n - b|$  is bigger than  $\epsilon$  in magnitude. Now, for a sum of two numbers to be bigger than  $\epsilon$  in magnitude, it has to be the case that either one of them is larger than  $\epsilon/2$  or the other number is bigger in magnitude than  $\epsilon/2$ .

So if this event happens, this event must also happen. This means that this event is a subset of this event. This is a smaller one. If this happens, then this one happens. So since it's a smaller event, it means that its probability is less than or equal to the probability of that event.

Now we use the union bound. The probability that something happens or something else is happening is less than or equal to the sum of their probabilities. And now, since  $X_n$  converges to  $a$  in probability,

then by definition, we know that this quantity converges to 0 as  $n$  goes to infinity. Similarly, since  $Y_n$  converges to  $b$  in probability, this quantity converges to 0 as  $n$  goes to infinity.

This is a sequence of numbers that converges to 0. This is another sequence of numbers that converges to 0. Therefore, the sum of these two sequences also converges to 0. In essence, here we're applying what we established earlier about convergence of numbers. If a sequence converges to 0 and another sequence converges to 0, then the sum of these sequences also converges to 0 as  $n$  goes to infinity.

But this is exactly what we need to show in order to establish convergence in probability of  $X_n$  plus  $Y_n$ . We have shown that if I fix any epsilon, positive, no matter how small, the probability that I am more than epsilon away, the probability that  $X_n$  plus  $Y_n$  is more than epsilon away from the supposed target or the limit, this probability must go to 0. And that's exactly what we established here, and this completes the derivation.