4. Uniform random variables

Expectation, variance and probabilities

4/4 points (graded)

Let X be a uniform random variable in the interval $[2, 8.5]$. Find the following quantities (if needed, round to the nearest 10^{-4}):

$\mathbb{E}[X] =$ 5.25 ✔ **Answer:** 21/4

$\text{Var}[X] =$ 3.520833 ✔ **Answer:** 169/48

$\mathbf{P}(X > 4) =$ 0.6923077 ✔ **Answer:** 9/13

$\mathbf{P}(\log(X) \le 1) =$ 0.11050489668600694 ✔ **Answer:** 0.11054897

(Note that the logarithm is the natural one to base $\mathbf{e}$)

STANDARD NOTATION

Solution:

We can write $X = 6.5Z + 2$, where Z follows a uniform distribution on $[0, 1]$. By properties of the uniform distribution, we then conclude:

$$\mathbb{E}[X] = 2 + 6.5\mathbb{E}[Z] = 2 + 6.5 \times \frac{1}{2} = \frac{21}{4},$$

$$\text{Var}[X] = 6.5^2 \times \text{Var}[Z] = \frac{169}{4} \times \frac{1}{12} = \frac{169}{48},$$

$$\mathbf{P}(X \ge 4) = \mathbf{P}(Z \ge \frac{4}{13}) = 1 - \frac{4}{13} = \frac{9}{13},$$

$$\mathbf{P}(\log(X) \le 1) = \mathbf{P}(X \le e) = \mathbf{P}(Z \le \frac{2(\mathbf{e}-2)}{13}) \approx 0.110549.$$

: Uniform PDF in Lecture 8, *Probability density functions*.

提交 你已经尝试了2次（总共可以尝试4次）

Answers are displayed within the problem

Two independent copies

2/3 points (graded)

Let U, V be i.i.d. random variables uniformly distributed in $[0,1]$. Compute the following quantities:

$\mathbb{E}[|U-V|] =$ 0.125 ✖ **Answer:** 1/3

$\mathbf{P}(U=V) =$ 0 ✔ **Answer:** 0

$\mathbf{P}(U \le V) =$ 0.5 ✔ **Answer:** 1/2

STANDARD NOTATION

Solution:

For the first quantity, we write the joint expectation as an iterated expectation and conditional expectation,

$$\mathbb{E}[|U-V|] = \mathbb{E}[\mathbb{E}[|U-V||V]].$$

By independence, we can compute the inner expectation as

$$\begin{aligned}
\mathbb{E}[|U-V||V=v] &= \int_0^1 |u-v|\, du \\
&= \int_0^v (v-u)\, du + \int_v^1 (u-v)\, du \\
&= \left[vu - \frac{1}{2}u^2\right]_0^v + \left[\frac{1}{2}u^2 - vu\right]_v^1 = v^2 - \frac{1}{2}v^2 + \frac{1}{2} - v - \frac{1}{2}v^2 + v^2 \\
&= v^2 - v + \frac{1}{2},
\end{aligned}$$

so

$$\mathbb{E}[|U-V|] = \mathbb{E}\left[V^2 - V + \frac{1}{2}\right] = \frac{1}{3} - \frac{1}{2} + \frac{1}{2} = \frac{1}{3}.$$

For the probability $\mathbf{P}(U=V)$, just write this as double expectation as well and notice that

$$\mathbf{P}(U=V) = \mathbb{E}[\mathbb{E}[\mathbf{1}(U=V)|V]] = \mathbb{E}[0] = 0,$$

because the probability of a uniform random variable being equal to any fixed number between 0 and 1 is zero.

For $\mathbf{P}(U \le V)$, write it again as a double expectation,

$$\mathbf{P}(U \le V) = \mathbb{E}[\mathbb{E}[\mathbf{1}(U \le V)|V]] = \mathbb{E}[\mathbf{P}(U \le V)|V] = \mathbb{E}[V] = \frac{1}{2}.$$

Alternatively, this can also be seen by symmetry of the two variables, i.e., $P(U \le V) = P(V \le U)$ and either one of the two must be true, counting double the zero-set of $\mathbf{P}(U=V)$.
: Uniform PDF in Lecture 8, *Probability density functions*.

提交 你已经尝试了3次（总共可以尝试3次）

Answers are displayed within the problem

Maximum and sum of independent copies

0/1 point (graded)
Let X, Y be independent random variables uniformly distributed in $[0,1]$. In the graph below, sketch

1. the probability density $f_{X+Y}(z)$ of $X+Y$;
2. the probability density $f_{\max(X,Y)}(z)$ of $\max(X,Y)$.

(Be sure to sketch on the **entire domain** shown on the graph.)

Drawing tip: The spline tool draws a smooth curve connecting the points you click. To draw sharp corners, click on the point where the corner would be, then click again very close to it, and then continue onto the next point of your function.

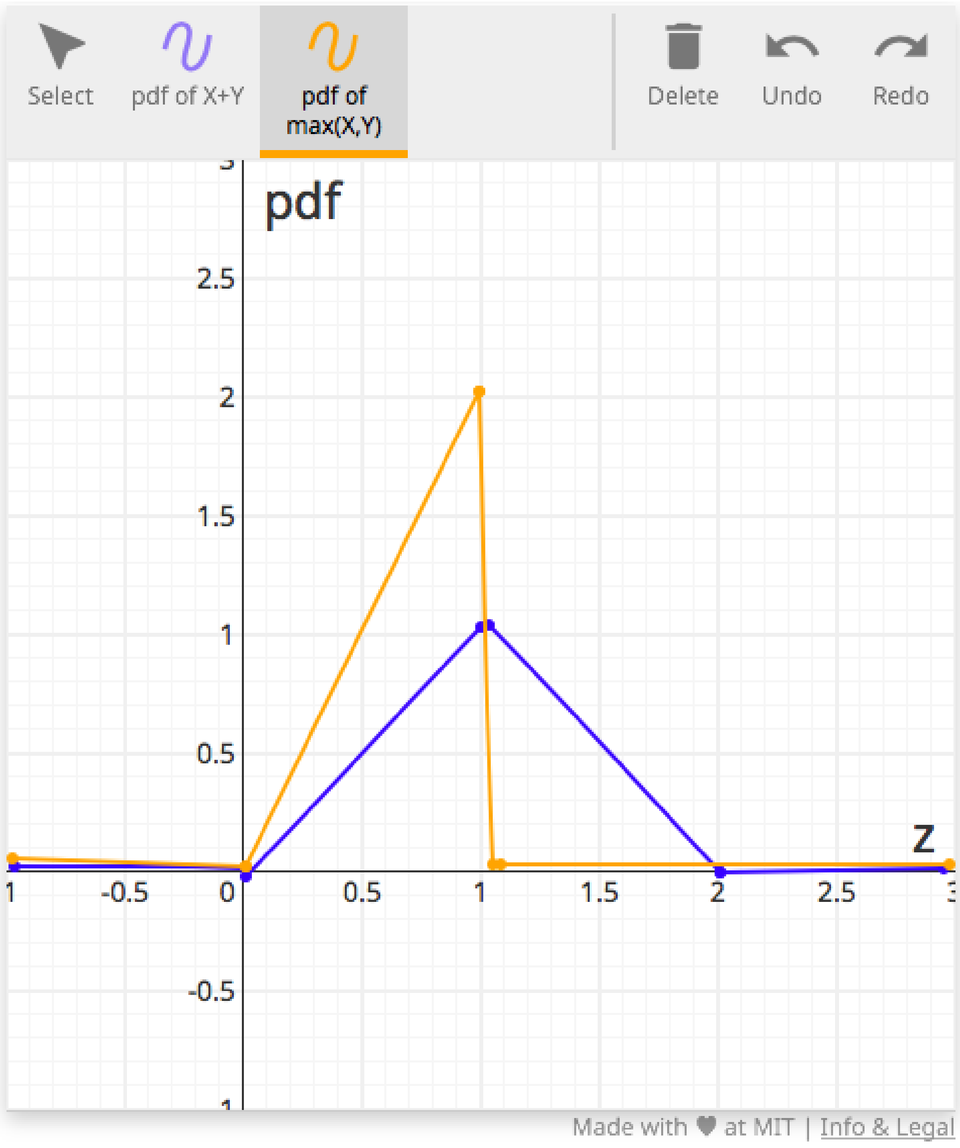
Answer: See solution.

✖

The pdf for X+Y does not have the correct shape. . The pdf for max(X,Y) does not have the correct shape. .

STANDARD NOTATION

Solution:



The density of $X+Y$ is given by the convolution of the density of a uniform random variable,

$$f(x) = \mathbf{1}[0,1] = \begin{cases} 1, & \text{if } x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

The density g of $X+Y$ therefore is

$$\begin{aligned} g(z) &= \int_{\mathbb{R}} f(x) f(z-x)\, dx \\ &= \int_{\mathbb{R}} \mathbf{1}(x \in [0,1])\, \mathbf{1}(z-x \in [0,1])\, dx \\ &= \int_0^1 \mathbf{1}(z-1 \le x \le z)\, dx \\ &= \mathbf{1}(z \le 2) \int_{\max\{0,z-1\}}^{\min\{1,z\}} dx \\ &= \begin{cases} 0, & z < 0 \\ z, & 0 < z < 1 \\ 2-z, & 1 < z < 2 \\ 0, & z > 2. \end{cases} \end{aligned}$$

For the density of $\max\{X,Y\}$, first note that it is supported in $[0,1]$. Now, first compute the cdf on that interval:

$$\begin{aligned} \mathbf{P}(\max\{X,Y\} \le y) = &\ \mathbf{P}(X \le y)\, \mathbf{P}(Y \le y) \quad \text{(by independence)} \\ = &\ t^2. \end{aligned}$$

Hence, the density h of $\max\{X,Y\}$ is given by

$$h(z) = \begin{cases} 0, & z < 0 \\ 2z, & 0 \le z \le 1 \\ 0, & z > 1. \end{cases}$$

提交 你已经尝试了2次（总共可以尝试10次）

Answers are displayed within the problem

Maximum of uniform random variables

1/2 points (graded)

Let $U_1, \ldots, U_n$ be i.i.d. random variables uniformly distributed in $[0,1]$ and let $M_n = \max_{1 \le i \le n} U_i$.

Find the cdf of M_n, which we denote by $G(t)$, for $t \in [0,1]$.

For $t \in [0,1]$,

$G(t) =$ t^n ✔ **Answer:** t^n

t^n

Now, let $F_n(t)$ denote the cdf of $n(1 - M_n)$; for $t > 0$, compute

$\lim_{n \to \infty} F_n(t) =$ t ✘ **Answer:** 1-exp(-t)

t

STANDARD NOTATION

Solution:

First, we compute the cdf. Let $t \in [0,1]$. Then,

$$\mathbf{P}(M_n \le t) = \mathbf{P}\left(\max_{i=1,\ldots,n} U_i \le t\right) = \mathbf{P}\left(\cap_{i=1}^n \{U_i \le t\}\right) = \prod_{i=1}^n \mathbf{P}(U_i \le t) = t^n,$$

where we used the independence of the U_i to write the intersection as a product.

Now,

$$\begin{aligned}\mathbf{P}(n(1-M_n) \le t) = \ & \mathbf{P}\left(1 - M_n \le \frac{t}{n}\right) = \mathbf{P}\left(M_n \ge 1 - \frac{t}{n}\right) \\ = \ & 1 - \mathbf{P}\left(M_n < 1 - \frac{t}{n}\right) = 1 - \left(1 - \frac{t}{n}\right)^n \xrightarrow{n \to \infty} 1 - \mathbf{e}^{-t}.\end{aligned}$$

Hence, $n(1 - M_n)$ converges in distribution to $\mathrm{Exp}(1)$.

提交 你已经尝试了3次（总共可以尝试3次）

Answers are displayed within the problem

讨论

显示讨论

主题： Unit 0. Course Overview, Syllabus, Guidelines, and Homework on Prerequisites:Homework 0: Probability and Linear algebra Review / 4. Uniform random variables

认证证书是什么？