8. Moments Estimator

Mapping Parameters to Moments I

2/2 points (graded)

Let $X_1, \ldots, X_n \stackrel{iid}{\sim} N\left(\mu^*, (\sigma^*)^2\right)$ and let $\left(\mathbb{R}, \{N(\mu, \sigma^2)\}_{\mu\in\mathbb{R}, \sigma>0}\right)$ denote the corresponding statistical model. Let

$$m_k(\mu, \sigma) = \mathbb{E}\left[X^k\right]$$

denote the k-th moment of $X \sim N\left(\mu^*, (\sigma^*)^2\right)$. Let $\psi : \mathbb{R} \times (0, \infty) \to \mathbb{R}^2$ be defined by $\psi(\mu, \sigma) = (m_1(\mu, \sigma), m_2(\mu, \sigma))$. (Since we have two parameters of interest,μ and σ, it makes sense to work with the first two moments. The hope is that the two moments will uniquely determine the paramters of interest μ and σ.)

Express $m_1(\mu, \sigma)$ and $m_2(\mu, \sigma)$ in terms of μ and σ.

$m_1(\mu, \sigma) =$ mu **Answer:** mu

μ

$m_2(\mu, \sigma) =$ mu^2 + sigma^2 **Answer:** mu^2 +sigma^2

$\mu^2 + \sigma^2$

STANDARD NOTATION

Solution:

Note that

$$\begin{aligned} m_1(\mu, \sigma^2) &= \mathbb{E}[X] = \mu \\ m_2(\mu, \sigma^2) &= \mathbb{E}[X^2] = (\mathbb{E}[X])^2 + \left(\mathbb{E}[X^2] - (\mathbb{E}[X])^2\right) = \mu^2 + \sigma^2. \end{aligned}$$

Hence, $\psi(\mu, \sigma) = (\mu, \mu^2 + \sigma^2)$.

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Answers are displayed within the problem

Mapping Parameters to Moments II

3/3 points (graded)

Let

$$\begin{aligned}\psi : \mathbb{R} &\times (0,\infty) \to \mathbb{R}^2 \\ (\mu,\sigma) &\mapsto (m_1(\mu,\sigma), m_2(\mu,\sigma)).\end{aligned}$$

denote the moments map considered in the previous problem, where $m_k(\mu,\sigma)$ denotes the k-th moment of the distribution $N(\mu,\sigma^2)$.

Is ψ one-to-one on the domain $\mathbb{R} \times (0,\infty)$? (Equivalently, given the outputs m_1 and m_2, can we use them to uniquely reconstruct $\mu \in \mathbb{R}$ and $\sigma > 0$?)

- (selected) Yes
- No

If ψ is one-to-one on the given domain and $\psi(\mu,\sigma) = (m_1, m_2)$, what is μ expressed in terms of m_1 and m_2? (If ψ is not one-to-one, enter 0.)

$\mu =$ m_1 **Answer:** m_1

m_1

If ψ is one-to-one on the given domain and $\psi(\mu,\sigma) = (m_1, m_2)$, what is σ expressed in terms of m_1 and m_2? (If ψ is not one-to-one, enter 0.)

$\sigma =$ sqrt(m_2 - m_1^2) **Answer:** sqrt(m_2 - m_1^2)

$\sqrt{m_2 - m_1^2}$

Type **m_1** for m_1 and **m_2** for m_2.

STANDARD NOTATION

Solution:

Note that

$$\begin{aligned} m_1(\mu,\sigma^2) &= \mathbb{E}[X] = \mu \\ m_2(\mu,\sigma^2) &= \mathbb{E}[X^2] = (\mathbb{E}[X])^2 + \left(\mathbb{E}[X^2] - (\mathbb{E}[X])^2\right) = \mu^2 + \sigma^2. \end{aligned}$$

Hence, $\psi(\mu,\sigma) = (\mu, \mu^2+\sigma^2)$. This function is one-to-one on the domain $\mathbb{R}\times(0,\infty)$. Since $m_1(\mu,\sigma)=\mu$, we can reconstruct the first parameter directly from the first moment: $\mu = m_1$.

Next, since we know $m_2(\mu,\sigma) = \sigma^2+\mu^2$, we can back-solve for σ:

$$\sigma = \sqrt{m_2-\mu^2} = \sqrt{m_2-m_1^2}.$$

Above we took the positive square-root because we have insisted a priori that $\sigma > 0$.

Remark: Assuming that m_1 and m_2 are one of the outputs of ψ, we have essentially shown how to construct $\psi^{-1}(m_1,m_2)$. In general, computing inverses is a computationally difficult problem, but in this particular example, the function ψ is simple enough that it is possible to invert by hand.

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Answers are displayed within the problem

Method of Moments Concept Question I

2/2 points (graded)

Let $(E, \{\mathbf{P}_\theta\}_{\theta\in\Theta})$ denote a statistical model associated to a statistical experiment $X_1,\ldots,X_n \overset{iid}{\sim} \mathbf{P}_{\theta^*}$ where $\theta^*\in\Theta$ is the true parameter. Assume that $\Theta\subset\mathbb{R}^d$ for some $d\geq 1$. Let $m_k(\theta) := \mathbb{E}[X^k]$ where $X\sim\mathbf{P}_\theta$. $m_k(\theta)$ is referred to as the **k-th moment of $\mathbf{P}_\theta$**. Also define the moments map:

$$\begin{aligned} \psi:\Theta &\to \mathbb{R}^d \\ \theta &\mapsto (m_1(\theta), m_2(\theta),\ldots,m_d(\theta)). \end{aligned}$$

Assume that ψ is one-to-one (and hence, invertible).

Which of the following is equal to θ^*?

- $(m_1(\theta^*), m_2(\theta^*),\ldots,m_d(\theta^*))$
- $\psi(m_1(\theta^*), m_2(\theta^*),\ldots,m_d(\theta^*))$
- $\psi^{-1}(m_1(\theta^*), m_2(\theta^*),\ldots,m_d(\theta^*))$ (selected, correct)
- $\psi^{-1}\left(\frac{1}{n}\sum_{k=1}^n X_i, \frac{1}{n}\sum_{k=1}^n X_i^2,\ldots,\frac{1}{n}\sum_{k=1}^n X_i^d\right)$

Which of the following is the method of moments estimator for θ^*?

- $(m_1(\theta^*), m_2(\theta^*),\ldots,m_d(\theta^*))$
- $\psi(m_1(\theta^*), m_2(\theta^*),\ldots,m_d(\theta^*))$
- $\psi^{-1}(m_1(\theta^*), m_2(\theta^*),\ldots,m_d(\theta^*))$
- $\psi^{-1}\left(\frac{1}{n}\sum_{k=1}^n X_i, \frac{1}{n}\sum_{k=1}^n X_i^2,\ldots,\frac{1}{n}\sum_{k=1}^n X_i^d\right)$ (selected, correct)

Solution:

Observe that $\psi(\theta^*) = (m_1(\theta^*), m_2(\theta^*), \ldots, m_d(\theta^*))$ by definition of ψ. Since ψ is invertible, then we know that $\psi^{-1}(m_1(\theta^*), m_2(\theta^*), \ldots, m_d(\theta^*)) = \theta^*$. Hence, $\psi^{-1}(m_1(\theta^*), m_2(\theta^*), \ldots, m_d(\theta^*))$ is the correct response to the first question.

The remaining choices are incorrect.

- $(m_1(\theta^*), m_2(\theta^*), \ldots, m_d(\theta^*))$ is the list of moments of $\mathbf{P}_{\theta^*}$, not the parameter θ^* itself.
- $\psi(m_1(\theta^*), m_2(\theta^*), \ldots, m_d(\theta^*))$ is incorrect, as this is really $\psi^2(\theta^*)$, which is not necessarily θ^*.
- This is the method of moments estimator, not the true parameter θ^*.

The method of moments estimator is given by

$$\hat{\theta}_n^{\text{MM}} = \psi^{-1}\left(\frac{1}{n}\sum_{k=1}^{n} X_i, \frac{1}{n}\sum_{k=1}^{n} X_i^2, \ldots, \frac{1}{n}\sum_{k=1}^{n} X_i^d\right),$$

so this is the correct response to the second question.

Remark: The above expression is consistent with the procedure we followed in the previous problems that we used to construct the method of moments estimator for a Gaussian statistical model with unknown mean and variance. Namely, we expressed the true parameters in terms of the true moments, and then plugged in the sample means into that expression. Informally, since we expect the sample means to give a good approximation for the true moments, plugging in the sample moments into the expression for the true parameters (in terms of the moments) should also give a good approximation for the true parameters. This is the strategy of the method of moments, and in general, the strategy of replacing expectations with averages is a recurring theme in statistics and in this course.

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Answers are displayed within the problem

Applying the Method of Moments to a Gaussian Statistical Model

2/2 points (graded)

We let $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu^*, (\sigma^*)^2)$ and consider the associated statistical model $(\mathbb{R}, \{N(\mu, \sigma^2)\}_{\mu\in\mathbb{R}, \sigma>0})$. Let

$$\begin{aligned} \psi: \mathbb{R} &\times (0,\infty) \to \mathbb{R}^2 \\ (\mu, \sigma) &\mapsto (m_1(\mu,\sigma), m_2(\mu,\sigma)). \end{aligned}$$

denote the moments map considered in the previous problem, where $m_k(\mu, \sigma)$ denotes the k-th moment of the distribution $N(\mu, \sigma^2)$.

To answer the next question, you should recall:

1. your result on writing μ and σ in terms of m_1 and m_2 obtained in the previous problem, and
2. the estimators $\widehat{m_1}$ and $\widehat{m_2}$ (the sample moments) for the true moments m_1 and m_2.

Suppose we observe the data-set $X_1 = 0.5, X_2 = 1.8, X_3 = -2.3, X_4 = 0.9$.

What is the method of moments estimator $\widehat{\mu}^{\text{MM}}$ for μ^* evaluated on this data-set? (You are encouraged to use whatever computational tools may be helpful.)

$\widehat{\mu}^{\text{MM}}(0.5, 1.8, -2.3, 0.9) =$ 0.225 Answer: 0.225

What is the method of moments $\widehat{\sigma}^{\text{MM}}$ estimator for σ^* evaluated on this data-set? (You are encouraged to use whatever computational tools may be helpful.)

$\widehat{\sigma}^{\text{MM}}(0.5, 1.8, -2.3, 0.9) =$ 1.532 Answer: 1.532

STANDARD NOTATION

Solution:

Since we computed ψ^{-1} explicitly in the previous problem, we can apply the method of moments to estimate the true parameters μ^* and σ^*. Let

$$\widehat{m}_1 = \frac{1}{n}\sum_{k=1}^{n} X_i, \quad \widehat{m}_2 = \frac{1}{n}\sum_{k=1}^{n} X_i^2$$

denote the first and second sample moments, respectively. Then the **method of moments estimator** is defined to be

$$(\widehat{\mu}_n^{\text{MM}}, \widehat{\sigma}_n^{\text{MM}}) := \psi^{-1}(\widehat{m}_1, \widehat{m}_2) = (\widehat{m}_1, \sqrt{\widehat{m}_2 - \widehat{m}_1^2}).$$

Using a calculator (or other computational software) we can compute,

$$\widehat{m}_1(0.5, 1.8, -2.3, 0.9) = \frac{0.5 + 1.8 - 2.3 + 0.9}{4} \approx 0.225$$

and

$$\widehat{m}_2(0.5, 1.8, -2.3, 0.9) = \frac{(0.5)^2 + (1.8)^2 + (-2.3)^2 + (0.9)^2}{4} \approx 2.3975.$$

Applying the method of moments,

$$\widehat{\mu}_n^{\text{MM}}(0.5, 1.8, -2.3, 0.9) = \widehat{m}_1(0.5, 1.8, -2.3, 0.9) \approx 0.225.$$

and

$$\widehat{\sigma}_n^{\text{MM}}(0.5, 1.8, -2.3, 0.9) = \sqrt{\widehat{m}_2 - (\widehat{m}_1)^2} \approx \sqrt{2.3975 - (0.225)^2} \approx 1.532.$$

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Answers are displayed within the problem

讨论

显示讨论

主题: Unit 3 Methods of Estimation:Lecture 11: Fisher Information, Asymptotic Normality of MLE; Method of Moments / 8. Moments Estimator