1. Confidence Intervals for Curved Gaussian Family

(a)

0/1 point (graded)

Let $X_1, \ldots, X_n$ be i.i.d. random variables with distribution $\mathcal{N}(\theta, \theta)$, for some unknown parameter $\theta > 0$.

True or False: The sample average $\overline{X}_n$ follows a normal distribution for any integer $n \geq 1$.

- True ✔
- False ✖

Solution:

As a sum of independent normal variables, $\overline{X}_n$ again follows a normal distribution. This is a special property of normal variables.

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Answers are displayed within the problem

(b)

2/2 points (graded)

What is the expectation and the variance of $\overline{X}_n$?

$\mathbb{E}[\overline{X}_n] =$ theta ✔ **Answer:** theta

θ

$\textsf{Var}(\overline{X}_n) =$ theta/n ✔ **Answer:** theta/n

$\frac{\theta}{n}$

STANDARD NOTATION

Solution:

As a sum of independent normal variables, $\overline{X}_n$ again follows a normal distribution, that is in turn completely characterized by its expectation and variance,

$$\mathbb{E}[\overline{X}_n] = \theta, \quad \textsf{Var}(\overline{X}_n) = \frac{1}{n^2}\sum_{i=1}^{n} \textsf{Var}(X_i) = \frac{\theta}{n}.$$

Hence,

$$\sqrt{\frac{n}{\theta}}\left(\overline{X}_n-\theta\right)\sim\mathcal{N}\left(0,1\right).$$

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Answers are displayed within the problem

(c)

0/2 points (graded) 只是interval不是confidence interval，被坑了

Find an interval $\mathcal{I}_\theta$ (that depends on θ) centered about $\overline{X}_n$ such that

$$\mathbf{P}\left(\mathcal{I}_\theta\ni\theta\right)=0.9\qquad\text{for all } n\text{(i.e, not only for large } n\text{)}.$$

(Write barX_n for $\overline{X}_n$. Use the estimate $q_{0.05}\approx 1.6448$ for best results.)

$\mathcal{I}_\theta=[A_\theta,B_\theta]$ for

$A_\theta=$ barX_n - 1.6448*sqrt(barX_n)/n ✖ **Answer:** barX_n - 1.6448 * sqrt(theta)/sqrt(n)

$B_\theta=$ barX_n - 1.6448*sqrt(barX_n)/n ✖ **Answer:** barX_n + 1.6448 * sqrt(theta)/sqrt(n)

STANDARD NOTATION

Solution:

By parts (a) and (b),

$$\sqrt{\frac{n}{\theta}}\left(\overline{X}_n-\theta\right)\sim\mathcal{N}\left(0,1\right),$$

so together with looking up the quantile value for a symmetric 90% confidence interval for a Gaussian random variable $Z\sim\mathcal{N}\left(0,1\right)$,

$$\mathbf{P}\left(|Z|\leq 1.6448\right)\approx 0.9,$$

we obtain

$$\mathbf{P}\left(\left|\sqrt{\frac{n}{\theta}}\left(\overline{X}_n-\theta\right)\right|\leq 1.6448\right)=0.9,$$

and hence can set

$$\mathcal{I}_1=\left[\overline{X}_n-\frac{1.6448\sqrt{\theta}}{\sqrt{n}},\overline{X}_n+\frac{1.6448\sqrt{\theta}}{\sqrt{n}}\right].$$

提交 你已经尝试了2次（总共可以尝试2次）

Answers are displayed within the problem

(d)

2/2 points (graded)

Again, use the estimate $q_{0.05} \approx 1.6448$ for best results.

Now, find a confidence interval $\mathcal{I}_{\text{plug-in}}$ with **asymptotic** confidence level 90% by plugging in $\overline{X}_n$ for all occurrences of θ in $\mathcal{I}_\theta$.

$\mathcal{I}_{\text{plug-in}} = [A_{\text{plug-in}}, B_{\text{plug-in}}]$ for

$A_{\text{plug-in}} =$ barX_n - 1.6448*sqrt(barX_n/n) ✔ $B_{\text{plug-in}} =$

barX_n + 1.6448*sqrt(barX_n/n) ✔

STANDARD NOTATION

提交 你已经尝试了2次（总共可以尝试2次）

(e)

2/2 points (graded)

Finally, find a confidence interval $\mathcal{I}_{\text{solve}}$ for θ with **nonasymptotic** level 90% solving the bounds in $\mathcal{I}_\theta$ for θ.

$\mathcal{I}_{\text{solve}} = [A_{\text{solve}}, B_{\text{solve}}]$ for

$A_{\text{solve}} =$ barX_n + (1.6448^2-1.6448*sqrt(4*n*barX_n + 1.6448^2)) ✔

Answer: barX_n + 1.6448^2/(2*n) - 1/2*sqrt(1.6448^4/n^2 + 4*1.6448^2*barX_n/n)

$B_{\text{solve}} =$ barX_n + (1.6448^2+1.6448*sqrt(4*n*barX_n + 1.6448^2) ✔

Answer: barX_n + 1.6448^2/(2*n) + 1/2*sqrt(1.6448^4/n^2 + 4*1.6448^2*barX_n/n)

STANDARD NOTATION

Solution:

From part (c), we have

$$\mathbf{P}\left(\left|\sqrt{\frac{n}{\theta}}\left(\overline{X}_n - \theta\right)\right| \leq 1.65\right) = 90\%.$$

With $t = 1.6448$, the constraint on θ is equivalent to

$$\begin{aligned}
& \left|\sqrt{\frac{n}{\theta}}\left(\overline{X}_n - \theta\right)\right| && \leq t \\
\iff \quad & \frac{n}{\theta}\left(\overline{X}_n - \theta\right)^2 && \leq t \\
\iff \quad & \theta^2 - 2\theta\overline{X}_n + \overline{X}_n^2 && \leq \frac{t^2\theta}{n} \\
\iff \quad & \theta^2 - \left(2\overline{X}_n + \frac{t^2}{n}\right)\theta + \overline{X}_n^2 && \leq 0 \\
\iff \quad & \theta \in \left[\overline{X}_n + \frac{t^2}{2n} - \sqrt{\Delta},\ \overline{X}_n + \frac{t^2}{2n} + \sqrt{\Delta}\right], && \text{where } \Delta = \frac{t^4}{4n^2} + \frac{t^2\overline{X}_n}{n}
\end{aligned}$$

by the quadratic formula.Substituting $t = 1.65$ gives

$$\mathcal{I}_{\text{solve}} = \left[\overline{X}_n + \frac{1.6448^2}{2n} - \sqrt{\frac{1.6448^4}{4n^2} + \frac{1.6448^2\overline{X}_n}{n}}, \overline{X}_n + \frac{1.6448^2}{2n} + \sqrt{\frac{1.6448^4}{4n^2} + \frac{1.6448^2\overline{X}_n}{n}}\right].$$

提交 你已经尝试了2次（总共可以尝试3次）

Answers are displayed within the problem

讨论

显示讨论

主题： Unit 2 Foundation of Inference:Homework 2: Statistical Models, Estimation, and Confidence Intervals / 1. Confidence Intervals for Curved Gaussian Family

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