Indicator variables: the problem of joint lives. Let X_i be the random variable taking the value 1 or 0 depending on whether the first partner of the ith couple has survived or not. Let Y_i be the corresponding random variable for the second partner of the ith couple. Then, we have $S = \sum_{i=1}^{m} X_i Y_i$, and by using linearity of expectations and the total expectation theorem,

$$\begin{aligned}
\mathbf{E}[S \mid A = a] &= \sum_{i=1}^{m} \mathbf{E}[X_i Y_i \mid A = a] \\
&= m\mathbf{E}[X_1 Y_1 \mid A = a] \\
&= m\mathbf{E}[Y_1 = 1 \mid X_1 = 1,\, A = a]\mathbf{P}(X_1 = 1 \mid A = a) \\
&= m\mathbf{P}(Y_1 = 1 \mid X_1 = 1,\, A = a)\mathbf{P}(X_1 = 1 \mid A = a).
\end{aligned}$$

We have

$$\mathbf{P}(Y_1 = 1 \mid X_1 = 1,\, A = a) = \frac{a-1}{2m-1}, \qquad \mathbf{P}(X_1 = 1 \mid A = a) = \frac{a}{2m}.$$

Thus

$$\mathbf{E}[S \mid A = a] = m\,\frac{a-1}{2m-1} \cdot \frac{a}{2m} = \frac{a(a-1)}{2(2m-1)}.$$

Note that $\mathbf{E}[S \mid A = a]$ does not depend on p.

A somewhat other elaboration

discussion posted 7 days ago by **FonsD**

I want to elaborate $P(X_1 = 1, Y_1 = 1 | A = a)$ by calculating explicitly the numerator and denominator, and using the probability p in the formulae.

First we note: $P(A = a) = \binom{2m}{a} p^a (1-p)^{2m-a}$. Then: $P(\{X_1 = 1, Y_1 = 1\} \cap A = a) = p^2 \binom{2m-2}{a-2} p^{a-2} (1-p)^{2m-2-(a-2)}$ $= \binom{2m-2}{a-2} p^a (1-p)^{2m-a}$ where we apply independence.

Therefore $P(X_1 = 1, Y_1 = 1 | A = a) = \frac{\binom{2m-2}{a-2}}{\binom{2m}{a}} = \frac{a(a-1)}{2m(2m-1)}$. We see explicitly that the terms with p cancel out (this is related to the symmetry in the problem: every living person / couple is probabilistically equivalent to every other living person /couple).

PS I must say: symmetry arguments are sometimes very tricky and prone to mistakes in the reasoning. Perhaps it is clearer to see how symmetry is working by first elaborating the common formulae.