Problem 6: Maximum Likelihood Estimation of Phase Noise

Phase Noise Estimation under Gaussian Noise: Setup

This problem is motivated by estimation in communication systems (Wi-Fi, cellphones, etc). The solution obtained in this problem is implemented real-time in many communication systems. For example, your laptop Wi-Fi adapter, which is downloading and uploading all the content that you are consuming in this course, is performing this estimation (albeit in a more complicated statistical model) tens of hundreds of times every second.

Let

$$\mathbf{x} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

be a known vector, i.e. **we assume that we know $\boldsymbol{\theta}$**. Let $\theta \in [0, \pi/2]$.

Let $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ be defined as follows:

$$\mathbf{Y}_i = \begin{bmatrix} Y_i^{(1)} \\ Y_i^{(2)} \end{bmatrix} = \begin{bmatrix} \cos(\theta+\phi) \\ \sin(\theta+\phi) \end{bmatrix} + \mathbf{Z}_i, \quad i = 1, \ldots, n,$$

where $\mathbf{Z}_i \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_2)$ for a known σ^2 and ϕ is an unknown constant. Assume that $\mathbf{Z}_i, i = 1, \ldots, n$ are independent.

Objective: Upon observing $\mathbf{Y}_i, i = 1, \ldots, n$, we wish to produce an estimate $\widehat{\phi}$ of $\phi \in [-\pi, \pi]$.

(a) True or False

1/1 point (graded)
Select whether the following statement is **true or false**: "$\mathbf{Y}_i$ are iid.""

- (•) True ✔
- ( ) False

Solution:

The statement is true. The multivariate Gaussian vectors $\mathbf{Z}_i$ are iid. Therefore, $\mathbf{Y}_i$, which are deterministic functions of the $\mathbf{Z}_i$'s, respectively for $i = 1, \ldots, n$, are iid.

Submit   You have used 1 of 3 attempts

Answers are displayed within the problem

(b) The Underlying Problem

1/1 point (graded)
Referring to the **objective** in the problem setup given above, select from the following the statements that are correct. (Choose all that apply.)

- [ ] We are trying to estimate the **magnitude** by which $\mathbf{x}$ is scaled (in the presence of vector Gaussian noise).
- [x] We are trying to estimate the **phase rotation** undergone by $\mathbf{x}$ (in the presence of vector Gaussian noise). ✔
- [ ] We are trying to estimate the **magnitude and phase changes** undergone by $\mathbf{x}$ (in the presence of vector Gaussian noise).

✔

Grading Note: Partial credit is given.

Solution:

The objective of the problem states that we wish to produce an estimate of ϕ, which is the phase rotation undergone by $\mathbf{x}$ under an additive Gaussian noise.

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Answers are displayed within the problem

(c) Observation Under Zero Noise

1.0/1 point (graded)

For a moment, assume that there is **no Gaussian noise in the problem**. That is, let $\mathbf{Y}_i = [\cos(\theta+\phi) \quad \sin(\theta+\phi)]^T \triangleq \begin{pmatrix} Y^{(1)} \\ Y^{(2)} \end{pmatrix}$, for all i, in this sub-problem.

For simplicity, assume that $\theta \in [0, \pi/2]$, $\theta+\phi \in [0, \pi/2]$.

What is ϕ?

(Express your answer in terms of $Y^{(1)}$, $Y^{(2)}$, θ, and the $\arctan(x)$ function. Use **Y_1** for $Y^{(1)}$ and **Y_2** for $Y^{(2)}$. Type **arctan(x)** for $\arctan(x)$ (where x can be any expression). **Do not use** any trigonometric function other than $\arctan$.)

$\phi =$ arctan(Y_2/Y_1) - theta ✔ **Answer:** -theta + arctan(Y_2/Y_1)

STANDARD NOTATION

Solution:

If there is no noise, the value of ϕ is $\arctan\left(\frac{Y^{(2)}}{Y^{(1)}}\right) - \theta$, where we assume that $\theta \in [0, \pi/2]$, $\theta+\phi \in [0, \pi/2]$ for simplicity.

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

(d) Maximum Likelihood Estimator of the Phase Noise - Log Likelihood

1.0/1 point (graded)
Now, let us return to the original setup. What is the log-likelihood $\ell_n(\mathbf{Y}_1, \ldots, \mathbf{Y}_n; \phi)$?

For the answer box below, ignore the term $\ln\left(\frac{1}{(\sqrt{2\pi\sigma^2})^{2n}}\right)$ **in the log-likelihood and input the rest of the log-likelihood expression.**

(Use **Sigma_i(X_i)** for $\sum_{i=1}^{n}(X_i)$ (where X_i can be any quantity in a series indexed by i), **Y_1** for $Y_i^{(1)}$, and **Y_2** for $Y_i^{(2)}$. Enter **sin(x)** for $\sin(x)$, **cos(x)** for $\cos(x)$.)

- (Sigma_i((Y_1 - cos(theta+phi))^2 + (Y_2 - sin(theta+phi))^2))/(2*sigma^2) ✔

Answer: -(1/(2*sigma^2))*Sigma_i((Y_1 - cos(theta + phi))^2 + (Y_2 - sin(theta + phi))^2)

STANDARD NOTATION

Solution:

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

(e) Maximum Likelihood Estimator of the Phase Noise

1.0/1 point (graded)

Let $\widehat{\mu}_1 = \sum_{i=1}^{n} \frac{Y_i^{(1)}}{n}$ and $\widehat{\mu}_2 = \sum_{i=1}^{n} \frac{Y_i^{(2)}}{n}$.

Compute the maximum likelihood estimator $\widehat{\phi}_{n,\mathrm{MLE}}$ of ϕ upon observing $\mathbf{Y}_i, i = 1, \ldots, n$.

Note: Again for simplicity, assume while entering the expression in the following box that $\theta \in [0, \pi/2]$, $\widehat{\mu}_1 > 0$, and $\widehat{\mu}_2 > 0$.

(Use **hatmu_1** for $\widehat{\mu}_1$ and **hatmu_2** for $\widehat{\mu}_2$. Type **arctan(x)** for $\arctan(x)$ (where x can be any expression). **Do not use** any trigonometric function other than $\arctan$.)

$\widehat{\phi}_{n,\mathrm{MLE}} =$ arctan(hatmu_2/hatmu_1) - theta ✔ **Answer:** -theta + arctan(hatmu_2/hatmu_1)

STANDARD NOTATION

Solution:

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

(f) Geometry of the MLE of Phase Noise

0.5/1 point (graded)

Select from the following all statements that are true. (Choose all that apply.)

- [x] The MLE of ϕ does not change if we scaled $\mathbf{x}$ by $r > 0$. ✔
- [ ] The MLE of ϕ does not change if the covariance matrix of the multivariate Gaussian is scaled by $s > 0$. ✔

✱

Grading Note: Partial credit is given.

Solution:

Submit You have used 2 of 3 attempts

Answers are displayed within the problem

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