6. Deterministic Design with Gaussian Noise

Review of Multi-Dimensional Gaussians

1/1 point (graded)

The n-dimensional Gaussian $\mathcal{N}_n(\mu, \Sigma)$ with mean μ and covariance matrix Σ has density

$$f(\mathbf{x}) = \frac{\exp\left(-\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)\right)}{\sqrt{(2\pi)^n \det\Sigma}}$$

for all $\mathbf{x} \in \mathbb{R}^n$.

Let $\mathbf{X} \sim \mathcal{N}_n(0, \Sigma)$, so that it is centered at the origin. If we have $\mathbf{Y} = M\mathbf{X}$ for some matrix M, it turns out that $\mathbf{Y}$ is also an n-dimensional Gaussian, $\mathcal{N}_n(0, \Sigma_\mathbf{Y})$. Which of the following provides a correct formula for the Covariance $\Sigma_\mathbf{Y}$ of $\mathbf{Y}$?

(*Hint: Recall the formula* $\Sigma_\mathbf{Y} = \mathbb{E}\left[(\mathbf{Y}-\mathbb{E}[\mathbf{Y}])(\mathbf{Y}-\mathbb{E}[\mathbf{Y}])^T\right]$.)

- $M\Sigma M^{-1}$
- $M^{-1}\Sigma M$
- $M\Sigma M^T$ ✔
- $M^T\Sigma M$

Solution:

This can be directly computed as hinted.

$$\begin{aligned}
\Sigma_\mathbf{Y} &= \mathbb{E}\left[(\mathbf{Y}-\mathbb{E}[\mathbf{Y}])(\mathbf{Y}-\mathbb{E}[\mathbf{Y}])^T\right] \\
&= \mathbb{E}\left[(M\mathbf{X}-\mathbb{E}[M\mathbf{X}])(M\mathbf{X}-\mathbb{E}[M\mathbf{X}])^T\right] \\
&= \mathbb{E}\left[M(\mathbf{X}-\mathbb{E}[\mathbf{X}])(\mathbf{X}-\mathbb{E}[\mathbf{X}])^T M^T\right] \\
&= M\mathbb{E}\left[(\mathbf{X}-\mathbb{E}[\mathbf{X}])(\mathbf{X}-\mathbb{E}[\mathbf{X}])^T\right] M^T \\
&= M\Sigma M^T.
\end{aligned}$$

Submit You have used 3 of 3 attempts

Answers are displayed within the problem

The Least Square Estimator is the MLE in Deterministic Design