Problem 2

Setup :

Let $X_1, \ldots, X_n$ be i.i.d. random variable with pdf f_θ defined as follows:

$$f_\theta(x) = \theta x^{\theta-1} \mathbf{1}(0 \le x \le 1)$$

偶函数一定没有反函数
这个函数换一下，发现有反函数

where θ is some positive number.

(a)

1/1 point (graded)
Is the parameter θ identifiable?

- (x) Yes
- ( ) No

Solution:

Yes it is identifiable. If $\theta_1 \ne \theta_2$, then the pdfs $f_{\theta_1}(x) \ne f_{\theta_2}(x)$.

提交 你已经尝试了1次（总共可以尝试3次）

Answers are displayed within the problem

(b)

2.0/2.0 points (graded)
Compute the maximum likelihood estimator $\hat{\theta}$ of θ.

(Enter **Sigma_i(g(X_i))** for the sum $\sum_{i=1}^n g(X_i)$, e.g. enter **Sigma_i(X_i^2)** for $\sum_{i=1}^n X_i^2$, enter **Sigma_i(ln(X_i))** for $\sum_{i=1}^n \ln(X_i)$. Do not forget any necessary n in your answer, e.g. $\bar{X}_n$ will need to be entered as **Sigma_i(X_i)/n** . Do not worry about the parser not rendering correctly, as the grader will still work independently. If you would like proper rendering, enclose $\Sigma_i(g(X_i))$ in parentheses i.e. use $(\Sigma_i(g(X_i)))$.)

Maximum likelihood estimator $\hat{\theta} =$ -n/(Sigma_i(ln(X_i))) **Answer:** -n/Sigma_i(ln(X_i))

STANDARD NOTATION

Solution:

The likelihood of $X_1, \ldots, X_n$ given a parameter θ is

$$L(X_1, \ldots, X_n; \theta) .= \theta^n \prod_{i=1}^n X_i^{\theta-1}.$$

Taking the logarithm we find that log-likelihood

$$\ell_n(\theta) = n \ln(\theta) + (\theta - 1) \sum_{i=1}^{n} \ln(X_i).$$

Setting $\ell'(\theta) = 0$ we find that

$$\hat{\theta} = \frac{-n}{\sum_{i=1}^{n} \ln X_i}.$$

This is the unique maximum as

$$\ell_n''(\theta) = \frac{-n}{\theta^2} < 0.$$

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Answers are displayed within the problem

(c)

2.0/2.0 points (graded)
Compute the Fisher information.

$I(\theta) =$ 1/theta^2 Answer: 1/theta^2

$\frac{1}{\theta^2}$

STANDARD NOTATION

Solution:

By definition, the Fisher information is defined as

$$I(\theta) = -\mathbb{E}[\ell''(X;\theta)]$$

where $\ell(\theta) = \ln L(X;\theta)$ is the log-likelihood defined using a sample of size 1.
The likelihood of X given a parameter θ is

$$L(X;\theta) = \theta(X^{\theta-1}).$$

Taking the logarithm we find that log-likelihood

$$\ell(\theta) = \ln(\theta) + (\theta - 1)\ln(X_i).$$

Taking the second derivative we find that

$$\ell''(\theta) = \frac{-1}{\theta^2}$$

and therefore we have that

$$I(\theta) = -\mathbb{E}\left[\ell''(X;\theta)\right] = \frac{1}{\theta^2}.$$

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Answers are displayed within the problem

(d)

1/1 point (graded)
What kind of distribution does the distribution of $\sqrt{n}\hat{\theta}$ approach as n grows large?

- Bernoulli
- Poisson
- Normal
- Exponential

Solution:

The theorem for MLE applies in this example as the following conditions hold:

- θ is identifiable
- $I(\theta)$ is invertible
- Support of f_θ does not depend on θ

Hence $\hat{\theta}$ is asymptotically normal:

$$\sqrt{n}\left(\hat{\theta} - \theta\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}\left(0, I^{-1}(\theta)\right).$$

This means that as n grows large,

$$\hat{\theta} \overset{\text{approx}}{\sim} \mathcal{N}\left(\theta, \frac{I^{-1}(\theta)}{n}\right)$$

and hence $\sqrt{n}\hat{\theta}$ is also approximately normal.

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Answers are displayed within the problem

(e)

1.0/1 point (graded)
What is the asymptotic variance $V(\hat{\theta})$ of $\hat{\theta}$?

To avoid double jeopardy, you may use I for the Fisher information $I(\theta)$ evaluated at $\theta,$ or you may enter your answer without using I.

$V(\hat{\theta}) =$ 1/I **Answer:** theta^2

STANDARD NOTATION

Solution:

By the theorem for the MLE the asymptotic variance of the estimator is $I(\theta)^{-1} = \frac{1}{\theta^2}$.

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Answers are displayed within the problem

(f)

2.0/2.0 points (graded)

Using the MLE $\hat{\theta}$, find the shortest confidence interval for θ with asymptotic level 85% using the plug-in method.

To avoid double jeopardy, you may use V for the appropriate estimator of the asymptotic variance $V(\hat{\theta})$, and/or I for the Fisher information $I(\hat{\theta})$ evaluated at $\hat{\theta}$, or you may enter your answer without using V or I.

(Enter **hattheta** for $\hat{\theta}$. If applicable, enter **Phi(z)** for the cdf $\Phi(z)$ of a normal variable Z, **q(alpha)** for the quantile q_α for any numerical value α. Recall the convention in this course that $\mathbf{P}(Z \leq q_\alpha) = 1 - \alpha$ for $Z \sim \mathcal{N}(0,1)$.)

$\mathcal{I}_{\text{plug-in}} = [A, B]$ where

$A =$ hattheta - (q((1-0.85)/2))/sqrt(n*I) **Answer:** hattheta-q(0.075)*sqrt(V/n)

$B =$ hattheta + (q((1-0.85)/2))/sqrt(n*I) **Answer:** hattheta+q(0.075)*sqrt(V/n)

STANDARD NOTATION

Solution:

Using the previous question on the asymptotic normality of the MLE it follows that

$$\lim_{n\to\infty} \mathbf{P}[\hat{\theta} \in [\theta - 1.44\frac{\theta}{\sqrt{n}}, \theta + 1.44\frac{\theta}{\sqrt{n}}]] = .85.$$

Therefore it follows that

$$\lim_{n\to\infty} \mathbf{P}[\theta \in [\hat{\theta} - 1.44\frac{\theta}{\sqrt{n}}, \hat{\theta} + 1.44\frac{\theta}{\sqrt{n}}]] = .85$$

and since $\hat{\theta}$ approaches θ almost surely we get the confidence interval

$$[\hat{\theta} - 1.44\frac{\hat{\theta}}{\sqrt{n}}, \hat{\theta} + 1.44\frac{\hat{\theta}}{\sqrt{n}}]$$

via the plug-in method.

提交 你已经尝试了1次（总共可以尝试3次）

Answers are displayed within the problem

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