

For those of you who are curious, we will go through an argument that establishes that the set of real numbers is an uncountable set. It's a famous argument known as Cantor's diagonalization argument. Actually, instead of looking at the set of all real numbers, we will first look at the set of all numbers, x , that belong to the open unit interval-- so numbers between 0 and 1-- and such that their decimal expansion involves only threes and fours. Now, the choice of three and four is somewhat arbitrary. It doesn't matter. What really matters is that we do not have long strings of nines.

So suppose that this set was countable. If the set was countable, then that set could be written as equal to a set of this form, x_1, x_2, x_3 and so on, where each one of these is a real number inside that set. Now, suppose that this is the case. Let us take those numbers and write them down in decimal notation.

For example, one number could be this one, and it continues forever. Since we're talking about real numbers, their decimal expansion will go on forever. Suppose that the second number is of this kind, and it has its own decimal expansion. Suppose that the third number is, again, with some decimal expansion and so on.

So we have assumed that our set is countable and therefore, the set is equal to that sequence. So this sequence exhausts all the numbers in that set. Can it do that?

Let's construct a new number in the following fashion. The new number looks at this digit and does something different. Looks at this digit, the second digit of the second number, and does something different. Looks at the third digit of the third number and does something different. And we continue this way.

This number that we have constructed here is different from the first number. They differ in the first digit. It's different from the second number. They differ in the second digit. It's different from the third number because it's different in the third digit and so on. So this is a number, and this number is different from x_i for all i .

So we have an element of this set which does not belong to this sequence. Therefore, it cannot be true that this set is equal to the set formed by that sequence. And so this is a contradiction to the initial assumption that this set could be written in this form, and this contradiction establishes that since this is

not possible, that the set that we have here is an uncountable set.

Now, this set is a subset of the set of real numbers. Since this one is uncountable, it is not hard to show that the set of real numbers, which is a bigger set, will also be uncountable. And so this is this particular famous argument. We will not need it or make any arguments of this type in this class, but it's so beautiful that it's worth for everyone to see it once in their lifetime.