Probability models often involve infinite sample spaces, that is, infinite sets. But not all sets are of the same kind. Some sets are discrete and we call them countable, and some are continuous and we call them uncountable. But what exactly is the difference between these two types of sets? How can we define it precisely?

Well, let us start by first giving a definition of what it means to have a countable set. A set will be called countable if its elements can be put into a 1-to-1 correspondence with the positive integers. This means that we look at the elements of that set, and we take one element-- we call it the first element. We take another element-- we call it the second. Another, we call the third element, and so on.

And this way we will eventually exhaust all of the elements of the set, so that each one of those elements corresponds to a particular positive integer, namely the index that appears underneath. More formally, what's happening is that we take elements of that set that are arranged in a sequence. We look at the set, which is the entire range of values of that sequence, and we want that sequence to exhaust the entire set omega. Or in other words, in simpler terms, we want to be able to arrange all of the elements of omega in a sequence.

So what are some examples of countable sets? In a trivial sense, the positive integers themselves are countable, because we can arrange them in a sequence. This is almost tautological, by the definition. For a more interesting example, let's look at the set of all integers. Can we arrange them in a sequence? Yes, we can, and we can do it in this manner, where we alternate between positive and negative numbers.

And this way, we're going to cover all of the integers, and we have arranged them in a sequence. How about the set of all pairs of positive integers? This is less clear. Let us look at this picture. This is the set of all pairs of positive integers, which we understand to continue indefinitely. Can we arrange this sets in a sequence?

It turns out that we can. And we can do it by tracing a path of this kind. So you can probably get the sense of how this path is going. And by continuing this way, over and over, we're going to cover the entire set of all pairs of positive integers. So we have managed to arrange them in a sequence. So the set of all such pairs is indeed a countable set.

And the same argument can be extended to argue for the set of all triples of positive integers, or the set of all quadruples of positive integers, and so on. This is actually not just a trivial mathematical point that we discuss for some curious reason, but it is because we will often have sample spaces that are of this kind. And it's important to know that they're countable.

Now for a more subtle example. Let us look at all rational numbers within the range between 0 and 1. What do we mean by rational numbers? We mean those numbers that can be expressed as a ratio of two integers. It turns out that we can arrange them in a sequence, and we can do it as follows. Let us first look at rational numbers that have a denominator term of 2. Then, look at the rational numbers that have a denominator term of 3. Then, look at the rational numbers, always within this range of interest, that have a denominator of 4.

And then we continue similarly-- rational numbers that have a denominator of 5, and so on. This way, we're going to exhaust all of the rational numbers. Actually, this number here already appeared there. It's the same number. So we do not need to include this in a sequence, but that's not an issue. Whenever we see a rational number that has already been encountered before, we just delete it.

In the end, we end up with a sequence that goes over all of the possible rational numbers. And so we conclude that the set of all rational numbers is itself a countable set. So what kind of set would be uncountable? An uncountable set, by definition, is a set that is not countable. And there are examples of uncountable sets, most prominent, continuous subsets of the real line. Whenever we have an interval, the unit interval, or any other interval that has positive length, that interval is an uncountable set.

And the same is true if, instead of an interval, we look at the entire real line, or we look at the two-dimensional plane, or three-dimensional space, and so on. So all the usual sets that we think of as continuous sets turn out to be uncountable. How do we know that they are uncountable? There is actually a brilliant argument that establishes that the unit interval is uncountable.

And then the argument is easily extended to other cases, like the reals and the plane. We do not need to know how this argument goes, for the purposes of this course. But just because it is so beautiful, we will actually be presenting it to you.