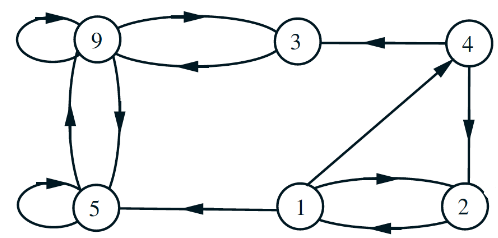
# Markov processes – III

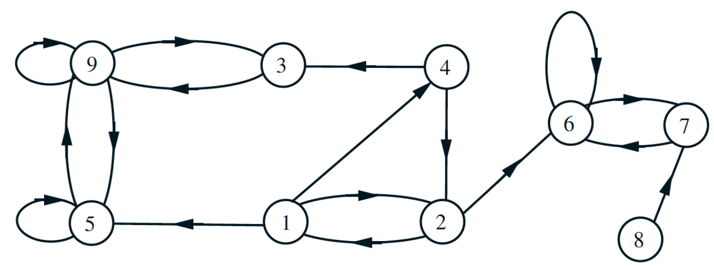
- review of steady-state behavior
- probability of blocked phone calls
- calculating absorption probabilities
- calculating expected time to absorption

## review of steady state behavior




- Markov chain with a single class of recurrent states, aperiodic; and some transient states; then,

$$\boxed{\lim_{n\to\infty} r_{ij}(n) = \lim_{n\to\infty} \mathbf{P}(X_n = j \mid X_0 = i) = \pi_j, \quad \forall\ i}$$

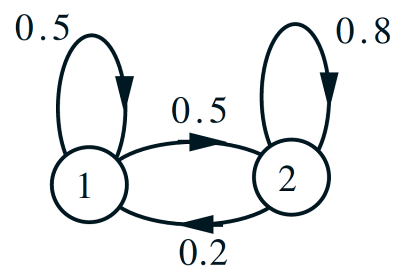



- can be found as the unique solution to the balance equations

$$\pi_j = \sum_k \pi_k p_{kj}, \qquad j = 1, \ldots, m,$$

- together with $\sum_j \pi_j = 1$

## on the use of steady state probabilities, example




$\pi_1 = 2/7, \ \pi_2 = 5/7$

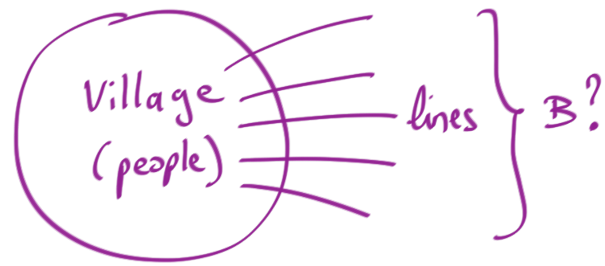
assume process starts in state 1

$\mathbf{P}(X_1 = 1 \text{ and } X_{100} = 1 \mid X_0 = 1) =$

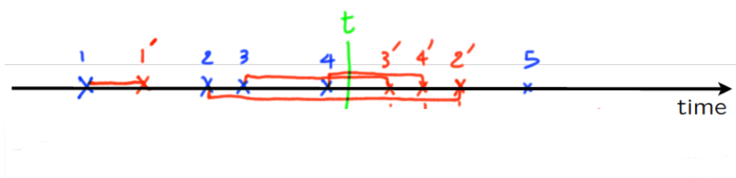
$\mathbf{P}(X_{100} = 1 \text{ and } X_{101} = 2 \mid X_0 = 1) =$

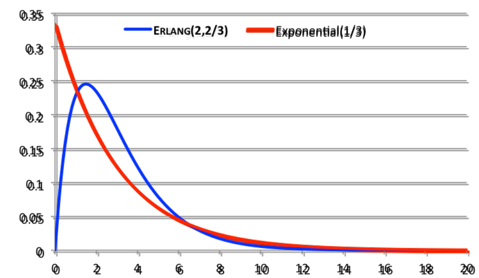
$\mathbf{P}(X_{100} = 1 \text{ and } X_{200} = 1 \mid X_0 = 1) =$

# design of a phone system

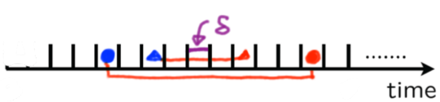



- calls originate as a Poisson process, rate $\lambda$
- each call duration is exponential (parameter $\mu$)
- need to decide on how many lines, B?



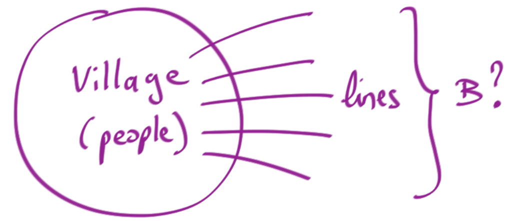




- for time slots of small duration $\delta$



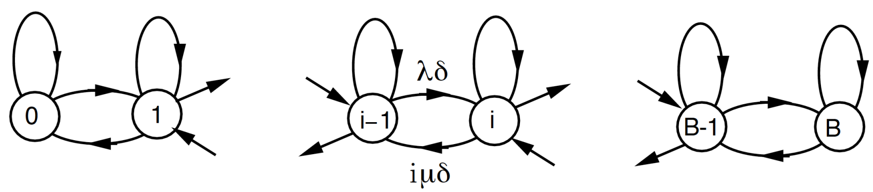

  - P(a new call arrives) $\approx \lambda\delta$
  - if you have $i$ active calls, then P(a departure) $\approx i\mu\delta$

# design of a phone system, a discrete time approximation


- approximation: discrete time slots of (small) duration $\delta$

$\mathbb{P}(1 \text{ new call}) \approx \lambda\delta$ ; $\mathbb{P}(1 \text{ call ends} \mid i \text{ busy}) \approx i\mu\delta$



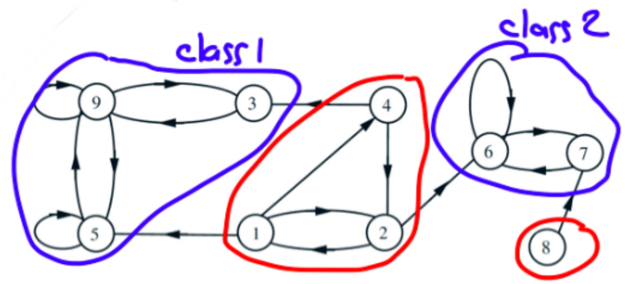

- balance equations

$$\lambda\pi_{i-1} = i\mu\pi_i$$

$$\pi_i = \pi_0 \frac{\lambda^i}{\mu^i i!} \qquad \pi_0 = 1 / \sum_{i=0}^{B} \frac{\lambda^i}{\mu^i i!}$$

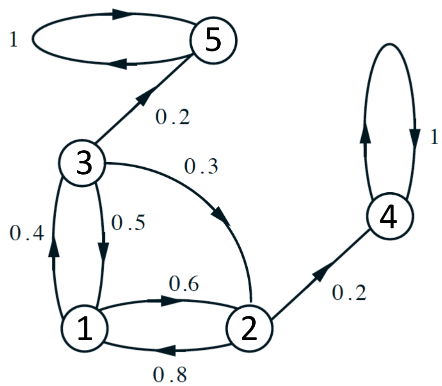
- P(arriving customer finds busy system) is $\pi_B$

## calculating absorption probabilities




- absorbing state: recurrent state $k$ with $p_{kk} = 1$

- what is the probability $a_i$ that the chain eventually settles in 4 given it started in $i$?

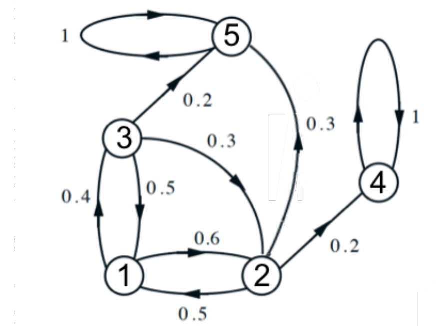


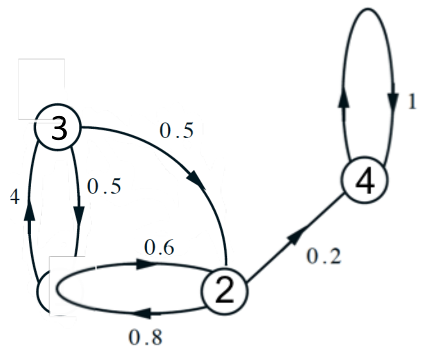

$$i = 4,\ a_i =$$
$$i = 5,\ a_i =$$
$$\text{otherwise},\ a_i =$$

- unique solution from $a_i = \sum_j p_{ij} a_j$

## expected time to absorption

- find expected number of transitions $\mu_i$ until reaching 4, given that the initial state is $i$




$\mu_i = 0$ for $i =$

for all others, $\mu_i =$

- unique solution from $\mu_i = 1 + \sum_j p_{ij}\mu_j$

## mean first passage and recurrence times

- chain with one recurrent class

- mean first passage time from $i$ to $s$ :

$$t_i = \mathbf{E}[\text{min}\{n \geq 0 \text{ such that } X_n = s\} \mid X_0 = i]$$

  – unique solution to:

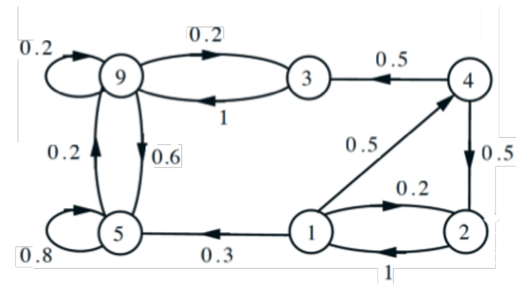
$$\begin{aligned} t_s &= 0, \\ t_i &= 1 + \sum_j p_{ij} t_j, \qquad \text{for all } i \neq s \end{aligned}$$

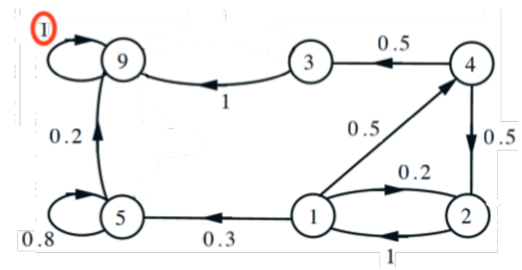
- mean recurrence time of $s$

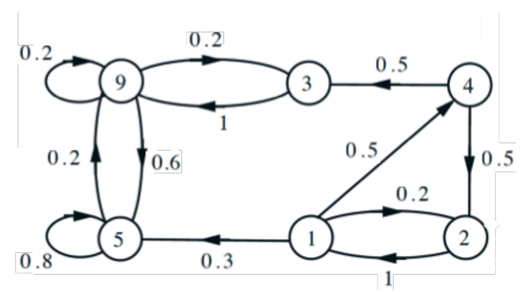
$$t_s^* = \mathbf{E}[\text{min}\{n \geq 1 \text{ such that } X_n = s\} \mid X_0 = s]$$

  – solution to:

$$t_s^* = 1 + \textstyle\sum_j p_{sj} t_j$$

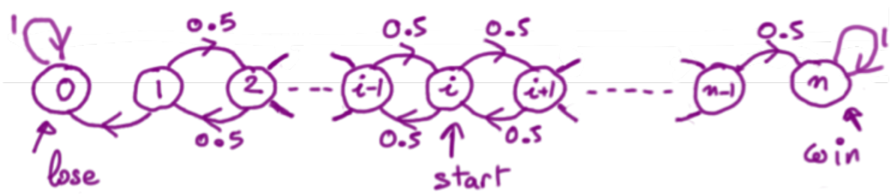


## gambler's example

- a gambler starts with $i$ dollars; each time, she bets \$1 in a fair game, until she either has 0 or $n$ dollars.
- what is the probability $a_i$ that she ends up with having $n$ dollars?




$i = 0,\ a_i =$ $\qquad$ $i = n,\ a_i =$

$0 < i < n,\ a_i =$

- expected wealth at the end? $0 \cdot (1 - a_i) + n \cdot a_i =$

- how long does the gambler expect to stay in the game?
  - $\mu_i =$ expected number of plays, starting from $i$
  - for $i = 0, n$: $\mu_i =$
  - in general

$$\mu_i = 1 + \sum_j p_{ij} \mu_j$$

- in case of unfavorable odds?