5. Improper Prior: Example

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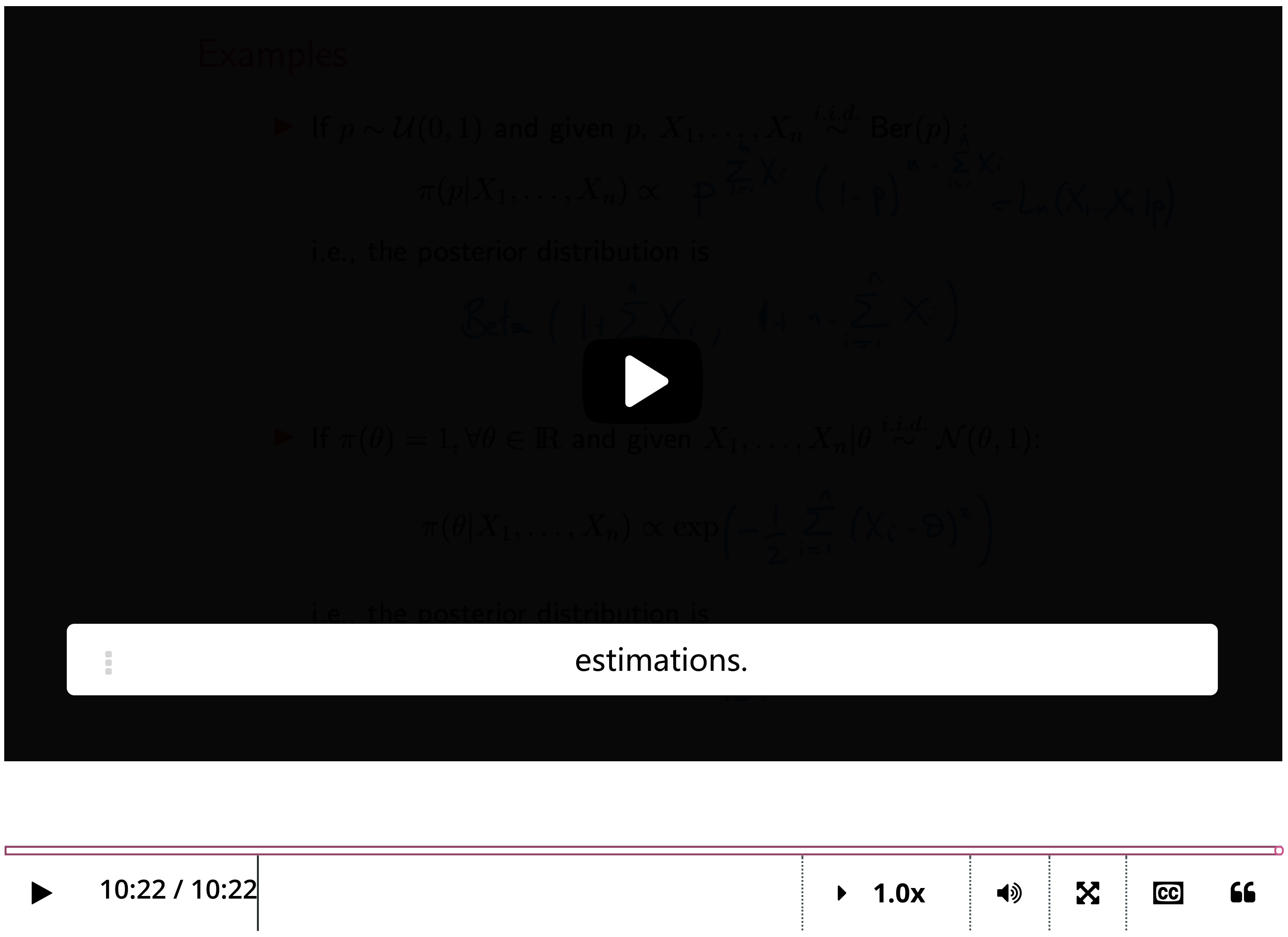

that were just in replacing the posterior by the rescaled log

likelihood so that it actually looks

like a density with respect to your parameters.

That's all it is.

There's nothing in there.

And so it's normal that it's coming back

to results that we know when we did maximum likelihood

estimations.

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Gaussian Prior on Gaussian Observations

2/2 points (graded)

Let $X_1, X_2, \ldots, X_n$ be random variables; which are i.i.d., conditional on θ, and such that,

$$p\left(X_i|\theta\right) = N\left(\theta, 1\right),$$

where $p\left(X_i|\theta\right)$ is the conditonal density of X_i given θ. Furthermore, we assume the prior $\pi\left(\theta\right) \sim N\left(\mu, 1\right)$. Let

$$\pi\left(\theta|X_1, \ldots, X_n\right) \sim N\left(\alpha, \beta^2\right).$$

Find, α and β^2.

- $\alpha =$

● $\frac{1}{n+1}((\sum_{i=1}^{n} X_i) + \mu)$ ✓

○ $\frac{1}{n}(\sum_{i=1}^{n} X_i + \mu)$

○ $\frac{1}{n+1}(\sum_{i=1}^{n} X_i) + \mu$

○ $\frac{1}{n}\left(\sum_{i=1}^{n} X_i\right)$

- $\beta^2 =$

● $\frac{1}{n+1}$ ✓

○ $\frac{1}{n}$

○ $\frac{\mu}{n}$

○ $\frac{\mu}{n+1}$

Solution:

We begin by recalling that,

$$\begin{aligned}\pi\left(\theta|X_1,\ldots,X_n\right) &\propto p_n\left(X_1,\ldots,X_n|\theta\right)\pi\left(\theta\right)\\ &\propto \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\left(X_i-\theta\right)^2\right)\exp\left(-\frac{1}{2}(\theta-\mu)^2\right).\end{aligned}$$

We now study the last quantity, keeping in mind that, Gaussian distribution is a conjugate prior of itself; hence, we expect the resulting distribution to be a Gaussian. For this, we need to arrive at a formula of the form,

$$\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\left(X_i-\theta\right)^2\right)\exp\left(-\frac{1}{2}(\theta-\mu)^2\right)\propto\exp\left(-\frac{1}{2\beta^2}(\theta-\alpha)^2\right).$$

Now, we begin doing the algebra, after removing $\exp$'s from both sides.

$$\begin{aligned}&-\frac{1}{2}\sum_{i=1}^{n}\left(X_i-\theta\right)^2-\frac{1}{2}(\theta-\mu)^2\\ &=-\frac{1}{2}\left((n+1)\,\theta^2-2\theta\left(\left[\sum_{i=1}^{n}X_i\right]+\mu\right)\right)+C\\ &=-\frac{n+1}{2}\left(\theta^2-2\theta\frac{\left[\sum_{i=1}^{n}X_i\right]+\mu}{n+1}\right)+C\\ &=-\frac{1}{2(n+1)^{-1}}(\theta-\alpha)^2+C',\end{aligned}$$

where C and C''s are constants; and,

$$\alpha=\frac{1}{n+1}\left(\left[\sum_{i=1}^{n}X_i\right]+\mu\right)\quad\text{and}\quad\beta^2=\frac{1}{n+1}.$$

This is, essentially, the same formula as derived in the lecture, except that n is replaced with $n+1$ (since we have a prior now), and the summation also takes μ into account. In a sense, you may view this as, $n+1$ observations, where the first n are $X_1,\ldots,X_n$ and the last one is from $N\left(\mu,1\right)$.

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You have used 2 of 3 attempts

Answers are displayed within the problem