5. True or False

Problem 5. True or False

3/3 points (graded)

Determine whether each of the following statement is true (i.e., always true) or false (i.e., not always true).

1. Let X be a random variable that takes values between 0 and c only, for some $c \geq 0$, so that $\mathbf{P}(0 \leq X \leq c) = 1$. Then, $\mathbf{Var}(X) \leq c^2/4$.

   True ▼ ✔ **Answer:** True

2. Let X and Y be continuous random variables. If $X \sim N(\mu, \sigma^2)$ (i.e., normal with mean μ and variance σ^2), $Y = aX + b$, and $a > 0$, then $Y \sim N(a\mu + b, a\sigma^2)$.

   False ▼ ✔ **Answer:** False

3. The expected value of a non-negative continuous random variable X, which is defined by $\mathbf{E}[X] = \int_0^\infty x f_X(x) dx$, also satisfies $\mathbf{E}[X] = \int_0^\infty \mathbf{P}(X > t) \mathrm{d}t$.

   True ▼ ✔ **Answer:** True

Solution:

1. The statement is true. Since $0 \leq X \leq c$,

$$\begin{aligned} \mathbf{E}[X^2] &= \mathbf{E}[XX] \\ &\leq \mathbf{E}[cX] \\ &= c\mathbf{E}[X]. \end{aligned}$$

   Therefore,

$$\begin{aligned} \mathbf{Var}(X) &= \mathbf{E}[X^2] - (\mathbf{E}[X])^2 \\ &\leq c\mathbf{E}[X] - (\mathbf{E}[X])^2 \\ &= c^2 \left(\frac{\mathbf{E}[X]}{c}\right) - c^2 \left(\frac{\mathbf{E}[X]}{c}\right)^2 \end{aligned}$$

$$
\begin{aligned}
&= c^2 \left(\frac{\mathbf{E}[X]}{c} \left(1 - \frac{\mathbf{E}[X]}{c} \right) \right) \\
&= c^2 [\alpha(1-\alpha)] \\
&\leq c^2/4,
\end{aligned}
$$

where $\alpha = \mathbf{E}[X]/c$. The last inequality is obtained by noticing that the function $\alpha(1-\alpha)$ is largest at $\alpha = 1/2$, where it takes a value of $1/4$.

2. The statement is false. The correct statement is: $Y \sim N(a\mu + b, a^2\sigma^2)$.

3. The statement is true. By changine the order of integration, we obtain

$$
\begin{aligned}
\int_0^\infty \mathbf{P}(X > t) \mathrm{d}t &= \int_0^\infty \int_t^\infty f_X(x) dx\, dt \\
&= \int_0^\infty \int_0^x f_X(x) dt\, dx \\
&= \int_0^\infty x f_X(x) dx \\
&= \mathbf{E}[X].
\end{aligned}
$$

This result is analogous to the result for discrete random variables that was the subject of a Unit 4 solved problem.

提交 You have used 1 of 1 attempt

Answers are displayed within the problem

讨论

显示讨论

Topic: Unit 5 / Problem Set / 5. True or False

Learn About Verified Certificates