Problem 4

(a)

2/2 points (graded)

Let $X_1, \ldots, X_n \overset{i.i.d.}{\sim}$ Ber (p) for some $p \in (0,1)$. Which of the following is the maximum likelihood estimator $\hat{p}$ for p?

- ○ X_1
- ◉ $\frac{\sum_{i=1}^{n} X_i}{n}$ ✔
- ○ $\frac{n}{\sum_{i=1}^{n} X_i}$
- ○ $\frac{n+1}{\sum_{i=1}^{n} X_i}$

Is the maximum likelihood estimator for p unbiased?

- ◉ Yes ✔
- ○ No

Solution:

Note that the likelihood of $X_1, \ldots, X_n$ is

$$L(X_1, \ldots, X_n | p) = (1-p)^{\sum_{i=1}^{n} X_i} (p)^{n - \sum_{i=1}^{n} X_i}.$$

Now note that the log-likelihood is

$$\ell(p) = \sum_{i=1}^{n} X_i \log(1-p) + n - \sum_{i=1}^{n} X_i \log(p).$$

Setting $\ell'(p) = 0$ it follows that

$$\hat{p} = \frac{\sum_{i=1}^{n} x_i}{n}.$$

Note that

$$\ell''(p) < 0 \qquad \text{Concave}$$

so the maximum is unique.

Note that $\mathbb{E}\left[\frac{\sum_{i=1}^{n} x_i}{n}\right] = p$ so the estimator is unbiased.

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

(b)

2/2 points (graded)
Compute the bias of the estimator $\hat{p}\left(1-\hat{p}\right)$ for $p\left(1-p\right)$.

bias of $\hat{p}\left(1-\hat{p}\right)$: -(p*(1-p))/n **Answer:** -(p-p^2)/n

$-\frac{p\cdot(1-p)}{n}$

There exists a constant C such that $C\hat{p}\left(1-\hat{p}\right)$ is unbiased. Compute C.

$C =$ n/(n-1) **Answer:** n/(n-1)

$\frac{n}{n-1}$

STANDARD NOTATION

Correction Note: May 23 An earlier version of the problem statement asked for the bias of the estimator $\hat{p}\left(1-\hat{p}\right)$ for $p\left(1-p\right)$, but in the prompt to the answer box, the word "bias" was missing.

Solution:

Note that

$$\mathbb{E}\left[\frac{\sum_{i=1}^{n} x_i}{n}\left(1-\frac{\sum_{i=1}^{n} x_i}{n}\right)\right] = p - \mathbb{E}\left[\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^2\right].$$

To compute the second term note that by symmetry it is the same as

$$\mathbb{E}\left[\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^2\right] = \frac{n\left(n-1\right)\mathbb{E}\left[x_1x_2\right]+n\mathbb{E}\left[x_1^2\right]}{n^2} = \frac{p^2\left(n-1\right)+p}{n}.$$

Thus

$$\mathbb{E}\left[\hat{p}\left(1-\hat{p}\right)\right] - p\left(1-p\right) = \frac{-\left(p-p^2\right)}{n}.$$

Next, we write that

$$\mathbb{E}\left[\hat{p}\left(1-\hat{p}\right)\right] = \frac{\left(n-1\right)p\left(1-p\right)}{n}$$

and the result follows.

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

(c)

2/2 points (graded)
Which of the following methods can be used to show that $\hat{p}\left(1-\hat{p}\right)$ is asymptotically normal?

- Central Limit Theorem
- Theorem on MLE
- Delta Method along with the Central Limit Theorem ✔

What is the asymptotic variance of $\hat{p}\left(1-\hat{p}\right)$?
(Express your answer as a function of p only.)

(1-2*p)^2*p*(1-p) ✔ **Answer:** p*(1-p)*(1-2*p)^2

$(1-2\cdot p)^2\cdot p\cdot(1-p)$

STANDARD NOTATION

Solution:

If one know that a random variable y is asymptotically normal then generally one uses the Delta method to prove that $f\left(y\right)$ is also asymptotically normal.

Note that by we apply the Delta Method to $\hat{p}$ for the function $f\left(x\right)=x\left(1-x\right)$. Note that $f'\left(x\right)=1-2x$ so it follows that

$$\sqrt{(n)}\left(\hat{p}\left(1-\hat{p}\right)-p\left(1-p\right)\right)\to\mathcal{N}\left(0,p\left(1-p\right)\left(1-2p\right)^2\right).$$

Therefore the asymptotic variance is $p\left(1-p\right)\left(1-2p\right)^2$.

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

(d)

0/1 point (graded)
Using the plug-in method determine $A>0$ such $\left[\hat{p}\left(1-\hat{p}\right)\left(1-A\right),\hat{p}\left(1-\hat{p}\right)\left(1+A\right)\right]$is a confidence interval for $p\left(1-p\right)$ with asymptotic level 95%. Note that A should only depend on n and $\hat{p}$.

(Enter **hatp** for $\hat{p}$. If applicable, enter **Phi(z)** for the cdf $\Phi\left(z\right)$ of a normal variable Z, **q(alpha)** for the quantile q_α for any numerical value α. Recall the convention in this course that $\mathbf{P}\left(Z\leq q_\alpha\right)=1-\alpha$ for $Z\sim\mathcal{N}\left(0,1\right)$.)

$A=$ 1.96*(1-2*hatp)/sqrt(n*hatp*(1-hatp)) ✖ **Answer:** q(0.025)*abs(1-2*hatp)/sqrt(n*hatp*(1-hatp))

傻了，没有注意符号

STANDARD NOTATION

Solution:

Note that

$$\sqrt{(n)}\left(\hat{p}\left(1-\hat{p}\right)-p\left(1-p\right)\right)\to\mathcal{N}\left(0,p\left(1-p\right)\left(1-2p\right)^2\right).$$

Thus $p\left(1-p\right)$ is in

$$\left[\hat{p}\left(1-\hat{p}\right)-\frac{1.96\sqrt{p\left(1-p\right)\left(1-2p\right)^2}}{\sqrt{n}},\hat{p}\left(1-\hat{p}\right)+\frac{1.96\sqrt{p\left(1-p\right)\left(1-2p\right)^2}}{\sqrt{n}}\right]$$

with probability 95% asymptotically and using the plug-in method we can replace the above with

$$\left[\hat{p}\left(1-\hat{p}\right)-\frac{1.96\sqrt{\hat{p}\left(1-\hat{p}\right)\left(1-2\hat{p}\right)^2}}{\sqrt{n}},\hat{p}\left(1-\hat{p}\right)+\frac{1.96\sqrt{\hat{p}\left(1-\hat{p}\right)\left(1-2\hat{p}\right)^2}}{\sqrt{n}}\right].$$

This gives A equal to $\frac{1.96\sqrt{\left(1-2\hat{p}\right)^2}}{\sqrt{n\hat{p}\left(1-\hat{p}\right)}}$.

Submit You have used 2 of 3 attempts

Answers are displayed within the problem

Error and Bug Reports/Technical Issues

Show Discussion

Topic: Final exam:Final Exam / Problem 4