1. The LMS estimator is

$$g(X) = \mathbf{E}[Y \mid X] = \begin{cases} \frac{1}{2}X, & 0 \le X < 1, \\ X - \frac{1}{2}, & 1 \le X \le 2, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

2. If $x \in [0,1]$, the conditional PDF of Y is uniform over the interval $[0,x]$, and

$$\mathbf{E}\left[(Y - g(X))^2 \mid X = x\right] = \frac{x^2}{12}.$$

Similarly, if $x \in [1,2]$, the conditional PDF of Y is uniform over $[1-x, x]$, and

$$\mathbf{E}\left[(Y - g(X))^2 \mid X = x\right] = \frac{1}{12}.$$

3. The expectations $\mathbf{E}\left[(\,Y - g(X)\,)^2\right]$ and $\mathbf{E}\left[\text{var}(Y \mid X)\right]$ are equal because by the law of iterated expectations,

$$\mathbf{E}\left[(\,Y - g(X)\,)^2\right] = \mathbf{E}\left[\mathbf{E}\left[(Y - g(X))^2 \mid X\right]\right] = \mathbf{E}[\text{var}(Y \mid X)].$$

Recall from part (b) that

$$\text{var}(Y \mid X = x) \quad = \quad \begin{cases} \frac{x^2}{12}, & 0 \le x < 1, \\ \frac{1}{12}, & 1 \le x \le 2. \end{cases}$$

It follows that

$$\mathbf{E}[\text{var}(Y \mid X)] = \int_x \text{var}(Y \mid X = x) f_X(x)\, dx = \int_0^1 \frac{x^2}{12} . \frac{2}{3} x\, dx + \int_1^2 \frac{1}{12} . \frac{2}{3}\, dx = \frac{5}{72}.$$

Note that

$$f_X(x) \quad = \quad \begin{cases} 2x/3, & 0 \le x < 1, \\ 2/3, & 1 \le x \le 2. \end{cases}$$

4. The linear LMS estimator is

$$L(X) = \mathbf{E}[Y] + \frac{\text{cov}(X,Y)}{\text{var}(X)}[X - \mathbf{E}[X]].$$

In order to calculate $\text{var}(X)$ we first calculate $\mathbf{E}[X^2]$ and $\mathbf{E}[X]^2$:

$$\begin{aligned} \mathbf{E}[X^2] &= \int_0^1 x^3 \frac{2}{3} dx + \int_1^2 x^2 \frac{2}{3} dx, \\ &= \frac{31}{18}, \\ \mathbf{E}[X] &= \int_0^1 x^2 \frac{2}{3} dx + \int_1^2 x \frac{2}{3} dx, \\ &= \frac{11}{9}. \end{aligned}$$

Thus, $\text{var}(X) = \mathbf{E}[X^2] - \mathbf{E}[X]^2 = \frac{37}{162}$. Also,

$$\mathbf{E}[Y] = \int_0^1 \int_0^x \frac{2}{3} y \; dydx + \int_1^2 \int_{x-1}^x \frac{2}{3} y \; dydx = \frac{1}{9} + \frac{2}{3} = \frac{7}{9}.$$

To determine $\text{cov}(X, Y)$ we need to evaluate $\mathbf{E}[XY]$:

$$\begin{aligned} \mathbf{E}[XY] &= \int_x \int_y xy f_{X,Y}(x,y) dydx \\ &= \int_0^1 \int_0^x yx \, \frac{2}{3} dydx + \int_1^2 \int_{x-1}^x yx \, \frac{2}{3} dydx \\ &= \frac{41}{36}, \end{aligned}$$

and so $\text{cov}(X, Y) = \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y] = \frac{61}{324}$. Therefore,

$$L(X) = \frac{7}{9} + \frac{61}{74} \left(X - \frac{11}{9} \right).$$

5. The LMS estimator is the one that minimizes mean squared error (among all estimators of Y based on X). The linear LMS estimator, therefore, cannot perform better than the LMS estimator, i.e., we expect $\mathbf{E}[(Y - L(X))^2] \geq \mathbf{E}[(Y - g(X))^2]$. In fact,

$$\begin{aligned} \mathbf{E}[(Y - L(X))^2] &= \sigma_Y^2 (1 - \rho^2), \\ &= \sigma_Y^2 \left(1 - \frac{\text{cov}(X,Y)^2}{\sigma_X^2 \sigma_Y^2} \right), \\ &= \frac{37}{162} \left(1 - \left(\frac{61}{74} \right)^2 \right), \\ &= 0.073 \\ &\geq \frac{5}{72}. \end{aligned}$$

6. For a single observation x of X, the MAP estimate is not unique since all possible values of Y for this x are equally likely. Therefore, the MAP estimator does not give meaningful results.