5. P-Values Formulas

In each of the following questions, you are given an i.i.d. sample and two hypotheses. For any $\alpha \in (0,1)$, define a test with asymptotic level α, then give a formula for the asymptotic p-value of your test.

(a)

1.0/1 point (graded)

$X_1,\ldots,X_n \stackrel{i.i.d.}{\sim}$ Poiss (λ) for some unknown $\lambda > 0$;

$$H_0 : \lambda = \lambda_0 \quad \text{v.s.} \quad H_1 : \lambda \neq \lambda_0 \qquad \text{where } \lambda_0 > 0.$$

(Type **barX_n** for $\overline{X}_n$, **lambda_0** for λ_0. . If applicable, type **abs(x)** for $|x|$, **Phi(x)** for $\Phi(x) = \mathbf{P}(Z \le x)$ where $Z \sim \mathcal{N}(0,1)$, and **q(alpha)** for q_α, the $1-\alpha$ quantile of a standard normal variable, e.g. enter **q(0.01)** for $q_{0.01}$.)

Asymptotic p-value = 2*(1-Phi(sqrt(n/lambda_0)*abs(barX_n-lambda_0)))

Answer: 2*(1-Phi(sqrt(n)*abs(barX_n-lambda_0)/sqrt(lambda_0)))

STANDARD NOTATION

Solution:

Since $X_i \sim$ Poiss (λ), $\mathbb{E}[X_i] = \lambda$ and $\sigma = \sqrt{\lambda}$. Hence, under $H_0 : \lambda = \lambda_0$, the central limit theorem gives

$$T_{n,\lambda_0}\left(\overline{X}_n\right) = \sqrt{n}\left(\frac{\overline{X}_n - \lambda_0}{\sqrt{\lambda_0}}\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,1).$$

A test ψ with asymptotic level α is therefore

$$\psi_{n,\lambda_0,\alpha} = \mathbf{1}\left(\left|T_{n,\lambda_0}\left(\overline{X}_n\right)\right| > q_{\alpha/2}\right).$$

The asymptotic p-value is

$$\begin{aligned} p\text{-value} &= \mathbf{P}\left(|Z| > \left|T_{n,\lambda_0}\left(\overline{X}_n\right)\right|\right) \qquad \text{where} Z \sim \mathcal{N}(0,1) \\ &= 2\left(1 - \Phi\left(T_{n,\lambda_0}\left(\overline{X}_n\right)\right)\right). \end{aligned}$$

Alternatively, define the test ψ and the p-value usig

$$T_{n,\lambda_0}\left(\overline{X}_n\right) = \sqrt{n}\left(\frac{\overline{X}_n - \lambda_0}{\sqrt{\overline{X}_n}}\right).$$

By Slutsky and CLT, $T_{n,\lambda_0}\left(\overline{X}_n\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,1)$.

提交 你已经尝试了2次（总共可以尝试3次）

Answers are displayed within the problem

(b)

1.0/1 point (graded)

$X_1,\ldots,X_n \stackrel{i.i.d.}{\sim}$ Poiss (λ) for some unknown $\lambda>0$;

$$H_0:\lambda\geq\lambda_0 \quad \text{v.s.} \quad H_1:\lambda<\lambda_0 \qquad \text{where } \lambda_0>0.$$

(Type **barX_n** for $\overline{X}_n$, **lambda_0** for λ_0. . If applicable, type **abs(x)** for $|x|$, **Phi(x)** for $\Phi(x)=\mathbf{P}(Z\leq x)$ where $Z\sim\mathcal{N}(0,1)$, and **q(alpha)** for q_α, the $1-\alpha$ quantile of a standard normal variable.)

Asymptotic p-value = 1-Phi(sqrt(n/lambda_0)*abs(barX_n-lambda_0))

Answer: Phi(sqrt(n)*(barX_n-lambda_0)/sqrt(lambda_0))

STANDARD NOTATION

Solution:

As in the previous problem, since $X_i\sim$ Poiss (λ), $\mathbb{E}[X_i]=\lambda$ and $\sigma=\sqrt{\lambda}$. Hence, assuming $\lambda=\lambda_0$, which is at the boundary of Θ_0 and Θ_1, the central limit theorem gives again

$$T_{n,\lambda_0}\left(\overline{X}_n\right)=\sqrt{n}\left(\frac{\overline{X}_n-\lambda_0}{\sqrt{\lambda_0}}\right)\xrightarrow[n\to\infty]{(d)}\mathcal{N}(0,1)\,.$$

A candidate test ψ with asymptotic level α is therefore

$$\psi_{n,\lambda_0,\alpha} \;=\; \mathbf{1}\left(T_{n,\lambda_0}\left(\overline{X}_n\right)<-q_\alpha\right).$$

This is because as argued in lecture exercises, $\mathbf{P}_\lambda\left(T_{n,\lambda_0}\left(\overline{X}_n\right)<-q_\alpha\right)$ Recall the (asymptotic) level α is an upper bound of the type 1 error. As argued in lecture and lecture exercises, the maximum of the type 1 error is achieved at the boundary of Θ_0 and Θ_1 for a one-sided tests where the parameter space is 1-dimensional.

The asymptotic p-value is

$$\begin{aligned} p\text{–value} &= \mathbf{P}\left(Z<T_{n,\lambda_0}\left(\overline{X}_n\right)\right) right) \qquad \text{where} Z\sim\mathcal{N}(0,1)\\ &= \Phi\left(T_{n,\lambda_0}\left(\overline{X}_n\right)\right). \end{aligned}$$

Alternatively, again define the test ψ and the p-value usig

$$T_{n,\lambda_0}\left(\overline{X}_n\right)=\sqrt{n}\left(\frac{\overline{X}_n-\lambda_0}{\sqrt{\overline{X}_n}}\right).$$

By Slutsky and CLT, $T_{n,\lambda_0}\left(\overline{X}_n\right)\xrightarrow[n\to\infty]{(d)}\mathcal{N}(0,1)$.

提交 你已经尝试了1次（总共可以尝试3次）

Answers are displayed within the problem

(c)

1.0/1 point (graded)

$X_1, \ldots, X_n \overset{i.i.d.}{\sim}$ Exp (λ) for some unknown $\lambda > 0$;

$$H_0 : \lambda = \lambda_0 \quad \text{v.s.} \quad H_1 : \lambda \neq \lambda_0 \quad \text{where } \lambda_0 > 0.$$

(Type **barX_n** for $\overline{X}_n$, **lambda_0** for λ_0. If applicable, type **abs(x)** for $|x|$, **Phi(x)** for $\Phi(x) = \mathbf{P}(Z \leq x)$ where $Z \sim \mathcal{N}(0,1)$, and **q(alpha)** for q_α, the $1-\alpha$ quantile of a standard normal variable.)

Asymptotic p-value = 2*(1-Phi(sqrt(n)*abs(barX_n*lambda_0-1)))

Answer: 2*(1-Phi(sqrt(n)*abs(1/barX_n-lambda_0)/lambda_0))

STANDARD NOTATION

Solution:

Since $X_i \sim$ Exp (λ), $\mathbb{E}[X_i] = \sigma = \frac{1}{\lambda}$. Hence, assuming $H_0 : \lambda = \lambda_0$, the central limit theorem and the delta method gives:

$$T_{n,\lambda_0}\left(\overline{X}_n\right) = \sqrt{n}\left(\frac{g\left(\overline{X}_n\right) - g(1/\lambda_0)}{|g'(1/\lambda_0)|(1/\lambda_0)}\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,1) \qquad \text{where } g(x) := 1/x.$$

$$\iff \qquad \sqrt{n}\left(\frac{1/\overline{X}_n - \lambda_0}{\lambda_0}\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,1) \qquad \text{since } g'(1/\lambda) = -\lambda^2.$$

As in Part (a), a test ψ with asymptotic level α is therefore

$$\psi_{n,\lambda_0,\alpha} = \mathbf{1}\left(|T_{n,\lambda_0}\left(\overline{X}_n\right)| > -q_{\alpha/2}\right).$$

with asymptotic p-value:

$$\begin{aligned} p\text{-value} &= \mathbf{P}\left(|Z| < |T_{n,\lambda_0}\left(\overline{X}_n\right)|\right) \qquad \text{where } Z \sim \mathcal{N}(0,1) \\ &= 2\left(1 - \Phi\left(|T_{n,\lambda_0}\left(\overline{X}_n\right)|\right)\right). \end{aligned}$$

Alternatively, define the test ψ and the p-value using

$$T_{n,\lambda_0}\left(\overline{X}_n\right) = \sqrt{n}\left(\frac{1/\overline{X}_n - \lambda_0}{1/\overline{X}_n}\right).$$

where we plug-in the estimator $1/\overline{X}_n$ for λ_0.

提交 你已经尝试了1次（总共可以尝试3次）

Answers are displayed within the problem

讨论

显示讨论

主题: Unit 2 Foundation of Inference:Homework 3: Introduction to Hypothesis Testing / 5. P-Values Formulas

认证证书是什么?