6. Covariance and Independence

Review: Independence

1/3 points (graded)

Let X be a random variable that takes on values 0 and 1 with **equal probability** . You decide to communicate this random variable to your friend through a medium that "adds" (defined below in individual sub-problems) a random noise Y that takes on values 0 and 1 with probabilities α_0 and α_1, respectively. Let Y be **independent** of X and let Z denote the result of this "addition", which is what your friend receives.

Note: This $Z = X \text{ add } Y$ model for random variables can be viewed as a problem on exploring functions of random variables. However, the problem is practically motivated by many real-world examples. For example, in a communication system the X is usually what a sender sends over the "medium" and Z is what the receiver receives. The "medium", which is also called channel, could range anything from a wired line to a wireless link connecting a sender and a receiver. The noise Y added by the medium is independent of what the sender sends over the channel.

In each of the following scenarios, indicate whether the statement in quotes is true or false.

Scenario 1: You treat your random variable as a binary digit (bit) and the medium adds (XORs) noise that takes on binary values 0 and 1 with probabilities $\alpha_0 = \frac{1}{2}$ and $\alpha_1 = \frac{1}{2}$, respectively. That is $Z = X \ \text{XOR} \ Y$. "In this scenario, X and Z are independent random variables."

Note: The XOR of two bits is a function that takes in two bits and produces an output of 1 if the two input bits are different and produces an output of 0 otherwise.

- ○ True □
- ◉ False □

Scenario 2: You treat your random variable as a binary digit (bit) and the medium adds (XORs) noise that takes on binary value 0 and 1 with probabilities $\alpha_0 = \frac{1}{3}$ and $\alpha_1 = \frac{2}{3}$, respectively. That is $Z = X \ \text{XOR} \ Y$. "In this scenario, X and Z are independent random variables."

- ○ True
- ◉ False □

Scenario 3: You treat your random variable as an integer and the medium adds (integer addition) noise that takes on integer values 0 and 1 with probabilities $\alpha_0 = \frac{1}{2}$ and $\alpha_1 = \frac{1}{2}$, respectively. That is $Z = X + Y$. "In this scenario, X and Z are independent random variables."

- ◉ True □
- ○ False □

Solution:

In the first two scenarios, Z takes on binary values 0 and 1 whose probabilities can be computed as follows (recall that X and Y are independent):

$$
\begin{aligned}
p_Z(z=0) &= p_X(x=0)\cdot p_Y(y=0)+p_X(x=1)\cdot p_Y(y=1)\\
&= \frac{1}{2}\alpha_0+\frac{1}{2}\alpha_1\\
&= \frac{1}{2},\\
p_Z(z=1) &= 1-p_Z(z=0)=\frac{1}{2}
\end{aligned}
$$

Furthermore, in the first two scenarios the joint pmf can be computed as

$$
p_{X,Z}(x,z)=p_X(x)\,p_Z(z\mid x)=\begin{cases}\frac{1}{2}p_{Y|X}(y=0\mid x)=\frac{1}{2}\alpha_0 & \text{if } x=z\\ \frac{1}{2}p_{Y|X}(y=1\mid x)=\frac{1}{2}\alpha_1 & \text{if } x\neq z.\end{cases}
$$

- **Scenario 1**: For scenario 1, the above joint pmf resolves to a value of $\frac{1}{4}$ for all values of x and z. Hence, X and Z are independent in scenario 1.
- **Scenario 2**: For scenario 2, X and Z are not independent. For example, when $x=0$, $z=0$, $p_{X,Z}(x,z)=\frac{1}{2}\alpha_0=\frac{1}{6}$, which is not equal to $p_X(x=0)\,p_Z(z=0)=\frac{1}{4}$.

Scenario 3: In this case, X takes on integer values 0 and 1 and Z takes on integer values $0,1,2$. X and Z are not independent. This can be seen by computing, for example, $p_{X,Z}(x=0,z=0)$ and $p_X(x=0)\,p_Z(z=0)$:

$$
\begin{aligned}
p_{X,Z}(x=0,z=0) &= p_X(x=0)\,p_{Z|X}(z=0\mid x=0)\\
&= \frac{1}{2}p_Y(y=0)=\frac{1}{4}\\
p_X(x=0)\,p_Z(z=0) &= \frac{1}{2}\left[p_{Z|X}(z=0\mid x=0)\,\frac{1}{2}+p_{Z|X}(z=0\mid x=1)\,\frac{1}{2}\right]\\
&= \frac{1}{2}\left[\frac{1}{4}+0\right]=\frac{1}{8}
\end{aligned}
$$

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Answers are displayed within the problem

Independence, Estimation

1/1 point (graded)
Problem setup as above.

Your friend decides to perform ML estimation of what you might have transmitted based on what they received.

Concretely, your friend wishes to obtain an ML estimate $\widehat{x}$ upon observing $z=x \text{ add } y$. They obtain a likelihood function $L(z,x)$, where z is the observed realization of random variable Z and x can be treated as the unknown parameter. Assume that your friend knows that X is equally likely to take on either 0 or 1 (in all scenarios).

Under which scenario(s) would your friend's ML estimate $\widehat{X}$ be equal to X with a probability more than 0.5?

Hint: Think heuristically, assuming you obtained the correct answers for the previous problem.

- [ ] Scenario 1
- [x] Scenario 2
- [x] Scenario 3

Solution:

Scenario 1: In scenario 1, Z is independent of X. This implies that even though Z is a function of what you send over the medium, your friend would not do better with an ML estimator than guessing randomly what the input X could have been. This can be seen from the following likelihood function:

$$L(z,x) = p_{Z|X}(z \mid x) = p_Z(z).$$

Upon observing Z, the above expression is maximized to have a value of $p_Z(z)$ with either $\widehat{x} = 0$ or $\widehat{x} = 1$ both being optimal irrespective of the received z.

Scenarios 2 and 3: In scenario 2 (and 3), Z is not independent of X. This implies that the dependence should intuitively make an ML estimator perform better than randomly guessing what might have been transmitted. We show that the ML estimator $\widehat{X}$ is equal to X with a probability more than 0.5 in the following for scenario 2, and the proof for scenario 3 is obtained in a similar fashion.

The likelihood function is

$$L(z,x) = p_{Z|X}(z \mid x) = \frac{p_{X|Z}(x \mid z)\, p_Z(z)}{p_X(x)},$$

where we can ignore $p_X(x)$ as it is equal to $\frac{1}{2}$ for any x and also ignore $p_Z(z)$ because it does not depend upon parameter x. Therefore, we need to perform the following optimization:

$$\max_{x \in \{0,1\}} p_{X|Z}(x \mid z)$$

upon observing z. If $z = 0$ is observed, then $\widehat{x} = 1$ maximizes the above expression with a value of $\frac{2}{3}$. Similarly, if $z = 1$ is observed, $\widehat{x} = 0$ maximizes the above expression with a value of $\frac{2}{3}$.

Therefore, the ML estimator for scenario 2 is:

$$\widehat{x} = \begin{cases} 1 & \text{if } \; z = 0 \\ 0 & \text{if } \; z = 1. \end{cases}$$

The probability of error $P[\widehat{X} \neq X]$ for this estimator is $\frac{1}{3}$.

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Answers are displayed within the problem

Covariance and Independence

2/2 points (graded)

For each of the following statements, indicate whether it is true or false.

"X, Y are independent $\implies$ Cov$(X,Y) = 0$."

Hint: $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$ whenever X, Y are independent.)

- (•) True
- ( ) False

"Cov$(X,Y) = 0 \implies X, Y$ are independent."

Hint: If this were false, there should be an easy counterexample. Is there an easy example where $\mathbb{E}[XY] = 0$ and $\mathbb{E}[Y] = 0$ but X,Y are **not independent** ?

- ( ) True
- (•) False

Solution:

- **True.** If X, Y are independent, **Cov** $(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[X]\mathbb{E}[Y] = 0$.

- **False.** Consider X which is Bernoulli$(1/2)$. Let Y be a random variable which is always 0 if $X=0$, and uniformly distributed over $\{\pm 1\}$ if $X=1$. Notice that $\mathbb{E}[Y] = \frac{1}{2}0 + \frac{1}{4}1 + \frac{1}{4}(-1) = 0$. On the other hand, $\mathbb{E}[XY] = (0\cdot 0)\cdot \frac{1}{2} + (1\cdot 1)\,\frac{1}{4} + (1\cdot -1)\,\frac{1}{4} = 0$. However, X and Y are not independent.

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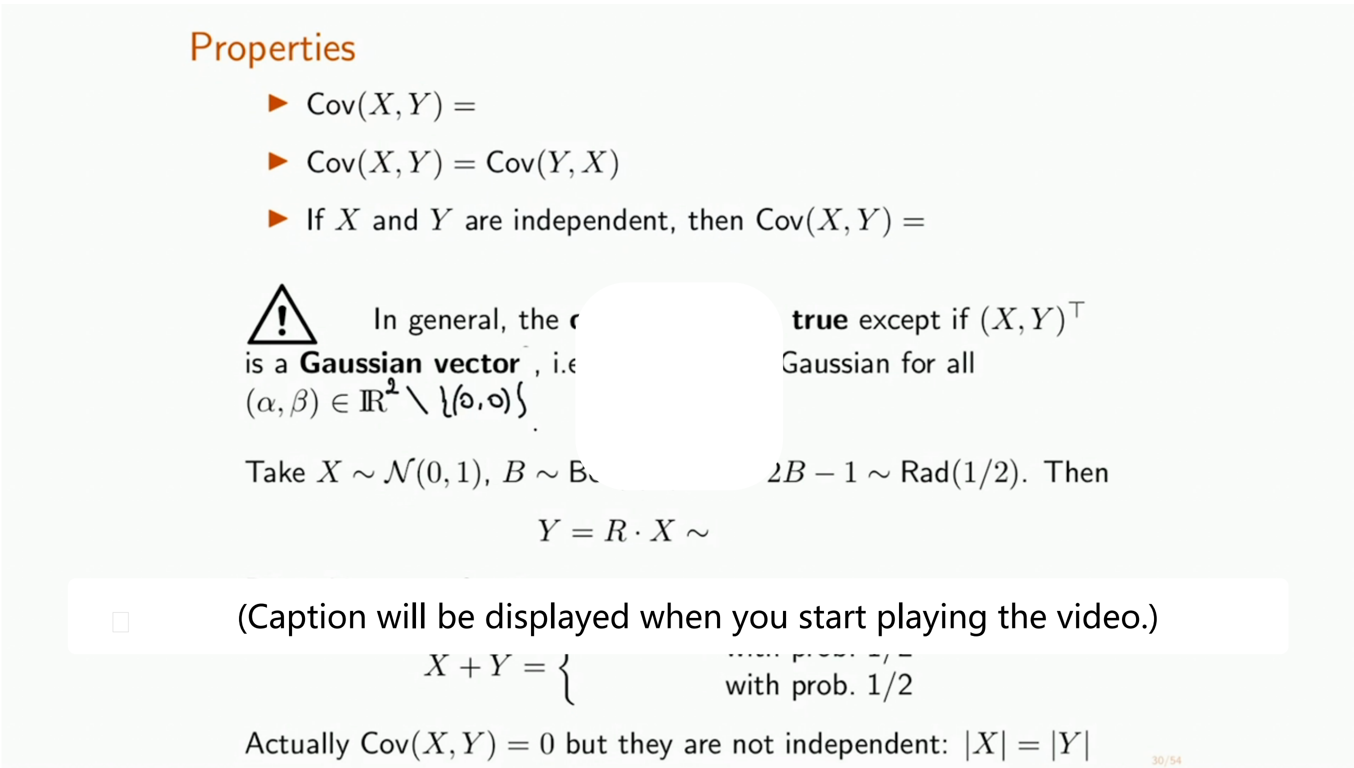
Answers are displayed within the problem

Covariance, Independence, and a Counter Example



want to say about the first one.

I'm not really sure.

Actually no, I know what I wanted to say.

It's actually a typo.

Maybe it's-- no, let me go look further.

No, it's actually what I wanted to do, OK.

So this is actually the covariance of x and x.

What is the covariance of x and x?

Variance of x.

So the covariance is just something that's essentially

extending the notion of variance to a two dimensional function.

So when I have a function of x y, and I look just at f of x x,

so I have a function that's defined on, say, the plane.

And when I look only at f of x x,

I'm really looking only at the values

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