(a) Showing that two random variables are equal entails showing that no matter what the outcome of the experiment, the two random variables take the same values. Therefore, we need to show that for any $y^*$ for which $p_Y(y^*) > 0$, the resulting values $\mathbf{E}\big[Xg(Y) \,|\, Y = y^*\big]$ and $g(y^*)\mathbf{E}[X \,|\, Y = y^*]$ of the two random variables of interest are the same.

Recall the expected value rule:

$$\mathbf{E}\big[Xg(Y)\big] = \sum_x \sum_y xg(y)p_{X,Y}(x,y).$$

We apply this to a conditional model in which the event $Y = y^*$ is known to have occurred:

$$\mathbf{E}\big[Xg(Y) \,|\, Y = y^*\big] = \sum_x \sum_y xg(y)p_{X,Y|Y}(x,y \,|\, y^*).$$

Now, note that for $y \neq y^*$, $\mathbf{P}(X = x, Y = y \,|\, Y = y^*) = 0$; in PMF notation,

$$p_{X,Y|Y}(x,y \,|\, y^*) = 0, \qquad \text{if } y \neq y^*.$$

On the other hand, for $y = y^*$, we have

$$\mathbf{P}(X = x, Y = y^* \,|\, Y = y^*) = \mathbf{P}(X = x \,|\, Y = y^*);$$

in PMF notation,

$$p_{X,Y|Y}(x,y^* \,|\, y^*) = p_{X|Y}(x \,|\, y^*).$$

Using these facts, the double summation in the expected value rule, becomes a single summation, and

$$\begin{aligned}
\mathbf{E}\big[Xg(Y) \,|\, Y = y^*\big] &= \sum_x xg(y^*)p_{X|Y}(x \,|\, y^*) \\
&= g(y^*) \sum_x xp_{X|Y}(x \,|\, y^*) \\
&= g(y^*)\mathbf{E}[X \,|\, Y = y^*].
\end{aligned}$$

We have therefore verified the desired equality.

*Note:* These results are true for all kinds of random variables (continuous, mixed, etc.). A general proof would require an appropriate variant of the expected value rule, but would be awkward because one would have to consider all possible combinations of cases. In a more advanced treatment, these issues are bypassed by working with an even more abstract definition of the concept of the conditional expectation. However, this is beyond the scope of a first class.

(b) Fix some $y$ with $p_Y(y) > 0$ and let $z = h(y)$. Since $h$ is an invertible function, the events $\{Y = y\}$ and $\{h(Y) = z\}$ are identical, and

$$\mathbf{P}(X = x \,|\, Y = y) = \mathbf{P}(X = x \,|\, h(Y) = z).$$

Multiplying both sides by $x$, and then summing over all $x$, we obtain

$$\mathbf{E}[X \mid Y = y] = \mathbf{E}[X \mid h(Y) = z].$$

Now, if the event $Y = y$ is known to have occurred, then $\mathbf{E}[X \mid Y]$ takes the value $\mathbf{E}[X \mid Y = y]$. In that case, the random variable $h(Y)$ takes the value $h(y) = z$, so that the random variable $\mathbf{E}[X \mid h(Y)]$ takes the value $\mathbf{E}[X \mid h(Y) = z]$. We have already shown that $\mathbf{E}[X \mid Y = y]$ is the same as $\mathbf{E}[X \mid h(Y) = z]$. Hence, the random variables $\mathbf{E}[X \mid Y]$ and $\mathbf{E}[X \mid h(Y)]$ always take the same values, and are therefore equal.