14. Example: a Non-Asymptotic Test for the Support of a Uniform Variable

Testing the Support of a Uniform Variable: Designing a Test

4/4 points (graded)

The next few problems cover a test that is not motivated by the clt.

Let $X_1, \ldots, X_n \stackrel{iid}{\sim} \text{Unif}[0, \theta]$ where θ is an unknown parameter. Let $\left(\mathbb{R}_{\geq 0}, \{\text{Unif}[0, \theta]\}_{\theta > 0}\right)$ denote the associated statistical model. (Here, $\mathbb{R}_{\geq 0}$ denotes the nonnegative real numbers.)

You want to answer the **question of interest**: **"Is $\theta \leq 1/2$?"**. To do so you formulate a hypothesis test with

$$\begin{aligned} H_0 &: \theta \leq 1/2 \qquad (\text{null hypothesis}) \\ H_1 &: \theta > 1/2 \qquad (\text{alternative hypothesis}). \end{aligned}$$

You also design the test

$$\psi_n = \mathbf{1}\left(\max_{1 \leq i \leq n} X_i > 1/2\right).$$

(If $\psi_n = 1$, then we will **reject** the null hypothesis. Note the dependence of ψ_n on the sample size.)

We use Θ_0 to denote the region of Θ defined by the null hypothesis. In this example, Θ_0 can be written as an interval $(A, B]$. What are A and B?

$A =$ 0 ✔ **Answer:** 0.0

$B =$ 1/2 ✔ **Answer:** 1/2

Similarly, we let Θ_1 denote the region of Θ defined by the alternative hypothesis. In this example, Θ_1 can be written as an interval (C, ∞). What is C?

$C =$ 1/2 ✔ **Answer:** 1/2

Suppose you observe the sample

$$0.1, 0.53, 0.002, 0.1234, 0.24, 0.48.$$

Should you **reject** or **fail to reject** the null hypothesis given this data?

- (selected) Reject ✔
- Fail to reject

Solution:

The parameter space is $\Theta = \{\theta : \theta > 0\}$. Since the null hypothesis is $H_0 : \theta < 1/2$, then $\Theta_0 = (0, 1/2)$. Similarly, $\Theta_1 = [1/2, \infty)$.

On observing the sample

$$0.1, 0.53, 0.002, 0.1234, 0.24, 0.48,$$

the null hypothesis $H_0 : \theta \leq 1/2$ should be rejected. Recall the test is $\psi_n = \mathbf{1}(\max_{1\leq i\leq n} X_i > 1/2)$ which evaluates to 1 on the given sample. (Here $n = 6$.)

提交 你已经尝试了1次（总共可以尝试3次）

Answers are displayed within the problem

Testing the Support of a Uniform Variable: Complement of the Rejection Region of a Test

3/3 points (graded)

As above, let $X_1, \ldots, X_n \stackrel{iid}{\sim} \mathbf{Unif}[0, \theta]$ for an unknown parameter θ and we designed the statistical test

$$\psi_n = \mathbf{1}(\max_{1\leq i\leq n} X_i > 1/2)$$

to decide between the null and alternative hypotheses

$$\begin{aligned} H_0 : \theta &\leq 1/2 \\ H_1 : \theta &> 1/2. \end{aligned}$$

(Going forward we will simply write the null and alternative hypotheses and omit the motivating yes/no question.)

Recall from lecture that the **rejection region** for a test ψ_n is

$$R_{\psi_n} := \{(x_1, \ldots, x_n) \in E^n : \psi_n(x_1, \ldots, x_n) = 1\}.$$

where E is the sample space of the i.i.d. variables X_i, which is $\mathbb{R}$ in this example since X_i are uniform random variables.

Consider the complement C_n of the rejection region: this is all the points in $(\mathbb{R}_{\geq 0})^n$ that do not lie in R_{ψ_n}. Note that the dimension of C_n is determined by the value of n.

What is the length of C_1?

1/2 ✔ **Answer:** 1/2

What is the area of C_2?

1/4 ✔ **Answer:** 1/4

What is the volume of C_3?

1/8 ✔ **Answer:** 1/8

Solution:

The complement C_n of the rejection region is the set of all $(x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0}$ such that $\max_{1\leq i\leq n} x_i \leq 1/2$. (Equivalently, it is the set of all $(x_1, \ldots, x_n)$ such that $\psi_n = \mathbf{1}(\max_{1\leq i\leq n} x_i > 1/2) = 0$). The region defined by the constraint $x_i \leq 1/2$ for all $1 \leq i \leq n$ is the set $[0, 1/2]^n$.

In one dimension, this is the interval $[0, 1/2]$ which has length $1/2$. In two dimensions, this is the square $[0, 1/2] \times [0, 1/2]$, which has area $(1/2)^2 = 1/4$. Finally in three dimensions, C_3 is a cube $[0, 1/2] \times [0, 1/2] \times [0, 1/2]$, which has volume $(1/2)^3 = 1/8$.

提交 你已经尝试了1次（总共可以尝试3次）

Answers are displayed within the problem

Testing the Support of a Uniform Variable: : Type 1 Error of a Test

1/1 point (graded)

As above, $X_1, \ldots, X_n \stackrel{iid}{\sim} \text{Unif}\,[0,\theta]$ for an unknown parameter θ and we designed the statistical test

$$\psi_n = \mathbf{1}(\max_{1\le i\le n} X_i > 1/2)$$

to decide between the null and alternative hypotheses

$$\begin{aligned} H_0 : \theta &\le 1/2 \\ H_1 : \theta &> 1/2. \end{aligned}$$

The region defined by the null hypothesis is $\Theta_0 = (0, 1/2]$. Therefore, the **type 1 error (or error rate)** of the test ψ_n is the **function**

$$\begin{aligned} \alpha_{\psi_n} : (0,1/2] &\to \mathbb{R} \\ \theta &\mapsto P_\theta\left(\psi_n = 1\right) \end{aligned}$$

where $P_\theta = \text{Unif}\,[0,\theta]$, and $P_\theta\left(\psi_n = 1\right)$ is the probability of the event $\psi_n = 1$ under the probability distribution P_θ when $\theta \in \Theta_0$, i.e. the probability of rejecting H_0 when H_0 is true.

What is $\alpha_{\psi_n}(\theta)$?

$\alpha_{\psi_n}(\theta) =$ 0 ✔ Answer: 0

0

Solution:

By definition,

$$\alpha_{\psi_n}(\theta) = P_\theta\left(\max_{1\le i\le n} X_i > 1/2\right)$$

where $P_\theta = \text{Unif}\,[0,\theta]$ and we restrict $\theta \in \Theta_0 = \{\theta : \theta < 1/2\}$. Observe that if $\theta < 1/2$, then there is a 0% chance of generating an observation which is larger than $1/2$. Hence, the type 1 error $\alpha_{\psi_n}(\theta)$ is 0 for all $\theta \in \Theta_0$.

Remark: In general, the type 1 error will be a function of θ, but in this special case it is constant.

提交 你已经尝试了1次（总共可以尝试3次）

Answers are displayed within the problem

讨论

显示讨论

主题: Unit 2 Foundation of Inference:Lecture 6: Introduction to Hypothesis Testing, and Type 1 and Type 2 Errors / 14. Example: a Non-Asymptotic Test for the Support of a Uniform Variable

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