3. Asymptotic Variance

a)

2/2 points (graded)

Note: *This question is the ungraded problem from homework 2.*

Let $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2)$, for some $\sigma^2 > 0$. Let

$$\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n} X_i^2, \quad \text{and} \quad \widetilde{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2.$$

Argue that both proposed estimators $\widehat{\sigma^2}$ and $\widetilde{\sigma^2}$ below are consistent and asymptotically normal.

Then, give their asymptotic variances $V(\widehat{\sigma^2})$ and $V(\widetilde{\sigma^2})$ and decide if one of them is always bigger than the other.

Hint: Use the multivariate Delta method. Also see Recitation 5 *Inference for the Variance of a Gaussian distribution*.

$V(\widehat{\sigma^2}) =$ 2*sigma^4 **Answer:** 2*(sigma^2)^2

$2 \cdot \sigma^4$

$V(\widetilde{\sigma^2}) =$ 2*sigma^4 **Answer:** 2*(sigma^2)^2

$2 \cdot \sigma^4$

STANDARD NOTATION

Solution:

Note that

$$\widehat{\sigma^2} = \overline{Y}_n, \quad \text{for } Y_i = X_i^2.$$

By the Law of Large Numbers,

$$\overline{Y}_n \xrightarrow[n\to\infty]{\mathbf{P}} \mathbb{E}[Y_1] = \sigma^2.$$

By the Central Limit Theorem,

$E[X^4] - E^2[X^2]$

$$\sqrt{n}(\overline{Y}_n - \sigma^2) \sim \mathcal{N}(0, \mathsf{Var}(Y_1)) = \mathcal{N}(0, 2(\sigma^2)^2),$$

不需要delta method，直接用方差的CLT

hence

$$V(\widehat{\sigma^2}) = 2(\sigma^2)^2.$$

For $\widetilde{\sigma^2}$, first observe that we can write it as

$$\begin{aligned}\widetilde{\sigma^2} &= \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}_n\right)^2 \\ &= \frac{1}{n}\sum_{i=1}^{n}\left(X_i^2 - 2\overline{X}_n X_i + \overline{X}_n^2\right) \\ &= \underbrace{\frac{1}{n}\left(\sum_{i=1}^{n} X_i^2\right) - \overline{X}_n^2}_{\overline{x^2} - \overline{x}^2} = \widehat{\sigma^2} - \overline{X}_n^2.\end{aligned}$$

Again, by the Law of Large Numbers,

这里才有两个变量

$$\overline{X}_n^2 \xrightarrow[n\to\infty]{\mathbf{P}} \mathbb{E}[X_1]^2 = 0,$$

so

$$\widetilde{\sigma^2} = \widehat{\sigma^2} - \overline{X}_n^2 \xrightarrow[n\to\infty]{\mathbf{P}} \sigma^2.$$

Now, we can consider $\widetilde{\sigma^2}$ as

$$\widetilde{\sigma^2} = g\left(\frac{1}{n}\sum_{i=1}^{n}\begin{pmatrix} X_i \\ X_i^2 \end{pmatrix}\right),$$

where

$$g(x, y) = y - x^2.$$

By the above, we have a multidimensional Central Limit Theorem for the first and second moments of a Gaussian together,

$$\sqrt{n}\left[\begin{pmatrix} \overline{X}_n \\ \overline{Y}_n \end{pmatrix} - \begin{pmatrix} 0 \\ V \end{pmatrix}\right] \xrightarrow[n\to\infty]{(\mathrm{D})} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 \\ 0 & 2(\sigma^2)^2 \end{pmatrix}\right),$$

where the 0s on the diagonal come from the fact that

$$\mathbb{E}\left[X_i \times X_i^2\right] = \mathbb{E}\left[X_i^3\right] = 0.$$

Now, apply the multidimensional Delta Method, computing

jacobian $$Dg(x, y) = \begin{pmatrix} -2x & 1 \end{pmatrix},$$

to obtain

$$\begin{aligned}\sqrt{n}\left(\widetilde{\sigma^2} - \sigma^2\right) &\xrightarrow[n\to\infty]{(\mathrm{D})} \mathcal{N}\left(0, Dg(0, \sigma^2)\begin{pmatrix} \sigma^2 & 0 \\ 0 & 2(\sigma^2)^2 \end{pmatrix} Dg(0, \sigma^2)^\top\right) \\ &= \mathcal{N}\left(0, \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} \sigma^2 & 0 \\ 0 & 2(\sigma^2)^2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \mathcal{N}\left(0, 2(\sigma^2)^2\right).\end{aligned}$$

Combined, we see that both estimators have the same asymptotic variance.

提交 你已经尝试了1次（总共可以尝试3次）

Answers are displayed within the problem

讨论

显示讨论

主题： Unit 3 Methods of Estimation:Homework 5: Maximum Likelihood Estimation / 3. Asymptotic Variance