5. Hats in a box

Problem 5. Hats in a box

4/5 points (graded)

Each one of n persons, indexed by $1, 2, \ldots, n$, has a clean hat and throws it into a box. The persons then pick hats from the box, at random. Every assignment of the hats to the persons is equally likely. In an equivalent model, each person picks a hat, one at a time, in the order of their index, with each one of the remaining hats being equally likely to be picked. Find the probability of the following events.

(You need to answer all 5 questions before you can submit.)

1. Every person gets his or her own hat back.

- (selected) $\frac{1}{n!}$ ✔
- $\frac{1}{(n+1)!}$
- $\frac{1}{n}$
- $\frac{1}{n+1}$

2. Each one of persons $1, \ldots, m$ gets his or her own hat back, where $1 \leq m \leq n$.

- $\frac{(n+m)!}{n!}$

- [x] $\dfrac{(n-m)!}{n!}$ ✓

- [ ] $\dfrac{n!}{(n+m)!}$

- [ ] $\dfrac{m!}{n!}$

3. Each one of persons $1, \ldots, m$ gets back a hat belonging to one of the last m persons (persons $n-m+1, \ldots, n$), where $1 \leq m \leq n$.

- [x] $\dfrac{1}{\binom{n}{m}}$ ✓

- [ ] $\dfrac{m}{\binom{n}{m}}$

- [ ] $\dfrac{n-m}{\binom{n}{m}}$

- [ ] $\dfrac{n}{\binom{n}{m}}$

Now assume, in addition, that every hat thrown into the box has probability p of getting dirty (independently of what happens to the other hats or who has dropped or picked it up). Find the probability that:

4. Persons $1, \ldots, m$ will pick up clean hats.

- $(1-p)^{n-m}$
- $m(1-p)^m$ ✖
- $(1-p)^m$ ✔
- $m(1-p)^{n-m}$

5. Exactly m persons will pick up clean hats.

- $\frac{\binom{n}{m}}{n!}(1-p)^m p^{n-m}$
- $(1-p)^m p^{n-m}$
- $\binom{n}{m}(1-p)^{n-m}p^m$
- $\binom{n}{m}(1-p)^m p^{n-m}$ ✔

Solution:

1. Consider the sample space of all possible hat assignments. It has $n!$ elements (n hat selections for the first person, after that $n-1$ for the second, etc.), with every assignment equally likely; hence each assignment has probability $1/n!$. The event that everyone gets his or her own hat back corresponds to exactly one of these $n!$ assignments. Therefore, the answer is $1/n!$.

2. Consider the same sample space and probabilities as in the solution of part 1. The event of interest assigns the first m people to their own hats and allows for an arbitrary assignment of hats to the remaining $n-m$ persons, so that there are $(n-m)!$ possible assignments. The

probability of an event with $(n-m)!$ elements is $(n-m)!/n!$.

3. Consider the m hats belonging to the last m persons. There are $m!$ ways to distribute these m hats among the first m persons. Then, there are $(n-m)!$ ways to distribute the remaining $n-m$ hats to everyone else. The probability of an event with $m!(n-m)!$ elements is $m!(n-m)!/n!$, which is equal to $1/\binom{n}{m}$.

4. The probability of a given person picking up a clean hat is $1-p$. By the independence assumption, the probability of m specific persons picking up clean hats is $(1-p)^m$.

5. Think of picking a clean hat as an independent Bernoulli trial with success probability $1-p$. The probability of m successes out of n trials is $\binom{n}{m}(1-p)^m p^{n-m}$.

提交 You have used 2 of 2 attempts

Answers are displayed within the problem

讨论

显示讨论

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