2. Oscar's running shoes

Problem 2. Oscar's running shoes

6/6 points (ungraded)

Oscar goes for a run each morning. When he leaves his house for his run, he is equally likely to use either the front or the back door; and similarly, when he returns, he is equally likely to use either the front or the back door. Assume that his choice of the door through which he leaves is independent of his choice of the door through which he returns, and also assume that these choices are independent across days.

Oscar owns only five pairs of running shoes, each pair placed at one of the two doors. If there is at least one pair of shoes at the door through which he leaves, he wears a pair for his run; otherwise, he runs barefoot. When he returns from his run, if he wore shoes for that run, he takes off the shoes after the run and leaves them at the door through which he returns.

We wish to determine the long-term proportion of time that Oscar runs barefoot.

1. We consider a Markov chain with states $\{0,1,2,3,4,5\}$, where state i indicates that there are i pairs of shoes available at the front door in the morning, before Oscar leaves for his run. Specify the numerical values of the following transition probabilities.

   - For $i \in \{0,1,2,3,4\}$,

     $p_{i,i+1} =$ 1/4 ✔ **Answer:** 0.25

   - For $i \in \{1,2,3,4,5\}$,

     $p_{i,i-1} =$ 1/4 ✔ **Answer:** 0.25

   - For $i \in \{1,2,3,4\}$,

     $p_{ii} =$ 1/2 ✔ **Answer:** 0.5

   - $p_{00} =$ 3/4 ✔ **Answer:** 0.75

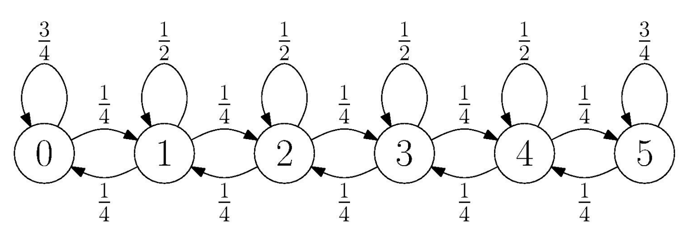
   - $p_{55} =$ 3/4 ✔ **Answer:** 0.75

2. Determine the steady-state probability that Oscar runs barefoot.

   1/6 ✔ **Answer:** 0.16667

Solution:

1. The transition probabilities are shown in the diagram below and can be justified as follows.



- If $i < 5$ pairs of shoes are initially at the front door, the new state will be $i+1$ if and only if he exits from the back door and enters from the front door. This has probability $1/4 = 0.25$. Thus, $p_{i,i+1} = 0.25$, for $i < 5$.
- By an entirely symmetrical argument, $p_{i,i-1} = 0.25$, for $i > 0$.
- $p_{ii} = 1 - p_{i,i-1} - p_{i,i+1} = 0.5$, for $i \in \{1,2,3,4\}$.
- $p_{00} = 1 - p_{01} = 0.75$, where we used the result from bullet 1.
- $p_{55} = 1 - p_{54} = 0.75$, where we used the result from bullet 2.

2. When there are either 0 or 5 pairs of shoes at the front door, there is probability $\frac{1}{2}$ that Oscar will leave on his run from the door with 0 shoes and hence run barefoot. To find the long-term probability of Oscar running barefoot, we first find the steady-state probabilities of being in states 0 and 5, π_0 and π_5, respectively.

   Since this is a birth-death process, we can use the local balance equations. We have $\pi_0 p_{01} = \pi_1 p_{10}$, implying that $\pi_1 = \pi_0$ and similarly, $\pi_5 = \cdots = \pi_1 = \pi_0$. As $\sum_{i=0}^{5} \pi_i = 1$, it follows that $\pi_i = \frac{1}{6}$ for $i = 0, 1, \ldots, 5$. Hence,

$$\mathbf{P}(\text{Oscar runs barefoot in the long-term}) = \frac{1}{2}(\pi_0 + \pi_5) = \frac{1}{6}.$$

提交 你已经尝试了3次（总共可以尝试4次）

Answers are displayed within the problem

讨论

显示讨论

主题: Unit 10 / Problem Set / 2. Oscar's running shoes