Probability–The Science of Uncertainty and Data

edX Course: 6.431x

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1. Exam Rules

Exam Rules

1. You have opened a timed exam with a **48 hours** time limit. Please use the timer to see the time remaining. If you had opened this exam too close to the exam **closing time, November 19 23:59UTC** , you will not have the full 48 hours, and the exam will close at the closing time.

2. This is an **open book exam** and you are allowed to refer back to all course material and use (online) calculators. However, you must abide by the honor code, and not ask for answers directly from any aide.

3. You will be given **no feedback** during the exam. This means that unlike in the problem sets, you will not be shown whether any of your answers are correct or not. This is to test your understanding, to prevent cheating, and to encourage you to try your very best before submitting. Solutions will be available after the exam closes.

4. You will be given **2 attempts** for each, multipart problem, except for True or False problems. Since you will be given no feedback, the extra attempt will be useful only in case you hit the "submit" button in a haste and wish to reconsider. With no exception, **your last submission will be the one that counts** . The exam will only be graded after the due date, and the Progress Page will show fake scores while the exam is open.

5. **Error and bug reports:** While the exam is open, you are **not allowed to post on the discussion forum on anything related to the exam, except to report bugs/platform difficulties** . If you think you have found a bug, please state on the forum only what needs to be checked on the forum. You can still go through Unit 7b and 8, and post on its contents, but the post must not shed any light on the contents or concepts in the exam. **Violators will receive a 0 score in Exam 2** .

6. **Clarification:** If you need clarification on a problem, please first **check the discussion forum** , where staff may have posted notes. After that, if you still need clarification that will **strictly not lead to hints of the solution** , you can email staff at 6431exam@mit.edu. If we see that the issue is indeed not addressed already on the forum, we will respond within 28 hours and post a note on the forum; otherwise–if the issue has been addressed on the forum, we will **not** respond and assume your responsibility to check the forum for answers.

2. Independent normal random variables

Problem 1. Independent normal random variables

Let U, V, and W be independent standard normal random variables (that is, independent normal random variables, each with mean 0 and variance 1), and let $X = 5U + 12V$ and $Y = U - W$.

Showhide: (ID: Standard Normal Table):./static/resources/normaltable2.xml

1. What is the probability that $X \geq 2.6$? (Give 3 decimal digits.) $\mathbb{P}(X \geq 2.6) =$
2. $\mathbf{E}[XY] =$
3. $\mathsf{Var}(X + Y) =$
4. Let H be a normal random variable with mean zero, and variance equal to 2. Let $a = \mathbf{E}[|H|]$. Find a.
   $a = \mathbf{E}[|H|] =$
5. In this final part of the problem, we will find $\mathbf{E}[\max\{U, V\}]$, using the following argument. First, note that,
   $$\max\{U,V\} - \min\{U,V\} = |U - V|,$$
   $$\max\{U,V\} + \min\{U,V\} = U + V.$$
   Using your answer to previous part, and using the formulas above, obtain the answer, symbolically, as a function of the constant a defined in previous part.
   $\mathbf{E}[\max\{U, V\}] =$

Solution:

1. Since X is a sum of independent normal random variables, X is also normal. Its mean and variance are, $\mathbf{E}[X] = \mathbf{E}[5U + 12V] = 5\mathbf{E}[U] + 12\mathbf{E}[V] = 0$, and $\mathsf{Var}(X) = \mathsf{Var}(5U + 12V) = 25 \cdot \mathsf{Var}(U) + 144 \cdot \mathsf{Var}(V) = 169$. Hence, letting N be a standard normal,
   $$\begin{aligned} \mathbb{P}(X \geq 2.6) &= \mathbb{P}\left(\frac{X-0}{13} \geq \frac{2.6-0}{13}\right) \\ &= \mathbb{P}(N \geq \frac{2.6}{13}) \\ &= 1 - \Phi(0.2) \\ &\approx 1 - 0.579 \\ &= 0.421. \end{aligned}$$

2. Since U, V, and W are zero-mean and independent, we have,

$$\begin{aligned}
\mathbf{E}[XY] &= \mathbf{E}[(5U + 12V)(U - W)] \\
&= \mathbf{E}[5U^2 - 5UW + 12UV - 12VW] \\
&= 5\mathbf{E}[U^2] - 5\mathbf{E}[U]\mathbf{E}[W] + 12\mathbf{E}[U]\mathbf{E}[V] - 12\mathbf{E}[V]\mathbf{E}[W] \\
&= 5.
\end{aligned}$$

3. For this part, note that $X + Y = 6U + 12V - W$. Since U, V, and W are independent, we have,

$$\begin{aligned}
\mathsf{Var}(X + Y) &= \mathsf{Var}(6U + 12V - W) \\
&= \mathsf{Var}(6U) + \mathsf{Var}(12V) + \mathsf{Var}(-W) \\
&= 36 \cdot \mathsf{Var}(U) + 144 \cdot \mathsf{Var}(V) + \mathsf{Var}(W) \\
&= 181.
\end{aligned}$$

4. For this part, we will integrate $|H|$ with respect to the density of H.

$$\begin{aligned}
\mathbf{E} &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi}} e^{-h^2/4} |h| \, dh \\
&= 2 \int_0^{\infty} \frac{1}{\sqrt{4\pi}} h e^{-h^2/4} \, dh \\
&= \frac{1}{\sqrt{\pi}} \int_0^{\infty} h e^{-h^2/4} \, dh.
\end{aligned}$$

Using a change of variables, $y = h^2/2$, we have, $dy = h \, dh$, and the integral becomes,

$$\frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-y/2} \, dy = \frac{1}{\sqrt{\pi}} 2e^{-y/2}|_0^{\infty} = \frac{2}{\sqrt{\pi}}(1 - 0) = \frac{2}{\sqrt{\pi}}.$$

5. Finally, for this part, notice that $U - V$ is a normal distribution with mean 0, and variance 2. Hence, from the previous part, $\mathbf{E}[|U - V|] = 2/\sqrt{\pi} = a$. Then, using the given formulas, we have,

$$2\mathbf{E}[\max\{U, V\}] = \mathbf{E}[|U - V|] + \mathbf{E}[U + V] = a.$$

Therefore,

$$\mathbf{E}[\max\{U, V\}] = a/2.$$

3. Independent exponential random variables

Problem 2. Independent exponential random variables

Let X and Y be two independent, exponentially distributed random variables with parameters λ, and μ, respectively.

For each question below, enter your answers using standard notation; enter **mu** for mu and **lambda** for λ.

1. Find the probability that $X \leq Y$.
   $\mathbb{P}(X \leq Y) =$
2. Let $Z = 1/(1+X)$. For $0 < z < 1$: $f_Z(z) =$

Solution:

1. Using the law of total probability theorem, and independence of X and Y,

$$\begin{aligned}\mathbb{P}(X \leq Y) &= \int_0^\infty \mathbb{P}(X \leq y) f_Y(y)\, dy \\ &= \int_0^\infty \mathbb{P}(X \leq y)\mu e^{-\mu y}\, dy \qquad = \int_0^\infty (1 - e^{-\lambda y})\mu e^{-\mu y}\, dy \\ &= \frac{\lambda}{\mu + \lambda}.\end{aligned}$$

2. We have, for $0 < z < 1$,

$$\begin{aligned}\mathbb{P}(Z \leq z) &= \mathbb{P}\left(\frac{1}{1+X} \leq z\right) \\ &= \mathbb{P}\left(1 + X \geq \frac{1}{z}\right) \\ &= \mathbb{P}\left(X \geq \frac{1}{z} - 1\right) \\ &= e^{-\lambda(1/z-1)} \\ &= e^{-\lambda/z} \cdot e^{\lambda}.\end{aligned}$$

   Differentiating the expression above with respect to z yields,

$$f_Z(z) = \frac{\lambda}{z^2} e^{-\lambda(1/z-1)} = \frac{\lambda e^{\lambda}}{z^2} e^{-\lambda/z}.$$

4. Breaking a stick twice

Problem 3. Breaking a stick twice

Let X be uniformly distributed on $[0,1]$. Given the value x of X, we let Y be uniformly distributed on $[0,x]$.

In lecture, we have seen that the PDF $f_Y(y)$ of Y is,

$$\begin{aligned} f_Y(y) &= \int_0^1 f_{Y|X}(y|x) f_X(x)\, dx \\ &= \int_y^1 \frac{1}{x}\, dx \\ &= -\ln(y), \end{aligned}$$

for $0 < y < 1$.

1. Find the conditional PDF of X, given that $Y = y$. For $0 < y < x < 1$:
   $f_{X|Y}(x \mid y) =$
2. The conditional expectation of X given Y, namely $\mathbf{E}[X|Y]$ is of the form $h(Y)$ for some function $h(\cdot)$. Find $h(\cdot)$. For $0 < y < 1$:
   $h(y) =$

Solution:

1. Note that conditioned on $X = x$, the PDF of Y is constant, and equal to $1/x$, for $0 < y < x$. Using the Bayes' rule, and for $0 < y < x < 1$,

$$\begin{aligned} f_{X|Y}(x \mid y) &= \frac{f_{Y|X}(y \mid x) f_X(x)}{f_Y(y)} \\ &= \frac{\frac{1}{x} \cdot 1}{-\ln(y)} \\ &= -\frac{1}{x\ln(y)}. \end{aligned}$$

2.

$$\begin{aligned}
\mathbf{E}[X|Y=y] &= \int_0^1 x f_{X|Y}(x \mid y)\, dx \\
&= \int_y^1 -x \frac{1}{x \ln(y)}\, dx \\
&= \int_y^1 -\frac{1}{\ln(y)}\, dx \\
&= \frac{y-1}{\ln(y)}.
\end{aligned}$$

5. Mixed Bayes rule - discrete unknown and continuous measurements

Problem 4. Mixed Bayes rule - discrete unknown and continuous measurements

Let S be a discrete random variable that takes the value 1 with probability $p \in (0, 1)$, and the value -1 with probability $1 - p$. Let X be a continuous random variable whose conditional distribution given S is as follows:

(i) If $S = 1$, then X is exponential with parameter $\alpha > 0$, i.e., $f_{X|S}(x \mid 1) = \alpha e^{-\alpha x}$, for $x \geq 0$.

(ii) If $S = -1$, then $-X$ is exponential with parameter $\beta > 0$, i.e., $f_{X|S}(x \mid -1) = \beta e^{\beta x}$, for $x \leq 0$.

Note that S can be viewed as the "sign" of X. Let $Z = |X|$.

1. Give an expression for $f_X(x)$. (Enter your answer using standard notation; type **alpha** for α, **beta** for β.)

   For $x > 0$:
   $f_X(x) =$
   For $x < 0$:
   $f_X(x) =$

2. Give an expression for $\mathbb{P}(S = 1 \mid Z = z)$, as a function of z. (Enter your answer using standard notation; type **alpha** for α, **beta** for β.)

   $\mathbb{P}(S = 1|Z = z) =$

Solution:

1. The answer is
$$f_X(x) = f_{X|S}(x \mid 1)p_S(1) + f_{X|S}(x \mid -1)p_S(-1),$$
and the reasoning is as follows. We are dealing with a mixture of two distributions. Hence, when $x > 0$ only the first is nonzero and we obtain $p\alpha e^{-\alpha x}$. When $x < 0$, only the second term is nonzero and we obtain $(1 - p)\beta e^{\beta x}$.

2. $Z = |X|$ is always non-negative, and $Z = X$, when $X \geq 0$, and $Z = -X$, when $X \leq 0$. Thus,

$$f_{Z|S}(z \mid s) = \begin{cases} f_{X|S}(z|1) = \alpha e^{-\alpha z}, & \text{if } s = 1 \\ f_{X|S}(-z|s) = \beta e^{-\beta z}, & \text{if } s = -1 \end{cases}$$

Now,

$$\begin{aligned} \mathbb{P}(S = 1 \mid Z = z) &= \frac{f_{Z|S}(z|1)\mathbb{P}(S = 1)}{f_{Z|S}(z|1)\mathbb{P}(S = 1) + f_{Z|S}(z|-1)\mathbb{P}(S = -1)} \\ &= \frac{p\alpha e^{-\alpha z}}{p\alpha e^{-\alpha z} + (1-p)\beta e^{-\beta z}}. \end{aligned}$$

6. Image Corrupted with noise

Problem 5. Image Corrupted with noise

Consider an image, in which every pixel takes a value of 1, with probability q, and a value 0, with probability $1-q$, where q is the realized value of a random variable Q which is distributed uniformly over the interval $[0,1]$. The realized value q is the same for every pixel.

Let X_i be the value of pixel i. We observe, for each pixel the value of $Y_i = X_i + N$, where N is normal with mean 2 and unit variance. (Note that we have the same noise at each pixel.) Assume that, conditional on Q, the X_i's are independent, and that the noise N is independent of Q and the X_i's.

1. Find $\mathbf{E}[Y_i]$. (Give a numerical answer.)

   $\mathbf{E}[Y_i] =$

2. Find $\mathsf{Var}[Y_i]$. (Give a numerical answer.)

   $\mathsf{Var}[Y_i] =$

3. Let A be the event that the actual values X_1 and X_2 of pixels 1 and 2, respectively, are zero. Find the conditional probability of Q given A.
   (Enter your answer in terms of q in standard notation.)

   For $0 \le q \le 1$:

   $f_{Q|A}(q) =$

Solution:

1. Notice that
   $$\mathbf{E}[Y_i] = \mathbf{E}[X_i + N] = \mathbf{E}[X_i] + \mathbf{E}[N] = \mathbf{E}[X_i] + 2.$$
   In order to find $\mathbf{E}[X_i]$, we will use the law of iterated expectations.
   $$\mathbf{E}[X_i] = \mathbf{E}[\mathbf{E}[X_i \mid Q]] = \mathbf{E}[Q] = 0.5.$$
   Hence, $\mathbf{E}[Y_i] = 2.5$.

2. Since X_i and N are independent,

$$\mathsf{Var}(X_i + N) = \mathsf{Var}(X_i) + \mathsf{Var}(N) = \mathsf{Var}(X_i) + 1.$$

In order to compute $\mathsf{Var}(X_i)$, we use the law of total variance as follows:

$$\begin{aligned}
\mathsf{Var}(X_i) &= \mathbf{E}[\mathsf{Var}(X_i \mid Q)] + \mathsf{Var}(\mathbf{E}[X_i \mid Q]) \\
&= \mathbf{E}[Q(1-Q)] + \mathsf{Var}(Q) \\
&= \mathbf{E}[Q] - \mathbf{E}[Q^2] + \mathbf{E}[Q^2] - (\mathbf{E}[Q])^2 \\
&= \mathbf{E}[Q] - \mathbf{E}[Q]^2 \\
&= 0.5 - 0.25 \\
&= 0.25.
\end{aligned}$$

A shorter derivation uses the fact $X_i = X_i^2$ and proceeds as follows.

$$\begin{aligned}
\mathsf{Var}(X_i) &= \mathbf{E}[X_i^2] - (\mathbf{E}[X_i])^2 \\
&= \mathbf{E}[X_i] - (\mathbf{E}[X_i])^2 \\
&= 0.5 - 0.25 \\
&= 0.25.
\end{aligned}$$

Given $\mathsf{Var}(X_i) = 0,25$,

$$\mathsf{Var}(Y_i) = \mathsf{Var}(X_i + N) = \mathsf{Var}(X_i) + \mathsf{Var}(N) = 1 + \frac{1}{4} = \frac{5}{4}.$$

3. Using Bayes' rule, we have for $0 \le q \le 1$,

$$\begin{aligned}
f_{Q|A}(q) &= \frac{f_Q(q)\mathbf{P}(A \mid Q = q)}{\mathbf{P}(A)} \\
&= \frac{f_Q(q)\mathbf{P}(A \mid Q = q)}{\int_0^1 f_Q(q')\mathbf{P}(A \mid Q = q')\, dq'} \\
&= \frac{1 \cdot (1-q)^2}{\int_0^1 (1-q')^2\, dq'} \\
&= 3(1-q)^2.
\end{aligned}$$

7. A random circle

Problem 6. A random circle

Terminology:

(i) A **circle** of radius r is a curve that consists of all points at distance r from the center of the circle.

(ii) A **disk** of radius r is the set of all points whose distance from its center is **less than or equal** to r.

Thus, a circle is the boundary of a disk.

`<img src="/static/images/exam2p6_randomcircle.jpg" width="400px" style= "margin: 10px 25px 25px 25px" alt= "Two concentric circles of radius 10 and 5. A circle of radius one whose interior is sha "/>`

Circles of radius 10 and 5. A random circle of radius 1, whose center is inside the larger circle, may or may not intersect the circle of radius 5. In the figure, the circle whose interior is shaded intersects the circle of radius 5. The other two circles of radius 1 do not intersect the circle of radius 5.

(a). We generate a random circle of radius 1, whose center is uniformly distributed inside a disk of radius 10 centered at the origin; see figure above.
Find the probability that the random circle intersects a circle of radius $r = 5$, also centered at the origin. (Give a numerical answer.)

(b). Answer the same question as in Part (a) but for the case where r, instead of being 5, is the realized value of a random variable R that is uniformly distributed between 2 and 5. (Give a numerical answer.)

Solution:

(a). A circle of radius 1 will intersect a circle of radius 5 if its center is contained within the annulus with inner radius 4 and outer radius 6. Since the probability is uniformly distributed, it is sufficient to compute the area of this annulus and divide by the total area of the disk to find the probability. Letting E denote this event, we have

$$\mathbf{P}(E) = \frac{\pi 6^2 - \pi 4^2}{\pi 10^2} = \frac{20}{100} = \frac{1}{5}$$

Note that this is equivalent to working out the total area, 100π, and integrating over the annulus normalized by the total area

$$\begin{aligned}
\mathbf{P}(E) &= \int_0^{2\pi}\int_4^6 \frac{1}{100\pi}\, r\, dr d\theta \\
&= 2\pi \int_4^6 \frac{1}{100\pi}\, r\, dr \\
&= \frac{1}{50} \times \frac{r^2}{2}\Big|_4^6 \\
&= \frac{20}{100} = \frac{1}{5}
\end{aligned}$$

(b). Note that conditional on $R = r$, the probability $\mathbf{P}(E|R = r)$ is computed as

$$\mathbf{P}(E|R = r) = \frac{\pi \cdot (r+1)^2 - \pi \cdot (r-1)^2}{\pi \cdot 10^2} = \frac{r}{25}.$$

Hence, using the continuous version of the law of total probability, we have,

$$\mathbf{P}(E) = \int_2^5 \mathbf{P}(E|R = r) f_R(r)\, dr = \int_2^5 \frac{r}{25} \cdot \frac{1}{3}\, dr = 0.14.$$