13. Testing Implicit Hypotheses II

Testing Implicit Hypotheses III: Slutsky's Theorem

2/2 points (graded)

As above, we have that

$$\sqrt{n}\left(\hat{\theta}_n-\theta^*\right)\xrightarrow[n\to\infty]{(d)}\mathcal{N}\left(\mathbf{0},\Sigma\left(\theta^*\right)\right),\quad \Sigma\left(\theta^*\right)\in\mathbb{R}^{d\times d}.$$

and

$$\sqrt{n}\left(g\left(\hat{\theta}_n\right)-g\left(\theta^*\right)\right)\xrightarrow[n\to\infty]{(d)}\mathcal{N}_{k}\left(\mathbf{0},\Gamma\left(\theta^*\right)\right),\quad \Gamma\left(\theta^*\right)\in\mathbb{R}^{\boxed{k\times k}}.$$

In particular, $\hat{\theta}_n$ is a consistent estimator for θ^*.

Assume that $\Gamma(\theta)^x$ is a continuous function of $\theta\in\mathbb{R}^d$ for all $x\in\mathbb{R}$.

Which of the following is a consistent estimator for $\Gamma(\theta^*)^{-1/2}$?

○ $\Gamma\left(\theta^*\right)$

○ $\Gamma\left(\hat{\theta}_n\right)$

○ $\Gamma\left(\hat{\theta}_n^{-1/2}\right)$

● $\Gamma(\hat{\theta}_n)^{-1/2}$ □

Applying Slutsky's theorem and the result of the previous problem, to what distribution does the random vector

$$\sqrt{n}\Gamma(\hat{\theta}_n)^{-1/2}\left(g\left(\hat{\theta}_n\right)-g\left(\theta^*\right)\right)$$

converge to as $n\to\infty$?

● $\mathcal{N}\left(\mathbf{0},I_k\right)$ □

○ $\mathcal{N}\left(\mathbf{0},I_d\right)$

○ χ^2_d

○ χ^2_k

Solution:

Since $\hat{\theta}_n$ is a consistent estimator for θ^*, by continuity of $\theta \mapsto \Gamma(\theta)^{-1/2}$, this implies that $\Gamma(\hat{\theta}_n)^{-1/2}$ is a consistent estimator for $\Gamma(\theta)^{-1/2}$.

By the result of the previous problem,

$$\sqrt{n}\Gamma(\theta^*)^{-1/2}\left(g\left(\hat{\theta}_n\right) - g\left(\theta^*\right)\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}\left(\mathbf{0}, I_k\right).$$

So by Slutsky's theorem,

$$\sqrt{n}\Gamma(\hat{\theta}_n)^{-1/2}\left(g\left(\hat{\theta}_n\right) - g\left(\theta^*\right)\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}\left(\mathbf{0}, I_k\right).$$

提交 你已经尝试了2次（总共可以尝试3次）

☐ Answers are displayed within the problem

Testing Implicit Hypotheses IV: Performing the Test

2/2 points (graded)

We would like to hypothesis test between the following null and alternative:

$$\begin{aligned} H_0 : g\left(\theta^*\right) &= 0 \\ H_1 : g\left(\theta^*\right) &\neq 0. \end{aligned}$$

To do so, we consider the test statistic

$$T_n := \left|\sqrt{n}\Gamma(\hat{\theta}_n)^{-1/2}\left(g\left(\hat{\theta}_n\right)\right)\right|_2^2 = ng(\hat{\theta}_n)^T\Gamma(\hat{\theta}_n)^{-1}g\left(\hat{\theta}_n\right)$$

and design the test

$$\psi = \mathbf{1}\left(T_n > C\right)$$

where C is a threshold to be determined.

Under the null hypothesis, to what distribution does the test-statistic T_n converge?

○ $\mathcal{N}\left(\mathbf{0}, I_k\right)$

○ $\mathcal{N}\left(\mathbf{0}, I_d\right)$

○ χ^2_d

● χ^2_k ☐

Supposing that $d = 6$ and $k = 3$, what value of C should be chosen so that ψ is a test of asymptotic level 5%?

(You should consult a table, e.g. https://people.richland.edu/james/lecture/m170/tbl-chi.html) or use software, e.g. R.)

7.815 ☐ **Answer:** 7.815

Solution:

Under the null-hypothesis, we have that $g(\theta^*) = 0$, so by the previous problem,

$$\sqrt{n}\Gamma(\hat{\theta}_n)^{-1/2} g(\hat{\theta}_n) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(\mathbf{0}, I_k).$$

By definition, $|\mathcal{N}(\mathbf{0}, I_k)|_2^2 \sim \chi^2_k$, so we have by continuity that

$$\left|\sqrt{n}\Gamma(\hat{\theta}_n)^{-1/2} g(\hat{\theta}_n)\right|_2^2 = ng(\hat{\theta}_n)^T \Gamma(\hat{\theta}_n)^{-1} g(\hat{\theta}_n) \xrightarrow[n\to\infty]{(d)} \chi^2_k.$$

Indeed, the test statistic T_n converges to χ^2_k in distribution.

When $k = 3$, then $T_n \xrightarrow[n\to\infty]{(d)} \chi^2_3$. The test $\psi = \mathbf{1}(T_n > C)$ will have asymptotic level 5% precisely when C is the 5%-quantile $q_{0.05}$ of χ^2_3. Consulting a table, we have that $q_{0.05} = 7.815$.

提交 你已经尝试了1次（总共可以尝试3次）

☐ Answers are displayed within the problem

讨论

显示讨论

主题: Unit 4 Hypothesis testing:Lecture 14: Wald's Test, Likelihood Ratio Test, and Implicit Hypothesis Test / 13. Testing Implicit Hypotheses II