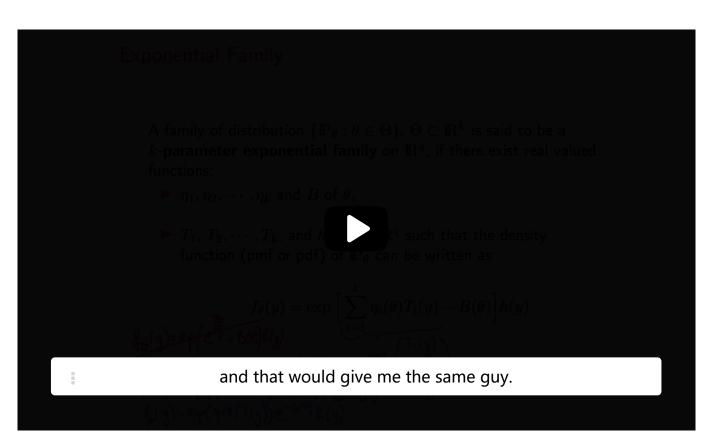


Lecture 21: Introduction to Generalized Linear Models;

<u>Course</u> > <u>Unit 7 Generalized Linear Models</u> > <u>Exponential Families</u>

> 6. The Exponential Family

6. The Exponential Family Exponential Families: Definition



to this very general family.

It's actually quite general, but it does not allow everything.

But the fact that I can take any function eta

and any function t here just gives me a lot of flexibility.

It's not uniquely defined, right?

I could multiply my etas by 2 and divide my t by 2,

and that would give me the same guy.

▶ 8:09 / 8:09 ▶ 1.0x ◆ ▼ © 66

End of transcript. Skip to the start.

Video Download video file Transcripts

<u>Download SubRip (.srt) file</u>

<u>Download Text (.txt) file</u>

Recall from lecture that a family of distribution $\{\mathbf{P}_{\theta}: \theta \in \Theta\}$, where the parameter space $\Theta \subset \mathbb{R}^k$ is k-dimensional, is called a k-parameter exponential family on \mathbb{R}^1 if the pmf or pdf $f_{\theta}: \mathbb{R}^q \to \mathbb{R}$ of \mathbf{P}_{θ} can be written in the form

$$f_{oldsymbol{ heta}}\left(\mathbf{y}
ight) = h\left(\mathbf{y}
ight) \exp\left(oldsymbol{\eta}\left(oldsymbol{ heta}
ight) \cdot \mathbf{T}\left(\mathbf{y}
ight) - B\left(oldsymbol{ heta}
ight)
ight) \qquad ext{where} egin{dcases} oldsymbol{\eta}\left(oldsymbol{ heta}\left(oldsymbol{ heta}
ight) \\ oldsymbol{T}\left(\mathbf{y}
ight) = egin{pmatrix} \eta_{1}\left(oldsymbol{ heta}
ight) \\ \vdots \\ T_{k}\left(oldsymbol{ heta}
ight) \\ \vdots \\ T_{k}\left(oldsymbol{ heta}
ight) \\ h\left(oldsymbol{ heta}
ight) & : \mathbb{R}^{q}
ightarrow \mathbb{R}^{k} \\ h\left(oldsymbol{ heta}
ight) & : \mathbb{R}^{q}
ightarrow \mathbb{R}. \end{cases}$$

When k=1, this reduces to

$$f_{\theta}(y) = h(y) \exp(\eta(\theta) T(y) - B(\theta)).$$

Note: The following exercises are similar to what will be presented in lecture, but we encourage you to first attempt these yourselves.

Practice: Decomposing the exponent

4/4 points (graded)

For the two following pmfs with one parameter $oldsymbol{ heta}$ that are written in the form

$$f_{ heta}\left(y
ight) \; = \; h\left(y
ight)e^{w\left(heta,y
ight)},$$

first decompose $w\left(heta,y
ight)$ as

$$w(\theta, y) = \eta(\theta) T(y) - B(\theta),$$

then enter the product $\eta\left(heta
ight)T\left(y
ight)$ below. Select the distribution that $f_{ heta}$ defines.

1. For
$$f_{ heta}\left(y
ight) = e^{w\left(heta,y
ight)}$$
 where

$$w(\theta, y) = y \ln(\theta) + (1 - y) \ln(1 - \theta)$$

and $y = 0, 1, \, \theta \in (0, 1)$:

$$\eta\left(heta
ight)T\left(y
ight)= egin{array}{c} y^{*}(\ln(heta)-\ln(1- heta)) \ \hline y\cdot(\ln\left(heta
ight)-\ln\left(1- heta
ight)) \end{array}$$

What distribution does the pmf $f_{ heta}\left(y
ight)$ define?

- \circ $\mathcal{N}\left(heta,1
 ight)$
- \circ $\mathcal{N}\left(1, heta
 ight)$
- Ber (θ)
- \bigcirc Poiss (θ)
- onone of the above

^{2.} For
$$f_{ heta}\left(y
ight)=rac{1}{y!}e^{w\left(heta,y
ight)}$$
 where $w\left(heta,y
ight)=- heta+y\ln\left(heta
ight)$, and $y=0,1,2,\ldots,\, heta\in\left(0,1
ight)$:

What distribution does the pmf $f_{ heta}\left(y
ight)$ define?

- \circ $\mathcal{N}\left(heta,1
 ight)$
- \circ $\mathcal{N}\left(1, heta
 ight)$
- \bigcirc Ber (θ)
- Poiss (θ)
- onone of the above

STANDARD NOTATION

Solution:

1. For
$$f_{ heta}\left(y
ight)=e^{w(heta,y)}$$
 where $w\left(heta,y
ight)=y\ln\left(heta
ight)+\left(1-y
ight)\ln\left(1- heta
ight)$ and $y\in\left\{0,1
ight\},\, heta\in\left(0,1
ight)$:

$$w\left(heta,y
ight) \,=\, y \ln \left(heta
ight) + \left(1-y
ight) \ln \left(1- heta
ight) \,=\, y \left(\ln \left(heta
ight) - \ln \left(1- heta
ight)
ight) + \ln \left(1- heta
ight)$$

Hence, $\eta\left(\theta\right)T\left(y\right)=y\left(\ln\left(\theta\right)-\ln\left(1-\theta\right)\right)$ and $B\left(\theta\right)=-\ln\left(1-\theta\right)$. Rewriting f_{θ} :

$$f_{\theta}(y) = e^{y \ln(\theta) + (1-y) \ln(1-\theta)} = \theta^{y} (1-\theta)^{(1-y)},$$

we see that $f_{ heta}$ is the pmf of a Bernoulli distribution with parameter heta.

2. For $f_{ heta}\left(y
ight)=rac{1}{y!}e^{w(heta,y)}$ where $w\left(heta,y
ight)=- heta+y\ln\left(heta
ight)$, and $y=0,1,2,\ldots,\, heta\in\left(0,1
ight)$ Hence, $\eta\left(heta
ight)T\left(y
ight)=y\ln\left(heta
ight)$ and $B\left(heta
ight)= heta$. Rewriting $f_{ heta}$

$$f_{ heta}\left(y
ight) \,=\, rac{1}{y!} e^{- heta+y\ln(heta)} \,=\, e^{- heta} rac{ heta^y}{y!},$$

we recognize $f_{ heta}$ as the pmf of a Poisson distribution with parameter heta.

Submit

You have used 2 of 3 attempts

• Answers are displayed within the problem

Practice: Normal distribution with known variance

1/1 point (graded)

The normal distribution $\mathcal{N}\left(heta,1
ight)$ with with mean heta and known variance $\sigma^2=1$ has pdf

$$f_{ heta}\left(y
ight) \; = \; rac{1}{\sqrt{2\pi}}e^{-rac{\left(y- heta
ight)^{2}}{2}} \, .$$

Rewrite $f_{ heta}$ in the form

$$f_{ heta}\left(y
ight) \; = \; h\left(y
ight)e^{\eta\left(heta
ight)T\left(y
ight)-B\left(heta
ight)} \qquad ext{where } \eta\left(heta
ight), \, T\left(y
ight): \mathbb{R}
ightarrow \mathbb{R},$$

and enter the product $\eta\left(heta\right) T\left(y
ight)$ below.

$$\eta\left(heta
ight)T\left(y
ight)=$$
 theta*y $egin{align*} oldsymbol{\phi}\cdot y \end{bmatrix}$ Answer: y*theta

STANDARD NOTATION

Solution:

$$egin{aligned} f_{ heta}\left(y
ight) &= rac{1}{\sqrt{2\pi}}e^{-rac{(y- heta)^2}{2}} \ &= rac{1}{\sqrt{2\pi}}e^{-rac{(y^2-2y heta+ heta^2)}{2}} \ &= rac{1}{\sqrt{2\pi}}e^{-y^2/2}e^{+rac{y heta- heta^2}{2}} \ &= h\left(y
ight)e^{\eta(heta)T(y)-B(heta)} \qquad ext{where} egin{aligned} \eta\left(heta
ight)T\left(y
ight) &= \left(y
ight)\left(heta
ight) \ B\left(heta
ight) &= rac{ heta^2}{2} \ h\left(y
ight) &= \left(e^{-rac{y^2}{2}}
ight)/\sqrt{2\pi} \end{aligned}$$

Submit

You have used 1 of 3 attempts

1 Answers are displayed within the problem