6. Image Corrupted with noise

Problem 5. Image Corrupted with noise

3/3 points (graded)

Consider an image, in which every pixel takes a value of 1, with probability q, and a value 0, with probability $1-q$, where q is the realized value of a random variable Q which is distributed uniformly over the interval $[0,1]$. The realized value q is the same for every pixel.

Let X_i be the value of pixel i. We observe, for each pixel the value of $Y_i = X_i + N$, where N is normal with mean 2 and unit variance. (Note that we have the same noise at each pixel.) Assume that, conditional on Q, the X_i's are independent, and that the noise N is independent of Q and the X_i's.

1. Find $\mathbf{E}[Y_i]$. (Give a numerical answer.)

   $\mathbf{E}[Y_i] =$ 5/2 ✔ **Answer:** 2.5

2. Find $\text{Var}[Y_i]$. (Give a numerical answer.)

   $\text{Var}[Y_i] =$ 5/4 ✔ **Answer:** 1.25

3. Let A be the event that the actual values X_1 and X_2 of pixels 1 and 2, respectively, are zero. Find the conditional probability of Q given A.
   (Enter your answer in terms of q in standard notation.)

   For $0 \le q \le 1$:

   $f_{Q|A}(q) =$ 3*(1-q)^2 ✔ **Answer:** 3*(1-q)^2

   $3 \cdot (1-q)^2$

STANDARD NOTATION

Solution:

1. Notice that

$$\mathbf{E}[Y_i] = \mathbf{E}[X_i + N] = \mathbf{E}[X_i] + \mathbf{E}[N] = \mathbf{E}[X_i] + 2.$$

   In order to find $\mathbf{E}[X_i]$, we will use the law of iterated expectations.

$$\mathbf{E}[X_i] = \mathbf{E}[\mathbf{E}[X_i \mid Q]] = \mathbf{E}[Q] = 0.5.$$

   Hence, $\mathbf{E}[Y_i] = 2.5$.

2. Since X_i and N are independent,

$$\text{Var}(X_i + N) = \text{Var}(X_i) + \text{Var}(N) = \text{Var}(X_i) + 1.$$

   In order to compute $\text{Var}(X_i)$, we use the law of total variance as follows:

$$\begin{aligned}\text{Var}(X_i) &= \mathbf{E}[\text{Var}(X_i \mid Q)] + \text{Var}(\mathbf{E}[X_i \mid Q]) \\ &= \mathbf{E}[Q(1-Q)] + \text{Var}(Q)\end{aligned}$$

$$= \mathbf{E}[Q] - \mathbf{E}[Q^2] + \mathbf{E}[Q^2] - (\mathbf{E}[Q])^2$$
$$= \mathbf{E}[Q] - \mathbf{E}[Q]^2$$
$$= 0.5 - 0.25$$
$$= 0.25.$$

A shorter derivation uses the fact $X_i = X_i^2$ and proceeds as follows.

$$\begin{aligned} \mathsf{Var}(X_i) &= \mathbf{E}[X_i^2] - (\mathbf{E}[X_i])^2 \\ &= \mathbf{E}[X_i] - (\mathbf{E}[X_i])^2 \\ &= 0.5 - 0.25 \\ &= 0.25. \end{aligned}$$

Given $\mathsf{Var}(X_i) = 0, 25$,

$$\mathsf{Var}(Y_i) = \mathsf{Var}(X_i + N) = \mathsf{Var}(X_i) + \mathsf{Var}(N) = 1 + \frac{1}{4} = \frac{5}{4}.$$

3. Using Bayes' rule, we have for $0 \leq q \leq 1$,

$$\begin{aligned} f_{Q|A}(q) &= \frac{f_Q(q)\mathbf{P}(A \mid Q = q)}{\mathbf{P}(A)} \\ &= \frac{f_Q(q)\mathbf{P}(A \mid Q = q)}{\int_0^1 f_Q(q')\mathbf{P}(A \mid Q = q')\, dq'} \\ &= \frac{1 \cdot (1-q)^2}{\int_0^1 (1-q')^2\, dq'} \\ &= 3(1-q)^2. \end{aligned}$$

提交 You have used 1 of 3 attempts

Answers are displayed within the problem

Error and Bug Reports/Technical Issues

显示讨论

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