4. Maximum Likelihood Estimator for Curved Gaussian

(a)

1.0/1 point (graded)

Note: *To avoid too much double jeopardy, the solution to part (a) will be available once you have either answered it correctly or reached the maximum number of attempts.*

Let $X_1, \ldots, X_n$ be n i.i.d. random variables with distribution $\mathcal{N}(\theta, \theta)$ for some unknown $\theta > 0$.

Compute the maximum likelihood estimator $\hat{\theta}$ for θ in terms of the sample averages of the linear and quadratic means, i.e. $\overline{X}_n$ and $\overline{X_n^2}$.

(Enter **barX_n** for $\overline{X}_n$ and **bar(X_n^2)** for $\overline{X_n^2}$.)

$\hat{\theta} =$ 1/2*sqrt(1+4*bar(X_n^2) **Answer:** (sqrt(4 * bar(X_n^2) + 1) - 1)/2

STANDARD NOTATION

Solution:

To compute the maximum likelihood estimator, we write the log likelihood and maximize it by setting its derivative to zero. First,

$$\begin{aligned}
\ell_n(\theta) &= \sum_{i=1}^{n} \log\left[\frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{(X_i-\theta)^2}{2\theta}\right)\right] \\
&= -\frac{n}{2}(\log(2) + \log(\pi) + \log(\theta)) - \sum_{i=1}^{n} \frac{(X_i-\theta)^2}{2\theta} \\
&= -\frac{n}{2}(\log(2) + \log(\pi) + \log(\theta)) - \sum_{i=1}^{n} \left[\frac{1}{2\theta} X_i^2 - X_i + \frac{1}{2}\theta\right].
\end{aligned}$$

Differentiating yields

$$\frac{d}{d\theta}\ell(\theta) = -\frac{n}{2\theta} + \frac{1}{2\theta^2}\sum_{i=1}^{n} X_i^2 - \frac{n}{2},$$

which we set to zero to obtain the equation

$$\hat{\theta}^2 + \hat{\theta} - \frac{1}{n}\sum_{i=1}^{n} X_i^2 = 0.$$

Employing the quadratic formula and picking the result that gives a positive $\hat{\theta}$ then leads to

$$\hat{\theta} = -\frac{1}{2} + \frac{1}{2}\sqrt{4\overline{X_n^2} + 1}.$$

提交 你已经尝试了1次（总共可以尝试3次）

Answers are displayed within the problem

(b)

2/4 points (graded)

We want to compute the asymptotic variance of $\hat{\theta}$ via two methods.

In this problem, we apply the Central Limit Theorem and the 1-dimensional Delta Method. We will compare this with the approach using the Fisher information next week.

First, compute the limit and asymptotic variance of $\overline{X_n^2}$.

The limit to which $\overline{X_n^2}$ converges in probability, also known as its $\mathbf{P}$**-limit**, is

$\overline{X_n^2} \xrightarrow[n\to\infty]{\mathbf{P}}$ theta^2 + theta **Answer:** theta + theta^2

$\theta^2+\theta$

The asymptotic variance $V\left(\overline{X_n^2}\right)$ of $\overline{X_n^2}$, which is equal to $\mathsf{Var}\left(X_1^2\right)$, is

$V\left(\overline{X_n^2}\right) = \mathsf{Var}\left(X_1^2\right) =$ 2*theta^4+8*theta^3+4* **Answer:** 2*theta^2*(2*theta + 1)

$2\cdot\theta^4+8\cdot\theta^3+4\cdot\theta^2$

两个相减，我加起来了，太蠢了

Now, write $\hat{\theta}$ as the function of $\overline{X_n^2}$ you found in part (a),

$$\hat{\theta} = g\left(\overline{X}_n^2\right)$$

and give its first derivative, $g'(x)$,

$g'(x) =$ 1/sqrt(4*x+1) **Answer:** 1/sqrt(4*x+1)

$\frac{1}{\sqrt{4\cdot x+1}}$

What can you conclude about the asymptotic variance $V\left(\hat{\theta}\right)$ of $\hat{\theta}$?

$V\left(\hat{\theta}\right) =$ 1/(4*(theta^2 + theta)+1) **Answer:** 2*theta^2/(2*theta + 1)

$\frac{1}{4\cdot(\theta^2+\theta)+1}\cdot\left(2\cdot\theta^4+8\cdot\theta^3+4\cdot\theta^2\right)$

Integrate$\left[\frac{x^4}{\sqrt{2\pi\theta}}\, e^{-(x-\theta)^2/(2\theta)}, \{x, -\infty, \infty\}, \text{Assumptions} \to \theta \geq 0\right]$

STANDARD NOTATION

Solution:

First, by the Law of Large Numbers,

$$\overline{X_n^2} \xrightarrow[n\to\infty]{\mathbf{P}} \mathbb{E}\left[X_1^2\right] = \mathsf{Var}\left(X_1\right) + \mathbb{E}[X_1]^2 = \theta+\theta^2.$$

Its asymptotic variance can be found by the Central Limit Theorem that gives us

$$\sqrt{n}\left(\overline{X_n^2} - (\theta+\theta^2)\right) \xrightarrow[n\to\infty]{(\mathrm{D})} \mathcal{N}\left(0, \mathsf{Var}\left(X_1^2\right)\right),$$

and

$$\begin{aligned}\mathsf{Var}\left(X_1^2\right) &= \mathbb{E}\left[X_1^4\right] - \mathbb{E}[X_1^2]^2\\ &= \mathbb{E}\left[(\theta+\sqrt{\theta}Z)^4\right] - (\theta+\theta^2)^2\end{aligned}$$

$$
\begin{aligned}
&= \theta^4 + 4\theta^3\sqrt{\theta}\underbrace{\mathbb{E}[Z]}_{=0} + 6\theta^2\theta\underbrace{\mathbb{E}[Z^2]}_{=0} + 4\theta\sqrt{\theta}^3\underbrace{\mathbb{E}[Z^3]}_{=0} + \theta^2\underbrace{\mathbb{E}[Z^4]}_{=3} - \theta^4 - 2\theta^3 + \theta^2 \\
&= 2\theta^2(2\theta+1),
\end{aligned}
$$

where $Z \sim \mathcal{N}(0,1)$ is a standard Normal variable.

From the previous part, we get

$$g(x) = \frac{1}{2}\left(\sqrt{4x+1} - 1\right),$$

so

$$g'(x) = \frac{1}{\sqrt{4x+1}}.$$

Finally, by the Delta Method,

$$\sqrt{n}\left(g\left(\overline{X_n^2}\right) - g\left(\theta+\theta^2\right)\right) \xrightarrow[n\to\infty]{(\mathrm{D})} \mathcal{N}\left(0, 2\theta^2(2\theta+1)\, g'\left(\theta+\theta^2\right)^2\right) = \mathcal{N}\left(0, \frac{2\theta^2}{2\theta+1}\right).$$

提交 你已经尝试了3次（总共可以尝试3次）

Answers are displayed within the problem

讨论

显示讨论

主题： Unit 3 Methods of Estimation:Homework 5: Maximum Likelihood Estimation / 4. Maximum Likelihood Estimator for Curved Gaussian