7. Sum of a random number of r.v.'s

Problem 7. Sum of a random number of r.v.'s

2/2 points (graded)

A fair coin is flipped independently until the first Heads is observed. Let the random variable K be the number of tosses until the first Heads is observed **plus 1**. For example, if we see TTTHTH, then $K=5$. For $k=1,2,\ldots,K$, let X_k be a continuous random variable that is uniform over the interval $[0,5]$. The X_k are independent of one another and of the coin flips. Let $X=\sum_{k=1}^{K} X_k$. Find the mean and variance of X. You may use the fact that the mean and variance of a geometric random variable with parameter p are $1/p$ and $(1-p)/p^2$, respectively.

$\mathbf{E}[X]=$ 15/2 ✔ **Answer:** 7.5

Var$(X)=$ 75/4 ✔ **Answer:** 18.75

Solution:

Since X_k is uniform over $[0,5]$, we have $\mathbf{E}[X_k]=5/2$ and Var$(X_k)=5^2/12=25/12$.

Note that $K-1$ is the number of tosses until the first Heads, and is therefore geometric with parameter $p=1/2$. In particular, $\mathbf{E}[K-1]=2$ and $\mathrm{Var}(K-1)=2$, which implies that $\mathbf{E}[K]=3$ and Var$(K)=2$.

Since $X=\sum_{k=1}^{K} X_k$ is the sum of a random number of independent and identically distributed random variables, we have

$$\mathbf{E}[X]=\mathbf{E}[X_1]\mathbf{E}[K]=\frac{5}{2}\cdot 3=15/2,$$

and

$$\mathrm{Var}(X)=\mathrm{Var}(X_1)\mathbf{E}[K]+(\mathbf{E}[X_1])^2\mathrm{Var}(K)=\frac{25}{12}\cdot 3+\frac{25}{4}\cdot 2=75/4.$$