Problem 5

(a) Canonical Exponential Family and Distribution Statistics

3/3 points (graded)

Consider a distribution with parameter $\lambda > 0$ that has density

$$f_\lambda(x) = \frac{x^4}{24\lambda^5} e^{\frac{-x}{\lambda}}, x > 0.$$

Let $X_1, \ldots, X_n$ be n independent random variables drawn from this distribution.
Does this distribution belong to the canonical exponential family?

● Yes ✓

○ No

Compute the expectation and variance of X_1 in terms of λ.

$\mathbb{E}[X_1] =$ 5*lambda ✓ **Answer:** 5*lambda

$5 \cdot \lambda$

Var $(X_1) =$ 5*lambda^2 ✓ **Answer:** 5*lambda^2

$5 \cdot \lambda^2$

STANDARD NOTATION

Solution:

Note that $f_\lambda(x) = exp(\frac{-x}{\lambda} - \log(24\lambda^5) + 4\log(x))$. This is a distribution with the form of a canonical exponential family. Note that using the notation from the class we have that (the dispersion parameter) $\phi = -1, \theta = \lambda, b(\lambda) = \log(24\lambda^5)$ and $c(x) = 4\log(x)$.

Using the properties of a distribution in the canonical exponential family $\mathbb{E}[X_1] = b'(\lambda) = \frac{5}{\lambda}$. We also get that the canonical exponential family $Var[X_1] = \phi b"(\lambda) = \frac{5}{\lambda^2}$.

Submit You have used 2 of 3 attempts

Answers are displayed within the problem

(b) Estimation of Lambda

2.0/2 points (graded)

Compute the maximum likelihood estimate $\hat{\lambda}^{\text{MLE}}$ of λ.

(If applicable, enter **barX_n** for $\bar{X}_n$.)

$\hat{\lambda}^{\text{MLE}} =$ barX_n/5 ✓ **Answer:** barX_n/5

Compute the method of moments estimate $\hat{\lambda}^{\mathrm{MM}}$ of λ.

(If applicable, enter **barX_n** for $\bar{X}_n$ and **bar(X_n^2)** for the second moment $\bar{X_n^2}$.)

$\hat{\lambda}^{\mathrm{MM}} =$ barX_n/5 ✓ **Answer:** barX_n/5

STANDARD NOTATION

Solution:

Note that the log-likelihood of $X_1, \ldots, X_n$ given λ is

$$\ell\left(X_1, \ldots, X_n|\lambda\right) = \frac{-\left(X_1 + \ldots + X_n\right)}{\lambda} + 4\sum_{i=1}^{n} \log\left(X_i\right) - 5n\log\left(\lambda\right).$$

Setting the first derivative equal to zero it follows that $\hat{\lambda} = \frac{5n}{X_1 + \ldots + X_n}$.

Since $\mathbb{E}\left[X_1\right] = \frac{5}{\lambda}$ it follows that that the method of moments estimate satisfies $\frac{5}{\hat{\lambda}} = \frac{X_1 + \ldots + X_n}{n}$ and thus $\hat{\lambda} = \frac{5n}{X_1 + \ldots + X_n}$.

Submit You have used 1 of 3 attempts

ℹ Answers are displayed within the problem

(c) Hypothesis Testing

0/1 point (graded)
You test the hypotheses $H_0 : \lambda = 1$ vs $H_1 : \lambda \neq 1$ by using the test

$$\psi_n = \mathbf{1}\left(\left|\bar{X}_n - 5\right| > C_{\alpha,n}\right)$$

Find the smallest threshold $C_{\alpha,n}$ so that the test ψ has asymptotic level α.

Note that here we are choosing a test so that $C_{\alpha,n}$ does not depend on the estimate of λ.

(If applicable, enter **Phi(z)** for the cdf $\Phi\left(z\right)$ of a normal variable Z, **q(alpha)** for the quantile q_α. Recall the convention in this course that $\mathbf{P}\left(Z \leq q_\alpha\right) = 1 - \alpha$ for $Z \sim \mathcal{N}\left(0, 1\right)$.)

$C_{\alpha,n} =$ (5*sqrt(5)*q(0.05/2))/sqrt(n) ✗ **Answer:** q(alpha/2)*sqrt(5/n)

这里傻了，后面又乘了那个5，本就是converge到N(0,5)了

Correction note: May 23 An earlier version of the problem statement does not contain the sentence "Note that here we are choosing a test so that $C_{\alpha,n}$ does not depend on the estimate of λ."

Grading Note: Before the note was added, there were choices as to what to use in the argument of the asymptotic variance, but the way the grader was set up only allows for the choice with λ_0 as the argument.

STANDARD NOTATION

Solution:

Note that $\hat{\lambda} = \frac{5n}{X_1 + \ldots + X_n}$ and note that a test for $\lambda = 1$ is equivalent to testing how close $\frac{X_1 + \ldots + X_n}{n}$ is to 5. Using the Central Limit Theorem it follows that

$$\sqrt{n}\,(\frac{X_1 + \ldots +_n}{n} - 5) \to \mathcal{N}\left(0, 5\right)$$

if H_0 is true. Therefore the asymptotic test at level α is

$$\phi = 1_{\left|\frac{X_1+\ldots+X_n}{n}-5\right| \geq \frac{q_{\frac{\alpha}{2}}\sqrt{5}}{\sqrt{n}}}.$$

Alternatively, use the theorem on MLE to obtain the same result.

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

Error and Bug Reports/Technical Issues

Show Discussion

Topic: Final exam:Final Exam / Problem 5