7. Friendship and happiness

Problem 7. Friendship and happiness

10.0/12.0 points (graded)

Consider a group of $n \geq 4$ people, numbered from 1 to n. For each pair (i, j) with $i \neq j$, person i and person j are friends, with probability p. (Assume that friendship is a symmetric relation, i.e. if i is friends with j, then j is also friends with i.) Friendships are independent for different pairs. These n people are seated around a round table. For convenience, assume that the chairs are numbered from 1 to n, clockwise, with n located next to 1, and that person i seated in chair i. In particular, person 1 and person n are seated next to each other.

If a person is friends with both people sitting next to him/her, we say this person is **happy**. Let H be the total number of happy people.

We will find $\mathbf{E}[H]$ and $\mathsf{Var}(H)$ by carrying out a sequence of steps. Express your answers below in terms of p and/or n using standard notation (or click on "STANDARD NOTATION" button below). Remember to use "*" for multiplication and to include parentheses where necessary.

We first work towards finding $\mathbf{E}[H]$.

1. Let I_i be a random variable indicating whether the person seated in chair i is **happy** or not (i.e., $I_i = 1$ if person i is happy and $I_i = 0$ otherwise). Find $\mathbf{E}[I_i]$.

   For $i = 1, 2, \ldots, n$,

   $\mathbf{E}[I_i] =$ p^2 ✔ **Answer:** p^2

   p^2

2. Find $\mathbf{E}[H]$.
   (Note: The notation $a \triangleq \mathbf{E}[H]$ means that a is defined to be $\mathbf{E}[H]$. The simpler variable names will be used in the last question of this problem.)

   $a \triangleq \mathbf{E}[H] =$ n*p^2 ✔ **Answer:** n*(p^2)

   $n \cdot p^2$

   Since $I_1, I_2, \ldots, I_n$ are not independent, the variance calculation is more involved.

3. For any $k \in \{1, 2, \ldots, n\}$, find $\mathbf{E}[I_k^2]$.

$b \triangleq \mathbf{E}[I_k^2] =$ p^2 ✔ Answer: p^2

p^2

4. For any $i \in \{1, 2, \ldots, n\}$, and under the convention $I_{n+1} = I_1$, find $\mathbf{E}[I_i I_{i+1}]$.

   $c \triangleq \mathbf{E}[I_i I_{i+1}] =$ p^3 ✔ Answer: p^3

   p^3

5. Suppose that $i \neq j$ and that persons i and j are not seated next to each other. Find $\mathbf{E}[I_i I_j]$.

   $d \triangleq \mathbf{E}[I_i I_j] =$ p^4 ✔ Answer: p^4

   p^4

6. Give an expression for $\text{Var}(H)$, in terms of n, and the quantities a, b, c, d defined in earlier parts.

   $\text{Var}(H) =$

   n*b+2*(n-1)*c+(n-2)*(n-1)*d-a^2 ✖

   Answer: n*b+2*n*c+(n^2-3*n)*d-a^2

   $n \cdot b + 2 \cdot (n-1) \cdot c + (n-2) \cdot (n-1) \cdot d - a^2$

STANDARD NOTATION

Solution:

As the seating is circular, it is convenient to use the following notational convention: $I_0 = I_n$, $I_1 = I_{n+1}$, $I_2 = I_{n+2}$.

1. Recall that $I_i = 1$, namely, the i^{th} person is happy, if and only if he/she is friends with both neighbors, which happens with probability p^2.

2. The total number of happy people is

$$H = \sum_{i=1}^{n} I_i.$$

   Therefore,

$$a = \mathbf{E}[H] = \sum_{i=1}^{n} \mathbf{E}[I_i] = np^2.$$

3. Since I_k is either 0 or 1, we have $I_k^2 = I_k$. Therefore, $\mathbf{E}[I_k^2] = \mathbb{P}(I_k = 1) = p^2$.

4. The random variable $I_k I_{k+1}$ is 1, if and only if both persons k and $k+1$ are happy. Equivalently, the pairs $(k-1, k)$, $(k, k+1)$, and $(k+1, k+2)$ are pairs of friends. As each of these events are independent, the probability that $I_k I_{k+1} = 1$ is p^3.

5. We note that $I_i I_j = 1$ if and only if both i and j are happy, i.e., when the four pairs

$$(i, i-1),\ (i, i+1),\ (j, j-1),\ (j, j+1)$$

are pairs of friends, which happens with probability p^4.

6. We have $\mathsf{Var}(H) = \mathbf{E}[H^2] - \mathbf{E}[H]^2 = \mathbf{E}[(I_1 + \cdots + I_n)^2] - a^2$. When we expand the product, $(I_1 + \cdots + I_n)^2$, we obtain n^2 terms, of different types:

(i) n terms of the form I_i^2.
(ii) $2n$ terms of the form $I_i I_{i+1}$, including terms such as $I_n I_{n+1}$, which is interpreted as $I_n I_1$.
(iii) The remaining $n^2 - 3n$ terms of the form $I_i I_j$, where i and j are not seated next to each other.

The expected value of a term of the form (i), (ii), (iii), is b, c, d, respectively. Therefore,

$$\mathbf{E}[(I_1 + \cdots + I_n)^2] = nb + 2nc + (n^2 - 3n)d,$$

which gives,

$$\mathsf{Var}(H) = nb + 2nc + (n^2 - 3n)d - a^2.$$

提交 You have used 1 of 2 attempts

Answers are displayed within the problem

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