Inverse-gamma distribution

In probability theory and statistics, the **inverse gamma distribution** is a two-parameter family of continuous probability distributions on the positive <u>real line</u>, which is the distribution of the <u>reciprocal</u> of a variable distributed according to the <u>gamma distribution</u>. Perhaps the chief use of the inverse gamma distribution is in <u>Bayesian statistics</u>, where the distribution arises as the marginal posterior distribution for the unknown <u>variance</u> of a <u>normal distribution</u>, if an <u>uninformative prior</u> is used, and as an analytically tractable <u>conjugate prior</u>, if an informative prior is required.

However, it is common among Bayesians to consider an alternative <u>parametrization</u> of the <u>normal distribution</u> in terms of the <u>precision</u>, defined as the reciprocal of the variance, which allows the gamma distribution to be used directly as a conjugate prior. Other Bayesians prefer to parametrize the inverse gamma distribution differently, as a <u>scaled inverse chi-squared distribution</u>.

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Characterization

Probability density function

The inverse gamma distribution's <u>probability density function</u> is defined over the <u>support</u> x>0

$$f(x;lpha,eta)=rac{eta^lpha}{\Gamma(lpha)}(1/x)^{lpha+1}\exp(-eta/x)$$

with shape parameter α and scale parameter β .^[1] Here $\Gamma(\cdot)$ denotes the gamma function.

Unlike the <u>Gamma distribution</u>, which contains a somewhat similar exponential term, β is a scale parameter as the distribution function satisfies:

$$f(x;lpha,eta)=rac{f(x/eta;lpha,1)}{eta}$$

Cumulative distribution function

The <u>cumulative distribution function</u> is the <u>regularized gamma function</u>

$$F(x;lpha,eta)=rac{\Gamma\left(lpha,rac{eta}{x}
ight)}{\Gamma(lpha)}=Q\left(lpha,rac{eta}{x}
ight)$$

where the numerator is the upper incomplete gamma function and the denominator is the gamma function. Many math packages allow direct computation of Q, the regularized gamma function.

Moments

The *n*-th moment of the inverse gamma distribution is given by [2]

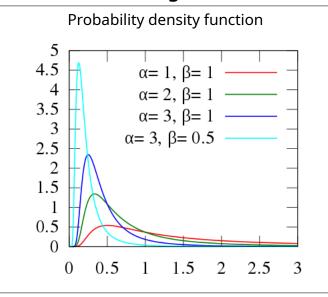
$$\mathrm{E}[X^n] = rac{eta^n}{(lpha-1)\cdots(lpha-n)}.$$

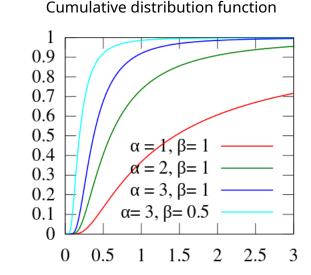
Characteristic function

 $K_{\alpha}(\cdot)$ in the expression of the <u>characteristic function</u> is the modified <u>Bessel function</u> of the 2nd kind.

Properties

Inverse-gamma





Ü	0.6 1 1.6 2 2.6 5
Parameters	lpha>0 shape (real)
	eta>0 scale (real)
Support	$x\in(0,\infty)$
PDF	β^{α} $\alpha^{-\alpha-1}$ over β
	$\left rac{eta^{lpha}}{\Gamma(lpha)}x^{-lpha-1}\exp\!\left(-rac{eta}{x} ight)$
CDF	$\Gamma(lpha,eta/x)$
	$\Gamma(lpha)$
Mean	$rac{eta}{lpha-1}$ for $lpha>1$
	$\frac{1}{\alpha-1}$ for $\alpha>1$
Mode	β
	$\alpha + 1$
Variance	$\left rac{eta^2}{(lpha-1)^2(lpha-2)}$ for $lpha>2$
	$\left \ \overline{(\alpha-1)^2(\alpha-2)} ight ^{101} lpha > 2$
Skewness	$4\sqrt{lpha-2}$
	$\left rac{4\sqrt{lpha-2}}{lpha-3}$ for $lpha>3$
Ex. kurtosis	$oxed{rac{6(5lpha-11)}{(lpha-3)(lpha-4)}}$ for $lpha>4$
	$\left \frac{(\alpha-3)(\alpha-4)}{(\alpha-3)(\alpha-4)} \right ^{1}$ for $\alpha>4$
Entropy	$\alpha + \ln(\beta\Gamma(lpha)) - (1 + lpha)\psi(lpha)$
	(see digamma function)
MGF	Does not exist.
CF	$2(-ieta t^{rac{lpha}{2}}$
	$rac{2(-ieta t)^{\!\!\!/\!\!\!\!2}}{\Gamma(lpha)}K_lpha\left(\sqrt{-4ieta t} ight)$

For $\alpha > 0$ and $\beta > 0$,

$$\mathbb{E}[\ln(X)] = \ln(\beta) - \psi(\alpha)$$

and

$$\mathbb{E}[X^{-1}] = \frac{lpha}{eta},$$

The information entropy is

$$egin{aligned} \mathrm{H}(X) &= \mathrm{E}[-\ln(p(X))] \ &= \mathrm{E}\left[-lpha\ln(eta) + \ln(\Gamma(lpha)) + (lpha+1)\ln(X) + rac{eta}{X}
ight] \ &= -lpha\ln(eta) + \ln(\Gamma(lpha)) + (lpha+1)\ln(eta) - (lpha+1)\psi(lpha) + lpha \ &= lpha + \ln(eta\Gamma(lpha)) - (lpha+1)\psi(lpha). \end{aligned}$$

where $\psi(\alpha)$ is the digamma function.

The Kullback-Leibler divergence of Inverse-Gamma(α_{p_i}, β_{p}) from Inverse-Gamma(α_{q_i}, β_{q}) is the same as the KL-divergence of Gamma(α_{p_i}, β_{p}) from Gamma(α_{q_i}, β_{q}):

$$D_{ ext{KL}}(lpha_p,eta_p;lpha_q,eta_q) = \mathbb{E}\left[\lograc{
ho(X)}{\pi(X)}
ight] = \mathbb{E}\left[\lograc{
ho(1/Y)}{\pi(1/Y)}
ight] = \mathbb{E}\left[\lograc{
ho_G(Y)}{\pi_G(Y)}
ight],$$

where ρ, π are the pdfs of the Inverse-Gamma distributions and ρ_G, π_G are the pdfs of the Gamma distributions, Y is Gamma(α_p, β_p) distributed.

Related distributions

- $lacksquare ext{If } X \sim ext{Inv-Gamma}(lpha,eta) ext{ then } kX \sim ext{Inv-Gamma}(lpha,keta)$
- lacktriangledown If $X \sim ext{Inv-Gamma}(lpha, rac{1}{2})$ then $X \sim ext{Inv-}\chi^2(2lpha)$ (inverse-chi-squared distribution)
- $lacksquare If X \sim ext{Inv-Gamma}(rac{lpha}{2},rac{1}{2})$ then $X \sim ext{Scaled Inv-}\chi^2(lpha,rac{1}{lpha})$ (scaled-inverse-chi-squared distribution)
- $lacksquare ext{Inv-Gamma}(rac{1}{2},rac{c}{2})$ then $X \sim ext{Levy}(0,c)$ (Lévy distribution)
- If $X \sim \operatorname{Gamma}(\alpha, \beta)$ (Gamma distribution) then $\frac{1}{X} \sim \operatorname{Inv-Gamma}(\alpha, \beta)$ (see derivation in the next paragraph for details)
- Inverse gamma distribution is a special case of type 5 Pearson distribution
- A multivariate generalization of the inverse-gamma distribution is the inverse-Wishart distribution.
- For the distribution of a sum of independent inverted Gamma variables see Witkovsky (2001)

Derivation from Gamma distribution

Let $X \sim \operatorname{Gamma}(\alpha, \beta)$, and recall that the pdf of the gamma distribution is

$$f(x) = rac{eta^lpha}{\Gamma(lpha)} x^{lpha-1} e^{-eta x}.$$

Define the transformation $Y=g(X)=rac{1}{X}$. Then, the pdf of Y is

$$egin{aligned} f_Y(y) &= f_X\left(g^{-1}(y)
ight) \left| rac{d}{dy} g^{-1}(y)
ight| \ &= rac{eta^lpha}{\Gamma(lpha)} \left(rac{1}{y}
ight)^{lpha-1} \exp\left(rac{-eta}{y}
ight) rac{1}{y^2} \ &= rac{eta^lpha}{\Gamma(lpha)} \left(rac{1}{y}
ight)^{lpha+1} \exp\left(rac{-eta}{y}
ight) \ &= rac{eta^lpha}{\Gamma(lpha)} (y)^{-lpha-1} \exp\left(rac{-eta}{y}
ight) \end{aligned}$$

Occurrence

See also

- gamma distribution
- inverse-chi-squared distribution
- normal distribution

References

- 1. "InverseGammaDistribution—Wolfram Language Documentation" (http://reference.wolfram.com/language/ref/InverseGammaDistribution.html). reference.wolfram.com. Retrieved 9 April 2018.
- 2. John D. Cook (Oct 3, 2008). "InverseGammaDistribution" (https://www.johndcook.com/inverse_gamma.pdf) (PDF). Retrieved 3 Dec 2018.

■ Witkovsky, V. (2001). "Computing the Distribution of a Linear Combination of Inverted Gamma Variables". *Kybernetika*. **37** (1): 79–90. MR 1825758 (https://www.ams.org/mat hscinet-getitem?mr=1825758). Zbl 1263.62022 (https://zbmath.org/?format=complete&q=an:1263.62022).

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