7. Modes of convergence

Convergence in distribution

4.0/4 points (graded)

Let T_n be a sequence of random variables that converges to $\mathcal{N}(0,1)$ in distribution. What family of distribution does the limit of $2T_n+1$ belong to?

- ○ χ^2 distribution
- ◉ Normal distribution ✔

Call this limit Y. Compute:

$\mathbb{E}[Y] =$ 1 ✔ **Answer:** 1

Var $[Y] =$ 4 ✔ **Answer:** 4

Let Φ be the cumulative distribution function (cdf) of the standard Gaussian distribution. In terms of Φ, what is the limit, as $n \to \infty$, of $\mathbf{P}(|T_n+2| \le 8)$?

(Write Phi, with capital P, for Φ).

Phi(6)+Phi(10)-1 ✔ **Answer:** Phi(6) - Phi(-10)

STANDARD NOTATION

Solution:

Since convergence in distribution is equivalent to convergence for all continuous bounded test functions f, let $f : \mathbb{R} \to \mathbb{R}$ be a continuous bounded function. Then, let $g(x) = 2x+1$ and observe

$$\mathbb{E}[f(2T_n+1)] = \mathbb{E}[f(g(T_n))] \xrightarrow[n\to\infty]{} \mathbb{E}[f(g(Z))],$$

where $Z \sim \mathcal{N}(0,1)$. Now, $g(Z) \sim \mathcal{N}(1,4)$, and therefore $2T_n+1$ converges to $\mathcal{N}(1,4)$ in distribution.

To calculate $\mathbf{P}(|T_n+2| \le 8)$, write

$$\mathbf{P}(|T_n+2| \le 8) \to \mathbf{P}(|Z+2| \le 8)$$

by convergence in distribution, and then

$$\mathbf{P}(|Z+2| \le 8) = \mathbf{P}(-10 \le Z \le 6) = \Phi(6) - \Phi(10).$$

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ℹ Answers are displayed within the problem

Convergence in probability and variance

3/3 points (graded)

For $n \geq 2$, let X_n be a random variable such that $\mathbf{P}\left(X_n = \frac{1}{n}\right) = 1 - \frac{1}{n^2}$ and $\mathbf{P}(X_n = n) = \frac{1}{n^2}$.

Does X_n converge in probability? If yes, enter the value of the limit; if no, enter DNE.

$X_n \xrightarrow[n\to\infty]{\mathbf{P}}$ 0 ✔ Answer: 0

0

Compute $\lim_{n\to\infty} \mathbb{E}[X_n]$ and $\lim_{n\to\infty} \mathsf{Var}(X_n)$. Enter DNE if the limit diverges or does not exist.

$\lim_{n\to\infty} \mathbb{E}[X_n] =$ 0 ✔ Answer: 0

0

$\lim_{n\to\infty} \mathsf{Var}(X_n) =$ 1 ✔ Answer: 1

1

STANDARD NOTATION

Solution:

$X_n \xrightarrow[n\to\infty]{\mathbf{P}} 0$ in probability: It is enough to check that for every $\varepsilon > 0$, $\mathbf{P}(|X_n| \leq \varepsilon) \to 1$ as $n \to \infty$, which is true since

$$\begin{aligned} \mathbf{P}(|X_n| \leq \varepsilon) = \quad & \mathbf{P}(X_n = n) && \text{if } n > \frac{1}{\varepsilon} \\ = \quad & 1 - \frac{1}{n^2} \to 1 && \text{as } n \to \infty. \end{aligned}$$

Now, compute $\lim_{n\to\infty} \mathbb{E}[X_n]$:

$$\mathbb{E}[X_n] = \frac{1}{n}\left(1 - \frac{1}{n^2}\right) + \frac{n}{n^2} \xrightarrow{n\to\infty} 0.$$

For the variance, the computation yields:

$$\mathsf{Var}(X_n) = \mathbb{E}\left[|X_n|^2\right] = \left(\frac{1}{n}\right)^2 \left(1 - \frac{1}{n^2}\right) + \frac{n^2}{n^2} \xrightarrow{n\to\infty} 1.$$

Remark: Convergence in probability does not necessarily imply convergence in variance.

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ℹ Answers are displayed within the problem

Modes of convergence

3/3 points (graded)

Let X_n and Y_n be two sequences of random variables. For each of the following statement, say whether it is true or false. When your answer is "false", try to think of a counter example.

1. If $X_n \xrightarrow[n\to\infty]{\text{a.s.}} X$ and $Y_n \xrightarrow[n\to\infty]{\text{a.s.}} Y$, then $X_n + Y_n \xrightarrow[n\to\infty]{\text{a.s.}} X + Y$.

- [x] True ✔
- [ ] False

2. If $X_n \xrightarrow[n\to\infty]{\mathbf{P}} X$ and $Y_n \xrightarrow[n\to\infty]{\mathbf{P}} Y$, then $X_n + Y_n \xrightarrow[n\to\infty]{\mathbf{P}} X + Y$.

- [x] True ✔
- [ ] False

3. If $X_n \xrightarrow[n\to\infty]{(\mathrm{d})} X$ and $Y_n \xrightarrow[n\to\infty]{(\mathrm{d})} Y$, then $X_n + Y_n \xrightarrow[n\to\infty]{(\mathrm{d})} X + Y$.

- [ ] True
- [x] False ✔

Solution:

The first statement is true. To prove it, let the variables all be defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. (Remember that this means that Ω denotes an abstract set and we consider all random variables X_n, Y_n, X, Y as functions from Ω to $\mathbb{R}$ that are measurable with respect to the sigma algebra $\mathcal{F}$.) Let $\mathcal{A}$ be the set where the convergence $X_n(\omega) \to X(\omega)$ holds, and similarly $\mathcal{B}$ the set where $Y_n(\omega) \to Y(\omega)$. Then on $\mathbf{P}(\mathcal{A} \cap \mathcal{B}) = 1 - \mathbf{P}(\mathcal{A}^c \cup \mathcal{B}^c) \geq 1 - \mathbf{P}(\mathcal{A}^c) - \mathbf{P}(\mathcal{B}^c)$, but $\mathbf{P}(\mathcal{A}^c) = \mathbf{P}(\mathcal{B}^c) = 0$ by the assumption of almost sure convergence, so $\mathbf{P}(\mathcal{A} \cap \mathcal{B}) = 1$. Therefore, $X_n + Y_n \to X + Y$ almost surely.

The second statement is true as well. To show convergence of $X_n + Y_n$ in probability, let $\varepsilon, \delta > 0$. By definition of this mode of convergence, we can choose n_1 and n_2 such that

$$\mathbf{P}\left(|X_n - X| > \frac{\varepsilon}{2}\right) < \frac{\delta}{2} \qquad \text{if } n \geq n_1 \mathbf{P}\left(|Y_n - Y| > \frac{\varepsilon}{2}\right) < \frac{\delta}{2} \qquad \text{if } n \geq n_2$$

Hence, by triangle inequality and sub-additivity of $\mathbf{P}$, if $n \geq \max\{n_1, n_2\}$, we have

$$\mathbf{P}\left(|X_n + Y_n - (X+Y)| > \varepsilon\right) \leq \mathbf{P}\left(|X_n - X| > \frac{\varepsilon}{2}\right) + \mathbf{P}\left(|Y_n - Y| > \frac{\varepsilon}{2}\right) < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

which shows the desired convergence.

The last statement is not true. The intuition is that random variables can be **coupled** in strange ways to make this statement false. In particular, there can be multiple different constructions of X and Y that exhibit counterexamples. This is an important feature of the definition of random variables as a function on the underlying probability space Ω.

To demonstrate this point, consider the following: let Z and $Z_1, Z_2, \ldots$ be a sequence of i.i.d. standard Gaussian RVs $\mathcal{N}(0,1)$. Using (Z_n), we now define a pair of sequences (X_n) and (Y_n): let $X_n = Z_n$ and $Y_n = -Z_n$. Let $X = Y = Z$. It is clear that $X_n \to Z$ in probability; and (even though it looks bizarre) by symmetry of the Gaussian, $Y_n \to Z$ in probability as well. However, $X_n + Y_n = 0$, so the sequence $(X_n + Y_n)$ converges to the constant 0 in probability. This is decidedly not the same as $X + Y = 2Z$, which has a Gaussian distribution.

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