

Markov processes can be very general. They can run in continuous or discrete time, can have a discrete or a continuous state space. In this class, we'll restrict ourselves to discrete time discrete state Markov chains. These are the simplest cases and are the best to build our intuition. So the state space is discrete, here, finite with m states, and time is discrete. That is, at any discrete point in time, the process is in one of these m states and let's say here, at any given time s . And again, time is discrete, so think about the following process.

You have someone hitting a drum, indicating that a transition occurs. And what it means is that the chain that was here will then jump. Let's say to another state j at the next time. So when the Markov chain jumps it can jump on itself or jump to another state like here or here. And then at time s plus 1 someone hit the drum and you jump again, and so on and so forth. You can think of a very active frog jumping from lilies to lilies on the pond and following a regular drumbeat.

So what is left to define are the various probabilities of transitions, such as the transition from i to j , and the notation we're going to use is p_{ij} , which by definition is that transition probability here. So given that you are in state i at time s , what is the probability that you end up in state j at time s plus 1. Notice that these transition probabilities here, p_{ij} , are not function of s . So irrespective of what the time s that we're talking about, these transition probabilities are the same.

So this is what we mean by a time-homogeneous Markov chain. In other words, these are valid for s equal 0, 1, 2, and so on and so forth. So the defining feature of a Markov chain is the Markov property. And the Markov property essentially says that the past is not really important in order to predict the future, as long as you know where you are now. Another way of saying it is that if you look at the probability of going next in state j given that you are now in state i and that I give you, in addition, the entire trajectories.

So I tell you that it was in i_0 at that time, and so on and so forth, all the way up to time s minus 1, where it was in i_{s-1} . So it gives you the entire trajectory of the chain up to s , and now I'm asking you, what is the probability that you're going to go to s plus 1? The Markov property here simply says that that probability here is again, p_{ij} .

So in other words, all this information here is of no use to compute this probability. Now, note that these

transition probabilities are really probabilities, in the following sense. Right? So you are in i and then at the next time step, you will definitely jump with probability 1. And where you're going to jump will depend, but the summation of all possibilities have to sum up to 1. So from j equals 1 to n has to be 1.

So now that we have introduced the main ingredients, usually we are very interested in knowing what a Markov chain is going to do in the long run. We are interested in finding the probability that the chain is in a state j after n transitions, given that it is now in state i . Now because of the time-homogeneous, this is the same thing as that. In other words, the current time could be in any time s , we just have to add s here. And again, that is nothing else than this property.

So we are interested in calculating $p_{ij}^{(n)}$ for any n . For n equals 1, this is nothing else than p_{ij} of 1 is the same as this transition probabilities that we have defined. But for n greater than or equals to 2, what we are seeing is the introduction of a key recursion here. And this is how you would be able to calculate these probabilities. Now, how did we come up with this recursion? Well, it's based on a classical divide and conquer and essentially, the use of the total property theorem.

Essentially, you have the time step here. This is the current time s . You are interested in what's going to happen at n plus s and n steps later. Here you are in state i . You are interested in knowing what is the probability of being in state j at that time. And what you simply do is you look at the step n plus s minus 1, just before the last one. And then you say, well, let me do a divide and conquer. This is k here, and I'm going to look at evaluating that probability.

And then once I have that, I will simply multiply it by this transition here. And what happened is that this here is nothing else than this calculation that we have here. And that's the same thing. And here, this is the probability of one step transition. And, of course, we have conditioned on the fact that we would be in a state k here, but k could be any of these states, right? And they are m of them, and this is why we saw from k equals 1 to m . So essentially this is how this key recursion has been put together, and we have used, of course, the Markov property in order to do that. Let's do now a little bit of warm up in terms of calculation and apply these concepts.