3. Likelihood Ratio Test

In the problems on this page, we consider a sample $X_1, \ldots, X_n \stackrel{iid}{\sim} \mathrm{ShiftExp}(\lambda, a)$, where $\mathrm{ShiftExp}(\lambda, a)$ is a continuous probability distribution with parameters $\lambda > 0, a \in \mathbb{R}$ and pdf

$$f_{\lambda,a}(x) = \lambda e^{-\lambda(x-a)} \mathbf{1}_{x \geq a}.$$

Likelihood for Shifted Exponential

1/1得分 (计入成绩)

Which of the following is the likelihood function $L(X_1, \ldots, X_n; \lambda, a)$ for the shifted exponential statistical model?

$L(X_1, \ldots, X_n; \lambda, a) =$

- $\lambda^n \exp\left(\lambda \sum_{i=1}^n (X_i - a)\right) \mathbf{1}_{\min(X_i) \geq a}$
- $\lambda^n \exp\left(-\lambda \sum_{i=1}^n (X_i - a)\right)$
- $\lambda^n \exp\left(-\lambda \sum_{i=1}^n (X_i - a)\right) \mathbf{1}_{\min(X_i) \geq a}$ □
- $\exp\left(-\lambda \sum_{i=1}^n (X_i - a)\right) \mathbf{1}_{\min(X_i) \geq a}$

Solution:

By definition, the likelihood is computed to be

$$\begin{aligned} L(X_1, \ldots, X_n; \lambda, a) &= \prod_{i=1}^n \lambda e^{-\lambda(X_i - a)} \mathbf{1}_{X_i \geq a} \\ &= \lambda^n \exp\left(-\lambda \sum_{i=1}^n (X_i - a)\right) \mathbf{1}_{\min_{i=1,\ldots,n}(X_i) \geq a}. \end{aligned}$$

The third choice is correct.

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□ Answers are displayed within the problem

MLE for Shifted Exponential

2/2得分 (计入成绩)

Let $(\hat{\lambda}, \hat{a})$ denote the MLE for the shifted exponential model.

What is $\hat{a}$?

- $\max_{i=1,\ldots,n} X_i$
- $\min_{i=1,\ldots,n} X_i$ □

- $-\min_{i=1,\ldots,n} X_i$
- None of the above.

What is $\hat{\lambda}$? Your answer should be expressed in terms of the sample mean $\overline{X}_n$ and $\hat{a}$.

(Enter **barX_n** for $\overline{X}_n$ and **hata** for $\hat{a}$.)

$\hat{\lambda} =$ 1/(barX_n - hata)

Answer: 1/(barX_n-hata)

STANDARD NOTATION

Solution:

Observe that the likelihood $L = 0$ if $a > \min_i (X_i)$, so let's restrict to $a \leq \min_i (X_i)$. Taking the log, then we need to maximize the function

$$\ell(\lambda, a) := n \ln \lambda - \lambda \sum_{i=1}^{n} X_i + n\lambda a$$

with respect to λ and a.

Since $\lambda > 0$, we see that this function is monotone increasing in a, so we choose a to be as large as possible given the constraint $a \leq \min_i (X_i)$. Accordingly, we set

$$\hat{a} = \min_{i=1,\ldots,n} (X_i).$$

To compute $\hat{\lambda}$, we set $a = \min_{i=1,\ldots,n} (X_i)$, and need to maximize the function

$$f(\lambda) = n \ln \lambda - \lambda \sum_{i=1}^{n} X_i + n\lambda \min_i (X_i).$$

Observe that

$$f'(\lambda) = \frac{n}{\lambda} - \sum_{i=1}^{n} X_i + n \min_i (X_i)$$

which is 0 if we set

$$\lambda = \frac{n}{\sum_{i=1}^{n} (X_i - \min_j (X_j))} = \frac{1}{\overline{X}_n - \hat{a}}.$$

Hence, we have

$$\hat{\lambda} = \frac{1}{\overline{X}_n - \hat{a}}.$$

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Answers are displayed within the problem

Likelihood Ratio Test for Shifted Exponential I

2/2得分 (计入成绩)

While we cannot take the log of a negative number, it makes sense to define the log-likelihood of a shifted exponential to be

$$\ell(\lambda, a) = \left(n \ln \lambda - \lambda \sum_{i=1}^{n} (X_i - a)\right) \mathbf{1}_{\min_i(X_i) \geq a}.$$

We will use this definition in the remaining problems.

Assume now that a is known and that $a = 0$. Consider the hypotheses

$$\begin{aligned} H_0 &: \lambda = 1 \\ H_1 &: \lambda \neq 1. \end{aligned}$$

What is the likelihood-ratio test statistic T_n?

(Enter **barX_n** for $\overline{X}_n$.)

$T_n =$ 2*n*(barX_n-1-ln(barX_n)) **Answer:** 2*(-n*ln(barX_n)-n+n*barX_n)

Assume that Wilks's theorem applies.

What is true about the distribution of T_n?

- T_n is distributed as $N(0,1)$.
- T_n is asymptotically distributed as χ_2^2.
- T_n is distributed as χ_1^2.
- T_n is asymptotically distributed as χ_1^2.

Solution:

Since we are given that $a = 0$ is known, we may write

$$\ell(\lambda, 0) = \left(n \ln \lambda - \lambda \sum_{i=1}^{n} (X_i)\right) \mathbf{1}_{\min_i(X_i) \geq 0} = n \ln \lambda - \lambda \sum_{i=1}^{n} (X_i),$$

because the generated data will certainly satisfy $\min_i (X_i) \geq 0$.

The likelihood-ratio test statistic is

$$\begin{aligned} T_n &= 2\left(\ell(\hat{\lambda}, 0) - \ell(1, 0)\right) \\ &= 2\left(n \ln\left(1/\overline{X}_n\right) - n - 0 + n\overline{X}_n\right) \\ &= 2\left(-n \ln\left(\overline{X}_n\right) - n + n\overline{X}_n\right). \end{aligned}$$

By Wilks's theorem,

$$T_n \xrightarrow[(d)]{n\to\infty} \chi_1^2,$$

because the parameter λ is 1-dimensional.

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Answers are displayed within the problem

Likelihood Ratio Test for Shifted Exponential II

1/1得分 (计入成绩)

In this problem, we assume that $\lambda = 1$ and is known. The parameter $a \in \mathbb{R}$ is now unknown.

As in the previous problem, you should use the following definition of the log-likelihood:

$$\ell(\lambda, a) = \left(n \ln \lambda - \lambda \sum_{i=1}^{n} (X_i - a)\right) \mathbf{1}_{\min_i (X_i) \geq a}.$$

Consider the following null and alternative hypotheses:

$$\begin{aligned} H_0 &: a \leq 1 \\ H_1 &: a > 1. \end{aligned}$$

Assuming that H_0 holds, compute the test statistic $\widetilde{T_n}$ for the log-likelihood ratio test for the above hypotheses.

(Enter **hata** for $\hat{a}$.)

$\widetilde{T_n} =$ 2*n*(hata - 1) **Answer:** 2*n*(hata - 1)

STANDARD NOTATION

Solution:

We compute that

$$\begin{aligned} \widetilde{T_n} &= 2\left(\ell(1, \hat{a}) - \ell(1, 1)\right) \\ &= 2\left(n \ln 1 - (1) \sum_{i=1}^{n} (X_i - \hat{a})\right) \mathbf{1}_{\min_i (X_i) \geq \hat{a}} - 2\left(n \ln 1 - (1) \sum_{i=1}^{n} (X_i - 1)\right) \mathbf{1}_{\min_i (X_i) \geq 1}. \end{aligned}$$

Recall that $\hat{a} = \min_i (X_i)$. Hence, $\mathbf{1}_{\min_i (X_i) \geq \hat{a}} = 1$. Moreover, if $H_0 : a \leq 1$ holds, then $\mathbf{1}_{\min_i (X_i) \geq 1} = 1$. We may further simplify

$$\begin{aligned} \widetilde{T_n} &= 2\left(n \ln 1 - (1) \sum_{i=1}^{n} (X_i - \hat{a})\right) - 2\left(n \ln 1 - (1) \sum_{i=1}^{n} (X_i - 1)\right) \\ &= 2n(\hat{a} - 1). \end{aligned}$$

提交 你已经尝试了4次（总共可以尝试4次）

Answers are displayed within the problem

P-value for Likelihood Ratio Test for Shifted Exponential

1/2得分 (计入成绩)

What is the distribution of

$$\hat{a} = \min_{i=1,\ldots,n} (X_i)$$

assuming that $a = 1$ and $\lambda = 1$?

- ◉ $\text{ShiftExp}\,(n, 1)$
- ○ $\text{ShiftExp}\,(1, 1)$
- ○ $\text{ShiftExp}\,(n, n)$
- ○ $\text{ShiftExp}\,(1, -n)$

Recall the test statistic $\widetilde{T_n}$ from the previous question. Suppose that $n = 100$ and $\widetilde{T_{100}} = 1.03$.

What is the p-value associated to this observation?

0.995

Answer: 0.5975

Solution:

We will compute the cdf of $\hat{a} = \min_i (X_i)$. Observe that by independence,

$$\begin{aligned} P\left(\min_i (X_i) \geq t\right) &= \left(\int_t^\infty e^{-(x-1)}\,dx\right)^n \\ &= e^{-n(t-a)} \\ &= \int_t^\infty -ne^{-n(x-1)}\,dx \\ &= \int_t^\infty f_{n,1}(x)\,dx. \end{aligned}$$

Therefore, $\hat{\mathbf{a}} \sim \textbf{ShiftExp}\,(\mathbf{n}, \mathbf{1})$, if we assume that $a = 1$ and $\lambda = 1$.

For the second question, we use this result to compute the p-value:

$$\begin{aligned} P\left(\widetilde{T_{100}} > 1.03\right) &= P\left(\hat{a} > 1 + \frac{1.03}{200}\right) \\ &= \int_{1+\frac{1.03}{200}}^\infty 100e^{-100(x-1)}\,dx \\ &= e^{-1.03/2} \\ &\approx 0.5975. \end{aligned}$$

提交 你已经尝试了3次（总共可以尝试3次）

Answers are displayed within the problem

讨论

显示讨论

主题: Unit 4 Hypothesis testing:Homework 7 / 3. Likelihood Ratio Test