2. Linear Independence and Rank

Note: Below are exercises from homework 0 that cover the ideas of linear independence, dimensions, and rank, which will be used in this lecture. These exercises were optional in homework 0, but is graded in this unit.

Linear Independence

Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are said to be **linearly dependent** if there exist scalars $c_1, \ldots, c_n$ such that

1. not all c_i's are zero, i.e. there is i such that $c_i \neq 0$;

2. $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = 0$ 方向一致

Vectors that are **not** linearly dependent are said to be **linearly independent** . In other words, vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent if the only scalars $c_1, \ldots, c_n$ such that $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = 0$ are $c_1 = \cdots = c_n = 0$, i.e. 方向不一致

$$\boxed{c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = 0 \quad \Longrightarrow \quad c_i = 0 \qquad \text{for all } i \qquad \text{(linear independence)}\,.}$$

Two non-zero vectors $\mathbf{v}_1,\ \mathbf{v}_2$ are linear dependent if and only if $\mathbf{v}_1 = c\mathbf{v}_2$, i.e. if one is a scalar multiple of the other.

Examples:

1. $\begin{pmatrix}1\\0.5\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}$ are linear dependent.

2. $\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}$ are linear independent.

3. $\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-2\end{pmatrix}$ are linear independent.

4. $\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\-1\end{pmatrix}$ are linear dependent, because

$$\begin{pmatrix}2\\-1\end{pmatrix} = 2\begin{pmatrix}1\\0\end{pmatrix} - 1\begin{pmatrix}0\\1\end{pmatrix}$$

   or written in a more symmetric form:

$$1\begin{pmatrix}2\\-1\end{pmatrix} - 2\begin{pmatrix}1\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\end{pmatrix} = 0.$$

5. $\begin{pmatrix}0\\1\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix}$ are linearly dependent, because

$$\begin{pmatrix}0\\1\\1\\0\\1\end{pmatrix} - \begin{pmatrix}0\\1\\2\\0\\1\end{pmatrix} + \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} = 0.$$

Span and dimension

The collection of non-zero vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^m$ determines a subspace of $\mathbb{R}^m$. This subspace **subspace of** $\mathbb{R}^m$, also known as the **span** of the vector $\mathbf{v}_1, \ldots, \mathbf{v}_n$, is the set of all linear combinations $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ over different choices of $c_1, \ldots, c_n \in \mathbb{R}$. denoted by

$$\boxed{\langle \mathbf{v}_1, \ldots, \mathbf{v}_n \rangle = \{\mathbf{v} \in \mathbb{R}^m : \mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n\} \qquad (\text{span of } \mathbf{v}_1, \ldots, \mathbf{v}_n)\,.}$$

The **dimension** of this subspace $\langle \mathbf{v}_1, \ldots, \mathbf{v}_n \rangle$ is the size of the **largest possible, linearly independent** sub-collection of the (non-zero) vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

Referring to the examples above:

1. $\left\langle \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 \\ 0.5 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle$, that is, either vector spans the entire subspace. Hence, this is a 1-dimensional subspace of $\mathbb{R}^{\boxed{2}}$.

2. $\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = \mathbb{R}^2$, and is 2-dimensional.

3. $\left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\rangle = \mathbb{R}^2$, and is 2-dimensional.

4. $\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\rangle = \mathbb{R}^2$, and again is 2-dimensional.
   That is, any 2 of the 3 given vectors span all of $\mathbb{R}^2$.

5. $$\left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \right\rangle \subset \mathbb{R}^{\boxed{5}}.$$
   That is, any two of the first three vectors along with the fourth vector span the subspace; hence. this is a $\boxed{\text{3-dimensional}}$ subspace of $\mathbb{R}^{\boxed{5}}$.

Rank

The **column space** and **row space** of matrix is the subspace spanned by its columns and its rows respectively. It is a fact from linear algebra that the dimension of the column space of a matrix $\mathbf{M}$ is equal to the dimension of its row space (try to show it by row-reduction). This dimension is the **rank** of the matrix, and denoted $\mathbf{rank}\,(\mathbf{M})$. Note that $\mathbf{rank}\,(\mathbf{M}) = \mathbf{rank}\,(\mathbf{M}^T)$.

Examples

Refer to the examples above. For each example, define a matrix $\mathbf{M}$ whose columns are the given vectors.

1. $\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 0.5 & 1 \end{pmatrix}$ has column rank 1 because the column space $\left\langle \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 \\ 0.5 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle$ is 1-dimensional.
   Check that the row space, $\left\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} \right\rangle$, spanned by the rows of the matrix, is also 1-dimensional.

2. $\mathbf{M} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is of rank 2 since the column space $\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$ is 2-dimensional. The row space and column space are both R^2.

3. $\mathbf{M} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$ is of rank 2 since the column space $\left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\rangle$ is 2-dimensional. The row space and column space of this matrix is equal.

4. $\mathbf{M} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}$ is of rank 2, since the dimension of the column space $\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\rangle$ is 2. The row space $\left\langle \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle$ is a subspace of $\mathbb{R}^3$ of dimension 2.

5. $\mathbf{M} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 4 \\ 1 & 1 & 0 & 5 \end{pmatrix}$ has rank 3. We have seen that tts column space is a 3-dimensional subspace of $\mathbb{R}^5$. This means its row space is also 3-dimensional; it is a subspace of $\mathbb{R}^4$.

Row and Column Rank

2/2 points (graded)

Suppose $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$. The rows of the matrix, $(1, 3)$ and $(2, 6)$, span a subspace of dimension

1

✔ **Answer:** 1 . This is the **row rank** of $\mathbf{A}$.

The columns of the matrix, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$ span a subspace of dimension

1

✔ **Answer:** 1 . This is the **column rank** of $\mathbf{A}$.

We will be using these ideas throughout this lecture, where we will work with larger, possibly rectangular matrices.

Solution:

In both cases, the two vectors are linearly dependent.

$$2 \cdot (1, 3) - (2, 6) = (0, 0)$$

$$3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

The rank of a matrix

3/3 points (graded)

In general, row rank is always equal to the column rank, so we simply refer to this common value as the **rank** of a matrix.

What is the largest possible rank of a 2×2 matrix?

2

✔ **Answer:** 2

What is the largest possible rank of a 5×2 matrix?

2

✔ **Answer:** 2

In general, what is the largest possible rank of an $m \times n$ matrix?

- ○ m
- ○ n
- ◉ $\min(m, n)$ ✔

- $\max(m, n)$
- None of the above

Solution:

In general, the rank of any $m \times n$ matrix can be at most $\min(m, n)$, since rank = column rank = row rank. For example, if there are five columns and three rows, the column rank cannot be larger than the largest possible row rank – the largest possible row rank for three rows is, unsurprisingly, 3. The opposite is also true if there are more rows than columns. If a matrix has two columns and six rows, then the row rank cannot exceed the column rank, which is at most 2.
In general, a matrix $\mathbf{A}$ is said to have **full rank** if $\operatorname{rank}(\mathbf{A}) = \min(m, n)$. (note the $=$, instead of $\leq$).

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

Examples of rank

5/5 points (graded)

What is the rank of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$?

1 Answer: 1

What is the rank of $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$?

2 Answer: 2

What is the rank of $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$?

0 Answer: 0

What is the rank of $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$?

2 Answer: 2

What is the rank of $\begin{pmatrix} 1 & 1 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \end{pmatrix}$?

3 Answer: 3

Solution:

1. The set of rows describe a subspace of dimension 1, spanned by $(1, 1)$.
2. This matrix has rank 2, since $(1, -1)$ and $(1, 0)$ are linearly independent.
3. This matrix has rank zero. By definition, the rank is equal to the number of nonzero linearly independent vectors.
4. The second and third rows are independent. However, the sum of the second and third rows are equal to the first: $(1, 0, 1) + (0, 1, 0) = (1, 1, 1)$. So this matrix has rank 2.
5. All three rows are independent. An easy way to check is to notice that this matrix is **upper triangular** , with nonzero entries along the diagonal.

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

The rank of a matrix continued

2/2 points (graded)

This question is meant to serve as an answer to the following: *If you sum two rank-1 matrices, do you get a rank-2 matrix? What about products? More generally, what rank is the sum of a rank-r_1 and a rank-r_2 matrix*?"

Let $\mathbf{A} = \begin{pmatrix} -1 & 1 \\ -3 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Observe that all four of these matrices are rank 1.

There are many ways to determine rank. Here is one useful fact that you could use for this problem:

"Every rank-1 matrix can be written as an outer product. Conversely, every outer product $\mathbf{uv}^T$ is a rank-1 matrix."

For example, $\mathbf{A} = \mathbf{uv}^T$, $\mathbf{B} = \mathbf{vv}^T$, $\mathbf{C} = \mathbf{ww}^T$ and $\mathbf{D} = \mathbf{xx}^T$, where

$$\mathbf{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Which combination of these matrices has rank 2? Choose all that apply.

- [ ] $\mathbf{A} + \mathbf{A}$
- [ ] $\mathbf{A} + \mathbf{B}$
- [x] $\mathbf{A} + \mathbf{C}$ ✔
- [ ] $\mathbf{AB}$
- [ ] $\mathbf{AC}$
- [ ] $\mathbf{BD}$

✔

Which combination of these matrices has rank 1? Choose all that apply.

- [x] $\mathbf{A} + \mathbf{A}$ ✔
- [x] $\mathbf{A} + \mathbf{B}$ ✔
- [ ] $\mathbf{A} + \mathbf{C}$
- [x] $\mathbf{AB}$ ✔
- [x] $\mathbf{AC}$ ✔
- [ ] $\mathbf{BD}$

✔

Solution:

The choices are of two general types: sums of matrices, and products of matrices.

$v^T v \neq v v^T$

内积 外积

- $\mathbf{A} + \mathbf{A} = 2\mathbf{A}$, which has rank 1.
- $\mathbf{A} + \mathbf{B} = \mathbf{u}\mathbf{v}^T + \mathbf{v}\mathbf{v}^T = (\mathbf{u} + \mathbf{v})\,\mathbf{v}^T$, which has rank 1.
- $\mathbf{A} + \mathbf{C} = \begin{pmatrix} -1 & 1 \\ -3 & 4 \end{pmatrix}$. This has two linearly independent rows, hence its rank is 2.

The last three choices $\mathbf{AB}$, $\mathbf{AC}$, $\mathbf{BD}$ cannot have rank 2 since they are products of rank-1 matrices.

- $\mathbf{AB} = \mathbf{u}\mathbf{v}^T\mathbf{v}\mathbf{v}^T = \mathbf{u}\langle\mathbf{v},\mathbf{v}\rangle\mathbf{v}^T = \langle\mathbf{v},\mathbf{v}\rangle\mathbf{u}\mathbf{v}^T$. Note that the inner product $\boxed{\mathbf{v}^T\mathbf{v} = \langle\mathbf{v},\mathbf{v}\rangle}$ "floats" to the front because it is a scalar. This is an outer product of two vectors, which has rank 1.
- $\mathbf{AC} = \mathbf{u}\mathbf{v}^T\mathbf{w}\mathbf{w}^T = \langle\mathbf{v},\mathbf{w}\rangle\mathbf{u}\mathbf{w}^T$, which again has rank 1.
- $\mathbf{BD} = \mathbf{v}\mathbf{v}^T\mathbf{x}\mathbf{x}^T = \langle\mathbf{v},\mathbf{x}\rangle\mathbf{v}\mathbf{x}^T$. Notice that $\mathbf{v}$ is orthogonal to $\mathbf{x}$, so $\mathbf{BD} = 0\mathbf{v}\mathbf{x}^T$ is the zero matrix. Its rank is zero.

In general, the sum of two matrices can have a varying range of ranks, and they can be greater **or** less than the ranks of matrices that are being summed up. On the other hand, it is a general fact that if $\mathbf{A}$ and $\mathbf{B}$ are arbitrary (possibly rectangular) matrices, $\boxed{\operatorname{rank}(\mathbf{AB}) \le \min(\operatorname{rank}(\mathbf{A}), \operatorname{rank}(\mathbf{B}))}$. It is possible to use **determinants** to reason about rank. For choices such as $\mathbf{A} + \mathbf{B} = \begin{pmatrix} 0 & 0 \\ -2 & 2 \end{pmatrix}$, the rank is obviously 1. Sometimes, it is easier if you know how to factor matrices – in this problem, we gave you the factorizations of rank-1 matrices into outer products of vectors. Other times, one may resort to using Gaussian Elimination – the rank of any upper triangular matrix is **at least** the number of non-zero entries along the diagonal.

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

Invertibility of a matrix

1/1 point (graded)

An $n \times n$ matrix $\mathbf{A}$ is invertible if and only if $\mathbf{A}$ has full rank, i.e. $\operatorname{rank}(\mathbf{A}) = n$.

Which of the following matrices are invertible? Choose all that apply.

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \\ \mathbf{B} &= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \\ \mathbf{C} &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\ \mathbf{D} &= \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \end{aligned}$$

- [ ] $\mathbf{A}$
- [x] $\mathbf{B}$ ✔
- [x] $\mathbf{C}$ ✔
- [ ] $\mathbf{D}$

✔

Solution:

We saw in a previous exercise that the rank of $\mathbf{A}$ is 1. The rank of $\mathbf{B}$ is 2, since $(1, 2)$ and $(2, 1)$ are linearly independent, since e.g. by Gaussian Elimination one obtains the reduced upper triangular matrix $\begin{pmatrix} 1 & 2 \\ 0 & 3/2 \end{pmatrix}$. In general, an upper triangular matrix with nonzero entries along the diagonal has full rank.

By the same reasoning, $\mathbf{C}$ also has full rank. Finally, $\mathbf{D}$ does not have full rank, since $(\text{row } 1) + (\text{row } 2) + (\text{row } 3) = \vec{0}$.