2. Planes

A hyperplane in n dimensions is a $n-1$ dimensional subspace. For instance, a hyperplane in 2-dimensional space can be any line in that space and a hyperplane in 3-dimensional space can be any plane in that space. A hyperplane separates a space into two sides.

In general, a hyperplane in n-dimensional space can be written as $\theta_0 + \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n = 0$. For example, a hyperplane in two dimensions, which is a line, can be expressed as $Ax_1 + Bx_2 + C = 0$.

Using this representation of a plane, we can define a plane given an n-dimensional vector $\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix}$ and offset θ_0. This vector and offset combination would define the plane $\theta_0 + \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n = 0$. One feature of this representation is that the vector θ is normal to the plane.

2. (a)

1/1 point (graded)
Given a d-dimensional vector θ and offset θ_0 which describe a hyperplane p, how many alternative descriptions θ' and θ'_0 are there for p?

- 0
- 1
- ∞ ✔

STANDARD NOTATION

Solution:

Given a normal vector θ and an offset θ_0 that uniquely determine the plane $\theta \cdot x + \theta_0 = 0$, we can scale θ and θ_0 by $\alpha > 0, \alpha \in \mathbb{R}$ without changing the orientation of the plane. Notice that if we only scale the normal $\theta' = \alpha\theta$ without affecting the offset $\theta'_0 = \theta_0$, then for $\alpha > 1$ the value of the θ'_0 must decrease for $\theta' \cdot x + \theta'_0 = 0$. Thus, there is an infinite number of possible parameter vectors that can describe the plane.

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Answers are displayed within the problem

2. (b)

0/1 point (graded)
To check if a vector x is orthogonal to a plane p characterized by θ and θ_0, we check whether

- $x = \alpha\theta$ for some $\alpha \in \mathbb{R}$ ✔
- $x \cdot \theta = 0$
- $x \cdot \theta + \theta_0 = 0$ ✖

STANDARD NOTATION

Solution:

A vector x is orthogonal to the plane if and only if it is collinear with the normal vector θ of the plane.

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Answers are displayed within the problem

2. (c)

1.0/1 point (graded)

Given a point x in n-dimensional space and a hyperplane described by θ and θ_0, find the **signed distance between the hyperplane and** x. This is equal to the perpendicular distance between the hyperplane and x, and is positive when x is on the same side of the plane as θ points and negative when x is on the opposite side.

(Enter **theta_0** for the offset θ_0.)
(To enter the norm of a vector, for instance $||\theta||$, type **norm(theta).**)

(To enter dot product of two vectors, for example $v \cdot w$, use the equivalent definition of the dot product $v^T w$ (or $w^T v$) where v^T is the transpose of the vector v. Type **trans(v)** for the transpose v^T. Then type **trans(v)*w** for the dot product $v^T w = v \cdot w$.)

(trans(theta)*x+theta_0)/norm(theta) **Answer:** (trans(theta)*x+theta_0)/norm(theta)

STANDARD NOTATION

Solution:

The distance from a point x_1 to a plane $\theta \cdot x + \theta_0$ is equal to $|\theta \cdot x_1 + \theta_0|/||\theta||$. If $\theta \cdot x_1 + \theta_0 > 0$, then x_1 belongs to a half-space in the direction of θ. Therefore, we can define the signed distance as:

$$d_{x_1} = \frac{\theta \cdot x_1 + \theta_0}{||\theta||}$$

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Answers are displayed within the problem

2. (d)

1.0/1 point (graded)

Find the expression for the **orthogonal projection** of a point v onto a plane p characterized by θ and θ_0.

(Enter **theta_0** for the offset θ_0.)
(To enter the norm of a vector, for instance $||\theta||$, type **norm(theta).**)
(To enter dot product of two vectors, for example $v \cdot w$, use the equivalent definition of the dot product $v^T w$ (or $w^T v$) where v^T is the transpose of the vector v. Type **trans(v)** for the transposev^T. Then type **trans(v)*w** for the vector product $v^T w = v \cdot w$.)

v - (trans(theta)*v+theta_0)/norm(theta)*(theta/norm(theta))

Answer: v-(((trans(v)*theta)+theta_0)/(norm(theta))^2)*theta

STANDARD NOTATION

Solution:

Since $v - x$ is collinear with the normal, $v - x = \lambda\theta$ for some λ. Also, x lies in the plane, so $\theta \cdot x + \theta_0 = 0$. Solve this to get the value of λ and plug it back to find the orthogonal projection:

$$\boxed{\begin{aligned}(v - \lambda\theta) \cdot \theta + \theta_0 &= 0 \\ \lambda &= \frac{v \cdot \theta + \theta_0}{\|\theta\|^2} \\ x &= v - \frac{v \cdot \theta + \theta_0}{\|\theta\|}\hat{\theta}\end{aligned}}$$

Submit You have used 4 of 5 attempts

Answers are displayed within the problem

2. (e)

4/4 points (graded)

Let p_1 be the hyperplane (a line, since it is 1-dimensional) consisting of the set of points $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for which $3x_1 + x_2 - 1 = 0$.

What is the signed perpendicular distance of point $a = [-1, -1]$ from p_1?

-5/sqrt(10) **Answer:** -5/sqrt(10)

What is the signed perpendicular distance of the origin from p_1?

-1/sqrt(10) **Answer:** -1/sqrt(10)

What is the orthogonal projection of point $a = [-1, -1]$ onto p_1?

First coordinate:

1/2 **Answer:** 1/2

Second coordinate:

-1/2 **Answer:** -1/2

STANDARD NOTATION

Solution:

1. For $a = [-1, -1]^T$ the signed distance is:

$$\frac{\theta \cdot a + \theta_0}{\|\theta\|} = \frac{(3)(-1) + (1)(-1) - 1}{\sqrt{(3)^2 + (1)^2}} = -\frac{5}{\sqrt{10}}$$

2. For $a = [0, 0]^T$ the signed distance is:

$$\frac{\theta \cdot 0 + \theta_0}{\|\theta\|} = \frac{-1}{\sqrt{(3)^2 + (1)^2}} = -\frac{1}{\sqrt{10}}$$

3. For $a = [-1, -1]^T$ the orthogonal projection is:

$$x = v - \frac{v \cdot \theta + \theta_0}{\|\theta\|}\hat{\theta}$$

$$= [-1,-1]^T - \frac{[-1,-1]^T \cdot [3,1]^T + (-1)}{\sqrt{(3)^2 + (1)^2}} [3/\sqrt{10}, 1/\sqrt{10}]^T$$

$$= [1/2, -1/2]^T$$

Submit You have used 2 of 3 attempts

Answers are displayed within the problem

2. (f)

1/1 point (graded)

Consider a hyperplane in a d-dimensional space. If we project a point onto the plane, can we recover the original point from this projection?

no ✔ **Answer:** no

STANDARD NOTATION

Solution:

Given a projection on a plane, there are infinitely many points that project to that point. They all lie along the normal to the plane which passes through that point.

Submit You have used 1 of 1 attempt

Answers are displayed within the problem

Discussion

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Topic: Unit 0. Course Overview, Homework 0, Project 0 (1 week):Homework 0 / 2. Planes

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