edX

1. Convergence in probability

Problem 1. Convergence in probability

8/8 points (graded)

For each of the following sequences, determine whether it converges in probability to a constant. If it does, enter the value of the limit. If it does not, enter the number "999".

1. Let $X_1, X_2, \ldots$ be independent continuous random variables, each uniformly distributed between -1 and 1.

- Let $U_i = \dfrac{X_1 + X_2 + \cdots + X_i}{i}$, $i = 1, 2, \ldots$. What value does the sequence U_i converge to in probability? (If it does not converge, enter the number "999". Similarly in all below.)

  0

  Answer: 0

- Let $\Sigma_i = X_1 + X_2 + \cdots + X_i$, $i = 1, 2, \ldots$. What value does the sequence Σ_i converge to in probability?

  999

  Answer: 999

- Let $I_i = 1$ if $X_i \geq 1/2$, and $I_i = 0$, otherwise. Define,

$$S_i = \frac{I_1 + I_2 + \cdots + I_i}{i}.$$

  What value does the sequence S_i converge to, in probability?

  1/4

  Answer: 0.25

- Let $W_i = \max\{X_1, \ldots, X_i\}$, $i = 1, 2, \ldots$. What value does the sequence W_i converge to in probability?

  1

  Answer: 1

- Let $V_i = X_1 \cdot X_2 \cdots X_i$, $i = 1, 2, \ldots$. What value does the sequence V_i converge to in probability?

  0

  Answer: 0

2. Let $X_1, X_2, \ldots,$ be independent identically distributed random variables with $\mathbf{E}[X_i] = 2$ and Var$(X_i) = 9$, and let $Y_i = X_i / 2^i$.

- What value does the sequence Y_i converge to in probability?

  0

  Answer: 0

- Let $A_n = \dfrac{1}{n} \displaystyle\sum_{i=1}^{n} Y_i$. What value does the sequence A_n converge to in probability?

  0

  Answer: 0

-

Let $Z_i = \frac{1}{3}X_i + \frac{2}{3}X_{i+1}$ for $i = 1, 2, \ldots$, and let $M_n = \frac{1}{n}\sum_{i=1}^{n} Z_i$ for $n = 1, 2, \ldots$. What value does the sequence M_n converge to in probability?

2

✔ **Answer:** 2

Solution:

1.

- The sequence U_i converges to 0. From the weak law of large numbers, we have convergence in probability to $\mathbf{E}[X_i]$, which is zero in this case.

- The sequence S_i does not converge in probability to any number. Let $\Sigma_n = X_1 + \cdots + X_n$, where the X_i are i.i.d. uniform random variables. Suppose that Σ_n converges, in probability, to a constant c. It then follows that Σ_{n-1} also converges, in probability, to a constant c. But this implies that $X_n = \Sigma_n - \Sigma_{n-1}$ converges in probability to $c - c = 0$, where we are using a fact shown in the additional theoretical material. But the sequence X_n does not converge to zero in probability. This contradiction establishes that Σ_n does not converge.

- Observe that, I_i's are i.i.d. random variables, and $\mathbf{P}(I_i = 1) = \mathbf{P}(X_i \geq 1/2) = 1/4$. Therefore, $\mathbf{E}[I_i] \triangleq \mu = 1$, hence, S_i converges to μ in probability, by the weak law of large numbers.

- The sequence converges to 1. Since $-1 \leq W_i \leq 1$, we have $|W_i - 1| \leq 2$ and so for $\epsilon > 2$, we trivially have $\lim_{i\to\infty} \mathbf{P}(|W_i - 1| \geq \epsilon) = \lim_{i\to\infty} 0 = 0$.

  Assuming $\epsilon \in (0, 2]$, we have,

$$\begin{aligned}
\lim_{i\to\infty} \mathbf{P}(|W_i - 1| \geq \epsilon) &= \lim_{i\to\infty} \mathbf{P}(1 - W_i \geq \epsilon) \\
&= \lim_{i\to\infty} \mathbf{P}(W_i \leq 1 - \epsilon) \\
&= \lim_{i\to\infty} \mathbf{P}(\max\{X_1, \ldots, X_i\} \leq 1 - \epsilon) \\
&= \lim_{i\to\infty} \mathbf{P}(X_1 \leq 1 - \epsilon) \cdots \mathbf{P}(X_i \leq 1 - \epsilon) \\
&= \lim_{i\to\infty} \left(1 - \frac{\epsilon}{2}\right)^i \\
&= 0.
\end{aligned}$$

- The sequence converges to 0. Note that $|X_k| \leq 1$ for all k, and so $|V_i| = |X_1||X_2|\cdots|X_i| \leq \min\{|X_1|, |X_2|, \ldots, |X_i|\} \leq 1$.

  Hence, for any $\epsilon > 1$, we trivially have $\lim_{i\to\infty} \mathbf{P}(|V_i - 0| \geq \epsilon) = \lim_{i\to\infty} 0 = 0$.

  For $\epsilon \in (0, 1]$, we have

$$\begin{aligned}
\lim_{i\to\infty} \mathbf{P}(|V_i - 0| \geq \epsilon) &= \lim_{i\to\infty} \mathbf{P}(|X_1 X_2 \cdots X_i| \geq \epsilon) \\
&= \lim_{i\to\infty} \mathbf{P}(|X_1||X_2|\cdots|X_i| \geq \epsilon) \\
&\leq \lim_{i\to\infty} \mathbf{P}(\min\{|X_1|, |X_2|, \ldots, |X_i|\} \geq \epsilon) \\
&= \lim_{i\to\infty} \mathbf{P}(|X_1| \geq \epsilon)\mathbf{P}(|X_2| \geq \epsilon) \cdots \mathbf{P}(|X_i| \geq \epsilon) \\
&= \lim_{i\to\infty} (1 - \epsilon)^i \\
&= 0.
\end{aligned}$$

2.

- The sequence converges to 0. We have $\mathbf{E}[Y_i] = \mathbf{E}[X_i]/2^i = 2/2^i = 1/2^{i-1}$ and $\mathsf{Var}(Y_i) = \mathsf{Var}(X_i)/(2^i)^2 = 9/2^{2i}$. By the Chebyshev inequality, for any $\epsilon > 0$,

$$\mathbf{P}\left(\left|Y_i - \frac{1}{2^{i-1}}\right| \geq \epsilon\right) \leq \frac{9}{2^{2i} \cdot \epsilon^2}.$$

  Taking the limit as $i \to \infty$, we have

$$\lim_{i\to\infty} \mathbf{P}(|Y_i - 0| \geq \epsilon) = 0.$$

- The sequence converges to 0. We have,

$$\begin{aligned}
\mathbf{E}[A_n] &= \left[\frac{1}{n}\sum_{i=1}^{n} Y_i\right] \\
&= \frac{1}{n}\left[\sum_{i=1}^{n} \frac{X_i}{2^i}\right] \\
&= \frac{1}{n}\left(\sum_{i=1}^{n} \frac{2}{2^i}\right) \\
&= \frac{1}{n}\left(2 - \frac{2}{2^n}\right),
\end{aligned}$$

and

$$\begin{aligned}
\text{Var}(A_n) &= \text{Var}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) \\
&= \frac{1}{n^2}\text{Var}\left(\sum_{i=1}^{n} \frac{X_i}{2^i}\right) \\
&= \frac{1}{n^2}\left(\sum_{i=1}^{n} \frac{9}{2^{2i}}\right) \\
&= \frac{1}{n^2}\left(3 - \frac{3}{2^{2n}}\right).
\end{aligned}$$

Note that $\lim_{n\to\infty} \mathbf{E}[A_n] = 0$ and $\lim_{n\to\infty} \text{Var}(A_n) = 0$.

By the Chebyshev inequality, for any $\epsilon > 0$,

$$\mathbf{P}\left(\left|A_n - \frac{1}{n}\left(2 - \frac{2}{2^n}\right)\right| \geq \epsilon\right) \leq \frac{1}{n^2\epsilon^2}\left(3 - \frac{3}{2^{2n}}\right).$$

Taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \mathbf{P}(|A_n - 0| \geq \epsilon) = 0.$$

- The sequence converges to 2. Note that

$$M_n = \frac{1}{3}\cdot\frac{1}{n}\sum_{i=1}^{n} X_i + \frac{2}{3}\cdot\frac{1}{n}\sum_{i=1}^{n} X_{i+1}.$$

By the weak law of large numbers, the first term converges in probability to $(1/3)\cdot\mathbf{E}[X_i]$ and the second term converges in probability to $(2/3)\cdot\mathbf{E}[X_i]$. As discussed in lecture, if two sequences of random variables each converge in probability, then their sum also converges in probability to the sum of the two limits. Therefore, M_n converges in probability to $(1/3)\cdot\mathbf{E}[X_i] + (2/3)\cdot\mathbf{E}[X_i] = 2$.