

4. Interlude: Square Roots of Matrices

Interlude: Square root of a positive semi-definite matrix

Recall that a matrix \mathbf{A} of size $d \times d$ is **positive semi-definite** if $\mathbf{x}^T \mathbf{A} \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^d$. Two example classes of positive semi-definite matrices are:

- Diagonal matrices with non-negative entries: $\mathbf{D} = \begin{pmatrix} c_1 & 0 & \dots & 0 \\ 0 & c_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & c_d \end{pmatrix}$ where $c_i \geq 0$ for all i . (You have shown in exercise in a previous lecture that indeed $\mathbf{x}^T \mathbf{D} \mathbf{x} \geq 0$ for all \mathbf{x} .)
- Matrix products $\mathbf{P}^T \mathbf{D} \mathbf{P}$ where \mathbf{P} is an invertible (square) matrix and \mathbf{D} is a diagonal matrix with non-negative entries (as above).
Proof: $\mathbf{x}^T (\mathbf{P}^T \mathbf{D} \mathbf{P}) \mathbf{x} = (\mathbf{P} \mathbf{x})^T \mathbf{D} (\mathbf{P} \mathbf{x}) = \mathbf{y}^T \mathbf{D} \mathbf{y} \geq 0$ for all vectors \mathbf{x} .

The **positive semi-definite square root** (or simply the square root) of a positive semi-definite matrix \mathbf{A} is another positive semi-definite matrix, denoted by $\mathbf{A}^{1/2}$, satisfying $\mathbf{A}^{1/2} \mathbf{A}^{1/2} = \mathbf{A}$. It is the case that for any positive semi-definite matrix (positive definite matrix, respectively), the positive semi-definite square root (positive definite square root, respectively) is unique.

Square Root of a Matrix

1/1 point (graded)

Using the definition above of the square root of a matrix, find the square root $\mathbf{D}^{1/2}$ of $\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$.

(Enter your answer as a matrix, e.g. by typing `[[1,2],[5,1]]` for the matrix $\begin{pmatrix} 1 & 2 \\ 5 & 1 \end{pmatrix}$. Note the square brackets, and the commas as separators.)

$\mathbf{D}^{1/2} =$



Answer: `[[sqrt(2),0],[0,0]]`

[STANDARD NOTATION](#)

Solution:

Since

$$\begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$

We have $\mathbf{D}^{1/2} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}$.

提交

你已经尝试了1次 (总共可以尝试3次)

$$\mathbf{A} = \mathbf{P}^T \mathbf{D} \mathbf{P} \quad \text{where} \quad \mathbf{D} = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\mathbf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Note that $\mathbf{P}^T = \mathbf{P}^{-1}$.

Find the square root $\mathbf{A}^{1/2}$ of the matrix \mathbf{A} .
Hint: $\mathbf{P}^T \mathbf{B}^2 \mathbf{P} = \mathbf{P}^T \mathbf{B} (\mathbf{P} \mathbf{P}^T) \mathbf{B} \mathbf{P}$.

(Enter your answer as a matrix, e.g. by typing `[[1,2],[5,-1]]` for the matrix $\begin{pmatrix} 1 & 2 \\ 5 & -1 \end{pmatrix}$. Note the square brackets, and commas as separators.)

$$\mathbf{A}^{1/2} =$$

`[[1.22474,1.22474*sqrt(-1)],[1.22474*sqrt(-1),1.22474]]`

Answer: sqrt(3)/2*[[1,-1],[-1,1]]

STANDARD NOTATION

Solution:

$$\begin{aligned} (\mathbf{P}^T \mathbf{D}^{1/2} \mathbf{P}) (\mathbf{P}^T \mathbf{D}^{1/2} \mathbf{P}) &= \mathbf{P}^T \mathbf{D}^{1/2} (\mathbf{P} \mathbf{P}^T) \mathbf{D}^{1/2} \mathbf{P} \\ &= \mathbf{P}^T \mathbf{D}^{1/2} (\mathbf{P} \mathbf{P}^{-1}) \mathbf{D}^{1/2} \mathbf{P} \quad \text{since } \mathbf{P}^T = \mathbf{P}^{-1} \\ &= \mathbf{P}^T \mathbf{D}^{1/2} \mathbf{D}^{1/2} \mathbf{P} \\ &= \mathbf{P}^T \mathbf{D} \mathbf{P} \end{aligned}$$

Hence $\mathbf{A}^{1/2} = \mathbf{P}^T \mathbf{D}^{1/2} \mathbf{P}$. Plugging in the values of \mathbf{D} and \mathbf{P} , we get

$$\begin{aligned} \mathbf{A}^{1/2} = \mathbf{P}^T \mathbf{D}^{1/2} \mathbf{P} &= \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \right) \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 0 \end{pmatrix} \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \right) \\ &= \frac{1}{2} \begin{pmatrix} \sqrt{3} & 0 \\ -\sqrt{3} & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{\sqrt{3}}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{aligned}$$

提交

你已经尝试了3次（总共可以尝试3次）

☐ Answers are displayed within the problem

讨论

显示讨论

主题： Unit 4 Hypothesis testing:Lecture 14: Wald's Test, Likelihood Ratio Test, and Implicit Hypothesis Test / 4. Interlude: Square Roots of Matrices