3. PMF, expectation, and variance

Problem 3. PMF, expectation, and variance

6.0/6.0 points (graded)

The random variables X and Y have the joint PMF

$$p_{X,Y}(x,y) = \begin{cases} c\cdot(x+y)^2, & \text{if } x \in \{1,2,4\} \text{ and } y \in \{1,3\}, \\ 0, & \text{otherwise.} \end{cases}$$

All answers in this problem should be numerical.

1. Find the value of the constant c.

   $c =$ 1/128 ✔ **Answer:** 1/128

2. Find $\mathbf{P}(Y < X)$.

   $\mathbf{P}(Y < X) =$ 83/128 ✔ **Answer:** 83/128

3. Find $\mathbf{P}(Y = X)$.

   $\mathbf{P}(Y = X) =$ 4/128 ✔ **Answer:** 4/128

4. Find the following probabilities.

   (*Hint*: To avoid double jeopardy with later problem sets, the answers are $\frac{74}{128}$, $\frac{34}{128}$, $\frac{20}{128}$, 0, not necessarily in that order.)

   $\mathbf{P}(X = 1) =$ 20/128 ✔ **Answer:** 20/128

   $\mathbf{P}(X = 2) =$ 34/128 ✔ **Answer:** 34/128

   $\mathbf{P}(X = 3) =$ 0 ✔ **Answer:** 0

   $\mathbf{P}(X = 4) =$ 74/128 ✔ **Answer:** 74/128

5. Find the expectations $\mathbf{E}[X]$ and $\mathbf{E}[XY]$.

$\mathbf{E}[X] =$ 384/128 Answer: 3

$\mathbf{E}[XY] =$ 57/8 Answer: 227/32

6. Find the variance of X.

Var$(X) =$ 1.46875 Answer: 47/32

Solution:

1. From the joint PMF, there are six (x, y) pairs with nonzero probability mass. These pairs are $(1,1), (1,3), (2,1), (2,3), (4,1), (4,3)$. Because the probability of the entire sample space must equal 1, we have:

$$c(1+1)^2 + c(1+3)^2 + c(2+1)^2 + c(2+3)^2 + c(4+1)^2 + c(4+3)^2 = 1.$$

Solving for c, we get $c = \frac{1}{128}$.

2. There are three possible outcomes for which $y < x$: $(2,1), (4,1), (4,3)$.

$$\mathbf{P}(Y < X) = p_{X,Y}(2,1) + p_{X,Y}(4,1) + p_{X,Y}(4,3) = \frac{9}{128} + \frac{25}{128} + \frac{49}{128} = \frac{83}{128}.$$

3. There is only one possible outcome for which $y = x$: $(1,1)$.

$$\mathbf{P}(Y = X) = p_{X,Y}(1,1) = \frac{4}{128}.$$

4. We use the formula $p_X(x) = \sum_y p_{X,Y}(x,y)$.

For example, $p_X(2) = p_{X,Y}(2,1) + p_{X,Y}(2,3) = \frac{34}{128}$. More generally, we find that

$$p_X(x) = \begin{cases} 20/128, & \text{if } x = 1, \\ 34/128, & \text{if } x = 2, \\ 74/128, & \text{if } x = 4, \\ 0, & \text{otherwise.} \end{cases}$$

5. We have

$$\mathbf{E}[X] = \sum_x x p_X(x) = 1 \cdot \frac{20}{128} + 2 \cdot \frac{34}{128} + 4 \cdot \frac{74}{128} = 3.$$

Using the expected value rule,

$$\begin{aligned}\mathbf{E}[XY] &= \sum_x \sum_y xy p_{X,Y}(x,y) \\ &= 1 \cdot \frac{4}{128} + 2 \cdot \frac{9}{128} + 4 \cdot \frac{25}{128} + 3 \cdot \frac{16}{128} + 6 \cdot \frac{25}{128} + 12 \cdot \frac{49}{128} \\ &= \frac{227}{32}.\end{aligned}$$

6. The variance of a random variable X can be computed as $\mathbf{E}[X^2] - (\mathbf{E}[X])^2$ or as $\mathbf{E}[(X - \mathbf{E}[X])^2]$. We use the second approach here. We have

$$\mathrm{Var}(X) = (1-3)^2 \frac{20}{128} + (2-3)^2 \frac{34}{128} + (4-3)^2 \frac{74}{128} = \frac{47}{32}.$$

提交 You have used 2 of 5 attempts

Answers are displayed within the problem

讨论

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