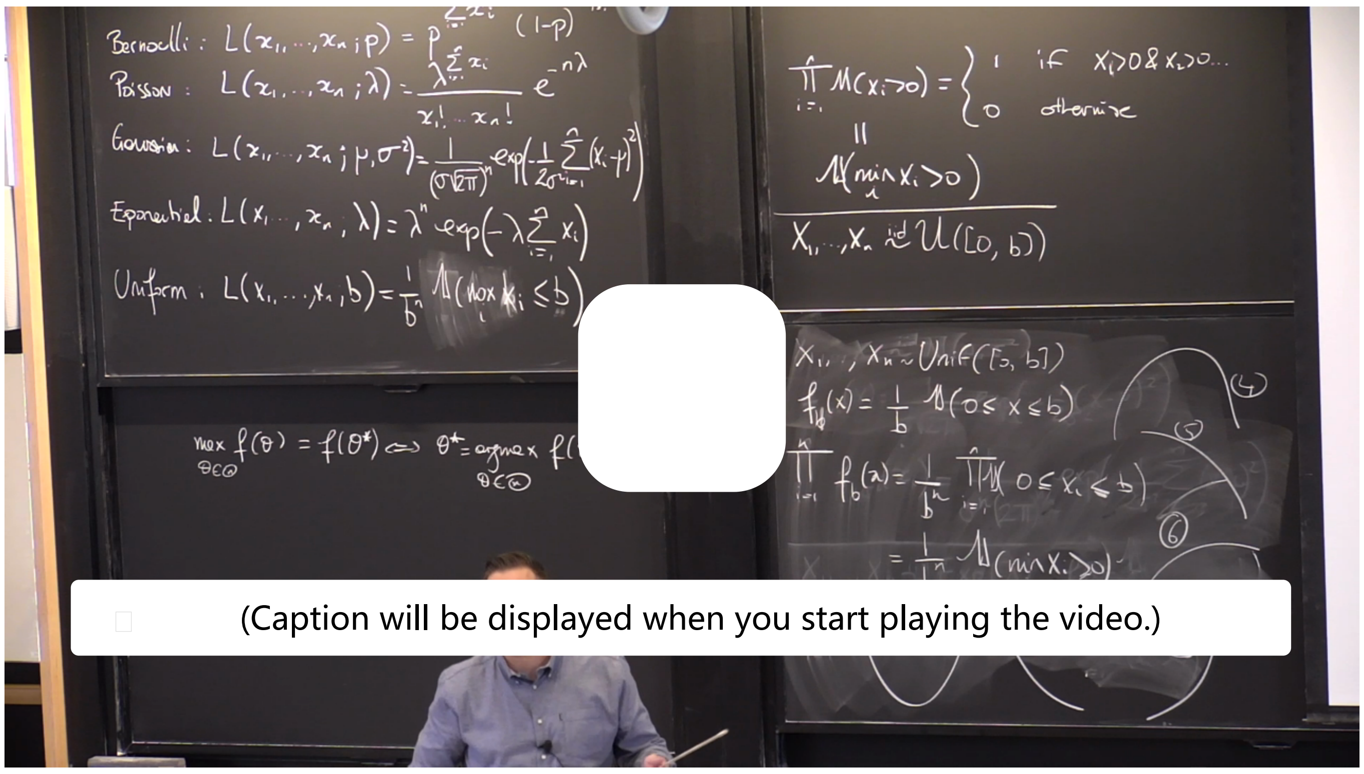
8. Review: Gradients and Hessians; Concavity in Higher dimensions

Concavity in Higher Dimensions: Gradients, Hessians, Semi-Definiteness



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So here, you can see we have one example where

the function we're looking at is actually

a function of two parameters-- one living on the real line

and one living on the positive real line.

And so we're going to have to talk about convex and concave

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Multivariable Calculus Review: Compute the Gradient

1/1 point (graded)

Let

$$\begin{aligned} f: \quad & \mathbb{R}^d \quad \to \mathbb{R} \\ \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_d \end{pmatrix} & \mapsto f(\theta) . \end{aligned}$$

denote a **differentiable** function. The **gradient** of f is the vector-valued function

$$\begin{aligned} \nabla f: \quad & \mathbb{R}^d \quad \to \mathbb{R}^d \\ \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_d \end{pmatrix} & \mapsto \left. \begin{pmatrix} \frac{\partial f}{\partial \theta_1} \\ \frac{\partial f}{\partial \theta_2} \\ \vdots \\ \frac{\partial f}{\partial \theta_d} \end{pmatrix} \right|_{\theta} . \end{aligned}$$

Consider $f(\theta) = -c_1\theta_1^2 - c_2\theta_2^2 - c_3\theta_3^2$ where $c_1, c_2, c_3 > 0$ are positive real numbers.

Compute the gradient ∇f.

(Enter your answer as a vector, e.g., type **[3,2,x]** for the vector $\begin{pmatrix} 3 \\ 2 \\ x \end{pmatrix}$. Note the square brackets, and commas as separators. Enter **c_i** for c_i, **theta_i** for θ_i.)

$\nabla f =$ [-2*c_1*theta_1,-2*c_2*theta_2,-2*c_3*theta_3]

Answer: [-2*c_1*theta_1,-2*c_2*theta_2,-2*c_3*theta_3]

STANDARD NOTATION

Solution:

$$\begin{aligned} f(\theta) &= -c_1\theta_1^2 - c_2\theta_2^2 - c_3\theta_3^2 \\ \nabla f(\theta) &= \left.\begin{pmatrix} \frac{\partial f}{\partial \theta_1} \\ \frac{\partial f}{\partial \theta_2} \\ \frac{\partial f}{\partial \theta_3} \end{pmatrix}\right|_{\theta} = \begin{pmatrix} -2c_1\theta_1 \\ -2c_2\theta_2 \\ -2c_3\theta_3 \end{pmatrix}. \end{aligned}$$

提交 你已经尝试了1次（总共可以尝试3次）

Answers are displayed within the problem

Multivariable Calulus Review: Compute the Hessian Matrix

1/1 point (graded)
As above, let

$$f:\ \mathbb{R}^d \to \mathbb{R} \quad \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_d \end{pmatrix} \mapsto f(\theta).$$

denote a **twice-differentiable** function.

The **Hessian** of f is the matrix

$$\mathbf{H} f:\ \mathbb{R}^d \to \mathbb{R}^{d\times d}$$

whose entry in the i-th row and j-th column is defined by

$$(\mathbf{H} f)_{ij} := \frac{\partial^2}{\partial\theta_i\partial\theta_j} f, \quad 1 \le i,j \le d.$$

Consider the same function $f(\theta) = -c_1\theta_1^2 - c_2\theta_2^2 - c_3\theta_3^2$ where $c_1, c_2, c_3 > 0$ as in the previous problem. Compute the Hessian matrix $\mathbf{H} f$.

(Enter your answer as a matrix, e.g. by typing **[[1,2],[5*x,y-1]]** for the matrix $\begin{pmatrix} 1 & 2 \\ 5x & y-1 \end{pmatrix}$. Note the square brackets, and commas as separaters.)

$\mathbf{H} f =$ [[-2*c_1,0,0],[0,-2*c_2,0],[0,0,-2*c_3]]

Answer: [[-2*c_1,0,0],[0,-2*c_2,0],[0,0,-2*c_3]]

STANDARD NOTATION

Solution:

Recall from the previous problem:

$$
\begin{aligned}
f(\theta) &= -c_1\theta_1^2 - c_2\theta_2^2 - c_3\theta_3^2 \\
\nabla f(\theta) &= \left.\begin{pmatrix} \frac{\partial f}{\partial \theta_1} \\ \frac{\partial f}{\partial \theta_2} \\ \frac{\partial f}{\partial \theta_3} \end{pmatrix}\right|_{\theta} = \begin{pmatrix} -2c_1\theta_1 \\ -2c_2\theta_2 \\ -2c_3\theta_3 \end{pmatrix}.
\end{aligned}
$$

One way to compute the Hessian is to start will in j-th column of the Hessian matrix by the gradient of the j-th component of ∇f. We obtain:

$$
\begin{aligned}
f(\theta) &= -c_1\theta_1^2 - c_2\theta_2^2 - c_3\theta_3^2 \\
\mathbf{H}f(\theta) &= \begin{pmatrix} | & | & | \\ \nabla(-2c_1\theta_1) & \nabla(-2c_2\theta_2) & \nabla(-2c_3\theta_3) \\ | & | & | \end{pmatrix} \\
&= \begin{pmatrix} -2c_1 & 0 & 0 & \\ 0 & 2c_2 & 0 & \\ 0 & 0 & & -2c_3 \end{pmatrix}.
\end{aligned}
$$

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Answers are displayed within the problem

Semi-Definiteness

2/3 points (graded)

A symmetric (real-valued) $d \times d$ matrix $\mathbf{A}$ is **positive semi-definite** if

$$\mathbf{x}^T \mathbf{A}\,\mathbf{x} \geq 0 \qquad \text{for all } \mathbf{x} \in \mathbb{R}^d.$$

If the inequality above is strict, i.e. if $\mathbf{x}^T \mathbf{A}\,\mathbf{x} > 0$ for all non-zero vectors $\mathbf{x} \in \mathbb{R}^d$, then $\mathbf{A}$ is **positive definite** .

Analogously, a symmetric (real-valued) $d \times d$ matrix $\mathbf{A}$ is **negative semi-definite** (*resp.* **negative definite**) if $\mathbf{x}^T \mathbf{A}\,\mathbf{x}$ is **non-positive** (*resp.* **negative**) for all $\mathbf{x} \in \mathbb{R}^d - \{\mathbf{0}\}$.

Note that by definition, positive (or negative) definiteness implies positive (or negative) semi-definiteness.

Consider the same function as in the problems above:

$$f(\theta) = -c_1\theta_1^2 - c_2\theta_2^2 - c_3\theta_3^2 \quad \text{where } c_1, c_2, c_3 > 0.$$

Compute $\mathbf{x}^T (\mathbf{H}f)\,\mathbf{x}$ where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

$\mathbf{x}^T (\mathbf{H}f)\,\mathbf{x} =$ -2*c_1*x_1^2 - 2*c_2*x_2^2 - 2*c_3*x_3^2

Answer: -2*c_1*x_1^2-2*c_2*x_2^2-2*c_3*x_3^2

$-2 \cdot c_1 \cdot x_1^2 - 2 \cdot c_2 \cdot x_2^2 - 2 \cdot c_3 \cdot x_3^2$

The matrix $\mathbf{H}f$ is (Choose all that apply.)

- [ ] positive semi-definite
- [ ] positive definite
- [ ] negative semi-definite
- [x] negative definite

Hence, the function f is
(Choose all that apply.)

- [x] concave
- [x] strictly concave
- [ ] convex
- [ ] strictly convex

Solution:

Recall from the previous problem that

$$\mathbf{H}f(\theta) = \begin{pmatrix} -2c_1 & 0 & 0 \\ 0 & 2c_2 & 0 \\ 0 & 0 & -2c_3 \end{pmatrix}.$$

Then

$$\begin{aligned} \mathbf{x}^T(\mathbf{H}f)\,\mathbf{x} &= \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} -2c_1 & 0 & 0 & \\ 0 & 2c_2 & 0 & \\ 0 & 0 & & 2c_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= -2c_1x_1^2 - 2c_2x_2^2 - 2c_3x_3^2 < 0. \end{aligned}$$

Since $c_1, c_2, c_3 > 0$, this means the $\mathbf{H}f$ is negative definite, (also negative semi-definite), and hence f is strictly concave (also concave).

提交 你已经尝试了2次（总共可以尝试2次）

Answers are displayed within the problem

讨论

显示讨论

主题: Unit 3 Methods of Estimation:Lecture 9: Introduction to Maximum Likelihood Estimation / 8. Review: Gradients and Hessians; Concavity in Higher dimensions