4. Proving binomial identities via counting

Problem 4. Proving binomial identities via counting

4/4 points (graded)

Binomial identities (i.e., identities involving binomial coefficients) can often be proved via a counting interpretation. For each of the binomial identities given below, select the counting problem that can be used to prove it.

Hint: You may find it useful to review the lecture exercise on counting committees before attempting the problem.

(You need to answer all 4 questions before you can submit.)

1. $n\binom{2n}{n} = 2n\binom{2n-1}{n-1}$.

- ○ In a group of $2n$ people, consisting of n boys and n girls, we want to select a committee of n people. In how many ways can this be done?
- ○ How many subsets does a set with $2n$ elements have?
- ○ Out of n people, we want to form a committee consisting of a chair and other members. We allow the committee size to be any integer in the range $1, 2, \ldots, n$. How many choices do we have in selecting a committee-chair combination?
- ◉ Out of $2n$ people, we want to choose a committee of n people, one of whom will be its chair. In how many different ways can this be done? ✔

2. $\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2 = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i}$.

- ◉ In a group of $2n$ people, consisting of n boys and n girls, we want to select a committee of n people. In how many ways can this be done? ✔
- ○ How many subsets does a set with $2n$ elements have?

- Out of n people, we want to form a committee consisting of a chair and other members. We allow the committee size to be any integer in the range $1, 2, \ldots, n$. How many choices do we have in selecting a committee-chair combination?

- Out of $2n$ people, we want to choose a committee of n people, one of whom will be its chair. In how many different ways can this be done?

3. $$2^{2n} = \sum_{i=0}^{2n} \binom{2n}{i}.$$

- In a group of $2n$ people, consisting of n boys and n girls, we want to select a committee of n people. In how many ways can this be done?

- (selected) How many subsets does a set with $2n$ elements have? ✔

- Out of n people, we want to form a committee consisting of a chair and other members. We allow the committee size to be any integer in the range $1, 2, \ldots, n$. How many choices do we have in selecting a committee-chair combination?

- Out of $2n$ people, we want to choose a committee of n people, one of whom will be its chair. In how many different ways can this be done?

4. $$n2^{n-1} = \sum_{i=0}^{n} \binom{n}{i} i.$$

- In a group of $2n$ people, consisting of n boys and n girls, we want to select a committee of n people. In how many ways can this be done?

- How many subsets does a set with $2n$ elements have?

- (selected) Out of n people, we want to form a committee consisting of a chair and other members. We allow the committee size to be any integer in the range $1, 2, \ldots, n$. How many choices do we have in selecting a committee-chair combination? ✔

- Out of $2n$ people, we want to choose a committee of n people, one of whom will be its chair. In how many different ways can this be done?

Solution:

1. “Out of $2n$ people, we want to choose a committee of n people, one of whom will be its chair. In how many different ways can this be done?" The reasoning is as follows.

   Among $2n$ people, we can select n people in $\binom{2n}{n}$ different ways. Having selected n such people, a chair can be selected in n different ways, leading to an overall count of $n\binom{2n}{n}$. Arguing alternatively, we can first select a chair in $2n$ different ways, and then, among the remaining $2n-1$ people, $n-1$ people can be selected in $\binom{2n-1}{n-1}$ different ways. Thus, the overall count is $2n\binom{2n-1}{n-1}$, proving that,

$$n\binom{2n}{n} = 2n\binom{2n-1}{n-1}.$$

2. “In a group of $2n$ people, consisting of n boys and n girls, we want to select a committee of n people. In how many ways can this be done?" The reasoning is as follows.

   Among $2n$ people, n people can be selected in $\binom{2n}{n}$ different ways. Alternatively, the committee can consist of i boys, and $n-i$ girls, for $i = 0, 1, 2, \ldots, n$. For each i, the number of committees with i boys and $n-i$ girls is $\binom{n}{i}\binom{n}{n-i}$. Hence,

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i}.$$

3. “How many subsets does a set with $2n$ elements have?" The reasoning is as follows.

   The total number of all subsets of a set of $2n$-elements is 2^{2n}. Arguing differently, we can consider the number of subsets with i elements, which is $\binom{2n}{i}$, and then sum over all i, proving that

$$2^{2n} = \sum_{i=0}^{2n} \binom{2n}{i}.$$

4. “Out of n people, we want to form a committee consisting of a chair and other members. We allow the committee size to be any integer in the range $1, 2, \ldots, n$. How many choices do we have in selecting a committee-chair combination?". The reasoning is as follows.

   Among n people, we first select a chair in n different ways. Having fixed the chair, each one of the remaining $n-1$ people can either belong to the committee or not, yielding 2^{n-1} choices. Multiplying these two numbers, we obtain, $n2^{n-1}$ for the overall count.

Arguing differently, we can first count the number of committees with i people (one of which is the chair). There are $\binom{n}{i}$ choices for the members. Once the members are chosen, there are i choices for the chair, leading to an overall count (for fixed i) of $\binom{n}{i} i$. Then, summing over i gives us the desired number of committees. Hence,

$$n2^{n-1} = \sum_{i=0}^{n} \binom{n}{i} i.$$

提交 You have used 1 of 2 attempts

Answers are displayed within the problem

讨论

显示讨论

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