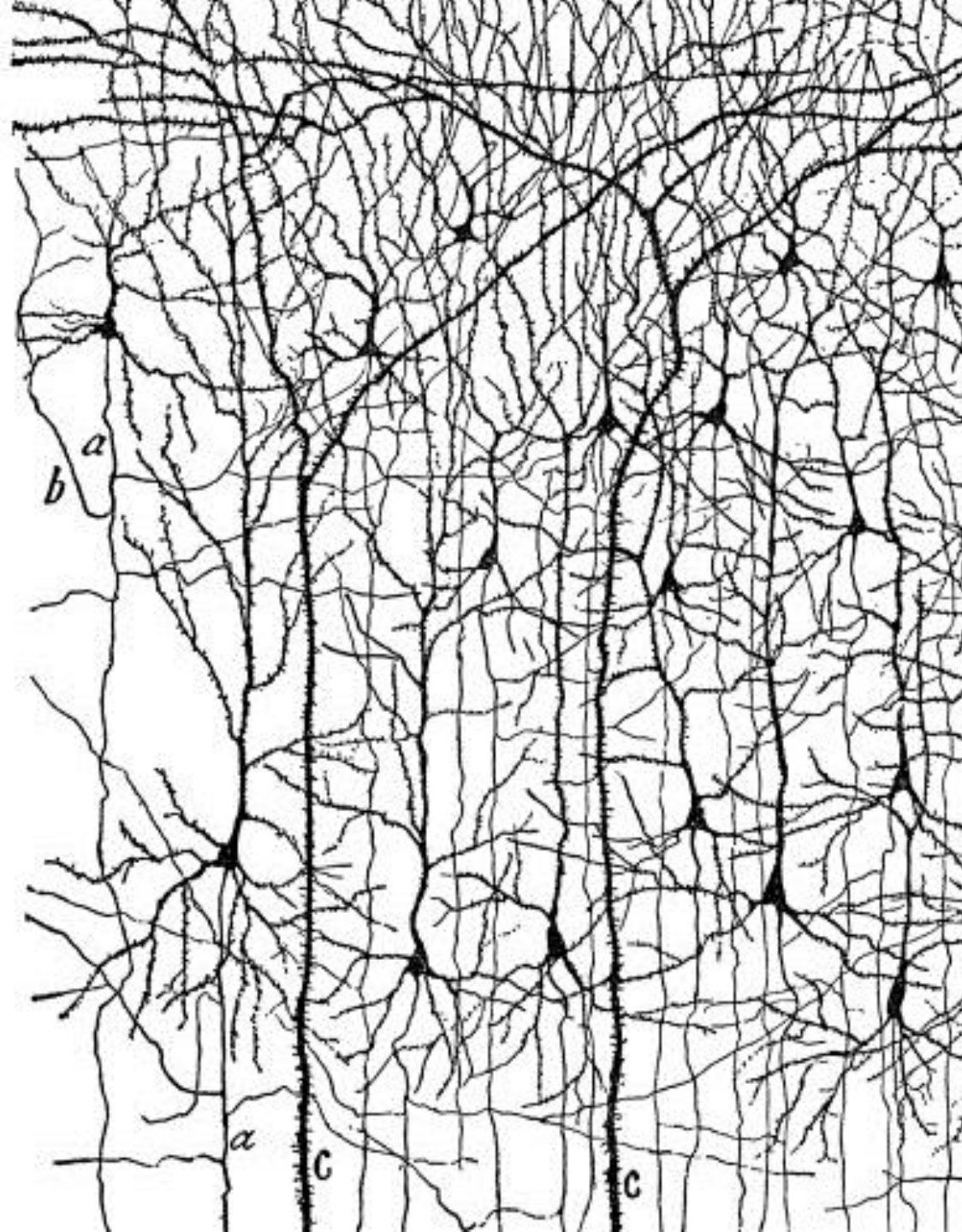
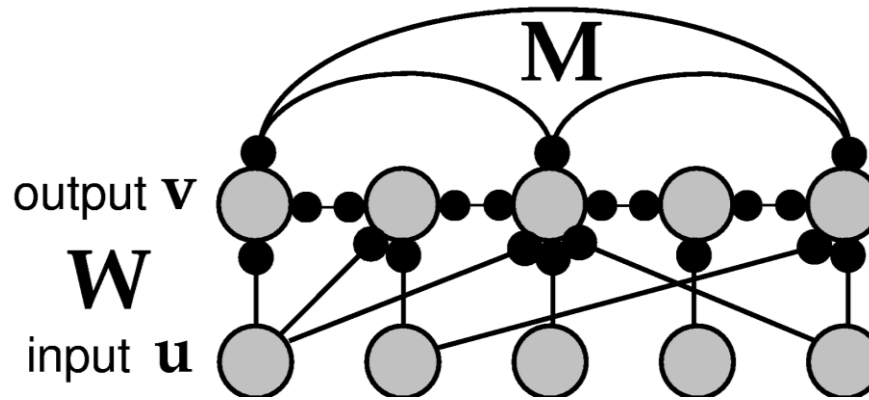


Supplementary Material for Recurrent Networks

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Solving the Differential Equation for a Linear Recurrent Network

$$\tau \frac{d\mathbf{v}}{dt} = -\mathbf{v} + \underbrace{\mathbf{W}\mathbf{u}}_{\mathbf{h}} + \mathbf{M}\mathbf{v}$$

Using Eigenvectors to Solve for $\mathbf{v}(t)$

$$\tau \frac{d\mathbf{v}}{dt} = -\mathbf{v} + \mathbf{h} + \mathbf{M}\mathbf{v}$$

- ◆ Suppose $N \times N$ matrix \mathbf{M} is *symmetric*
- ◆ \mathbf{M} has N *orthogonal* eigenvectors \mathbf{e}_i and N eigenvalues λ_i which satisfy:

$$\mathbf{M}\mathbf{e}_i = \lambda_i \mathbf{e}_i$$

- ◆ Normalize eigenvectors to have length 1
⇒ Divide each by its length
- ◆ Then we have a set of *orthonormal* vectors (a new “basis” or coordinate system) such that:

$$\mathbf{e}_i \cdot \mathbf{e}_j = 0 \text{ for } i \neq j \text{ and } 1 \text{ otherwise}$$

Using Eigenvectors to Solve for $\mathbf{v}(t)$

- ◆ We can represent any N -dimensional vector, including our output vector $\mathbf{v}(t)$, using the orthonormal eigenvectors of \mathbf{M} :

$$\mathbf{v}(t) = \sum_{j=1}^N c_j(t) \mathbf{e}_j$$

- ◆ Substituting above in the diff. equation for \mathbf{v} : $\tau \frac{d\mathbf{v}}{dt} = -\mathbf{v} + \mathbf{h} + \mathbf{M}\mathbf{v}$

$$\tau \frac{d \sum_{j=1}^N c_j \mathbf{e}_j}{dt} = - \sum_{j=1}^N c_j \mathbf{e}_j + \mathbf{h} + \mathbf{M} \sum_{j=1}^N c_j \mathbf{e}_j$$

$$\tau \sum_{j=1}^N \frac{dc_j}{dt} \mathbf{e}_j = - \sum_{j=1}^N c_j (\mathbf{e}_j - \mathbf{M} \mathbf{e}_j) + \mathbf{h}$$

$$\tau \sum_{j=1}^N \frac{dc_j}{dt} \mathbf{e}_j = - \sum_{j=1}^N c_j (\mathbf{e}_j - \lambda_j \mathbf{e}_j) + \mathbf{h}$$

Using eigenvector equation

$$\mathbf{M} \mathbf{e}_j = \lambda_j \mathbf{e}_j$$

Using Eigenvectors to Solve for $\mathbf{v}(t)$

- ◆ Taking the dot product of both sides with any \mathbf{e}_i :

$$\left(\tau \sum_{j=1}^N \frac{dc_j}{dt} \mathbf{e}_j = -\sum_{j=1}^N c_j (\mathbf{e}_j - \lambda_j \mathbf{e}_j) + \mathbf{h}\right) \cdot \mathbf{e}_i$$

Using orthonormality
of the \mathbf{e}_i

$$\tau \frac{dc_i}{dt} = -c_i(1 - \lambda_i) + \mathbf{h} \cdot \mathbf{e}_i$$

- ◆ Solve differential equation for c_i (and use it to get $\mathbf{v}(t)$!):

$$c_i(t) = \frac{\mathbf{h} \cdot \mathbf{e}_i}{1 - \lambda_i} \left(1 - \exp\left(\frac{-t(1 - \lambda_i)}{\tau}\right)\right) + c_i(0) \exp\left(\frac{-t(1 - \lambda_i)}{\tau}\right)$$

$$\mathbf{v}(t) = \sum_{i=1}^N c_i(t) \mathbf{e}_i$$

Eigenvalues determine Network Stability!

$$\mathbf{v}(t) = \sum_{i=1}^N c_i(t) \mathbf{e}_i \quad c_i(t) = \frac{\mathbf{h} \cdot \mathbf{e}_i}{1 - \lambda_i} (1 - \exp(\frac{-t(1 - \lambda_i)}{\tau})) + c_i(0) \exp(\frac{-t(1 - \lambda_i)}{\tau}))$$

If any $\lambda_i > 1$, $\mathbf{v}(t)$ explodes \Rightarrow network is unstable!

If all $\lambda_i < 1$, network is stable and $\mathbf{v}(t)$ converges to steady state value :

$$\mathbf{v}_{ss} = \sum_i \frac{\mathbf{h} \cdot \mathbf{e}_i}{1 - \lambda_i} \mathbf{e}_i$$

Amplification of Inputs in a Recurrent Network

$$\mathbf{v}_{ss} = \sum_i \frac{\mathbf{h} \cdot \mathbf{e}_i}{1 - \lambda_i} \mathbf{e}_i$$

If all $\lambda_i < 1$ and one λ_i (say λ_1) is close to 1 with others much smaller :

$$\mathbf{v}_{ss} \approx \frac{\mathbf{h} \cdot \mathbf{e}_1}{1 - \lambda_1} \mathbf{e}_1$$

Amplification of input
projection by a factor of $\frac{1}{1 - \lambda_i}$

Memory in Linear Recurrent Networks

$$\tau \frac{d\mathbf{v}}{dt} = -\mathbf{v} + \mathbf{h} + \mathbf{M}\mathbf{v} \quad \mathbf{v}(t) = \sum_{i=1}^N c_i(t) \mathbf{e}_i$$

Suppose $\lambda_1 = 1$ and all other $\lambda_i < 1$. Then, $\tau \frac{dc_1}{dt} = \mathbf{h} \cdot \mathbf{e}_1$

$$\text{Solving for } c_1, \text{ we get } c_1(t) = c_1(0) + \frac{1}{\tau} \int_0^t \mathbf{h}(t') \cdot \mathbf{e}_1 dt' \tau$$

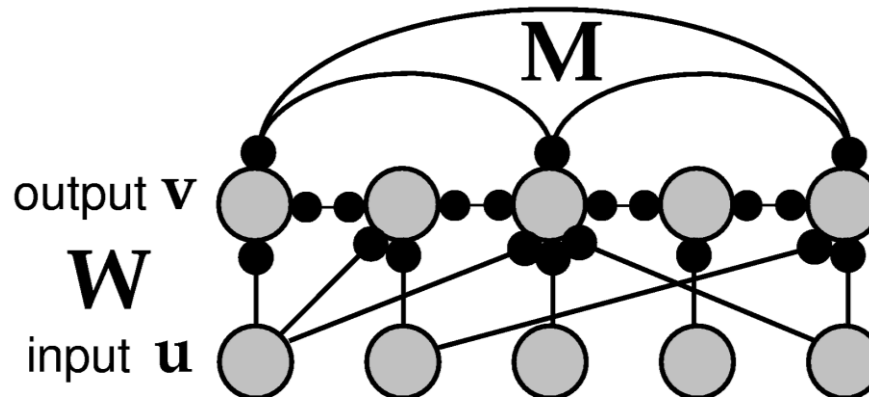
If input \mathbf{h} is turned on and then off, can show that even after $\mathbf{h} = 0$:

$$\mathbf{v}(t) = \sum_i c_i(t) \mathbf{e}_i$$

$$\approx c_1 \mathbf{e}_1 = \frac{\mathbf{e}_1}{\tau} \int_0^t \mathbf{h}(t') \cdot \mathbf{e}_1 dt' \quad (\text{assuming } c_1(0) = 0)$$

Sustained activity without any input!

Networks keeps a memory of **integral** of past input



Stability of **Nonlinear** Recurrent Networks

$$\tau \frac{d\mathbf{v}}{dt} = -\mathbf{v} + F(\mathbf{h} + \mathbf{M}\mathbf{v})$$

Stability Analysis of Nonlinear Recurrent Networks

General case : $\frac{d\mathbf{v}}{dt} = \mathbf{f}(\mathbf{v})$

Suppose \mathbf{v}_∞ is a fixed point (i.e., $\mathbf{f}(\mathbf{v}_\infty) = 0$)

We want to know: How stable is the network near \mathbf{v}_∞ ?

Near \mathbf{v}_∞ , $\mathbf{v}(t) = \mathbf{v}_\infty + \boldsymbol{\varepsilon}(t)$. Differentiating both sides wrt t : $\frac{d\mathbf{v}}{dt} = \frac{d\boldsymbol{\varepsilon}}{dt}$.

Taylor expansion near \mathbf{v}_∞ : $\mathbf{f}(\mathbf{v}(t)) \approx \mathbf{f}(\mathbf{v}_\infty) + \left. \frac{\partial \mathbf{f}}{\partial \mathbf{v}} \right|_{\mathbf{v}_\infty} \boldsymbol{\varepsilon}(t) = \left. \frac{\partial \mathbf{f}}{\partial \mathbf{v}} \right|_{\mathbf{v}_\infty} \boldsymbol{\varepsilon}(t)$

i.e. $\frac{d\mathbf{v}}{dt} = \mathbf{f}(\mathbf{v}(t)) = \left. \frac{\partial \mathbf{f}}{\partial \mathbf{v}} \right|_{\mathbf{v}_\infty} \boldsymbol{\varepsilon}(t) = J \cdot \boldsymbol{\varepsilon}(t)$ *J is the stability or “Jacobian” matrix*

Also, since $\frac{d\mathbf{v}}{dt} = \frac{d\boldsymbol{\varepsilon}}{dt}$, we have $\frac{d\boldsymbol{\varepsilon}}{dt} = J \cdot \boldsymbol{\varepsilon}(t)$

Assuming J is real with N linearly independent eigenvectors \mathbf{e}_i , we can write :

$$\boldsymbol{\varepsilon}(t) = \sum_i c_i(t) \mathbf{e}_i$$

Stability Analysis of Nonlinear Recurrent Networks

Continued from previous page :

Substituting $\mathbf{\epsilon}(t) = \sum_i c_i(t) \mathbf{e}_i$ into $\frac{d\mathbf{\epsilon}}{dt} = J \cdot \mathbf{\epsilon}(t)$, we find that the coefficients must satisfy :

$$\frac{dc_i}{dt} = \lambda_i c_i. \text{ The solution is : } c_i(t) = c_i(0) \exp(\lambda_i t)$$

Therefore, $\mathbf{\epsilon}(t) = \sum_i c_i(0) \exp(\lambda_i t) \mathbf{e}_i$ i.e., evolution of $\mathbf{v}(t)$ near \mathbf{v}_∞ depends on eigenvalues of J .

Each individual term in the above sum is called a mode.

Eigenvalues of J can be complex, e.g., $\lambda_k = a_k + ib_k$, which means that :

$\exp(\lambda_k t) = \exp(a_k t) \exp(ib_k t) = \exp(a_k t) (\cos(b_k t) + i \sin(b_k t))$. This implies :

For complex λ_k ($b_k \neq 0$), the mode will oscillate with frequency b_k .

If $a_k < 0$ for all k , the oscillations will be damped exponentially to 0 and the network will be stable near the fixed point \mathbf{v}_∞ .

If $a_k > 0$ for any k , the oscillations for that mode will grow exponentially and the network may not be stable (unless the network's nonlinearity curbs the growth as in the following example).

Example: Non-Symmetric Nonlinear Recurrent Network

- ♦ Example: Network of Excitatory (E) and Inhibitory (I) Neurons
 - ⇒ Connections can't be symmetric: Why?

$$10 \text{ ms} \rightarrow \tau_E \frac{dv_E}{dt} = -v_E + \left[\overset{1.25}{M_{EE}} v_E + \overset{-1}{M_{EI}} v_I - \overset{-10}{\gamma_E} \right]^+$$

$$\tau_I \frac{dv_I}{dt} = -v_I + \left[\overset{0}{M_{II}} v_I + \overset{1}{M_{IE}} v_E - \overset{10}{\gamma_I} \right]^+$$

Parameter
we will vary to
study the network

We want to analyze stability of this network near fixed point \mathbf{v}_∞ i.e., near the (v_E, v_I)

which results in $\frac{dv_E}{dt} = \frac{dv_I}{dt} = 0$.

Linear Stability Analysis near Fixed Point

$$\frac{dv_E}{dt} = \frac{-v_E + [M_{EE}v_E + M_{EI}v_I - \gamma_E]^+}{\tau_E}$$

Take derivatives of right hand side with respect to both v_E and v_I

$$\frac{dv_I}{dt} = \frac{-v_I + [M_{II}v_I + M_{IE}v_E - \gamma_I]^+}{\tau_I}$$

Stability Matrix (aka the “Jacobian” Matrix):

$$J = \begin{bmatrix} \frac{(M_{EE} - 1)}{\tau_E} & \frac{M_{EI}}{\tau_E} \\ \frac{M_{IE}}{\tau_I} & \frac{(M_{II} - 1)}{\tau_I} \end{bmatrix}$$

Compute the Eigenvalues

♦ Jacobian Matrix:

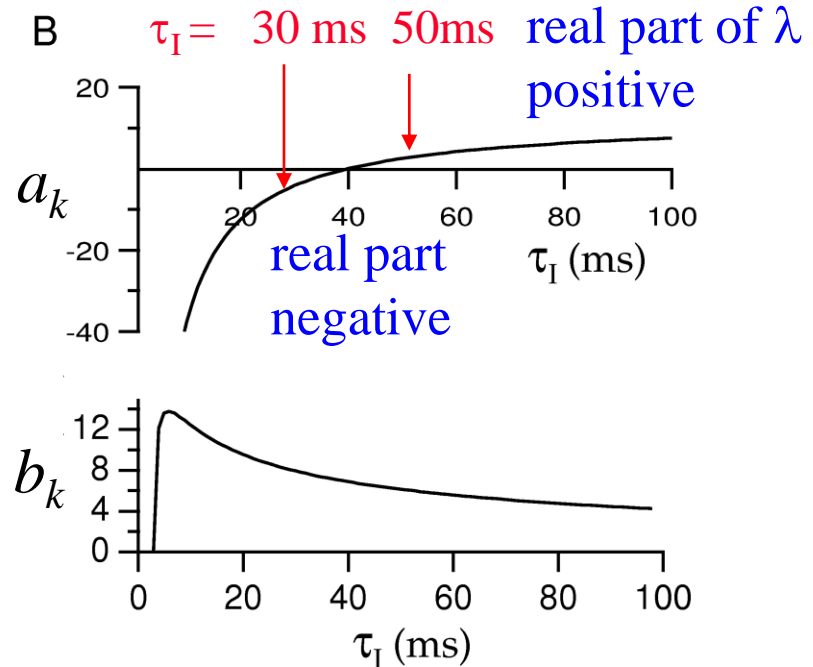
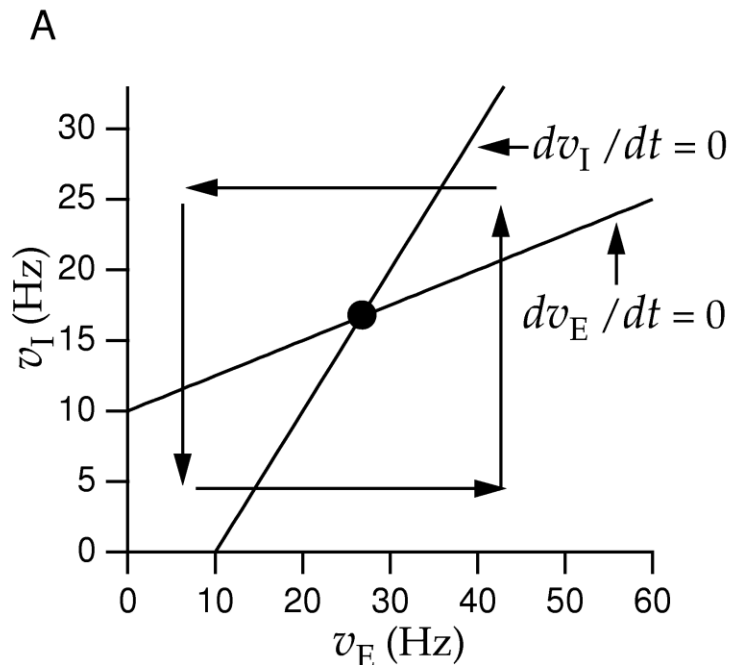
$$J = \begin{bmatrix} \frac{(M_{EE} - 1)}{\tau_E} & \frac{M_{EI}}{\tau_E} \\ \frac{M_{IE}}{\tau_I} & \frac{(M_{II} - 1)}{\tau_I} \end{bmatrix}$$

♦ Its two eigenvalues (obtained by solving $\det(J - \lambda I) = 0$):

$$\lambda = \frac{1}{2} \left(\frac{\overset{1.25}{(M_{EE} - 1)}}{\underset{10 \text{ ms}}{\tau_E}} + \frac{\overset{0}{(M_{II} - 1)}}{\tau_I} \pm \sqrt{\left(\frac{M_{EE} - 1}{\tau_E} - \frac{M_{II} - 1}{\tau_I} \right)^2 + 4 \frac{\overset{-1}{M_{EI}} \overset{1}{M_{IE}}}{\tau_E \tau_I}} \right)$$

Next page plots real and imaginary parts of λ as a function of τ_I

Phase Plane and Eigenvalue Analysis



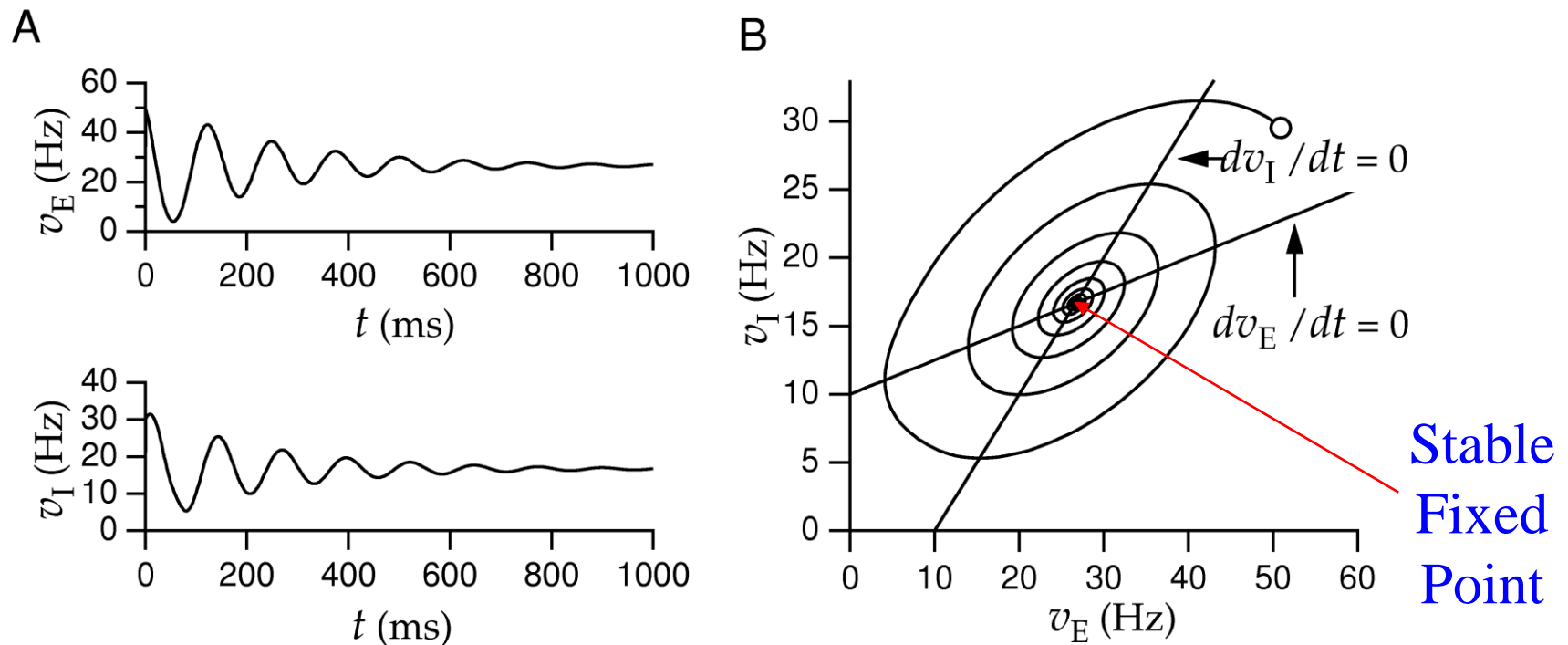
$$10 \frac{dv_E}{dt} = -v_E + [1.25v_E - v_I + 10]^+$$

$$\tau_I \frac{dv_I}{dt} = -v_I + [0 \cdot v_I + v_E - 10]^+$$

Real and imaginary parts
(a_k and b_k) of λ ($= a_k + ib_k$)
as a function of τ_I

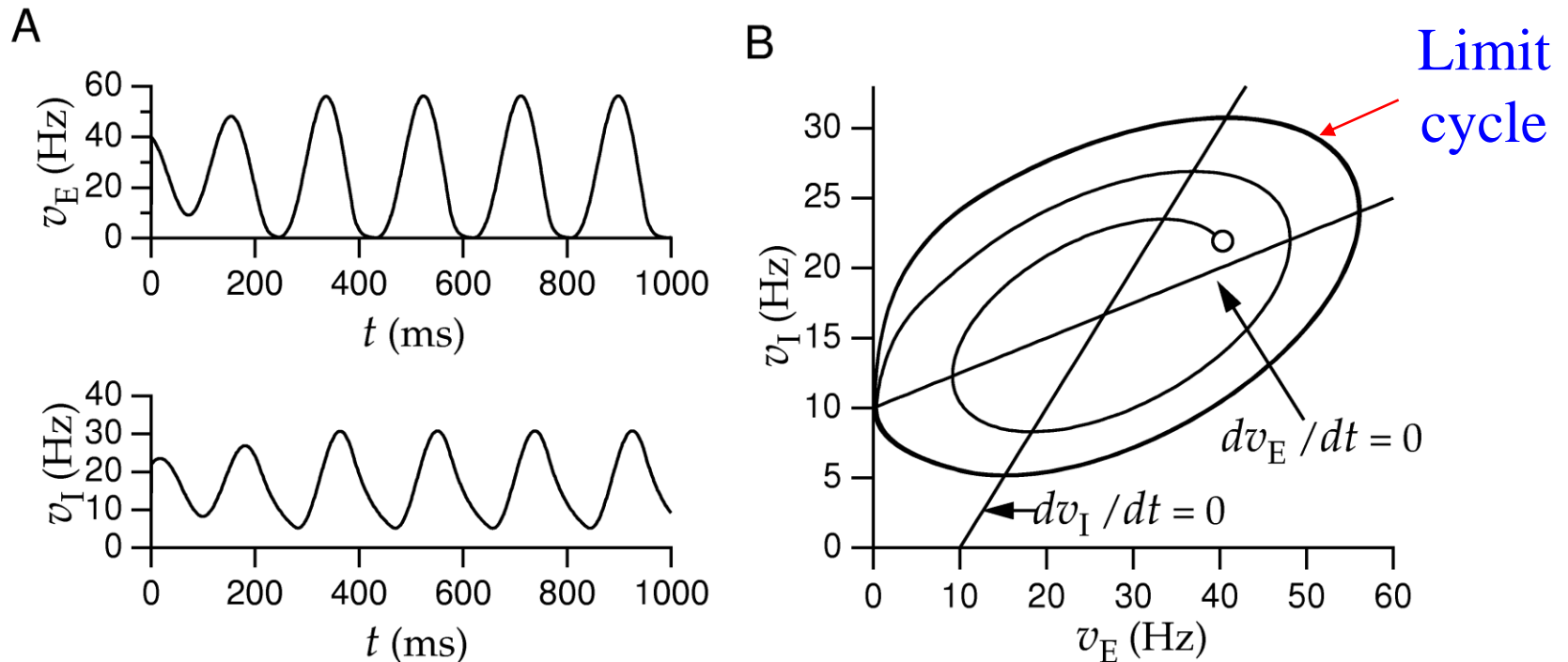
Damped Oscillations in the Network

Choose $\tau_I = 30$ ms (makes real part of eigenvalues negative)



Unstable Behavior and Limit Cycle

Choose $\tau_I = 50$ ms (makes real part of eigenvalues positive)



Oscillations grow initially but
curbed by rectification nonlinearity