8. A Union-Intersection Test

Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli random variables with unknown parameter $p \in (0,1)$. Suppose we want to test

$$H_0 : p \in [0.48, 0.51] \quad \text{vs} \quad H_1 : p \notin [0.48, 0.51]$$

We want to construct an asymptotic test ψ for these hypotheses using $\overline{X}_n$. For this problem, we specifically consider the family of tests ψ_{c_1,c_2} where we reject the null hypothesis if either $\overline{X}_n < c_1 \leq 0.48$ or $\overline{X}_n > c_2 \geq 0.51$ for some c_1 and c_2 that may depend on n, i.e.

$$\psi_{c_1,c_2} = \mathbf{1}\left((\overline{X}_n < c_1) \cup (\overline{X}_n > c_2)\right) \qquad \text{where } c_1 < 0.48 < 0.51 < c_2.$$

Throughout this problem, we will discuss possible choices for constants c_1 and c_2, and their impact to both the asymptotic and non-asymptotic level of the test.

(a)

1/1 point (graded)

Which expression represents the (smallest asymptotic) level α of this test? Recall the (smallest asymptotic) level equals the maximum Type 1 error rate.

- (selected, correct) $\alpha = \max_{p \in [0.48, 0.51]} \left(\mathbf{P}_p(\overline{X}_n < c_1) + \mathbf{P}_p(\overline{X}_n > c_2)\right)$
- $\alpha = \max_{p \in [0.48, 0.51]} \left(\max\left(\mathbf{P}_p(\overline{X}_n < c_1), \mathbf{P}_p(\overline{X}_n > c_2)\right)\right)$
- $\alpha = \max_{p \in [0.48, 0.51]} \mathbf{P}_p(\overline{X}_n < c_1)$
- $\alpha = \max_{p \in [0.48, 0.51]} \mathbf{P}_p(\overline{X}_n > c_2)$
- $\alpha = \max_{p \in [0.48, 0.51]} \left(\mathbf{P}_p(\overline{X}_n < c_1) \cdot \mathbf{P}_p(\overline{X}_n > c_2)\right)$

Solution:

A Type I error occurs when $\psi = 1$ but H_0 is true; hence the type 1 error rate is

$$\alpha_\psi(p) = \mathbf{P}_p\left((\overline{X}_n < c_1) \cup (\overline{X}_n > c_2)\right)$$

Since $c_1 < 0.48 < 0.51 < c_2$, we have

$$\mathbf{P}_p\left((\overline{X}_n < c_1) \cup (\overline{X}_n > c_2)\right) = \mathbf{P}_p\left(\overline{X}_n < c_1\right) + \mathbf{P}_p\left(\overline{X}_n > c_2\right).$$

Maximizing over this over $p \in [0.48, 0.51]$, we get that the maximum Type 1 error rate of this test, i.e. the smallest level, is

$$\boxed{\alpha = \max_{p\in[0.48,0.51]} \left(\mathbf{P}_p\left(\overline{X}_n < c_1\right) + \mathbf{P}_p\left(\overline{X}_n > c_2\right)\right)}.$$

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Answers are displayed within the problem

(b)

4.0/4 points (graded)

Use the central limit theorem and the approximation $\sqrt{p(1-p)} \approx \frac{1}{2}$ for $p \in [0.48, 0.51]$ to approximate $\mathbf{P}_p\left(\overline{X}_n < c_1\right)$ and $\mathbf{P}_p\left(\overline{X}_n > c_2\right)$ for large n. Express your answers as a formula in terms of c_1, c_2, n and p.

(Write **Phi** for the cdf of a Normal distribution, **c_1** for c_1, and **c_2** for c_2.)

$\mathbf{P}_p\left(\overline{X}_n < c_1\right) \approx$ Phi(sqrt(n)*(c_1-p)*2) **Answer:** Phi(2*(c_1 - p)*sqrt(n))

For what value of $p \in [0.48, 0.51]$ is the expression above for $\mathbf{P}_p\left(\overline{X}_n < c_1\right)$ maximized?

$\mathbf{P}_p\left(\overline{X}_n < c_1\right)$ is max at $p =$ 0.48 **Answer:** 0.48

$\mathbf{P}_p\left(\overline{X}_n > c_2\right) \approx$ 1 - Phi(sqrt(n)*(c_2-p)*2) **Answer:** 1 - Phi(2*(c_2 - p)*sqrt(n))

For what value of $p \in [0.48, 0.51]$ is the expression above for $\mathbf{P}_p\left(\overline{X}_n > c_2\right)$ maximized?

$\mathbf{P}_p\left(\overline{X}_n > c_2\right)$ is max at $p =$ 0.51 **Answer:** 0.51

Solution:

Consider a specific $p \in [0.48, 0.51]$. Then,

$$\mathbf{P}_p\left(\overline{X}_n < c_1\right) = \mathbf{P}_p\left(\frac{\overline{X}_n - p}{\sqrt{p(1-p)}}\sqrt{n} < \frac{c_1 - p}{\sqrt{p(1-p)}}\sqrt{n}\right).$$

By the Central Limit Theorem and noting that the variance of X_1 is $\sqrt{p(1-p)}$, we see that $\frac{\overline{X}_n - p}{\sqrt{p(1-p)}}\sqrt{n}$ has a standard Gaussian distribution, so

$$\mathbf{P}_p\left(\overline{X}_n < c_1\right) = \Phi\left(\frac{c_1 - p}{\sqrt{p(1-p)}}\sqrt{n}\right) \approx \Phi\left(2\left(c_1 - p\right)\sqrt{n}\right).$$

As $\Phi(x)$ is an increasing function, $\Phi\left(2\left(c_1 - p\right)\sqrt{n}\right)$ is maximized at the minimum possible p in the range, which is $p = 0.48$. Hence, $\max_{p\in[0.48,0.51]} \mathbf{P}_p\left(\overline{X}_n < c_1\right) = \Phi\left(2\left(c_1 - 0.48\right)\sqrt{n}\right)$.

Similarly for a specific $p \in [0.48, 0.51]$,

$$\mathbf{P}_p\left(\overline{X}_n > c_2\right) = \mathbf{P}_p\left(\frac{\overline{X}_n - p}{\sqrt{p(1-p)}}\sqrt{n} > \frac{c_2 - p}{\sqrt{p(1-p)}}\sqrt{n}\right)$$

Applying the Central Limit Theorem as in the previous part and then the approximation $\sqrt{p(1-p)} \approx \frac{1}{2}$ gives

$$\mathbf{P}_p\left(\overline{X}_n > c_2\right) \approx 1 - \Phi\left(2\left(c_2 - p\right)\sqrt{n}\right).$$

As $\Phi(x)$ is an increasing function, $1 - \Phi\left(2\left(c_2 - p\right)\sqrt{n}\right)$ is maximized at the maximum possible p in the range, which is $p = 0.51$. Hence,

$$\max_{p \in [0.48, 0.51]} \mathbf{P}_p\left(\overline{X}_n > c_1\right) = 1 - \Phi\left(2\left(c_2 - 0.51\right)\sqrt{n}\right)$$

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Answers are displayed within the problem

(c)

1.0/1 point (graded)

Next, we combine the results from parts (a) and (b).

Apply the inequality $\max_x\left(f(x) + g(x)\right) \leq \max_x f(x) + \max_x g(x)$ to the expression for the (asymptotic) level α obtained in part (a) and use the results from part (b) to give an upper bound on α.

Express your answer as a formula in terms of c_1, c_2, and n.
(Write **Phi** for the cdf of a Normal distribution, **c_1** for c_1, and **c_2** for c_2.)

$\alpha \leq$ Phi(sqrt(n)*(c_1-0.48)*2) + 1 - Phi(sqrt(n)*(c_2-0.51)*2) **Answer:** 1+Phi(2*(c_1-0.48)*sqrt(n))-Phi(2*(c_2-0.51)*sqrt(n))

(*Food for thought:* Is this upper bound tight? A bound is tight if equality may be achieved.)

Solution:

Recall that the (smallest) asymptotic level α of a test is equal to the maximum Type 1 error rate. Recalling from part (a) the expression for (smallest) asymptotic level α, applying the given inequality $\max_x\left(f(x) + g(x)\right) \leq \max_x f(x) + \max_x g(x)$, and using all the results from part (b), we have

$$\begin{aligned}\max_{p \in [0.48, 0.51]}\left(\mathbf{P}_p\left(\overline{X}_n < c_1\right) + \mathbf{P}\left(\overline{X}_n > c_2\right)\right) &\leq \max_{p \in [0.48, 0.51]} \mathbf{P}\left(\overline{X}_n < c_1\right) + \max_{p \in [0.48, 0.51]} \mathbf{P}\left(\overline{X}_n > c_2\right) \\ &\approx \boxed{\Phi\left(2\left(c_1 - 0.48\right)\sqrt{n}\right) + 1 - \Phi\left(2\left(c_2 - 0.51\right)\sqrt{n}\right)}.\end{aligned}$$

(This bound is not tight because the the maxima for the two summands are not obtained at the same p.)

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Answers are displayed within the problem

(d)

2.0/2 points (graded)

Suppose that we wish to have a level $\alpha = 0.05$. What c_1 and c_2 will achieve $\alpha = 0.05$? Choose c_1 and c_2 by setting the expressions you obtained above for $\max_{p \in [0.48, 0.51]} \mathbf{P}_p\left(\overline{X}_n < c_1\right)$ and $\max_{p \in [0.48, 0.51]} \mathbf{P}_p\left(\overline{X}_n > c_2\right)$ to both be 0.025.

(If applicable, enter **q(alpha)** for q_α, the $1-\alpha$-quantile of a standard normal distribution, e.g. enter **q(0.01)** for $q_{0.01}$.)

$c_1 =$ q(0.975)*(1/(2*sqrt(n)))+ **Answer:** -q(0.025)/(2* sqrt(n)) + 0.48

$c_2 =$ q(0.025)*(1/(2*sqrt(n)))+ Answer: q(0.025)/(2* sqrt(n)) + 0.51

Solution:

To have a test of level 0.05 and equal left and right tails, according to the description above, we want to set

$$\Phi\left(2 * (c_1 - 0.48) * \sqrt{n}\right) = 0.025$$

and

$$\Phi\left(2 * (c_2 - 0.51) * \sqrt{n}\right) = 0.975.$$

Taking the inverse Phi function to both equations gives

$$2 * (c_1 - 0.48) * \sqrt{n} = -1.96$$

and

$$2 * (c_2 - 0.51) * \sqrt{n} = 1.96,$$

respectively. Solving the two equations (independently) gives

$$c_1 = \boxed{-\frac{0.98}{\sqrt{n}} + 0.48}$$

$$c_2 = \boxed{\frac{0.98}{\sqrt{n}} + 0.51}$$

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Answers are displayed within the problem

(e)

2/2 points (graded)

We will now show that the values we just derived for c_1 and c_2 are in fact too conservative.

Recall the expression from part (b) for $\mathbf{P}_p\left(\overline{X}_n < c_1\right)$ for large n. For $p > 0.48$ (note the strict inequality), find $\lim_{n\to\infty} \mathbf{P}_p\left(\overline{X}_n < c_1\right)$.

$\lim_{n\to\infty} \mathbf{P}_{p>0.48}\left(\overline{X}_n < c_1\right) =$ 0 Answer: 0

Similarly, for $p < 0.51$ (note the strict inequality), find $\lim_{n\to\infty} \mathbf{P}_p\left(\overline{X}_n > c_2\right)$. Use the expression you found in part (b) for $\mathbf{P}_p\left(\overline{X}_n > c_2\right)$.

$\lim_{n\to\infty} \mathbf{P}_{p<0.51}\left(\overline{X}_n > c_2\right) =$ 0 Answer: 0

Solution:

Recall from part (b)

$$\mathbf{P}\left(\overline{X}_n < c_1\right) = \Phi\left(\frac{c_1 - p}{\sqrt{p(1-p)}}\sqrt{n}\right)\right) \approx \boxed{\Phi\left(2\left(c_1 - p\right)\sqrt{n}\right)}.$$

If $p > 0.48$, then

$$\Phi\left(2\left(c_1 - p\right)\sqrt{n}\right) < \Phi\left(2\left(0.48 - p\right)\sqrt{n}\right).$$

This argument in Φ on the right is a negative constant times $\sqrt{n}$, so the argument tends to negative infinity as $n \to \infty$ and thus

$$\Phi\left(2\left(c_1 - p\right)\sqrt{n}\right) \to \boxed{0}.$$

For the other side, c_2, we obtained in part (b) that

$$\mathbf{P}\left(\overline{X}_n > c_2\right) \approx \boxed{1 - \Phi\left(2\left(c_2 - p\right)\sqrt{n}\right)}.$$

If $p < 0.51$, then

$$1 - \Phi\left(2\left(c_2 - p\right)\sqrt{n}\right) < 1 - \Phi\left(2\left(c_2 - 0.51\right)\sqrt{n}\right).$$

Taking $n \to \infty$, as $c_2 > 0.51$,

$$2\left(c_2 - 0.51\right)\sqrt{n} \to +\infty;$$

so

$$1 - \Phi\left(2\left(c_2 - 0.51\right)\sqrt{n}\right) \to 0$$

and thus

$$1 - \Phi\left(2\left(c_2 - p\right)\sqrt{n}\right) \to \boxed{0}.$$

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Answers are displayed within the problem

(f)

2.0/2 points (graded)

Note: This part of the problem will contain multiple steps but you would only enter answers to the final step. Also refer to the last video in recitation 3 for related ideas.

Next, we analyze the asymptotic test given different possible values of p, in order to choose suitable and sufficiently-tight c_1 and c_2. Looking more closely at part (d), we may note that the asymptotic behavior of the expressions for the errors are different depending on whether $p = 0.48$, $0.48 < p < 0.51$, or $p = 0.51$.

Based on your answers and work from the previous part, evaluate the asymptotic Type 1 error

$$\mathbf{P}\left(\overline{X}_n < c_1\right) + \mathbf{P}\left(\overline{X}_n > c_2\right).$$

on each of the three cases for the value of p in terms of c_1, c_2, and n, and determine in each case which component(s) of the Type 1 error will converge to zero.

This would allow you to come up with a new set of conditions for c_1 and c_2 in terms of n, given the desired level of 5%. Enter these values (in terms of n) below.

(If applicable, enter **q(alpha)** for q_α, the $1-\alpha$-quantile of a standard normal distribution, e.g. enter **q(0.01)** for $q_{0.01}$. Do not worry about the parser not rendering **q(alpha)** properly; the grader will work nonetheless. You could also enclose **q(alpha)** by brackets for the rendering to show properly.)

$c_1 =$ 0.48 - q(0.05)*(1/(2*sqrt(n))) **Answer:** -q(0.05)/(2* sqrt(n)) + 0.48

$c_2 =$ 0.51 + q(0.05)*(1/(2*sqrt(n))) **Answer:** q(0.05)/(2* sqrt(n)) + 0.51

STANDARD NOTATION

Solution:

For this solution, we write $c_1(n)$ and $c_2(n)$ for c_1 and c_2 respectively as they are in practice functions of n.

From the previous part, $\mathbf{P}\left(\overline{X}_n < c_1(n)\right)$ for any $c_1(n) < 0.48$ will definitely converge to 0 for $0.48 < p < 0.51$ and for $p = 0.51$, but not for $p = 0.48$. When $p = 0.48$, we could write

$$\mathbf{P}\left(\overline{X}_n < c_1(n)\right) = \Phi\left(2\left(c_1(n) - 0.48\right)\sqrt{n}\right).$$

Similarly, $\mathbf{P}\left(\overline{X}_n > c_2(n)\right)$ for any $c_2(n) > 0.51$ will converge to 0 for $p = 0.48$ and $0.48 < p < 0.51$, while for $p = 0.51$,

$$\mathbf{P}\left(\overline{X}_n > c_2(n)\right) = 1 - \Phi\left(2\left(c_2(n) - 0.51\right)\sqrt{n}\right).$$

Summarizing the above observations, we get that for $p = 0.48$,

$$\lim_{n\to\infty}\left(\mathbf{P}(\overline{X}_n < c_1(n)) + \mathbf{P}(\overline{X}_n > c_2(n))\right) = \lim_{n\to\infty}\Phi\left(2\left(c_1(n) - 0.48\right)\sqrt{n}\right);$$

for $0.48 < p < 0.51$,

$$\lim_{n\to\infty}\left(\mathbf{P}(\overline{X}_n < c_1(n)) + \mathbf{P}(\overline{X}_n > c_2(n))\right) = 0;$$

while for $p = 0.51$,

$$\lim_{n\to\infty}\left(\mathbf{P}(\overline{X}_n < c_1(n)) + \mathbf{P}(\overline{X}_n > c_2(n))\right) = \lim_{n\to\infty} 1 - \Phi\left(2\left(c_2(n) - 0.51\right)\sqrt{n}\right);$$

Hence, our constraints (from the first and third cases above) are

$$\lim_{n\to\infty}\Phi\left(2\left(c_1(n) - 0.48\right)\sqrt{n}\right) \le 0.05, \ \lim_{n\to\infty}\Phi\left(2\left(c_2(n) - 0.51\right)\sqrt{n}\right) \ge 0.95.$$

Taking the simplest (or broadest) case, we could set equality everywhere, which gives

$$\Phi\left(2\left(c_1\left(n\right)-0.48\right)\sqrt{n}\right)=0.05,\quad \Phi(\left(2\left(c_2\left(n\right)-0.51\right)\sqrt{n}\right)=0.95.$$

Finally, taking Φ^{-1} of both sides then rearranging gives our answer:

$$\boxed{c_1\left(n\right)=\frac{-q_{0.05}}{2\sqrt{n}}+0.48,\quad c_2\left(n\right)=\frac{q_{0.05}}{2\sqrt{n}}+0.51}$$

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Answers are displayed within the problem

讨论

显示讨论

主题： Unit 2 Foundation of Inference:Homework 3: Introduction to Hypothesis Testing / 8. A Union-Intersection Test

认证证书是什么？