(a) Let X be the time until the first bulb failure. Let A (respectively, B) be the event that the first bulb is of type A (respectively, B). Since the two bulb types are equally likely, the total expectation theorem yields

$$\mathbf{E}[X] = \mathbf{E}[X \mid A]\mathbf{P}(A) + \mathbf{E}[X \mid B]\mathbf{P}(B) = 1 \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} = \frac{2}{3}.$$

(b) Let D be the event of no bulb failures before time t. Using the total probability theorem, and the exponential distributions for bulbs of the two types, we obtain

$$\mathbf{P}(D) = \mathbf{P}(D \mid A)\mathbf{P}(A) + \mathbf{P}(D \mid B)\mathbf{P}(B) = \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t}.$$

(c) We have

$$\mathbf{P}(A \mid D) = \frac{\mathbf{P}(A \cap D)}{\mathbf{P}(D)} = \frac{\frac{1}{2}e^{-t}}{\frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t}} = \frac{1}{1 + e^{-2t}}.$$

(d) We first find $\mathbf{E}[X^2]$. We use the fact that the second moment of an exponential random variable T with parameter λ is equal to $\mathbf{E}[T^2] = \mathbf{E}[T]^2 + \text{var}(T) = 1/\lambda^2 + 1/\lambda^2 = 2/\lambda^2$. Conditioning on the two possible types of the first bulb, we obtain

$$\mathbf{E}[X^2] = \mathbf{E}[X^2 \mid A]\mathbf{P}(A) + \mathbf{E}[X^2 \mid B]\mathbf{P}(B) = 2 \cdot \frac{1}{2} + \frac{2}{9} \cdot \frac{1}{2} = \frac{10}{9}.$$

Finally, using the fact $\mathbf{E}[X] = 2/3$ from part (a),

$$\text{var}(X) = \mathbf{E}[X^2] - \mathbf{E}[X]^2 = \frac{10}{9} - \frac{2^2}{3^2} = \frac{2}{3}.$$

(e) This is the probability that out of the first 11 bulbs, exactly 3 were of type A and that the 12th bulb was of type A. It is equal to

$$\binom{11}{3}\left(\frac{1}{2}\right)^{12}.$$

(f) This is the probability that out of the first 12 bulbs, exactly 4 were of type A, and is equal to

$$\binom{12}{4}\left(\frac{1}{2}\right)^{12}.$$

(h) Let Y be the total period of illumination provided by the first two type-B bulbs. This has an Erlang distribution of order 2, and its PDF is

$$f_Y(y) = 9ye^{-3y}, \qquad y \geq 0.$$

Let T be the period of illumination provided by the first type-A bulb. Its PDF is

$$f_T(t) = e^{-t}, \qquad t \geq 0.$$

We are interested in the event $T < Y$. We have

$$\mathbf{P}(T < Y \mid Y = y) = 1 - e^{-y}, \qquad y \geq 0.$$

Thus,

$$\mathbf{P}(T < Y) = \int_0^\infty f_Y(y)\mathbf{P}(T < Y \mid Y = y)\,dy = \int_0^\infty 9ye^{-3y}\big(1-e^{-y}\big)\,dy = \frac{7}{16},$$

as can be verified by carrying out the integration.

We now describe an alternative method for obtaining the answer. Consider merging a type-A and a type-B process. We are asked for the probability that the first arrival in the type-A process occurs before the 2nd arrival of the type-B process. The probability of the complementary event is that the first two arrivals in a merged process were type-B bulbs, hence is $(3/4) \cdot (3/4) = 9/16$. Thus the required probability is $7/16$.

(i) Let V be the total period of illumination provided by type-B bulbs while the process is in operation. Let N be the number of light bulbs, out of the first 12, that are of type B. Let X_i be the period of illumination from the ith type-B bulb. We then have $V = Y_1 + \cdots + Y_N$. Note that N is a binomial random variable, with parameters $n = 12$ and $p = 1/2$, so that

$$\mathbf{E}[N] = 6, \qquad \mathrm{var}(N) = 12 \cdot \frac{1}{2} \cdot \frac{1}{2} = 3.$$

Furthermore, $\mathbf{E}[X_i] = 1/3$ and $\mathrm{var}(X_i) = 1/9$. Using the formulas for the mean and variance of the sum of a random number of random variables, we obtain

$$\mathbf{E}[V] = \mathbf{E}[N]\mathbf{E}[X_i] = 2,$$

and

$$\mathrm{var}(V) = \mathrm{var}(X_i)\mathbf{E}[N] + \mathbf{E}[X_i]^2\mathrm{var}(N) = \frac{1}{9} \cdot 6 + \frac{1}{9} \cdot 3 = 1.$$

(j) Using the notation in parts (a)-(c), and the result of part (c), we have

$$\begin{aligned}
\mathbf{E}[T \mid D] &= t + \mathbf{E}[T - t \mid D \cap A]\mathbf{P}(A \mid D) + \mathbf{E}[T - t \mid D \cap B]\mathbf{P}(B \mid D) \\
&= t + 1 \cdot \frac{1}{1 + e^{-2t}} + \frac{1}{3}\left(1 - \frac{1}{1 + e^{-2t}}\right) \\
&= t + \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{1 + e^{-2t}}.
\end{aligned}$$