5. Bayesian estimator

Instructions

On this page, you will be given a distribution and another distribution conditional on the first one. Then, you will find the posterior distribution in a Bayesian approach. You will compute the Bayesian estimator, which is defined in lecture as the mean of the posterior distribution. Then, determine if the Bayesian estimator is consistent and/or asymptotically normal.

We recall that the Gamma distribution with parameters $q > 0$ and $\lambda > 0$ is the continuous distribution on $(0, \infty)$ whose density is given by $f(x) = \frac{\lambda^q x^{q-1} e^{-\lambda x}}{\Gamma(q)}$, where Γ is the Euler Gamma function $\Gamma(q) = \int_0^\infty t^{q-1} e^{-t} dt$, and its mean is q/λ.

We also recall that the $\text{Beta}(a, b)$ distribution has the density $f(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}$ and expectation $a/(a+b)$, where $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

(a)

3.0/3 points (graded)

$p \sim \text{Beta}(a, b)$ for some $a, b > 0$ and conditional on p, $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \text{Ber}(p)$.

What is the Bayesian estimator $\hat{p}^{\text{Bayes}}$?

(If applicable, enter **barX_n** for $\bar{X}_n$, **max(X_i)** for $\max_i X_i$. Do not worry if the parser does not render properly; the grader works independently. If you wish to have proper rendering, enclose **max(X_i)** by brackets.))

$\hat{p}^{\text{Bayes}} =$ (a+n*barX_n)/(a+b+n) ✔ **Answer:** (barX_n+a/n)/(1+(a+b)/n)

Determine whether the Bayesian estimator is consistent, and whether it is asymptotically normal.

- (●) Consistent and asymptotically normal ✔
- ( ) Consistent but not asymptotically normal
- ( ) Asymptotically normal but not consistent
- ( ) Neither consistent nor asymptotically normal

If it is asymptotically normal, what is its asymptotic variance $V(a, b, p)$? If it is not asymptotically normal, type in 0.

$V(a, b, p) =$ (p*(1-p)) ✔ **Answer:** p*(1-p)

$(p \cdot (1-p))$

STANDARD NOTATION

Solution:

1. Find the posterior distribution in a Bayesian approach.

$$\pi(p|x_1,\ldots,x_n) \propto \pi(p) L_n(x_1,\ldots,x_n|p) \propto p^{\sum_i x_i+a-1}(1-p)^{n-\sum_i x_i+b-1}$$

We recognize the posterior distribution as $\text{Beta}\left(\sum_i X_i + a, n - \sum_i X_i + b\right)$.

2. Compute the Bayesian estimator.

$$\hat{p} = \int_0^1 p\pi(p|x_i,\ldots,x_n)\,dp = \frac{\sum_i X_i + a}{n+a+b} = \frac{\bar{X}_n + a/n}{1+(a+b)/n}$$

3. Determine whether the Bayesian estimator is consistent.

看一个estimator是不是consistent就看n goes to infinity的时候的极限。

$$\boxed{\lim_{n\to\infty}\hat{p}} = \lim_{\boxed{n\to\infty}} \frac{\bar{X}_n + a/n}{1+(a+b)/n} = \bar{X}_n$$

4. Determine whether the Bayesian estimator is asymptotically normal.

From CLT,

$$\sqrt{n}\left(\bar{X}_n - p\right) \xrightarrow[\boxed{n\to\infty}]{(d)} \mathcal{N}(0, p(1-p))$$

Therefore, we find $\hat{p}$ is asymptotically normal.

$$\sqrt{n}(\hat{p}-p) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0, p(1-p))$$

5. If it is asymptotically normal, what is its asymptotic variance?

From the above equation, we see that the asymptotic variance is $p(1-p)$.

$$\sqrt{n}(\hat{p}-p) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0, p(1-p))$$

Submit You have used 2 of 3 attempts

Answers are displayed within the problem

(b)

2.0/3 points (graded)

$\pi(\theta) = 1, \forall\theta > 0$ and conditional on θ, $X_1,\ldots,X_n \overset{i.i.d.}{\sim} \mathcal{U}([0,\theta])$.

What is the Bayesian estimator $\hat{\theta}^{\text{Bayes}}$?

(If applicable, enter **barX_n** for $\bar{X}_n$, **max(X_i)** for $\max_i X_i$. Do not worry if the parser does not render properly; the grader works independently. If you wish to have proper rendering, enclose **max(X_i)** by brackets.)

$\hat{\theta}^{\text{Bayes}} =$ (max(X_i)^(1-n))/(1-n) ✖ **Answer:** (n-1)/(n-2)*max(X_i)

Determine whether the Bayesian estimator is consistent, and whether it is asymptotically normal.

- ○ Consistent and asymptotically normal
- ◉ Consistent but not asymptotically normal ✔
- ○ Asymptotically normal but not consistent
- ○ Neither consistent nor asymptotically normal

If it is asymptotically normal, what is its asymptotic variance $V(\theta)$? If it is not asymptotically normal, type in 0.

$V(\theta) =$ 0 ✔ **Answer:** 0

0

STANDARD NOTATION

Solution:

1. Find the posterior distribution in a Bayesian approach.

$$\pi(\theta|X_1,\ldots,X_n) \propto \pi(\theta) L_n(X_1,\ldots,X_n|\theta) \propto \boxed{\theta^{-n}\mathbf{1}\{\max_i X_i \le \theta\}}$$

To find the distribution, we set the scale parameter as C.

$$\int_0^\infty \pi(\theta|X_1,\ldots,X_n)\,d\theta = C\int_{\boxed{\max_i X_i}}^{\boxed{\infty}} \theta^{-n}\,d\boxed{\theta} = \frac{C}{n-1}(\max_i X_i)^{-n+1} = 1$$

这里theta是要求的
theta一定把Xi的最大值大

Solving this, we get the full distribution function.

$$C = \frac{n-1}{(\max_i X_i)^{-n+1}}, \quad \pi(\theta|X_1,\ldots,X_n) = \boxed{\frac{n-1}{(\max_i X_i)^{-n+1}}\theta^{-n}\mathbf{1}\{\max_i X_i \le \theta\}}$$

2. Compute the Bayesian estimator.

$$\hat\theta = \int_{\max_i X_i}^\infty \boxed{\theta}\pi(\theta|X_i,\ldots,X_n)\,d\theta = \frac{n-1}{(\max_i X_i)^{1-n}}\int_{\max_i X_i}^\infty \theta^{-n+1}\,d\theta = \frac{n-1}{n-2}\max_i X_i$$

total expectation theorem

3. Determine whether the Bayesian estimator is consistent.

Since we know $\hat\theta^{MLE} = \max_i X_i \xrightarrow[n\to\infty]{(P)} \theta$, we can find that the Bayesian estimator is consistent.

$$\hat\theta = \frac{n-1}{n-2}\max_i X_i \xrightarrow[n\to\infty]{(P)} \theta$$

However, it is not asymptotically normal.

4. Determine whether the Bayesian estimator is asymptotically normal.

It is not asymptotically normal.

5. If it is asymptotically normal, what is its asymptotic variance?

It is not asymptotically normal.

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Answers are displayed within the problem

(c)

3.0/3 points (graded)

$\lambda \sim$ Exp (α) for some $\alpha > 0$ and conditional on λ, $X_1, \ldots, X_n \overset{i.i.d.}{\sim}$ Exp (λ).

What is the Bayesian estimator $\hat{\lambda}^{\text{Bayes}}$?

(If applicable, enter **barX_n** for $\bar{X}_n$, **max(X_i)** for $\max_i X_i$. Do not worry if the parser does not render properly; the grader works independently. If you wish to have proper rendering, enclose **max(X_i)** by brackets.)

$\hat{\lambda}^{\text{Bayes}} =$ (n+1) / (n*barX_n + alpha) ✔ **Answer:** (1+1/n)/(alpha/n+barX_n)

Determine whether the Bayesian estimator is consistent, and whether it is asymptotically normal.

- (●) Consistent and asymptotically normal ✔
- ( ) Consistent but not asymptotically normal
- ( ) Asymptotically normal but not consistent
- ( ) Neither consistent nor asymptotically normal

If it is asymptotically normal, what is its asymptotic variance $V(\lambda)$? If it is not asymptotically normal, type in 0. You may use the variable λ.

$V(\lambda) =$ lambda^2 ✔ **Answer:** lambda^2

λ^2

STANDARD NOTATION

Solution:

1. Find the posterior distribution in a Bayesian approach.

$$\pi(\lambda|X_1, \ldots, X_n) \propto \pi(\lambda) L_n(X_1, \ldots, X_n|\lambda) \propto \alpha e^{-\alpha\lambda} \lambda^n e^{-\lambda \sum_i X_i} \propto \alpha \lambda^n e^{-(\alpha + \sum_i X_i)\lambda}$$

We recognize the posterior distribution as $\boxed{\text{Gamma}\left(n+1, \alpha + \sum_i X_i\right)}$.

2. Compute the Bayesian estimator.

$$\hat{\lambda} = \frac{n+1}{\alpha + \sum_i X_i} = \frac{1 + 1/n}{\alpha/n + \bar{X}_n}$$

3. Determine whether the Bayesian estimator is consistent.

$$\hat{\lambda} = \frac{1 + 1/n}{\alpha/n + \bar{X}_n} \xrightarrow[n\to\infty]{(a.s.)} \frac{1}{\bar{X}_n} \xrightarrow[n\to\infty]{(P)} \lambda$$

In the last transition, we used the knowledge that we already have: $\hat{\lambda}^{MLE} = \frac{1}{\bar{X}_n}$. We conclude that the Bayesian estimator is consistent.

4. Determine whether the Bayesian estimator is asymptotically normal.
   By CLT and Delta method,

$$\sqrt{n}\,(\hat{\lambda} - \lambda) \xrightarrow[n\to\infty]{(d)} \mathcal{N}\,(0, \lambda^2)$$

   Therefore, it is asymptotically normal.

5. If it is asymptotically normal, what is its asymptotic variance?

   The asymptotic variance is λ^2.

$$\sqrt{n}\,(\hat{\lambda} - \lambda) \xrightarrow[n\to\infty]{(d)} \mathcal{N}\,(0, \lambda^2)$$

Submit You have used 3 of 3 attempts

Answers are displayed within the problem

(d)

2.0/3 points (graded)

$\lambda \sim$ Exp (α) for some $\alpha > 0$ and conditional on λ, $X_1, \ldots, X_n \overset{i.i.d.}{\sim}$ Poiss (λ).

What is the Bayesian estimator $\hat{\lambda}^{\text{Bayes}}$?

(If applicable, enter **barX_n** for $\bar{X}_n$, **max(X_i)** for $\max_i X_i$. Do not worry if the parser does not render properly; the grader works independently. If you wish to have proper rendering, enclose **max(X_i)** by brackets.)

$\hat{\lambda}^{\text{Bayes}} =$ (n*barX_n + alpha)/(n+1) ✖ **Answer:** (barX_n+1/n)/(1+alpha/n)

Determine whether the Bayesian estimator is consistent, and whether it is asymptotically normal.

- (●) Consistent and asymptotically normal ✔
- ( ) Consistent but not asymptotically normal
- ( ) Asymptotically normal but not consistent
- ( ) Neither consistent nor asymptotically normal

If it is asymptotically normal, what is its asymptotic variance $V(\lambda)$? If it is not asymptotically normal, type in 0.

$V(\lambda) =$ lambda ✔ **Answer:** lambda

λ

Solution:

1. Find the posterior distribution in a Bayesian approach.

$$\pi(\lambda|X_1,\ldots,X_n) \propto \pi(\lambda)\,L_n(X_1,\ldots,X_n|\lambda) \propto \alpha e^{-\alpha\lambda}\frac{\lambda^{\sum_i X_i}e^{-n\lambda}}{\prod_i X_i!} \propto \alpha\frac{\lambda^{\sum_i X_i}e^{-(\alpha+n)\lambda}}{\prod_i X_i!}$$

We recognize the posterior distribution as $\boxed{\textsf{Gamma}\left(\sum_i X_i + 1, \alpha + n\right)}$.

2. Compute the Bayesian estimator.

$$\hat{\lambda} = \frac{\sum_i X_i + 1}{\alpha + n} = \frac{\bar{X}_n + 1/n}{1 + \alpha/n}$$

3. Determine whether the Bayesian estimator is consistent.

By LLN,

$$\boxed{\hat{\lambda} = \frac{\bar{X}_n + 1/n}{1 + \alpha/n} \xrightarrow[n\to\infty]{(a.s.)} \bar{X}_n \xrightarrow[n\to\infty]{(P)} \lambda}$$

Therefore, the Bayesian estimator is consistent.

4. Determine whether the Bayesian estimator is asymptotically normal.

By CLT,

$$\sqrt{n}\left(\bar{X}_n - \lambda\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,\lambda)$$

Since the estimator is consistent,

$$\sqrt{n}\left(\hat{\lambda} - \lambda\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,\lambda)$$

Therefore, it is asymptotically normal.

5. If it is asymptotically normal, what is its asymptotic variance?

The asymptotic variance is λ.

$$\sqrt{n}\left(\hat{\lambda} - \lambda\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,\lambda)$$

Submit You have used 3 of 3 attempts

Answers are displayed within the problem

Discussion

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