LECTURE 12: Sums of independent random variables; Covariance and correlation

- The PMF/PDF of $X + Y$ (X and Y independent)
  - the discrete case
  - the continuous case
  - the mechanics
  - the sum of independent normals

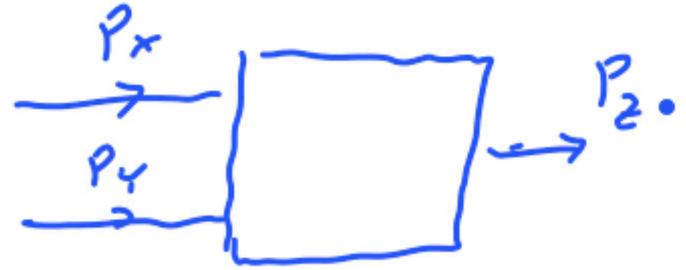
- Covariance and correlation
  - definitions
  - mathematical properties
  - interpretation

The distribution of $X + Y$: the discrete case

$$p_Z(z) = \sum_x p_X(x)\, p_Y(z - x)$$

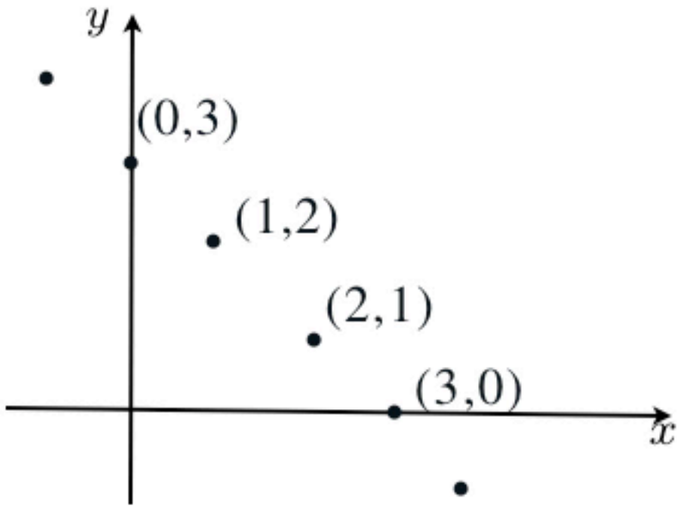
- $Z = X + Y$; X, Y independent, discrete

$g(x,y)$ known PMFs



$$p_Z(3) = \cdots + P(X=0, Y=3) + P(X=1, Y=2) + \cdots$$

$$= \cdots + p_X(0)\, p_Y(3) + p_X(1)\, p_Y(2) + \cdots$$

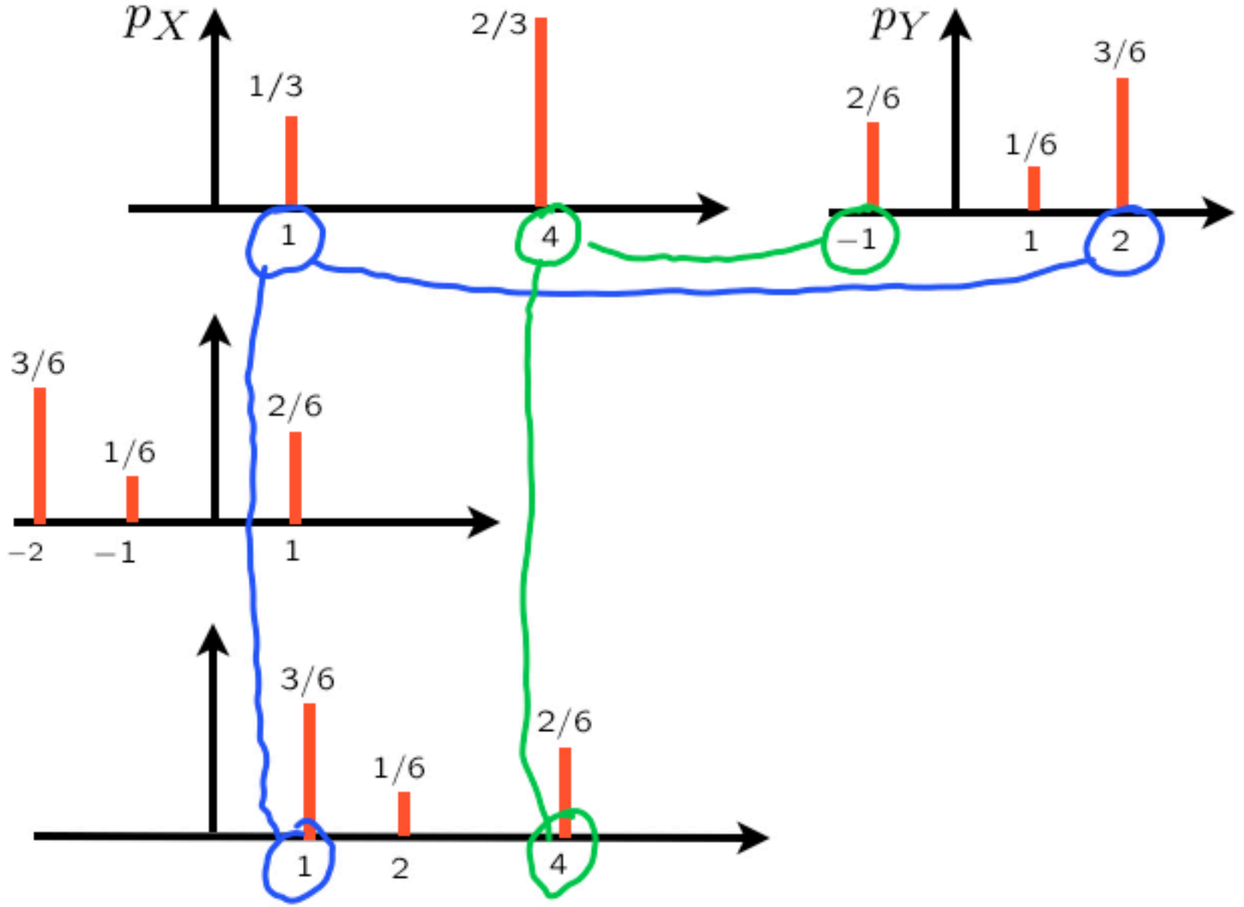


$$p_Z(z) = \sum_x P(X = x, Y = z - x)$$

$$= \sum_x p_X(x)\, p_Y(z - x)$$

Discrete convolution mechanics

$$p_Z(z) = \sum_x p_X(x)\, p_Y(z - x)$$



- To find $p_Z(3)$:

– Flip (horizontally) the PMF of Y

– Put it underneath the PMF of X

– Right-shift the flipped PMF by 3

– Cross-multiply and add

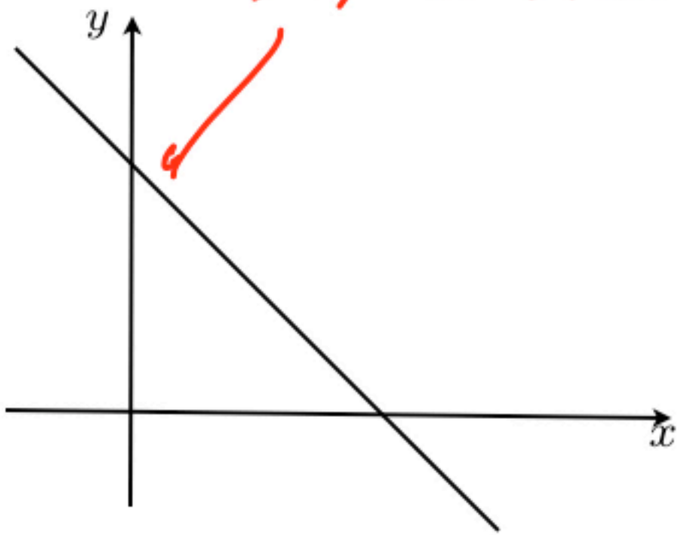
– Repeat for other values of z

The distribution of $X + Y$: the continuous case

$$p_Z(z) = \sum_x p_X(x)\, p_Y(z - x)$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx$$

- $Z = X + Y$; X, Y independent, continuous known PDFs



Conditional on $X = x$: $Z = x + Y$ $x = 3$ $Z = Y + 3$

$f_{Z|X}(z|3) = f_{Y+3|X}(z|3) = f_{Y+3}(z) = f_Y(z-3)$

$f_{Z|X}(z|x) = f_Y(z - x)$ $f_{x+b}(x) = f_x(x-b)$

Joint PDF of Z and X:

$f_{X,Z}(x,z) = f_X(x)\, f_Y(z-x)$

From joint to the marginal: $f_Z(z) = \int_{-\infty}^{\infty} f_{X,Z}(x, z)\, dx$

- Same mechanics as in discrete case (flip, shift, etc.)

The sum of independent normal r.v.'s

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx$$

- $X \sim N(\mu_x, \sigma_x^2)$, $Y \sim N(\mu_y, \sigma_y^2)$, independent $\qquad Z = X + Y$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-(x-\mu_x)^2/2\sigma_x^2} \qquad f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma_y} e^{-(y-\mu_y)^2/2\sigma_y^2}$$

$X + Y + W$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left\{ -\frac{(x-\mu_x)^2}{2\sigma_x^2} \right\} \frac{1}{\sqrt{2\pi}\,\sigma_y} \exp\left\{ -\frac{(z-x-\mu_y)^2}{2\sigma_y^2} \right\} dx$$

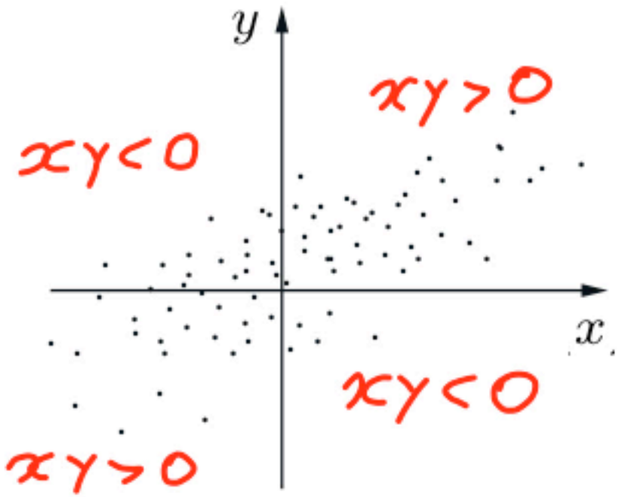
(algebra) $$= \frac{1}{\sqrt{2\pi(\sigma_x^2+\sigma_y^2)}} \exp\left\{ -\frac{(z-\mu_x-\mu_y)^2}{2(\sigma_x^2+\sigma_y^2)} \right\}$$

$N(\mu_x+\mu_y, \sigma_x^2+\sigma_y^2)$

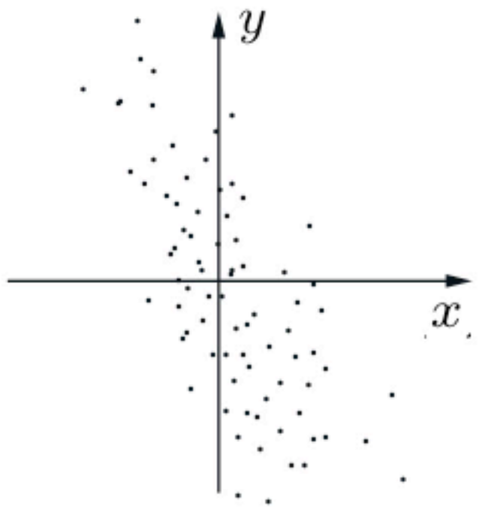
The sum of finitely many independent normals is normal

Covariance

- Zero-mean, discrete X and Y
  - if independent: $\mathbf{E}[XY] =$ $= E[X]\,E[Y] = 0$



$\mathbf{E}[XY] > 0$



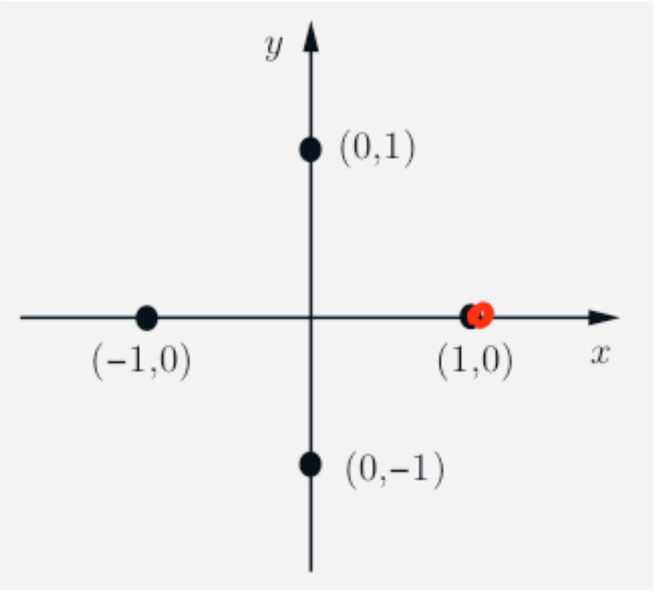
$\mathbf{E}[XY] < 0$

Definition for general case:

$$\operatorname{cov}(X,Y) = \mathbf{E}\Big[(X - \mathbf{E}[X]) \cdot (Y - \mathbf{E}[Y])\Big]$$

and $0 = E[(X - E[X])]\,E[Y - E[Y]]$

- independent $\Rightarrow$ $\operatorname{cov}(X,Y) = 0$
  (converse is not true)



$XY = 0$

$Cov = 0$

$X = 1 \Rightarrow Y = 0$

Covariance properties

$$\text{cov}(X, X) = E[(X - E[X])^2]$$

$$= \text{var}(X) = E[X^2] - (E[X])^2$$

$$\text{cov}(aX + b, Y) =$$

(assume 0 means)

$$= E[(aX+b)Y] = aE[XY] + bE[Y]$$

$$= a \cdot \text{cov}(X, Y)$$

$$\text{cov}(X, Y + Z) = E[X(Y+Z)]$$

$$= E[XY] + E[XZ] = \text{cov}(X, Y) + \text{cov}(X, Z)$$

$$\text{cov}(X, Y) = \mathbf{E}\Big[(X - \mathbf{E}[X]) \cdot (Y - \mathbf{E}[Y])\Big]$$

$$= E[XY] - E[XE[Y]]$$

$$- E[E[X]Y] + E[E[X]E[Y]]$$

$$= E[XY] - E[X]E[Y]$$

$$- E[X]E[Y] + E[X]E[Y]$$

$$\text{cov}(X, Y) = \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y]$$

The variance of a sum of random variables

$$\mathrm{var}(X_1 + X_2) = E\left[(X_1 + X_2 - E[X_1 + X_2])^2\right]$$

$$= E\left[\left((X_1 - E[X_1]) + (X_2 - E[X_2])\right)^2\right]$$

$$= E\Big[(X_1 - E[X_1])^2 + (X_2 - E[X_2])^2 + 2\,(X_1 - E[X_1])(X_2 - E[X_2])\Big]$$

$$= \mathrm{var}(X_1) + \mathrm{var}(X_2) + 2\,\mathrm{cov}(X_1, X_2).$$

The variance of a sum of random variables

$$\mathrm{var}(X_1 + X_2) = \mathrm{var}(X_1) + \mathrm{var}(X_2) + 2\,\mathrm{cov}(X_1, X_2)$$

$$\mathrm{var}(X_1 + \cdots + X_n) =$$

(assume 0 means)

$$= E\left[(X_1 + \cdots + X_n)^2\right]$$

$$= E\left[\sum_{i=1}^{n} X_i^2 + \sum_{\substack{i=1,\ldots,n \\ j=1,\ldots,n \\ i \neq j}} X_i X_j\right] \quad \left.\begin{array}{l}\\\\\\\end{array}\right\} n^2 - n \text{ terms}$$

$$= \sum_i \mathrm{Var}(X_i) + \sum_{i \neq j} \mathrm{Cov}(X_i, X_j)$$

$$\mathrm{var}(X_1 + \cdots + X_n) = \sum_{i=1}^{n} \mathrm{var}(X_i) + \sum_{\{(i,j):\, i \neq j\}} \mathrm{cov}(X_i, X_j)$$

The Correlation coefficient

- Dimensionless version of covariance:

$$\rho(X,Y) = \mathbf{E}\left[\frac{(X-\mathbf{E}[X])}{\sigma_X}\cdot\frac{(Y-\mathbf{E}[Y])}{\sigma_Y}\right] = \frac{\mathrm{cov}(X,Y)}{\sigma_X\sigma_Y}$$

$$-1 \le \rho \le 1$$

- Measure of the degree of "association" between X and Y
- Independent $\Rightarrow$ $\rho = 0$, "uncorrelated" (converse is not true)
- $\rho(X,X) = \dfrac{var(X)}{\sigma_x^2} = 1$
- $|\rho| = 1 \Leftrightarrow (X-\mathbf{E}[X]) = c(Y-\mathbf{E}[Y])$ (linearly related)
- $\mathrm{cov}(aX+b,Y) = a\cdot\mathrm{cov}(X,Y) \Rightarrow \rho(aX+b,Y) = \dfrac{a\,cov(X,Y)}{|a|\sigma_x\sigma_y} = sign(a)\cdot\rho(X,Y)$

Proof of key properties of the correlation coefficient

$$\rho(X,Y) = \mathbf{E}\left[\frac{(X-\mathbf{E}[X])}{\sigma_X}\cdot\frac{(Y-\mathbf{E}[Y])}{\sigma_Y}\right]$$

$$-1 \le \rho \le 1$$

- Assume, for simplicity, zero means and unit variances, so that $\rho(X,Y) = \mathbf{E}[XY]$

$$\mathbf{E}\left[(X-\rho Y)^2\right] = E[X^2] - 2\rho E[XY] + \rho^2 E[Y^2]$$

$$0 \le = 1 - 2\rho^2 + \rho^2 = 1-\rho^2 \qquad 1-\rho^2 \ge 0 \Rightarrow \rho^2 \le 1$$

If $|\rho| = 1$, then $X = \rho Y \Rightarrow X = Y$ or $X = -Y$

Interpreting the correlation coefficient

$$\rho(X,Y) = \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y}$$

- Association does not imply causation or influence

  X: math aptitude
  Y: musical ability

- Correlation often reflects underlying, common, hidden factor
  - Assume, Z, V, W are independent

    $X = Z + V \qquad Y = Z + W$

    Assume, for simplicity, that Z, V, W have zero means, unit variances

$$\rho(X,Y) = \frac{1}{\sqrt{2}\cdot\sqrt{2}} = \frac{1}{2}$$

$$\text{var}(X) = \text{var}(Z) + \text{var}(V) = 2 \Rightarrow \sigma_X = \sqrt{2} \qquad \sigma_Y = \sqrt{2}$$

$$\text{cov}(X,Y) = E[(Z+V)(Z+W)] = E[Z^2] + E[VZ] + E[ZW] + E[VW]$$

$$= 1 + 0 + 0 + 0$$

Correlations matter...

- A real-estate investment company invests \$10M in each of 10 states. At each state i, the return on its investment is a random variable X_i, with mean 1 and standard deviation 1.3 (in millions).

$$\text{var}(X_1 + \cdots + X_{10}) = \sum_{i=1}^{10} \text{var}(X_i) + \sum_{\{(i,j):\, i \neq j\}} \text{cov}(X_i, X_j)$$

$E[X_1 + \cdots + X_{10}] = 10$

- If the X_i are uncorrelated, then:

$\text{var}(X_1 + \cdots + X_{10}) = 10 \cdot (1.3)^2 = 16.9$ $\sigma(X_1 + \cdots + X_{10}) = 4.1$

- If for $i \neq j$, $\rho(X_i, X_j) = 0.9$:

$\text{cov}(X_i, X_j) = \rho\, \sigma_{X_i} \sigma_{X_j} = 0.9 \times 1.3 \times 1.3 = 1.052$

$\text{var}(X_1 + \cdots + X_{10}) = 10 \cdot (1.3)^2 + 90 \cdot 1.052 = 154$

$\sigma(X_1 + \cdots + X_{10}) = 12.4$