9. Performing Wald's Test on a Gaussian Data Set

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3/3 points (graded)

Suppose $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. Your goal is to hypothesis test between

$$\begin{aligned} H_0 &: (\mu, \sigma^2) = (0, 1) \\ H_1 &: (\mu, \sigma^2) \neq (0, 1). \end{aligned}$$

Recall Wald's test from a previous problem, which, under the above hypotheses, takes the form

$$\psi_\alpha := \mathbf{1}\left(W_n > q_\alpha\left(\chi_2^2\right)\right) = \mathbf{1}\left(n\left(\hat{\theta}_n^T - \begin{pmatrix} 0 & 1 \end{pmatrix}\right)\mathcal{I}((0,1))\left(\hat{\theta}_n - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) > q_\alpha\left(\chi_2^2\right)\right)$$

where $q_\alpha\left(\chi_2^2\right)$ is the α-quantile of χ_2^2. You are given that the technical conditions required for the MLE to be asymptotically normal are satisfied for a Gaussian statistical model with unknown mean and variance.

What is the smallest value of $q_\alpha\left(\chi_2^2\right)$ so that ψ_α is a test with asymptotic level 5%?
(You should use a table (e.g. https://people.richland.edu/james/lecture/m170/tbl-chi.html or software (e.g. R) to answer this question.)

For ψ_α to have level 5%:

$q_\alpha\left(\chi_2^2\right) \geq$ 5.991 **Answer:** 5.991

Suppose you observe the data set

$$0.2, -0.1, -1.9, -0.4, -1.8$$

What is the value of the test statistic W_5 for this data set?

Hint: Recall that the MLE of a Gaussian $\mathcal{N}(\mu, \sigma^2)$ is given by

$$\begin{pmatrix} \widehat{\mu}_n^{MLE} \\ \left(\widehat{\sigma^2}\right)_n^{MLE} \end{pmatrix} = \begin{pmatrix} \overline{X}_n \\ \frac{1}{n}\sum_{i=1}^n \left(X_i - \overline{X}_n\right)^2 \end{pmatrix}$$

and the Fisher information is given by

$$\mathcal{I}(\mu, \sigma^2) = \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^4} \end{pmatrix}.$$

$W_5 =$ 3.33 **Answer:** 3.33

Will Wald's test **reject** or **fail to reject** for this data set?

- Reject

Fail to reject

Solution:

Since we have assumed that the MLE is asymptotically normal, we have

$$W_n \xrightarrow[n\to\infty]{(d)} \chi_2^2.$$

There are precisely two degrees of freedom since we have two unknowns. The test ψ_α has asymptotic level 5% if $\alpha = 5\%$. Consulting a table, we see that the 0.05-quantile for χ_2^2 is $q_\alpha = 5.991$.

For the given data set, we compute

$$\begin{aligned} \widehat{\mu}_n^{MLE} &= \overline{X}_n \approx -0.8 \\ \widehat{\sigma^2}_n^{MLE} &= \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}_n\right)^2 \approx 0.772. \end{aligned}$$

The Fisher information, under the null hypothesis $(\mu, \sigma^2) = (0, 1)$, is

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Therefore,

$$W_5 = 5 \cdot ((-0.8, 0.772) - (0, 1)) \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \left(\begin{pmatrix} -0.8 \\ 0.772 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)^T \approx 3.33.$$

Since $q_{0.05} = 5.991 > 3.33$, we would **fail to reject** the null hypothesis for the given sample.

提交 你已经尝试了3次（总共可以尝试3次）

Answers are displayed within the problem

讨论

显示讨论

主题： Unit 4 Hypothesis testing:Lecture 14: Wald's Test, Likelihood Ratio Test, and Implicit Hypothesis Test / 9. Performing Wald's Test on a Gaussian Data Set