6. Biased coin

Problem 5. Biased coin

5.0/5.0 points (graded)

We are given a biased coin, where the probability of Heads is q. The bias q is itself the realization of a random variable Q which is uniformly distributed on the interval $[0,1]$. We want to estimate the bias of this coin. We flip it 5 times, and define the (observed) random variable N as the number of Heads in this experiment.

Throughout this problem, you may find the following formula useful:
For every positive integers n, k,

$$\int_0^1 x^n(1-x)^k\,dx = \frac{n!k!}{(n+k+1)!}.$$

1. Given the observation $N=3$, calculate the posterior distribution of the bias Q. That is, find the conditional distribution of Q, given $N=3$.

   For $0 \le q \le 1$,

   $f_{Q|N}(q \mid N=3) =$ 60*q^3*(1-q)^2 ✔ **Answer:** 60*q^3*(1-q)^2

   $60 \cdot q^3 \cdot (1-q)^2$

2. What is the LMS estimate of Q, given $N=3$?

   $\hat{Q}_{\text{LMS}} =$ 4/7 ✔ **Answer:** 4/7

3. What is the resulting conditional mean squared error of the LMS estimator, given $N=3$?

   3/98 ✔ **Answer:** 3/98

STANDARD NOTATION

Solution:

1. Using the Bayes' rule, we have for $0 \le q \le 1$,

$$\begin{aligned}
f_{Q|N}(q \mid 3) &= \frac{p_{N|Q}(3 \mid q) f_Q(q)}{p_N(n)} \\
&= \frac{\binom{5}{3} q^3(1-q)^2}{\int_0^1 p_{N|Q}(3 \mid q) f_Q(q)\,dq} \\
&= \frac{10q^3(1-q)^2}{\int_0^1 \binom{5}{3} q^3(1-q)^2\,dq} \\
&= \frac{10q^3(1-q)^2}{10\int_0^1 q^3(1-q)^2\,dq} \\
&= \frac{10q^3(1-q)^2}{10\frac{3!2!}{6!}}
\end{aligned}$$

$$= 60q^3(1-q)^2.$$

2. In order to compute the LMS estimate, we have to compute the conditional expectation of Q, given $N=3$, namely, we have to evaluate the quantity $\mathbf{E}[Q \mid N=3]$.

$$\begin{aligned}
\mathbf{E}[Q \mid N=3] &= \int_0^1 q f_{Q|N}(q \mid 3)\, dq \\
&= \int_0^1 60q^4(1-q)^2\, dq \\
&= 60\frac{4!2!}{7!} \\
&= \frac{4}{7}.
\end{aligned}$$

3. The conditional mean squared error of the LMS estimator is the conditional variance:

$$\begin{aligned}
\mathrm{var}(Q \mid N=3) &= \mathbf{E}[Q^2 \mid N=3] - (\mathbf{E}[Q \mid N=3])^2 \\
&= \int_0^1 q^2 f_{Q|N}(q \mid 3)\, dq - \left(\frac{4}{7}\right)^2 \\
&= \int_0^1 60q^5(1-q)^2\, dq - \frac{16}{49} \\
&= 60\frac{5!2!}{8!} - \frac{16}{49} \\
&= \frac{5}{14} - \frac{16}{49} \\
&= \frac{3}{98}.
\end{aligned}$$

提交 你已经尝试了1次（总共可以尝试2次）

Answers are displayed within the problem

Error and Bug Reports/Technical Issues

显示讨论

主题: Final Exam / 6. Biased coin