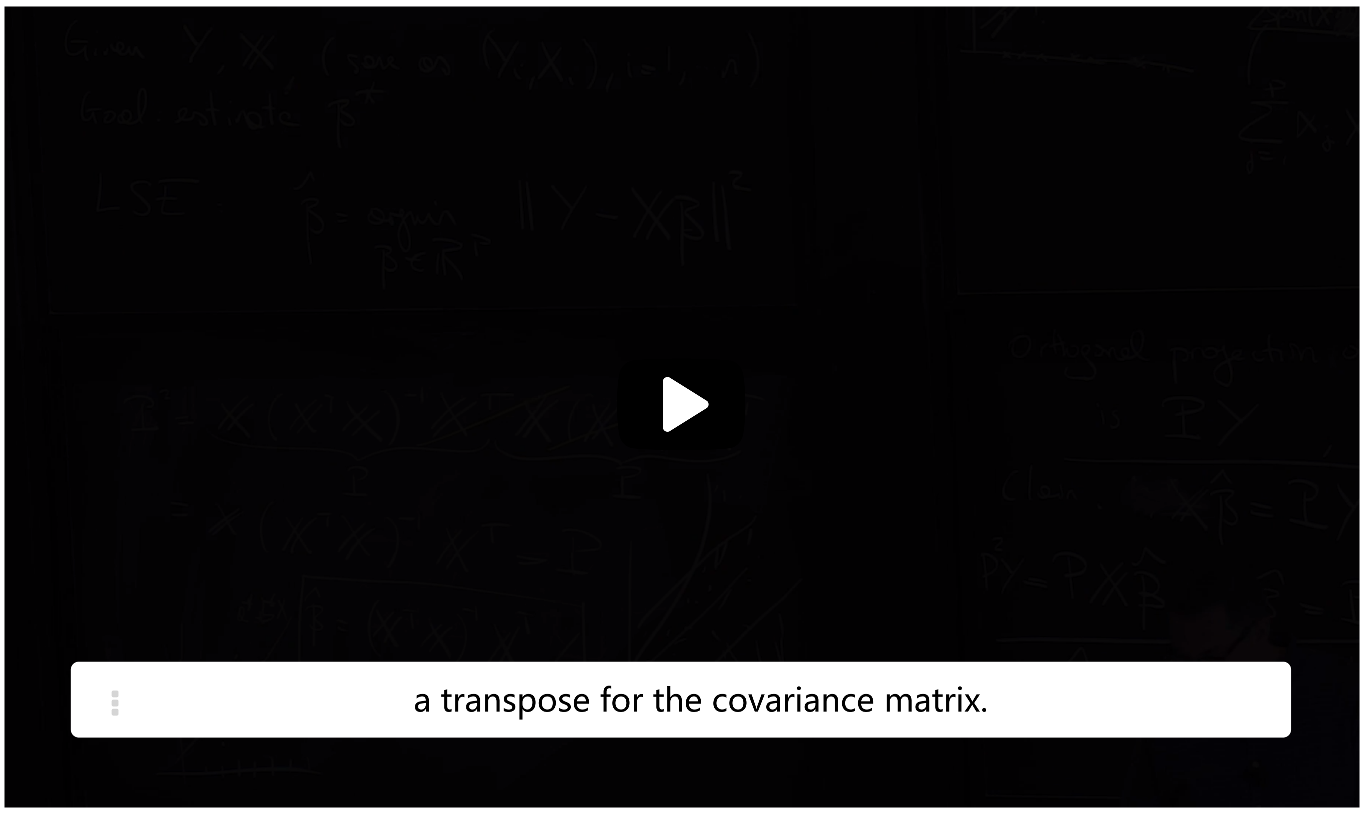
5. Linear Regression with Deterministic Design

Linear Regression with Deterministic Design



8:56 / 8:56 1.0x

So I know that y has this form under this assumption.

So I certainly know what the distribution of this guy

is, because it's just taking a Gaussian

and hitting it with a matrix.

And we know this, right?

We know that if I take an n mu capital sigma

and I hit it with a matrix a, get an n a times mu and a sigma

a transpose for the covariance matrix.

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Deterministic Design

1/1 point (graded)

In the setting of **deterministic design** for linear regression, we assume that the design matrix $\mathbb{X}$ is deterministic instead of random. The **model** still prescribes $\mathbf{Y} = \mathbb{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)$ is a random vector that represents noise. Take note that the only random object on the right hand side is ε, and that Y is **still random**.

For the rest of this section, we will always assume $\left(\mathbb{X}^T\mathbb{X}\right)^{-1}$ exists; i.e. $\mathrm{rank}\,(\mathbb{X}) = p$.

Recall that the Least-Squares Estimator $\hat{\boldsymbol{\beta}}$ has the formula

$$\hat{\boldsymbol{\beta}} = \left(\mathbb{X}^T\mathbb{X}\right)^{-1}\mathbb{X}^T\mathbf{Y}.$$

If we assume that the vector $\boldsymbol{\varepsilon}$ is a random variable with mean $\mathbb{E}\left[\boldsymbol{\varepsilon}\right] = \mathbf{0}$, then in the deterministic design setting: "The LSE $\hat{\boldsymbol{\beta}}$ is a random variable, with mean..." (choose all that apply)

- [ ] 0
- [x] $\left(\mathbb{X}^T\mathbb{X}\right)^{-1}\mathbb{X}^T\mathbb{E}\left[\mathbf{Y}\right]$ ✔
- [ ] $\mathbb{X}^T\mathbb{X}\beta$

- [x] β ✔
- [ ] ϵ

✔

Solution:

- The model is $\mathbf{Y} = \mathbb{X}\beta + \varepsilon$, and ε is a random variable. So $\mathbf{Y}$ should in fact be considered as a random variable.
- Using the formula for $\hat{\beta}$ and applying linearity of expectation, we obtain:

$$\begin{aligned}\mathbb{E}[\hat{\beta}] &= \mathbb{E}\left[(\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T\mathbf{Y}\right] \\ &= \mathbb{E}\left[(\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T\mathbb{X}\beta + (\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T\epsilon\right] \\ &= \beta + (\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T\mathbb{E}[\varepsilon] \\ &= \beta\end{aligned}$$

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Answers are displayed within the problem

Uniform Noise

3/3 points (graded)

Assume that $n = p$, so that the number of samples matches the number of covariates, and that $\mathbb{X}$ has rank n. Recall that the Least-Squares Estimator $\hat{\boldsymbol{\beta}}$ has the formula

$$\hat{\boldsymbol{\beta}} = (\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T\mathbf{Y}.$$

If we assume that the vector $\boldsymbol{\varepsilon} = (\epsilon_1, \ldots, \epsilon_n)$ is uniformly distributed in the n-dimensional box $[-1, +1]^n$, then:

"The model is **homoscedastic** ; i.e. $\varepsilon_1, \ldots, \varepsilon_n$ are i.i.d."

- (x) True ✔
- ( ) False

"In the deterministic design setting, $\mathbf{Y}$ is also deterministic."

- ( ) True
- (x) False ✔

"In the deterministic design setting, the LSE $\hat{\boldsymbol{\beta}}$ is a uniformly distributed random variable."

- (x) True ✔
- ( ) False

(You may use the following fact: in the 1-dimensional case, consider $a \sim \text{Uniform}([0,1])$ and let $\lambda > 0$. Intuitively enough, the distribution of $b = \lambda a$ is uniform over the interval $[0, \lambda]$. More generally, if a is uniformly distributed over a rectangular region $R \subset \mathbb{R}^n$ and M is an $n \times n$ matrix of full rank, then b is uniformly distributed over the region $M(R) \subset \mathbb{R}^n$, the image of R under the transformation M.)

Solution:

- "**The model is homoscedastic ; i.e. $\varepsilon_1, \ldots, \varepsilon_n$ are i.i.d.**" is true. The uniform distribution over $[-1, +1]^n$ is the product distribution of n uniform distributions over $[-1, +1]$. Therefore, each component is i.i.d.

- "**In the deterministic design setting, $\mathbf{Y}$ is also deterministic**" is false. The model is $\mathbf{Y} = \mathbb{X}\beta + \varepsilon$, so $\mathbf{Y}$ is a random variable that is a translation of ε by $\mathbb{X}\beta$.

- "**In the deterministic design setting, the LSE $\hat{\beta}$ is a uniformly distributed random variable**" is true. Note that

$$(\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T\mathbf{Y} = (\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T\mathbb{X}\beta + (\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T\epsilon) = \beta + \boxed{(\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T}\epsilon$$

把误差从n维变换成p维

The random variable $(\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T\epsilon$ is uniformly distributed over the region $(\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T\left([-1, +1]^n\right)$. Uniformity is preserved under translation by β, as well.

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