We will condition on X and use the law of total variance:

$$\text{var}(X+Y) = \mathbf{E}[\text{var}(X+Y \mid X)] + \text{var}(\mathbf{E}[X+Y \mid X]).$$

Given a value x of X, the random variable Y is uniformly distributed in the interval $[x, x+1]$, and the random variable $X+Y$ is uniformly distributed in the interval $[2x, 2x+1]$. Therefore,

$$\mathbf{E}[X+Y \mid X] = 2X + \frac{1}{2}$$

and

$$\text{var}(X+Y \mid X) = \frac{1}{12}.$$

Thus,

$$\text{var}(X+Y) = \mathbf{E}\left[\frac{1}{12}\right] + \text{var}\left(2X + \frac{1}{2}\right) = \frac{1}{12} + 4 \cdot \text{var}(X) = \frac{5}{12}.$$