2. Find the limits

Problem 2. Find the limits

3/3 points (graded)

Let S_n be the number of successes in n independent Bernoulli trials, where the probability of success at each trial is $1/3$. Provide a numerical value, to a precision of 3 decimal places, for each of the following limits. You may want to refer to the standard normal table.

Normal Table Show

1. $$\lim_{n\to\infty} \mathbf{P}\left(\frac{n}{3} - 10 \le S_n \le \frac{n}{3} + 10\right) =$$

0 ✔ **Answer:** 0

2. $$\lim_{n\to\infty} \mathbf{P}\left(\frac{n}{3} - \frac{n}{6} \le S_n \le \frac{n}{3} + \frac{n}{6}\right) =$$

1 ✔ **Answer:** 1

3. $$\lim_{n\to\infty} \mathbf{P}\left(\frac{n}{3} - \frac{\sqrt{2n}}{5} \le S_n \le \frac{n}{3} + \frac{\sqrt{2n}}{5}\right) =$$

0.4514 ✔ **Answer:** 0.4514

Solution:

First, notice that $S_n = X_1 + \cdots + X_n$, where the X_i are independent Bernoulli random variables with parameter $1/3$. Hence, $\mathbf{E}[S_n] = n/3$, and $\mathsf{Var}(S_n) = 2n/9$.

1. Fix an $\epsilon > 0$. No matter how small ϵ is, we have, for sufficiently large n, $\epsilon\sqrt{n} > 10$. For any such large enough n,

$$\begin{aligned}
\mathbf{P}\left(\frac{n}{3} - 10 \le S_n \le \frac{n}{3} + 10\right) &\le \mathbf{P}\left(\frac{n}{3} - \epsilon\sqrt{n} \le S_n \le \frac{n}{3} + \epsilon\sqrt{n}\right) \\
&= \mathbf{P}\left(-\epsilon\sqrt{n} \le S_n - \frac{n}{3} \le \epsilon\sqrt{n}\right) \\
&= \mathbf{P}\left(-\frac{\epsilon\sqrt{n}}{\sqrt{2n/9}} \le \frac{S_n - \frac{n}{3}}{\sqrt{2n/9}} \le \frac{\epsilon\sqrt{n}}{\sqrt{2n/9}}\right) \\
&= \mathbf{P}\left(-\frac{3}{\sqrt{2}}\epsilon \le \frac{S_n - \frac{n}{3}}{\sqrt{2n/9}} \le \frac{3}{\sqrt{2}}\epsilon\right).
\end{aligned}$$

By the Central Limit Theorem,

$$\lim_{n\to\infty} \mathbf{P}\left(-\frac{3}{\sqrt{2}}\epsilon \le \frac{S_n - \frac{n}{3}}{\sqrt{2n/9}} \le \frac{3}{\sqrt{2}}\epsilon\right) = \Phi\left(\frac{3}{\sqrt{2}}\epsilon\right) - \Phi\left(-\frac{3}{\sqrt{2}}\epsilon\right).$$

Since this is true for every $\epsilon > 0$, it is also true in the limit as $\epsilon \downarrow 0$. The final answer then follows from the fact that,

$$\lim_{\epsilon \downarrow 0} \left[\Phi\left(\frac{3}{\sqrt{2}}\epsilon\right) - \Phi\left(-\frac{3}{\sqrt{2}}\epsilon\right) \right] = \Phi(0) - \Phi(0) = 0.$$

2. The given event, after some algebraic manipulations, is equivalent to the following event:

$$\left| \frac{S_n}{n} - \frac{1}{3} \right| \leq \frac{1}{6}.$$

   Since $\mathbf{E}[S_n/n] = n/3$, by the weak law of large numbers, the probability of the event above converges to 1 as $n \to \infty$.

3. By the Central Limit Theorem,

$$\begin{aligned} \lim_{n\to\infty} \mathbf{P}\left(\frac{n}{3} - \frac{\sqrt{2n}}{5} \leq S_n \leq \frac{n}{3} + \frac{\sqrt{2n}}{5} \right) &= \mathbf{P}\left(\left| \frac{S_n - \frac{n}{3}}{\sqrt{2n/9}} \right| \leq \frac{\sqrt{2n}/5}{\sqrt{2n/9}} \right) \\ &= \Phi(0.6) - \Phi(-0.6) \\ &\approx 0.4514. \end{aligned}$$

提交 你已经尝试了2次（总共可以尝试3次）

Answers are displayed within the problem

讨论

显示讨论

主题： Unit 8 / Problem Set / 2. Find the limits