3. Gaussian random variables

Moments of Gaussian random variables

5.0/5 points (graded)

Let X be a Gaussian random variable with mean μ and variance σ^2 . Compute the following moments:

Remember that we use the terms **Gaussian random variable** and **normal random variable** interchangeably.

(Enter your answers in terms of μ and σ.)

$\mathbb{E}[X^2] =$ mu^2 + sigma^2 ✔ **Answer:** sigma^2 + mu^2

$\mu^2 + \sigma^2$

$\mathbb{E}[X^3] =$ mu^3 + 3*mu*sigma^2 ✔ **Answer:** 3 * sigma^2 * mu + mu^3

$\mu^3 + 3 \cdot \mu \cdot \sigma^2$

$\mathbb{E}[X^4] =$ mu^4 + 6*mu^2*sigma^ ✔ **Answer:** 3 * sigma^4 + 6 * sigma^2 * mu^2 + mu^4

$\mu^4 + 6 \cdot \mu^2 \cdot \sigma^2 + 3 \cdot \sigma^4$

$\mathsf{Var}(X^2) =$ mu^4 + 6*mu^2*sigma^ ✔ **Answer:** 2 * sigma^4 + 4 * sigma^2 * mu^2

$\mu^4 + 6 \cdot \mu^2 \cdot \sigma^2 + 3 \cdot \sigma^4 - \mu^4 - 2 \cdot \mu^2 \cdot \sigma^2 - \sigma^4$

Write $\mathbf{P}(X > 0)$ in terms of the **cumulative distribution function (cdf)** Φ of the standard Gaussian distribution, that is,

$$\Phi(x) = \mathbf{P}(Z \leq x), \quad x \in \mathbb{R},$$

where $Z \sim \mathcal{N}(0,1)$ is a standard normal variable. (Enter Phi for Φ.)

$\mathbf{P}(X > 0) =$ 1- Phi((0 - mu)/sigma) ✔ **Answer:** 1 - Phi(-mu/sigma)

STANDARD NOTATION

Solution:

We can write a general Gaussian variable $X \sim \mathcal{N}(\mu, \sigma^2)$ as $X = \sigma Z + \mu$, where $Z \sim \mathcal{N}(0,1)$ is a standard normal variable. Hence, the calculation can be made by factoring out the corresponding polynomials and calculating (or looking up) the moments of Z:

$$\begin{aligned} \mathbb{E}[Z] =& \ 0 \\ \mathbb{E}[Z^2] =& \ 1 \\ \mathbb{E}[Z^3] =& \ 0 \\ \mathbb{E}[Z^4] =& \ 3. \end{aligned}$$

As an example, let us compute $\mathbb{E}[X^3]$. Denote the density of a standard normal distribution by $\varphi(z)$, i.e.,

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}.$$

With this, we calculate

$$\begin{aligned}\mathbb{E}[X^3] &= \int_{-\infty}^{\infty} (\sigma z + \mu)^3 \phi(z)\, dz \\ &= \mathbb{E}[Z^3] + 3\sigma^2 \mu \mathbb{E}[Z^2] + 3\sigma\mu^2 \mathbb{E}[Z] + \mu^3 \\ &= 3\sigma^2\mu + \mu^3.\end{aligned}$$

For $\text{Var}(X^2)$, we can use the formula $\text{Var}(X^2) = \mathbb{E}[X^4] - (\mathbb{E}[X^2])^2$.

Similarly, we can express the probability $\mathbf{P}(X > 0)$ as

$$\begin{aligned}\mathbf{P}(X > 0) = & \ \mathbf{P}(\sigma Z + \mu > 0) = \mathbf{P}(\sigma Z > -\mu) \\ = & \ \mathbf{P}\left(Z > -\frac{\mu}{\sigma}\right) = 1 - \Phi\left(-\frac{\mu}{\sigma}\right).\end{aligned}$$

提交 你已经尝试了4次（总共可以尝试4次）

Answers are displayed within the problem

Covariance of Gaussians

4/4 points (graded)

Recall that **i.i.d.** stands for **independent and identically distributed** . A collection of random variables $X_1, \ldots, X_n$ are **i.i.d.** if all of them follow the same distribution, and each X_i does not contain information about the other realizations.

Let X, Y be i.i.d. **standard** normal random variables, that is, $X, Y \sim \mathcal{N}(0,1)$.

Recall that the **covariance** of two random variables X and Y, denoted by $\text{Cov}(X,Y)$, is defined as

$$\text{Cov}(X,Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]. \tag{1.3}$$

Compute the following variances and covariances.

$\text{Var}(X+Y) =$ 2 Answer: 2

$\text{Var}(XY) =$ 1 Answer: 1

$\text{Cov}(X, X+Y) =$ 1 Answer: 1

$\text{Cov}(X, XY) =$ 0 Answer: 0

STANDARD NOTATION

Solution:

Note that by the definition of a standard Gaussian random variable,

$$\mathbb{E}[X] = \mathbb{E}[Y] = 0 \quad \mathbb{E}[X^2] = \mathbb{E}[Y^2] = 1.$$

With this, compute

$$\begin{aligned}\text{Var}(X+Y) = & \ \text{Var}(X) + \text{Var}(Y) && \text{(independence)} \\ = & \ 1 + 1 = 2,\end{aligned}$$

$$\begin{aligned}(XY) = \; & \mathbb{E}\left[(XY)^2\right] - \left(\mathbb{E}\left[XY\right]\right)^2 & \\ = \; & \mathbb{E}\left[X^2\right]\mathbb{E}\left[Y^2\right] - \mathbb{E}[X]^2\mathbb{E}[Y]^2 & \text{(independence)} \\ = \; & 1 \times 1 - 0 = 1, &\end{aligned}$$

$$\begin{aligned}(X, X+Y) = \; & \mathbb{E}\left[X\left(X+Y\right)\right] - \mathbb{E}\left[X\right]\mathbb{E}\left[X+Y\right] & \\ = \; & \mathbb{E}\left[X^2\right] + \mathbb{E}\left[XY\right] - \mathbb{E}\left[X\right]\left(\mathbb{E}\left[X\right] + \mathbb{E}\left[Y\right]\right) & \text{(linearity of expectation)} \\ = \; & \mathbb{E}\left[X^2\right] + \mathbb{E}\left[X\right]\mathbb{E}\left[Y\right] - \mathbb{E}[X]^2 - \mathbb{E}\left[X\right]\mathbb{E}\left[Y\right] & \text{(independence)} \\ = \; & 1, &\end{aligned}$$

$$\begin{aligned}(X, XY) = \; & \mathbb{E}\left[X\left(XY\right)\right] - \mathbb{E}\left[X\right]\mathbb{E}\left[XY\right] & \\ = \; & \mathbb{E}\left[X^2\right]\mathbb{E}\left[Y\right] - \mathbb{E}[X]^2\mathbb{E}\left[Y\right] & \text{(independence)} \\ = \; & 1 \cdot 0 - 0 \cdot 0 = 0. &\end{aligned}$$

: 8. Covariance, 9. Covariance properties, and 10. the variance of a sum in Lecture 12, *Sums of independent random variables; covariance, and correlation*.

提交 你已经尝试了1次（总共可以尝试3次）

Answers are displayed within the problem

True or False: Variance, covariance and independence

2/2 points (graded)
For each of the statements below, determine whether it is true (meaning, always true) or false (meaning, not always true).

- For any two random variables, $\textsf{Var}\left(X+Y\right) = \textsf{Var}\left(X\right) + \textsf{Var}\left(Y\right)$.

  - True
  - False ✔

- If the covariance, $\text{Cov}\left(X, Y\right)$ between two random variables X, Y are 0, then X and Y are independent.

  - True
  - False ✔

STANDARD NOTATION

Solution:

- The first item is False. For any two random variables, it is known that,

$$\textsf{Var}\left(X+Y\right) = \textsf{Var}\left(X\right) + \textsf{Var}\left(Y\right) + 2\text{Cov}\left(X, Y\right).$$

  In particular, if $\text{Cov}\left(X, Y\right) \neq 0$, this does not hold.

- The second item is also false. As a simple example, let $X \sim \text{Unif}\left[-1, 1\right]$ and let $Y = X^2$. Then,

$$\text{Cov}\left(X, Y\right) = \left[XY\right] - \mathbb{E}\left[X\right]\mathbb{E}\left[Y\right] = \mathbb{E}\left[X^3\right] - \mathbb{E}\left[X\right]\mathbb{E}\left[X^2\right] = 0,$$

using the fact that X is centered and symmetric around 0, and its odd moments vanish. Even though they are uncorrelated, they are (highly) dependent, Y is obtained from X, intuitively!

提交 你已经尝试了1次（总共可以尝试3次）

Answers are displayed within the problem

讨论

显示讨论

主题： Unit 0. Course Overview, Syllabus, Guidelines, and Homework on Prerequisites:Homework 0: Probability and Linear algebra Review / 3. Gaussian random variables

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