LECTURE 13: Conditional expectation and variance revisited;
Application: Sum of a random number of independent r.v.'s

- A more abstract version of the conditional expectation
  - view it as a random variable
  - the law of iterated expectations
- A more abstract version of the conditional variance
  - view it as a random variable
  - the law of total variance
- Sum of a random number
  of independent r.v.'s
  - mean
  - variance

Conditional expectation as a random variable

- Function h

  e.g., $h(x) = x^2$, for all x

- Random variable X; what is $h(X)$?
  $= X^2$

- $h(X)$ is the r.v. that takes the value x^2, if X happens to take the value x

- $g(y) = \mathbf{E}[X \mid Y = y] = \sum_x x p_{X|Y}(x \mid y)$
  (integral in continuous case)

- $g(Y)$: is the r.v. that takes the value $\underbrace{\mathbf{E}[X \mid Y = y]}_{g(y)}$, if Y happens to take the value y

- Remarks:
  - It is a function of Y
  - It is a random variable
  - Has a distribution, mean, variance, etc.

Definition: $\mathbf{E}[X|Y] = g(Y)$

The mean of $\mathrm{E}[X \mid Y]$: Law of iterated expectations

- $g(y) = \mathbf{E}[X \mid Y = y]$

$$\mathbf{E}\big[\mathbf{E}[X \mid Y]\big] = \mathbf{E}[X]$$

$$E[X \mid Y] \triangleq g(Y)$$

$$\mathbf{E}\big[\underbrace{\mathbf{E}[X \mid Y]}\big] = E[g(Y)]$$

$$= \sum_y g(y)\, p_Y(y)$$

exp. value rule

$$= \sum_y E[X \mid Y = y]\, p_Y(y)$$

- total exp thm

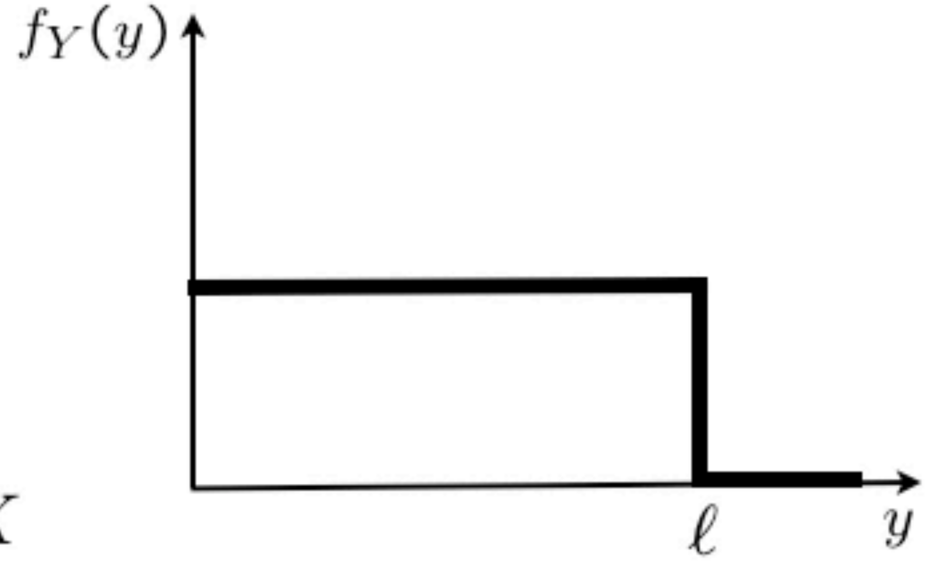
$$= E[X]$$

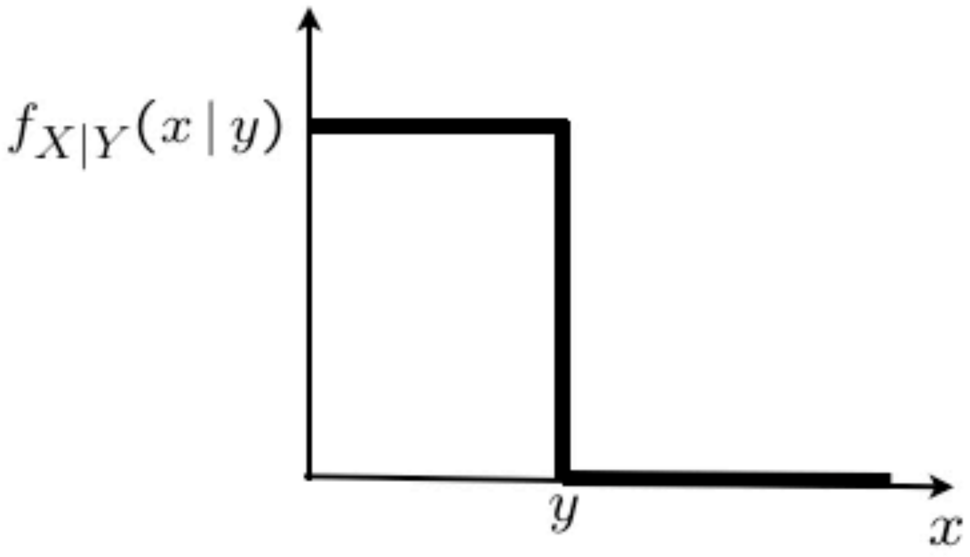
Stick-breaking example

- Stick example: stick of length ℓ

  break at uniformly chosen point Y

  break what is left at uniformly chosen point X





- $\mathbf{E}[X \mid Y = y] =$ $Y/2$

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  $\mathbf{E}[X] = E\big[E[X|Y]\big] = E[Y/2] = \frac{1}{2}E[Y] = \frac{1}{2}\cdot\frac{\ell}{2} = \frac{\ell}{4}$

Forecast revisions

$$\boxed{\mathbf{E}\big[\mathbf{E}[X \mid Y]\big] = \mathbf{E}[X]}$$

- Suppose forecasts are made by calculating expected value, given any available information

- X: February sales

- Forecast in the beginning of the year: $E[X]$

- End of January: will get new information, value y of Y

  Revised forecast: $E[X \mid Y=y]$ $E[X \mid Y]$

- Law of iterated expectations:

  $E[\text{revised forecast}] = E[X] = \text{original forecast}$

The conditional variance as a random variable

$\text{var}(X) = \mathbf{E}\Big[(X - \mathbf{E}[X])^2\Big]$

$\text{var}(X \mid Y = y) = \mathbf{E}\left[(X - \underline{\mathbf{E}[X \mid Y = y]})^2 \mid Y = y\right]$

> $\text{var}(X \mid Y)$ is the r.v. that takes the value $\text{var}(X \mid Y = y)$, when $Y = y$

- Example: X uniform on $[0, Y]$

$\text{var}(X \mid Y = y) = y^2/12$

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> **Law of total variance:** $\text{var}(X) = \mathbf{E}\Big[\text{var}(X \mid Y)\Big] + \text{var}\Big(\mathbf{E}[X \mid Y]\Big)$

Derivation of the law of total variance

$$\boxed{\text{var}(X) = \mathbf{E}\big[\text{var}(X \mid Y)\big] + \text{var}\big(\mathbf{E}[X \mid Y]\big)}$$

- $\text{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2$

$$\text{var}(X \mid Y = y) = E[X^2 \mid Y=y] - \big(E[X \mid Y=y]\big)^2 \quad \text{for all } y$$

$$\text{var}(X \mid Y) = E[X^2 \mid Y] - \big(E[X \mid Y]\big)^2$$

$$\mathbf{E}\big[\text{var}(X \mid Y)\big] = E[X^2] - \cancel{E\Big[\big(E[X \mid Y]\big)^2\Big]}$$

$+$

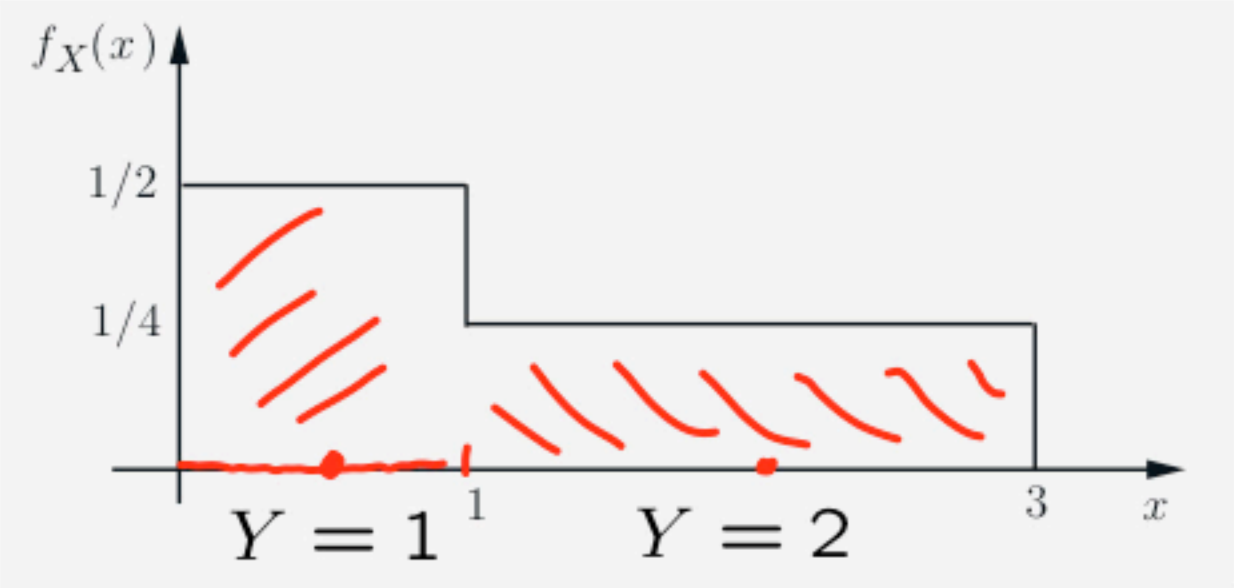
$$\text{var}\big(\mathbf{E}[X \mid Y]\big) = E\Big[\cancel{\big(E[X \mid Y]\big)^2}\Big] - \big(E\big[E[X \mid Y]\big]\big)^2$$

$$\big(E[X]\big)^2$$

A simple example

$$\boxed{\mathrm{var}(X) = \mathbf{E}\big[\mathrm{var}(X \mid Y)\big] + \mathrm{var}\big(\mathbf{E}[X \mid Y]\big)} = \frac{37}{48}$$

$$= 5/24 \qquad + 9/16$$



$$\mathrm{var}(X \mid Y) = \begin{cases} \mathrm{var}(X \mid Y = 1) = 1/12 & 1/2 \\ \mathrm{var}(X \mid Y = 2) = 2^2/12 = \frac{4}{12} & 1/2 \end{cases}$$

$$\mathbf{E}\big[\mathrm{var}(X \mid Y)\big] = \frac{1}{2} \cdot \frac{1}{12} + \frac{1}{2} \cdot \frac{4}{12} = \frac{5}{24}$$

$$\mathbf{E}[X \mid Y] = \begin{cases} \mathbf{E}[X \mid Y = 1] = \frac{1}{2} & 1/2 \\ \mathbf{E}[X \mid Y = 2] = 2 & 1/2 \end{cases}$$

$$\mathrm{var}\big(\mathbf{E}[X \mid Y]\big) = \frac{1}{2}\left(\frac{1}{2} - \frac{5}{4}\right)^2 + \frac{1}{2}\left(2 - \frac{5}{4}\right)^2 = \frac{9}{16}$$

$$\mathbf{E}\big[\mathbf{E}[X \mid Y]\big] = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 2 = \frac{5}{4} = E[X]$$

Section means and variances

- Two sections of a class: $y = 1$ (10 students); $y = 2$ (20 students)

  x_i: score of student i

- Experiment: pick a student at random (uniformly)
  random variables: X and Y

- Data: $y = 1: \ \frac{1}{10}\sum_{i=1}^{10} x_i = 90 \qquad y = 2: \ \frac{1}{20}\sum_{i=11}^{30} x_i = 60$

- $\mathbf{E}[X] = \frac{1}{30}\sum_{i=1}^{30} x_i = \frac{1}{30}(90 \cdot 10 + 60 \cdot 20) = 70$

  $\mathbf{E}[X \mid Y = 1] = 90$

  $\mathbf{E}[X \mid Y = 2] = 60$

  $\mathbf{E}[X \mid Y] = \begin{cases} 90 & 1/3 \\ 60 & 2/3 \end{cases}$

- $\mathbf{E}\big[\mathbf{E}[X \mid Y]\big] = \frac{1}{3} \cdot 90 + \frac{2}{3} \cdot 60 = 70$

Section means and variances (ctd.)

$$\mathbf{E}[X \mid Y] = \begin{cases} 90, & \text{w.p. } 1/3 \\ 60, & \text{w.p. } 2/3 \end{cases} \qquad \mathbf{E}\big[\mathbf{E}[X \mid Y]\big] = 70 = \mathbf{E}[X]$$

$$\mathrm{var}\big(\mathbf{E}[X \mid Y]\big) = \frac{1}{3}(90-70)^2 + \frac{2}{3}(60-70)^2 = 200$$

- More data: $\displaystyle \frac{1}{10}\sum_{i=1}^{10}(x_i - 90)^2 = 10 \qquad \frac{1}{20}\sum_{i=11}^{30}(x_i - 60)^2 = 20$

$$\mathrm{var}(X \mid Y = 1) = 10 \qquad \mathrm{var}(X \mid Y) = \begin{cases} 10, & 1/3 \\ 20, & 2/3 \end{cases}$$

$$\mathrm{var}(X \mid Y = 2) = 20$$

$$\mathbf{E}\big[\mathrm{var}(X \mid Y)\big] = \frac{1}{3}\cdot 10 + \frac{2}{3}\cdot 20 = \frac{50}{3}$$

$$\mathrm{var}(X) = \mathbf{E}\big[\mathrm{var}(X \mid Y)\big] + \mathrm{var}\big(\mathbf{E}[X \mid Y]\big) = 50/3 + 200$$

$$\boxed{\mathrm{var}(X) = (\text{average variability } \textbf{within} \text{ sections}) + (\text{variability } \textbf{between} \text{ sections})}$$

Sum of a random number of independent r.v.'s

$$\boxed{\mathbf{E}[Y] = \mathbf{E}[N] \cdot \mathbf{E}[X]}$$

- N: number of stores visited
  (N is a nonnegative integer r.v.)
- Let $Y = X_1 + \cdots + X_N$
- X_i: money spent in store i
  - X_i independent, identically distributed
  - independent of N

$$\mathbf{E}[Y \mid N = n] = E[X_1 + \cdots + X_N \mid N = n] = E[X_1 + \cdots + X_n \mid N = n]$$
$$= E[X_1 + \cdots + X_n] = n\,E[X]$$

$$E[Y \mid N] = N E[X]$$

- Total expectation theorem:

$$\mathbf{E}[Y] = \sum_n p_N(n)\,\mathbf{E}[Y \mid N = n] = \underbrace{\sum_n p_N(n)\, n}\, E[X] = E[N]\,E[X]$$

- Law of iterated expectations:

$$\mathbf{E}[Y] = \mathbf{E}\big[\mathbf{E}[Y \mid N]\big] = E\big[N E[X]\big] = E[N]\,E[X]$$

Variance of sum of a random number of independent r.v.'s

$Y = X_1 + \cdots + X_N$

- $\boxed{\text{var}(Y) = \mathbf{E}\big[\text{var}(Y \mid N)\big] + \text{var}\big(\mathbf{E}[Y \mid N]\big)}$

$$\boxed{\text{var}(Y) \;=\; \mathbf{E}[N]\,\text{var}(X) + (\mathbf{E}[X])^2\,\text{var}(N)}$$

- $\mathbf{E}[Y \mid N] = N\,\mathbf{E}[X]$

- $\text{var}\big(\mathbf{E}[Y \mid N]\big) = \text{var}(N E[X]) = (E[X])^2 \text{var}(N)$

- $\text{var}(Y \mid N = n) = \text{var}(X_1 + \cdots + X_n \mid N = n) = \text{var}(X_1 + \cdots + X_n)$
  $= n\,\text{var}(X)$

  $\text{var}(Y \mid N) = N \text{var}(X)$

- $\mathbf{E}\big[\text{var}(Y \mid N)\big] = E[N \text{var}(X)] = E[N]\,\text{var}(X)$