8. The One-Dimensional Delta Method

Applying Linear Functions to a Random Sequence

3/3 points (graded)

Let $(Z_n)_{n\geq 1}$ be a sequence of random variables such that

$$\sqrt{n}\,(Z_n - \theta) \xrightarrow[n\to\infty]{(d)} Z$$

for some $\theta \in \mathbb{R}$ and some random variable Z.

Let $g(x) = 5x$ and define another sequence by $Y_n = g(Z_n)$.

The sequence $\sqrt{n}\,(Y_n - g(\theta))$ converges. In terms of Z, what random variable does it converge to?

$$\sqrt{n}\,(Y_n - g(\theta)) \xrightarrow[n\to\infty]{(d)} \quad Y.$$

(Answer in terms of Z)

$Y =$ 5*Z ✔ **Answer:** 5*Z

$5 \cdot Z$

What theorem did we invoke to compute Y?
(There can be more than 1 acceptable answers.)

- ○ Laws of large number
- ○ Central Limit theorem
- ◉ Slutsky theorem ✔
- ○ Continuous mapping theorem ✔

If **Var** $(Z) = \sigma^2$, what is **Var** (Y)? This is the asymptotic variance of $(Y_n)_{n\geq 1}$.
(Answer in terms of σ^2.)

Var $(Y) =$ 25*sigma^2 ✔ **Answer:** 25*sigma^2

$25 \cdot \sigma^2$

STANDARD NOTATION

Solution:

1.

$$\begin{aligned}\sqrt{n}\,(Y_n - g(\theta)) \;=\; \sqrt{n}\,(g(Z_n) - g(\theta)) \;&=\; \sqrt{n}\,(5Z_n - 5\theta)\\ &=\; 5\left(\sqrt{n}\,(Z_n - \theta)\right) \xrightarrow[n\to\infty]{(d)} 5Z\end{aligned}$$

by the continuous mapping theorem because $5\left(\sqrt{n}\left(Z_n-\theta\right)\right)$ is a linear and hence continuous function of Z_n in the last step. Alternatively, since we were given that $\sqrt{n}\left(Z_n-\theta\right) \xrightarrow[n\to\infty]{(d)} Z$, and 5 converges trivially in probability to itself, we can also use Slutsky theorem to conclude.

2. Since $Y=5Z$, Var $(Y)=25$Var $(Z)=25\sigma^2$.

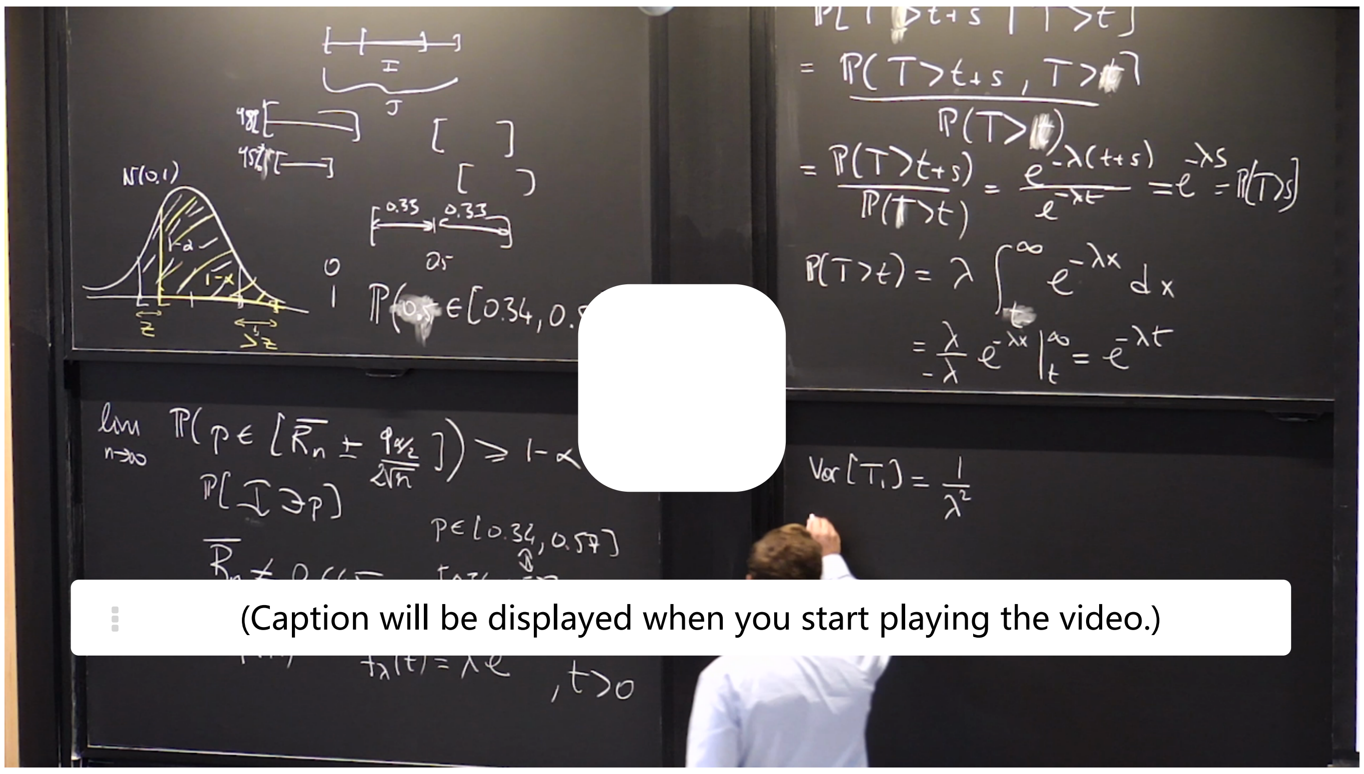
提交 你已经尝试了1次（总共可以尝试2次）

Answers are displayed within the problem

Video note: In the video below, there is an important misprint at roughly 1:26, which will be corrected in the video on the next page. The Central limit theorem applied to $\overline{T}_n$ should read

$$\sqrt{n}\left(\overline{T}_n-\frac{1}{\lambda}\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}\left(0,\frac{1}{\lambda^2}\right).$$

the Delta Method



Well, this is precisely the main contribution technical point

of this lecture.

And it's called the delta method.

So it looks like an innocuous slide.

So just remember this is an important slide.

OK?

Let me actually tell you.

This is important.

We can of course call it the delta method,

if you're into Greeks.

And so this is what it says.

It says, OK, I have a central limit theorem.

So what I have is square of n T n bar minus lambda

goes to some n 0 one over lambda squared, right?

And what I would want is to take a function

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(Optional) Proof of the Delta Method

For simplicity, we will prove only the case when g is continuously differentiable everywhere in $\mathbb{R}$, ie. g and g' exist and are continuous everywhere. Let μ be arbitrary. The mean value theorem (which is the first order statement of Taylor's theorem) states that for any $z>\mu$,

$$g(z)=g(\mu)+g'(c_z)(z-\mu) \qquad \text{for some } c_z\in(\mu,z)$$

Note that c_z is a function of z. This works also for the case $z<\mu$. The two cases together give the statement that for any z:

$$g(z)=g(\mu)+g'(c_z)(z-\mu) \qquad \text{for some } c_z \text{ such that } |c_z-\mu|<|z-\mu|.$$

For each z, we can make a choice of c_z that makes the above statement true: we now think of c as being a function of z (but we will continue to write c_z to denote $c(z)$). This implies that for a random variable Z,

$$g(Z) - g(\mu) = g'(c_Z)(Z - \mu) \qquad \text{for some } c \text{ such that } |c_Z - \mu| < |Z - \mu|.$$

Now, given an arbitrary sequence $(Z_n)_{n\geq 1}$ and for any μ, the above statement is true for each random variable Z_n in the sequence:

$$g(Z_n) - g(\mu) = g'(c_{Z_n})(Z_n - \mu) \qquad \text{for some } c \text{ such that } |c_{Z_n} - \mu| < |Z_n - \mu|.$$

We return to the statistical context. Let $X_1, X_2, \ldots, X_n \stackrel{\text{i.i.d}}{\sim} X$, and let $Z_n = \overline{X}_n$ and $\mu = \mathbb{E}[X]$. Plugging these into the equation above and multiplying by $\sqrt{n}$, we have

$$\sqrt{n}\left(g(\overline{X}_n) - g(\mu)\right) \quad = \quad g'\left(c_{\overline{X}_n}\right)\left(\sqrt{n}(\overline{X}_n - \mu)\right) \qquad \text{where } \left|c_{\overline{X}_n} - \mu\right| < \left|\overline{X}_n - \mu\right|$$

We deal with the two factors on the right hand side separately. By CLT, we know the second factor is asymptotically normal:

$$\left(\sqrt{n}(\overline{X}_n - \mu)\right) \xrightarrow[n\to\infty]{(d)} N(0, \sigma^2).$$

For the second factor, observe that since $\left|c_{\overline{X}_n} - \mu\right| < \left|\overline{X}_n - \mu\right|$, we have for any $\epsilon > 0$,

$$\mathbf{P}\left(\left|c_{\overline{X}_n} - \mu\right| > \epsilon\right) \leq \mathbf{P}\left(\left|\overline{X}_n - \mu\right| > \epsilon\right).$$

Together with the fact that $\overline{X}_n \xrightarrow[n\to\infty]{\mathbf{P}} \mu$, this implies

$$c_{\overline{X}_n} \xrightarrow[n\to\infty]{\mathbf{P}} \mu.$$

Since g' is continuous, by the continous mapping theorem,

$$g'(c_{\overline{X}_n}) \xrightarrow[n\to\infty]{\mathbf{P}} g'(\mu).$$

Finally, by Slutsky Theorem,

$$\sqrt{n}\left(g(\overline{X}_n) - g(\mu)\right) \xrightarrow[n\to\infty]{(d)} N(0, (g'(\mu))^2\sigma^2).$$

Remark: Notice that g' is only needed to be continuously differentiable close to μ.

Hide

讨论

显示讨论

主题: Unit 2 Foundation of Inference:Lecture 5: Delta Method and Confidence Intervals / 8. The One-Dimensional Delta Method

认证证书是什么?