7. Wald's Test in 1 Dimension

Comparing Quantiles

0/1 point (graded)

Let $Z \sim \mathcal{N}(0,1)$. Then $Z^2 \sim \chi_1^2$.

The **quantile** $q_\alpha(\chi_1^2)$ of the χ_1^2-distibution is the number such that

$$\mathbf{P}\left(Z^2 > q_\alpha(\chi_1^2)\right) = \alpha.$$

Find the quantiles of the χ_1^2 distribution in terms of the quantiles of the normal distribution. That is, write $q_\alpha(\chi_1^2)$ in terms of $q_{\alpha'}(\mathcal{N}(0,1))$ where α' is a function of α.

(Enter **q(alpha)** for the quantile $q_\alpha(\mathcal{N}(0,1))$ of the normal distribution.)

$q_\alpha(\chi_1^2) =$ (q(2*alpha))^2 **Answer:** (q(alpha/2))^2

STANDARD NOTATION

Solution:

Since $Z^2 > q$ for any $q > 0$ if and only if $|Z| > \sqrt{q}$, we have

$$P\left(Z^2 > q_\alpha(\chi_1^2)\right) = P\left(|Z| > \sqrt{q_\alpha(\chi_1^2)}\right) = \alpha.$$

Since $Z \sim \mathcal{N}(0,1)$, $P\left(\boxed{|Z|} > \sqrt{q_\alpha(\chi_1^2)}\right) = \alpha$ if and only if

inherently two-side

因为有绝对值 $\boxed{\sqrt{q_\alpha(\chi_1^2)} = q_{\alpha/2}(\mathcal{N}(0,1))}$

Hence $q_\alpha(\chi_1^2) = q_{\alpha/2}(\mathcal{N}(0,1))^2$.

For example, for $\alpha = 5\%$, using a table (e.g. https://people.richland.edu/james/lecture/m170/tbl-chi.html) or software (e.g. R), we have

$$\begin{aligned} q_{0.05}(\chi_1^2) &\approx 3.84. \\ \left(q_{0.025}(\mathcal{N}(0,1))\right)^2 &\approx (1.96)^2 \approx 3.84. \end{aligned}$$

提交 你已经尝试了3次（总共可以尝试3次）

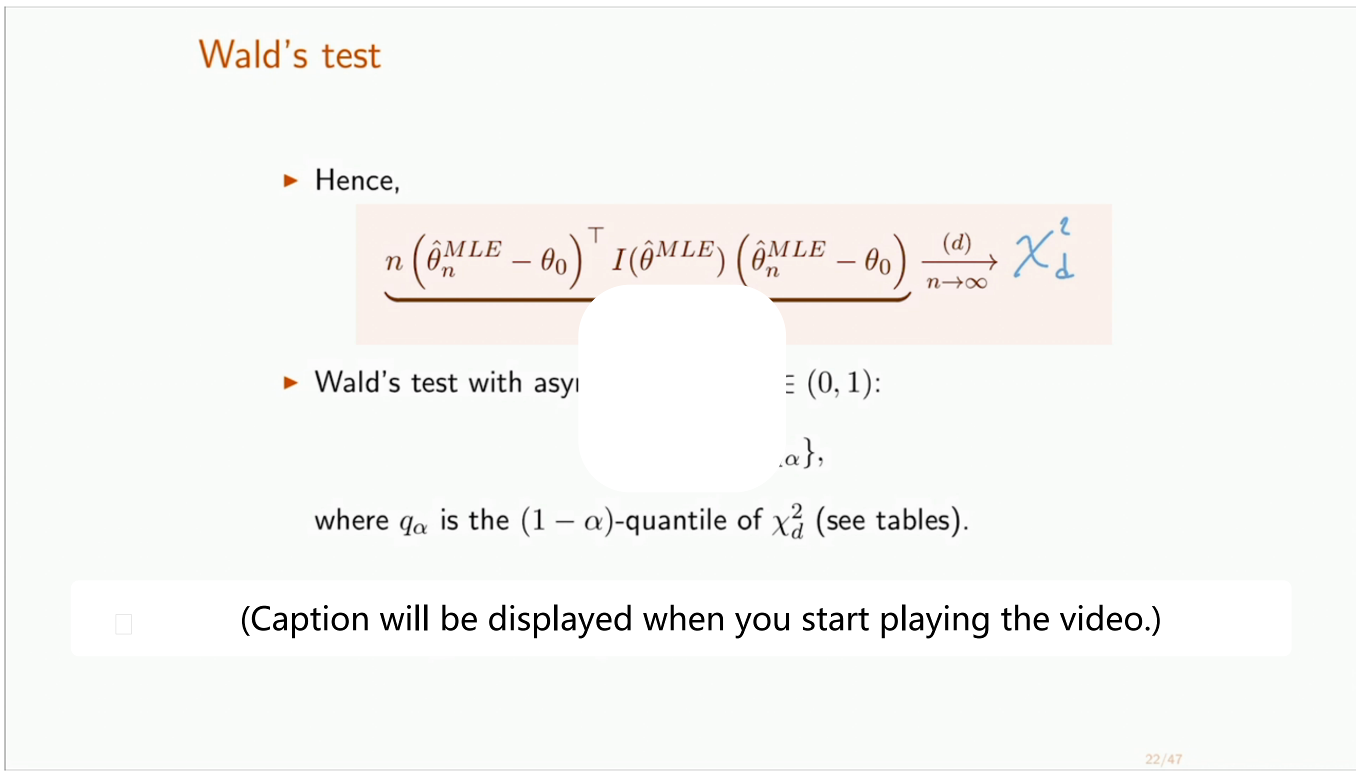
Answers are displayed within the problem

Video note: In the video below at 5:27, Prof Rigollet misprinted on the board: the bottom inequality should read:

$$\sqrt{n}\frac{|\hat{\theta} - \theta|}{\sigma} > q_{\alpha/2}(\mathcal{N}(0,1)).$$

Wald's Test Continued

Start of transcript. Skip to the end.



So now I'm in good shape.

Because I'm going to want to reject when this thing, which

measures proximity from theta-hat to theta-0,

is large or small?

It's large, right?

This is really measuring how far theta-hat is from theta-star

in the right geometry.

视频

下载视频文件

字幕

下载 SubRip (.srt) file

下载 Text (.txt) file

Wald's Test in 1 dimension

In 1 dimension, Wald's Test coincides with the two-sided test based on on the asymptotic normality of the MLE.

Given the hypotheses

$$H_0 : \theta^{\text{MLE}} = \theta_0$$
$$H_1 : \theta^{\text{MLE}} \neq \theta_0,$$

a two-sided test of level α, based on the asymptotic normality of the MLE, is

$$\psi_\alpha = \mathbf{1}\left(\sqrt{nI(\theta_0)}\left|\hat{\theta}^{\text{MLE}} - \theta_0\right| > q_{\alpha/2}\left(\mathcal{N}(0,1)\right)\right)$$

where the Fisher information $I(\theta_0)$ is the asymptotic variance of $\hat{\theta}^{\text{MLE}}$ under the null hypothesis.

On the other hand, a Wald's test of level α is

$$\begin{aligned}\psi_\alpha^{\text{Wald}} &= \mathbf{1}\left(nI(\theta_0)\left(\hat{\theta}^{\text{MLE}} - \theta_0\right)^2 > q_\alpha(\chi_1^2)\right)\\ &= \mathbf{1}\left(\sqrt{nI(\theta_0)}\left|\hat{\theta}^{\text{MLE}} - \theta_0\right| > \sqrt{q_\alpha(\chi_1^2)}\right).\end{aligned}$$

Using the result from the problem above, we see that the two-sided test of level α is the same as Wald's test at level α.

讨论

显示讨论