# 3. Concave functions

## (a)

3/3 points (graded)

Are the following functions concave, convex, or neither?

$$f_1(x) = \ln x, \quad x > 0.$$

- (•) Concave □
- ( ) Convex
- ( ) Not concave and not convex

$$f_2(x) = -x^4 + x^2 - 40x, \quad x \in \mathbb{R}$$

- ( ) Concave
- ( ) Convex
- (•) Not concave and not convex □

$$f_3(x) = \frac{1}{\exp(x) - 1}, \quad x > 0$$

- ( ) Concave
- (•) Convex □
- ( ) Not concave and not convex

**Solution:**

Recall that for a twice continuously differentiable function $f$, we can check concavity by testing whether $f''(x) \leq 0$ for all $x$ in the (convex) domain in question.

To begin, compute

$$f_1'(x) = \frac{1}{x}$$

$$f_1''(x) = -\frac{1}{x^2} < 0, \quad \text{for } x > 0,$$

so $f_1$ is concave.

$$\begin{aligned} f_2'(x) &= -4x^3 + 2x - 40 \\ f_2''(x) &= -12x^2 + 2, \end{aligned}$$

which means $f_2''(x) > 0$ for $x \in \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$, but $f_2''(x) < 0$ for $x \notin \left[-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right]$, hence $f_2$ is neither concave or convex.

$$\begin{aligned} f_3'(x) &= -\frac{e^x}{(e^x - 1)^2} \\ f_3''(x) &= -\frac{e^x(e^x - 1) - 2e^{2x}}{(e^x - 1)^3} \\ &= \frac{e^{2x} + e^x}{(e^x - 1)^3} > 0, \quad \text{for } x > 0. \end{aligned}$$

That means that $f$ is convex for $x > 0$.

提交 你已经尝试了1次（总共可以尝试1次）

Answers are displayed within the problem

## (b)

2/2 points (graded)

A symmetric $2 \times 2$ matrix $\mathbf{A}$ (i.e. $\mathbf{A}^T = \mathbf{A}$) is negative semi- definite, i.e. $\mathbf{x}^T \mathbf{A} \mathbf{x} \leq 0$ for all $\mathbf{x} \in \mathbb{R}^2$, if and only if both of the following is true:

- $\mathbf{tr}(\mathbf{A}) \leq 0$
- $\det(\mathbf{A}) \geq 0$

(This fact can be explained in terms the eigenvalues of $\mathbf{A}$. Let $\lambda_1$ and $\lambda_2$ be the eigenvalues of $\mathbf{A}$, then $\mathbf{tr}(\mathbf{A}) = \lambda_1 + \lambda_2$ while $\det(\mathbf{A}) = \lambda_1 \lambda_2$. The two conditions above ensure that $\lambda_1, \lambda_2 \leq 0$.)

Use the fact given above to determine whether the following functions concave, convex, or neither.

$$f_4(\theta_1, \theta_2) = -\theta_1^2 + \frac{1}{2}(\theta_1 - \theta_2)^2 - 3\theta_2^2, \quad (\theta_1, \theta_2) \in \mathbb{R}^2$$

- (•) Concave
- ( ) Convex
- ( ) Not concave and not convex

$$f_5(\theta_1, \theta_2) = -\theta_1^4 - \theta_2^4 - (\theta_2 - \theta_1)^3, \quad (\theta_1, \theta_2) \in \mathbb{R}^2, \ \text{with } \theta_1 < \theta_2$$

- (selected) Concave
- Convex
- Not concave and not convex

**Solution:**

If $f$ is function from $\Omega \subseteq \mathbb{R}^d \to \mathbb{R}$, then it is concave if the Hessian of $f$ is negative semi-definite. In the special case of two dimensions, this can be checked by testing whether both $\mathbf{tr}\,\nabla^2 f \leq 0$ and $\det \nabla^2 f \geq 0$ are true.

$$\nabla f_4(\theta_1, \theta_2) = \begin{pmatrix} -\theta_1 - \theta_2 \\ -\theta_1 - 5\theta_2 \end{pmatrix}$$
$$H f_4(\theta_1, \theta_2) = \begin{pmatrix} -1 & -1 \\ -1 & -5 \end{pmatrix}.$$

Since $\mathbf{tr}\,\nabla^2 f_4 = -6 < 0$ and $\det \nabla^2 f_4 = 4 > 0$, we have $\nabla^2 f$ is negative semi-definite for all $\theta$, and in turn, $f_4$ is concave.

$$\nabla f_5(\theta_1, \theta_2) = \begin{pmatrix} -4\theta_1^3 + 3(\theta_2 - \theta_1)^2 \\ -4\theta_2^3 - 3(\theta_2 - \theta_1)^2 \end{pmatrix}$$
$$H f_5(\theta_1, \theta_2) = \begin{pmatrix} -12\theta_1^2 - 6(\theta_2 - \theta_1) & 6(\theta_2 - \theta_1) \\ 6(\theta_2 - \theta_1) & -12\theta_2^2 - 6(\theta_2 - \theta_1) \end{pmatrix}.$$

We again check

$$\mathrm{tr}\,\nabla^2 f_5(\theta_1, \theta_2) = -12\theta_1^2 - 12(\theta_2 - \theta_1) - 12\theta_2^2 < 0, \quad \text{if } \theta_1 < \theta_2,$$

$$\begin{aligned} \det \nabla^2 f_5(\theta_1, \theta_2) = &\ (12\theta_1^2 + 6(\theta_2 - \theta_1))(12\theta_2^2 + 6(\theta_2 - \theta_1)) - 36(\theta_2 - \theta_1)^2 \\ = &\ 144\theta_1^2\theta_2^2 + 72(\theta_1^2 + \theta_2^2)(\theta_2 - \theta_1) > 0, \quad \text{if } \theta_1 < \theta_2. \end{aligned}$$

Combined, $f_5$ is concave on $\{\theta_1 < \theta_2\}$.

提交 你已经尝试了1次（总共可以尝试1次）

Answers are displayed within the problem

## 讨论

显示讨论

**主题:** Unit 3 Methods of Estimation:Homework 4: TV distance, KL-Divergence, and Introduction to MLE / 3. Concave functions

认证证书是什么?