4. LLMS estimation with random sums

Problem 4. LLMS estimation with random sums

4/4 points (graded)

Let N be a random variable with mean $\mathbf{E}[N]=m$, and $\textsf{Var}(N)=v$; let $A_1, A_2, \ldots$ be a sequence of i.i.d random variables, all independent of N, with mean 1 and variance 1; let $B_1, B_2, \ldots$ be another sequence of i.i.d. random variables, all independent of N and of $A_1, A_2, \ldots$, also with mean 1 and variance 1. Let $A=\sum_{i=1}^{N} A_i$ and $B=\sum_{i=1}^{N} B_i$.

1. Find the following expectations using the law of iterated expectations. Express each answer in terms of m and v, using standard notation.

   $\mathbf{E}[AB] =$

   v+m^2

   $v+m^2$

   Answer: m^2+v

   $\mathbf{E}[NA] =$

   v+m^2

   $v+m^2$

   Answer: m^2+v

2. Let $\hat{N} = c_1 A + c_2$ be the LLMS estimator of N given A. Find c_1 and c_2 in terms of m and v.

   $c_1 =$ v/(m+v)

   $\frac{v}{m+v}$

   Answer: v/(m+v)

   $c_2 =$ (m^2)/(m+v)

   $\frac{m^2}{m+v}$

   Answer: m^2/(m+v)

STANDARD NOTATION

Solution:

1. We begin by finding $\mathbf{E}[AB]$.

$$
\begin{aligned}
\mathbf{E}[AB] &= \mathbf{E}[(A_1 + \cdots + A_N)(B_1 + \cdots + B_N)] \\
&= \mathbf{E}[\mathbf{E}[(A_1 + \cdots + A_N)(B_1 + \cdots + B_N) mid N]] \\
&= \mathbf{E}[\mathbf{E}[(A_1 + \cdots + A_N) mid N]\mathbf{E}[(B_1 + \cdots + B_N) mid N]] \\
&= \mathbf{E}[N\mathbf{E}[A_1]N\mathbf{E}[B_1]] \\
&= \mathbf{E}[N^2] \\
&= \mathsf{Var}(N) + (\mathbf{E}[N])^2 \\
&= m^2 + v.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathbf{E}[NA] &= \mathbf{E}[\mathbf{E}[N(A_1 + \cdots + A_N) mid N]] \\
&= \mathbf{E}[N\mathbf{E}[A_1 + \cdots + A_N mid N]] \\
&= \mathbf{E}[N(N\mathbf{E}[A_1])] \\
&= \mathbf{E}[N^2] \\
&= m^2 + v.
\end{aligned}
$$

2. A is the sum of a random number, N, of independent and identically distributed random variables $A_1, \ldots, A_N$. Therefore,

$$
\mathbf{E}[A] = \mathbf{E}[\mathbf{E}[A\cancel{mid}N]] = \mathbf{E}[\mathbf{E}[A_1]N] = m,
$$

and

constant

$$
\mathsf{Var}(A) = \mathsf{Var}(A_i)\mathbf{E}[N] + (\mathbf{E}[A_i])^2\mathsf{Var}(N) = m + v.
$$

Law of total variance: $\mathsf{var}(X) = \mathbf{E}[\mathsf{var}(X \mid Y)] + \mathsf{var}(\mathbf{E}[X \mid Y])$

If X, Y are independent: $\mathsf{var}(X + Y) = \mathsf{var}(X) + \mathsf{var}(Y)$

不适用！因为variance的来源还有A的数量

Similarly, $\mathbf{E}[B] = m$, and $\mathsf{Var}(B) = m + v$. Furthermore,

$$
\begin{aligned}
\mathrm{cov}(N, A) &= \mathbf{E}[NA] - \mathbf{E}[N]\mathbf{E}[A] \\
&= (m^2 + v) - m^2 \\
&= v.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\hat{N} &= \mathbf{E}[N] + \frac{\mathrm{cov}(N, A)}{\mathsf{Var}(A)}(A - \mathbf{E}[A]) \\
&= m + \frac{v}{m + v}(A - m) \\
&= \frac{m^2}{m + v} + \frac{v}{m + v}A.
\end{aligned}
$$