5. Exponential random variables

Sums and products

3/3 points (graded)

Let X be an exponential random variable with parameter $\lambda > 0$ and Y be a Poisson random variable with parameter $\mu > 0$. Assume that X and Y are independent. Compute the following quantities:

$\mathbb{E}[X^2 + Y^2] =$ 2/lambda^2 + mu^2 + m ✔ **Answer:** 2/(lambda^2)+ mu + mu^2

$\frac{2}{\lambda^2} + \mu^2 + \mu$

$\mathbb{E}[X^2 Y] =$ 2*mu/lambda^2 ✔ **Answer:** 2 * mu / (lambda^2)

$\frac{2\cdot\mu}{\lambda^2}$

$\text{Var}(2X + 3Y) =$ 4/lambda^2 + 9*mu ✔ **Answer:** 4/(lambda^2) + 9 * mu

$\frac{4}{\lambda^2} + 9\cdot\mu$

STANDARD NOTATION

Solution:

First, let us review the moments of the Exponential and Poisson distribution:

If $X \sim \text{Exp}(\lambda)$ with $\lambda > 0$, then

$$\mathbb{E}[X] = \lambda, \quad \mathbb{E}[X^2] = \frac{2}{\lambda^2}, \quad \text{Var}(X) = \frac{1}{\lambda^2}. \tag{1.4}$$

If $Y \sim \text{Poi}(\mu)$, again with $\mu > 0$, then

$$\mathbb{E}[Y] = \mu, \quad \mathbb{E}[Y^2] = \mu + \mu^2, \quad \text{Var}(Y) = \mu. \tag{1.5}$$

Now, we can use the rules for expectation and variance to calculate:

$$\begin{aligned}
\mathbb{E}[X^2 + Y^2] &= \mathbb{E}[X^2] + \mathbb{E}[Y^2] && \text{(linearity of expectation)} \\
&= \frac{2}{\lambda^2} + \mu + \mu^2 \\
\mathbb{E}[X^2 Y] &= \mathbb{E}[X^2]\,\mathbb{E}[Y] && \text{(multiplicativity of expectation for independent variables)} \\
&= \frac{2\mu}{\lambda^2} \\
\text{Var}(2X + 3Y) &= \text{Var}(2X) + \text{Var}(3Y) && \text{(additivity of variance for independent variables)} \\
&= 2^2\text{Var}(X) + 3^2\text{Var}(Y) && \text{(scaling property of variance)} \\
&= \frac{4}{\lambda^2} + 9\mu
\end{aligned}$$

提交 你已经尝试了2次（总共可以尝试3次）

Answers are displayed within the problem

Estimators

0/1 point (graded)
Let $X_1, \ldots, X_n$ be i.i.d exponential random variables with parameter λ and let $Z_i = \mathbf{1}(X_i \le 1), i = 1, \ldots, n$. Recall that $\mathbf{1}(X \le 1)$ denotes the **indicator function** that takes the value 1 when $X \le 1$ and 0 otherwise.

What is the limit in probability, as n goes to infinity, of $\frac{1}{n}\sum_{i=1}^{n} Z_i$?

$\frac{1}{n}\sum_{i=1}^{n} Z_i \xrightarrow[n\to\infty]{\mathbf{P}}$ exp(-1*lambda) ✖ **Answer:** 1 - exp(-lambda)

$\exp(-1 \cdot \lambda)$

STANDARD NOTATION

Solution:

Since the X_i are independent and identically distributed, so are the Z_i. By the Law of Large Numbers, we know that

$$\frac{1}{n}\sum_{i=1}^{n} Z_i \xrightarrow[n\to\infty]{\mathbf{P}} \mathbb{E}[Z_i],$$

PDF: $f(x) = \lambda \cdot e^{-\lambda x}$
CDF: $F(x) = \int f(x) = 1 - e^{-\lambda x} = P(X \le x)$

so it is enough to compute that quantity.

For this, note that

$$\mathbb{E}[Z_i] = \mathbf{P}(X_i \le 1) = 1 - \exp(-\lambda \times 1) = 1 - \exp(-\lambda),$$

which follows from the formula for the cdf of an Exponential distribution. Hence,

$$\frac{1}{n}\sum_{i=1}^{n} Z_i \xrightarrow[n\to\infty]{\mathbf{P}} 1 - \exp(-\lambda),$$

:

提交 你已经尝试了3次（总共可以尝试3次）

Answers are displayed within the problem

Properties of the exponential distribution

2/2 points (graded)
Let X be an exponential random variable with parameter 2 that models the lifetime (in years) of a lightbulb. Compute the probability that the lightbulb lasts for at least 2 years. Round your answer to the nearest 10^{-2}.

$\mathbf{P}(X \ge 2) =$ exp(-4) ✔ **Answer:** exp(-4)

Given the lightbulb has lasted 2 years, find the probability that it lasts for k more years for any positive integer k.

$\mathbf{P}(X \ge k+2 | X \ge 2) =$ exp(-2*k) ✔ **Answer:** exp(-2*k)

$\exp(-2 \cdot k)$

STANDARD NOTATION

Solution:

The exponential distribution with parameter λ has a continuous density on $(0, \infty)$ with cdf

$$F(x) = 1 - \exp(-\lambda x).$$

PDF: $f(x) = \lambda \cdot e^{-\lambda x}$
CDF: $F(x) = \int f(x) = 1 - e^{-\lambda x} = P(X \le x)$

Hence, for $\lambda = 2$,

$$\mathbf{P}\left(X \geq 2\right) = 1 - \mathbf{P}\left(X \leq 2\right) = 1 - \left(1 - \exp\left(-2 \times 2\right)\right) = e^{-4}.$$

For the second part, note that $\{X \geq k+2\} \subseteq \{X \geq 2\}$. Therefore,

$$\mathbf{P}\left(X \geq k+2 | X \geq 2\right) = \frac{\mathbf{P}\left(\{X \geq k+2\} \cap \{X \geq 2\}\right)}{\mathbf{P}\left(X \geq 2\right)} = \frac{\mathbf{P}\left(X \geq k+2\right)}{\mathbf{P}\left(X \geq 2\right)} = \frac{e^{-2(k+2)}}{e^{-4}} = e^{-2k}.$$

Remark: This is an example of the exponential distribution being **memoryless** : The probability of the lightulb lasting k more years given that it already lasted 2 years is exactly the same as the probability of it lasting k years in the first place.

提交 你已经尝试了2次（总共可以尝试3次）

Answers are displayed within the problem

讨论

显示讨论

主题: Unit 0. Course Overview, Syllabus, Guidelines, and Homework on Prerequisites:Homework 0: Probability and Linear algebra Review / 5. Exponential random variables

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