3. A joint PDF given by a simple formula

Problem 3. A joint PDF given by a simple formula

2/4 points (graded)

The random variables X and Y are distributed according to the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} ax^2, & \text{if } 1 \leq x \leq 2 \text{ and } 0 \leq y \leq x, \\ 0, & \text{otherwise.} \end{cases}$$

1. Find the constant a.

$a =$ 4/15 ✔ **Answer:** 4/15

2. Determine the marginal PDF $f_Y(y)$.
(Your answer can be either numerical or an algebraic function of y).

Useful fact: You may find the following fact useful: $\int_a^b x^2\ dx = \frac{1}{3}(b^3 - a^3)$.

If $0 \leq y \leq 1$:

$f_Y(y) =$ 28/45 ✔ **Answer:** 28/45

$\frac{28}{45}$

If $1 < y \leq 2$:

$f_Y(y) =$ 1/((-1/4)*y^4+(1/3)*y^3+4/3) ✘ **Answer:** (32-4*y^3)/45

$\frac{1}{\left(-\frac{1}{4}\right)\cdot y^4+\left(\frac{1}{3}\right)\cdot y^3+\frac{4}{3}}$

3. Determine the conditional expectation of $1/(X^2Y)$, given that $Y = 5/4$.

$\mathbf{E}\left[\frac{1}{X^2Y}\middle| Y = \frac{5}{4}\right] =$ 12/75 ✘ **Answer:** 64/215

STANDARD NOTATION

Solution:

1. The joint PDF has to integrate to 1. From

$$\int_1^2 \int_0^x ax^2 \, dy \, dx = \int_1^2 ax^3 \, dx = \frac{15}{4} a = 1,$$

we get $a = 4/15$.

2. To find the marginal PDF of Y, we integrate the joint PDF over x:

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx \\ &= \begin{cases} \int_1^2 \frac{4}{15} x^2 \, dx, & \text{if } 0 \le y \le 1, \\ \int_y^2 \frac{4}{15} x^2 \, dx, & \text{if } 1 < y \le 2, \\ 0, & \text{otherwise}, \end{cases} \\ &= \begin{cases} \frac{28}{45}, & \text{if } 0 \le y \le 1, \\ \frac{4}{45}(8 - y^3), & \text{if } 1 < y \le 2, \\ 0, & \text{otherwise}. \end{cases} \end{aligned}$$

3. We first find the conditional PDF of X given $Y = 5/4$:

$$f_{X|Y}\left(x \,\middle|\, \frac{5}{4}\right) = \frac{f_{X,Y}(x, \frac{5}{4})}{f_Y(\frac{5}{4})} = \frac{\frac{4}{15} x^2}{\frac{4}{45}\left(8 - \left(\frac{5}{4}\right)^3\right)} = \frac{64}{129} x^2, \text{ for } \frac{5}{4} \le x \le 2.$$

and equals 0 otherwise. Then,

$$\mathbf{E}\left[\frac{1}{X^2 Y} \,\middle|\, Y = \frac{5}{4}\right] = \mathbf{E}\left[\frac{4}{5X^2} \,\middle|\, Y = \frac{5}{4}\right] = \int_{-\infty}^{\infty} \frac{4}{5x^2} \cdot f_{X|Y}\left(x \,\middle|\, \frac{5}{4}\right) dx,$$

which evaluates to

$$\int_{5/4}^2 \frac{4}{5x^2} \cdot \frac{64}{129} x^2 \, dx = \frac{64}{215}.$$