5. A confidence interval for Poisson variables

(a)

2/2 points (graded)

Let $X_1, \ldots, X_n$ be i.i.d. Poisson random variables with parameter $\lambda > 0$ and denote by $\overline{X}_n$ their empirical average,

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Find two sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ such that $a_n\left(\overline{X}_n - b_n\right)$ converges in distribution to a standard Gaussian random variable $Z \sim N(0,1)$.

$a_n =$ sqrt(n/lambda) ✔ **Answer:** sqrt(n/lambda)

$\sqrt{\frac{n}{\lambda}}$

$b_n =$ lambda ✔ **Answer:** lambda

λ

STANDARD NOTATION

Solution:

We want to apply the Central Limit Theorem. Hence, we need to know the mean and variance of X_i,

$$\mathbb{E}\left[X_i\right] = \lambda, \quad \textsf{Var}\left(X_i\right) = \lambda.$$

Therefore,

$$\sqrt{\frac{n}{\lambda}}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \lambda\right) \xrightarrow{(d)} Z \sim \mathcal{N}(0,1).$$

Therefore, pick

$$a_n = \sqrt{\frac{n}{\lambda}}, \quad b_n = \lambda.$$

提交 你已经尝试了1次（总共可以尝试3次）

Answers are displayed within the problem

(b)

1.0/1 point (graded)

Secondly, express $\mathbf{P}\left(|Z| \leq t\right)$ in terms of $\Phi(r) = \mathbf{P}\left(Z \leq r\right)$ for $t > 0$.

Write Phi(t) (with capital P)for $\Phi(t)$.

$\mathbf{P}(|Z| \le t) =$ 2*Phi(t) - 1 ✔ **Answer:** 2*Phi(t) - 1

STANDARD NOTATION

Solution:

In order to express $\mathbf{P}(|Z| \le t)$ in terms of $\Phi(t)$, first observe that by substitution in the Gaussian integral and symmetry,

$$\begin{aligned}\mathbf{P}(Z \ge -t) &= \frac{1}{\sqrt{2\pi}} \int_{-t}^{\infty} \exp\left(-\frac{x^2}{2}\right)\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left(-\frac{(-x)^2}{2}\right)\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left(-\frac{x^2}{2}\right)\, dx = \mathbf{P}(Z \le t).\end{aligned}$$

Then, apply this to write

$$\begin{aligned}\mathbf{P}(|Z| \le t) &= \mathbf{P}(-t \le Z \le t) \\ &= \mathbf{P}(Z \le t) - \mathbf{P}(Z \le -t) \\ &= \mathbf{P}(Z \le t) - (1 - \mathbf{P}(Z \ge -t)) \\ &= \mathbf{P}(Z \le t) - 1 + \mathbf{P}(Z \le t) \\ &= 2\Phi(t) - 1.\end{aligned}$$

提交 你已经尝试了1次（总共可以尝试3次）

Answers are displayed within the problem

(c)

2/2 points (graded)

Using the previous questions, find an interval $\mathcal{I}_\lambda$ that **depends on** λ and that is centered around $\overline{X}_n$ such that

$$\mathbf{P}[\mathcal{I}_\lambda \ni \lambda] \to .95, \quad n \to \infty.$$

(In other words, the interval before applying any of the 3 methods.)

(Write barX_n for $\overline{X}_n$.)

(*Hint*: The 97.5% -quantile of the standard Gaussian distribution is 1.96.)

$\mathcal{I}_\lambda = [A, B]$ for

$A =$ barX_n - sqrt(lambda/n)* ✔ **Answer:** barX_n - 1.96 * sqrt(lambda/n) $B =$ barX_n + sqrt(lambda/n) ✔

Answer: barX_n + 1.96 * sqrt(lambda/n)

STANDARD NOTATION

Solution:

Combining the first two questions, by setting

$$q = \Phi^{-1}(0.975) = 1.96,$$

we see that

$$\mathbf{P}\left(\sqrt{\frac{n}{\lambda}}\left(\overline{X}_n - \lambda\right) \in [-q, q]\right) \to \mathbf{P}\left(Z \in [-q, q]\right) = 2\Phi\left(q\right) - 1 = 2 \times 0.975 - 1 = 0.95.$$

Hence, we have

$$\mathcal{I}_\lambda := \left[\overline{X}_n - 1.96\sqrt{\frac{\lambda}{n}}, \overline{X}_n + 1.96\sqrt{\frac{\lambda}{n}}\right],$$

where $\mathcal{I}_\lambda$ is centered about $\overline{X}_n$ and

$$\mathbf{P}\left(\lambda \in \mathcal{I}_\lambda\right) \to 0.95,$$

as desired.

提交 你已经尝试了3次（总共可以尝试3次）

Answers are displayed within the problem

(d)

0/1 point (graded)

Which of the following is a confidence interval $\mathcal{J}$ that fulfills

$$\mathbf{P}\left[\mathcal{J} \ni \lambda\right] \to .95, \quad n \to \infty.$$

(Choose all that apply.)

- [x] $\mathcal{J} = [\overline{X}_n - 1.96\sqrt{\lambda/n},\ \overline{X}_n + 1.96\sqrt{\lambda/n}]$ depends on lambda
- [x] $\mathcal{J} = [\overline{X}_n - 1.96\sqrt{\overline{X}_n/n^2},\ \overline{X}_n + 1.96\sqrt{\overline{X}_n/n^2}]$ narrower
- [x] $\mathcal{J} = [\overline{X}_n - 1.96\sqrt{\overline{X}_n/n},\ \overline{X}_n + 1.96\sqrt{\overline{X}_n/n}]$ ✔
- [ ] $\mathcal{J} = [\overline{X}_n - 1.96\sqrt{100/n},\ \overline{X}_n + 1.96\sqrt{100/n}]$

✖

Solution:

$\overline{X}_n$ is a consistent estimator of λ by the Law of Large Numbers, so $\sqrt{\frac{n}{\overline{X}_n}}\left(\overline{X}_n - \lambda\right) \to Z \sim \mathcal{N}\left(0, 1\right)$ by Slutsky's Theorem. Hence, we can obtain an interval that does not depend on λ as

$$\mathcal{J} = \left[\overline{X}_n - 1.96\sqrt{\frac{\overline{X}_n}{n}}, \overline{X}_n + 1.96\sqrt{\frac{\overline{X}_n}{n}}\right].$$

All the other choices either depend on λ or will not attain the right asymptotic confidence level 0.95. As a reminder, we wanted:

$$\mathbf{P}(\lambda \in \mathcal{J}) \to 0.95.$$

For some choices of λ, the band around $\overline{X}_n$ will be too small.

提交 你已经尝试了2次（总共可以尝试2次）

Answers are displayed within the problem

讨论

显示讨论

主题： Unit 2 Foundation of Inference:Homework 1: Estimation, Confidence Interval, Modes of Convergence / 5. A confidence interval for Poisson variables

认证证书是什么?