6. Review: Conditional Likelihood and Bayes' Rule

Note: The following two problems will be quick review of the concepts of **conditional probability and Bayes' rule** , which were covered in the previous course 6.431x. **If understanding these problems or their solutions pose a significant difficultly, please review these concepts before proceeding.**

An Observation Model

3/3 points (graded)

Let $\theta \sim \pi(\theta)$ be a parameter supported on $\mathbb{Z}$, the integers. Suppose that we observe random variables

$$Y_i = \theta X_i$$

for $i = 1, \ldots, n$. The outcomes of $X_1, \ldots, X_n$ are unknown to you, but you do know that they are i.i.d. and uniformly distributed on the set $\{-1, 0, 1\}$. Assume that X_i is independent of θ for all i.

Compute each of the probabilities below:

- $\mathbb{P}(Y_1 = 6 \text{and} Y_2 = 0 | \theta = 3) =$

  0 ✔ **Answer:** 0

- $\mathbb{P}(Y_1 = 7 \text{and} Y_2 = -7 \text{and} Y_3 \in \{0, 7\} | \theta = -7) =.$

  2/27 ✔ **Answer:** 2/27

- $\mathbb{P}(Y_1 = Y_2 + Y_3 | \theta = 5).$

  7/27 ✔ **Answer:** 7/27

Solution:

- Note that, conditional on $\theta = 3$,

  $$Y_1 = 6, Y_2 = 0 \implies X_1 = 2, X_2 = 0.$$

  Since $X_1 \in \{-1, 0, 1\}$, this probability is 0, as X_1 cannot be 2.

- Similar to the item above, we have, conditional on $\theta = -7$,

  $$Y_1 = 7, Y_2 = -7, Y_3 \in \{0, 7\} \implies X_1 = -1, X_2 = 1 \quad \text{and} \quad X_3 \in \{0, 1\}.$$

  In particular,

  $$\mathbb{P}(Y_1 = 7 \text{and} Y_2 = -7 \text{and} Y_3 \in \{0, 7\} | \theta = -7) = \mathbb{P}(X_1 = -1 \text{and} X_2 = 1 \text{and} X_3 \in \{0, 1\}),$$

  which, by using independence, is equal to,

$$\mathbb{P}(X_1 = -1 \text{and} X_2 = 1 \text{and} X_3 \in \{0,1\}) = \mathbb{P}(X_1 = -1)\,\mathbb{P}(X_2 = 1)\,\mathbb{P}(X_3 \in \{0,1\}) = \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{2}{3} = \frac{2}{27}.$$

- Note that,

$$Y_1 = Y_2 + Y_3 \iff X_1 = X_2 + X_3.$$

In particular, using the law of total probability,

$$\mathbb{P}(X_1 = X_2 + X_3) = \sum_{i=-1}^{1} \mathbb{P}(X_1 = X_2 + i | X_3 = i)\,\mathbb{P}(X_3 = i) = \frac{1}{3}\cdot\frac{1}{3} + \frac{2}{9}\cdot\frac{1}{3} + \frac{2}{9}\cdot\frac{1}{3} = \frac{7}{27}.$$

since, $\mathbb{P}(X_3 = i) = 1/3$ for each $i \in \{-1,0,1\}$ and

  - if $i = -1$, then $X_1 = X_2 - 1$ iff $(X_1, X_2) = (0,1)$ or $(X_1, X_2) = (-1,0)$;
  - if $i = 0$, then $X_1 = X_2$ in three possible ways: $X_1 = X_2 = j$ for $j \in \{-1,0,1\}$; and
  - if $i = 1$, then $X_1 = X_2 + 1$ iff $(X_1, X_2) = (1,0)$ or $(X_1, X_2) = (0,-1)$.

Submit You have used 3 of 3 attempts

Answers are displayed within the problem

Probability Review: Bayes' Rule

2/2 points (graded)
Assume that, each person is Republican or Democrat with probability $1/2$ for each; independent of any other person. If two persons are of same political view, they become friends with probability a; and if they are of opposite political view, they become friends with probability b.

What is the probability that Amy and Ben are friends?

1/2*(a+b) Answer: (a+b)/2

$\frac{1}{2}\cdot(a+b)$

Given that Amy and Ben are two friends, what is the probability that they have the same political views?

a/(a+b) Answer: a/(a+b)

$\frac{a}{a+b}$

STANDARD NOTATION

Solution:

- Let E be the event that Amy and Ben are friends, and let σ_A denote the view of Amy; and σ_B denote the view of Ben. Observe that,

$$\begin{aligned}\mathbb{P}(\sigma_A = \sigma_B) &= \mathbb{P}(\sigma_A = \sigma_B = \text{Republican}) + \mathbb{P}(\sigma_A = \sigma_B = \text{Democrat}) \\ &= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} \\ &= 1/2,\end{aligned}$$

where, the first line uses the definition (namely, Amy and Ben have the same political view, if and only if, either both are Democrat; or both are Republican), and the second line uses the independence, and uniformity of the distribution. Similarly, $\mathbb{P}(\sigma_A \neq \sigma_B) = 1/2$. With this,

$$\mathbb{P}(E) = \mathbb{P}(E|\sigma_A = \sigma_B)\,\mathbb{P}(\sigma_A = \sigma_B) + \mathbb{P}(E|\sigma_A \neq \sigma_B)\,\mathbb{P}(\sigma_A \neq \sigma_B) = \frac{a+b}{2},$$

using the law of total probability.

- Our goal is to compute,

$$\mathbb{P}(\sigma_A = \sigma_B|E),$$

which, by Bayes' rule;

$$\begin{aligned}\mathbb{P}(\sigma_A = \sigma_B|E) &= \frac{\mathbb{P}(E|\sigma_A = \sigma_B)\,\mathbb{P}(\sigma_A = \sigma_B)}{\mathbb{P}(E)} \\ &= \frac{a\cdot(1/2)}{((a+b)/2)} \\ &= \frac{a}{a+b}.\end{aligned}$$

Submit

You have used 2 of 3 attempts

Answers are displayed within the problem

Discussion

Show Discussion

Topic: Unit 5 Bayesian statistics:Lecture 17: Introduction to Bayesian Statistics / 6. Review: Conditional Likelihood and Bayes' Rule