3. Independent exponential random variables

Problem 2. Independent exponential random variables

2/2 points (graded)

Let X and Y be two independent, exponentially distributed random variables with parameters λ, and μ, respectively.

For each question below, enter your answers using standard notation; enter **mu** for mu and **lambda** for λ.

1. Find the probability that $X \le Y$.

$\mathbf{P}(X \le Y) =$ lambda/(lambda+mu) ✔ **Answer:** lambda/(mu+lambda)

$\frac{\lambda}{\lambda+\mu}$

2. Let $Z = 1/(1+X)$. For $0 < z < 1$:

$f_Z(z) =$ lambda*e^(-1*lambda*(✔ **Answer:** (lambda*exp(lambda)*exp(-lambda/z))/(z^2)

$\frac{\lambda \cdot e^{-1\cdot\lambda\cdot\left(\frac{1}{z}-1\right)}}{z^2}$

STANDARD NOTATION

Solution:

1. Using the law of total probability theorem, and independence of X and Y,

$$\begin{aligned}\mathbf{P}(X \le Y) &= \int_0^\infty \mathbf{P}(X \le y) f_Y(y)\, dy \\ &= \int_0^\infty \mathbf{P}(X \le y)\mu e^{-\mu y}\, dy = \int_0^\infty (1-e^{-\lambda y})\mu e^{-\mu y}\, dy \\ &= \frac{\lambda}{\mu+\lambda}.\end{aligned}$$

2. We have, for $0 < z < 1$,

$$\begin{aligned}\mathbf{P}(Z \le z) &= \mathbf{P}\left(\frac{1}{1+X} \le z\right) \\ &= \mathbf{P}\left(1+X \ge \frac{1}{z}\right) \\ &= \mathbf{P}\left(X \ge \frac{1}{z} - 1\right) \\ &= e^{-\lambda(1/z-1)} \\ &= e^{-\lambda/z} \cdot e^{\lambda}.\end{aligned}$$

Differentiating the expression above with respect to z yields,

$$f_Z(z) = \frac{\lambda}{z^2} e^{-\lambda(1/z-1)} = \frac{\lambda e^{\lambda}}{z^2} e^{-\lambda/z}.$$

提交 You have used 2 of 3 attempts

Answers are displayed within the problem

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I used convolution formula to solve Part 1. The answer is correct ,but am I just lucky?

question posted about 22 hours ago by **sakimarquis**

I am not very familiar with LATEX, please kindly bear with my poor handwriting. Here's my solution.I use $X - Z$ to substitute Y, but when i use $Y + Z$ to substitute X, the result is not defined. So i'm wondering if i was just lucky to coincide with the correct answer.

Suppose, $Z = X - Y$, $Y = X - Z$

$$P(X \leq Y) = P(X - Y \leq 0) = P(Z \leq 0)$$

$$= \int_{-\infty}^{0} f_Z(z)\, dz$$

$$= \int_{-\infty}^{0} \left[\int_{0}^{+\infty} f_X(x) \cdot f_Y(x-z)\, dx \right] dz$$

$$= \int_{-\infty}^{0} \left[\int_{0}^{+\infty} \lambda \cdot e^{-\lambda x} \cdot u \cdot e^{-u(x-z)}\, dx \right] dz$$

$$= \int_{-\infty}^{0} \frac{\lambda u}{\lambda + u} \cdot e^{uz} \cdot dz = \frac{\lambda}{\lambda + u}$$

此帖对所有人可见。

alexannan

about 19 hours ago - about 19 hours ago 前被 **sakimarquis** 标记为答案

Your reasoning looks pretty good to me!

Possibly the reason it didn't work with $y + z$ is that, when you take the outer integral, z takes on negative values. So $y + z$ might be negative, and so you'd need to explicitly take account of the fact that $f_X(x) = 0$ when $x < 0$.

You got lucky with $x - z$ in as much as you accidentally didn't have to take account of the piecewise nature of f_Y, because $x - z$ is always positive when $z < 0$.

Does that make any sense?

I think it's just an issue of the PDFs of X and Y being piecewise, and remembering to take account of that. That's always where I trip up on problems involving integrals. I find it helps to use Iverson brackets to represent piecewise functions. But then that might just be creating more problems!

Thank you for explanation!

sakimarquis 在about 18 hours ago前发表

Just to make sure, so in the case $X = Y + Z$,the inner integral should be $\int_{z}^{\infty} f_X(y+z) f_Y(y) dy$ so f_X would always be positive right?

De-Mai 在about 14 hours ago前发表

I redid the calculations, with limits of dy from -z to infinity instead of 0 to infinity. It leads to the same answer. And yeah, -z so that you exclude all values of y for which fX will be invalid. The reason being exponential PDFs, i.e. fX(x) do not default to 0 when x < 0.

LimYiLe 在about 13 hours ago前发表

That is how I solved it and it is legit.

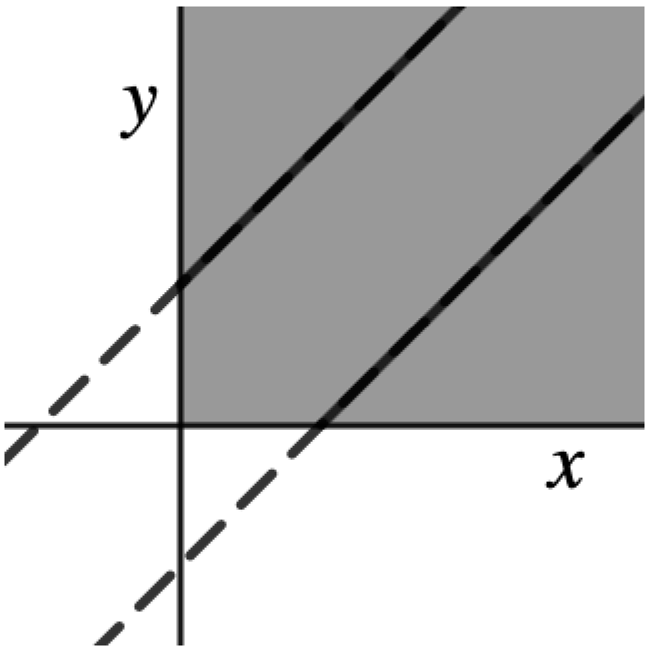
Faithinking 在about 11 hours ago前发表

$\int_{-z}^{\infty} f_X(y+z) f_Y(y) dy$ right, because z is negative.

De-Mai 在about 8 hours ago前发表

I thought I'd provide a more complete answer for anyone who's not clear.

Let's just focus on the inner part --- deriving the marginal PDF of Z. Here's what the line $x - y = z$ looks like when $z < 0$ (the upper line), and when $z > 0$ (the lower line):



Also highlighted is the region where f_X and f_Y are non-zero.

You can integrate over the joint PDF with respect to x or with respect to y, but the result is piecewise either way. You need to be careful about the limits of integration to stay within the shaded region:

[

$$\begin{aligned} f_Z(z) &= \begin{cases} \int_{x=z}^{\infty} f_{X,Y}(x, x-z)dx, & z \geq 0; \\ \int_{x=0}^{\infty} f_{X,Y}(x, x-z)dx, & z < 0; \end{cases} \\ &= \begin{cases} \int_{y=0}^{\infty} f_{X,Y}(y+z, y)dy, & z \geq 0; \\ \int_{y=-z}^{\infty} f_{X,Y}(y+z, y)dy, & z < 0; \end{cases} \\ &= \begin{cases} \frac{\lambda\mu}{\lambda+\mu} e^{-\lambda z}, & z \geq 0; \\ \frac{\lambda\mu}{\lambda+\mu} e^{\mu z}, & z < 0. \end{cases} \end{aligned}$$

]

So to find the answer you want to use the expression for $f_Z(z)$ when $z < 0$:

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$$\begin{aligned} \int_{-\infty}^{0} f_Z(z)dx &= \int_{-\infty}^{0} \frac{\lambda\mu}{\lambda+\mu} e^{\mu z} dz \\ &= \frac{\lambda}{\lambda+\mu}. \end{aligned}$$

]

(Sorry the text in the formulas is so tiny.)

alexannan 在about 7 hours ago前发表

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