Let $Z = X + Y$. Using the 2 step CDF method,

$$\begin{aligned} F_Z(z) &= \mathbf{P}(Z \leq z) \\ &= \mathbf{P}(X + Y \leq z) \end{aligned}$$

Using the Total Probability Theorem, we have

$$\begin{aligned} F_Z(z) &= \sum_x p_X(x)\mathbf{P}(x + Y \leq z) \\ &= \sum_x p_X(x)\mathbf{P}(Y \leq z - x) \\ &= \sum_x p_X(x)F_Y(z - x) \end{aligned}$$

Differentiating both sides with respect to z, we obtain

$$\begin{aligned} f_Z(z) &= \frac{d}{dz}F_Z(z) \\ &= \sum_x p_X(x)f_Y(z - x) \end{aligned}$$