12. Testing Implicit Hypotheses I
Implicit Hypothesis Testing and the Delta Method

Deriving a Test for Implicit Hypotheses

In the next few problems, we derive a general method for testing hypotheses of the form

$$H_0 : g(\theta^*) = 0$$

$$H_1 : g(\theta^*) \neq 0$$

where g is a function of an unknown parameter θ^*. We refer to such hypotheses as **implicit** since θ^* is not isolated in the equations defining the null and alternative hypotheses.

Let's suppose that

- $\theta^* \in \mathbb{R}^d$ is unknown.
- $g : \mathbb{R}^d \to \mathbb{R}^k$ has is continuously differentiable (*i.e.*, the gradient ∇g is continuous).
- $\hat{\theta}_n$ is an asymptotically normal estimator; *i.e.*,

$$\sqrt{n}\left(\hat{\theta}_n - \theta^*\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}\left(\mathbf{0}, \Sigma(\theta^*)\right), \quad \Sigma(\theta^*) \in \mathbb{R}^{d\times d}.$$

Testing Implicit Hypotheses I: The Delta Method

1/1 point (graded)

Recall that $\hat{\theta}_n$ is an asymptotically normal estimator; *i.e.,*

$$\sqrt{n}\left(\hat{\theta}_n - \theta^*\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}\left(\mathbf{0}, \Sigma\left(\theta^*\right)\right), \quad \Sigma\left(\theta^*\right) \in \mathbb{R}^{d\times d}.$$

This implies, by the Delta method, that $g\left(\hat{\theta}_n\right)$ is also asymptotically normal; *i.e.,*

$$\sqrt{n}\left(g\left(\hat{\theta}_n\right) - g\left(\theta^*\right)\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}\left(\mathbf{0}, \Gamma\left(\theta^*\right)\right), \quad \Gamma\left(\theta^*\right) \in \mathbb{R}^{k\times k}.$$

Which of the following is $\Gamma\left(\theta^*\right)$, the asymptotic covariance matrix?

- $\nabla g(\theta^*)^T \Sigma\left(\theta^*\right)$
- $\nabla g(\theta^*)^T \Sigma\left(\theta^*\right) \nabla g\left(\theta^*\right)$
- $\nabla g\left(\theta^*\right) \Sigma\left(\theta^*\right) \nabla g(\theta^*)^T$
- $\nabla g(\theta^*)^{-1} \Sigma\left(\theta^*\right) \left(\nabla g(\theta^*)^{-1}\right)^T$

Solution:

The Delta method states that if

$$\sqrt{n}\left(\hat{\theta}_n - \theta^*\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}\left(\mathbf{0}, \Sigma\left(\theta^*\right)\right),$$

then

$$\sqrt{n}\left(g\left(\hat{\theta}_n\right) - g\left(\theta^*\right)\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}\left(\mathbf{0}, \nabla g(\theta^*)^T \Sigma\left(\theta^*\right) \nabla g\left(\theta^*\right)\right) \in \mathbb{R}^{k\times k}$$

provided that g is continuously differentiable. Hence the second answer choice $\nabla g(\theta^*)^T \Sigma\left(\theta^*\right) \nabla g\left(\theta^*\right)$ is correct.

We can easily see that some of the other given answer choices are incorrect by inspecting the dimensions of the matrices involved. Note that ∇g is a $d \times k$ matrix and $\Sigma\left(\theta^*\right)$ is a $d \times d$ matrix.

- The matrix product $\nabla g(\theta^*)^T \Sigma\left(\theta^*\right)$ will exist, but it is not a square matrix unless $k = d$. Hence, this cannot be a covariance matrix, so the first answer choice is incorrect.
- The matrix product given by $\nabla g\left(\theta^*\right) \Sigma\left(\theta^*\right) \nabla g(\theta^*)^T$ will not exist if $k \neq d$, so the third answer choice is incorrect.
- The fourth answer choice is incorrect. Since ∇g is a $d \times k$ matrix, it will not be invertible if $d \neq k$. Hence, the matrix product $\nabla g(\theta^*)^{-1} \Sigma\left(\theta^*\right) \nabla g(\theta^*)^{-T}$ will not exist in general.

提交 你已经尝试了1次（总共可以尝试3次）

Answers are displayed within the problem

Testing Implicit Hypotheses II: Renormalizing

1/1 point (graded)

As above, by the Delta method, we have that

$$\sqrt{n}\left(g\left(\hat{\theta}_n\right)-g\left(\theta^*\right)\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}\left(\mathbf{0}, \Gamma\left(\theta^*\right)\right),$$

for some matrix $\Gamma\left(\theta^*\right) \in \mathbb{R}^{k\times k}$.

For some real number x,

$$\sqrt{n}\Gamma(\theta^*)^{x}\left(g\left(\hat{\theta}_n\right)-g\left(\theta^*\right)\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}\left(\mathbf{0}, I_k\right).$$

(You are allowed to assume $\Gamma(\theta^*)^{x}$ exists for any $x \in \mathbb{R}$.)

What is x?

-1/2

Answer: -0.5

Solution:

By the properties of multivariate Gaussians, if $\mathbf{Z} \sim \mathcal{N}\left(\mathbf{0}, \Gamma\left(\theta^*\right)\right)$, then

$$\Gamma(\theta^*)^{-1/2}\mathbf{Z} \sim \mathcal{N}\left(\mathbf{0}, I_k\right)$$

provided that $\Gamma(\theta^*)^{-1/2}$ exists. We proved this property in general in the problem "Review: Manipulating Multivariate Gaussians" in the vertical "Introduction to Wald's Test" from this lecture.

Remark: For a square matrix M, we are guaranteed that $M^{-1/2}$ exists if M is **positive-definite**. In particular, since $\Gamma\left(\theta^*\right)$ is a covariance matrix, it is guaranteed to be positive semidefinite. So then $\Gamma(\theta^*)^{-1/2}$ exists if and only if $\Gamma\left(\theta^*\right)$ is invertible. Moreover, by the previous problem,

$$\Gamma\left(\theta^*\right) = \nabla g(\theta^*)^{T}\Sigma\left(\theta^*\right)\nabla g\left(\theta^*\right).$$

Hence, $\Gamma\left(\theta^*\right)$ is invertible if Σ is invertible and $\nabla g\left(\theta^*\right)$ is rank k.

提交 你已经尝试了1次（总共可以尝试3次）

Answers are displayed within the problem

讨论

主题： Unit 4 Hypothesis testing:Lecture 14: Wald's Test, Likelihood Ratio Test, and Implicit Hypothesis Test / 12. Testing Implicit Hypotheses I

显示讨论