Bernoulli distribution

In probability theory and statistics, the **Bernoulli distribution**, named after Swiss mathematician Jacob Bernoulli,[1] is the discrete probability distribution of a random variable which takes the value 1 with probability p and the value 0 with probability $q = 1 - p$, that is, the probability distribution of any single experiment that asks a yes–no question; the question results in a boolean-valued outcome, a single bit of information whose value is success/yes/true/one with probability *p* and failure/no/false/zero with probability *q*. It can be used to represent a (possibly biased) coin toss where 1 and 0 would represent "heads" and "tails" (or vice versa), respectively, and *p* would be the probability of the coin landing on heads or tails, respectively. In particular, unfair coins would have $p \neq 1/2$.

The Bernoulli distribution is a special case of the binomial distribution where a single trial is conducted (so *n* would be 1 for such a binomial distribution). It is also a special case of the **two-point distribution**, for which the possible outcomes need not be 0 and 1.

Bernoulli

Parameters	$0 \leq p \leq 1$ $q = 1 - p$
Support	$k \in \{0, 1\}$
pmf	$\begin{cases} q = 1-p & \text{if } k = 0 \\ p & \text{if } k = 1 \end{cases}$
CDF	$\begin{cases} 0 & \text{if } k < 0 \\ 1-p & \text{if } 0 \leq k < 1 \\ 1 & \text{if } k \geq 1 \end{cases}$
Mean	p
Median	$\begin{cases} 0 & \text{if } p < 1/2 \\ [0,1] & \text{if } p = 1/2 \\ 1 & \text{if } p > 1/2 \end{cases}$
Mode	$\begin{cases} 0 & \text{if } p < 1/2 \\ 0, 1 & \text{if } p = 1/2 \\ 1 & \text{if } p > 1/2 \end{cases}$
Variance	$p(1-p) = pq$
Skewness	$\frac{1-2p}{\sqrt{pq}}$
Ex. kurtosis	$\frac{1-6pq}{pq}$
Entropy	$-q \ln q - p \ln p$
MGF	$q + pe^t$
CF	$q + pe^{it}$
PGF	$q + pz$
Fisher information	$\frac{1}{pq}$

Contents

Properties of the Bernoulli distribution

If X is a random variable with this distribution, then:

$$\Pr(X=1) = p = 1 - \Pr(X=0) = 1 - q.$$

The probability mass function f of this distribution, over possible outcomes *k*, is

$$f(k;p) = \begin{cases} p & \text{if } k = 1, \\ q = 1-p & \text{if } k = 0. \end{cases}$$ [2]

This can also be expressed as

$$f(k;p) = p^k (1-p)^{1-k} \quad \text{for } k \in \{0,1\}$$

or as

$$f(k;p) = pk + (1-p)(1-k) \quad \text{for } k \in \{0,1\}.$$

The Bernoulli distribution is a special case of the binomial distribution with $n = 1$.[3]

The kurtosis goes to infinity for high and low values of p, but for $p = 1/2$ the two-point distributions including the Bernoulli distribution have a lower excess kurtosis than any other probability distribution, namely −2.

The Bernoulli distributions for $0 \leq p \leq 1$ form an exponential family.

The maximum likelihood estimator of p based on a random sample is the sample mean.

Mean

The expected value of a Bernoulli random variable X is

$$\operatorname{E}(X) = p$$

This is due to the fact that for a Bernoulli distributed random variable X with $\Pr(X=1) = p$ and $\Pr(X=0) = q$ we find

$$\operatorname{E}[X] = \Pr(X=1) \cdot 1 + \Pr(X=0) \cdot 0 = p \cdot 1 + q \cdot 0 = p.$$ [2]

Variance

The variance of a Bernoulli distributed X is

$$\operatorname{Var}[X] = pq = p(1-p)$$

We first find

$$\mathrm{E}[X^2] = \Pr(X=1)\cdot 1^2 + \Pr(X=0)\cdot 0^2 = p \cdot 1^2 + q\cdot 0^2 = p$$

From this follows

$$\operatorname{Var}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2 = p - p^2 = p(1-p) = pq$$ [2]

Skewness

The skewness is $\frac{q-p}{\sqrt{pq}} = \frac{1-2p}{\sqrt{pq}}$. When we take the standardized Bernoulli distributed random variable $\frac{X-\mathrm{E}[X]}{\sqrt{\operatorname{Var}[X]}}$ we find that this random variable attains $\frac{q}{\sqrt{pq}}$ with probability p and attains $-\frac{p}{\sqrt{pq}}$ with probability q. Thus we get

$$\begin{aligned}
\gamma_1 &= \mathrm{E}\left[\left(\frac{X-\mathrm{E}[X]}{\sqrt{\operatorname{Var}[X]}}\right)^3\right] \\
&= p\cdot\left(\frac{q}{\sqrt{pq}}\right)^3 + q\cdot\left(-\frac{p}{\sqrt{pq}}\right)^3 \\
&= \frac{1}{\sqrt{pq}^3}\left(pq^3 - qp^3\right) \\
&= \frac{pq}{\sqrt{pq}^3}(q-p) \\
&= \frac{q-p}{\sqrt{pq}}
\end{aligned}$$

Related distributions

- If $X_1, \dots, X_n$ are independent, identically distributed (i.i.d.) random variables, all Bernoulli trials with success probability *p*, then their sum is distributed according to a binomial distribution with parameters *n* and *p*:

  $\sum_{k=1}^n X_k \sim \mathrm{B}(n,p)$ (binomial distribution).[2]

  The Bernoulli distribution is simply $\mathrm{B}(1,p)$, also written as $\mathrm{Bernoulli}(p)$.

- The categorical distribution is the generalization of the Bernoulli distribution for variables with any constant number of discrete values.
- The Beta distribution is the conjugate prior of the Bernoulli distribution.
- The geometric distribution models the number of independent and identical Bernoulli trials needed to get one success.
- If $Y \sim \mathrm{Bernoulli}\left(\frac{1}{2}\right)$, then $2Y-1$ has a Rademacher distribution.

See also

- Bernoulli trials, random variables distributed according to a Bernoulli distribution
- Bernoulli process, a random process consisting of a sequence of independent Bernoulli trials
- Bernoulli sampling
- Binary entropy function
- Binomial distribution
- Binary decision diagram

References

1. James Victor Uspensky: *Introduction to Mathematical Probability*, McGraw-Hill, New York 1937, page 45
2. Bertsekas, Dimitri P. (2002). *Introduction to Probability* (https://www.worldcat.org/oclc/51441829). Tsitsiklis, John N., Τσιτσικλής, Γιάννης N. Belmont, Mass.: Athena Scientific. ISBN 188652940X. OCLC 51441829 (https://www.worldcat.org/oclc/51441829).
3. McCullagh, Peter; Nelder, John (1989). *Generalized Linear Models, Second Edition*. Boca Raton: Chapman and Hall/CRC. Section 4.2.2. ISBN 0-412-31760-5.

Further reading

- Johnson, N. L.; Kotz, S.; Kemp, A. (1993). *Univariate Discrete Distributions* (2nd ed.). Wiley. ISBN 0-471-54897-9.
- Peatman, John G. (1963). *Introduction to Applied Statistics*. New York: Harper & Row. pp. 162–171.

External links

- Hazewinkel, Michiel, ed. (2001) [1994], "Binomial distribution" (https://www.encyclopediaofmath.org/index.php?title=p/b016420), *Encyclopedia of Mathematics*, Springer Science+Business Media B.V. / Kluwer Academic Publishers, ISBN 978-1-55608-010-4
- Weisstein, Eric W. "Bernoulli Distribution" (http://mathworld.wolfram.com/BernoulliDistribution.html). *MathWorld*.
- Interactive graphic: Univariate Distribution Relationships (http://www.math.wm.edu/~leemis/chart/UDR/UDR.html)