3. Application of Delta Method on Gamma Variables

The **Gamma distribution** $\mathrm{Gamma}\,(\alpha, \beta)$ with paramters $\alpha > 0$, and $\beta > 0$ is defined by the density

$$f_{\alpha,\beta}\,(x) = \frac{\beta^\alpha}{\Gamma\,(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad \text{for all} x \geq 0.$$

The Γ function is defined by

$$\Gamma\,(s) = \int_0^\infty x^{s-1} e^{-x}\,dx.$$

As usual, the constant $\frac{\beta^\alpha}{\Gamma(\alpha)}$ is a normalization constant that gives $\int_0^\infty f_{\alpha,\beta}\,(x)\,dx = 1$.

In this problem, let $X_1, \ldots, X_n$ be i.i.d. Gamma variables with

$$\beta = \frac{1}{\alpha} \text{for some } \alpha > 0.$$

That is, $X_1, \ldots, X_n \sim \mathrm{Gamma}\,\left(\alpha, \frac{1}{\alpha}\right)$ random variables for some $\alpha > 0$. The pdf for X_i is therefore

$$f_\alpha\,(x) = \frac{1}{\Gamma\,(\alpha)\,\alpha^\alpha} x^{\alpha-1} e^{-x/\alpha}, \quad \text{for all } x \geq 0.$$

(a)

1/1 point (graded)

What is the limit, in probability, of the sample average $\overline{X}_n$ of the sample in terms of α?

$\overline{X}_n \xrightarrow[n\to\infty]{\mathbf{P}}$ alpha^2

α^2

✔ **Answer:** alpha^2

STANDARD NOTATION

Solution:

By the weak law of large numbers

$$\overline{X}_n \xrightarrow[n\to\infty]{\mathbf{P}} \mathbb{E}\,[X_i]\,.$$

In general, the expectation for a Gamma variable with parameters α, β is $\dfrac{\alpha}{\beta}$, since

$$\begin{aligned}\int_0^\infty x f_{\alpha,\beta}\,(x)\,dx &= \frac{\beta^\alpha}{\Gamma\,(\alpha)} \int_0^\infty x^\alpha e^{-\beta x} \\ &= \frac{\beta^\alpha}{\Gamma\,(\alpha)} \left(\frac{x^\alpha e^{-\beta x}}{-\beta}\Bigg|_0^\infty - \int_0^\infty \left(\alpha x^{\alpha-1}\right) \left(\frac{e^{-\beta x}}{-\beta}\right) dx \right) = \frac{\alpha}{\beta}.\end{aligned}$$

Hence, for $X_i \sim \mathrm{Gamma}\,\left(\alpha, \frac{1}{\alpha}\right)$, we have

$$\mathbb{E}[X_i] = \frac{\alpha}{1/\alpha} = \alpha^2.$$

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Answers are displayed within the problem

(b)

1/1 point (graded)

Use the result from the previous problem to give a consistent estimator $\hat{\alpha}$ of α in terms of $\overline{X}_n$.

(Enter barX_n for $\overline{X}_n$)

$\hat{\alpha} =$ sqrt(barX_n) ✔ **Answer:** sqrt(barX_n)

Solution:

From the previous problem, we know that $\overline{X}_n \xrightarrow[\mathbf{P}]{n\to\infty} \alpha^2$. By the continuous mapping theorem, $\hat{\alpha} = \sqrt{\overline{X}_n} \xrightarrow[\mathbf{P}]{n\to\infty} \sqrt{\alpha^2} = \alpha$ since $\alpha > 0$
.

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(c)

3/3 points (graded)

For the Delta method to apply, at what value of x does g need to be continuously differentiable? (Your answer should be in terms of α.)

$x =$ alpha^2 ✔ **Answer:** alpha^2

α^2

What distribution does $\sqrt{n}\hat{\alpha}$ converge to as $n \to \infty$?

- Gamma distribution
- Normal distribution ✔
- None of the above

What is its asymptotic variance of $\hat{\alpha}$?

$\mathsf{Var}(\sqrt{n}\hat{\alpha}) = \mathsf{Var}(\sqrt{n}(\hat{\alpha} - \alpha)) =$ alpha/4 ✔ **Answer:** alpha/4

$\frac{\alpha}{4}$

STANDARD NOTATION

Solution:

The Delta method would give

$$\sqrt{n}(\hat{\alpha} - \alpha) = \sqrt{n}\left(\sqrt{\overline{X}_n} - \alpha\right) \xrightarrow[d.]{n\to\infty} \mathcal{N}\left(0, (g'(\mathbb{E}[X_i]))^2 \mathsf{Var}(X_i)\right) = \mathcal{N}\left(0, (g'(\alpha^2))^2 \mathsf{Var}(X)\right) \qquad where\, g(x) = \sqrt{x}$$

if g is continuously differentiable at α^2. Indeed, since $g'(x) = \frac{1}{2\sqrt{x}}$ exists and is continuous for all $x > 0$, g' is continuously differentiable at any α^2 value. Hence, the Delta method does apply.

To compute the asymptotic variance $\left(g'(\alpha^2)\right)^2 \mathsf{Var}(X_i)$, we need to compute $g'(\alpha^2)$ and $\mathsf{Var}(X_i)$.

$$g'(\alpha^2) = \frac{1}{2\sqrt{\alpha^2}} = \frac{1}{2\alpha}$$

In general, the variance for a Gamma variable X with parameters α, β is $\frac{\alpha}{\beta^2}$, since

$$\begin{aligned}
\mathbb{E}[X^2] = \int_0^\infty x^2 f_{\alpha,\beta}(x)\,dx &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+1} e^{-\beta x} \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{x^{\alpha+1} e^{-\beta x}}{-\beta} \Big|_0^\infty - \int_0^\infty \left((\alpha+1) x^\alpha\right) \left(\frac{e^{-\beta x}}{-\beta}\right) dx \right) \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{\alpha+1}{\beta} \int_0^\infty x^\alpha e^{-\beta x}\,dx \right) \\
&= \frac{\alpha+1}{\beta} (\mathbb{E}[X]) = \frac{\alpha+1}{\beta} \left(\frac{\alpha}{\beta}\right) \\
\mathsf{Var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\
&= \frac{\alpha+1}{\beta}\left(\frac{\alpha}{\beta}\right) - \left(\frac{\alpha}{\beta}\right)^2 = \frac{\alpha}{\beta^2}
\end{aligned}$$

In this problem, $\beta = 1/\alpha$, hence

$$\mathsf{Var}(X_i) = \alpha^3.$$

Putting these together, the asymptotic variance is

$$\left(g'(\alpha^2)\right)^2 \mathsf{Var}(X_i) = \frac{1}{4\alpha^2}(\alpha^3) = \frac{\alpha}{4}.$$

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Answers are displayed within the problem

(d)

4.0/4.0 points (graded)

Using the previous part, find confidence intervals for α with asymptotic level 90% using both the "solving" and the "plug-in" methods. Use $n = 25$, and $\overline{X}_n = 4.5$.

(Enter your answers accurate to 2 decimal places. Use the Gaussian estimate $q_{0.05} \approx 1.6448$ for best results.)

$\mathcal{I}_{\text{solve}} =$ [1.89506 ✔ Answer: 1.89 , 2.37459 ✔ Answer: 2.37]

$\mathcal{I}_{\text{plug-in}} =$ [2.1213-1.6648*0.7282/5 ✔ Answer: 1.88 , 2.1213+1.6648*0.7282/5 ✔ Answer: 2.36]

STANDARD NOTATION

Solution:

Recall from the last part that

$$\sqrt{n}(\hat{\alpha} - \alpha) \xrightarrow[d.]{n\to\infty} \mathcal{N}(0, \tau^2) \qquad \text{where } \tau^2 = \frac{\alpha}{4}$$

This implies

$$\frac{\sqrt{n}}{\tau}(\hat{\alpha}-\alpha) \xrightarrow[d.]{n\to\infty} \mathcal{N}(0,1) \text{ where } \tau^2 = \frac{\alpha}{4}$$

Therefore, following the usual procedure for confidence intervals, for large n, approximately

$$\mathbf{P}\left(\hat{\alpha} - q_{0.05}\frac{\tau}{\sqrt{n}} < \alpha < \hat{\alpha} + q_{0.05}\frac{\tau}{\sqrt{n}}\right) = 0.9.$$

Plugging in the asymptotic variance $\tau = \sqrt{\alpha}/2$ gives

$$\mathbf{P}\left(\hat{\alpha} - q_{0.05}\frac{\sqrt{\alpha}}{2\sqrt{n}} < \alpha < \hat{\alpha} + q_{0.05}\frac{\sqrt{\alpha}}{2\sqrt{n}}\right) = 0.9.$$

We now go through the three methods of solving for the confidence interval:

1. Conservative bound: Since $\sqrt{\alpha}$ is not bounded, the conservative bound method does not give a confidence interval.
2. Solving for α: we need to solve the following for α:

$$\begin{aligned} & |\hat{\alpha}-\alpha| && < q_{0.05}\frac{\tau}{\sqrt{n}} = q_{0.05}\frac{\sqrt{\alpha}}{2\sqrt{n}} \\ \Longleftrightarrow\quad & (\hat{\alpha}-\alpha)^2 && < q_{0.05}^2\frac{\alpha}{4n} \\ \Longleftrightarrow\quad & \alpha^2 - \left(2\hat{\alpha} + \frac{q_{0.05}^2}{4n}\right) + \hat{\alpha}^2 && = 0 \end{aligned}$$

   where $\hat{\alpha}^2 = \overline{X}_n = 4.5$, and $q_{0.05} = 1.6448$. Using the quadratic formula or software, we get the confidence interval

$$\mathcal{I}_{\text{solve}} = [1.89, 2.37]$$

3. Plug-in: Since $\hat{\alpha}^2 = \overline{X}_n = 4.5$, the plug-in confidence interval is

$$\begin{aligned} \mathcal{I}_{\text{plug-in}} &= \left[\hat{\alpha} - q_{0.05}\frac{\sqrt{\hat{\alpha}}}{2\sqrt{n}}, \hat{\alpha} + q_{0.05}\frac{\sqrt{\hat{\alpha}}}{2\sqrt{n}}\right] \\ &= [1.88, 2.36] \end{aligned}$$

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Answers are displayed within the problem

讨论

显示讨论

主题: Unit 2 Foundation of Inference:Homework 2: Statistical Models, Estimation, and Confidence Intervals / 3. Application of Delta Method on Gamma Variables

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