11. Properties of the Kullback-Leibler (KL) Divergence

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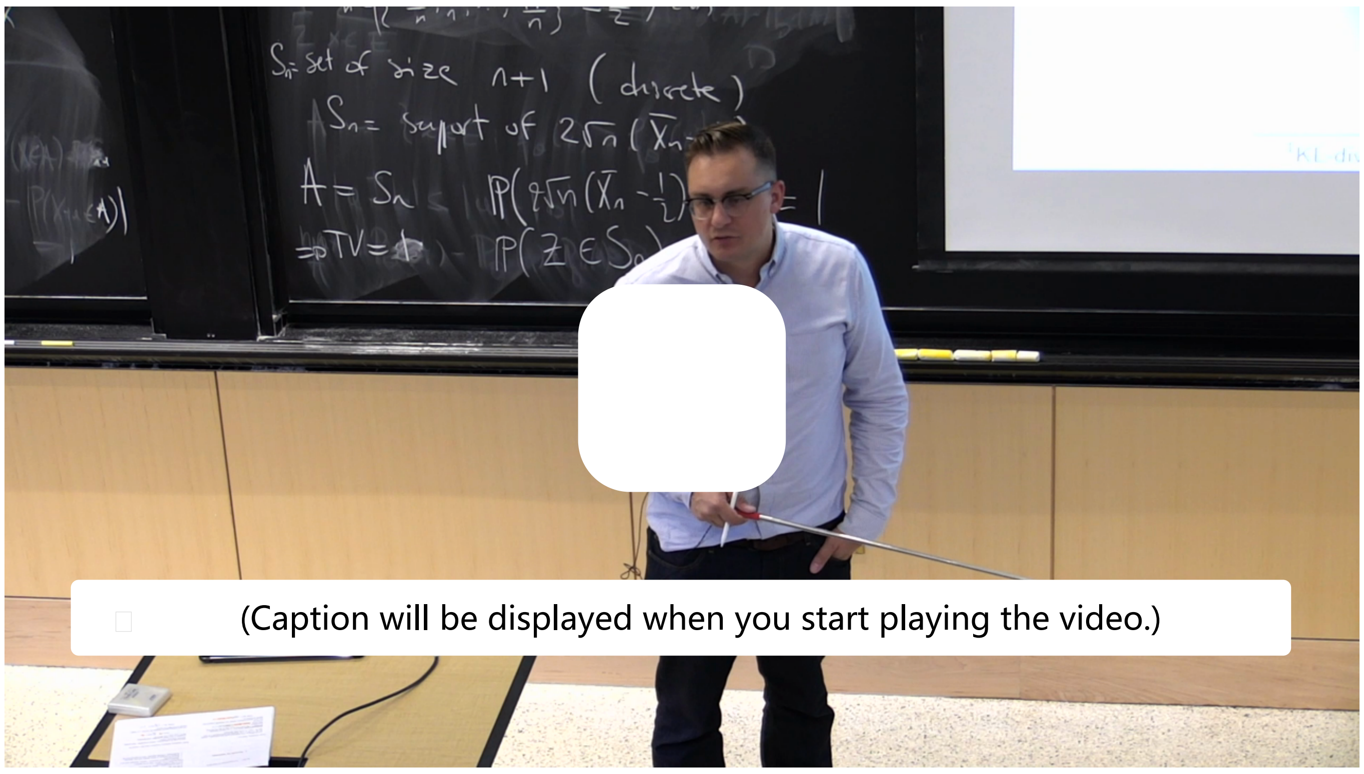

Start of transcript. Skip to the end.

So now It's not even clear that this thing is actually

just like, you know, when I tell you, oh take

the absolute value of the difference,

clearly when those things become closed,

the absolute value of the difference becomes smaller.

It's not even clear that this is happening here.

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Computing KL Divergence II

2/3 points (graded)

Let $X \sim \mathbf{P}_X = \mathrm{Ber}\,(1/2)$ and let $Y \sim \mathbf{P}_Y = \mathrm{Ber}\,(1/3)$. What is $\mathrm{KL}\,(\mathbf{P}_X, \mathbf{P}_Y)$?

(If applicable, enter **ln(x)** for $\ln\,(x)$.)

$\mathrm{KL}\,(\mathbf{P}_X, \mathbf{P}_Y) =$ 1/2*ln(9/8) **Answer:** 0.0588915

What is $\mathrm{KL}\,(\mathbf{P}_Y, \mathbf{P}_X)$?

$\mathrm{KL}\,(\mathbf{P}_Y, \mathbf{P}_X) =$ 1/3*ln(3/4)+2/3*ln(3/2) **Answer:** 0.05663301

Is $\mathrm{KL}\,(\mathbf{P}_X, \mathbf{P}_Y) = \mathrm{KL}\,(\mathbf{P}_Y, \mathbf{P}_X)$?

- Yes
- No

STANDARD NOTATION

Solution:

Let f and g denote the pmfs of $\mathrm{Ber}\,(1/2)$ and $\mathrm{Ber}\,(1/3)$, respectively. Note that the sample space is $E = \{0, 1\}$. Then

$$\begin{aligned}\mathrm{KL}\left(\mathbf{P}_X, \mathbf{P}_Y\right) &= \sum_{x\in\{0,1\}} f(x)\ln\left(f(x)/g(x)\right)\\ &= (1/2)\ln(3/2) + (1/2)\ln(3/4) \approx 0.0588915\end{aligned}$$

Next,

$$\begin{aligned}\mathrm{KL}\left(\mathbf{P}_Y, \mathbf{P}_X\right) &= \sum_{x\in\{0,1\}} g(x)\ln\left(g(x)/f(x)\right)\\ &= (1/3)\ln(2/3) + (2/3)\ln(4/3) \approx 0.05663301\end{aligned}$$

Remark: In general, we have the formula

$$\mathrm{KL}\left(\mathrm{Ber}(p), \mathrm{Ber}(q)\right) = p\ln\left(\frac{p}{q}\right) + (1-p)\ln\left(\frac{1-p}{1-q}\right).$$

提交 你已经尝试了3次（总共可以尝试3次）

Answers are displayed within the problem

Properties of KL Divergence I

2/2 points (graded)

Let $\mathbf{P}$ be a distribution such that $\mathbf{KL}\left(\mathrm{Ber}(1/2), \mathbf{P}\right) = 0$. What can we conclude about $\mathbf{P}$?

- (selected, correct) $\mathbf{P} = \mathrm{Ber}(1/2)$.
- It is possible that $\mathbf{P} = \mathrm{Ber}(p)$ for any $0 \le p \le 1$.
- $\mathbf{P}$ could be any Gaussian distribution with mean 0 and variance $1/4$.
- None of the above.

What property of the KL divergence did you use to make your conclusion?

- Symmetric
- Nonnegative
- (selected, correct) Definite
- Triangle Inequality

Solution:

The definite property of the KL divergence implies that if $\mathrm{KL}(\mathbf{P}, \mathbf{Q}) = 0$, then $\mathbf{P}$ and $\mathbf{Q}$ are the same distribution. Hence, we use this property to conclude that $\mathbf{P} = \mathrm{Ber}(1/2)$.

Note that while the KL divergence is nonnegative and definite, it is not a distance because it does not satisfy the triangle inequality nor is it symmetric.

提交 你已经尝试了2次（总共可以尝试2次）

Answers are displayed within the problem

(Optional) Why does the KL divergence take only non-negative values?

Here is the proof of the positive semi-definiteness of the KL-divergence. For simplicity, we only prove the case when the two distributions are given by pdfs.

$\mathrm{KL}(\mathbf{P}_X, \mathbf{P}_Y) \geq 0$ for all distributions $\mathbf{P}_Y$ and $\mathbf{P}_X$ (positive semi-definiteness).

Proof. The main idea is to use Jensen's inequality (which you could review in lecture 4).
Let $p_X\, p_Y$ (with suitable condition) be the pdfs defining the distribution $\mathbf{P}_X$ and $\mathbf{P}_Y$ respectively. Define another random variable $Z = \dfrac{p_Y(X)}{p_X(X)}$, which is a function of the random variable X. Observe that the function $-\ln$ is convex. Then Jensen's inequality gives

$$\begin{aligned}
\mathrm{KL}(\mathbf{P}_X, \mathbf{P}_Y) = \mathbb{E}_X[-\ln(Z)] &\geq -\ln(\mathbb{E}_X[Z]) \qquad \textbf{(Jensen's Inequality)} \\
&= -\ln\left(\mathbb{E}_X\left[\frac{p_Y(X)}{p_X(X)}\right]\right) \\
&= -\ln(1) = 0.
\end{aligned}$$

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讨论

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主题: Unit 3 Methods of Estimation:Lecture 8: Distance measures between distributions / 11. Properties of the Kullback-Leibler (KL) Divergence