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1. Determining the type of a lightbulb

Problem 1. Determining the type of a lightbulb

3/3 points (graded)

The lifetime of a type-A bulb is exponentially distributed with parameter λ. The lifetime of a type-B bulb is exponentially distributed with parameter μ, where $\mu > \lambda > 0$. You have a box full of lightbulbs of the same type, and you would like to know whether they are of type A or B. Assume an **a priori** probability of $1/4$ that the box contains type-B lightbulbs.

1. Assume that $\mu \geq 3\lambda$. You observe the value t_1 of the lifetime, T_1, of a lightbulb. A MAP decision rule decides that the lightbulb is of type A if and only if $t_1 \geq \alpha$. Find α, and express your answer in terms of μ and λ. Use 'mu" and 'lambda" and 'ln" to denote μ, λ, and the natural logarithm function, respectively. For example, $\ln \frac{2\mu}{\lambda}$ should be entered as 'ln((2*mu)/lambda)".

   $\alpha =$

   ln((mu)/(3*lambda))/(mu-lambda) ✔

   Answer: (1/(mu-lambda))*ln(mu/(3*lambda))

   $$\frac{\ln\left(\frac{\mu}{3\cdot\lambda}\right)}{\mu-\lambda}$$

2. Assume again that $\mu \geq 3\lambda$. What is the probability of error of the MAP decision rule?

   - (●) $\frac{1}{4}e^{-\mu\alpha} + \frac{3}{4}(1 - e^{-\lambda\alpha})$ ✔
   - ( ) $\frac{3}{4}e^{-\mu\alpha} + \frac{1}{4}(1 - e^{-\lambda\alpha})$
   - ( ) $\frac{1}{4}(1 - e^{-\mu\alpha}) + \frac{3}{4}e^{-\lambda\alpha}$
   - ( ) $\frac{3}{4}(1 - e^{-\mu\alpha}) + \frac{1}{4}e^{-\lambda\alpha}$

3. Assume that $\lambda = 3$ and $\mu = 4$. Find the LMS estimate of T_2, the lifetime of another lightbulb from the same box, based on observing $T_1 = 2$. Assume that conditioned on the bulb type, bulb lifetimes are independent. (For this part, you will need a calculator. Provide an answer with an accuracy of three decimal places.)

   LMS estimate of T_2:

0.3286

✔ **Answer:** 0.328

Solution:

1. With some abuse of notation, we let A and B be the events that the box contains lightbulbs of type A and type B, respectively. A MAP rule decides in favor of type A if and only if

$$\begin{aligned} \mathbf{P}(A \mid T_1 = t_1) &\geq \mathbf{P}(B \mid T_1 = t_1) \\ \frac{f_{T_1|A}(t_1)\mathbf{P}(A)}{f_{T_1}(t_1)} &\geq \frac{f_{T_1|B}(t_1)\mathbf{P}(B)}{f_{T_1}(t_1)}. \end{aligned}$$

Equivalently, we decide that the bulb is of type A if and only if

$$\begin{aligned} f_{T_1|A}(t_1)\mathbf{P}(A) &\geq f_{T_1|B}(t_1)\mathbf{P}(B), \\ \lambda e^{-\lambda t_1}\frac{3}{4} &\geq \mu e^{-\mu t_1}\frac{1}{4}, \\ \frac{\lambda}{\mu}e^{(\mu-\lambda)t_1} &\geq \frac{1}{3}, \\ (\mu-\lambda)t_1 \geq \ln\Big(\frac{\mu}{3\lambda}\Big). \end{aligned}$$

Thus, since $\mu - \lambda > 0$, a MAP rule decides in favor of type A if and only if $t_1 \geq \ln\Big(\frac{\mu}{3\lambda}\Big) \cdot \frac{1}{\mu-\lambda}$. Hence, we deduce that,

$$\alpha = \frac{1}{\mu-\lambda}\ln\Big(\frac{\mu}{3\lambda}\Big).$$

2. Let events A and B be defined as in part (1). Let $\hat{A}$ be the event that the MAP rule decides in favor of type A, and let $\hat{B}$ be the event that the MAP rule decides in favor of type B. An error occurs whenever the decision is different from the actual type of the bulb. Thus,

$$\begin{aligned} \mathbf{P}(\text{error}) &= \mathbf{P}(\hat{A} \cap B) + \mathbf{P}(A \cap \hat{B}) \\ &= \mathbf{P}(\hat{A} \mid B) \cdot \mathbf{P}(B) + \mathbf{P}(\hat{B} \mid A) \cdot \mathbf{P}(A) \\ &= \mathbf{P}(T_1 \geq \alpha \mid B) \cdot \frac{1}{4} + \mathbf{P}(T_1 < \alpha \mid A) \cdot \frac{3}{4} \\ &= e^{-\mu\alpha} \cdot \frac{1}{4} + (1 - e^{-\lambda\alpha}) \cdot \frac{3}{4}. \end{aligned}$$

3. The LMS estimate of T_2 based on observing $T_1 = t_1$ is

$$\begin{aligned} \mathbf{E}[T_2 \mid T_1 = t_1] &= \mathbf{E}[T_2 \mid T_1 = t_1, A] \cdot \mathbf{P}(A \mid T_1 = t_1) + \mathbf{E}[T_2 \mid T_1 = t_1, B] \cdot \mathbf{P}(B \mid T_1 = t_1) \\ &= \mathbf{E}[T_2 \mid A] \cdot \mathbf{P}(A \mid T_1 = t_1) + \mathbf{E}[T_2 \mid B] \cdot \mathbf{P}(B \mid T_1 = t_1) \\ &= \frac{1}{\lambda} \cdot \left(\frac{f_{T_1|A}(t_1) \cdot \mathbf{P}(A)}{f_{T_1}(t_1)}\right) + \frac{1}{\mu} \cdot \left(\frac{f_{T_1|B}(t_1) \cdot \mathbf{P}(B)}{f_{T_1}(t_1)}\right) \end{aligned}$$

$$= \frac{\frac{1}{\lambda}\frac{3}{4}\lambda e^{-\lambda t_1} + \frac{1}{\mu}\frac{1}{4}\mu e^{-\mu t_1}}{\frac{3}{4}\lambda e^{-\lambda t_1} + \frac{1}{4}\mu e^{-\mu t_1}}.$$

Inserting the values $\lambda = 3$,$\mu = 4$, and $t_1 = 2$, we obtain $\mathbf{E}[T_2 \mid T_1 = 2] = 0.328$.

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Answers are displayed within the problem

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Additional questions for the lightbulb problem.

question posted 16 days ago by SergK (Community TA)

1. In part 1, we assume $\mu \geqslant 3\lambda$. What is MAP decision if for example $\mu = 2\lambda$?

2. We consider the problem as hypothesis testing problem and use MAP decision rule. Does it make sense to consider the problem as estimation problem (of the unknown parameter of exponential distribution) and use LMS rule? If it does make sense, what is the meaning of LMS estimate?

此帖对所有人可见。

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1 response

e_kizildag (Staff)
16 days ago

1) If $\mu = 2\lambda$, then the MAP rule will always favor [since the threshold value will be a negative number, thresholded against a positive number, hence we trivially have inequality] one of the bulbs (that will be clear from the solution), hence there is no need to consider that case.

2) The way its given, it seems hypothesis testing is somehow a better fit. The reason is, you already know the parameter of exponential (in fact, you know your parameter is one of the two), and all you are doing is just to identify whether it is one, or the other (as opposed to a set of many possibilities, as recovered by estimator).

However, in general, you may also view estimation as a hypothesis testing problem, where in some sense, you have continuously many number of hypothesis (say a parameter of exponential can be anything on $(0, \infty)$), and you are simply trying to find whichever is more plausible. I'll invest more thoughts on this, and will update my post accordingly.

I think the purpose of this post (by a CTA) was as a challenge to students to do some additional thinking on the problem, not for the question to be answered.

markweitzman (Community TA) 在16 days ago前发表

Indeed, I suggested to think a bit more on the problem after solving it. :) My point is: the difference between Bayesian hypothesis testing and estimation is the way we treat the unknown parameter. Sometimes the unknown parameter can be treated both as an index of hypothesis, or as a value to be estimated. The parts 1-2 of the problem use hypothesis testing setup, while the part 3 use parameter estimation setup.

SergK (Community TA) 在15 days ago前发表

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