10. Multivariate Central Limit Theorem

Note: The following exercise will be presented in the video that follows. We encourage you to attempt it before watching the video.

Vector Version of the Central Limit Theorem

1/1 point (graded)

Let $\mathbf{X}$ be a random vector of dimension $d \times 1$ and let μ and Σ be its mean and covariance. Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be i.i.d. copies of $\mathbf{X}$. Let $\overline{\mathbf{X}}_n \triangleq \frac{1}{n}\sum_{i=1}^{n} \mathbf{X}_i$.

Based on your knowledge of the central limit theorem for a single random variable, select from the following the correct shift and scale factor for $\overline{\mathbf{X}}_n$ so that $\overline{\mathbf{X}}_n$ could potentially converge to the Gaussian random vector $\mathcal{N}(0, I_{d\times d})$.

- $\sqrt{d} \cdot \Sigma^{-\frac{1}{2}} \left(\overline{\mathbf{X}}_n - \mu\right)$
- $\sqrt{d} \cdot \Sigma^{-1} \left(\overline{\mathbf{X}}_n - \mu\right)$
- $\sqrt{n} \cdot \Sigma^{-1} \left(\overline{\mathbf{X}}_n - \mu\right)$
- $\sqrt{n} \cdot \Sigma^{-\frac{1}{2}} \left(\overline{\mathbf{X}}_n - \mu\right)$
- None of the above

Solution:

The shift of $\mathbf{X}$ by μ is the correct shift that needs to be applied in order to center the random vector.

The scaling factor should be $\sqrt{n}\Sigma^{-\frac{1}{2}}$ because it mimics the single variable CLT case most closely. In particular, the division by $\sqrt{\sigma^2}$ in the single variable CLT case is being taken care of by the inverse of the square root of Σ.

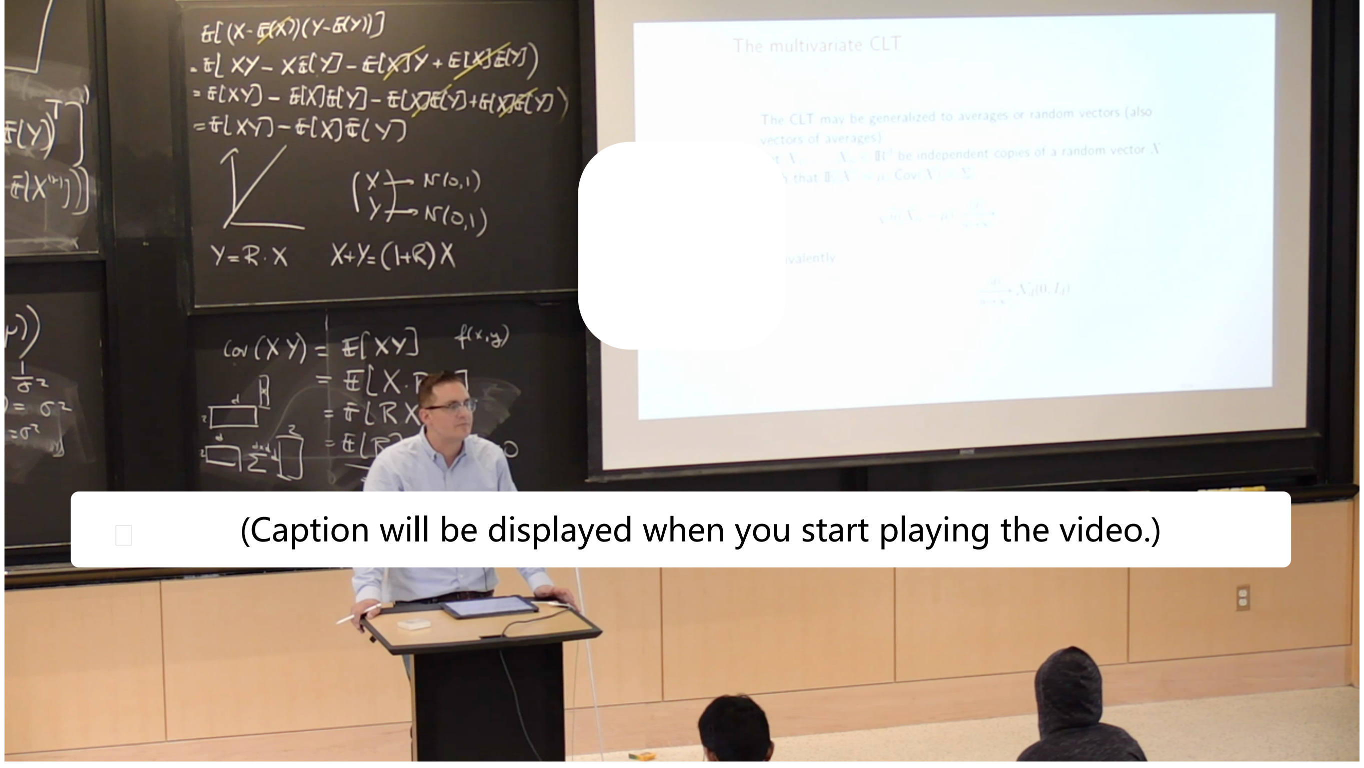
Note: Of course, this is only a heuristic discussion that is meant to test how you can potentially generalize the single variable CLT. This is not a proof and the solution is also written as guesswork.

提交 你已经尝试了1次（总共可以尝试3次）

Answers are displayed within the problem

Multivariate Central Limit Theorem

Start of transcript. Skip to the end.



So now that I have this, I can talk
about multivariate Gaussians as being
limits of a multivariate central limit theorem.
And this is how it goes.
Rather than looking at averages of random variables,
I'm going to look at averages of random vectors.
And this is understood in the most natural sense.

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(Optional) Multivariate Convergence in Distribution and Proof of Multivariate CLT

Convergence in Distribution in Higher Dimensions

Convergence in distribution of a random vector is **not implied** by convergence in distribution of each of its components.

A sequence $\mathbf{T}_1, \mathbf{T}_2, \ldots$ of random vectors in $\mathbb{R}^d$ **converges in distribution** to a random vector $\mathbf{T}$ if

$$\mathbf{v}^T\mathbf{T}_n \xrightarrow[(d)]{n\to\infty} \mathbf{v}^T\mathbf{T} \qquad \text{for all } \mathbf{v}\in\mathbb{R}^d \qquad (\text{multivariate convergence in distribution})\,.$$

That is, the vector sequence $(\mathbf{T}_n)_{n\geq 1}$ converges in distribution only if its dot product $\mathbf{v}^T\mathbf{T}_n$ with **any** constant vector $\mathbf{v}$, which is a scalar random variable, converges in distribution (or equivalently, if the projection of the vector sequence onto **any** line converges in distribution.)

Univariate CLT Implies Multivariate CLT

Let $\mathbf{X}_1, \ldots, \mathbf{X}_n \overset{i.i.d.}{\sim} \mathbf{X}$ be random vectors in $\mathbb{R}^d$ with (vector) mean $\mathbb{E}[\mathbf{X}] = \mu_{\mathbf{X}}$ and covariance matrix $\Sigma_{\mathbf{X}}$.

Let $\mathbf{v}\in\mathbb{R}^d$ and define $Y_i = \mathbf{v}^T\mathbf{X}_i$. Then

- Y_i is a scalar random variable;
- Its mean and variance are $\mathbb{E}[Y_i] = \mathbf{v}^T\mathbb{E}[\mathbf{X}_i]$ and $\sigma^2_{Y_i} = \mathbf{v}^T\Sigma_{\mathbf{X}_i}\mathbf{v}$ (you can check that the variance is indeed a scalar).

Hence Y_i satisfies the univariate CLT:

$$\sqrt{n}\left(\overline{Y_n} - \mathbf{v}^T\mu_{\mathbf{X}}\right) \xrightarrow[(d)]{n\to\infty} \mathcal{N}\left(0, \mathbf{v}^T\Sigma_{\mathbf{X}}\mathbf{v}\right)$$

On the other hand, consider a multivariate Gaussian variable $\mathbf{Z}\sim\mathcal{N}(\mathbf{0}, \Sigma_{\mathbf{X}})$. For any constant vector $\mathbf{v}\in\mathbb{R}^d$, $\mathbf{v}^T\mathbf{Z}$ is a univariate Gaussian with variance $\mathbf{v}^T\Sigma_{\mathbf{X}}\mathbf{v}.$ Hence, $\mathbf{v}^T\mathbf{Z}\sim\mathcal{N}\left(0, \mathbf{v}^T\Sigma_{\mathbf{X}}\mathbf{v}\right)$, which is the distribution on the right hand side above. Therefore, $\overline{\mathbf{X}_n}$ converges in distribution:

$$\begin{aligned} \sqrt{n}\left(\mathbf{v}^T\overline{\mathbf{X}_n} - \mathbf{v}^T\mu_{\mathbf{X}}\right) = \sqrt{n}\left(\overline{Y_n} - \mathbf{v}^T\mu_{\mathbf{X}}\right) &\xrightarrow[(d)]{n\to\infty} \mathcal{N}\left(0, \mathbf{v}^T\Sigma_{\mathbf{X}}\mathbf{v}\right) = \mathbf{v}^T\mathcal{N}\left(0, \Sigma_{\mathbf{X}}\right) \\ \iff \qquad \sqrt{n}\left(\overline{\mathbf{X}_n} - \mu_{\mathbf{X}}\right) &\xrightarrow[(d)]{n\to\infty} \mathcal{N}\left(0, \Sigma_{\mathbf{X}}\right). \end{aligned}$$

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讨论

主题： Unit 3 Methods of Estimation:Lecture 10: Consistency of MLE, Covariance Matrices, and Multivariate Statistics / 10. Multivariate Central Limit Theorem

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