Problem 6: Maximum Likelihood Estimation of Phase Noise

Phase Noise Estimation under Gaussian Noise: Setup

This problem is motivated by estimation in communication systems (Wi-Fi, cellphones, etc). The solution obtained in this problem is implemented real-time in many communication systems. For example, your laptop Wi-Fi adapter, which is downloading and uploading all the content that you are consuming in this course, is performing this estimation (albeit in a more complicated statistical model) tens of hundreds of times every second.

Let

$$\mathbf{x} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

be a known vector, i.e. **we assume that we know $\boldsymbol{\theta}$**. Let $\theta \in [0, \pi/2]$.

Let $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ be defined as follows:

$$\mathbf{Y}_i = \begin{bmatrix} Y_i^{(1)} \\ Y_i^{(2)} \end{bmatrix} = \begin{bmatrix} \cos(\theta+\phi) \\ \sin(\theta+\phi) \end{bmatrix} + \mathbf{Z}_i, \quad i = 1, \ldots, n,$$

where $\mathbf{Z}_i \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_2)$ for a known σ^2 and ϕ is an unknown constant. Assume that $\mathbf{Z}_i, i = 1, \ldots, n$ are independent.

Objective: Upon observing $\mathbf{Y}_i, i = 1, \ldots, n$, we wish to produce an estimate $\widehat{\phi}$ of $\phi \in [-\pi, \pi]$.

(a) True or False

1/1 point (graded)
Select whether the following statement is **true or false**: "$\mathbf{Y}_i$ are iid.""

- (•) True ✔
- ( ) False

Solution:

The statement is true. The multivariate Gaussian vectors $\mathbf{Z}_i$ are iid. Therefore, $\mathbf{Y}_i$, which are deterministic functions of the $\mathbf{Z}_i$'s, respectively for $i = 1, \ldots, n$, are iid.

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

(b) The Underlying Problem

1/1 point (graded)
Referring to the **objective** in the problem setup given above, select from the following the statements that are correct. (Choose all that apply.)

- [ ] We are trying to estimate the **magnitude** by which $\mathbf{x}$ is scaled (in the presence of vector Gaussian noise).
- [x] We are trying to estimate the **phase rotation** undergone by $\mathbf{x}$ (in the presence of vector Gaussian noise).
- [ ] We are trying to estimate the **magnitude and phase changes** undergone by $\mathbf{x}$ (in the presence of vector Gaussian noise).

Grading Note: Partial credit is given.

Solution:

The objective of the problem states that we wish to produce an estimate of ϕ, which is the phase rotation undergone by $\mathbf{x}$ under an additive Gaussian noise.

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

(c) Observation Under Zero Noise

1.0/1 point (graded)

For a moment, assume that there is **no Gaussian noise in the problem**. That is, let $\mathbf{Y}_i = [\cos(\theta+\phi) \quad \sin(\theta+\phi)]^T \triangleq \begin{pmatrix} Y^{(1)} \\ Y^{(2)} \end{pmatrix}$, for all i, in this sub-problem.

For simplicity, assume that $\theta \in [0, \pi/2]$, $\theta + \phi \in [0, \pi/2]$.

What is ϕ?

(Express your answer in terms of $Y^{(1)}$, $Y^{(2)}$, θ, and the $\arctan(x)$ function. Use **Y_1** for $Y^{(1)}$ and **Y_2** for $Y^{(2)}$. Type **arctan(x)** for $\arctan(x)$ (where x can be any expression). **Do not use** any trigonometric function other than $\arctan$.)

$\phi =$ arctan(Y_2/Y_1) - theta **Answer:** -theta + arctan(Y_2/Y_1)

STANDARD NOTATION

Solution:

If there is no noise, the value of ϕ is $\arctan\left(\frac{Y^{(2)}}{Y^{(1)}}\right) - \theta$, where we assume that $\theta \in [0, \pi/2]$, $\theta + \phi \in [0, \pi/2]$ for simplicity.

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

(d) Maximum Likelihood Estimator of the Phase Noise - Log Likelihood

1.0/1 point (graded)

Now, let us return to the original setup. What is the log-likelihood $\ell_n(\mathbf{Y}_1, \ldots, \mathbf{Y}_n; \phi)$?

For the answer box below, ignore the term $\ln\left(\frac{1}{(\sqrt{2\pi\sigma^2})^{2n}}\right)$ **in the log-likelihood and input the rest of the log-likelihood expression.**

(Use **Sigma_i(X_i)** for $\sum_{i=1}^{n}(X_i)$ (where X_i can be any quantity in a series indexed by i), **Y_1** for $Y_i^{(1)}$, and **Y_2** for $Y_i^{(2)}$. Enter **sin(x)** for $\sin(x)$, **cos(x)** for $\cos(x)$.)

- (Sigma_i((Y_1 - cos(theta+phi))^2 + (Y_2 - sin(theta+phi))^2))/(2*sigma^2)

Answer: -(1/(2*sigma^2))*Sigma_i((Y_1 - cos(theta + phi))^2 + (Y_2 - sin(theta + phi))^2)

STANDARD NOTATION

Solution:

$$\ell_n(\mathbf{Y}_1,\ldots,\mathbf{Y}_n;\phi) = \log L_n(\mathbf{Y}_1,\ldots,\mathbf{Y}_n;\phi) = \log\left(\prod_{i=1}^{n}\frac{\exp\left(-\frac{1}{2}\left[\frac{(Y_i^{(1)}-\cos(\theta+\phi))^2}{\sigma^2}+\frac{(Y_i^{(2)}-\sin(\theta+\phi))^2}{\sigma^2}\right]\right)}{2\pi\sigma^2}\right)$$

$$= \log\left(\frac{1}{(2\pi\sigma^2)^n}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left[(Y_i^{(1)}-\cos(\theta+\phi))^2 + (Y_i^{(2)}-\sin(\theta+\phi))^2\right]$$

Note: My notes showed I also calculated the expression passed to the $\exp()$ function in detail, but I am too tired to type that up in LaTex now.

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

(e) Maximum Likelihood Estimator of the Phase Noise

1.0/1 point (graded)

Let $\widehat{\mu}_1 = \sum_{i=1}^{n}\frac{Y_i^{(1)}}{n}$ and $\widehat{\mu}_2 = \sum_{i=1}^{n}\frac{Y_i^{(2)}}{n}$.

Compute the maximum likelihood estimator $\widehat{\phi}_{n,\mathrm{MLE}}$ of ϕ upon observing $\mathbf{Y}_i, i = 1,\ldots,n$.

Note: Again for simplicity, assume while entering the expression in the following box that $\theta \in [0,\pi/2]$, $\widehat{\mu}_1 > 0$, and $\widehat{\mu}_2 > 0$.

(Use **hatmu_1** for $\widehat{\mu}_1$ and **hatmu_2** for $\widehat{\mu}_2$. Type **arctan(x)** for $\arctan(x)$ (where x can be any expression). **Do not use** any trigonometric function other than $\arctan$.)

$\widehat{\phi}_{n,\mathrm{MLE}} =$ arctan(hatmu_2/hatmu_1) - theta ✔ **Answer:** -theta + arctan(hatmu_2/hatmu_1)

STANDARD NOTATION

Solution:

$$\frac{\partial \ell_n}{\partial\phi} = \frac{\partial}{\partial\phi}\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left((Y_i^{(1)}-\cos(\theta+\phi))^2 + (Y_i^{(2)}-\sin(\theta+\phi))^2\right)\right]$$

$$= -\frac{1}{2\sigma^2}\sum_{i=1}^{n}(2(Y_i^{(1)}-\cos(\theta+\phi)\sin(\theta+\phi) - 2(Y_i^{(2)}-\sin(\theta+\phi))\cos(\theta+\phi)) = -\frac{1}{\sigma^2}\sum_{i=1}^{n}(Y_i^{(1)}\sin(\theta+\phi) - Y_i^{(2)}\cos(\theta+\phi))$$

Setting $\frac{\partial \ell_n}{\partial\phi} = 0$ and solving for ϕ, we have

$$\sin(\theta+\phi)\sum_{i=1}^{n}Y_i^{(1)} = \cos(\theta+\phi)\sum_{i=1}^{n}Y_i^{(2)}$$

$$\implies \frac{\sin(\theta+\phi)}{\cos(\theta+\phi)} = \tan(\theta+\phi) = \frac{\sum_{i=1}^{n}Y_i^{(2)}}{\sum_{i=1}^{n}Y_i^{(1)}} = \frac{\widehat{\mu}_2}{\widehat{\mu}_1}$$

$\implies \theta+\phi = \arctan\left(\frac{\widehat{\mu}_2}{\widehat{\mu}_1}\right)$ Note: This step assumes $\theta+\phi\in[0,\pi/2]$.

$$\implies \phi = \arctan\left(\frac{\widehat{\mu}_2}{\widehat{\mu}_1}\right) - \theta$$

So $\widehat{\phi}_{n,\mathrm{MLE}} = \arctan\left(\frac{\widehat{\mu}_2}{\widehat{\mu}_1}\right) - \theta$.

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

(f) Geometry of the MLE of Phase Noise

0.5/1 point (graded)

Select from the following all statements that are true. (Choose all that apply.)

- [x] The MLE of ϕ does not change if we scaled $\mathbf{x}$ by $r > 0$. ✔
- [ ] The MLE of ϕ does not change if the covariance matrix of the multivariate Gaussian is scaled by $s > 0$. ✔

✱

Grading Note: Partial credit is given.

Solution:

ptressel

11 days ago - endorsed 10 days ago by **sudarsanvsr_mit** (Staff)

Here's a solution for part f (perhaps too wordy...):

"The MLE of ϕ does not change if we scale x by $r > 0$."

This is true: The factor of r will appear anywhere x does, and will not affect the location of the maximum of the log likelihood over ϕ because it will drop out when we set the derivative to zero. The derivative of the log likelihood becomes:

$$\ell'(Y_1,\ldots,Y_n;\phi) = -\frac{n}{\sigma^2}\hat{\mu}_1 r\sin(\theta+\phi) + \frac{n}{\sigma^2}\hat{\mu}_2 r\cos(\theta+\phi)$$

The constant factor $\frac{nr}{\sigma^2}$ will cancel when we equate that to zero.

"The MLE of ϕ does not change if the covariance matrix of the multivariate Gaussian is scaled by $s > 0$."

This is true: σ^2 does not appear in the MLE. Multiplying σ^2 by $s > 0$ is the same as changing σ^2. Since σ^2 was arbitrary to begin with, we can just absorb the s into σ^2, and the derivation of the MLE remains unchanged.

Submit You have used 2 of 3 attempts

Answers are displayed within the problem

Error and Bug Reports/Technical Issues

Show Discussion

Topic: Final exam:Final Exam / Problem 6: Maximum Likelihood Estimation of Phase Noise

Just to add: The whole point of the problem is that it is **intuitive**. The case when we have no Gaussian noise says we just take the arctan of whatever point we see and we subtract off θ. The same principle carries over to the case when we have the Gaussian noise. We estimate the mean of the two Gaussian-added-$\mathbf{x}$ coordinates. With enough samples, this is a consistent estimator. We then take the arctan of the estimated point on the 2-D space and once again subtract off θ.