5. Estimating the parameter of a uniform r.v.

Problem 5. Estimating the parameter of a uniform r.v.

5/5 points (graded)

The random variable X is uniformly distributed over the interval $[\theta, 2\theta]$. The parameter θ is unknown and is modeled as the value of a continuous random variable Θ, uniformly distributed between zero and one.

1. Given an observation x of X, find the posterior distribution of Θ. Express your answers below in terms of θ and x. Use 'theta" to denote θ and 'ln" to denote the natural logarithm function. For example, $\ln(\theta)$ should be entered as 'ln(theta)'.

   For $0 \leq x \leq 1$ and $x/2 \leq \theta \leq x$:

   $f_{\Theta|X}(\theta \mid x) =$ 1/(theta*(ln(x) - ln(x/2))) ✔ **Answer:** 1/(theta*ln(2))

   $$\frac{1}{\theta \cdot \left(\ln(x) - \ln\left(\frac{x}{2}\right)\right)}$$

2. Find the MAP estimate of Θ based on the observation $X = x$ and assuming that $0 \leq x \leq 1$. Express your answer in terms of x.

   For $0 \leq x \leq 1$:

   $\hat{\theta}_{\text{MAP}}(x) =$ 1/2*x ✔ **Answer:** x/2

   $$\frac{1}{2} \cdot x$$

3. Find the LMS estimate of Θ based on the observation $X = x$ and assuming that $0 \leq x \leq 1$. Express your answer in terms of x.

   For $0 \leq x \leq 1$:

   $\hat{\theta}_{\text{LMS}}(x) =$ 1/2*x/(ln(x) - ln(x/2)) ✔ **Answer:** x/(2*ln(2))

   $$\frac{1}{2} \cdot \frac{x}{\ln(x) - \ln\left(\frac{x}{2}\right)}$$

4.

Find the linear LMS estimate $\hat{\theta}_{\text{LLMS}}$ of Θ based on the observation $X = x$. Specifically, $\hat{\theta}_{\text{LLMS}}$ is of the form $c_1 + c_2 x$. Find c_1 and c_2.

$c_1 =$ 2/31 ✔ **Answer:** 0.06452

$c_2 =$ 18/31 ✔ **Answer:** 0.58065

Solution:

1. The prior PDF of Θ is

$$f_\Theta(\theta) = \begin{cases} 1, & \text{if } 0 \le \theta \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

and the conditional PDF of the observation X is

$$f_{X|\Theta}(x \mid \theta) = \begin{cases} 1/\theta, & \text{if } \theta \le x \le 2\theta, \\ 0, & \text{otherwise.} \end{cases}$$

Using Bayes' rule, we find that for any $x \in [0,1]$ and for $\theta \in [x/2, x]$, the posterior PDF is

$$\begin{aligned} f_{\Theta|X}(\theta \mid x) &= \frac{f_\Theta(\theta) f_{X|\Theta}(x \mid \theta)}{\displaystyle\int_{x/2}^{x} f_\Theta(\tilde{\theta}) f_{X|\Theta}(x \mid \tilde{\theta}) d\tilde{\theta}} \\ &= \frac{1/\theta}{\displaystyle\int_{x/2}^{x} \frac{1}{\tilde{\theta}} d\tilde{\theta}} \\ &= \frac{1}{\theta \cdot (\ln(x) - \ln(x/2))} \\ &= \frac{1}{\theta \cdot \ln(2)}. \end{aligned}$$

2. In part (1), we saw that for $x \in [0,1]$ and $x/2 \le \theta \le x$, the posterior PDF is

$$f_{\Theta|X}(\theta \mid x) = \frac{1}{\theta \cdot \ln(2)},$$

which is decreasing in θ over the range $[x/2, x]$ of possible values of Θ. Thus, the MAP estimate for this case is equal to $x/2$.

3. The LMS estimate is the conditional expectation estimate. For $x \in [0, 1]$,

$$\mathbf{E}[\Theta \mid X = x] = \int_{x/2}^{x} \theta \frac{1}{\theta \cdot \ln(2)} \, d\theta = \frac{x}{2 \cdot \ln(2)}.$$

4. The LLMS estimate is of the form

$$\hat{\theta}_{LLMS}(x) = \mathbf{E}[\Theta] + \frac{\text{cov}(\Theta, X)}{\text{Var}(X)}(x - \mathbf{E}[X]).$$

Here,

$$\begin{aligned}
\mathbf{E}[\Theta] &= 1/2, \\
\mathbf{E}[X] &= \mathbf{E}[\mathbf{E}[X \mid \Theta]] \\
&= \mathbf{E}\left[\frac{3}{2}\Theta\right] \\
&= \frac{3}{4}, \\
\mathbf{E}[X^2] &= \mathbf{E}[\mathbf{E}[X^2 \mid \Theta]] \\
&= \mathbf{E}\left[\frac{7}{3}\Theta^2\right] \\
&= \frac{7}{9}.
\end{aligned}$$

$X \sim U(\theta, 2\theta) = \frac{1}{2\theta - \theta} = \frac{1}{\theta}$, so a direct calculation gives, $E[X^2] = \int_{\theta}^{2\theta} \frac{1}{\theta} x^2 \, dx = \frac{7}{3}\theta^2$.

Or you can simply use $E[X^2] = E[X]^2 + \mathrm{Var(X)} = \left(\frac{3}{2}\theta\right)^2 + \frac{(2\theta - \theta)^2}{12} = \frac{7}{3}\theta^2$.

Hence,

$$\begin{aligned}
\text{Var}(X) &= \mathbf{E}[X^2] - (\mathbf{E}[X])^2 \\
&= \frac{31}{144}, \\
\mathbf{E}[\Theta X] &= \mathbf{E}[\mathbf{E}[X\Theta \mid \Theta]] \\
&= \mathbf{E}\left[\frac{3}{2}\Theta^2\right] \\
&= \frac{1}{2}, \\
\text{cov}(\Theta, X) &= \mathbf{E}[\Theta X] - \mathbf{E}[\Theta]\mathbf{E}[X] \\
&= \frac{1}{2} - \frac{1}{2} \cdot \frac{3}{4} \\
&= \frac{1}{8}.
\end{aligned}$$

Finally, we have

$$\begin{aligned}\hat{\Theta}_{LLMS} &= \mathbf{E}[\Theta] + \frac{\text{cov}(\Theta, X)}{\text{Var}(X)}(x - \mathbf{E}[X]) \\ &= \frac{1}{2} + \frac{1/8}{31/144}\left(x - \frac{3}{4}\right) \\ &= \frac{2}{31} + \frac{18}{31}x.\end{aligned}$$

提交 You have used 2 of 4 attempts

Answers are displayed within the problem

讨论

显示讨论

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