We will now go through two examples of convergence in probability. Our first example is quite trivial. We're dealing with a sequence of random variables Yn that are discrete. Most of the probability is concentrated at 0. But there is also a small probability of a large value.

Because the bulk of the probability mass is concentrated at 0, it is a good guess that this sequence converges to 0. Do we have, indeed, convergence in probability to 0? We need to check the definition.

So we fix some epsilon, which is a positive number. And we look at the probability of the event that our random variable is epsilon or more away than what we think is the limit of that sequence. We look at that probability. And in this example, it is equal to 1 over n, which goes to 0 as n goes to infinity.

And this verifies that, indeed, in this example, Yn converges to 0, as n goes to infinity in probability. Now, we make the following observation. If we are to calculate the expected value of this random variable, what we get is the following. We get a value of 0 with this probability, no contribution to the expectation.

But we also get a value of n squared with probability 1 over n. And so the expected value is equal to n, which, actually, goes to infinity, as n goes to infinity. So we have a situation where the sequence of the random variables converges to 0. But the expectation does not converge to 0. In fact, it goes to infinity.

And this example serves to make the point that convergence in probability does not imply convergence of expectations. The reason is that convergence in probability has to do with the bulk of the distribution. It only cares that the tail of the distribution has small probability.

On the other hand, the expectation is highly sensitive to the tail of the distribution. It might be that the tail only has a small probability. But if that probability is assigned to a very large value, then the expectation will be strongly affected and can be quite different from the limit of the random variable.

Our second example is going to be less trivial and more interesting. Consider random variables that are independent and identically distributed and whose common distribution is uniform on the unit interval, so that the PDF takes this form. Are these random variables convergent to something?

The answer is no. And the reason is that as i increases, the distribution does not change. And it does not to get concentrated around a certain number. The distribution remains spread out over the entire unit interval. But let us look now at some related random variables.

Let us define Yn to be the minimum of the first n of the X's that we get. So if n is equal to 4, and we obtain these four values, Yn would be equal to this value. Notice that if we draw more values, then the new values might be above the minimum, in which case the minimum does not change. But we might also get a value that's below the minimum, in which case the minimum moves down.

The only thing that can happen is that the minimum goes down. It cannot go up. And this gives us this inequality. So the random variables Yn tends to go down. How far down? If n is very large, we expect that we will obtain some X's whose value happens to be very close to 0, which means that Yn will go down to values that are very close to 0.

And this leads us to conjecture that, perhaps, Yn does converge to 0. This is always the first step when dealing with convergence in probability. The first step is to make an educated guess about what the limit might be. And then we want to verify that this is, indeed, the correct limit.

To verify that, what we do is we fix some positive epsilon. And we look for the probability that the distance of the random variable Yn from the conjectured limit has a magnitude that's larger than or equal to epsilon. And what we need to show is that this quantity converges to 0 as n goes to infinity, no matter what epsilon is.

Now, because Yn is a non-negative random variable, this is the same as the probability that Yn is larger than or equal to epsilon. Now, let us distinguish between two cases. If epsilon is bigger than 1, we're asking for the probability that Yn is larger than or equal to a certain number epsilon that's out there. But this probability is 0.

There's no way that the minimum of these uniforms will take a value that's larger than some epsilon that's larger than 1. So in that case, this quantity is equal to zero. And we are done. But we need to check that this quantity becomes small no matter what epsilon is.

So now, let us consider taking a small epsilon that is some number that's less than or equal to 1. For that case, let us continue with the calculation. The minimum is going to be at least epsilon, if, and only if, all of the random variables are at least epsilon.

So this is an equivalent way of writing this particular event here. Now, because of independence, this is the product of the probabilities that each one of the random variables is larger than or equal to epsilon. The probability that X1 is larger than or equal to epsilon can be found as follows.

If we have here epsilon, the probability of being larger than or equal to epsilon is the probability of this event here. So it's the area of this rectangle. The base of that rectangle is 1 minus epsilon. And so we obtain 1 minus epsilon for this first term.

But because the Xi's are identically distributed, all the other terms that we multiply are also the same. And so the answer is this expression here. Now, epsilon is a positive number. So 1 minus epsilon is strictly less than 1.

And when we take higher powers of a number that's less than 1, we obtain something that converges to 0 as n goes to infinity. And that's what we needed to verify. Since this is the case for any epsilon, we conclude that the random variables Yn converge to zero in the sense that we have defined, in probability.

Generalizing from this example, when we want to show convergence in probability, the first step is to make a guess as to what is the value that the sequence converges to. In this example, that value was equal to 0. Once we have made that conjecture, then we write down an expression for the probability of being epsilon away from the conjectured limit. And then we calculate that probability either exactly, as in this example. Or we try to bound it somehow and still show that it goes to 0.