8. Some examples of convergence

Rescaled Poisson random variables

2/2 points (graded)

For $n \geq 1$, let X_n be a Poisson random variable with parameter $1/n$. Compute

$\mathbf{P}(X_n = 0) =$ exp(-1/n) ✔ **Answer:** exp(-1/n)

$\exp\left(-\frac{1}{n}\right)$

What can you conclude?

- ○ $X_n \to 0$ in probability, but nX_n does not converge in probability
- ○ $X_n \to 0$ in probability, $nX_n \to 0$ in probability, and $\mathbb{E}\left[(nX_n)^2\right]$ converges.
- ◉ $X_n \to 0$ and $nX_n \to 0$ in probability, but $\mathbb{E}\left[(nX_n)^2\right]$ does not converge. ✔

STANDARD NOTATION

Solution:

Using the probability mass function of the Poisson distribution, we compute

$$\mathbf{P}(X_n = 0) = \left(\frac{1}{n}\right)^0 \frac{1}{0!} \exp\left(-\frac{1}{n}\right) = \exp\left(-\frac{1}{n}\right).$$

As $n \to \infty$, this tends to 1, and therefore $X_n \to 0$ in probability. Moreover, the same calculation tells us the probability $\mathbf{P}(nX_n = 0)$, therefore we also obtain that $nX_n \to 0$ in probability.

However, the expectation of the square of nX_n does not go to zero:

$$\mathbb{E}\left[(nX_n)^2\right] = n^2 \mathbb{E}\left[X_n^2\right] = n^2\left(\frac{1}{n^2} + \frac{1}{n}\right) = n + 1 \to \infty.$$

Remark: We also say that nX_n does **not** "converge in L^2-norm". A sequence of random variables $(Y_n)_{n\geq 1}$ **converges in L^2-norm** to a random variable Y, denoted by $Y_n \xrightarrow[n\to\infty]{L^2} Y$, if $\lim_{n\to\infty} \mathbb{E}\left[|Y_n - Y|^2\right] = 0.$ Moreover, if $Y_n \xrightarrow[n\to\infty]{L^2} Y$, then $\lim_{n\to\infty} \mathbb{E}\left[|Y_n|^2\right] = \mathbb{E}\left[Y^2\right]$. Hence, in this example, since $\mathbb{E}\left[(nX_n)^2\right] \xrightarrow[n\to\infty]{} \infty$, nX_n does not converge in L^2-norm.

提交 你已经尝试了1次（总共可以尝试2次）

Answers are displayed within the problem

Limit of rescaled Binomials

1.0/1 point (graded)

Let X_n be a binomial random variable with parameters n and $p = \lambda/n$, where λ is a fixed positive number.

Let $k \in \mathbb{N}$ be fixed. As $n \to \infty$, the probability mass function $\mathbf{P}(X_n = k)$ converges to a number that only depends on λ and k. What is the limit?

(If necessary, enter **fact** to indicate the factorial function. For instance, **fact(10)** denotes $10!$.)

$\lim_{n\to\infty} \mathbf{P}(X_n = k) =$ exp(-lambda)*lambda^k/ (fact(k)) ✔ **Answer:** lambda^k/fact(k)*exp(-lambda)

(*Food for thought:* What can you conclude?) 泊松分布

STANDARD NOTATION

Solution:

The probability mass function of the Binomial random variable X_n is given by

$$\mathbf{P}(X_n = k) = \binom{n}{k}\left(\frac{\lambda}{n}\right)^n\left(1 - \frac{\lambda}{n}\right)^{n-k},$$

where k is an integer between 0 and n.

Writing the binomial coefficient as

$$\binom{n}{k} = \frac{1}{k!}\frac{n!}{(n-k)!},$$

we have

$$\mathbf{P}(X_n = k) = \frac{\lambda^k}{k!}\underbrace{\left(1 - \frac{\lambda}{n}\right)^n}_{=:A_n}\underbrace{\left(1 - \frac{\lambda}{n}\right)^{-k}}_{=:B_n}\underbrace{\frac{n!}{(n-k)!}}_{=:C_n}.$$

Term A_n can be handled by the exponential formula,

$$\left(1 + \frac{a}{n}\right)^n \xrightarrow[n\to\infty]{} \exp(a), \quad \text{for } a \in \mathbb{R}.$$

Hence,

$$A_n \to \exp(-\lambda), \quad \text{as } n \to \infty.$$

Since k is fixed and $\lambda/n \to 0$, we have $B_n \to 1$. Finally, write

$$\begin{aligned} C_n = \frac{n!}{(n-k)!} &= 1 \times \left(\frac{n-1}{n}\right) \times \cdots \times \left(\frac{n-k+1}{n}\right) \\ &= 1 \times \left(1 - \frac{1}{n}\right) \times \cdots \times \left(1 - \frac{k-1}{n}\right) \to 1, \quad \text{as } n \to \infty. \end{aligned}$$

Combined, we get that

$$\mathbf{P}(X_n = k) \to \frac{\lambda^k}{k!}\exp(-\lambda).$$

Since that entails the convergence of the cumulative mass function, $\mathbf{P}(X_n \leq m)$, for any $m \in \mathbb{Z}$ as well, we have just shown that X_n converges in distribution to a Poisson distribution with parameter λ.

提交 你已经尝试了1次（总共可以尝试3次）

Answers are displayed within the problem

讨论

显示讨论

主题： Unit 2 Foundation of Inference:Homework 1: Estimation, Confidence Interval, Modes of Convergence / 8. Some examples of convergence

认证证书是什么?