8. Example: Assessing the performance of planes

Setup:

An aerospace manufacturing company would like to assess the performance of its existing planes for its latest design. Based on a sample size of $n = 1000$ flights, each with an identically designed plane, it collects data of the form $(x_1, y_1), \ldots, (x_{1000}, y_{1000})$, where x represents the distance traveled and y represents liters of fuel consumed.

You, as a statistician hired by the company, decide to perform linear regression on the model $y = a + bx$ to predict the efficiency of the design. In the context of linear regression, recall that the mathematical model calls for:

$$\mathbf{Y} = \begin{pmatrix} y_1 \\ \vdots \\ y_{1000} \end{pmatrix} \in \mathbb{R}^{1000}, \quad \boldsymbol{\varepsilon} \in \mathbb{R}^{1000}, \quad \mathbb{X} = \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_{1000} \end{pmatrix} \in \mathbb{R}^{1000 \times 2}, \quad \boldsymbol{\beta} = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2.$$

Assume that $\boldsymbol{\varepsilon} \sim \mathcal{N}(0, \sigma^2 I_{1000})$ for some fixed σ^2, so that $\mathbf{Y} \sim \mathcal{N}(\mathbb{X}\boldsymbol{\beta}, \sigma^2 I_{1000})$.

Prediction using Regression

1/1 point (graded)
Using the setup as above, you compute the LSE, which comes out to

$$\hat{\boldsymbol{\beta}} = \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = \begin{pmatrix} 0.8 \text{ liters} \\ 15.0 \text{ liters / km} \end{pmatrix}.$$

Just from $\hat{\boldsymbol{\beta}}$, what is a reasonable prediction for the total amount of fuel a plane (in liters) consumes after 200 kilometers?

3000.8

✔ **Answer:** 3000.8

Solution:

The LSE gives the "best-fitting" model $y = 0.8 + 15x$. Plugging in $x = 200$ gives $y = 3000.8$.

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Answers are displayed within the problem

The Maximum Likelihood Estimator

2/2 points (graded)
Using the same setup as the previous problem:

$$\hat{\boldsymbol{\beta}} = \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = \begin{pmatrix} 0.8 \text{ liters} \\ 15.0 \text{ liters / km} \end{pmatrix}.$$

Using $n = 1000$ samples, by thinking of $\mathbf{Y}$ as the vector of observations, we might also consider the Maximum Likelihood Estimator β_{MLE}. As a reminder, $\boldsymbol{\beta}_{MLE}$ maximizes, over all choices of $\boldsymbol{\beta}$, the likelihood (or the log-likelihood) of $\mathbf{Y} \sim \mathcal{N}(\mathbb{X}\boldsymbol{\beta}, \sigma^2 I_{1000})$.

Numerically, $\beta_{MLE} = \begin{pmatrix} a_{MLE} \\ b_{MLE} \end{pmatrix}$, where:

$a_{MLE} =$

0.8 ✔ **Answer:** 0.8

$b_{MLE} =$

15 ✔ **Answer:** 15.0

Solution:

This scenario is the homoscedastic gaussian case, **so the MLE coincides with the LSE.** Therefore, $a_{MLE} = \hat{a} = 0.8$ and $b_{MLE} = \hat{b} = 15.0$.
To see why, recall that $\mathcal{N}\left(\mathbb{X}\beta, \sigma^2 I_{1000}\right)$ follows the density

$$f\left(\mathbf{Y}\right) = \frac{\exp\left(-\frac{1}{2}\left(\mathbf{Y} - \mathbb{X}\beta\right)^T \left(\sigma^2 I\right)^{-1}\left(\mathbf{Y} - \mathbb{X}\beta\right)\right)}{\sqrt{\left(2\pi\right)^{1000}\det\left(\sigma^2 I\right)}}$$

so the log-likelihood function can be written

$$\log f\left(\mathbf{Y}\right) = -500\log\left(2\pi\sigma^2\right) - \frac{1}{2\sigma^2}\left(\mathbf{Y} - \mathbb{X}\beta\right)^T\left(\mathbf{Y} - \mathbb{X}\beta\right)$$

The vector β only appears in the second term, so maximizing this expression is the same as minimizing

$$\left(\mathbf{Y} - \mathbb{X}\beta\right)^T\left(\mathbf{Y} - \mathbb{X}\beta\right) = \|\mathbf{Y} - \mathbb{X}\beta\|^2$$

which, by definition, is attained by the least-squares estimator $\hat{\beta}$.

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Discussion

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