Hello. We will now continue our discussion of the distribution of the sum of a random number of independent, identically distributed, random variables. We will look at some additional special cases, again, using ideas from the Bernoulli and the Poisson processes.

So what we have here is a sequence of random valuable X1, X2, and so on. Independent, identically distributed, they can be discrete or continuous. And we add a number capital N of them, with N being a discrete random variable which will be an integer greater or equal to 1. And it will be independent of all the Xi's.

So we are interested in properties of Y, the sum of this random number of independent random variables. Earlier in the course, we have derived this important formula about the mean and the variance of Y, which are special cases of the law of iterated expectations and the law of total variance. These involve just the expected values of N and X. X is the random variable which is identical to all those, so a generic symbol for any one of the X1, X2, and so on.

So the variance of Y depends not only on the means, but also on the variances of X and N. Now we are going to make some specific assumptions about X and N. And we are going to derive not just the mean and the values of Y, but also the entire distribution.

So we're going to obtain favorable results. In particular, we're going to show that if N is geometrically distributed, and X is geometrically distributed, all of these are geometrically distributed, then the sum is also geometrically distributed. So we have our result that has sort of a nice ring to it: the geometric sum of geometric random variables is geometric.

There's also another similar result that is sort of a limiting version of this result here. If N is geometric-- but instead of X being geometric, we have that X is exponential-- then Y is also exponential. So the geometric sum of exponentials random valuables is exponential.

We are going to use the properties of the Bernoulli and the Poisson processes, and the properties or processes obtained by splitting them, in order to obtain and verify these results.

As our working example, we're going to consider a store at which customers arrive, according to a Bernoulli process, with probability p. And then each customer may buy something from the store with probability q, or not buy with a complementary probability 1 minus q. So what we have here is this Bernoulli process of arrivals of customers into the store. And time here is discrete. And we are going to consider the interarrival times between customers.

Now what's going to happen here is that some customers, in the beginning, may not buy anything. But there will be a first customer that will buy. And this customer is denoted as this one here, and its index is denoted by capital N.

So let's define our variables here. Xi is the interarrival time between customer i minus 1 and i. Customer zero is hypothetical customer here. So as you see in the figure, X1 is the time up to the arrival of the first customer, X2 is the time up to arrival of the second customer, and so on.

And we're going to call N the index corresponding to the first sale. The first customer that comes in and buys something. So N is the index of the first buying customer.

So as an example, if N is equal to 3, that means that customer one and two did not buy, but customer three bought. Of course, it's possible that the very first customer bought, in which case, N is equal to 1.

OK. So now we have the customers buy with probability q and do not buy with probability 1 minus q. So we can consider a splitting of the Poisson process into two processes. The buying customers and the non buying customers. And then we recognize this time here as geometrically distributed because this is the time of the first arrival in the top process, which we know to be a Bernoulli process.

So we will call this time Y. And it's the time of the first sale. And we see that this time is precisely the sum of all these interarrival intervals, up to the time of the first sale. So this is X1 plus X2, all the way up to XN.

So what we have here in this example is a situation that matches this result. We have a sum of a random number of random variables, where N is geometrically distributed as specified there, because N is the time of the first arrival in a Bernoulli process. And we have that each one of these Xs are geometrically distributed, because in the Bernoulli process, the interarrival times are geometric.

So we have shown that the sum of a geometrically distributed number of geometric random variables is geometric, and that's what we wanted to prove. We will now show this result here corresponding to the case when X is exponential. We will use again this example, a variation in this example, where the customer arrival process is Poisson with parameter lambda rather than being Bernoulli.

And, again, Y will be the time of arrival of the first customer in this top q, and will correspond to the time of the first sale. The only difference between the Bernoulli case and the Poisson case is that these interarrival times are going to be exponentially distributed rather than being geometrically distributed. So let's correct this figure.

The customer arrivals are according to a Poisson process with rate lambda. The interarrival times between customer i minus 1 and i, before it was geometric. Now, because we have Poisson arrivals, it will be exponential

with parameter lambda.

The index of the first buying customer will continue to be geometrically distributed because it corresponds to the time of first arrival in the top process-- which by splitting, is Poisson. This is Poisson with parameter lamda. This is also Poisson with parameter lambda q. And this is Poisson with parameter lamda 1 minus q.

Why is the time of the first sale, N corresponds to the time of first arrival in the top process of customers that buy? We know that in the Poisson process, the time to get the first arrival is exponentially distributed and therefore this shows this result that Y is exponentially distributed.

So the only difference between these two examples, is that we're considering a Poisson process instead of a Bernoulli. And then, because the Bernoulli process in the limit becomes a Poisson process, it is natural to obtain that in the limit as the time slot length becomes smaller and smaller, the geometric random variable becomes an exponential.

We have seen that the sum of a geometrically distributed number of exponential random variables is exponential. But what would happen in the simpler case where the number N is not random, but instead is fixed and deterministic? Would the sum of a fixed number of exponential random variables still be exponential?

Well, the answer is that it would have been nice if it were, but unfortunately, it is not. We have already verified earlier that the sum has an Erlang distribution of order n when n is fixed. So, for this example, the distribution of the sum is simpler when the number of terms is random than when it is fixed.

In general, however, this is not the case. For example, we know that the sum of a fixed number of normal random variables is normal. But the sum of a random number of normal random variables is not normal. And you can easily verify this by example, even by taking N to be a Bernoulli random valuable, taking the value 1 with a certain probability or 0, with the complementary probability.

This concludes our discussion of the distribution of the sum of a random number of random variables. Generally the distribution is complicated but when you can relate it to the Bernoulli and Poisson processes, it becomes interesting and intuitive in many cases.

I hope you have enjoyed this discussion and you explore further the properties of various random variables involved in the Bernoulli and the Poisson processes. This study often brings out a lot of insight and it can be a lot of fun. I'm glad you joined us.