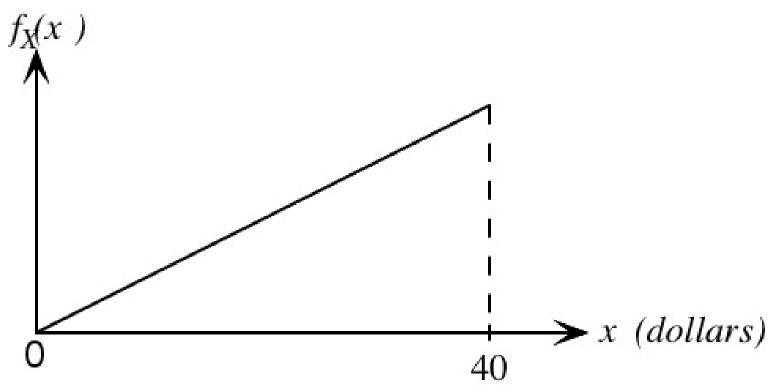
4. Sophia's vacation

Problem 4. Sophia's vacation

4/7 points (graded)

Sophia is vacationing in Monte Carlo. On any given night, she takes X dollars to the casino and returns with Y dollars. The random variable X has the PDF shown in the figure. Conditional on $X = x$, the continuous random variable Y is uniformly distributed between zero and $3x$.



1. Determine the joint PDF $f_{X,Y}(x, y)$.

   If $0 < x < 40$ and $0 < y < 3x$:

   $f_{X,Y}(x, y) =$ 1/2400 ✔ **Answer:** 1/2400

   If $y < 0$ or $y > 3x$:

   $f_{X,Y}(x, y) =$ 0 ✔ **Answer:** 0

2. On any particular night, Sophia makes a profit $Z = Y - X$ dollars. Find the probability that Sophia makes a positive profit, that is, find $\mathbf{P}(Z > 0)$.

   $\mathbf{P}(Z > 0) =$ 2/3 ✔ **Answer:** 2/3

3. Find the PDF of Z. Express your answers in terms of z using standard notation.

   Hint: Start by finding $f_{Z|X}(z \mid x)$.

If $-40 < z < 0$:

$f_Z(z) =$

✖ **Answer:** (40+z)/2400

If $0 < z < 80$:

$f_Z(z) =$

✖ **Answer:** (80-z)/4800

If $z < -40$ or $z > 80$:

$f_Z(z) =$ 0

0

✔ **Answer:** 0

4. What is $\mathbf{E}[Z]$?

$\mathbf{E}[Z] =$ 10

✖ **Answer:** 40/3

STANDARD NOTATION

Solution:

1. For this part, we will use the fact that $f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y \mid x)$. Let us start by revealing $f_X(x)$. Clearly, $f_X(x) = ax$ for some ax, as shown in figure. Hence,

$$1 = \int_{-\infty}^{\infty} f_X(x)\, dx = \int_0^{40} ax\, dx = 800a.$$

Hence, $f_X(x) = \frac{x}{800}$. Using $f_{Y|X}(y \mid x) = \frac{1}{3x}$, for $0 < y < 3x$, we obtain the following expression for the joint density:

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{2400}, & \text{if } 0 < x < 40 \text{ and } 0 < y < 3x \\ 0, & \text{otherwise.} \end{cases}$$

2. The first approach is to consider the region where Sophia makes positive profit. Notice that, this region consists of pairs (x, y), where $y > x$. Intersecting this region with the region where the joint density is non-negative, we need to consider

$$\{(x,y) : 0 < x < 40, x < y < 3x\}.$$

Thus,

$$\mathbf{P}(Y > X) = \int_0^{40} \int_x^{3x} f_{X,Y}(x,y)\, dy\, dx = \int_0^{40} \int_x^{3x} \frac{1}{2400}\, dy\, dx = \int_0^{40} \frac{x}{1200} = \frac{2}{3}.$$

We could have also arrived at this answer by realizing that for each possible value of X, there is a $2/3$ probability that Y>X, and therefore by the total probability theorem,

$$\begin{aligned} \mathbf{P}(Y > X) &= \int_0^{40} \mathbf{P}(Y > X \mid X = x) f_X(x)\, dx \\ &= \int_0^{40} \frac{2}{3} f_X(x)\, dx \\ &= \frac{2}{3}, \end{aligned}$$

where the last equality follows because a PDF always integrates to 1, over the region where it is nonzero.

3. Given $X = x$, Y is uniformly distributed on $[0, 3x]$, hence $Z = Y - x$ is uniform over $[-x, 2x]$. Thus,

$$f_{Z|X}(z \mid x) = \frac{1}{3x}, \qquad \text{for } -x \le z \le 2x.$$

Therefore,

$$f_{X,Z}(x,z) = f_X(x) f_{Z|X}(z \mid x) = \frac{x}{800}\frac{1}{3x} = \frac{1}{2400}, \text{ for } 0 < x < 40 \text{ and } -x \le z \le 2x.$$

Now, we will integrate over x to compute the marginal density $f_Z(z)$. Note that, $x \ge -z$ and $x \ge \frac{z}{2}$ must be satisfied at the same time (in order for $f_{X,Z}$ to be non-zero).

If $-40 < z < 0$, the range of integration is $-z < x < 40$. Hence,

$$f_Z(z) = \int_{-z}^{40} \frac{1}{2400}\, dx = \frac{40 + z}{2400}.$$

If $0 < z < 80$, the range of integration is $z/2 \le x \le 40$. Hence,

$$f_Z(z) = \int_{z/2}^{40} \frac{1}{2400}\, dx = \frac{80-z}{4800}.$$

Therefore, the pdf of Z is

$$f_Z(z) = \begin{cases} \dfrac{40+z}{2400}, & -40 < z < 0 \\ \dfrac{80-z}{4800}, & 0 < z < 80 \\ 0, & \text{otherwise.} \end{cases}$$

4. First, note that $\mathbf{E}[Y|X=x] = \frac{3x}{2}$, for any $x \in [0, 40]$. Thus, using the total expectation theorem,

$$\begin{aligned} \mathbf{E}[Y] &= \int_0^{40} \mathbf{E}[Y|X=x] f_X(x)\, dx \\ &= \frac{3}{2}\int_0^{40} x f_X(x)\, dx \\ &= \frac{3}{2}\mathbf{E}[X]. \end{aligned}$$

Since, $Z = Y - X$, we have, using linearity of expectation, $\mathbf{E}[Z] = \mathbf{E}[Y] - \mathbf{E}[X] = \frac{1}{2}\mathbf{E}[X]$.

Now,

$$\mathbf{E}[X] = \int_0^{40} x f_X(x)\, dx = \int_0^{40} \frac{x^2}{800}\, dx = \frac{80}{3}.$$

Hence, $\mathbf{E}[Z] = 40/3$.

提交 You have used 4 of 6 attempts

Answers are displayed within the problem

讨论

显示讨论

Topic: Unit 5 / Problem Set / 4. Sophia's vacation

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