## Homework 4: TV distance, KL-Divergence, and Introduction to

课程 □ Unit 3 Methods of Estimation □ MLE

☐ 3. Concave functions

## 3. Concave functions

(a)

3/3 points (graded)

Are the following functions concave, convex, or neither?

$$f_{1}\left( x
ight) =\ln x,\quad x>0.$$

● Concave □

Convex

Not concave and not convex

$$f_{2}\left( x
ight) =-x^{4}+x^{2}-40x,\quad x\in \mathbb{R}$$

Concave

Convex

lacktriangle Not concave and not convex  $\Box$ 

$$f_{3}\left( x
ight) =rac{1}{\exp \left( x
ight) -1},\quad x>0$$

Concave

● Convex □

Not concave and not convex

## Solution:

Recall that for a twice continuously differentiable function f, we can check concavity by testing whether  $f''(x) \leq 0$  for all x in the (convex) domain in question.

To begin, compute

$$f_{1}^{\prime}\left( x
ight) =rac{1}{x}$$

$$f_{1}^{\prime\prime}\left( x
ight) = ext{ }-rac{1}{x^{2}}<0, ext{ for }x>0,$$

so  $f_1$  is concave.

$$egin{array}{ll} f_2'\left(x
ight) = & -4x^3 + 2x - 40 \ f_2''\left(x
ight) = & -12x^2 + 2, \end{array}$$

which means  $f_2''\left(x
ight)>0$  for  $x\in\left(-rac{1}{\sqrt{6}},rac{1}{\sqrt{6}}
ight)$  , but  $f_2''\left(x
ight)<0$  for  $x
otin\left[-rac{1}{\sqrt{6}},rac{1}{\sqrt{6}}
ight]$  , hence  $f_2$  is neither concave or convex.

$$egin{align} f_3'\left(x
ight) &=& -rac{e^x}{\left(e^x-1
ight)^2} \ f_3''\left(x
ight) &=& -rac{e^x\left(e^x-1
ight)-2e^{2x}}{\left(e^x-1
ight)^3} \ &=& rac{e^{2x}+e^x}{\left(e^x-1
ight)^3} > 0, \quad ext{for } x > 0. \end{array}$$

That means that f is convex for x>0 .

提交

你已经尝试了1次(总共可以尝试1次)

☐ Answers are displayed within the problem

(b)

2/2 points (graded)

A symmetric  $2 \times 2$  matrix  $\mathbf{A}$  (i.e.  $\mathbf{A}^T = \mathbf{A}$ ) is negative semi- definite, i.e.  $\mathbf{x}^T \mathbf{A} \mathbf{x} \leq \mathbf{0}$  for all  $\mathbf{x} \in \mathbb{R}^2$ , if and only if both of the following is true:

- $\operatorname{tr}(\mathbf{A}) \leq 0$
- $\det(\mathbf{A}) \geq 0$

(This fact can be explained in terms the eigenvalues of  $\bf A$ . Let  $\lambda_1$  and  $\lambda_2$  be the eigenvalues of  $\bf A$ , then  ${\rm tr}\,({\bf A})=\lambda_1+\lambda_2$  while  ${\rm det}\,({\bf A})=\lambda_1\lambda_2$ . The two conditions above ensure that  $\lambda_1,\lambda_2\leq 0$ .)

Use the fact given above to determine whether the following functions concave, convex, or neither.

$$f_4\left( heta_1, heta_2
ight) = - heta_1^2 + rac{1}{2}( heta_1- heta_2)^2 - 3 heta_2^2, \quad ( heta_1, heta_2) \in \mathbb{R}^2$$

- Concave
- Convex
- Not concave and not convex

$$f_5\left( heta_1, heta_2
ight)=- heta_1^4- heta_2^4-\left( heta_2- heta_1
ight)^3,\quad \left( heta_1, heta_2
ight)\in\mathbb{R}^2, ext{ with } heta_1< heta_2$$

● Concave □
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- Convex
- Not concave and not convex

## **Solution:**

If f is function from  $\Omega \subseteq \mathbb{R}^d \to \mathbb{R}$ , then it is concave if the Hessian of f is negative semi-definite. In the special case of two dimensions, this can be checked by testing whether both  $\operatorname{tr} \nabla^2 f \leq 0$  and  $\operatorname{det} \nabla^2 f \geq 0$  are true.

$$egin{aligned} 
abla f_4\left( heta_1, heta_2
ight) &=& egin{pmatrix} - heta_1- heta_2 \ - heta_1-5 heta_2 \end{pmatrix} \ Hf_4\left( heta_1, heta_2
ight) &=& egin{pmatrix} -1 & -1 \ -1 & -5 \end{pmatrix}. \end{aligned}$$

Since  ${
m tr}\, 
abla^2 f_4 = -6 < 0$  and  ${
m det}
abla^2 f_4 = 4 > 0$  , we have  $abla^2 f$  is negative semi-definite for all  $\, heta$  , and in turn,  $\,f_4\,$  is concave.

$$egin{aligned} 
abla f_5\left( heta_1, heta_2
ight) = & egin{pmatrix} -4 heta_1^3 + 3( heta_2 - heta_1)^2 \ -4 heta_2^3 - 3( heta_2 - heta_1)^2 \end{pmatrix} \ Hf_5\left( heta_1, heta_2
ight) = & egin{pmatrix} -12 heta_1^2 - 6\left( heta_2 - heta_1
ight) & 6\left( heta_2 - heta_1
ight) \ 6\left( heta_2 - heta_1
ight) & -12 heta_2^2 - 6\left( heta_2 - heta_1
ight) \end{pmatrix}. \end{aligned}$$

We again check

$$\operatorname{tr}
abla^2 f_5\left( heta_1, heta_2
ight) = -12 heta_1^2 - 12\left( heta_2 - heta_1
ight) - 12 heta_2^2 < 0, \quad ext{if } heta_1 < heta_2,$$

$$egin{aligned} \det & 
abla^2 f_5 \left( heta_1, heta_2 
ight) = & \left( 12 heta_1^2 + 6 \left( heta_2 - heta_1 
ight) 
ight) \left( 12 heta_2^2 + 6 \left( heta_2 - heta_1 
ight) 
ight) - 36 ( heta_2 - heta_1)^2 \ & = & 144 heta_1^2 heta_2^2 + 72 \left( heta_1^2 + heta_2^2 
ight) \left( heta_2 - heta_1 
ight) > 0, & ext{if } heta_1 < heta_2. \end{aligned}$$

Combined,  $f_5$  is concave on  $\left\{ heta_1 < heta_2 
ight\}$  .

提交

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☐ Answers are displayed within the problem

讨论

显示讨论

主题: Unit 3 Methods of Estimation:Homework 4: TV distance, KL-Divergence, and Introduction to MLE / 3. Concave functions

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