(a) Let A_k be the event that the process enters S_2 for first time after the kth trial. The only way to enter state S_2 for the first time after the kth trial is to enter state S_3 on the first trial, remain in S_3 for the next $k-2$ trials, and finally enter S_2 on the last trial. Thus,

$$\mathbf{P}(A_k) = p_{03} \cdot (p_{33})^{k-2} \cdot p_{32} = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)^{k-2}\left(\frac{1}{4}\right) = \frac{1}{3}\left(\frac{1}{4}\right)^{k-1} \qquad \text{for } k = 2, 3, \ldots.$$

(b) Let B be the event that the process never enters S_4. There are three possible ways for B to occur. The first two are if the first transition is either from S_0 to S_1 or from S_0 to S_5. This occurs with probability $\frac{2}{3}$. The other is if the first transition is from S_0 to S_3, and that the next change of state *after* that is to state S_2. We know that the probability of going from S_0 to S_3 is $\frac{1}{3}$. Given this has occurred, and given a change of state occurs from state S_3, we know that the probability that the state transitioned to is state S_2 is simply $\frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{2}} = \frac{1}{3}$. Thus, the probability of transitioning from S_0 to S_3 and then eventually transitioning to S_2 is $\frac{1}{9}$. Thus, the probability of never entering S_4 is $\frac{2}{3} + \frac{1}{9} = \frac{7}{9}$.

(c) Let C be the event that the process enters S_2 and then leaves S_2 on the next trial.

$$\begin{aligned}
\mathbf{P}(C) &= \mathbf{P}(\text{process enters } S_2)\mathbf{P}(\text{leaves } S_2 \text{ on next trial} \mid \text{process enters } S_2) \\
&= \left[\sum_{k=2}^{\infty} \mathbf{P}(A_k)\right] \cdot \frac{1}{2} \\
&= \left[\sum_{k=2}^{\infty} \frac{1}{3}\left(\frac{1}{4}\right)^{k-1}\right] \cdot \frac{1}{2} \\
&= \frac{1}{6} \cdot \frac{\frac{1}{4}}{1-\frac{1}{4}} \\
&= \frac{1}{18}.
\end{aligned}$$

(d) Let D be the event that the process enters S_1 for the first time on the third trial. This event can happen only if the sequence of state transitions is as follows:

$$S_0 \longrightarrow S_3 \longrightarrow S_2 \longrightarrow S_1.$$

Thus, $\mathbf{P}(D) = p_{03} \cdot p_{32} \cdot p_{21} = \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{24}$.

(e) Let E be the event that the process is in S_3 immediately after the nth trials. This event can happen only if the process moves to S_3 after the first trial and then self-transitions to stay in S_3 for the next $n-1$ trials. Hence, for $n = 1, 2, 3, \ldots$,

$$\mathbf{P}(E) = \frac{1}{3}\left(\frac{1}{4}\right)^{n-1}.$$