Joint PMF drill #2

1.

$$\begin{aligned} p_X(1) &= \mathbf{P}(X=1, Y=1) + \mathbf{P}(X=1, Y=2) + \mathbf{P}(X=1, Y=3) \\ &= 1/12 + 2/12 + 1/12 = 1/3 \end{aligned}$$

2. The solution is a sketch of the following conditional PMF:

$$p_{Y|X}(y \mid 1) = \frac{p_{Y,X}(y,1)}{p_X(1)} = \begin{cases} 1/4, & \text{if } y=1, \\ 1/2, & \text{if } y=2, \\ 1/4, & \text{if } y=3, \\ 0, & \text{otherwise.} \end{cases}$$

3. $\mathbf{E}[Y \mid X=1] = \sum_{y=1}^{3} y\, p_{Y|X}(y \mid 1) = 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{4} = 2$

4. Notice that no matter what the starred values are, we have $p_{Y|X}(1|2) > 0$ but $p_{Y|X}(1|3) = 0$. Therefore information on whether X is 2 or 3 affects the distribution of Y. It follows that X and Y are not independent.

5. The conditional probabilities $p_{X,Y|B}(x,y)$ given B, are proportional to the given unconditional probabilities $p_{X,Y}(x,y)$, for $(x,y) \in B$. Furthermore if we have conditional independence, the conditional probabilities in the row where $x=1$ must be proportional to the conditional probabilities in the row where $x=2$. It follows that

$$\frac{p_{X,Y}(1,1)}{p_{X,Y}(1,2)} = \frac{p_{X,Y}(2,1)}{p_{X,Y}(2,2)}$$

since the (X,Y) pairs in the equality are all in B. Thus

$$p_{X,Y}(2,2) = \frac{p_{X,Y}(1,2)p_{X,Y}(2,1)}{p_{X,Y}(1,1)} = \frac{(2/12)(2/12)}{1/12} = \frac{4}{12} = \frac{1}{3}.$$

6. Since $\mathbf{P}(B) = 9/12 = 3/4$, we normalize to obtain $p_{X,Y|B}(2,2) = \frac{p_{X,Y}(2,2)}{\mathbf{P}(B)} = 4/9$.