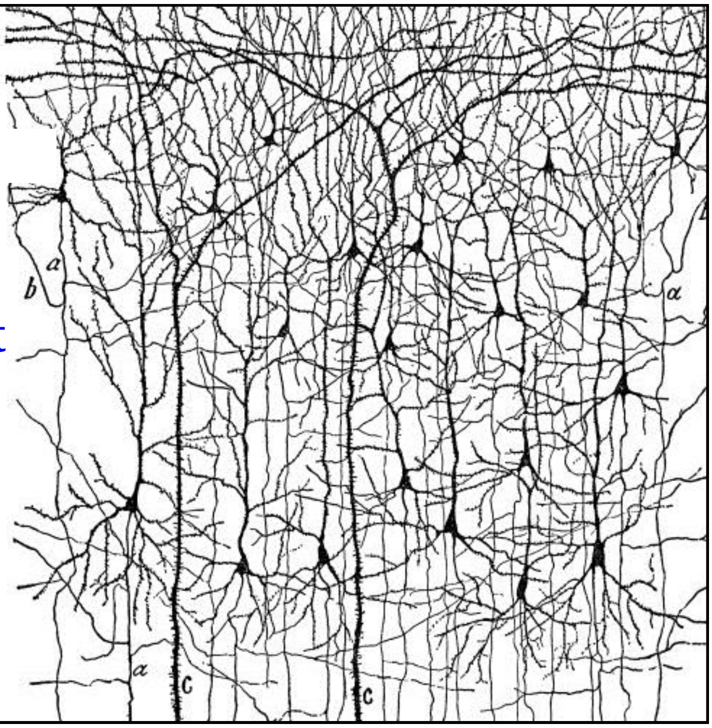
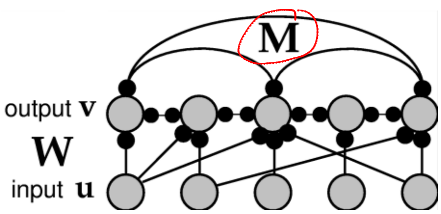
Recurrent Networks



Drawing by Ramón y Cajal



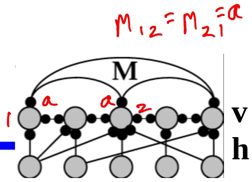
What can a Linear Recurrent Network do?

$$\tau \frac{d\mathbf{v}}{dt}^{N\times 1} = -\mathbf{v} + \underbrace{\overset{N\times 1}{W\mathbf{u}}}_{\mathbf{h}} + M\mathbf{v}$$

Want to find out how **v**(t) behaves for different M

Image Source: Dayan & Abbott textbook

Eigenvectors to the rescue!



$$\tau \frac{d\mathbf{v}}{dt} = -\mathbf{v} + \mathbf{h} + M\mathbf{v}$$ v(t)

- Idea: Use *eigenvectors* of M to *solve differential equation* for **v**
- Suppose $N \times N$ matrix M is symmetric
- M has *N orthogonal* eigenvectors $\mathbf{e}_i$ and *N* eigenvalues λ_i which satisfy:

$$M\mathbf{e}_i = \lambda_i \mathbf{e}_i$$

$e_i \cdot e_j = 0 \quad i \neq j$

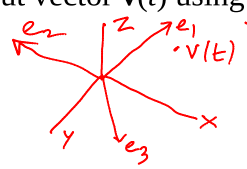
ORTHONORMAL

$e_i \cdot e_i = 1$

Using Eigenvectors to Solve for Network Output $\mathbf{v}(t)$

- We can represent output vector $\mathbf{v}(t)$ using eigenvectors of M:

$$\mathbf{v}(t) = \sum_{i=1}^{N} c_i(t)\mathbf{e}_i$$



$v(t) = v_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$= c_1 e_1 + c_2 e_2 + c_3 e_3$

- Substituting above in the diff. equation for **v**: $\tau \frac{d\mathbf{v}}{dt} = -\mathbf{v} + \mathbf{h} + M\mathbf{v}$

  using $M\mathbf{e}_i = \lambda_i \mathbf{e}_i$ and orthonormality of $\mathbf{e}_i$, we can solve for c_i (and therefore $\mathbf{v}(t)$!):

$$c_i(t) = \frac{\mathbf{h} \cdot \mathbf{e}_i}{1 - \lambda_i}\left(1 - \exp\left(\frac{-t(1-\lambda_i)}{\tau}\right)\right) + c_i(0)\exp\left(\frac{-t(1-\lambda_i)}{\tau}\right)$$

(For full derivation, see "Supplementary Materials" on course website)

Eigenvalues determine Network Stability!

$$\mathbf{v}(t) = \sum_{i=1}^{N} c_i(t)\mathbf{e}_i \qquad c_i(t) = \frac{\mathbf{h}\cdot\mathbf{e}_i}{1-\lambda_i}\left(1-\exp\left(\frac{-t(1-\lambda_i)}{\tau}\right)\right) + c_i(0)\exp\left(\frac{-t(1-\lambda_i)}{\tau}\right)$$

2

$\exp\left(\frac{t}{\tau}\right)$

If any $\lambda_i > 1$, $\mathbf{v}(t)$ explodes $\Rightarrow$ network is unstable!

If all $\lambda_i < 1$, network is stable and $\mathbf{v}(t)$ converges to steady state value :

$$\mathbf{v}_{ss} = \sum_i \frac{\mathbf{h}\cdot\mathbf{e}_i}{1-\lambda_i}\mathbf{e}_i$$

Amplification of Inputs in a Recurrent Network

$$\mathbf{v}_{ss} = \sum_i \frac{\mathbf{h}\cdot\mathbf{e}_i}{1-\lambda_i}\mathbf{e}_i$$

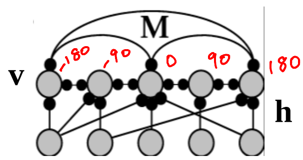
If all $\lambda_i < 1$ and one λ_i (say λ_1) is close to 1 with others much smaller :

$$\mathbf{v}_{ss} \approx \frac{\mathbf{h}\cdot\mathbf{e}_1}{1-\lambda_1}\mathbf{e}_1$$

Amplification of input projection by a factor of $\frac{1}{1-\lambda_1} = 10$

$\lambda_1 = 0.9$

Example of a Linear Recurrent Network

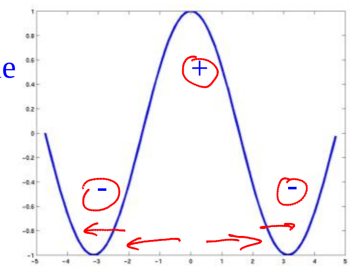


Each output neuron codes for an angle between -180 to +180 degrees

Recurrent connections M = cosine function of relative angle

$$M(\theta, \theta') \propto \cos(\theta - \theta')$$

Excitation nearby,
Inhibition further away



$(\theta - \theta')$

=

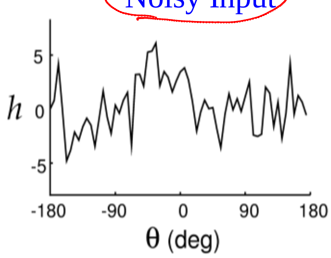
Is M symmetric? $M(\theta, \theta') = M(\theta', \theta)$?

Amplification in the Linear Recurrent Network

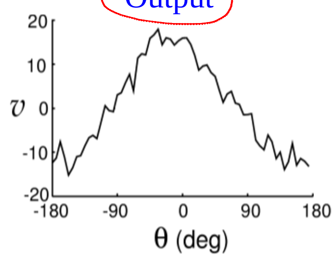
$M(\theta, \theta') \propto \cos(\theta - \theta')$, all eigenvalues = 0 except $\lambda_1 = 0.9$

Amplification $\mathbf{v}_{ss} \approx \dfrac{(\mathbf{e}_1 \cdot \mathbf{h})\mathbf{e}_1}{1 - \lambda_1} = 10 \times (\mathbf{e}_1 \cdot \mathbf{h})\mathbf{e}_1$

Noisy Input



Output



Preferred angle of neuron

(From section 7.4 in Dayan & Abbott textbook)

Memory in Linear Recurrent Networks

$$\tau \frac{d\mathbf{v}}{dt} = -\mathbf{v} + \mathbf{h} + M\mathbf{v} \qquad \mathbf{v}(t) = \sum_{i=1}^{N} c_i(t)\mathbf{e}_i$$

Suppose $\lambda_1 = 1$ and all other $\lambda_i < 1$. Then, $\tau \frac{dc_1}{dt} = \mathbf{h} \cdot \mathbf{e}_1$

If input $\mathbf{h}$ is turned on and then off, can show that even after $\mathbf{h} = 0$:

$$\mathbf{v}(t) = \sum_i c_i(t)\mathbf{e}_i$$

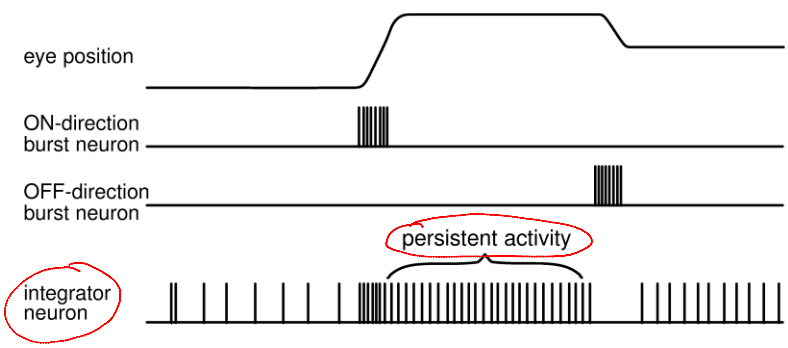
$$\approx c_1\mathbf{e}_1 = \frac{\mathbf{e}_1}{\tau}\int_0^t \mathbf{h}(t') \cdot \mathbf{e}_1 dt'$$

Sustained activity without any input!

Networks keeps a memory of integral of past input

(For full derivation, see "Supplementary Materials" on course website)

The Brain can do Calculus (Part II: Integration)*



Input: Bursts of spikes from brain stem oculomotor neurons

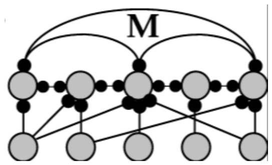
Output: Memory of eye position in medial vestibular nucleus

*For "Part I: Differentiation," see previous lecture

(Image: Dayan & Abbott based on (Seung et al., 2000))

Nonlinear Recurrent Networks

Input vector **h**



Output vector **v**

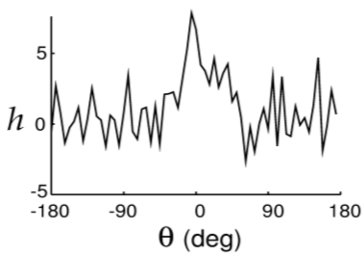
Example: Rectification nonlinearity:
$F(x) = [x]^+ = x$ if $x > 0$ and 0 o.w.

$$\tau \frac{d\mathbf{v}}{dt} = -\mathbf{v} + F(\mathbf{h} + \mathrm{M}\mathbf{v})$$

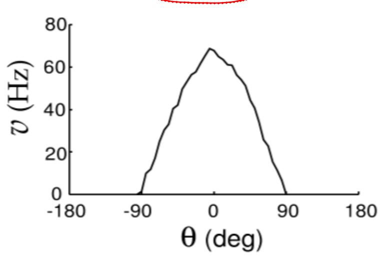
Output Decay Input Recurrent Feedback

Nonlinear Recurrent Network performs **Amplification**

Input



Output



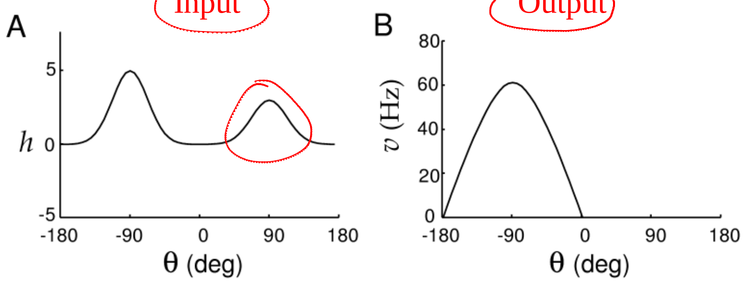
As before, recurrent connections $M(\theta, \theta') \propto \cos(\theta - \theta')$

All eigenvalues = 0 but $\lambda_1 = 1.9$ (yet stable due to rectification)

Image Source: Dayan & Abbott textbook

Same Nonlinear Network performs **Selective “Attention”**

Input Output

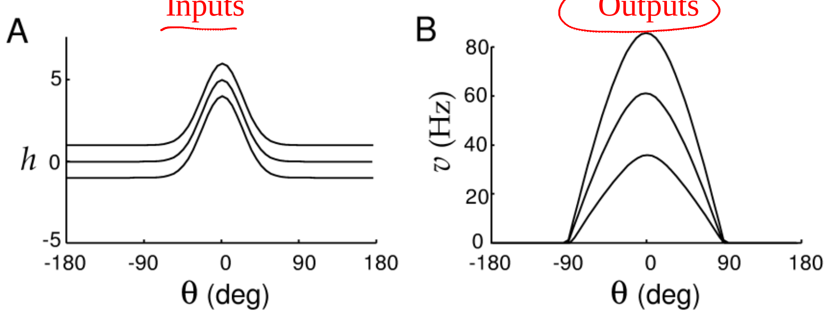


Network performs “Winner-Takes-All” input selection

Image Source: Dayan & Abbott textbook

Gain Modulation in the Nonlinear Network

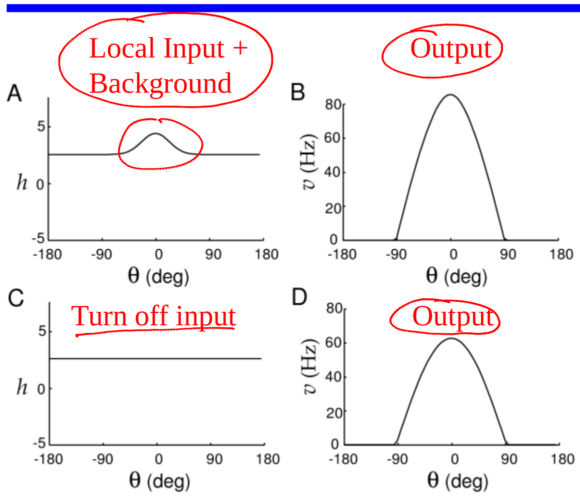
Inputs Outputs



Adding a constant amount to the input h **multiplies** the output

Image Source: Dayan & Abbott textbook

Memory in the Nonlinear Network



Network maintains a *memory of previous activity* when input is turned off.

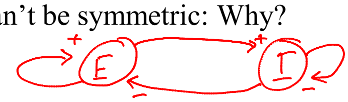
Similar to "short-term memory" or "working memory" in prefrontal cortex

Memory is maintained by recurrent activity

Image Source: Dayan & Abbott textbook

What about Non-Symmetric Recurrent Networks?

- Example: Network of Excitatory (E) and Inhibitory (I) Neurons
  - Connections can't be symmetric: Why?



10 ms

$$\tau_E \frac{dv_E}{dt} = -v_E + [\overset{1.25}{M_{EE}} v_E + \overset{-1}{M_{EI}} v_I - \overset{-10}{\gamma_E}]^+$$

$$\tau_I \frac{dv_I}{dt} = -v_I + [\overset{0}{M_{II}} v_I + \overset{1}{M_{IE}} v_E - \overset{10}{\gamma_I}]^+$$

Parameter we will vary to study the network

How do we analyze the dynamic behavior of such a network?

Linear Stability Analysis

$$\frac{dv_E}{dt} = \frac{-v_E + [M_{EE}v_E + M_{EI}v_I - \gamma_E]^+}{\tau_E}$$

$$\frac{dv_I}{dt} = \frac{-v_I + [M_{II}v_I + M_{IE}v_E - \gamma_I]^+}{\tau_I}$$

Take derivatives of right hand side with respect to both v_E and v_I

Stability Matrix (aka the "Jacobian" Matrix):

$$J = \begin{bmatrix} \frac{(\overset{1.25}{M_{EE}} - 1)}{\tau_E} \; \text{10 ms} & \frac{\overset{-1}{M_{EI}}}{\tau_E} \\ \frac{\overset{1}{M_{IE}}}{\tau_I} & \frac{(\overset{0}{M_{II}} - 1)}{\tau_I} \end{bmatrix}$$

- Eigenvalues of J can have real and imaginary parts
- These eigenvalues determine dynamics of the nonlinear network near a fixed point

(For all the gory details of this stability analysis, see "Supplementary Materials" on course website)

Damped Oscillations in the Network

Choose τ_I = 30 ms (makes real part of eigenvalues negative)

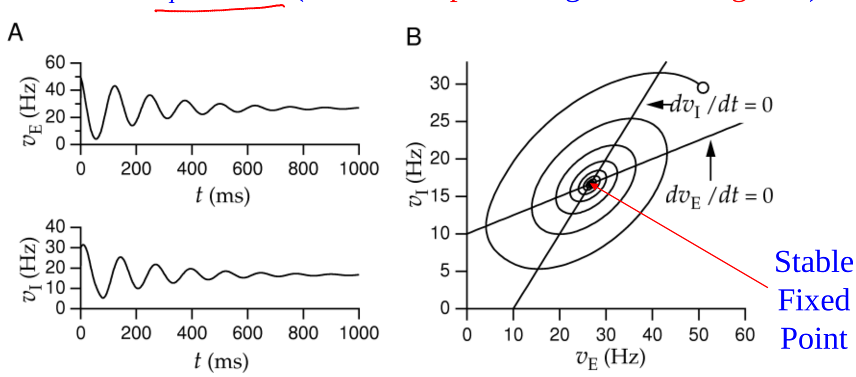


Image Source: Dayan & Abbott textbook

Unstable Behavior and Limit Cycle

Choose τ_I = 50 ms (makes real part of eigenvalues positive)

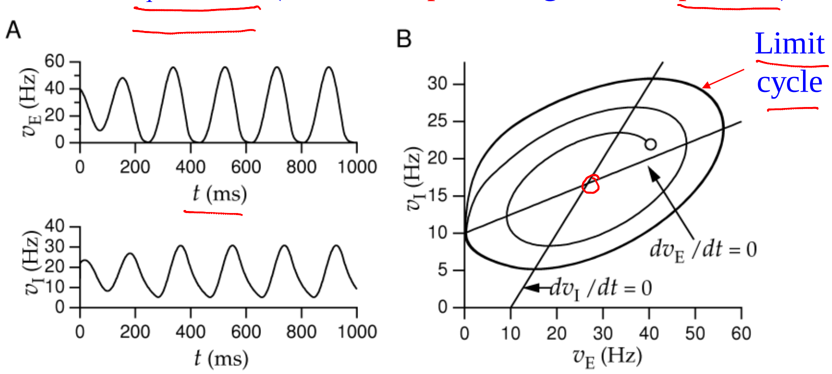


Image Source: Dayan & Abbott textbook

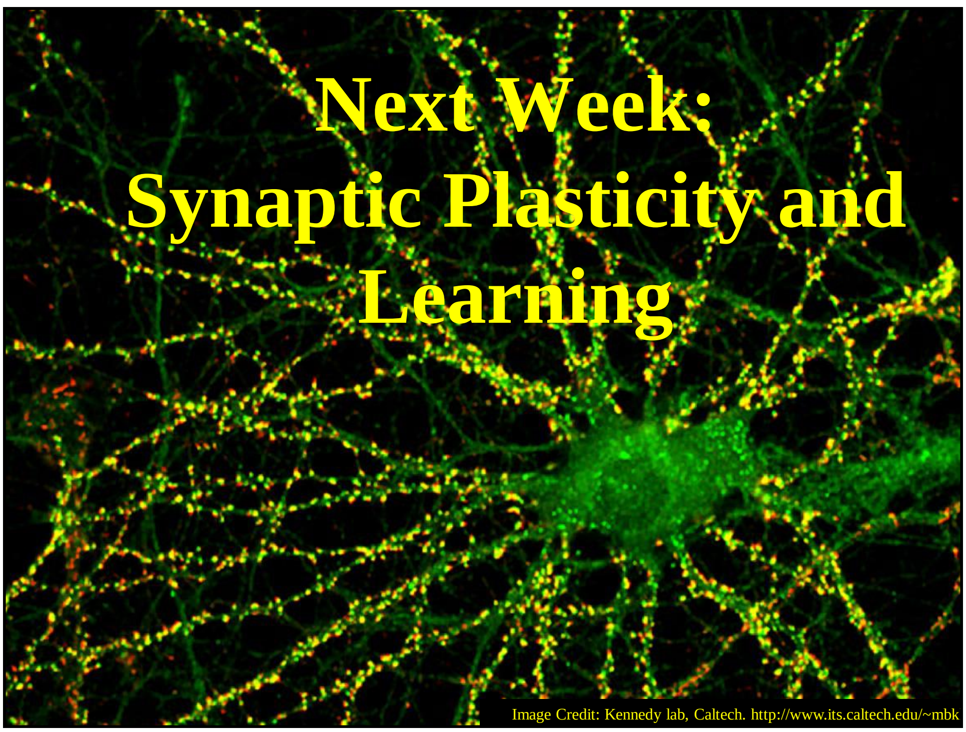


Image Credit: Kennedy lab, Caltech. http://www.its.caltech.edu/~mbk