# 10. Confidence Interval for an Exponential Statistical Model

## Confidence Interval for an Exponential Statistical Model

Consequence of the Delta method

- $\sqrt{n}\left(\hat{\lambda}-\lambda\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,\lambda^2)$.
- Hence, for $\alpha \in (0,1)$ and when $n$ is large enough,

with probability appr .

- Can $\left[\hat{\lambda}-\frac{q_{\alpha/2}\lambda}{\sqrt{n}}, \hat{\lambda}+\frac{q_{\alpha/2}\lambda}{\sqrt{n}}\right]$ be used as an asymptotic

(Caption will be displayed when you start playing the video.)

And now, I want to use this as an asymptotic confidence

interval, lambda hat plus or minus 2 alpha

over 2 lambda divided by root n.

Can I use this?

Is this a valid asymptotic confidence interval

at sympototic level 95%?

Who says yes?

Who says no?

No, because it depends on lambda.

So we have three things at our disposal

at this point to overcome this limitation.

What was the first technique that we had?

The conservative bound, all right?

So we said, oh, p1 minus p is bounded by 1/4.

And therefore, I can just take the most conservative

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## Confidence interval Concept Check

1/1 point (graded)

As in the previous section, let $X_1,\ldots,X_n \stackrel{iid}{\sim} \exp(\lambda)$. Let

$$\widehat{\lambda}_n := \frac{n}{\sum_{i=1}^n X_i}$$

denote an estimator for $\lambda$. We know by now that $\widehat{\lambda}_n$ is a **consistent** and **asymptotically normal** estimator for $\lambda$.

Recall $q_{\alpha/2}$ denote the $1-\alpha/2$ quantile of a standard Gaussian. By the Delta method:

$$\lambda \in \left[\widehat{\lambda}_n - \frac{q_{\alpha/2}\lambda}{\sqrt{n}}, \widehat{\lambda}_n + \frac{q_{\alpha/2}\lambda}{\sqrt{n}}\right] =: \mathcal{I}$$

with probability $1-\alpha$. However, $\mathcal{I}$ is still **not** a confidence interval for $\lambda$.

Why is this the case?

- ○ $\mathcal{I}$ is actually a confidence interval for $1/\lambda$, not $\lambda$.
- ◉ The endpoints of $\mathcal{I}$ depend on the true parameter. ✔

- A confidence interval is supposed to be random, but $\mathcal{I}$, as constructed, is not.

- As written, the left endpoint of $\mathcal{I}$ may be larger than the right endpoint of $\mathcal{I}$, in which case $\mathcal{I}$ would not even be a valid interval.

**Solution:**

The **second choice** is correct. The expression for the left and right endpoint of $\mathcal{I}$ both depend on the true parameter $\lambda$. By definition, a confidence interval must be computed only using the data and other known quantities, but not the true parameter, which is unknown. Therefor $\mathcal{I}$ is not a valid confidence interval.

Now we examine the incorrect choices.

- The first choice '$\mathcal{I}$ is actually a confidence interval for $1/\lambda$, not $\lambda$' is incorrect because, as already discussed, $\mathcal{I}$ cannot be a confidence in the first place because its endpoints depend on the true parameter.

- The third choice 'A confidence interval is supposed to be random, but $\mathcal{I}$, as constructed, is not' is also incorrect. The randomness for $\mathcal{I}$ comes from $\widehat{\lambda}_n$, which is random because it depends on the sample. Recall that $\widehat{\lambda}_n$ is the reciprocal of the sample mean.

- The fourth choice 'As written, the left endpoint of $\mathcal{I}$ may be larger than the right endpoint of $\mathcal{I}$, in which case $\mathcal{I}$ would not be a valid interval' is also incorrect. Since $q_{\alpha/2}$, $\lambda$, and $\sqrt{n}$ are all positive numbers, it follows that

$$\widehat{\lambda}_n - \frac{q_{\alpha/2}\lambda}{\sqrt{n}} < \widehat{\lambda}_n + \frac{q_{\alpha/2}\lambda}{\sqrt{n}}.$$

  Hence, $\mathcal{I}$ is always a valid interval, just not a valid **confidence** interval.

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Answers are displayed within the problem

## Conservative confidence interval for a exponential model

1/1 point (graded)

This problem illustrates the failure of the 'conservative method' for constructing confidence intervals for an exponential statistical model.

As above, let $X_1, \ldots, X_n \overset{iid}{\sim} \exp(\lambda)$. Let

$$\widehat{\lambda}_n := \frac{n}{\sum_{i=1}^n X_i}$$

denote an estimator for $\lambda$.

Previously, we used the Delta method to show that for $n$ sufficiently large

$$\lambda \in \left[\widehat{\lambda}_n - \frac{q_{\alpha/2}\lambda}{\sqrt{n}}, \widehat{\lambda}_n + \frac{q_{\alpha/2}\lambda}{\sqrt{n}}\right] =: \mathcal{I}$$

where $q_{\alpha/2}$ is the $1-\alpha/2$ quantile of a standard Gaussian.

Given an interval of the form $\mathcal{I}$, we can use the "conservative method" to find a confidence interval $\mathcal{I}_{cons}$ for $\lambda$ defined by

$$\mathcal{I}_{cons} := \left[\widehat{\lambda}_n - \max_{\lambda\in(0,\infty)} \frac{q_{\alpha/2}\lambda}{\sqrt{n}}, \widehat{\lambda}_n + \max_{\lambda\in(0,\infty)} \frac{q_{\alpha/2}\lambda}{\sqrt{n}}\right].$$

Which of the following is $\mathcal{I}_{cons}$?

- (•) $(-\infty, \infty)$ ✔
- ( ) the empty interval
- ( ) the point $\widehat{\lambda}_n$
- ( ) Cannot be determined, since the exact form of $\mathcal{I}_{cons}$ will depend on the particular sample.

**Solution:**

Observe that

$$\max_{\lambda \in (0,\infty)} \frac{q_{\alpha/2}\lambda}{\sqrt{n}} = \infty$$

. In this case, we take the max over the interval $(0, \infty)$, because a priori, $\lambda$ can be any number in this interval. Therefore,

$$\mathcal{I}_{cons} = (-\infty, \infty)\,.$$

**Remark**: Although $(-\infty, \infty)$ is technically still a confidence interval (it even has level $100\%$!), it is not useful for statistical purposes because such a confidence interval gives no information about the location of the true parameter.

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Answers are displayed within the problem

## Explicit Confidence Intervals for an Exponential Statistical Model

6/6 points (graded)

Suppose you observe a sample data set consisting of $n = 64$ inter-arrival times $X_1, \ldots, X_{64}$ for the subway, measured in minutes. As before, we assume the statistical model that $X_1, \ldots, X_{64} \stackrel{iid}{\sim} \exp(\lambda)$ for some unknown parameter $\lambda > 0$. In this data set, you observe that the sample mean is $\frac{1}{64}\sum_{i=1}^{64} X_i = 7.8$.

*Additional Instructions: For best results, please adhere to the following guidelines and reminders:*

1. *For the upcoming calculations, please truncate $q_{\alpha/2}$ at 2 decimal places, instead of a more exact value. For example, if $q_{\alpha/2} = 3.84941$, use $3.84$ instead of $3.85$ or $3.849$.*
2. *Input answers truncated at 4 decimal places. For example, if your calculations yield $11.327458$, use $11.3274$ instead of $11.3275$ or $11.32745$.*
3. *You will be computing CIs at **asymptotic level** 90%.*

Using the 'solve method' (**refer to the slide 'Three solutions'**), construct a confidence interval $\mathcal{I}_{solve}$ with asymptotic level $90\%$ for the unknown parameter $\lambda$.

[ 0.1063 ✔ **Answer:** 0.1063408 , 0.1615 ✔ **Answer:** 0.1613875 ]

Using the 'plug-in method' $\mathcal{I}_{plug-in}$ (**refer to the slide 'Three solutions'**), construct a confidence interval with asymptotic level $90\%$ for the unknown parameter $\lambda$.

[ 0.1018 ✔ **Answer:** 0.1018453 , 0.1546 ✔ **Answer:** 0.154565 ]

Which interval is narrower?

- ○ $\mathcal{I}_{solve}$
- ◉ $\mathcal{I}_{plug-in}$ ✔

Which of these confidence intervals is centered about the sample estimate, $\hat{\lambda}_n$?

- ○ $\mathcal{I}_{solve}$
- ◉ $\mathcal{I}_{plug-in}$ ✔
- ○ Both
- ○ Neither

**Solution:**

The formula for $\mathcal{I}_{solve}$ at asymptotic level $1-\alpha$ is given by

$$\mathcal{I}_{solve} = \left[\widehat{\lambda_n}\left(1+\frac{q_{\alpha/2}}{\sqrt{n}}\right)^{-1}, \widehat{\lambda_n}\left(1-\frac{q_{\alpha/2}}{\sqrt{n}}\right)^{-1}\right].$$

We need to construct a confidence interval of (asymptotic) level $90\%$, so this implies that $\alpha = 0.1$ and thus $q_{\alpha/2} = q_{0.05} \approx 1.64$ (consulting a table for the standard Gaussian). Hence, for this data set,

$$\begin{aligned}\mathcal{I}_{solve} &= \left[\frac{1}{7.8}\left(1+\frac{1.64}{\sqrt{64}}\right)^{-1}, \frac{1}{7.8}\left(1-\frac{1.64}{\sqrt{64}}\right)^{-1}\right] \\ &\approx [0.1064, 0.1613]\,.\end{aligned}$$

Next we compute $\mathcal{I}_{plug-in}$. The formula is given by

$$\mathcal{I}_{plug-in} = \left[\widehat{\lambda_n}\left(1-\frac{q_{\alpha/2}}{\sqrt{n}}\right), \widehat{\lambda_n}\left(1+\frac{q_{\alpha/2}}{\sqrt{n}}\right)\right].$$

Thus for this data set,

$$\begin{aligned}\mathcal{I}_{plug-in} &= \left[\frac{1}{7.8}\left(1-\frac{q_{0.05}}{\sqrt{64}}\right), \frac{1}{7.8}\left(1+\frac{q_{0.05}}{\sqrt{64}}\right)\right] \\ &\approx [0.1019, 0.1545]\,.\end{aligned}$$

Since

$$\begin{aligned}|0.1064 - 0.1613| &= 0.0549 \\ |0.1019 - 0.1545| &= 0.0526,\end{aligned}$$

this implies that $\mathcal{I}_{plug-in}$ is the **narrower** confidence interval.

Finally,

$$\begin{aligned}((0.1064 + 0.1613)\,/2) * 7.8 &= 1.04403 \\ (0.1019 + 0.1545)\,/2) * 7.8 &= 0.99996,\end{aligned}$$

so we see that $\mathcal{I}_{plug-in}$ is **centered** about $\widehat{\lambda}_n$, while $\mathcal{I}_{solve}$ is not. Alternatively, one can see directly from the formulas that $\mathcal{I}_{plug-in}$ is always centered about $\widehat{\lambda}_n$ whereas $\mathcal{I}_{solve}$ is **not** in general.