Inner product

Consider a vector space V. A positive definite, symmetric bilinear mapping $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ is called an **inner product** on V.

- **symmetric**: For all $\boldsymbol{x}, \boldsymbol{y} \in V$ it holds that $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{y}, \boldsymbol{x} \rangle$
- **positive definite**: For all $\boldsymbol{x} \in V \backslash \{\boldsymbol{0}\}$ it holds that
  $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, \quad \langle \boldsymbol{0}, \boldsymbol{0} \rangle = 0$
- **bilinear**: For all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in V, \lambda \in \mathbb{R}$

$$\langle \lambda \boldsymbol{x} + \boldsymbol{y}, \boldsymbol{z} \rangle = \lambda \langle \boldsymbol{x}, \boldsymbol{z} \rangle + \langle \boldsymbol{y}, \boldsymbol{z} \rangle$$
$$\langle \boldsymbol{x}, \lambda \boldsymbol{y} + \boldsymbol{z} \rangle = \lambda \langle \boldsymbol{x}, \boldsymbol{y} \rangle + \langle \boldsymbol{x}, \boldsymbol{z} \rangle$$