

2. Independent uniform random variables

Problem 1. Independent uniform random variables

5.0/5.0 points (graded)

Let X, Y, Z be three independent (i.e. mutually independent) random variables, each uniformly distributed on the interval $[0, 1]$.

1. Find the mean and variance of $1/(Z + 1)$.

$$\mathbf{E}[1/(Z + 1)] = \ln(2) \quad \checkmark \text{ Answer: } 0.693$$

$$\text{var}(1/(Z + 1)) = 1/2 - (\ln(2))^2 \quad \checkmark \text{ Answer: } 0.019$$

2. Find the mean of $XY/(Z + 1)$.

Hint: Use your answer to the previous part, together with the independence assumption.

$$\mathbf{E}[XY/(Z + 1)] = 1/4 * \ln(2) \quad \checkmark \text{ Answer: } 0.173$$

3. Find the probability that $XY/Z \leq 1$. Enter a numerical answer.

$$\mathbf{P}(XY/Z \leq 1) = 3/4 \quad \checkmark \text{ Answer: } 0.75$$

Solution:

1. Since Z is uniform on $[0, 1]$, we can compute the expected value of $1/(Z + 1)$ as follows:

$$\begin{aligned} \mathbf{E}\left[\frac{1}{Z + 1}\right] &= \int_0^1 \frac{1}{z + 1} f_Z(z) dz \\ &= \int_0^1 \frac{1}{z + 1} dz \\ &= \ln(z + 1) \Big|_0^1 = \ln(2) - \ln(1) \\ &= \ln(2) \\ &\approx 0.693. \end{aligned}$$

For the variance, we start by computing $\mathbf{E}[1/(Z + 1)^2]$.

$$\begin{aligned} \mathbf{E}\left[\frac{1}{(Z + 1)^2}\right] &= \int_0^1 \frac{1}{(z + 1)^2} f_Z(z) dz \\ &= \int_0^1 \frac{1}{(z + 1)^2} dz \\ &= -\frac{1}{z + 1} \Big|_0^1 = 1 - \frac{1}{2} \\ &= \frac{1}{2}. \end{aligned}$$

Hence,

$$\text{var}(1/(Z + 1)) = \mathbf{E}\left[\left(\frac{1}{(Z + 1)^2}\right)\right] - \left(\mathbf{E}\left[\frac{1}{Z + 1}\right]\right)^2 = \frac{1}{2} - (\ln(2))^2 \approx 0.019.$$

2. Using independence,

$$\mathbf{E}\left[\frac{XY}{Z+1}\right] = \mathbf{E}[X]\mathbf{E}[Y]\mathbf{E}\left[\frac{1}{Z+1}\right] = \frac{\ln(2)}{4} \approx 0.173.$$

3. $\mathbf{P}\left(\frac{XY}{Z} \leq 1\right)$ is the same as $\mathbf{P}(XY \leq Z) = 1 - \mathbf{P}(XY \geq Z)$. But since Z is a uniform r.v. in $[0, 1]$, $\mathbf{P}(XY \geq Z)$ is simply the expected value of XY , which is **0.25**, so the answer is **0.75**.

Alternate longer solution: We first write down the joint density of X, Y , and Z :

$$f_{X,Y,Z}(x,y,z) = \begin{cases} 1, & \text{if } 0 \leq x,y,z \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

To calculate the probability of interest, we find the region over which we should integrate this joint PDF. The region we are considering consists of the points (x,y,z) such that $0 \leq x,y \leq 1$, and $xy \leq z \leq 1$ (also notice that $xy \leq 1$ always holds).

Using this,

$$\begin{aligned} \mathbf{P}\left(\frac{XY}{Z} \leq 1\right) &= \mathbf{P}(XY \leq Z) \\ &= \int_0^1 \int_0^1 \int_{xy}^1 dz \, dy \, dx \\ &= \int_0^1 \int_0^1 (1 - xy) \, dy \, dx \\ &= \int_0^1 \left(y - \frac{xy^2}{2}\right) \Big|_{y=0}^1 \, dx \\ &= \int_0^1 \left(1 - \frac{x}{2}\right) \, dx \\ &= \left(x - \frac{x^2}{4}\right) \Big|_0^1 \\ &= \frac{3}{4}. \end{aligned}$$

提交

你已经尝试了1次（总共可以尝试2次）

i Answers are displayed within the problem

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显示讨论

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