11. Kolmogorov-Lilliefors Test I

Motivation: Goodness of Fit Testing for a Gaussian Distribution

0/1 point (graded)

Let $X_1, \ldots, X_n$ be iid random variables with continuous cdf F. Let $\{\mathcal{N}(\mu, \sigma^2)\}_{\mu \in \mathbb{R}, \sigma^2 > 0}$ denote the family of all **Gaussian** distributions. We want to test whether or not $F \in \{\mathcal{N}(\mu, \sigma^2)\}_{\mu \in \mathbb{R}, \sigma^2 > 0}$.

Let Φ_{μ,σ^2} denote the cdf of $\mathcal{N}(\mu, \sigma^2)$. We formulate the null and alternative hypotheses

$$\begin{aligned} H_0 &: F = \Phi_{\mu,\sigma^2} \text{ for some } \mu \in \mathbb{R}, \sigma^2 > 0 \\ H_1 &: F \neq \Phi_{\mu,\sigma^2} \text{ for any } \mu \in \mathbb{R}, \sigma^2 > 0. \end{aligned}$$

Motivated by the Kolmogorov-Smirnov test, you define a test-statistic using the sample mean $\hat{\mu}$ and sample variance $\hat{\sigma^2}$:

$$\widetilde{T}_n = \sup_{t \in \mathbb{R}} \sqrt{n} |F_n(t) - \Phi_{\hat{\mu},\hat{\sigma^2}}|.$$

Assume that the **null hypothesis** is **true**. Is it true that

$$\widetilde{T}_n \xrightarrow[n\to\infty]{(d)} \sup_{x \in [0,1]} |\mathbb{B}(x)|$$

where $\mathbb{B}(x)$ is a **Brownian bridge**? (Refer to the slides.)

- (•) True ✖
- ( ) False ✔

Solution:

This claim is **false**. It is true that for any fixed μ, σ^2 that

$$T_n = \sup_{t \in \mathbb{R}} \sqrt{n} |F_n(t) - \Phi_{\mu,\sigma^2}| \xrightarrow[n\to\infty]{(d)} \sup_{x \in [0,1]} |\mathbb{B}(x)|.$$

This result follows by **Donsker's theorem** as the Gaussian cdf is continuous over the real line.

But if we plug in **estimators** for μ and σ^2 (and not their true values), then this convergence result no longer holds.

Remark: However, it is true that under the null hypothesis

$$H_0 : F = \Phi_{\mu,\sigma^2} \text{ for some } \mu \in \mathbb{R}, \sigma^2 > 0$$

that the statistic

$$\widetilde{T}_n = \sup_{t \in \mathbb{R}} \sqrt{n} |F_n(t) - \Phi_{\hat{\mu}, \hat{\sigma^2}}|$$

is **pivotal**. Moreover, the statistic $\widetilde{T}_n$ converges in distribution as $n \to \infty$. The quantiles of $\widetilde{T}_n$ can be found in tables, and the test based on $\widetilde{T}_n$ is known as the **Kolmogorov-Lilliefors test** , which we discuss next.

Submit You have used 1 of 1 attempt

Answers are displayed within the problem

Kolmogorov-Lilliefors Test

4:55 / 4:55 1.0x

Kolmogorov-Lilliefors

says no, you're not Gaussian, then the Kolmogorov-Smirnov

test would say no, but it could be that one says yes

and the other one says no.

And that's when you're in danger.

OK, so you have some critical values for different sample

sizes and this is what they look like.

This is actually from Lillifors' original paper.

End of transcript. Skip to the start.

Video
Download video file

Transcripts
Download SubRip (.srt) file
Download Text (.txt) file

xuetangX.com 学堂在线

Video Note : In the above video, at the very end it is stated that *"So it will always be the case that if Kolmogorov Lilliefors says no, you're not Gaussian, then the Kolmogorov-Smirnov test would say no"*. The correct assertion is "**So it will always be the case that if Kolmogorov-Smirnov says no, you're not Gaussian, then the Kolmogorov-Lilliefors test would say no**". Put another way, the K-L test is more likely to reject than the K-S test.

Concept Check: Kolmogorov-Smirnov vs. Kolmogorov-Lilliefors Test

1/1 point (graded)

Let $X_1, \ldots, X_n \stackrel{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$, let

$$F_n(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(X_i \le t)$$

denote their empirical distribution, and let Φ_{μ,σ^2} denote the cdf of the distribution $\mathcal{N}(\mu, \sigma^2)$.

Recall that in the Kolmogorov-Smirnov test, we considered the test statistic

$$T_n = \sqrt{n} \sup_{t \in \mathbb{R}} |F_n(t) - \Phi_{\mu,\sigma^2}|$$

In the Kolmogorov-Lilliefors test, we consider the test statistic

$$\widetilde{T}_n = \sqrt{n} \sup_{t \in \mathbb{R}} |F_n(t) - \Phi_{\widehat{\mu},\widehat{\sigma}^2}|$$

It is true or false that T_n and $\widetilde{T}_n$ have the same distribution for for all $n \in \mathbb{N}$? (Refer to the slides.)

- ( ) True
- (•) False ✔

Solution:

In T_n, we plug in the actual values of parameters μ and σ^2. However, in $\widetilde{T}_n$, we plug in estimators $\widehat{\mu}$ and $\widehat{\sigma}^2$ for the true parameters. Therefore, T_n and $\widetilde{T}_n$ will have different distributions for all n.

Remark: Since the Kolmogorov-Lilliefors test uses a test statistic with a different distribution, when performing this goodness of fit test, we will have to consult a different table than what we used for the Kolmogorov-Smirnov test. We will compare the quantiles of the Kolmogorov-Smirnov test statistic and those of the Kolmogorov-Lilliefors test statistic in a problem below.

Submit You have used 1 of 1 attempt

Answers are displayed within the problem

Testing the Mean for a Sample with Unknown Distribution

2/2 points (graded)

Suppose that you observe a sample $X_1, \ldots, X_n \stackrel{iid}{\sim} \mathbf{P}$ for some distribution $\mathbf{P}$ with continuous cdf. Your goal is to decide between the null and alternative hypotheses

$$\begin{aligned} H_0 &: \mu = 0 \\ H_1 &: \mu \neq 0. \end{aligned}$$

Looking at a histogram, you suspect that $X_1, \ldots, X_n$ have a Gaussian distribution. You would like to first test this suspicion. Formally, you would like to decide between the following null and alternative hypotheses:

$$\begin{aligned} H_0' &: P \in \{\mathcal{N}(\mu, \sigma^2)\}_{\mu \in \mathbb{R}, \sigma^2 > 0} \\ H_1' &: P \notin \{\mathcal{N}(\mu, \sigma^2)\}_{\mu \in \mathbb{R}, \sigma^2 > 0}. \end{aligned}$$

Which of the following tests should you use to decide between H_0' and H_1'?

- ( ) Student's T test
- ( ) Kolmogorov-Smirnov test
- (•) Kolmogorov-Lilliefors test ✔

Suppose that the test you used in the previous part for H_0' and H_1' **fails to reject**.

If this test did not make an error, which of the following could you use to decide between the original hypotheses H_0 and H_1?

- [x] Student's T test ✔
- [ ] Kolmogorov-Smirnov test
- [ ] Kolmogorov-Lilliefors test

Solution:

We examine the choices for the first question in order.

- We have seen that the Kolmogorov-Lilliefors test can be used to decide between H_0' and H_1', so this is the correct choice.
- The second choice is incorrect. The Kolmogorov-Smirnov test was designed to test if the data has a specific distribution; it is not useful for deciding whether or not the true distribution $\mathbf{P}$ lies in a given **family of distributions**.
- The third choice is incorrect. The student's T test is only valid if we know that the data has a Gaussian distribution. Since we want to test whether or not the data has a Gaussian distribution, it would not make sense to apply the T test in this scenario.

Now consider the second question. If the Kolmogorov-Lilliefors test for H_0' and H_1' failed to reject and did not make an error, then this implies that our data likely to be Gaussian. Therefore, the student's T test can be applied to test if the true mean of our data is $\mu = 0$.

Remark: In practice, many of the methods for statistical inference, such as the student's T test, rely on the assumption the data is Gaussian. Hence, before performing such a test, we need to evaluate whether or not the data is Gaussian. This problem gives an example of such a procedure. First we tested for the Gaussianity of our data, and since the Kolmogorov-Lilliefors test failed to reject, assuming that there was no error, we could apply the student's T test to answer our original hypothesis testing question. Of course, in reality, there may be errors, which can be quantified using the tools from this Chapter as you have seen already.

Submit You have used 2 of 2 attempts

Answers are displayed within the problem

Concept Check: Kolmogorov-Lilliefors Test

1/1 point (graded)

Let $\{\mathcal{N}(\mu, \sigma^2)\}_{\mu\in\mathbb{R}, \sigma^2>0}$ denote the family of all Gaussian distributions. Let $X_1, \ldots, X_n \stackrel{iid}{\sim} \mathbf{P}$ for some distribution $\mathbf{P}$ with a continuous cdf. The goal of the Kolmogorov-Lilliefors test is to determine if $X_1, \ldots, X_n$ have a Gaussian distribution. Formally, we set the null and alternative hypotheses to be, respectively,

$$H_0 : P \in \{\mathcal{N}(\mu, \sigma^2)\}_{\mu,\sigma^2}$$
$$H_1 : P \notin \{\mathcal{N}(\mu, \sigma^2)\}_{\mu,\sigma^2}.$$

The Kolmogorov-Lilliefors test statistic is

$$T_n = \sqrt{n} \sup_{t\in\mathbb{R}} |F_n(t) - \Phi_{\widehat{\mu},\widehat{\sigma}^2}|$$

and the test is

$$\psi_n = \mathbf{1}(T_n > q_\eta)$$

where q_η denotes the $1-\eta$ quantile of the distribution T_n.

Which of the following are true about the Kolmogorov-Lilliefors test? (Refer to the slides. Choose all that apply.)

- [x] The test-statistic T_n is pivotal: if the null hypothesis holds and $P = \mathcal{N}(\mu, \sigma^2)$ for some μ, σ^2, then the distribution of T_n does not depend on the specific values of μ and σ^2. ✔

- [x] The quantiles of T_n can be found using a table or computational software. ✔

- [x] If the Kolmogorov-Lilliefors test fails to reject (and no error was made), then it is valid to perform further hypothesis tests that only work for Gaussian random variables (for example, the student's T test). ✔

✔

Solution:

We examine the choices in order.

- The first choice is correct. As stated in the slides, under the null hypothesis the test statistic T_n has a distribution that does **not** depend on the true (unknown) parameters.
- The second choice is also correct. Since the distribution T_n is pivotal, one can compute its quantiles using a table or computational software.
- The third choice is also correct. The logic behind this procedure was discussed in the previous problem.

Submit You have used 1 of 2 attempts

Answers are displayed within the problem

Comparing Quantiles for the Kolmogorov-Smirnov and Kolmogorov-Lilliefors Test Statistics

1/1 point (graded)

Let $X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$, let

$$F_n(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(X_i \leq t)$$

denote their empirical distribution, and let Φ_{μ,σ^2} denote the cdf of the distribution $\mathcal{N}(\mu, \sigma^2)$.

Recall that in the Kolmogorov-Smirnov test, we considered the test statistic

$$T_n = \sqrt{n} \sup_{t \in \mathbb{R}} |F_n(t) - \Phi_{\mu,\sigma^2}|$$

In the Kolmogorov-Lilliefors test, we consider the test statistic

$$\widetilde{T}_n = \sqrt{n} \sup_{t \in \mathbb{R}} |F_n(t) - \Phi_{\widehat{\mu},\widehat{\sigma}^2}|$$

Let q_η denote the $1-\eta$ quantile of T_n (*i.e.*, $P(T_n \geq q_\eta) = \eta$) and let q'_η denote the $1-\eta$ quantile of $\widetilde{T}_n$ (*i.e.*, $P(\widetilde{T}_n \geq q'_\eta) = \eta$) .

Which of the following do you expect to be true?
Hint: Start by arguing informally which of Φ_{μ,σ^2} and $\Phi_{\widehat{\mu},\widehat{\sigma}^2}$ better approximates F_n.

- (•) $q_\eta > q'_\eta$ ✔
- ( ) $q_\eta < q'_\eta$
- ( ) $q_\eta = q'_\eta$
- ( ) Whether $q_\eta > q'_\eta$ or $q_\eta < q'_\eta$ depends on n.

Solution:

Here is an informal argument to show that $q_\eta > q_{\eta'}$. Consider the two test statistics

$$\begin{aligned} T_n &= \sup_{t\in\mathbb{R}} \left|F_n(t) - \Phi_{\mu,\sigma^2}\right| \\ \widetilde{T}_n &= \sup_{t\in\mathbb{R}} \left|F_n(t) - \Phi_{\widehat{\mu},\widehat{\sigma}^2}\right| \end{aligned}$$

For $\widetilde{T}_n$, note that the mean and variance of the empirical distribution are $\widehat{\mu}$ and $\widehat{\sigma}^2$, respectively. Hence, it is natural to expect $\mathcal{N}(\widehat{\mu},\widehat{\sigma}^2)$ to be a good approximation to $F_n(t)$, at least amongst all Gaussian distributions.

For T_n, the mean and variance of the empirical distribution do **not** match the mean and variance of $\mathcal{N}(\mu,\sigma^2)$. This again lends reason to believe that the cdf $\Phi_{\widehat{\mu},\widehat{\sigma}^2}$ will approximate the empirical distribution $F_n(t)$ better than Φ_{μ,σ^2}.

Therefore, on average, we expect $\widetilde{T}_n = \sup_{t\in\mathbb{R}}\left|F_n(t) - \Phi_{\widehat{\mu},\widehat{\sigma}^2}\right|$ to be smaller than $T_n = \sup_{t\in\mathbb{R}}\left|F_n(t) - \Phi_{\mu,\sigma^2}\right|$. Hence, the cdf of $\widetilde{T}_n$ will be more shifted to the left, and the cdf of T_n will be more shifted to the right. This means that the quantiles of T_n will be **larger** than the quantiles of $\widetilde{T}_n$. It is in fact true that $q_\eta > q'_\eta$ for all η.

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

Comparing the Kolmogorov-Lilliefors and Kolmogorov-Smirnov Tests

0/1 point (graded)

Let $X_1,\ldots,X_n \stackrel{iid}{\sim} \mathbf{P}$ with continuous cdf F. Consider the following two hypothesis tests.

Hypothesis Test 1:(Kolmogorov-Smirnov) For the Kolmogorov-Smirnov test, our goal is to decide between a null and alternative hypothesis of the form

$$\begin{aligned} H_0: F &= \Phi_{0,1} \\ H_1: F &\neq \Phi_{0,1}. \end{aligned}$$

The Kolmogorov-Smirnov uses the test statistic

$$T_n = \sqrt{n}\sup_{t\in\mathbb{R}}\left|F_n(t) - \Phi_{0,1}\right|$$

and the test

$$\psi_n = \mathbf{1}(T_n > q_\eta)$$

where q_η denotes the $1-\eta$ quantile of T_n. You choose η such that $q_\eta = 0.5$.

Hypothesis Test 2:(Kolmogorov-Lilliefors) For the Kolmogorov-Lilliefors test, our goal is to decide between a null and alternative hypothesis of the form

$$\begin{aligned} \tilde{H}_0: \mathbf{P} &\in \{\mathcal{N}(\mu,\sigma^2)\}_{\mu\in\mathbb{R},\sigma^2>0} \\ \tilde{H}_1: \mathbf{P} &\notin \{\mathcal{N}(\mu,\sigma^2)\}_{\mu\in\mathbb{R},\sigma^2>0}. \end{aligned}$$

The Kolmogorov-Lilliefors test uses the test statistic

$$\widetilde{T}_n = \sqrt{n}\sup_{t\in\mathbb{R}}\left|F_n(t) - \Phi_{\widehat{\mu},\widehat{\sigma}^2}\right|$$

and the test

$$\tilde{\psi}_n = \mathbf{1}(\widetilde{T}_n > q'_\nu)$$

where q'_ν denotes the $1-\nu$ quantile of $\widetilde{T}_n$. You choose ν such that $q'_\nu = 0.5$.

Assume that the null hypotheses H_0 and $\tilde{H}_0$ hold for both hypothesis tests above.

Which test has a greater probability of rejecting the null hypothesis? 犯一类错误的概率

- Kolmogorov-Smirnov test ✔
- Kolmogorov-Lilliefors test

Solution:

The Kolmogorov-Smirnov test has a greater probability of rejection under its null hypothesis H_0. This is because, under both H_0 and $\tilde{H}_0$, the quantiles of the test statistic T_n are larger than the quantiles of the test statistic $\widetilde{T}_n$ (see e.g. the previous problem). Therefore, under both H_0 and $\tilde{H}_0$,

$$\boxed{P(\widetilde{T}_n > 0.5) \leq P(T_n > 0.5)}.$$

Submit You have used 1 of 1 attempt

Answers are displayed within the problem

Discussion

Topic: Unit 4 Hypothesis testing:Lecture 16: Goodness of Fit Tests Continued: Kolmogorov-Smirnov test, Kolmogorov-Lilliefors test, Quantile-Quantile Plots / 11. Kolmogorov-Lilliefors Test I

Show Discussion