4. Joint PMF

Problem 4. Joint PMF

5/5 points (graded)

The joint PMF, $p_{X,Y}(x,y)$, of the random variables X and Y is given by the following table:

$y=1$	$4c$	0	$2c$	$8c$
$y=0$	$3c$	$2c$	0	$2c$
$y=-1$	$2c$	0	c	$4c$
	$x=-2$	$x=-1$	$x=0$	$x=1$

1. Find the value of the constant c.

   $c=$ 1/28 ✔ **Answer:** 0.03571

2. Find $p_X(1)$.

   $p_X(1)=$ 14/28 ✔ **Answer:** 0.5

3. Consider the random variable $Z=X^2Y^3$. Find $\mathbf{E}[Z\mid Y=-1]$.

   $\mathbf{E}[Z\mid Y=-1]=$ -12/7 ✔ **Answer:** -1.71429

4. Conditioned on the event that $Y\neq 0$, are X and Y independent?

   Yes ▼ ✔ **Answer:** Yes

5. Find the conditional variance of Y given that $X=0$.

   Var$(Y\mid X=0)=$ 8/9 ✔ **Answer:** 0.88889

Solution:

1. We find c by using the fact that the probability of the entire sample space must equal 1.

$$
\begin{aligned}
1 &= \sum_{x=-2}^{1} \sum_{y=-1}^{1} p_{X,Y}(x,y) \\
&= 2c + 3c + 4c + 2c + c + 2c + 4c + 2c + 8c \\
&= 28c.
\end{aligned}
$$

Therefore, $c = \frac{1}{28}$.

2. $$p_X(1) = \sum_{y=-1}^{1} p_{X,Y}(1,y) = 4c + 2c + 8c = 14c = \frac{1}{2}.$$

3. $$
\begin{aligned}
\mathbf{E}[Z \mid Y = -1] &= \mathbf{E}[X^2 Y^3 \mid Y = -1] \\
&= \mathbf{E}[X^2 (-1)^3 \mid Y = -1] \\
&= -\mathbf{E}[X^2 \mid Y = -1]
\end{aligned}
$$

In order to calculate this conditional expectation, we need the conditional PMF of X given $Y = -1$:

$$
p_{X|Y}(x \mid -1) = \frac{p_{X,Y}(x,-1)}{p_Y(-1)} = \begin{cases} \frac{2c}{7c} = \frac{2}{7}, & \text{if } x = -2, \\ \frac{c}{7c} = \frac{1}{7}, & \text{if } x = 0, \\ \frac{4c}{7c} = \frac{4}{7}, & \text{if } x = 1, \\ 0, & \text{otherwise.} \end{cases}
$$

Therefore,

$$
\begin{aligned}
\mathbf{E}[Z \mid Y = -1] &= -\sum_{x=-2}^{1} x^2 p_{X|Y}(x \mid -1) \\
&= -\left((-2)^2 \cdot \frac{2}{7} + 1^2 \cdot \frac{4}{7} \right) \\
&= -\frac{12}{7}.
\end{aligned}
$$

4. Yes. Given $Y \neq 0$, the conditional distribution of Y given $X = x$ is the same for all $x \in \{-2, -1, 0, 1\}$:
$\mathbf{P}(Y = y \mid X = x, Y \neq 0) = \mathbf{P}(Y = y \mid Y \neq 0)$, for all $x \in \{-2, -1, 0, 1\}$.

For example,

$$
\begin{aligned}
\mathbf{P}(Y = 1 \mid X = -2, Y \neq 0) &= \mathbf{P}(Y = 1 \mid X = 0, Y \neq 0) \\
&= \mathbf{P}(Y = 1 \mid X = 1, Y \neq 0) \\
&= \mathbf{P}(Y = 1 \mid Y \neq 0) = \frac{2}{3}.
\end{aligned}
$$

5. We first find the conditional PMF of Y given $X=0$:

$$p_{Y|X}(y \mid 0) = \frac{p_{X,Y}(0,y)}{p_X(0)} = \begin{cases} \frac{c}{c+2c} = \frac{1}{3}, & \text{if } y=-1, \\ \frac{2c}{c+2c} = \frac{2}{3}, & \text{if } y=1, \\ 0, & \text{otherwise.} \end{cases}$$

We can then calculate the conditional expectation:

$$\mathbf{E}[Y \mid X=0] = \sum_{y=-1}^{1} y p_{Y|X}(y \mid 0) = (-1) \cdot \frac{1}{3} + (1) \cdot \frac{2}{3} = \frac{1}{3}.$$

Finally, the conditional variance can be calculated as

$$\begin{aligned} \text{Var}(Y \mid X=0) &= \mathbf{E}[(Y - \mathbf{E}[Y \mid X=0])^2 \mid X=0] \\ &= \mathbf{E}\left[\left(Y - \frac{1}{3}\right)^2 \mid X=0\right] \\ &= \sum_{y=-1}^{1} \left(y - \frac{1}{3}\right)^2 p_{Y|X}(y \mid 0) \\ &= \left(-1 - \frac{1}{3}\right)^2 \cdot \left(\frac{1}{3}\right) + \left(1 - \frac{1}{3}\right)^2 \cdot \left(\frac{2}{3}\right) \\ &= \frac{8}{9}. \end{aligned}$$

提交 You have used 2 of 5 attempts

Answers are displayed within the problem

讨论

显示讨论

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