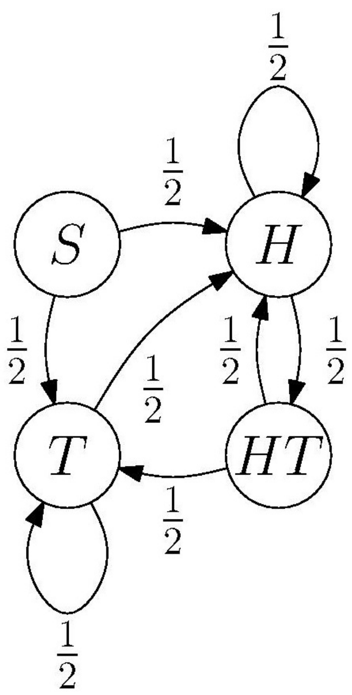
5. Coin tosses revisited

Problem 5. Coin tosses revisited

4/4 points (ungraded)

A fair coin is tossed repeatedly and independently. We want to determine the expected number of tosses until we first observe Tails immediately preceded by Heads. To do so, we define a Markov chain with four states, $\{S, H, T, HT\}$, where S is a starting state, H indicates Heads on the current toss, T indicates Tails on the current toss (without Heads on the previous toss), and HT indicates Heads followed by Tails over the last two tosses. This Markov chain is illustrated below:



Note: State S is in fact unnecessary, and is only included to facilitate understanding. Having the process start at state T, rather than S, makes no difference on what the next state will be; in both cases, the next state is equally likely to be H or T. Therefore S can be dispensed with.

1. What is the expected number of tosses until we first observe Tails immediately preceded by Heads? **Hint:** Solve the corresponding mean first passage time problem for our Markov chain.

4

Answer: 4

2. Assuming that we have just observed Tails immediately preceded by Heads, what is the expected number of additional tosses until we next observe Tails immediately preceded by Heads?

4

Answer: 4

Next, we want to answer similar questions for the event that Tails is immediately preceded by Tails. Set up a new Markov chain from which you can calculate the expected number of tosses until we first observe Tails immediately preceded by Tails.

3. What is the expected number of tosses until we first observe Tails immediately preceded by Tails?

6

Answer: 6

4. Assuming that we have just observed Tails immediately preceded by Tails, what is the expected number of additional tosses until we again observe Tails immediately preceded by Tails?

4

Answer: 4

Solution:

1. First, we observe that the Markov chain has one aperiodic recurrent class, which contains our target state HT. Let t_i be the expected time until the state HT is first reached, starting from state i, i.e., the mean first passage time to reach state HT starting from state i. Note that t_S is the expected number of tosses until first observing Tails immediately preceded by Heads. We have

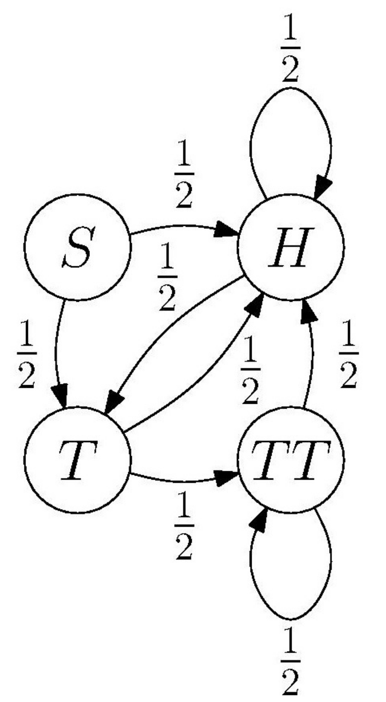
$$\begin{aligned} t_S &= 1 + \frac{1}{2} t_H + \frac{1}{2} t_T, \\ t_T &= 1 + \frac{1}{2} t_H + \frac{1}{2} t_T, \\ t_H &= 1 + \frac{1}{2} t_H. \end{aligned}$$

By solving these equations, we find that the expected number of tosses until first observing Tails immediately preceded by Heads is $t_S = 4$. Additionally, we have $t_H = 2$ and $t_T = 4$.

2. To find the expected number of additional tosses necessary to again observe Tails immediately preceded by Heads, we recognize that this is the mean recurrence time t^*_{HT} of state HT. It is

$$\begin{aligned} t^*_{HT} &= 1 + p_{HT,H} t_H + p_{HT,T} t_T \\ &= 1 + \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 4 \\ &= 4\,. \end{aligned}$$

3. Let us consider a Markov chain with states $\{S, H, T, TT\}$, where S is a starting state, H indicates Heads on the current toss, T indicates Tails on the current toss (without Tails on the previous toss), and TT indicates Tails over the last two tosses. The transition probabilities for this Markov chain are illustrated in the state transition diagram below:



Note: State S is in fact unnecessary, and is only included to facilitate understanding. Having the process start at state H, rather than S, makes no difference on what the next state will be; in both cases, the next state is equally likely to be H or T. Therefore S can be dispensed with.

Let t_i be the expected time until the state TT is first reached, starting from state i, i.e., the mean first passage time to reach state TT starting from state i. Note that t_S is the expected number of tosses until first observing Tails immediately preceded by Tails. We have

$$\begin{aligned} t_S &= 1 + \frac{1}{2} t_H + \frac{1}{2} t_T, \\ t_T &= 1 + \frac{1}{2} t_H, \\ t_H &= 1 + \frac{1}{2} t_H + \frac{1}{2} t_T. \end{aligned}$$

By solving these equations, we find that the expected number of tosses until first observing Tails immediately preceded by Tails is $t_S = 6$. We also have $t_H = 6$ and $t_T = 4$.

4. To find the expected number of additional tosses necessary to again observe Tails immediately preceded by Tails, we recognize that this is the mean recurrence time t^*_{TT} of state TT. It is

$$
\begin{aligned}
t^*_{TT} &= 1 + p_{TT,H}t_H + p_{TT,TT}t_{TT} \\
&= 1 + \frac{1}{2}\cdot 6 + \frac{1}{2}\cdot 0 \\
&= 4.
\end{aligned}
$$

提交 你已经尝试了4次（总共可以尝试4次）

Answers are displayed within the problem

讨论

显示讨论

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