Markov processes – II

- review and some warm-up
  - definitions, Markov property
  - calculating the probabilities of trajectories
- steady-state behavior
  - recurrent states, transient states, recurrent classes
  - periodic states
  - convergence theorem
  - balance equations
- birth-death processes

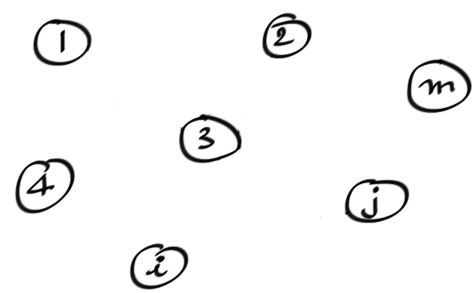
review

- discrete time, discrete state space, time-homogeneous
  - transition probabilities
  - Markov property

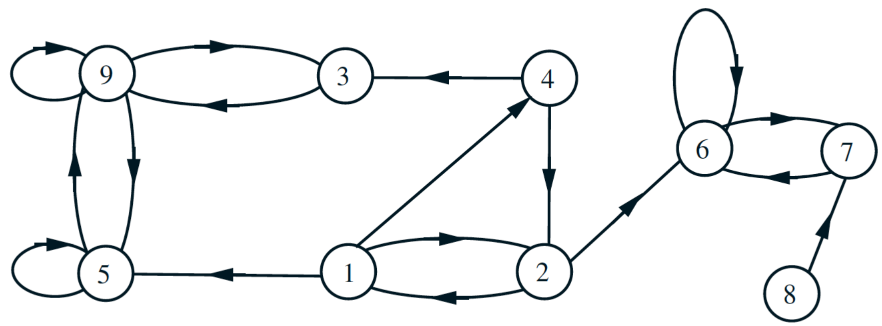
- $$\begin{aligned} r_{ij}(n) &= \mathbf{P}(X_n = j \mid X_0 = i) \\ &= \mathbf{P}(X_{n+s} = j \mid X_s = i) \end{aligned}$$

- key recursion:

$$\boxed{r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1)p_{kj}}$$



warmup

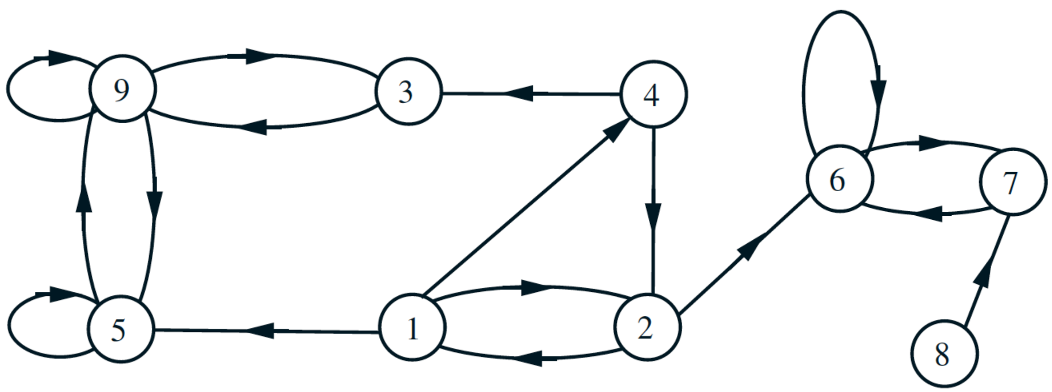


$$\mathbf{P}(X_1 = 2, X_2 = 6, X_3 = 7 \mid X_0 = 1) =$$

$$\mathbf{P}(X_4 = 7 \mid X_0 = 2) =$$

review: recurrent and transient states

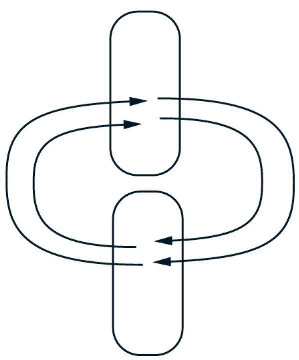
- state i is recurrent if "starting from i, and from wherever you can go, there is a way of returning to i"
- if not recurrent, called transient

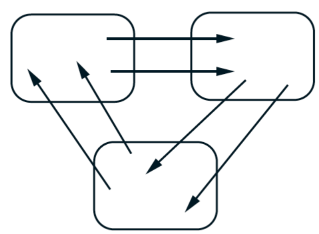


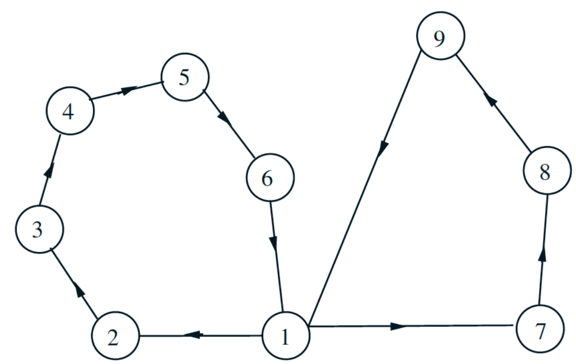
- recurrent class: a collection of recurrent states communicating only between each other

periodic states in a recurrent class

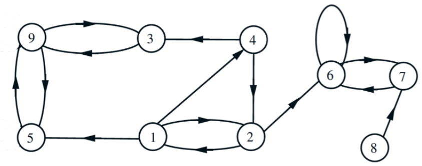
The states in a recurrent class are periodic if they can be grouped into d > 1 groups so that all transitions from one group lead to the next group

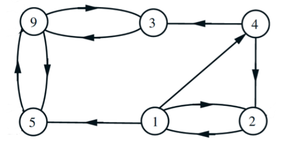




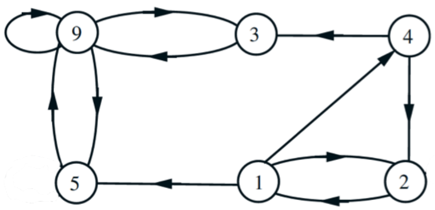


steady-state probabilities





- does $r_{ij}(n) \;=\; \mathrm{P}(X_n = j \mid X_0 = i)$ converge to some π_j ?
- theorem: yes, if:
  - recurrent states are all in a single class, and
  - single recurrent class is not periodic

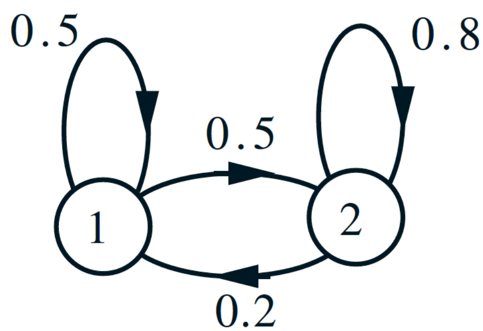


- assuming "yes", start from key recursion $r_{ij}(n) = \sum_k r_{ik}(n-1)p_{kj}$
  - take the limit as $n \to \infty$

$$\pi_j = \sum_k \pi_k p_{kj}$$

  - need also: $\sum_j \pi_j = 1$

example



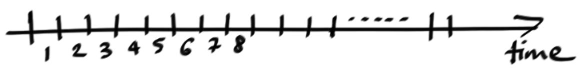
$$\pi_j = \sum_k \pi_k p_{kj}$$

visit frequency interpretation

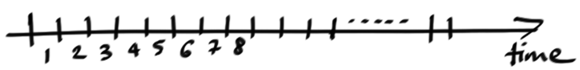
- balance equations

$$\pi_j = \sum_k \pi_k p_{kj}$$

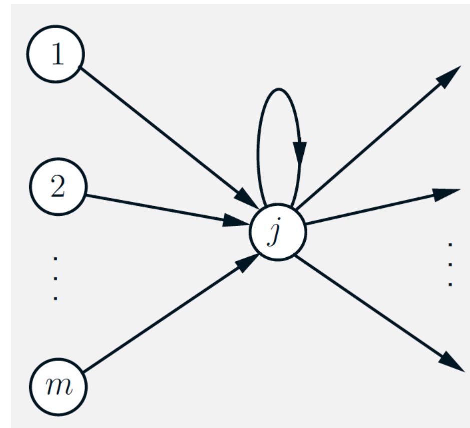
- (long run) frequency of being in j: π_j

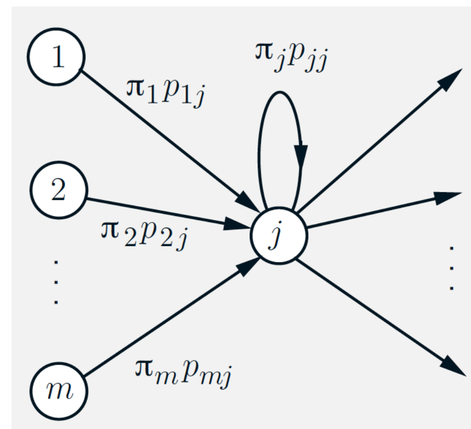


- frequency of transitions $1 \to j$: $\pi_1 p_{1j}$

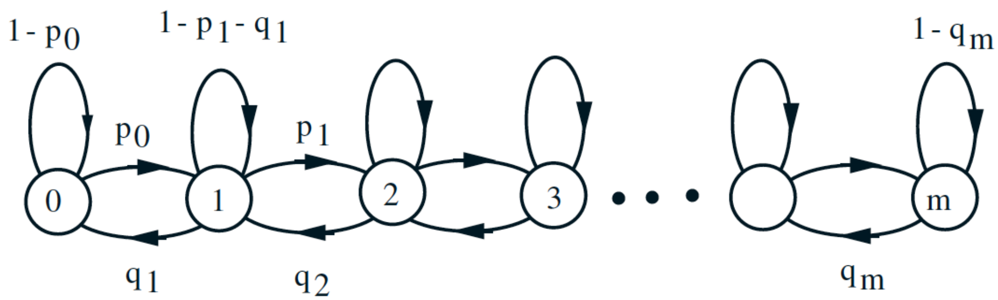


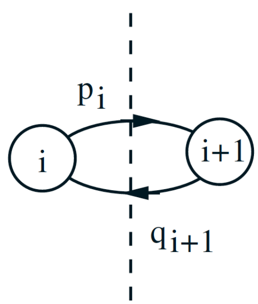
- frequency of transitions into j: $\sum_k \pi_k p_{kj}$



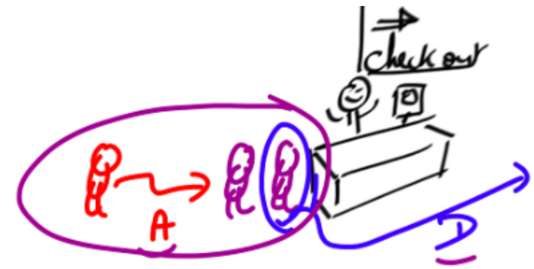


birth-death processes I

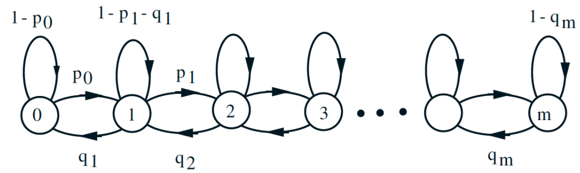


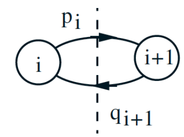


$$\pi_i p_i = \pi_{i+1} q_{i+1}$$



birth-death processes II





$$\pi_i p_i = \pi_{i+1} q_{i+1}$$

$$\sum_j \pi_j = 1$$

special case: $p_i = p$ and $q_i = q$ for all i

$$\rho = p/q \qquad \pi_{i+1} = \pi_i \frac{p}{q} = \pi_i \rho$$

$$\pi_i = \pi_0 \rho^i \qquad i = 0, 1, \ldots, m$$

- assume $p = q$

- assume $p < q$ and $m \approx \infty$

$$\pi_0 = 1 - \rho \qquad \mathbf{E}[X_n] = \frac{\rho}{1 - \rho} \text{ (in steady-state)}$$