3. Checking the Markov property

Problem 3. Checking the Markov property

7/7 points (ungraded)

For each one of the following definitions of the state X_k at time k (for $k = 1, 2, \ldots$), determine whether the Markov property is satisfied by the sequence $X_1, X_2, \ldots$.

1. A fair six-sided die (with sides labelled $1, 2, \ldots, 6$) is rolled repeatedly and independently.

   (a) Let X_k denote the largest number obtained in the first k rolls. Does the sequence $X_1, X_2, \ldots$ satisfy the Markov property?

   Yes ▾ ✔ **Answer:** Yes

   (b) Let X_k denote the number of 6's obtained in the first k rolls, up to a maximum of ten. (That is, if ten or more 6's are obtained in the first k rolls, then $X_k = 10$.) Does the sequence $X_1, X_2, \ldots$ satisfy the Markov property?

   Yes ▾ ✔ **Answer:** Yes

   (c) Let Y_k denote the result of the k^{th} roll. Let $X_1 = Y_1$, and for $k \geq 2$, let $X_k = Y_k + Y_{k-1}$. Does the sequence $X_1, X_2, \ldots$ satisfy the Markov property?

   No ▾ ✔ **Answer:** No

   (d) Let $Y_k = 1$ if the k^{th} roll results in an odd number; and $Y_k = 0$ otherwise. Let $X_1 = Y_1$, and for $k \geq 2$, let $X_k = Y_k \cdot X_{k-1}$. Does the sequence $X_1, X_2, \ldots$ satisfy the Markov property?

   Yes ▾ ✔ **Answer:** Yes

2. Let Y_k be the state of some Markov chain at time k (i.e., it is known that the sequence $Y_1, Y_2, \ldots$ satisfies the Markov property).

   (a) For a fixed integer $r > 0$, let $X_k = Y_{r+k}$. Does the sequence $X_1, X_2, \ldots$ satisfy the Markov property?

   Yes ▾ ✔ **Answer:** Yes

   (b) Let $X_k = Y_{2k}$. Does the sequence $X_1, X_2, \ldots$ satisfy the Markov property?

   Yes ▾ ✔ **Answer:** Yes

   (c) Let $X_k = (Y_k, Y_{k+1})$. Does the sequence $X_1, X_2, \ldots$ satisfy the Markov property?

   Yes ▾ ✔ **Answer:** Yes

Solution:

1. (a) Since the state X_k is the largest number obtained in k rolls, the set of states is $S = \{1, 2, 3, 4, 5, 6\}$. Given the largest number obtained in the first k rolls, the probability distribution of the largest number obtained in the first $k+1$ rolls no longer depends on what the largest number obtained was in the first $k-1$ rolls (or in the first $k-2$ rolls, etc.). Therefore the Markov property is satisfied.

   For $i, j \in \{1, 2, 3, 4, 5, 6\}$, the transition probabilities are

$$p_{ij} = \begin{cases} 0, & \text{if } j < i, \\ \frac{i}{6}, & \text{if } j = i, \\ \frac{1}{6}, & \text{if } j > i. \end{cases}$$

(b) Since the state X_k is the number of 6's in the first k rolls, the set of states is $S = \{0, 1, 2, \dots 10\}$. The probability of getting a 6 in a given trial is $1/6$. Given the number of 6's in the first k rolls, the probability distribution of the number of 6's in the first $k+1$ rolls no longer depends on the number of 6's in the first $k-1$ rolls (or in the first $k-2$ rolls, etc.). Therefore the Markov property is satisfied.

Thus, $p_{10,10} = 1$, and for $i \le 9$, the transition probabilities are

$$p_{ij} = \begin{cases} \frac{1}{6}, & \text{if } j = i+1, \\ \frac{5}{6}, & \text{if } j = i, \\ 0, & \text{otherwise.} \end{cases}$$

(c) We have

$$\begin{aligned} \mathbf{P}(X_3 = 2 \mid X_2 = 3, X_1 = 1) &= \mathbf{P}(Y_2 + Y_3 = 2 \mid Y_1 = 1, Y_2 = 2) \\ &= \mathbf{P}(Y_3 = 0 \mid Y_1 = 1, Y_2 = 2) \\ &= 0, \end{aligned}$$

but

$$\begin{aligned} \mathbf{P}(X_3 = 2 \mid X_2 = 3, X_1 = 2) &= \mathbf{P}(Y_2 + Y_3 = 2 \mid Y_1 = 2, Y_2 = 1) \\ &= \mathbf{P}(Y_3 = 1 \mid Y_1 = 2, Y_2 = 1) \\ &= \mathbf{P}(Y_3 = 1) \\ &= 1/6, \end{aligned}$$

and therefore the Markov property is violated.

(d) At each stage, Y_k has equal probability of being 0 or 1. Since $X_k = Y_k \cdot X_{k-1}$, and we assume independent rolls, clearly X_k depends only on the k^{th} roll and the value of X_{k-1}. Therefore the Markov property is satisfied.

The transition probabilities are $p_{00} = 1$, $p_{01} = 0$, $p_{10} = 1/2$, and $p_{11} = 1/2$.

2. (a) For $X_k = Y_{r+k}$, and because the sequence $\{Y_k\}$ satisfies the Markov property,

$$\begin{aligned} & \mathbf{P}(X_{k+1} = j \mid X_k = i, X_{k-1} = i_{k-1}, \dots, X_1 = i_1) \\ = \; & \mathbf{P}(Y_{r+k+1} = j \mid Y_{r+k} = i, Y_{r+k-1} = i_{k-1}, \dots, Y_{r+1} = i_1) \\ = \; & \mathbf{P}(Y_{r+k+1} = j \mid Y_{r+k} = i) \\ = \; & \mathbf{P}(X_{k+1} = j \mid X_k = i) \end{aligned}$$

Thus, the sequence $\{X_k\}$ satisfies the Markov property.

(b) For $X_k = Y_{2k}$, and because the sequence $\{Y_k\}$ satisfies the Markov property,

$$\begin{aligned} & \mathbf{P}(X_{k+1} = j \mid X_k = i, X_{k-1} = i_{k-1}, \dots, X_1 = i_1) \\ = \; & \mathbf{P}(Y_{2k+2} = j \mid Y_{2k} = i, Y_{2k-2} = i_{k-1}, \dots, Y_2 = i_1) \\ = \; & \mathbf{P}(Y_{2k+2} = j \mid Y_{2k} = i) \\ = \; & \mathbf{P}(X_{k+1} = j \mid X_k = i) \end{aligned}$$

Thus, X_k satisfies the Markov property. The transition probabilities p_{ij} are given by

$$\begin{aligned} p_{ij} &= \mathbf{P}(X_{k+1} = j \mid X_k = i) \\ &= \mathbf{P}(Y_{2k+2} = j \mid Y_{2k} = i) \\ &= r^y_{ij}(2), \end{aligned}$$

where $r^y_{ij}(n)$ are the n-step transition probabilities of the Markov chain $\{Y_k\}$.

(c) Note that

$$\mathbf{P}(X_{k+1} = (n, \ell) \mid X_1 = (i_1, i_2), X_2 = (i_2, i_3), \dots, X_k = (i_k, n))$$

$$
\begin{aligned}
&= \mathbf{P}(Y_{k+1} = n, Y_{k+2=\ell} \mid Y_1 = i_1, Y_2 = i_2, Y_3 = i_3, \dots, Y_k = i_k, Y_{k+1} = n) \\
&= \mathbf{P}(Y_{k+2} = \ell \mid Y_1 = i_1, Y_2 = i_2, \dots, Y_k = i_k, Y_{k+1} = n) \\
&= \mathbf{P}(Y_{k+2} = \ell \mid Y_k = i_k, Y_{k+1} = n) \\
&= \mathbf{P}(Y_{k+1} = n, Y_{k+2} = \ell \mid Y_k = i_k, Y_{k+1} = n) \\
&= \mathbf{P}(X_{k+1} = (n, \ell) \mid X_k = (i_k, n)).
\end{aligned}
$$

Therefore the Markov property is satisfied.

Letting $i = (i_k, i_{k+1})$ and $j = (n, \ell)$, the transition probabilities p_{ij} are given by

$$
p_{ij} = \mathbf{P}(X_{k+1} = (n, \ell) \mid X_k = (i_k, i_{k+1})) = \begin{cases} q_{n\ell}, & \text{if } i_{k+1} = n, \\ 0, & \text{if } i_{k+1} \neq n, \end{cases}
$$

where $q_{n\ell}$ are the transition probabilities associated with the Markov chain $\{Y_k\}$.

提交 你已经尝试了2次（总共可以尝试3次）

Answers are displayed within the problem

讨论

显示讨论

主题： Unit 10 / Problem Set / 3. Checking the Markov property