2. Estimating the parameter of a geometric r.v.

Problem 2. Estimating the parameter of a geometric r.v.

3/3 points (graded)

We have k coins. The probability of Heads is the same for each coin and is the realized value q of a random variable Q that is uniformly distributed on $[0,1]$. We assume that conditioned on $Q=q$, all coin tosses are independent. Let T_i be the number of tosses of the i^{th} coin until that coin results in Heads for the first time, for $i=1,2,\ldots,k$. (T_i includes the toss that results in the first Heads.)

You may find the following integral useful: For any non-negative integers k and m,

$$\int_0^1 q^k(1-q)^m\,dq=\frac{k!m!}{(k+m+1)!}.$$

1. Find the PMF of T_1. (Express your answer in terms of t using standard notation.)

   For $t=1,2,\ldots$,

   $p_{T_1}(t)=$ 1/((t+1)*t) ✔ **Answer:** 1/(t*(t+1))

   $\frac{1}{(t+1)\cdot t}$

2. Find the least mean squares (LMS) estimate of Q based on the observed value, t, of T_1. (Express your answer in terms of t using standard notation.)

   $\mathbf{E}[Q\mid T_1=t]=$ 2/(t+2) ✔ **Answer:** 2/(t+2)

   $\frac{2}{t+2}$

3. We flip each of the k coins until they result in Heads for the first time. Compute the maximum a posteriori (MAP) estimate $\hat{q}$ of Q given the number of tosses needed, $T_1=t_1,\ldots,T_k=t_k$, for each coin. Choose the correct expression for $\hat{q}$.

   ○ $\hat{q}=\dfrac{k-1}{\sum_{i=1}^k t_i}$

- ◉ $\hat{q} = \dfrac{k}{\sum_{i=1}^{k} t_i}$ ✔
- ○ $\hat{q} = \dfrac{k+1}{\sum_{i=1}^{k} t_i}$
- ○ none of the above

STANDARD NOTATION

Solution:

1. Note that, $p_{T_1|Q}(t \mid q) = (1-q)^{t-1}q$. Using the total probability theorem, we have

$$p_{T_1}(t) = \int_0^1 p_{T_1|Q}(t \mid q) f_Q(q)\, dq = \int_0^1 (1-q)^{t-1} q\, dq = \frac{1}{(t+1)t}, \quad \text{for } t = 1, 2, \ldots.$$

2. The LMS estimate is

$$\begin{aligned}
\mathbf{E}[Q \mid T_1 = t] &= \int_0^1 f_{Q|T_1}(q \mid t) q\, dq \\
&= \int_0^1 \frac{p_{T_1|Q}(t \mid q) f_Q(q)}{p_{T_1}(t)} q\, dq \\
&= \int_0^1 t(t+1)q(1-q)^{t-1} q\, dq \\
&= \int_0^1 t(t+1)q^2(1-q)^{t-1} dq \\
&= t(t+1)\frac{2(t-1)!}{(t+2)!} \\
&= \frac{2}{t+2}.
\end{aligned}$$

3. We compute the posterior distribution of Q given that $T_1 = t_1, \ldots, T_k = t_k$:

$$f_{Q|T_1,\ldots,T_k}(q \mid t_1, \ldots, t_k) = \frac{f_Q(q) \prod_{i=1}^{k} p_{T_i|Q}(t_i \mid q)}{\int_0^1 f_Q(q) \prod_{i=1}^{k} p_{T_i|Q}(t_i \mid q) dq}$$

$$= \frac{q^k (1-q)^{\sum_{i=1}^k t_i - k}}{c},$$

where c is a normalizing constant that does not depend on q.

To maximize the above expression, we set its derivative with respect to q to zero and obtain

$$kq^{k-1}(1-q)^{\sum_{i=1}^k t_i - k} - \left(\sum_{i=1}^k t_i - k\right) q^k (1-q)^{\sum_{i=1}^k t_i - k - 1} = 0,$$

or equivalently,

$$k(1-q) - \left(\sum_{i=1}^k t_i - k\right) q = 0,$$

which yields the MAP estimate

$$\hat{q} = \frac{k}{\sum_{i=1}^k t_i}.$$

(In an alternative derivation, we can first take the logarithm of the posterior, and then maximize.)

提交 You have used 2 of 3 attempts

Answers are displayed within the problem

讨论

显示讨论

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