## Hoeffding's inequality for $\mathbf{P}(X_1 + \cdots + X_n \geq na)$

$X_i$: **i.i.d.**


$Y = X_1 + \cdots + X_n$

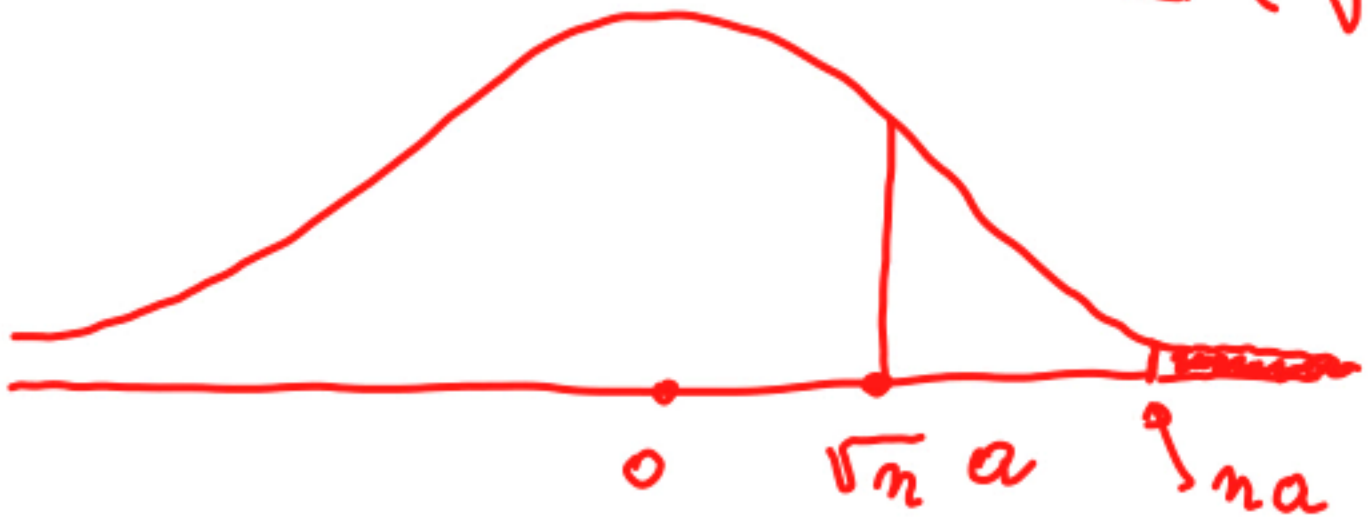
$E[X_i] = 0 \qquad E[Y] = 0$

$\text{var}(X_i) = 1 \qquad \text{var}(Y) = n$

st. normal CDF

$$P\left(\frac{Y}{\sqrt{n}} \geq a\right) \approx 1 - \Phi(a)$$

$$P(Y \geq na) \leq \frac{\text{var}(Y)}{n^2 a^2} = \frac{1}{na^2}$$




- Hoeffding's inequality: If $X_i$ is equally likely to be $-1$ or 1, and $a > 0$, then

$$\mathbf{P}(X_1 + \cdots + X_n \geq na) \leq e^{-na^2/2}.$$

**Hoeffding's inequality for $\mathbf{P}(X_1 + \cdots + X_n \geq na)$** $a>0$ $X_i$: **i.i.d.**

$$P\left(e^{s(X_1+\cdots+X_n)} \geq e^{sna}\right) \qquad s>0$$

$$\mathbf{P}(Z \geq c) \leq \frac{\mathbf{E}[Z]}{c}$$

$$\leq E\left[e^{s(X_1+\cdots+X_n)}\right] / e^{sna}$$

Chernoff bound

$$= E\left[e^{sX_1} \cdots e^{sX_n}\right] / e^{sna}$$

$$= \left(E\left[e^{sX_1}\right]\right)^n / e^{sna} = \left(\frac{E\left[e^{sX_1}\right]}{e^{sa}}\right)^n = \rho^n$$

$$\left[\frac{\frac{1}{2}\left(e^s + e^{-s}\right)}{e^{sa}}\right]^n$$


for "small" s

$\rho < 1$

$s = a$

$$\boxed{\leq e^{-na^2/2}}$$

**Hoeffding's inequality for $\mathrm{P}(X_1 + \cdots + X_n \geq na)$**

$X_i$: **i.i.d.**

$$\mathrm{P}(X_1 + \cdots + X_n - n\mu \geq na) \leq \left(\frac{(e^s + e^{-s})/2}{e^{sa}}\right)^n \leq \left(\frac{e^{s^2/2}}{e^{sa}}\right)^n = \left(e^{\frac{s^2}{2} - sa}\right)^n$$

$$s = a: \quad \boxed{e^{-na^2/2}}$$

$$e^s = 1 + s + \frac{s^2}{2!} + \frac{s^3}{3!} + \cdots = \sum_{i=0}^{\infty} \frac{s^i}{i!}$$

$$\frac{1}{2}(e^s + e^{-s}) = \frac{1}{2}\left(1 + s + \frac{s^2}{2!} + \frac{s^3}{3!} + \cdots\right) + \frac{1}{2}\left(1 - s + \frac{s^2}{2!} - \frac{s^3}{3!} + \cdots\right)$$

$$= \sum_{i=0}^{\infty} \frac{s^{2i}}{(2i)!} \leq \sum_{i=0}^{\infty} \frac{s^{2i}}{i!\,2^i} = \sum_{i=0}^{\infty} \frac{(s^2/2)^i}{i!} = e^{s^2/2}$$

$$(2i)! = \underbrace{1 \cdot 2 \cdot 3 \cdots i}\cdot(i+1)\cdots(2i) \geq i!\,2^i$$