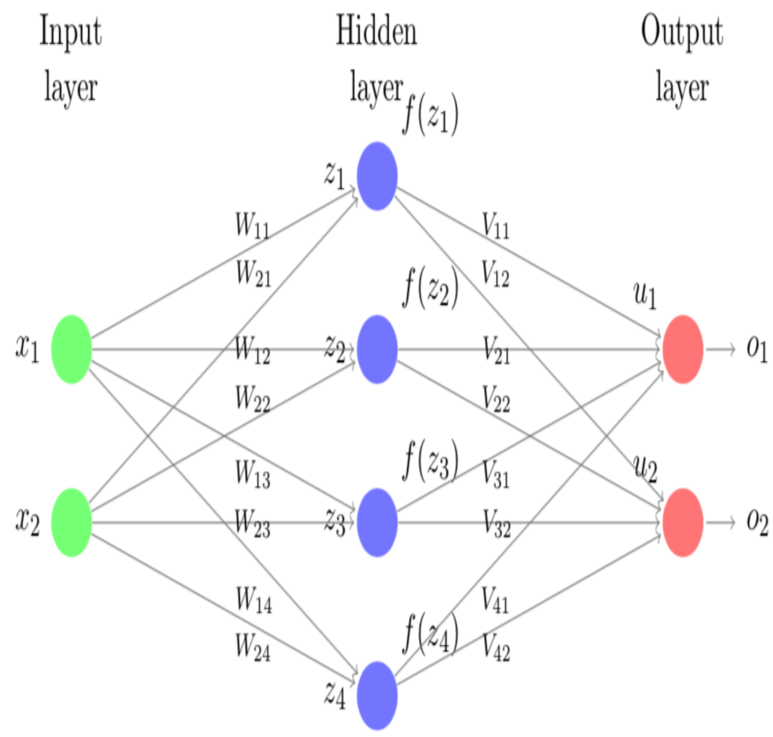
1. Neural Networks

Extension Note: Homework 4 due date has been extended by 1 day to **July 27 23:59UTC** .

In this problem we will analyze a simple neural network to understand its classification properties. Consider the neural network given in the figure below, with **ReLU activation functions (denoted by f) on all neurons**, and a **softmax activation function in the output layer**:



Given an input $x = [x_1, x_2]^T$, the hidden units in the network are activated in stages as described by the following equations:

$$\begin{aligned} z_1 &= x_1 W_{11} + x_2 W_{21} + W_{01} & f(z_1) &= \max\{z_1, 0\} \\ z_2 &= x_1 W_{12} + x_2 W_{22} + W_{02} & f(z_2) &= \max\{z_2, 0\} \\ z_3 &= x_1 W_{13} + x_2 W_{23} + W_{03} & f(z_3) &= \max\{z_3, 0\} \\ z_4 &= x_1 W_{14} + x_2 W_{24} + W_{04} & f(z_4) &= \max\{z_4, 0\} \end{aligned}$$

$$\begin{aligned} u_1 &= f(z_1) V_{11} + f(z_2) V_{21} + f(z_3) V_{31} + f(z_4) V_{41} + V_{01} & f(u_1) &= \max\{u_1, 0\} \\ u_2 &= f(z_1) V_{12} + f(z_2) V_{22} + f(z_3) V_{32} + f(z_4) V_{42} + V_{02} & f(u_2) &= \max\{u_2, 0\}. \end{aligned}$$

The final output of the network is obtained by applying the **softmax** function to the last hidden layer,

$$\begin{aligned} o_1 &= \frac{e^{f(u_1)}}{e^{f(u_1)} + e^{f(u_2)}} \\ o_2 &= \frac{e^{f(u_2)}}{e^{f(u_1)} + e^{f(u_2)}}. \end{aligned}$$

In this problem, we will consider the following setting of parameters:

$$\begin{bmatrix} W_{11} & W_{21} & W_{01} \\ W_{12} & W_{22} & W_{02} \\ W_{13} & W_{23} & W_{03} \\ W_{14} & W_{24} & W_{04} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & -1 \\ 0 & -1 & -1 \end{bmatrix},$$

$$\begin{bmatrix} V_{11} & V_{21} & V_{31} & V_{41} & V_{01} \\ V_{12} & V_{22} & V_{32} & V_{42} & V_{02} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 2 \end{bmatrix}.$$

Feed Forward Step

2/2 points (graded)
Consider the input $x_1 = 3, x_2 = 14$. What is the final output (o_1, o_2) of the network?

Important: Numerical outputs from the softmax function are sometimes extremely close to 0 or 1; if you choose to enter your answers numerically, make sure to report them to at least 9 decimal places! (Alternatively, you may enter your answers symbolically as a function of symbolically e.)

$o_1 =$ exp(15)/(exp(15)+exp(0)) ✔ **Answer:** e^15 / (e^15 + 1) $o_2 =$ exp(0)/(exp(15)+exp(0)) ✔ **Answer:** 1 / (e^15 + 1)

STANDARD NOTATION

Solution:

Plugging the formula, we see that

$$f(z_1) = \max\{z_1, 0\} = 2$$

$$f(z_2) = \max\{z_2, 0\} = 13$$

$$f(z_3) = \max\{z_3, 0\} = 0$$

$$f(z_4) = \max\{z_4, 0\} = 0$$

Going to the next layer, we see that

$$\begin{aligned} u_1 &= f(z_1)V_{11} + f(z_2)V_{21} + f(z_3)V_{31} + f(z_4)V_{41} + V_{01} \\ u_1 &= 2*1 + 13*1 + 0*1 + 0*1 \\ u_1 &= 15 \\ u_2 &= f(z_1)V_{12} + f(z_2)V_{22} + f(z_3)V_{32} + f(z_4)V_{42} + V_{02} \\ u_2 &= 2*-1 + 13*-1 + 0*-1 + 0*-1 \\ u_2 &= -15 \end{aligned}$$

Passing the values of u_1, u_2 through the function f gives:

$$\begin{aligned} f(u_1) &= \max\{u_1, 0\} \\ f(u_1) &= \max\{15, 0\} \\ f(u_1) &= 15 \\ f(u_2) &= \max\{u_2, 0\} \\ f(u_2) &= \max\{-15, 0\} \\ f(u_2) &= 0 \end{aligned}$$

Plugging these values into the following equations for o_1, o_2 gives:

$$\begin{aligned} o_1 &= \frac{e^{f(u_1)}}{e^{f(u_1)} + e^{f(u_2)}} \\ o_2 &= \frac{e^{f(u_2)}}{e^{f(u_1)} + e^{f(u_2)}} \end{aligned}$$

$$o_1 = \frac{e^{15}}{e^{15}+1}, \qquad o_2 = \frac{1}{e^{15}+1}$$

Submit You have used 1 of 4 attempts

Answers are displayed within the problem

Decision Boundaries

1/1 point (graded)

In this problem we visualize the “decision boundaries" in x-space, corresponding to the four hidden units. These are the lines where the input to the units z_1, z_2, z_3, z_4 are exactly zero. Plot the decision boundaries of the four hidden units using the parameters of W provided above.

Enter below the **area of the region** of your plot that corresponds to a negative (< 0) value for all of the four hidden units.

4 ✔ **Answer:** 4

Solution:

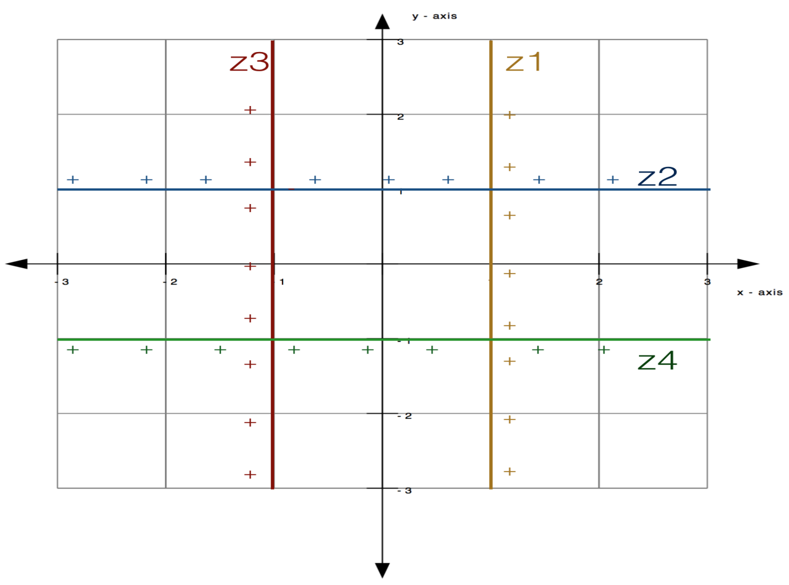
The four decision boundaries are given by the following four functions respectively.

$$\begin{aligned} z_1 &= x_1 W_{11} + x_2 W_{21} + W_{01} = 0 \\ z_2 &= x_1 W_{12} + x_2 W_{22} + W_{02} = 0 \\ z_3 &= x_1 W_{13} + x_2 W_{23} + W_{03} = 0 \\ z_4 &= x_1 W_{14} + x_2 W_{24} + W_{04} = 0 \end{aligned}$$

When the weight parameters are plugged in, the above equations simplify to the following expressions:

$$\begin{aligned} x_1 - 1 &= 0 \\ x_2 - 1 &= 0 \\ -x_1 - 1 &= 0 \\ -x_2 - 1 &= 0 \end{aligned}$$

Note that the four equations above correspond to four straight lines in the two-dimensional x-space. The four equations are visualized in the figure below.



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Answers are displayed within the problem

Output of Neural Network

3/3 points (graded)

Using the same matrix V as above, what is the value of o_1 (accurate to at least three decimal places if responding numerically) in the following three cases?

- Assuming that $f(z_1) + f(z_2) + f(z_3) + f(z_4) = 1$:

$o_1 =$ 1/2 ✔ **Answer:** 0.5

- Assuming that $f(z_1) + f(z_2) + f(z_3) + f(z_4) = 0$:

$o_1 =$ exp(0)/(exp(2)+exp(0)) ✔ **Answer:** 1/(1+e^2)

- Assuming that $f(z_1) + f(z_2) + f(z_3) + f(z_4) = 3$:

$o_1 =$ exp(3)/(exp(0)+exp(3)) ✔ **Answer:** 1/(1+e^-3)

STANDARD NOTATION

Solution:

Note that,

$$\begin{aligned} u_1 &= f(z_1)V_{11} + f(z_2)V_{21} + f(z_3)V_{31} + f(z_4)V_{41} + V_{01} \\ u_2 &= f(z_1)V_{12} + f(z_2)V_{22} + f(z_3)V_{32} + f(z_4)V_{42} + V_{02} \end{aligned}$$

Plugging in values of V and the assumption of the first case, we get:

$$\begin{aligned} u_1 &= f(z_1) + f(z_2) + f(z_3) + f(z_4) + 0 \\ u_1 &= 1 \\ u_2 &= -1(f(z_1) + f(z_2) + f(z_3) + f(z_4)) + 2 \\ u_2 &= 1 \end{aligned}$$

From the above we substitute the values of $u_1 = u_2 = 1$ into the equations for o_1, o_2 to get:

$$\begin{aligned} o_1 &= \frac{e^{f(1)}}{e^{f(1)} + e^{f(1)}} \\ o_1 &= \frac{e^1}{e^1 + e^1} \\ o_1 &= \frac{1}{2} \\ o_2 &= \frac{e^{f(1)}}{e^{f(1)} + e^{f(1)}} \\ o_2 &= \frac{e^1}{e^1 + e^1} \\ o_2 &= \frac{1}{2} \end{aligned}$$

The other two cases are solved similarly. Note that $\frac{e^3}{e^3+1} = \frac{1}{1+e^{-3}}$

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Answers are displayed within the problem

Inverse Temperature

3/3 points (graded)

Now, suppose we modify the network's softmax function as follows:

$$o_1 = \frac{e^{\beta f(u_1)}}{e^{\beta f(u_1)} + e^{\beta f(u_2)}}$$

$$o_2 = \frac{e^{\beta f(u_2)}}{e^{\beta f(u_1)} + e^{\beta f(u_2)}},$$

where $\beta > 0$ is a parameter. Note that our previous setting corresponded to the special case $\beta = 1$.
In the following, please write a numerical solution with an accuracy of at least 3 places.
For $\beta = 1$, in order to satisfy $o_2 \geq \frac{1}{1000}$, the value of $f(u_1) - f(u_2)$ should be smaller or equal than:

ln(999) ✔ **Answer:** 6.906754778648554

If we increase the value to $\beta = 3$, in order to satisfy $o_2 \geq \frac{1}{1000}$, the value of $f(u_1) - f(u_2)$ should be smaller or equal than:

ln(999)/3 ✔ **Answer:** 2.3022515928828513

In general, increasing the value of β can result in $f(u_1) - f(u_2)$ being:

- larger
- smaller ✔

Solution:

For $o_2 \geq \frac{1}{1000}$ we must have

$$\frac{1}{1 + e^{\beta(f(u_1) - f(u_2))}} \geq \frac{1}{1000}$$

which is equivalent to $e^{\beta(f(u_1)-f(u_2))} \leq 999$. In other words,

$$f(u_1) - f(u_2) \leq \frac{\ln(999)}{\beta}$$

As β increases from 1 to 3 the above condition becomes more strict, and hence the corresponding region in the x-space **shrinks** . (To see this more clearly, consider the boundaries $f(u_1) - f(u_2) = \ln(999)$ and $f(u_1) - f(u_2) = \ln(999)/3$.)

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Discussion

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