Problem 3

Setup: Let $X_1, \ldots, X_n \overset{i.i.d.}{\sim}$ Unif $([\theta, \theta^2])$ for some unknown $\theta > 1$. That is, the pdf of X_i is

$$f_\theta(x) = C\mathbf{1}_{[\theta,\theta^2]}(x).$$

where $C = \frac{1}{\theta^2 - \theta}$.

(a)

1/1 point (graded)
Is the parameter θ identifiable?

- (●) True ✔
- ( ) False

Solution:

Yes, θ is identifiable as the minimum of the support of the $f_\theta(x)$.

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Answers are displayed within the problem

(b)

1/1 point (graded)
Which of the following statements are true regarding the samples? Note that $X_{(i)}$ denote the order statistics, i.e. $X_{(i)}$ represents the $i^{(th)}$ smallest value of the sample. For example, $X_{(1)}$ is the smallest and $X_{(n)}$ is the greatest of a sample of size n.
(Check all that apply.)

- [x] $X_{(2)} \geq X_{(1)}$ ✔
- [ ] $X_{(2)} \geq X_{(3)}$
- [x] $X_{(1)}^2 \geq X_{(n)}$ ✔
- [ ] $X_{(2)} \geq X_{(3)}^2$

✔

Grading note: Partial credits are given.

Solution:

Since $X_{(i)}$ are the order statistics of the data $X_1, \ldots, X_n$, $X_{(i)}$ are the rearrangement of $X_1, \ldots, X_n$ such that $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$. Hence, the first choice is true while the second is not.
For the third choice, the key point is that since $\theta > 1$, we have $\theta^2 > \theta$, and therefore $X_{(1)} \geq \theta$ and $X_{(n)} \leq \theta^2$. Therefore $X_{(1)}^2 \geq X_{(n)}$.

The fourth choice is false, since $X_3 > \theta > 1,\ X_{(3)}^2 > X_{(3)} > X_{(2)}$.

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Answers are displayed within the problem

(c)

1/1 point (graded)

Compute the maximum likelihood estimator $\hat{\theta}^{\mathrm{MLE}}$ of θ.

(If applicable, enter **m** for the minimum $\min_i (X_i)$ of the X_i, **M** for the maximum $\max_i (X_i)$ of the X_i, and **barX_n** for $\bar{X}_n = \frac{\sum_{i=1}^n X_i}{n}$.)

$\hat{\theta}^{\mathrm{MLE}} =$ sqrt(M) ✔ **Answer:** sqrt(M)

STANDARD NOTATION

Solution:

Note that the likelihood of $X_1, \ldots, X_n$ is

$$L(X_1, \ldots, X_n; \theta) = \left(\frac{1}{\theta(\theta-1)}\right)^n \mathbf{1}(\theta \le X_{(1)} < X_{(n)} \le \theta^2)$$

In the region $\theta > 1$, the function $\frac{1}{(\theta(\theta-1))^n}$ is strictly decreasing, so $\hat{\theta}^{\mathrm{MLE}}$ is the smallest possible θ that can lead to the given data, i.e. a greatest lower bound on θ. Since $X_{(n)} < \theta^2$ can be arbitrarily close θ^2, we have $\sqrt{X_{(n)}} < \theta$. Therefore, the MLE is so $\hat{\theta}^{\mathrm{MLE}} = \sqrt{X_{(n)}}$
.

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Answers are displayed within the problem

(d)

1/2 points (graded)

Suppose that $X_{(1)} = 2.01$ and $n = 18$. Design a test for the hypotheses

$$H_0 : \theta = 2 \text{ vs. } H_1 : \theta < 2.$$

that uses $T_n = X_{(1)}$ as the test statistic.

这个test是怎么想出来的呢？
我觉得唯一的原因就在这句话了
但是这里没法解释的是一些波动的可能

Compute the p-value of this test.

(Enter a numerical value accurate to at least 3 decimal places.)

p-value: 0.47210 ✖ **Answer:** 0.086

Do we reject the null hypothesis at the level 5%?

- ○ Yes
- ◉ No ✔

STANDARD NOTATION

Correction Note (added May 5):. An earlier version of the problem statement was "Design a test for the hypotheses

$$H_0 : \theta = 2 \text{ vs. } H_1 : \theta < 2.$$

that uses $X_{(1)}$ as the test statistic."

Solution:

The test for the given hypotheses

$$H_0 : \theta = 2 \text{ vs. } H_1 : \theta < 2.$$

and the estimator $X_{(1)} = \min_i\{X_i\}$, is

$$\psi = \mathbf{1}\left(X_{(1)} < C\right)$$

for some threshold C. Hence, the p-value of this test is

$$\begin{aligned} p\text{-value} &= \mathbf{P}_{\theta=2}\left(X_{(1)} < 2.01\right) \\ &= 1 - \mathbf{P}_{\theta=2}\left(X_{(1)} \geq 2.01\right) \\ &= 1 - \left(\prod_{i=1}^{18} \mathbf{P}_{\theta=2}\left(X_i \geq 2.01\right)\right) \\ &= 1 - \left(1 - \frac{2.01-2}{2}\right)^{18} \approx .086. \end{aligned}$$

如果theta = 2：在18个data下，X(1) 会小于2.01的概率

这个概率如果特别小，我们就认为这是小概率事件，那么theta就很可能不等于2而是小于2，因此才会拒绝虚无假设。

Since the p-value is greater than $0.05,$ we fail to reject the null hypothesis.

Submit You have used 2 of 3 attempts

Answers are displayed within the problem

(e)

0/1 point (graded)

Again, $X_{(1)} = 2.01$ and $n = 18$, and consider the same hypotheses as above:

$$H_0 : \theta = 2 \text{ vs. } H_1 : \theta < 2$$

Now, design a test using the test statistic $T_n = X_{(n)}$. What is the largest value of $X_{(n)}$ that would lead to a rejection of H_0 at level 5%?

(Enter a numerical answer accurate to at least 2 decimal places.)

To reject H_0 at level 5%,

$X_{(n)} \leq$ 3.1066 ✖ **Answer:** 3.69

3.1066

Solution:

The test for the given hypotheses

$$H_0 : \theta = 2 \text{ vs. } H_1 : \theta < 2.$$

using the given test statistic $T_n = X_{(n)} = \max_i\{X_i\}$, is

$$\psi = \mathbf{1}\left(X_{(n)} < C\right)$$

for some threshold C. We want C such that

$$\begin{aligned} & \mathbf{P}_{\theta=2}\left(X_{(n)} < C\right) &= 0.05 \\ \iff & \prod_{i=1}^{n} \mathbf{P}_{\theta=2}\left(X_i < C\right) &= 0.05 \\ \iff & \left(\frac{C-2}{4-2}\right)^{18} &= 0.05 \\ \iff & C &= 2 + 2(0.05)^{\frac{1}{18}} \approx 3.69. \end{aligned}$$

This C is the largest value of $X_{(n)}$ so that H_0 will be rejected at 5% by this test.

Submit

You have used 1 of 3 attempts

Answers are displayed within the problem

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