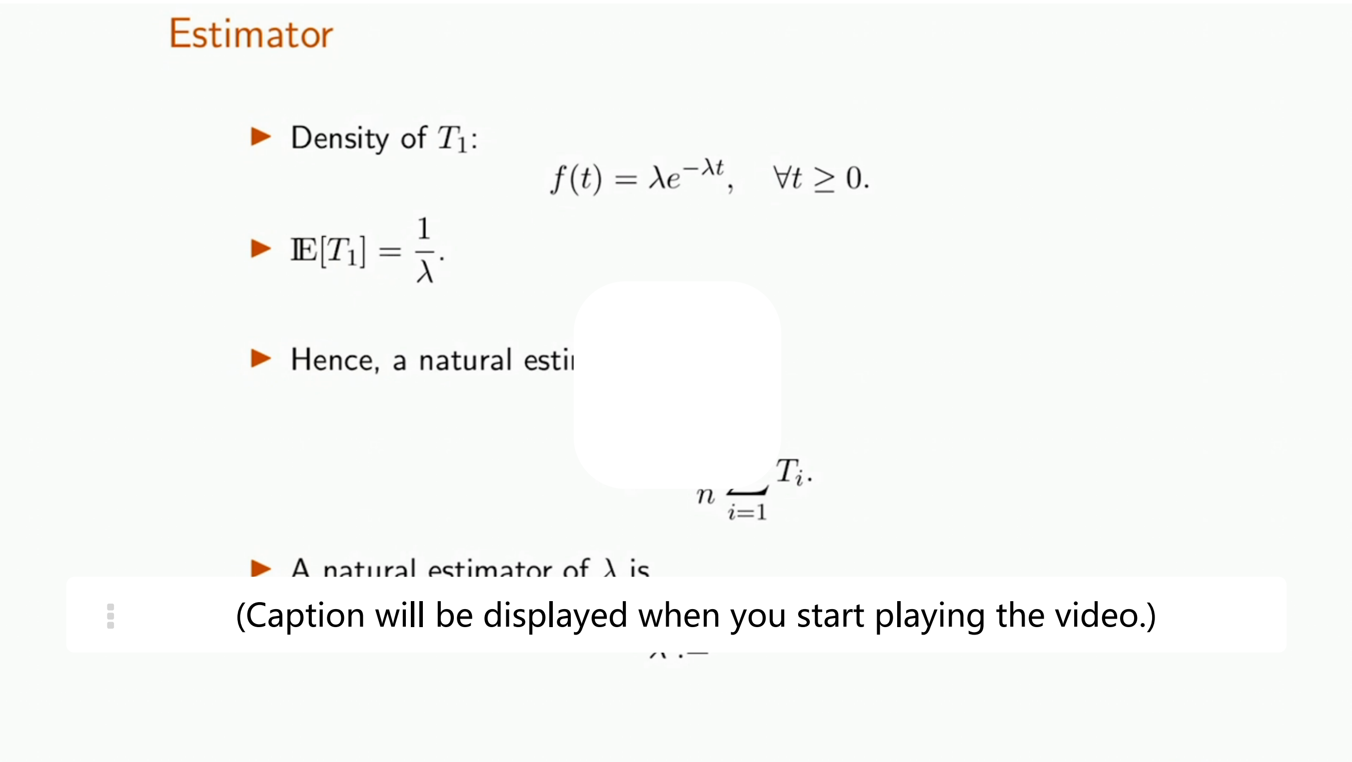
7. Estimating the Parameter for an Exponential Model

Estimating the Parameter for an Exponential Model



Start of transcript. Skip to the end.

OK, so as I said, the density is lambda e to the minus lambda T.

You can probably convince yourself just by looking at it

that this thing will integrate to one on zero infinity.

I won't do it, but you can check that the expectation is

one over lambda.

That's probably a pretty standard exercise.

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Consistency and Biasedness

3/4 points (graded)

Let $X_1, X_2, \ldots, X_n \stackrel{iid}{\sim} \exp(\lambda)$. Let $\overline{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i$ denote the sample mean of the data set.

To which value does $\overline{X}_n$ converge (both a.s. and in probability) as $n \to \infty$?
(Choose all that apply)

- [x] $\mathbb{E}[X_i]$ ✔
- [ ] $\frac{1}{\mathbb{E}[X_i]}$
- [ ] $\mathbb{E}\left[\frac{1}{X_i}\right]$
- [ ] λ
- [x] $\frac{1}{\lambda}$ ✔

✔

To which value does $\frac{1}{\overline{X}_n}$ converge (both a.s. and in probability) as $n \to \infty$? (Choose all that apply)

- [ ] $\mathbb{E}[X_i]$
- [x] $\dfrac{1}{\mathbb{E}[X_i]}$ ✔
- [x] $\mathbb{E}\left[\dfrac{1}{X_i}\right]$
- [x] λ ✔
- [ ] $\dfrac{1}{\lambda}$

✖

Which of the following is the bias of $\dfrac{1}{\overline{X}_n}$ as an estimator of λ? (Choose all that apply.)

- [x] $\mathbb{E}\left[\dfrac{1}{\overline{X}_n}\right] - \lambda$ ✔
- [x] $\mathbb{E}\left[\dfrac{1}{\overline{X}_n}\right] - \dfrac{1}{\mathbb{E}[X_i]}$ ✔
- [x] $\mathbb{E}\left[\dfrac{1}{\overline{X}_n}\right] - \dfrac{1}{\mathbb{E}[\overline{X}_n]}$ ✔
- [ ] $\dfrac{1}{\mathbb{E}[X_i]} - \lambda$
- [ ] $\dfrac{1}{\mathbb{E}[X_i]} - \dfrac{1}{\mathbb{E}[\overline{X}_n]}$

✔

Which of the following are properties of $\dfrac{1}{\overline{X}_n}$ as an estimator of λ? (Choose all that apply.)

- [x] consistent ✔
- [ ] unbiased

✔

Solution:

- By the (strong/weak) law of large numbers

$$\overline{X}_n = \frac{\sum_{i=1}^n X_i}{n} \xrightarrow[n\to\infty]{a.s/\mathbf{P}} \mathbb{E}[X_i] = \frac{1}{\lambda}.$$

- On the other hand, by the continuous mapping theorem

$$\frac{1}{\overline{X}_n} \xrightarrow[n\to\infty]{a.s/\mathbf{P}} \frac{1}{\mathbb{E}[X_i]} = \lambda.$$

- Hence, we can answer the last part immediately: $\frac{1}{\overline{X}_n}$ is a consistent estimator of λ.

- However,

$$\mathbb{E}\left[\frac{1}{\overline{X}_n}\right] \neq \frac{1}{\mathbb{E}\left[\overline{X}_n\right]} = \lambda.$$

So the bias of $\frac{1}{\overline{X}_n}$ as an estimator of $\lambda = \frac{1}{\mathbb{E}[X_i]} = \frac{1}{\mathbb{E}\left[\overline{X}_n\right]}$ is

$$\text{Bias} = \mathbb{E}\left[\frac{1}{\overline{X}_n}\right] - \frac{1}{\mathbb{E}\left[\overline{X}_n\right]}.$$

Remark: Since the function $\frac{1}{x}$ is convex (by the shape of its graph or by $\left(\frac{1}{x}\right)'' = \frac{2}{x^3} > 0$), Jensen's inequality gives $\mathbb{E}\left[\frac{1}{\overline{X}_n}\right] > \frac{1}{\mathbb{E}\left[\overline{X}_n\right]}$ and hence the bias is greater than zero.

提交 你已经尝试了2次（总共可以尝试2次）

Answers are displayed within the problem

Review: Central Limit Theorem

1/1 point (graded)

The **Central Limit Theorem** states that if $X_1, \ldots, X_n$ are i.i.d. and

$$\mathbb{E}[X_1] = \mu < \infty \quad ; \qquad \text{Var}(X_1) = \sigma^2 < \infty,$$

then

$$\sqrt{n}\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - \mu\right] \xrightarrow[n\to\infty]{(d)} Z \qquad \text{where } Z \sim \mathcal{N}(0, ?)\,.$$

What is $\text{Var}(Z)$? (Express your answer in terms of n, μ and σ).

$\text{Var}(Z) =$ sigma^2 ✔ **Answer:** sigma^2

σ^2

STANDARD NOTATION

Solution:

For any n,

$$\text{Var}\sqrt{n}\left(\overline{X}_n - \mu\right) = n\text{Var}\left(\overline{X}_n\right) = \text{Var}(X_i) = \sigma^2.$$

The central limit theorem states as $n \to \infty$, the distribution of $\sqrt{n}\left(\overline{X}_n - \mu\right)$ becomes Gaussian with the variance above (and mean 0); that is,

$$\sqrt{n}\left(\overline{X}_n - \mu\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0, \sigma^2)\,.$$

Note: The variance of Z is called the **asymptotic variance** of $\overline{X}_n$, even though it equals the variance of $\sqrt{n}\overline{X}_n$.