11. When g is Not Invertible

Delta Method for Non-invertible g: Asymptotic Variance

1/1 point (graded)

Let $(Z_n)_{n\geq 1}$ denote an asymptotically normal sequence with **known asymptotic variance** 1:

$$\sqrt{n}\,(Z_n - \mu) \xrightarrow[n\to\infty]{(d)} N\,(0,1)\,.$$

Let

$$\begin{aligned} g:\ & \mathbb{R} \to \mathbb{R} \\ & x \mapsto \sqrt{2}\sin{(x)}\,. \end{aligned}$$

Notice that g is not invertible, but the Delta method still applies.

Which of the following can be concluded by applying the Delta method? (Choose all that apply. Refer to the slides.)

- [x] The sequence $(g\,(Z_n))_{n\geq 1}$ is asymptotically normal. ✔
- [ ] For each fixed value of n, the random variable $g\,(Z_n)$ is a Gaussian.
- [ ] The asymptotic variance of the sequence $(g\,(Z_n))_{n\geq 1}$ is $\sqrt{2}\cos{(\mu)}$.
- [x] The asymptotic variance of the sequence $(g\,(Z_n))_{n\geq 1}$ is $2\cos^2{(\mu)}$. ✔
- [ ] The asymptotic variance of the sequence $(g\,(Z_n))_{n\geq 1}$ is $\sqrt{2}$.
- [ ] The asymptotic variance of the sequence $(g\,(Z_n))_{n\geq 1}$ is 2.

✔

Solution:

We examine the choices in order.

1. “The sequence $(g\,(Z_n))_{n\geq 1}$ is asymptotically normal." is correct. The Delta method states that continuously differentiable function applied to an asymptotically normal sequence of random variables is again asymptotically normal.

2. “For each fixed value of n, the random variable $g\,(Z_n)$ is a Gaussian." is incorrect. The Delta method only concerns **asymptotic** normality, and does not conclude for finite n that the random variable $g\,(Z_n)$ is normal (or Gaussian).

3. For the final four choices, let us find the asymptotic variance of $g\,(Z_n)$. The function g has derivative

$$g'\,(x) = \sqrt{2}\cos{(x)}\,.$$

By the Delta method, the asymptotic variance of $g\,(Z_n)$ is $g'\,(\mu)^2\mathsf{Var}\,(Z) = 2\cos^2{(\mu)}$.

Remark: Note that the asymptotic variance of $g\,(Z_n)$ depends on the unknown parameter μ, even though the asymptotic variance of Z_n is known to be 1.

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Answers are displayed within the problem

Delta Method for Non-invertible g: Confidence Interval

1/1 point (graded)

Continuing from above, let $Z_n = \overline{X}_n$ where $X_1, X_2, \ldots X_n$ are i.i.d. with mean μ and known variance 1. Then the CLT gives

$$\sqrt{n}\left(\overline{X}_n - \mu\right) \xrightarrow[n\to\infty]{(d)} N\left(0, 1\right).$$

As above, let $g = \sqrt{2}\sin\left(x\right)$. You estimates $\theta = g\left(\mu\right)$ by the consistent estimator $\hat{\theta} = g\left(\overline{X}_n\right)$. Use the "plug-in" method to construct a confidence interval for $\theta = g\left(\mu\right)$ at **asymptotic** level $1-\alpha$. (Choose all that apply. Some choices are equivalent to each other.)

- [x] $\left[\sqrt{2}\sin\left(\overline{X}_n\right) - q_{\alpha/2}\frac{\sqrt{2}|\cos\left(\overline{X}_n\right)|}{\sqrt{n}}, \sqrt{2}\sin\left(\overline{X}_n\right) + q_{\alpha/2}\frac{\sqrt{2}|\cos\left(\overline{X}_n\right)|}{\sqrt{n}}\right]$ ✔
- [x] $\left[g\left(\overline{X}_n\right) - q_{\alpha/2}\frac{|g'\left(\overline{X}_n\right)|}{\sqrt{n}}, g\left(\overline{X}_n\right) + q_{\alpha/2}\frac{|g'\left(\overline{X}_n\right)|}{\sqrt{n}}\right]$ ✔
- [ ] $\left[\sqrt{2}\sin\left(\mu\right) - q_{\alpha/2}\frac{\sqrt{2}|\cos\left(\mu\right)|}{\sqrt{n}}, \sqrt{2}\sin\left(\mu\right) + q_{\alpha/2}\frac{\sqrt{2}|\cos\left(\mu\right)|}{\sqrt{n}}\right]$
- [ ] $\left[g\left(\mu\right) - q_{\alpha/2}\frac{|g'\left(\mu\right)|}{\sqrt{n}}, g\left(\mu\right) + q_{\alpha/2}\frac{|g'\left(\mu\right)|}{\sqrt{n}}\right]$
- [ ] $\left[\sqrt{2}\arcsin\left(\overline{X}_n\right) - q_{\alpha/2}\frac{\sqrt{2}|\cos\left(\overline{X}_n\right)|}{\sqrt{n}}, \sqrt{2}\arcsin\left(\overline{X}_n\right) + q_{\alpha/2}\frac{\sqrt{2}|\cos\left(\overline{X}_n\right)|}{\sqrt{n}}\right]$
- [ ] $\left[g^{-1}\left(\overline{X}_n\right) - q_{\alpha/2}\frac{|g'\left(\overline{X}_n\right)|}{\sqrt{n}}, g^{-1}\left(\overline{X}_n\right) + q_{\alpha/2}\frac{|g'\left(\overline{X}_n\right)|}{\sqrt{n}}\right]$

✔

Solution:

Right off the bat, we can eliminate the middle two choices; they are not confidence intervals because they are in terms of the unknown true parameter μ.

From the last problem, we know that

$$\sqrt{n}\left(g\left(\overline{X}_n\right) - g\left(\mu\right)\right) \xrightarrow[d.]{n\to\infty} \mathcal{N}\left(0, \tau^2\right) \qquad \text{where } \tau^2 = \left(g'\left(\mu\right)\right)^2 \mathsf{Var}\left(X\right) = \left(g'\left(\mu\right)\right)^2.$$

This implies

$$\frac{\sqrt{n}}{\tau}\left(g\left(\overline{X}_n\right) - g\left(\mu\right)\right) \xrightarrow[d.]{n\to\infty} \mathcal{N}\left(0, 1\right) \text{ where } \tau^2 = \left(g'\left(\mu\right)\right)^2.$$

We follow the usual procedure for confidence intervals:

$$\mathbf{P}\left(\frac{\sqrt{n}}{\tau}\left|g\left(\overline{X}_n\right)-g\left(\mu\right)\right|<q_{\alpha/2}\right)=1-\alpha.$$

Manipulate the event with the probability above:

$$\begin{aligned}\frac{\sqrt{n}}{\tau}\left|g\left(\overline{X}_n\right)-g\left(\mu\right)\right|<q_{\alpha/2} &\iff -q_{\alpha/2}\frac{\tau}{\sqrt{n}}<g\left(\overline{X}_n\right)-g\left(\mu\right)<q_{\alpha/2}\frac{\tau}{\sqrt{n}}\\ &\iff g\left(\overline{X}_n\right)-q_{\alpha/2}\frac{\tau}{\sqrt{n}}<g\left(\mu\right)<g\left(\overline{X}_n\right)+q_{\alpha/2}\frac{\tau}{\sqrt{n}}\end{aligned}$$

Therefore:

$$g\left(\mu\right)\in\left[g\left(\overline{X}_n\right)-q_{\alpha/2}\frac{|g'\left(\mu\right)|}{\sqrt{n}},g\left(\overline{X}_n\right)+q_{\alpha/2}\frac{|g'\left(\mu\right)|}{\sqrt{n}}\right]\qquad\left(\tau=\sqrt{\left(g'\left(\mu\right)\right)^2}=|\,g'\left(\mu\right)|\right).$$

But this does not itself constitute a confidence interval, because μ shows up in both expressions. To remedy this, we apply the Plug-in method: substitute $g'\left(\mu\right)$ with $g'\left(\overline{X}_n\right)$ via Slutsky's theorem and the Continuous Mapping theorem.

This gives

$$g\left(\mu\right)\in\left(g\left(\overline{X}_n\right)-q_{\alpha/2}\frac{|g'\left(\overline{X}_n\right)|}{\sqrt{n}},g\left(\overline{X}_n\right)+q_{\alpha/2}\frac{|g'\left(\overline{X}_n\right)|}{\sqrt{n}}\right).$$

Which is the second choice. Equivalently, by plugging in $g\left(x\right)=\sqrt{2}\sin\left(x\right)$ and $g'\left(x\right)=\sqrt{2}\cos\left(x\right)$, we obtain the first choice.

Remark: Notice that the correct plug-in confidence intervals do not involve g^{-1}. What was necessary was to plug in $\hat{\mu}=\overline{X}_n$ into g and g'. The Delta method works even when g is non-invertible.

提交 你已经尝试了1次（总共可以尝试3次）

Answers are displayed within the problem

讨论

显示讨论

主题: Unit 2 Foundation of Inference:Lecture 5: Delta Method and Confidence Intervals / 11. When g is Not Invertible

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