The **multivariate linear model** can be described via the equation $Y = \mathbf{X}^T\boldsymbol{\beta} + \varepsilon$, where:

- $\mathbf{X} \in \mathbb{R}^p$ is the vector of **covariates** , also called **independent/explanatory** variables,
- $Y \in \mathbb{R}$ is the **dependent** variable,
- $\varepsilon \in \mathbb{R}$ is the noise, and
- $\boldsymbol{\beta} \in \mathbb{R}^p$ is the model parameter.

(**Note:** We may have also written $\boldsymbol{\beta}^T\mathbf{X}$ instead of $\mathbf{X}^T\boldsymbol{\beta}$. These are transposes of each other, but they are equal since they are both scalars. Recall that the transpose of a scalar is itself.)

If we have n observations $\{(\mathbf{X}_i, Y_i)\}$, then this determines n linear relationships, each of the form $Y_i = \mathbf{X}_i^T\boldsymbol{\beta} + \varepsilon_i$. We can stack these into a matrix equation:

n个数据，每个数据有i个x和i个y

$$\begin{aligned} Y_1 &= \mathbf{X}_1^T\boldsymbol{\beta} + \varepsilon_1 \\ Y_2 &= \mathbf{X}_2^T\boldsymbol{\beta} + \varepsilon_2 \\ &\vdots \\ Y_n &= \mathbf{X}_n^T\boldsymbol{\beta} + \varepsilon_n \end{aligned} \quad \leftrightarrow \quad \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1^T \\ \mathbf{X}_2^T \\ \vdots \\ \mathbf{X}_n^T \end{pmatrix} \boldsymbol{\beta} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix} \tag{10.1}$$

In this course, we typically condense the equation on the right into the form $\mathbf{Y} = \mathbb{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$.

“Model" versus “Regression":

The assumption that the random variable pair $(\mathbf{X}, Y)$ obeys the relationship $Y = \mathbf{X}^T\boldsymbol{\beta} + \varepsilon$ is an assumption on the *model*. Equivalently, we can assume that the regression function is linear: $\mu(x) = \mathbb{E}[Y|\mathbf{X} = \mathbf{x}] = \mathbf{x}^T\boldsymbol{\beta}$, with the understanding that $\mathbb{E}[\varepsilon] = 0$.

This allows us to perform **linear regression** , which consists of coming up with an estimator $\hat{\boldsymbol{\beta}}$ in an attempt to find the best-fitting guess $\hat{\boldsymbol{\beta}}$ for $\boldsymbol{\beta}$.

Note that we can always **perform** linear regression, even if the model is misspecified. There are many ways that things can go wrong! For example, the estimator may not be unique, or the estimator $\boldsymbol{\beta}$ may have huge variance. **This unit will help us understand when and why these issues occur.**

How does this relate to the single-variable setting?

只有一个x

Recall that in the previous section ($p = 1$), the model was $Y = a + bX + \varepsilon$ for scalar values of a, b, X, Y, ε. To write this down using the notation in the multivariate setting, take

$$\beta = \begin{pmatrix} a \\ b \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} 1 \\ X \end{pmatrix}.$$

To extrapolate from the single-variable case, consider the p-dimensional linear model with intercept β_0 which looks like

$$\boxed{Y = \beta_0 + \beta_1 X^{(1)} + \beta_2 X^{(2)} + \cdots + \beta_p X^{(p)} + \varepsilon.}$$

The natural analogy is to take $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_p)^T \in \mathbb{R}^{p+1}$ and $\mathbf{X} = (1, X^{(1)}, \ldots, X^{(p)}) \in \mathbb{R}^{p+1}$. Therefore, whenever we have an intercept in the model, we extend the dimension by 1 and take the first coordinate of $\mathbf{X}$ to always be 1.

(On the other hand, if we did not have an intercept in our model, then we would not need β_0. In this case, for a typical p-dimensional model, we usually write $\mathbf{X} = (X^{(1)}, \ldots, X^{(p)})$, a p-dimensional vector.)

This technical distinction won't affect theoretical analyses. **Unless otherwise specified, we will always take $\mathbf{X}$ and β to be generic vectors in $\mathbb{R}^{\boxed{p}}$.**

Linear Regression as a Statistical Model I

1/2 points (graded)
Consider the linear regression model introduced in the slides and lecture, restated below:

Linear regression model : $(\mathbf{X}_1, Y_1), \ldots, (\mathbf{X}_n, Y_n) \in \mathbb{R}^d \times \mathbb{R}$ are i.i.d from the linear regression model $Y_i = \boldsymbol{\beta}^\top \mathbf{X}_i + \varepsilon_i, \quad \varepsilon_i \stackrel{iid}{\sim} \mathcal{N}(0,1)$ for an unknown $\boldsymbol{\beta} \in \mathbb{R}^d$ and $\mathbf{X}_i \sim \mathcal{N}_d(0, I_d)$ independent of ε_i.

Suppose that $\boldsymbol{\beta} = \mathbf{1} \in \mathbb{R}^d$, which denotes the d-dimensional vector with all entries equal to 1.

What is the mean of Y_1?

$\mathbb{E}[Y_1] =$ 0 ✔ **Answer:** 0

What is the variance of Y_1? (Express your answer in terms of d.)

Var $(Y_1) =$ d^2 ✘ **Answer:** d+1

d^2

STANDARD NOTATION

Solution:

By definition of the model and setting $\boldsymbol{\beta} = \mathbf{1}$, we have

$$Y_1 = \boldsymbol{\beta}^T \mathbf{X}_1 + \varepsilon_1 = \mathbf{1}^T \mathbf{X}_1 + \varepsilon_1 = \boxed{\varepsilon_1 + \sum_{j=1}^{d} X_{1,j}.}$$

where $X_{i,j}$ denotes the j'th coordinate of $\mathbf{X}_i \sim \mathcal{N}(0, I_d)$. By linearity of expectation,

$$\mathbb{E}[Y_1] = \mathbb{E}[\varepsilon_1] + \sum_{j=1}^{d} \mathbb{E}[X_{1,j}] = 0$$

Next we compute the variance. Since $X_{1,1}, \ldots, X_{1,d}, \varepsilon_i$ are mutually independent, the variance is additive:

$$\mathbf{Var}[Y_1] = \boxed{\mathbf{Var}[\varepsilon_1] + \sum_{j=1}^{d} \mathbf{Var}[X_{1,j}] = d + 1}$$

每个dimension会有一个方差

because $X_{1,1}, \ldots, X_{1,d}, \varepsilon_1 \stackrel{iid}{\sim} \mathcal{N}(0,1)$.

Submit You have used 2 of 2 attempts

Answers are displayed within the problem

Linear Regression as a Statistical Model II

2/2 points (graded)
Recall the linear regression model as introduced above in the previous question. This model is parametric, although it is not written in the standard notation previously introduced for parametric statistical models. In this problem, you will explicitly write the linear regression model as a parametric statistical model.

We will represent the linear regression model as an ordered pair $(E, \{P_\beta\}_{\beta \in \Theta})$. Here E denotes the sample space associated to the distribution P_β, where P_β is defined as follows for $\boldsymbol{\beta} \in \mathbb{R}^d$:

The random ordered pair $(\mathbf{X}, Y) \subset \mathbb{R}^d \times \mathbb{R}$ *is distributed as* P_β *if:*

- $\mathbf{X} \sim \mathcal{N}(0, I_d)$, 有d个维度
- $Y \sim \beta^T X + \varepsilon$, *where* $\varepsilon \sim \mathcal{N}(0,1)$ *and* ε *is independent of* $\mathbf{X}$.

The set Θ in the ordered pair $(E, \{P_\beta\}_{\beta \in \Theta})$ denotes the parameter space for this model.

The sample space for the linear regression model can be written $E = \mathbb{R}^k$ for some integer k. What is k? (Express your answer in terms of d.)

Hint: You should use the fact that $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ for all integers $m, n \geq 0$.

$k =$ d+1 ✔ **Answer:** d+1

$d + 1$

The parameter space for the model can be written as $\Theta = \mathbb{R}^j$ for some integer j. What is j? (Express your answer in terms of d.)

$j =$ d ✔ **Answer:** d

d

STANDARD NOTATION

Solution:

The statistical experiment is given by the iid sample $(\mathbf{X}_1, Y_1), \ldots, (\mathbf{X}_n, Y_n)$. Where $\mathbf{X}_i \sim \mathcal{N}(0, I_d)$ and $Y_i = \boldsymbol{\beta}^T \mathbf{X}_i + \varepsilon_i$ for $\varepsilon_i \sim \mathcal{N}(0,1)$ and some true parameter $\boldsymbol{\beta} \in \mathbb{R}^d$. In particular, $\mathbf{X}_i \in \mathbb{R}^d$ and $Y_i \in \mathbb{R}$. Therefore, $(\mathbf{X}_i, Y_i) \in \mathbb{R}^{d+1}$, so indeed $E = R^{d+1}$ is the sample space for this model. We conclude that $k = d + 1$.

This model is parametrized by the vector $\boldsymbol{\beta} \in \mathbb{R}^d$. That is, specifying the value of $\boldsymbol{\beta}$ uniquely determines the distribution of $(\mathbf{X}_1, Y_1), \ldots, (\mathbf{X}_n, Y_n)$. Hence, the parameter is $\boldsymbol{\beta}$, and the parameter space is $\Theta = \mathbb{R}^d$. We conclude that $j = d$.

Submit You have used 2 of 2 attempts

Answers are displayed within the problem

Discussion

Show Discussion

Topic: Unit 6 Linear Regression:Lectures 19: Linear Regression 1 / 11. Multivariate Regression: Definitions, Modeling, and Matrix LSE