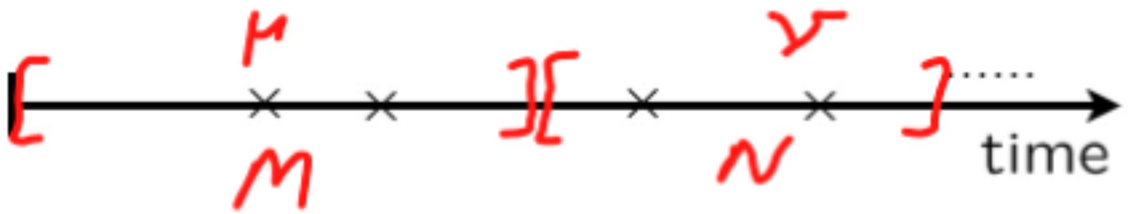
LECTURE 23: More on the Poisson process

- The sum of independent Poisson r.v.s
- Merging and splitting
- Random incidence

The sum of independent Poisson random variables

- Poisson process of rate $\lambda = 1$



- Consecutive intervals of length μ and ν

$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}$$

- Numbers of arrivals during these intervals: M and N Poisson($\lambda\tau$)

- M: Poisson(μ)
- N: Poisson(ν)
- Independent? Yes
- $M + N$: Poisson($\mu + \nu$)

The sum of independent Poisson random variables, with means/parameters μ and ν, is Poisson with mean/parameter $\mu + \nu$

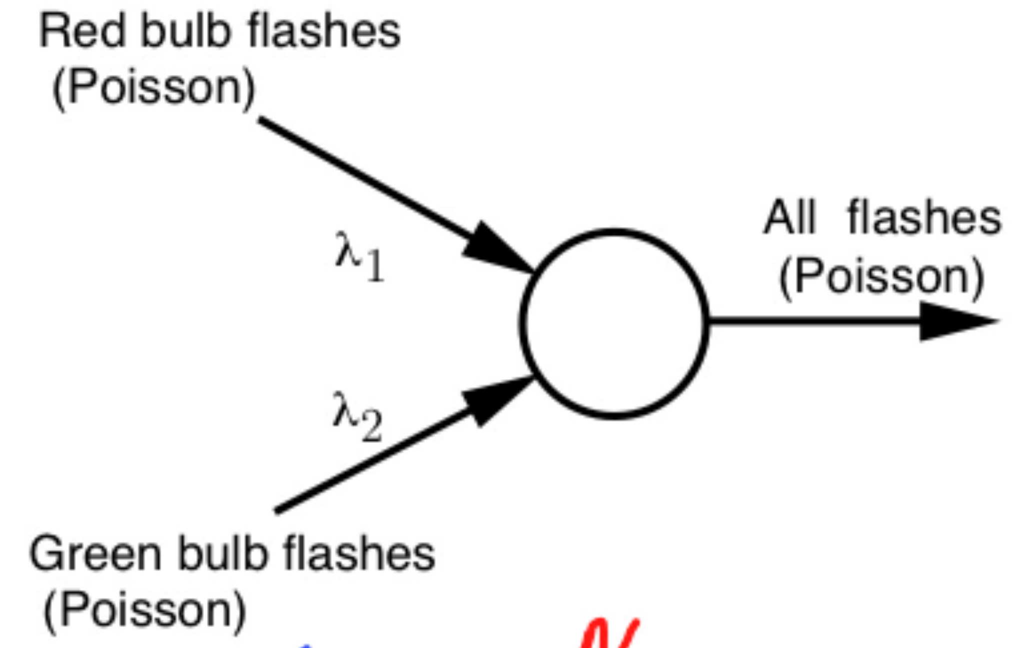
Merging of independent Poisson processes

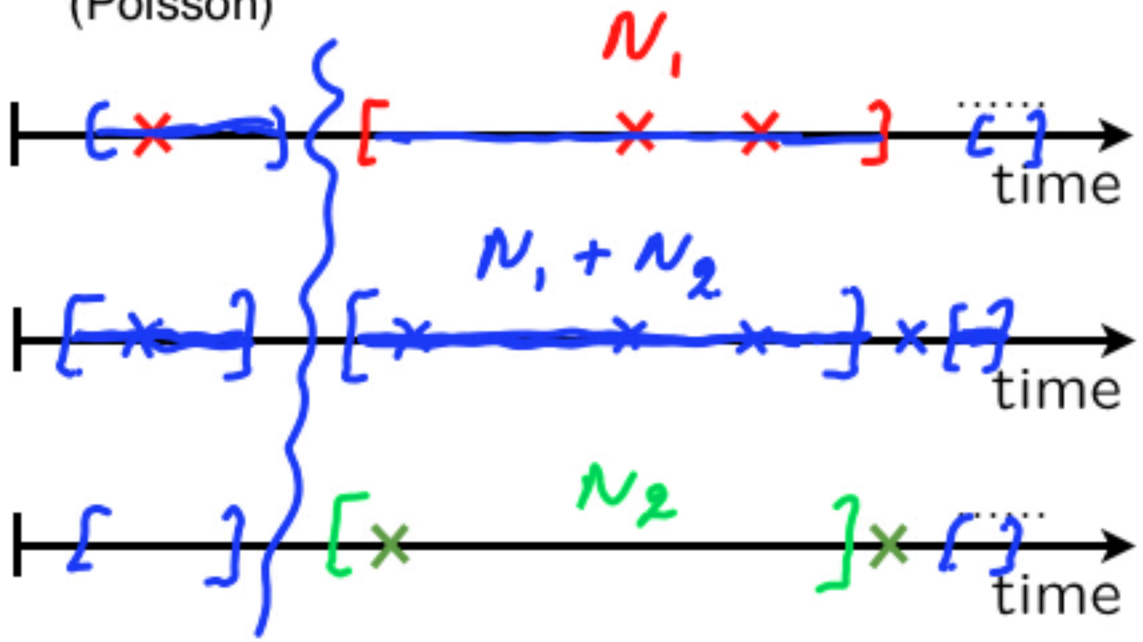
		$1-\lambda_1\delta$	$\lambda_1\delta$	$O(\delta^2)$
		0	1	≥ 2
$1-\lambda_2\delta$	0	$(1-\lambda_1\delta)\cdot(1-\lambda_2\delta)$	$\lambda_1\delta(1-\lambda_2\delta)$	
$\lambda_2\delta$	1	$\lambda_2\delta(1-\lambda_1\delta)$	$\lambda_1\lambda_2\delta^2$	
$O(\delta^2)$	≥ 2			

$\geq 2 : o(\delta^2)$

$0 : 1-(\lambda_1+\lambda_2)\delta$

$1 : (\lambda_1+\lambda_2)\delta$



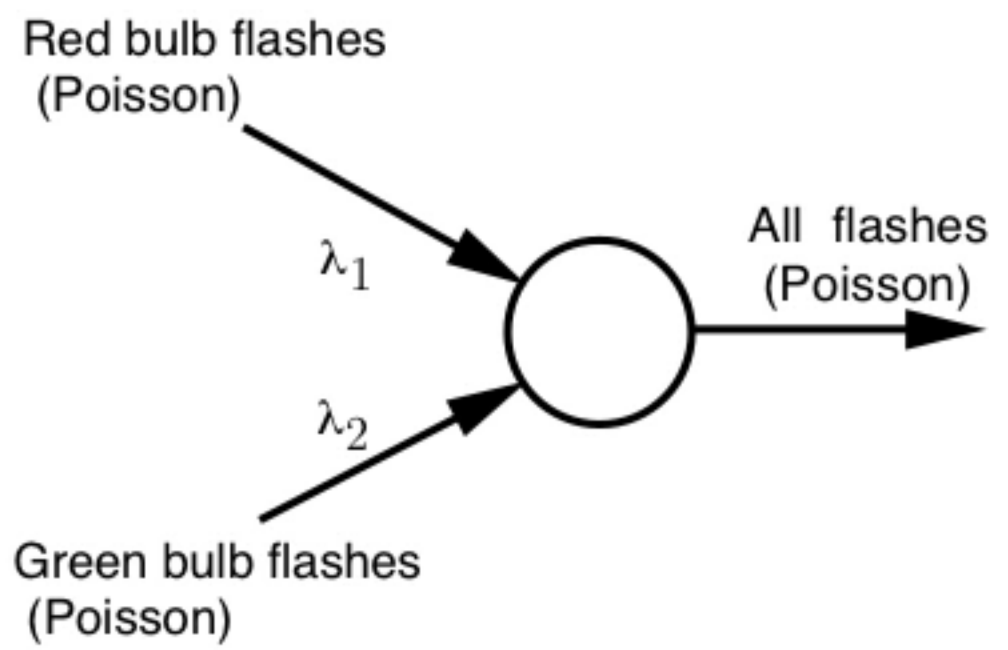


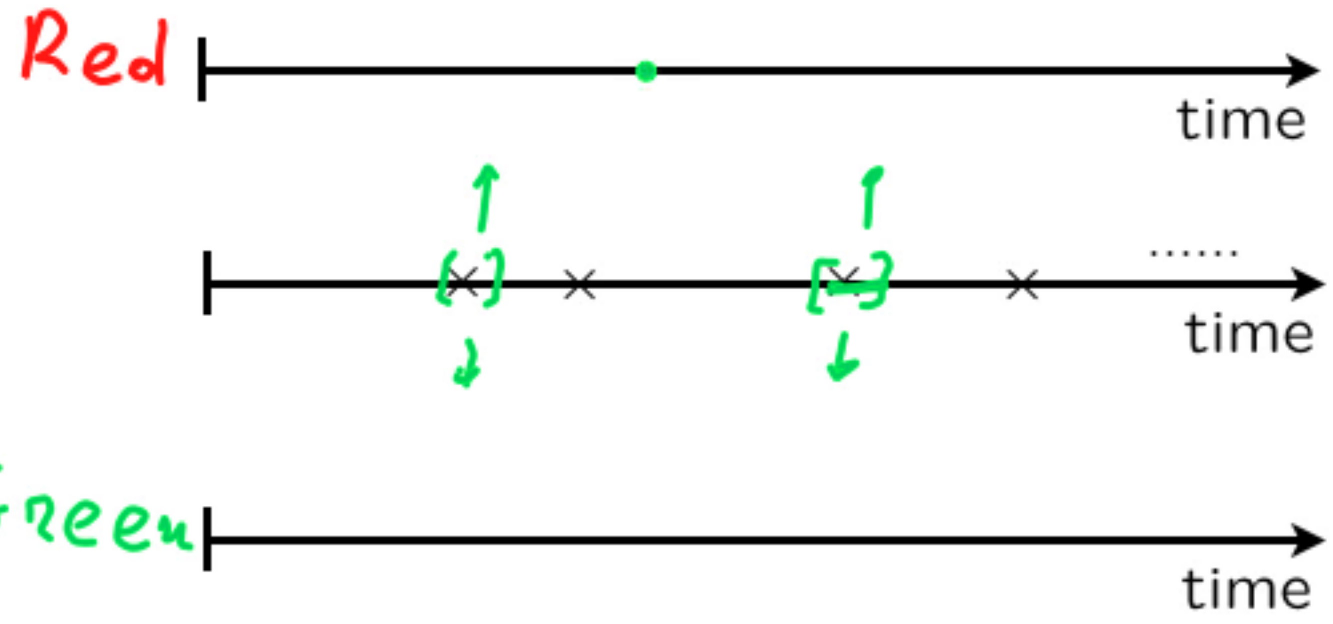
Merged process: Poisson$(\lambda_1+\lambda_2)$

Where is an arrival of the merged process coming from?

$\mathbf{P}(\text{Red} \mid \text{arrival at time } t) = \lambda_1/(\lambda_1+\lambda_2)$

		$1-\lambda_1\delta$	$\lambda_1\delta$	$O(\delta^2)$
		0	1	≥ 2
$1-\lambda_2\delta$	0	$1-(\lambda_1+\lambda_2)\delta$	$\lambda_1\delta$	
$\lambda_2\delta$	1	$\lambda_2\delta$	$O(\delta^2)$	
$O(\delta^2)$	≥ 2			



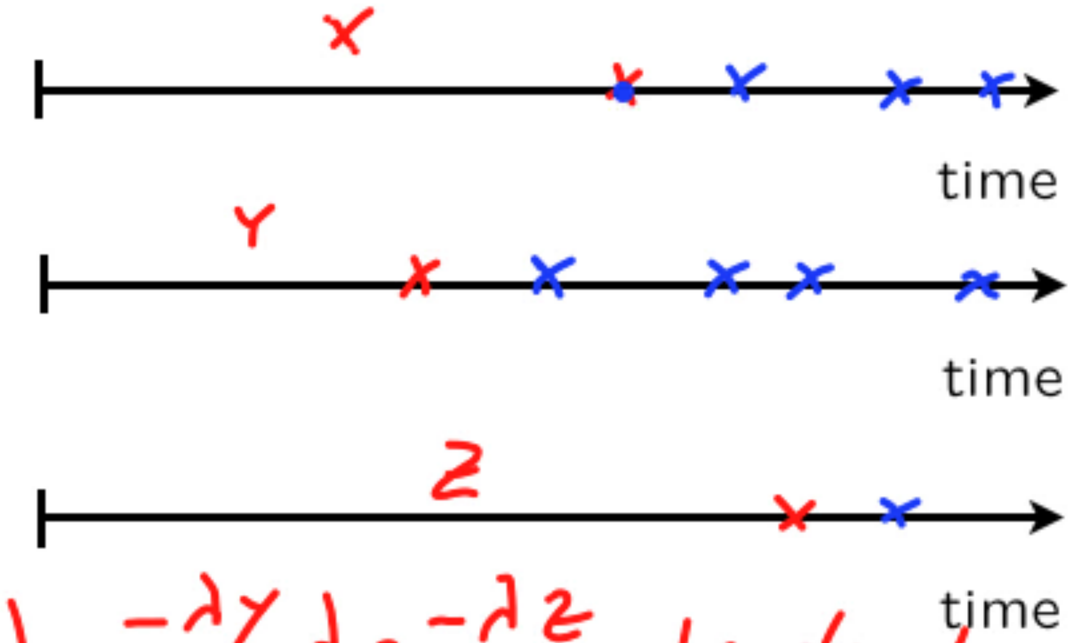


$\mathbf{P}(k\text{th arrival is Red}) = \lambda_1/(\lambda_1+\lambda_2)$

- Independence for different arrivals

$\mathbf{P}(\text{4 out of first 10 arrivals are Red}) = \binom{10}{4}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^4\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^6$

The time the first (or the last) light bulb burns out



- Three lightbulbs
  - independent lifetimes X, Y, Z; exponential(λ)
- Find expected time until first burnout $= \boxed{1/3\lambda}$

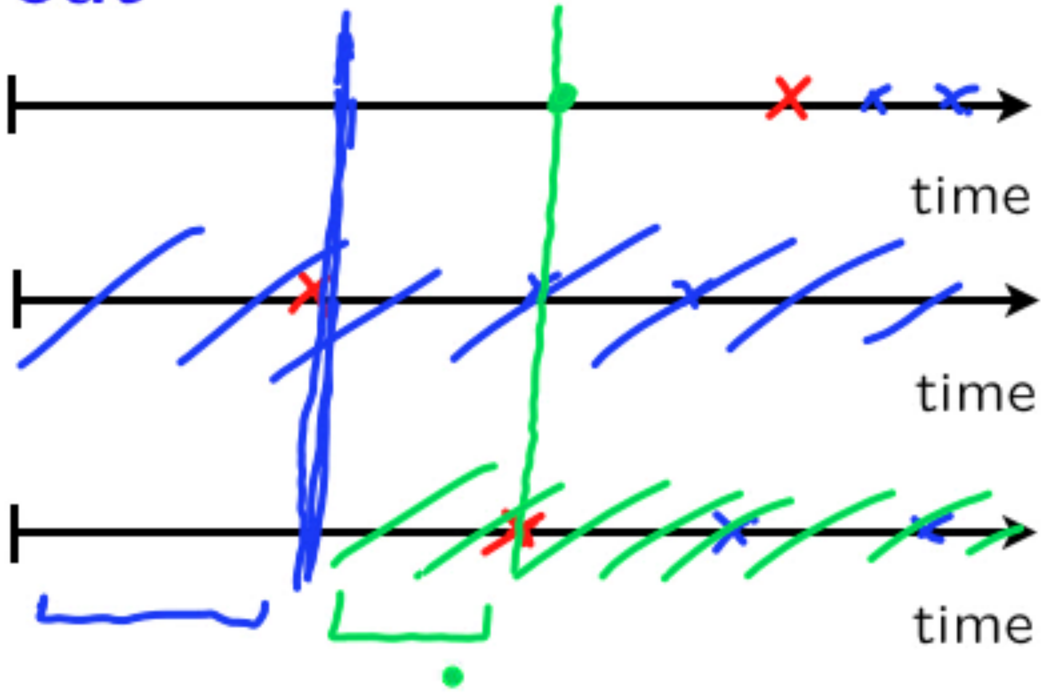
$$E[\min\{X,Y,Z\}] = \int_0^\infty\int_0^\infty\int_0^\infty \min\{x,y,z\}\,\lambda e^{-\lambda x}\lambda e^{-\lambda y}\lambda e^{-\lambda z}\,dx\,dy\,dz$$

$$P(\underbrace{\min\{X,Y,Z\}}_{\mathrm{Exp}(3\lambda)} \geq t) = P(X \geq t, Y \geq t, Z \geq t) = e^{-\lambda t}e^{-\lambda t}e^{-\lambda t} = e^{-3\lambda t}$$

- X, Y, Z: first arrivals in independent Poisson processes
- Merged process: Poisson(3λ)
- $\min\{X,Y,Z\}$: 1st arrival in merged process $\leftarrow \mathrm{Exp}(3\lambda)$

The time the first (or the last) light bulb burns out

- Three lightbulbs
  - independent lifetimes X, Y, Z; exponential(λ)
- Find expected time until all burn out

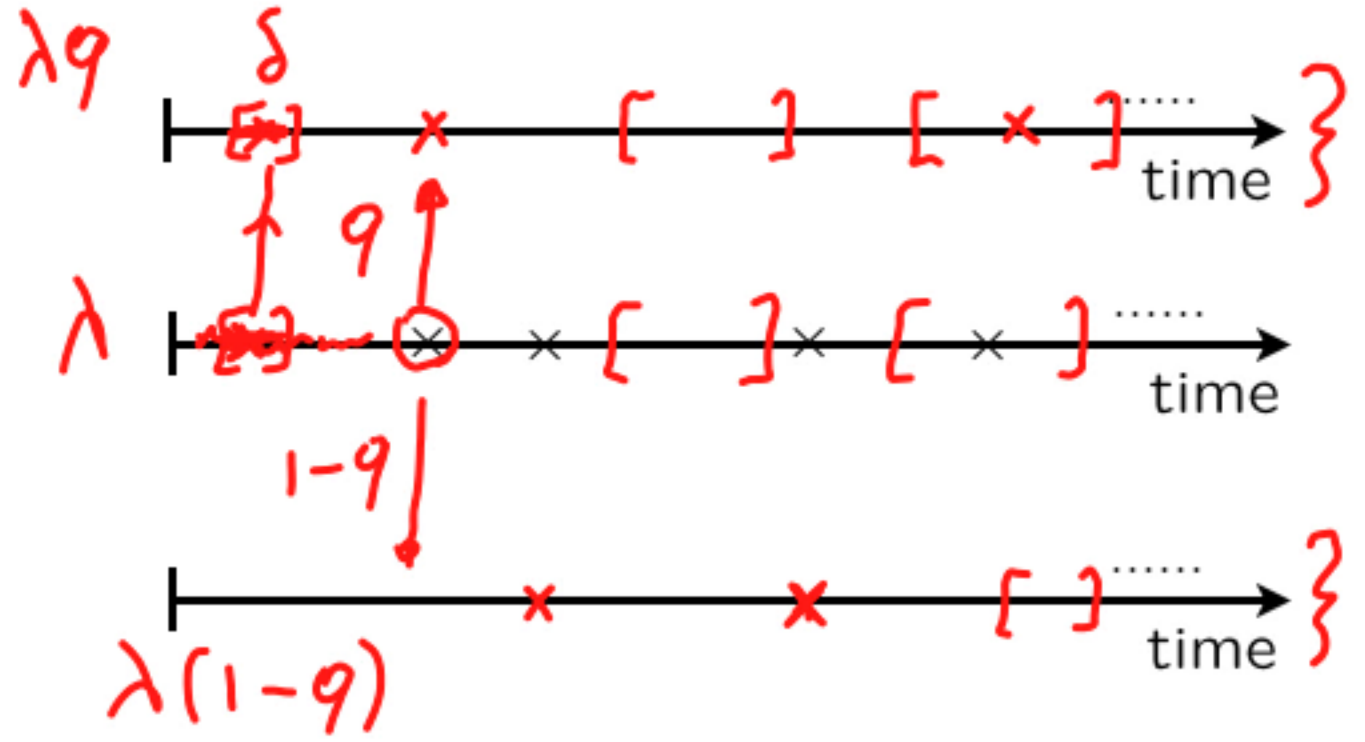


$\max\{X, Y, Z\}$

$\frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda}$

Splitting of a Poisson process

- Split arrivals into two streams, using independent coin flips of a coin with bias q
  - assume that coin flips are independent from the original Poisson process



≥ 2 $\quad o(\delta^2)$

$= 1$ $\quad \lambda \delta q$

> Resulting streams are Poisson, rates λq, $\lambda(1-q)$

- Are the two resulting streams independent?

  Surprisingly, yes!

“Random incidence” in the Poisson process

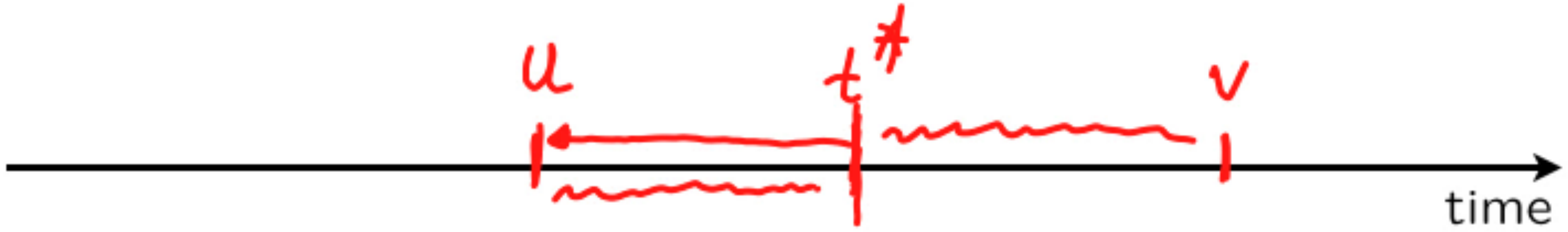
- Poisson process that has been running forever

- Believe that $\lambda = 4$/hour, so that $\mathbf{E}[T_k] = \frac{1}{4}$ hrs $= 15$ mins

- Show up at some time and measure interarrival time

  – do it many times, average results, see something around 30 mins! Why?

"Random incidence" in the Poisson process — analysis

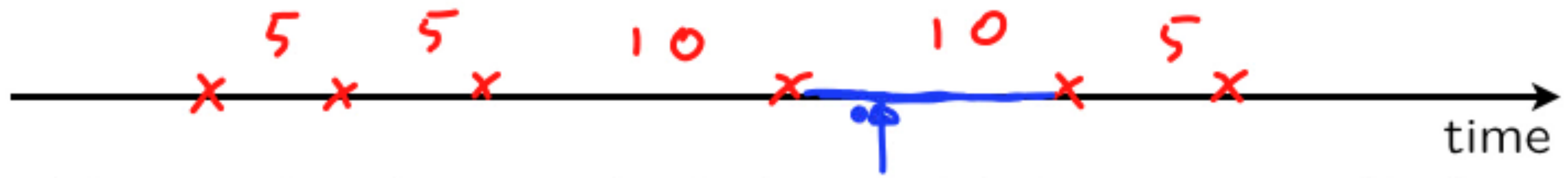


- Arrive at time t^*
- U: last arrival time
- V: next arrival time
- $V - U = \underbrace{(V - t^*)}_{Exp(\lambda)} + \underbrace{(t^* - U)}_{Exp(\lambda)}$
- $\mathbf{E}[V - U] = \frac{1}{\lambda} + \frac{1}{\lambda} = \frac{2}{\lambda}$

$\frac{1}{\lambda}$

- $V - U$: interarrival time you see, versus kth interarrival time

Random incidence "paradox" is not special to the Poisson process



- **Example:** interrarival times, i.i.d., equally likely to be 5 or 10 minutes

  expected value of kth interarrival time: $\frac{1}{2} \cdot 5 + \frac{1}{2} \cdot 10 = 7.5$

- you show up at a "random time"

  $\mathbf{P}$(arrive during a 5-minute interarrival interval) $= \frac{1}{3}$

  expected length of interarrival interval during which you arrive $= \frac{1}{3} \cdot 5 + \frac{2}{3} \cdot 10$

  ≈ 8.3

- Calculation generalizes to "renewal processes:"
  i.i.d. interarrival times, from some general distribution

- "Sampling method" matters

Different sampling methods can give different results

- Average family size? $\frac{3}{4}\cdot 1 + \frac{1}{4}\cdot 6$
  - look at a "random" family (uniformly chosen)
  - look at a "random" person's (uniformly chosen) family $\frac{3}{9}\cdot 1 + \frac{6}{9}\cdot 6$

- Average bus occupancy? $\boxed{0}$ $\boxed{50}$
  - look at a "random" bus (uniformly chosen)
  - look at a "random" passenger's bus

- Average class size?