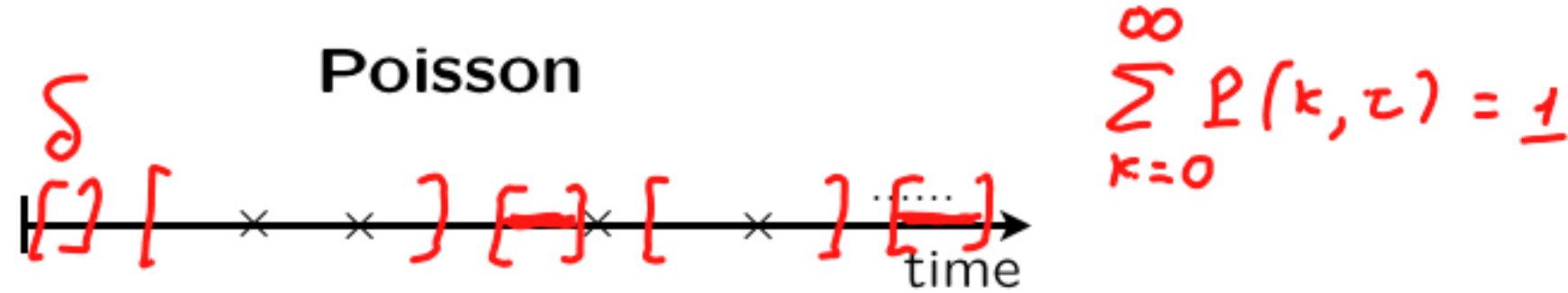


## LECTURE 22: The Poisson process

- Definition of the Poisson process
  - applications
- Distribution of number of arrivals
- The time of the  $k$ th arrival
- Memorylessness
- Distribution of interarrival times

# Definition of the Poisson process



- Numbers of arrivals in disjoint time intervals are **independent**

$P(k, \tau)$  = Prob. of  $k$  arrivals in interval of duration  $\tau$

In order to avoid multiple arrivals at the same time point, we introduce one more assumption.

- Small interval probabilities:**

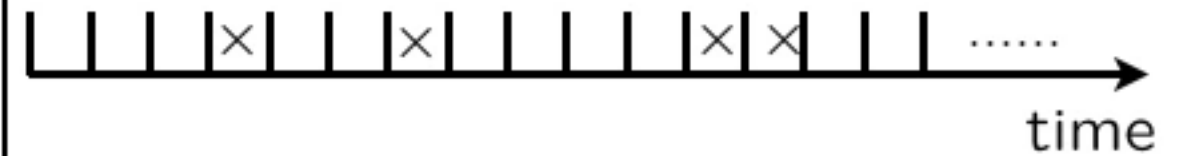
For VERY small  $\delta$ :

$$P(k, \delta) \approx \begin{cases} 1 - \lambda\delta & \text{if } k = 0 \\ \lambda\delta & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases} \quad P(k, \delta) = \begin{cases} 1 - \lambda\delta + O(\delta^2) & \text{if } k = 0 \\ \lambda\delta + O(\delta^2) & \text{if } k = 1 \\ 0 + O(\delta^2) & \text{if } k > 1 \end{cases}$$

$$\frac{O(\delta^2)}{\delta} \xrightarrow{\delta \rightarrow 0} 0$$

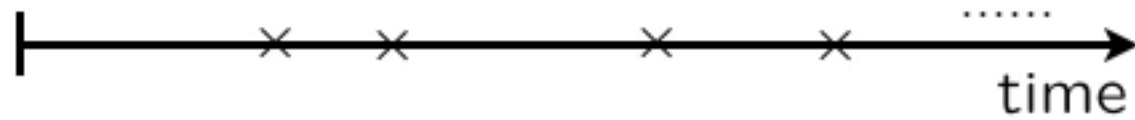
$\lambda$ : “arrival rate”

## Bernoulli



- Independence
- Time homogeneity:**  
Constant  $p$  at each slot

## Applications of the Poisson process



- Deaths from horse kicks in the Prussian army (1898)
- Particle emissions and radioactive decay
- Photon arrivals from a weak source
- Financial market shocks
- Placement of phone calls, service requests, etc. •



Siméon Denis Poisson  
(1781-1840)

## The Poisson PMF for the number of arrivals



- $N_\tau$ : arrivals in  $[0, \tau]$   $P(k, \tau) = \mathbf{P}(N_\tau = k)$

$n = \tau/\delta$  intervals/slots of length  $\delta$  *← small*

$\mathbf{P}(\text{some slot contains two or more arrivals})$

$$\leq \sum_i \mathbf{P}(\text{slot } i \text{ has } \geq 2 \text{ arrivals})$$

$$= \frac{\tau}{\delta} O(\delta^2) \xrightarrow{\delta \rightarrow 0} 0$$

$\mathbf{P}(k \text{ arrivals in Poisson}) \approx \mathbf{P}(k \text{ slots have arrival})$

$N_\tau \approx \text{binomial} \quad p = \lambda\delta + O(\delta^2)$

$np = \lambda\tau + O(\delta) \approx \lambda\tau$

### Bernoulli

$$p_S(k) = \frac{n!}{(n-k)!k!} \cdot p^k (1-p)^{n-k},$$

$$k = 0, \dots, n$$

$$\lambda = np \quad n \rightarrow \infty \quad p \rightarrow 0$$

For fixed  $k = 0, 1, \dots$ ,

$$p_S(k) \rightarrow \frac{\lambda^k}{k!} e^{-\lambda},$$

$$P(k, \tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \quad k = 0, 1, \dots$$

## Mean and variance of the number of arrivals

$$P(k, \tau) = \mathbf{P}(N_\tau = k) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \quad k = 0, 1, \dots$$

$$\mathbf{E}[N_\tau] = \sum_{k=0}^{\infty} k \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!} = \dots = \lambda\tau$$

$$N_\tau \approx \text{Binomial}(n, p)$$

$$n = \tau/\delta, \quad p = \lambda\delta + O(\delta^2)$$

$$\mathbf{E}[N_\tau] \approx np \approx \lambda\tau$$

$$\text{var}(N_\tau) \approx np(1-p) \approx \lambda\tau$$

$$\mathbf{E}[N_\tau] = \lambda\tau$$

$$\text{var}(N_\tau) = \lambda\tau$$

$$\lambda = \frac{\mathbf{E}[N_\tau]}{\tau}$$



## Example

- You get email according to a Poisson process, at a rate of  $\lambda = 5$  messages per hour.

$$E[N_\tau] = \lambda\tau$$

$$\text{var}(N_\tau) = \lambda\tau$$

- Mean and variance of mails received during a day =  $5 \cdot 24$
- $P(\text{one new message in the next hour}) = P(1,1) = 5e^{-5}$

$$P(k, \tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \quad k = 0, 1, \dots$$

- $P(\text{exactly two messages during each of the next three hours}) =$

$$\begin{array}{|c|c|c|} \hline 2 & 2 & 2 \\ \hline \end{array} \quad \left( P(2,1) \right)^3 = \left( \frac{5^2 e^{-5}}{2} \right)^3$$

## The time $T_1$ until the first arrival



$$P(k, \tau) = \frac{(\lambda \tau)^k e^{-\lambda \tau}}{k!}, \quad k = 0, 1, \dots$$

- Find the CDF:  $P(T_1 \leq t) =$

$$= 1 - P(T_1 > t) = 1 - P(0, t) = 1 - e^{-\lambda t}$$

$$f_{T_1}(t) = \lambda e^{-\lambda t}, \quad \text{for } t \geq 0$$

Exponential( $\lambda$ )

**Memorylessness:** conditioned on  $T_1 > t$ ,  
the PDF of  $T_1 - t$  is again exponential

## The time $Y_k$ of the $k$ th arrival

$$P(k, \tau) = \frac{(\lambda \tau)^k e^{-\lambda \tau}}{k!}, \quad k = 0, 1, \dots$$

- Can derive its PDF by first finding the CDF
- More intuitive argument:

$$P(Y_k \leq \gamma) = \sum_{n=k}^{\infty} P(n, \gamma)$$

$$f_{Y_k}(y) \delta \approx P(y \leq Y_k \leq y + \delta) =$$

$$\approx P(k-1, y) \lambda \delta + P(k-2, y) O(\delta^2) + P(k-3, y) O(\delta^2)$$

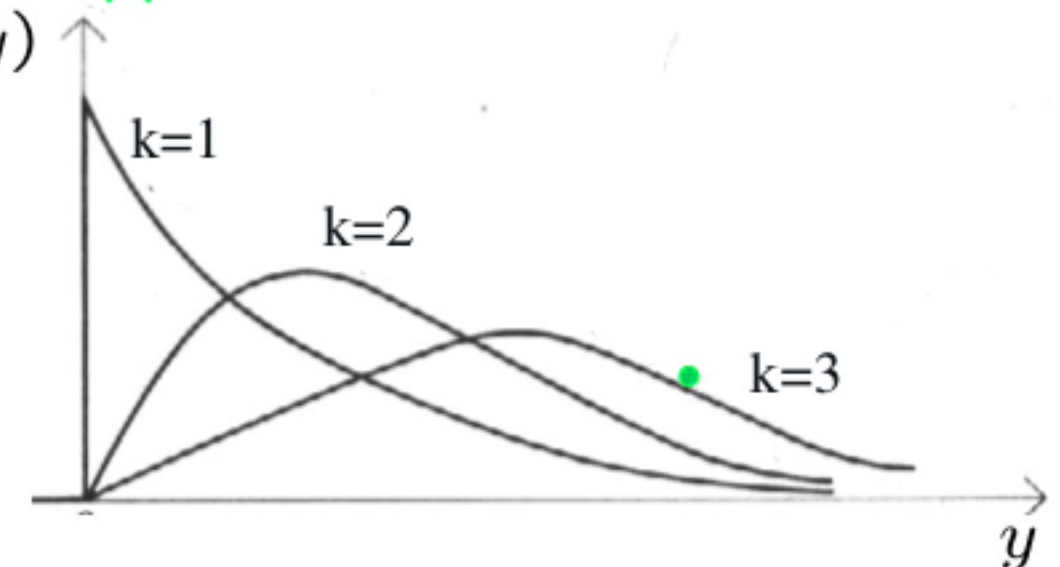
The third order term is specifically  $O(\delta^3)$  from the Poisson process  $P_K(K=3; \lambda \delta) = \frac{(\lambda \delta)^3}{3!} e^{-\lambda \delta}$ . Still  $O(\delta^3) \subset O(\delta^2)$  and for the proof it suffices that the additional terms are  $O(\delta^2)$

"The order of error is always the order of the largest term dropped off"

Indeed. This is the most fundamental observation.

$$\frac{(\lambda y)^{k-1} e^{-\lambda y}}{(k-1)!} \lambda$$

**Erlang distribution:**  $f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \quad y \geq 0$   
 order  $k$

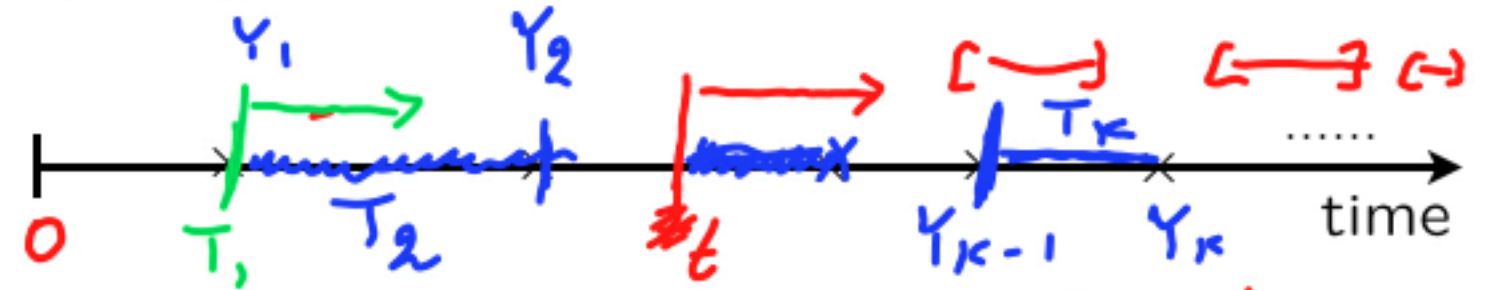




## Memorylessness and the fresh-start property

- Analogous to the properties for the Bernoulli process
  - plausible, given the relation between the two processes
  - use intuitive reasoning
  - can be proved rigorously

## Memorylessness and the fresh-start property



- If we start watching at time  $t$ ,

we see Poisson process, independent of the history until time  $t$

- time until next arrival:  $\text{Exp}(\lambda)$ , independent of past

- If we start watching at time  $T_1$ ,  $T_1 = 3$

we see Poisson process, independent of the history until time  $T_1$

- hence: time between first and second arrival,  $T_2 = Y_2 - Y_1$  is:  $\text{Exp}(\lambda)$
- similarly for all  $T_k = Y_k - Y_{k-1}$ ,  $k \geq 2$  incl. of  $T_1$

$Y_k = T_1 + \dots + T_k$  is sum of i.i.d. exponentials

$$\mathbf{E}[Y_k] = k/\lambda \quad \text{var}(Y_k) = k/\lambda^2$$

- An equivalent definition
- A simulation method

Bernoulli/Poisson relation



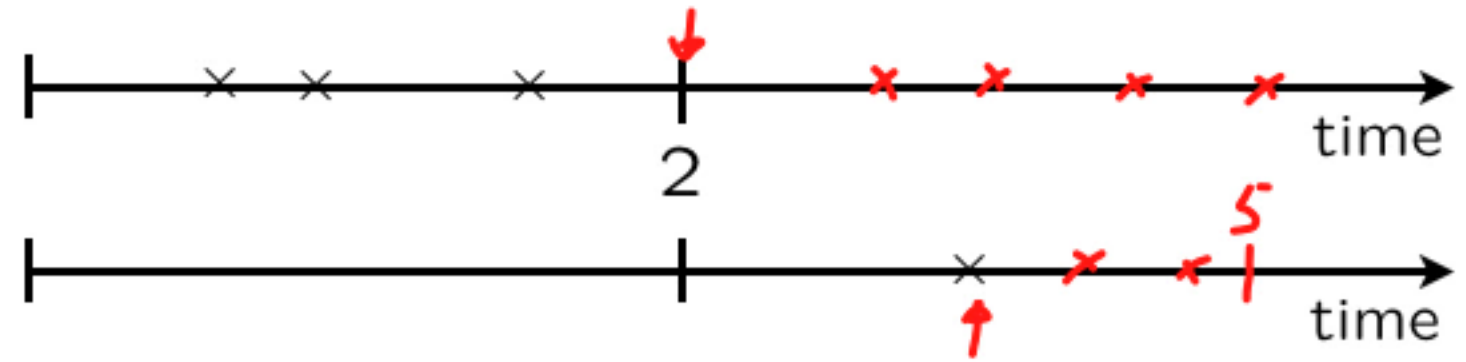
$$n = \tau/\delta,$$
$$p = \lambda\delta$$

$$np = \lambda\tau$$

	POISSON	BERNOULLI
Times of Arrival	Continuous	Discrete
Arrival Rate	$\lambda$ /unit time	$p$ /per trial
PMF of # of Arrivals	<div>•</div> Poisson	Binomial
Interarrival Time Distr.	Exponential	Geometric
Time to $k$ -th arrival	Erlang	Pascal

## Example: Poisson fishing

- Fish are caught as a Poisson process,  $\lambda = 0.6/\text{hour}$ 
  - fish for two hours;
  - if you caught at least one fish, stop
  - else continue until first fish is caught



$P(\text{fish for more than two hours}) = P(0, 2)$

$$P(\tau_1 > 2) = \int_2^{\infty} f_{\tau_1}(t) dt$$

$P(\text{fish for more than two and less than five hours}) =$

$$P(0, 2) (1 - P(0, 3))$$

$$P(2 < \tau_1 \leq 5) = \int_2^5 f_{\tau_1}(t) dt$$

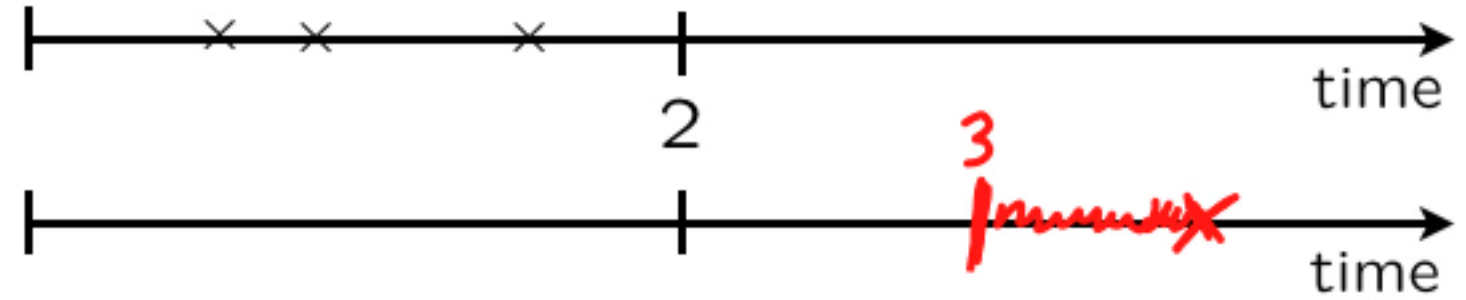
$$P(k, \tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}$$

$$E[N_\tau] = \lambda\tau$$

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}$$

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$P(\text{catch at least two fish}) =$

$$\sum_{k=2}^{\infty} P(k, 2) = 1 - P(0, 2) - P(1, 2)$$
$$P(Y_2 \leq 2) = \int_0^2 f_{Y_2}(y) dy$$

$E[\text{future fishing time} \mid \text{already fished for three hours}] = \frac{1}{\lambda}$

$$P(k, \tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}$$

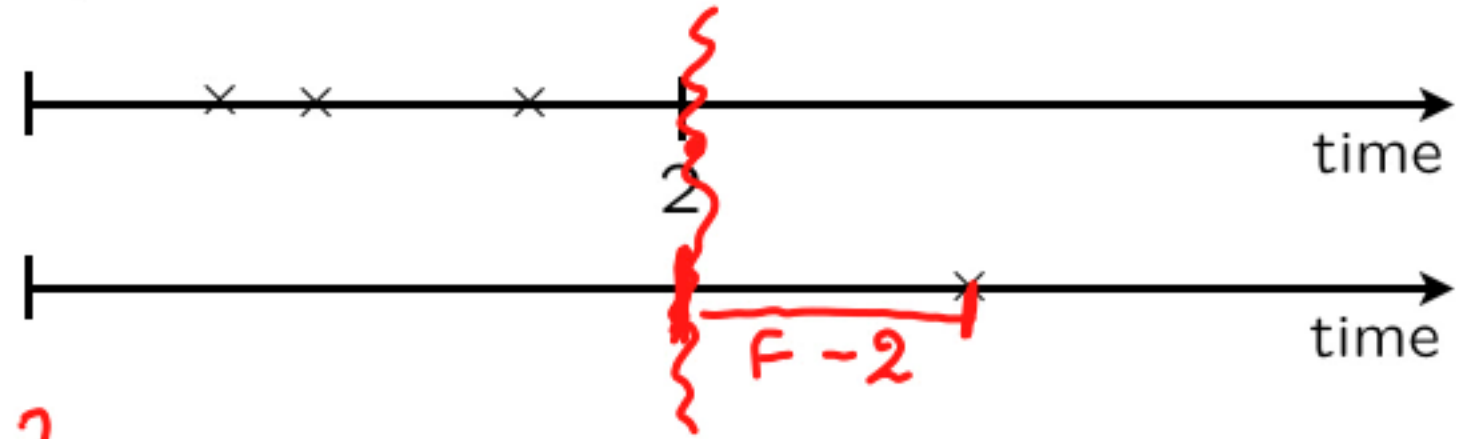
$$E[N_\tau] = \lambda\tau$$

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$$\begin{aligned} E[\text{total fishing time}] &= E[F] = 2 + E[F - 2] \\ &= 2 + P(F = 2) \cdot 0 + P(F > 2) E[F - 2 | F > 2] \\ &= 2 + P(0, 2) \cdot 1/\lambda \end{aligned}$$

$$E[\text{number of fish}] = \lambda \tau + P(0, 2) \cdot 1$$

$0.6 \times 2$

$$P(k, \tau) = \frac{(\lambda \tau)^k e^{-\lambda \tau}}{k!}$$

$$E[N_\tau] = \lambda \tau$$

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}$$