An interesting random variable associated with the Bernoulli process is the time of the kth success or the time of the kth arrival, depending on what kind of context we have in mind. So the picture is as follows.

The process starts and we wait until the first arrival occurs, and the time that it occurs, we call that time Y1. Then we keep observing the process, and there's a time at which a second arrival comes. We call that time Y2. The process continues, and there is a certain time that the third arrival comes. We call that time Y3.

Now, the time that the first arrival comes, this is also what we called T1. T1 is this length. It's the time until the first arrival. Let us give a name to the time it takes from the first to the second arrival, and we call that time T2, which is the second inter-arrival time. And similarly, we will call T3 the time between the second and the third arrival.

So we define in general Tk to be the difference between two consecutive arrival times. And of course, the time of the third arrival is the sum of T1 plus T2 plus T3, the first three inter-arrival times. And more generally, Yk is going to be the sum of these k inter-arrival times.

So in order to study the random variable Yk and its properties, the way that we can proceed is to understand first the random variables Ti. What kind of random variables are they? Well, we know that T1, the time until the first arrival, has a geometric distribution with parameter p.

Now, at the time of the first arrival, the process starts fresh. So after this time, there will be a sequence of independent Bernoulli trials, and T2 will be the number of Bernoulli trials it takes until an arrival. So T2 will also be geometric with the same parameter, p. Furthermore, because the process starts fresh, whatever happens in the future after this time is independent from whatever happened in the past, and so the random variable T2 will be independent from T1. And then by a similar argument, T3 will be independent from T1 and T2 and will also have the same geometric distribution.

Based on these properties, we can now go ahead and calculate properties of Yk. Yk is the sum of random variables. The expected value of Yk is the sum of the expected values of the Ts. Each one of the Ts has a geometric distribution with parameter p, and in particular has a mean of 1 over p. By adding those means, we obtain that the mean of Yk is k over p.

Similarly, the variance of Yk will be equal to the sum of the variances of the Tis, the reason being that the Tis are independent, and so to find the variance of the sum, it's enough to just add the variances. And we have a formula for the variance of a geometric, and using that formula and multiplying it by k, we obtain the variance of Yk.

Finally, we would like to calculate the PMF of Yk. So we would like to find this probability here, the probability that Yk takes on a specific value equal to t. Notice that in this argument, we're thinking of k as a fixed, given number. For example, we're interested in the time of the fifth arrival. This is a random variable that can take different values, t, and we want to find the probabilities of those different values, t. So think of k as being fixed and t as a parameter that varies, and we want to carry out this calculation for all possible choices of t.

Now, what is this event here? This is the event that the kth arrival occurs at time t. So this means that at time t, we have an arrival. But for this to be the kth arrival, we must have k minus 1 arrivals in the previous time slots, of which there's t minus 1 of them.

The probability that Yk is equal to t is the probability that these two events happen, k minus 1 arrivals in t minus 1 slots and one arrival at slot number t. So we are looking at the probability that these two events occur. Now this event, k minus 1 arrivals in t minus 1 slots, is an event that's completely determined by whatever happens in the first t minus 1 time slots, whereas the event of an arrival at slot time t refers to whatever happens during slot time t.

Because of our assumptions on the Bernoulli process, whatever happens during these t minus 1 time slots is independent from what happens in slot number t. So the probability of these two events happening, because of independence, will be the probability of the first event happening, k minus 1 arrivals in time t minus 1, times the probability of an arrival at time t.

Now, the first probability is given by the binomial formula. In t minus 1 time slots, we want to have k minus 1 arrivals. And the binomial formula gives us an exponent, p to this power times 1 minus p to the power that's the difference of these two numbers, which is t minus k. And then finally, we multiply with the probability of an arrival at time t, which is equal to p. This p will cancel the exponent of minus 1 up here and leads us to this formula for the probability that the kth arrival happens at time t.

Notice the range of the random variable Yk. The kth arrival cannot happen before time k. You need at least k time slots to obtain k arrivals, so this probability will be positive only starting at time k and for future times. So this random variable Yk, in general, will have a PMF of this form. It's zero for ts smaller than k, and then at time k, in general, it's going to be a positive entry. And for future values of t, it will also have positive entries.

And this PMF extends all the way to infinity because it is possible that the kth arrival takes an arbitrarily long time to occur. If we consider different values of k, of course we will get a different PMF. The PMF of Y3 is different than the PMF of Y2. And the PMF of Y3 will generally sit to the right of the PMF of Y2 because the third arrival generally will take longer to occur than the second arrival.