1. Covariance

Calculate the covariance of each of the following pairs of random variables. Please enter answers according to the standard notation.

(a)

1/1 point (graded)

$X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y = X^2$. Please enter in terms of μ and σ.

Cov $(X, Y) =$ 2*mu*sigma^2 Answer: 2*mu*sigma^2

$2 \cdot \mu \cdot \sigma^2$

STANDARD NOTATION

Solution:

The definition for the covariance of two random variables: Cov $(X,Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$. An alternative form for the covariance is Cov $(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$. This form is easier to work with to calculate covariances compared to the original definition.

$\mathbb{E}[X^2] = \sigma^2 + \mu^2$, $\mathbb{E}[X^3] = \mu^3 + 3\mu\sigma^2$.

$$\begin{aligned} \text{Cov}(X, X^2) &= \mathbb{E}[X^3] - \mathbb{E}[X]\mathbb{E}[X^2] \\ &= \mu^3 + 3\mu\sigma^2 - \mu(\mu^2 + \sigma^2) \\ &= 2\mu\sigma^2 \end{aligned}$$

提交 你已经尝试了2次（总共可以尝试3次）

Answers are displayed within the problem

(b)

1/1 point (graded)

X, Y have the joint probability density function $f(x,y) = 1$, $0 < x < 1$, $x < y < x+1$. Please enter a number.

Cov $(X, Y) =$ 1/12 Answer: 1/12

Solution:

Cov $(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$, so we need to find out the expectations of X, Y, and XY. From the joint distribution, we can derive the marginal distribution: $f_X(x) = \int_x^{x+1} 1\, dy = y|_x^{x+1} = 1$, $x \in (0,1)$ and the conditional distribution $f(y|x) = \frac{f(x,y)}{f(x)} = 1$, $y \in (x, x+1)$.

$$\begin{aligned} \mathbb{E}[Y|X] &= \int_x^{x+1} y\, dy \\ &= \left.\frac{y^2}{2}\right|_x^{x+1} \\ &= \frac{2x+1}{2} \end{aligned}$$

According to the law of iterated expectations,

$$\begin{aligned}\mathbb{E}[Y] &= \mathbb{E}[\mathbb{E}[Y|X]] \\ &= \mathbb{E}[\frac{2X+1}{2}] \\ &= \int_0^1 \frac{2x+1}{2}\,dx \\ &= 1\end{aligned}$$

$$\begin{aligned}\mathbb{E}[XY] &= \int_0^1 x\left[\int_x^{x+1} y\,dy\right]dx \\ &= \int_0^1 x\frac{y^2}{2}\Big|_x^{x+1} dx \\ &= \int_0^1 \frac{2x^2+x}{2}\,dx \\ &= \frac{7}{12}\end{aligned}$$

$\text{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] = \frac{7}{12} - \frac{1}{2}\times 1 = \frac{1}{12}$

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Answers are displayed within the problem

(c)

1/1 point (graded)
$X \sim f(x) = \frac{1}{2b}e^{-|x|/b},\ x \in \mathbb{R},\ b > 0$ and $Y = \text{sign}(X)$

$\text{Cov}(X,Y) =$ b Answer: b

b

Solution:

By symmetry, $\mathbb{E}[X] = \int_{-\infty}^{\infty} \frac{x}{2b}e^{-|x|/b}\,dx = 0$. $\mathbb{E}[Y] = (-1)\cdot P(X<0) + 1\cdot P(X>0) = -\frac{1}{2} + \frac{1}{2} = 0$

乘起来是正数

$$\begin{aligned}\text{Cov}(X,Y) = \mathbb{E}[XY] &= \int_{-\infty}^{\infty} \frac{x\cdot\text{sign}(x)}{2b}e^{-|x|/b}\,dx \\ &= \int_0^{\infty} \frac{x}{b}e^{-x/b}\,dx\end{aligned}$$

乘以2了

We can think of this as the expectation of an exponential random variable Z with parameter $\frac{1}{b}$. $\int_0^{\infty} \frac{x}{b}e^{-x/b}\,dx = \mathbb{E}[Z] = b$, where $Z \sim \text{Exp}\left(\frac{1}{b}\right)$.

提交 你已经尝试了2次（总共可以尝试3次）

Answers are displayed within the problem

(d)

1/1 point (graded)
$X \sim \text{Unif}(0,1)$ and given $X = x$, $Y \sim \text{Unif}(x,1)$

$\text{Cov}(X,Y) =$ 1/24 Answer: 1/24

Solution:

$\mathbb{E}[X] = \frac{1}{2}$
$\mathbb{E}[Y|X] = \frac{X+1}{2}$
According to the law of iterated expectations, $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[\frac{X+1}{2}] = \int_0^1 \frac{x+1}{2}\, dx = \frac{3}{4}$

$$f(x,y) = f(y|x)\, f(x) = \frac{1}{1-x}$$

$$\begin{aligned}\mathbb{E}[XY] &= \int_0^1 \int_x^1 \frac{1}{1-x} \cdot xy \, dydx \\ &= \int_0^1 \frac{x}{1-x} \cdot \frac{y^2}{2}\Big|_x^1 \, dx \\ &= \int_0^1 \frac{x}{1-x}\left(\frac{1}{2} - \frac{x^2}{2}\right) dx \\ &= \frac{1}{2}\int_0^1 (x + x^2)\, dx \\ &= \frac{5}{12}\end{aligned}$$

Cov $(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] = \frac{5}{12} - \frac{1}{2} \times \frac{3}{4} = \frac{1}{24}$

提交 你已经尝试了3次（总共可以尝试3次）

Answers are displayed within the problem

(e)

1/1 point (graded)
X and Y have the joint density function

$$f(x,y) = \begin{cases} x+y, & 0 \le x \le 1,\ 0 \le y \le 1, \\ 0 & \text{else.}\end{cases}$$

Cov $(X,Y) =$ -1/144 Answer: -1/144

Solution:

$f(x) = \int_0^1 (x+y)\, dy = x + \frac{1}{2}$
$\mathbb{E}[X] = \int_0^1 (x^2 + \frac{x}{2})\, dx = \frac{7}{12}$
$\mathbb{E}[Y] = \frac{7}{12}$ by symmetry

$$\begin{aligned}\mathbb{E}[XY] &= \int_0^1 \int_0^1 xy\,(x+y)\, dxdy \\ &= \int_0^1 \frac{x^3 y}{3} + \frac{x^2y^2}{2}\Big|_0^1 \, dy \\ &= \int_0^1 \left(\frac{y}{3} + \frac{y^2}{2}\right) dy \\ &= \frac{1}{3}\end{aligned}$$

Cov $(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] = \frac{1}{3} - \frac{7}{12} \times \frac{7}{12} = -\frac{1}{144}$

提交 你已经尝试了0次（总共可以尝试3次）

Answers are displayed within the problem

(f)

1/1 point (graded)

$X+Y$ and $X-Y$, where X and Y are independent $\mathcal{N}(\mu,\sigma^2)$.

$\mathsf{Cov}(X+Y, X-Y) =$ 0 Answer: 0

0

Solution:

$$\begin{aligned}\mathsf{Cov}(X+Y,X-Y) &= \mathbb{E}[(X+Y)(X-Y)] - \mathbb{E}[X+Y]\,\mathbb{E}[X-Y] \\ &= \mathbb{E}[X^2] - \mathbb{E}[Y^2] - (\mathbb{E}[X]+\mathbb{E}[Y])(\mathbb{E}[X]-\mathbb{E}[Y]) \\ &= (\sigma^2+\mu^2) - (\sigma^2+\mu^2) - ((\mathbb{E}[X])^2 - (\mathbb{E}[Y])^2) \\ &= 0\end{aligned}$$

提交 你已经尝试了1次（总共可以尝试3次）

Answers are displayed within the problem

讨论

显示讨论

主题： Unit 3 Methods of Estimation:Homework 5: Maximum Likelihood Estimation / 1. Covariance