Markov processes – II

- review and some warm-up
  - definitions, Markov property
  - calculating the probabilities of trajectories
- steady-state behavior
  - recurrent states, transient states, recurrent classes
  - periodic states
  - convergence theorem
  - balance equations
- birth-death processes

review

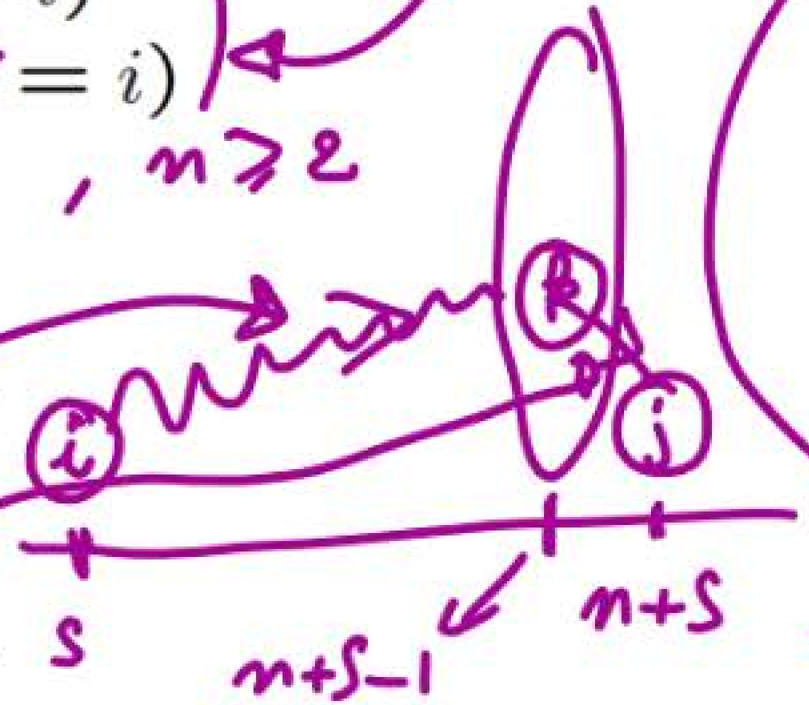
- discrete time, discrete state space, time-homogeneous
  - transition probabilities $p_{ij} \triangleq \mathbf{P}(X_{s+1}=j \mid X_s=i) \leftarrow$ for $s=0,1,2,\ldots$
  - Markov property $\mathbf{P}(X_{s+1}=j \mid X_s=i, X_0=i_0, \ldots, X_{s-1}=i_{s-1}) = p_{ij}$

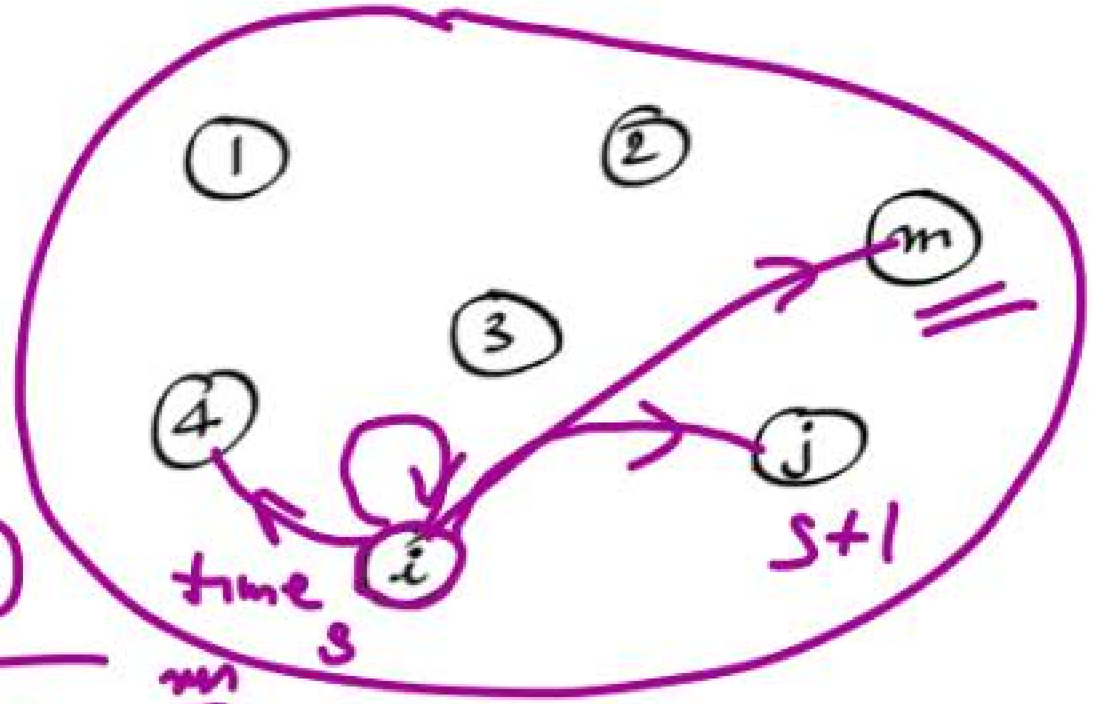
- $$\begin{aligned} r_{ij}(n) &= \mathbf{P}(X_n = j \mid X_0 = i) \\ &= \mathbf{P}(X_{n+s} = j \mid X_s = i) \end{aligned}$$

  $n=1$ $\quad r_{ij}(1) = p_{ij}$ $\quad , n \geq 2$

- key recursion:

$$\boxed{r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1)\, p_{kj}}$$



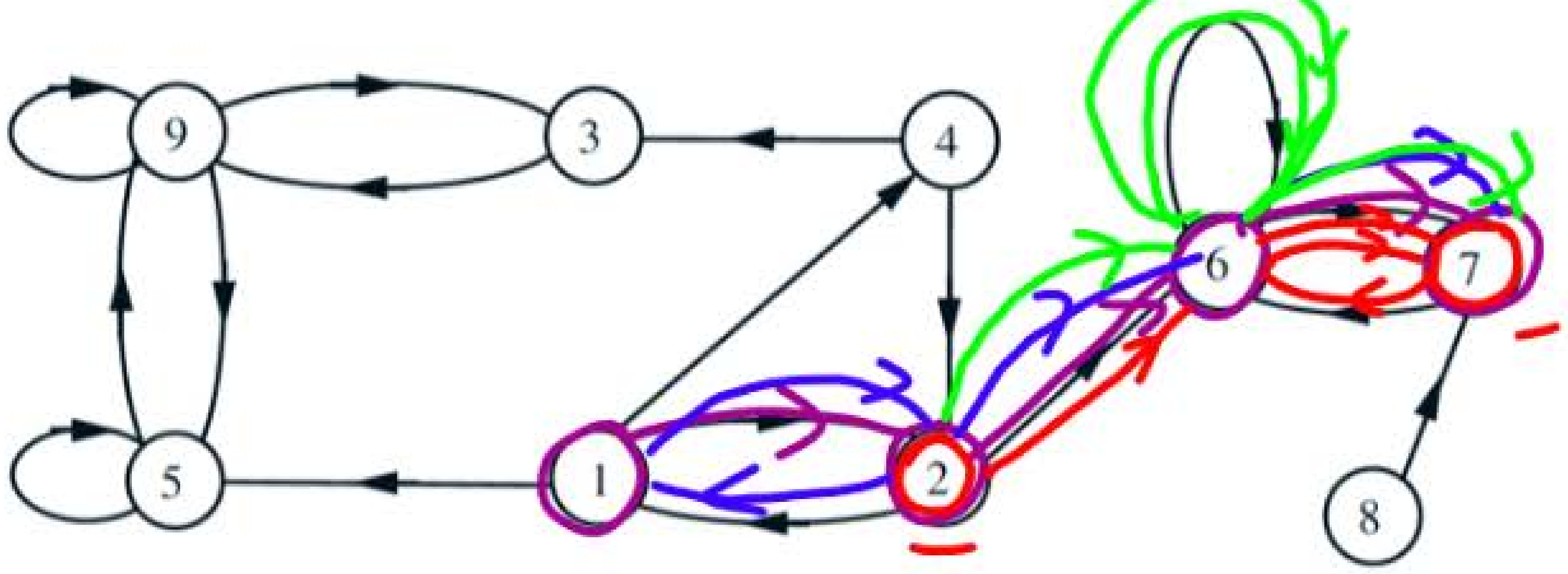


$$\sum_{j=1}^{m} p_{ij} = 1$$

warmup

"multiplicative rule"

$$\mathbb{P}(B\cap C\cap D\,|\,A) = \mathbb{P}(B\,|\,A)\times\mathbb{P}(C\,|\,A\cap B)\times\mathbb{P}(D\,|\,A\cap B\cap C)$$



$$\mathbf{P}(X_1 = 2, X_2 = 6, X_3 = 7 \mid X_0 = 1) =$$

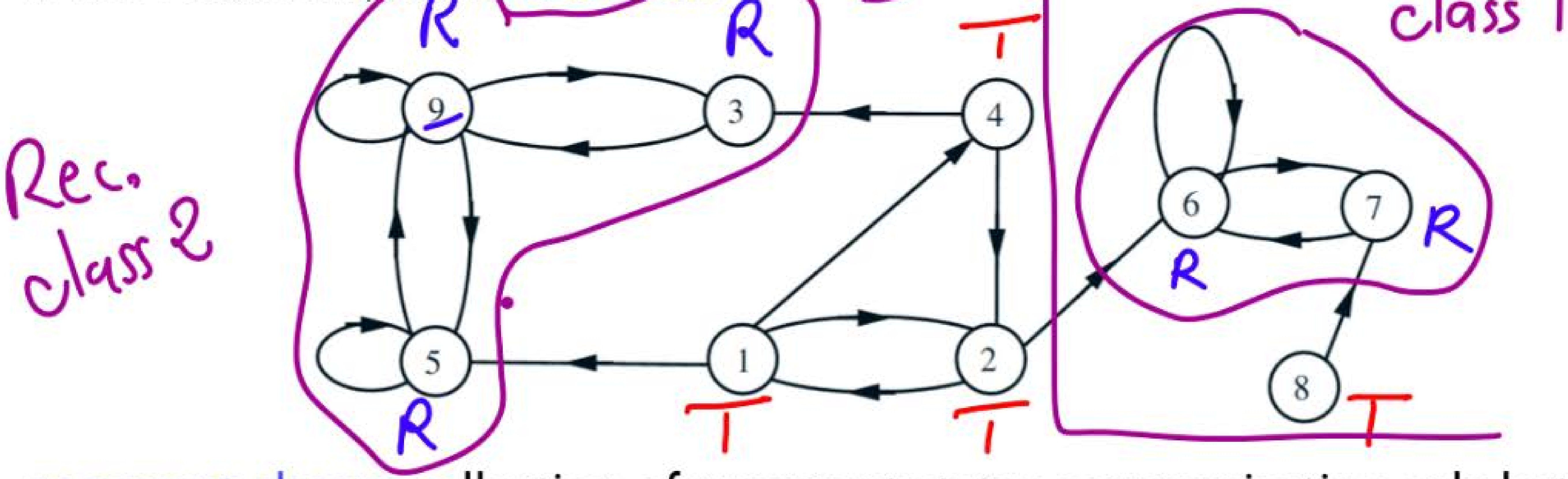
$$\underbrace{\mathbb{P}(X_1=2\,|\,X_0=1)}_{p_{12}}\times\underbrace{\mathbb{P}(X_2=6\,|\,X_0=1,X_1=2)}_{p_{26}}\times\underbrace{\mathbb{P}(X_3=7\,|\,X_0=1,X_1=2,X_2=6)}_{p_{67}}$$

$$\mathbf{P}(\underset{200}{X_4} = 7 \mid X_0 = 2) = p_{26}\times p_{67}\times p_{76}\times p_{67} + p_{21}\times p_{12}\times p_{26}\times p_{67} + p_{26}\times p_{66}\times p_{66}\times p_{67}$$

$m \quad m^2 \quad \underline{n\times m^2}$

review: recurrent and transient states

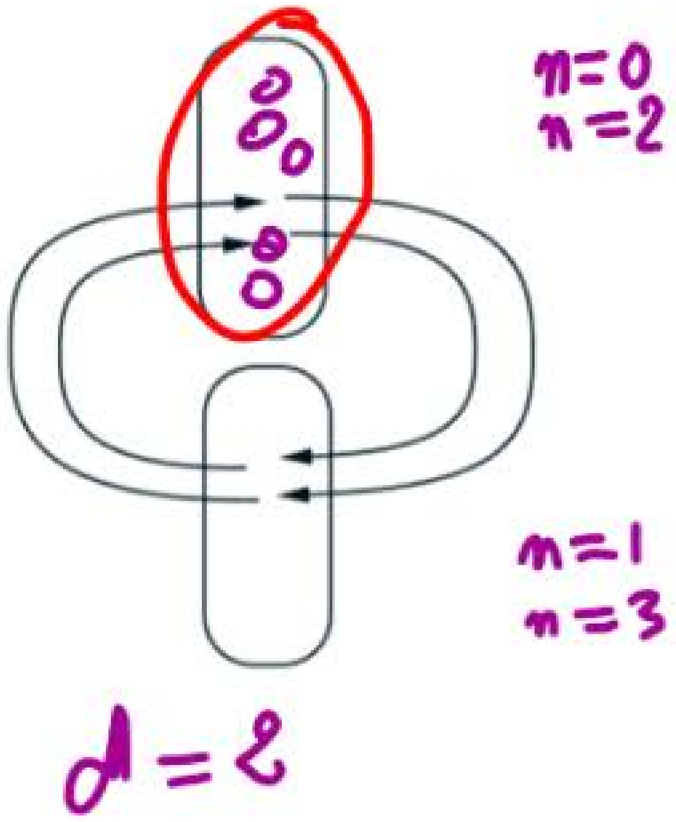
- state i is recurrent if "starting from i, and from wherever you can go, there is a way of returning to i"
- if not recurrent, called transient

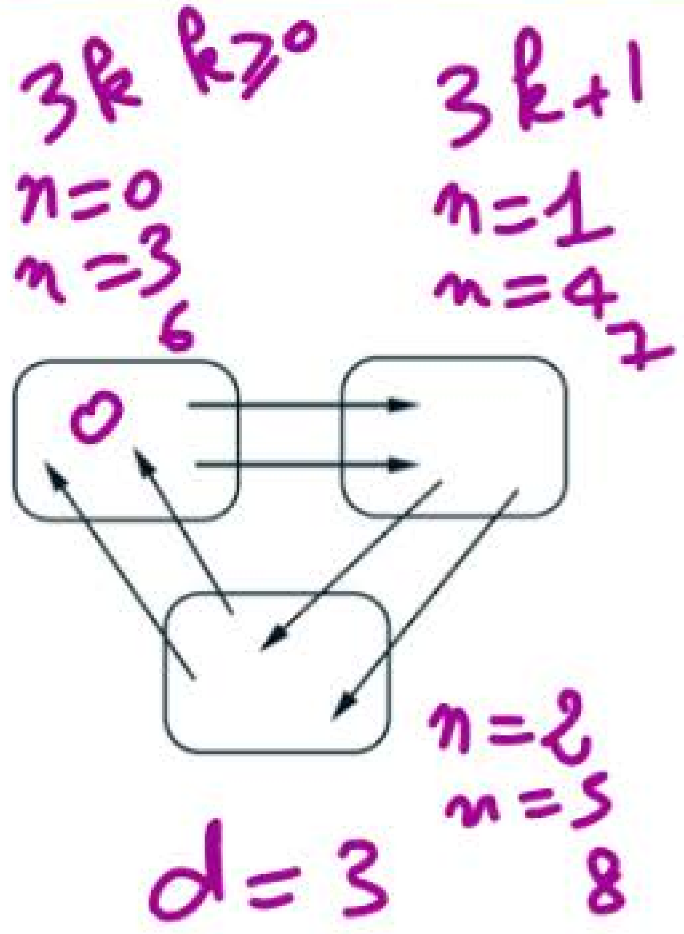


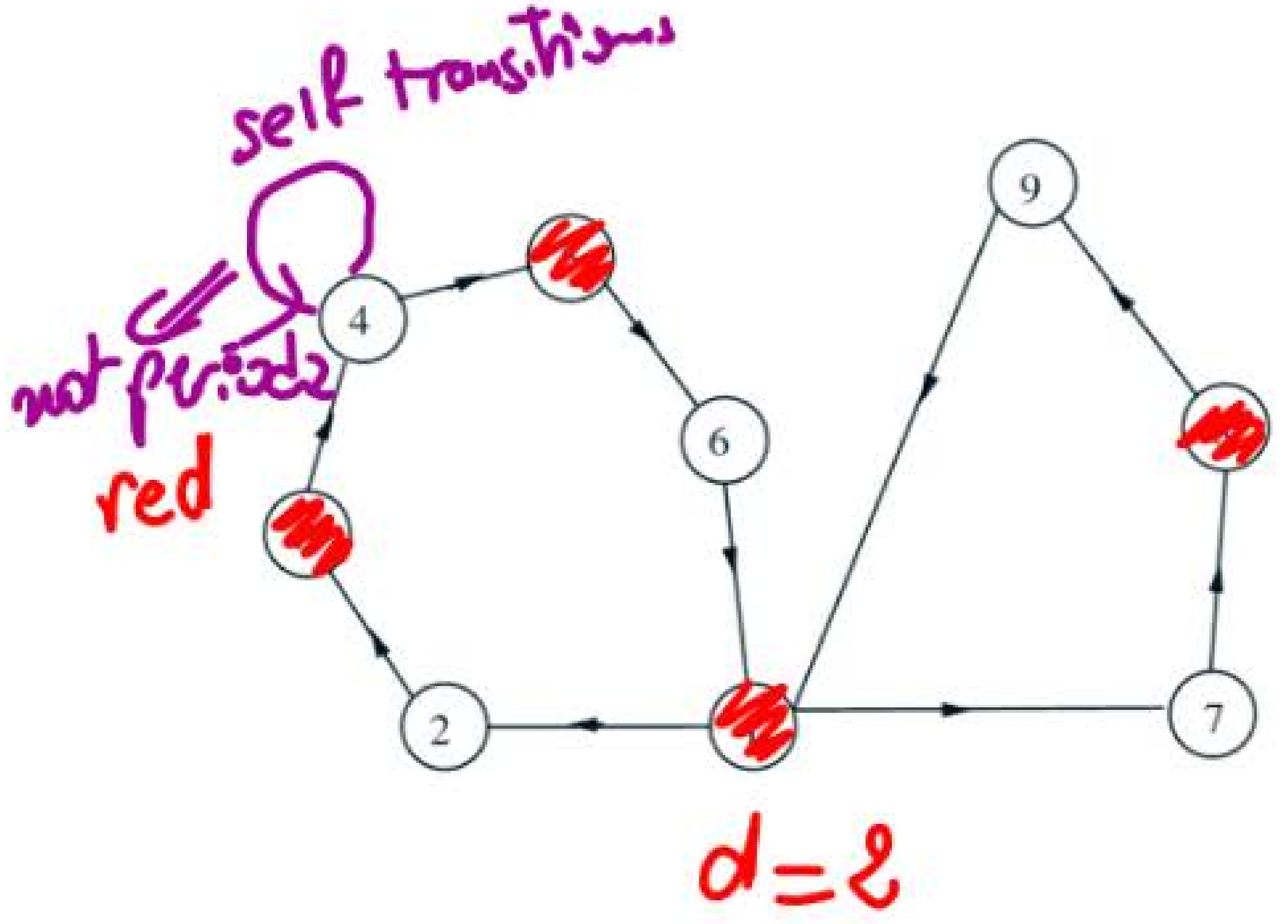
- recurrent class: a collection of recurrent states communicating only between each other

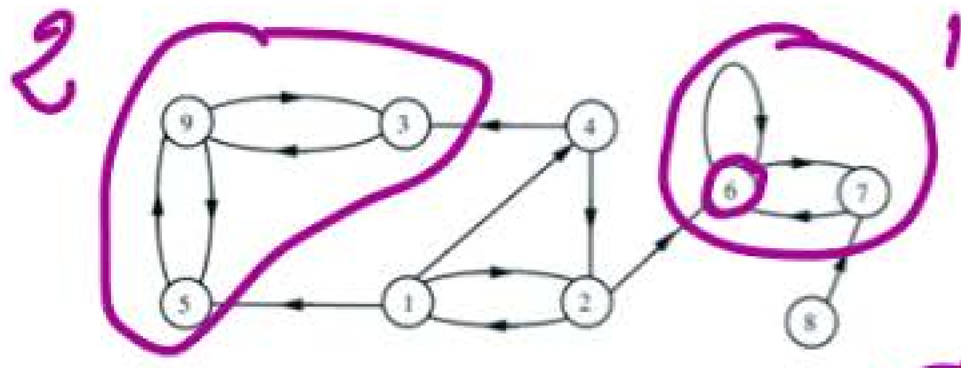
periodic states in a recurrent class

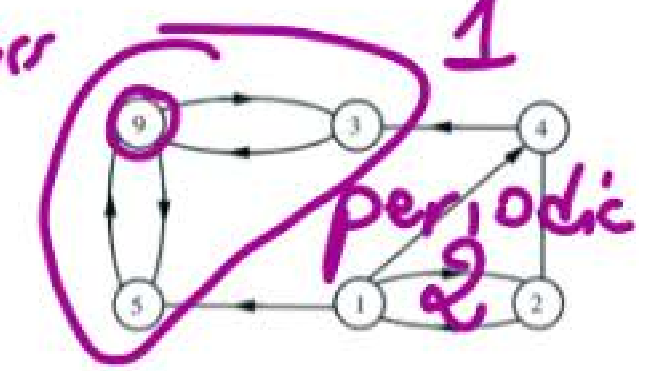
The states in a recurrent class are periodic if they can be grouped into d > 1 groups so that all transitions from one group lead to the next group







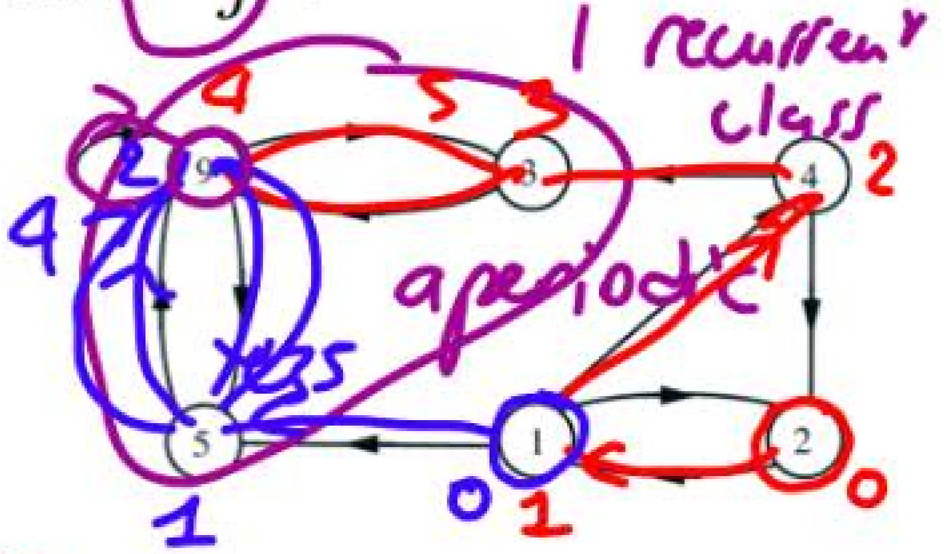




steady-state probabilities

- convergence?
- is independent of i?

- does $r_{ij}(n) = \mathrm{P}(X_n = j \mid X_0 = i)$ converge to some π_j ?
- theorem: yes, if:
  - recurrent states are all in a single class, and
  - single recurrent class is not periodic



- assuming "yes", start from key recursion $r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1)p_{kj}$
  - take the limit as $n \to \infty$

$n \to \infty$

$$\boxed{\pi_j = \sum_k \pi_k p_{kj} \quad \text{for } j = 1, \dots, m}$$

~~$\pi_j = 0 \ \forall j$~~

m equations
m unknowns — π_j's

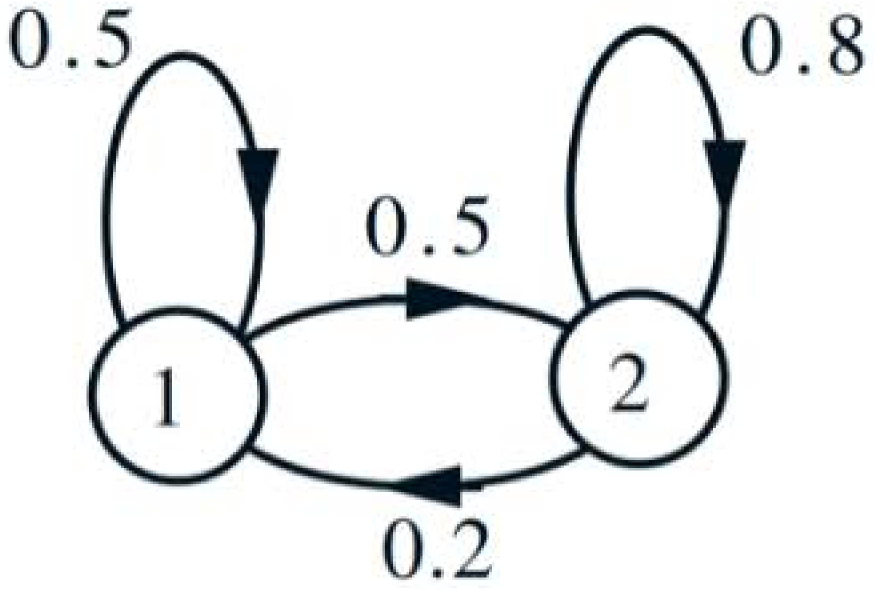
  - need also: $\sum_{j=1}^{m} \pi_j = 1$

and

$\Downarrow$

unique sol^n.

example



- single recurrent class
- not periodic

$$\begin{cases} \pi_j = \sum_{k=1}^{m} \pi_k p_{kj}, \quad j=1,\dots,m \\ m=2 \end{cases}$$

$$\begin{cases} \pi_1 = \pi_1 \times 0.5 + \pi_2 \times 0.2 \\ \pi_2 = \pi_1 \times 0.5 + \pi_2 \times 0.8 \end{cases} \Rightarrow \begin{cases} \pi_1 \times 0.5 = \pi_2 \times 0.2 \\ \pi_2 \times 0.2 = \pi_1 \times 0.5 \end{cases}$$

same

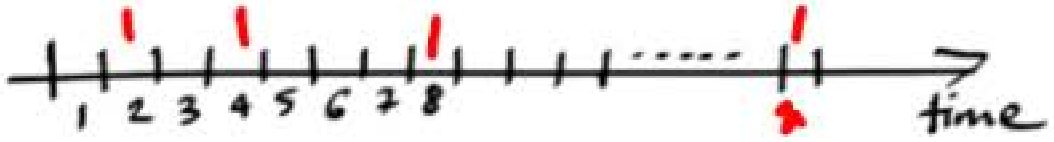
$$\begin{cases} \pi_1 \times \frac{1}{2} = \pi_2 \times \frac{1}{5} \\ \pi_1 + \pi_2 = 1 \end{cases} \Rightarrow \begin{cases} \pi_1 = \frac{2}{5}\pi_2 \\ \pi_2\left(\frac{2}{5} + 1\right) = 1 \end{cases} \Rightarrow \begin{cases} \pi_1 = \frac{2}{5} \times \frac{5}{7} = \frac{2}{7} \\ \pi_2 = \frac{5}{7} \end{cases}$$

visit frequency interpretation

- balance equations

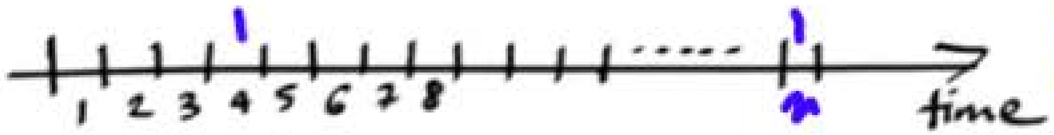
$$\pi_j = \sum_k \pi_k p_{kj}$$

- (long run) frequency of being in j: π_j



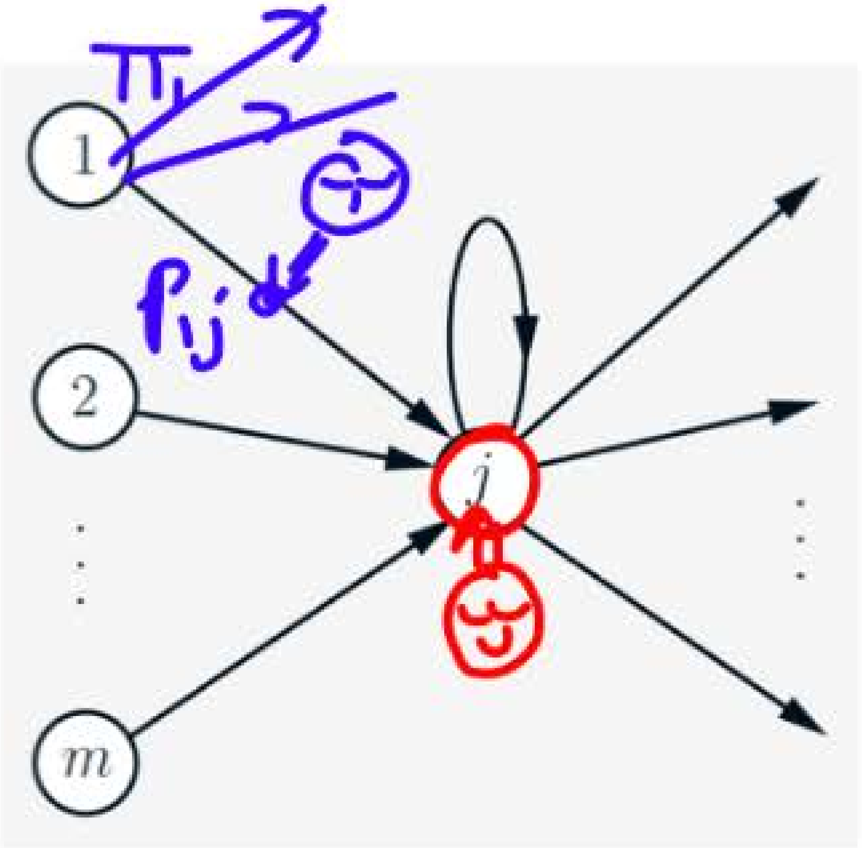
$\frac{Y_j(n)}{n} \underset{n\ \text{large}}{\approx} \pi_j$

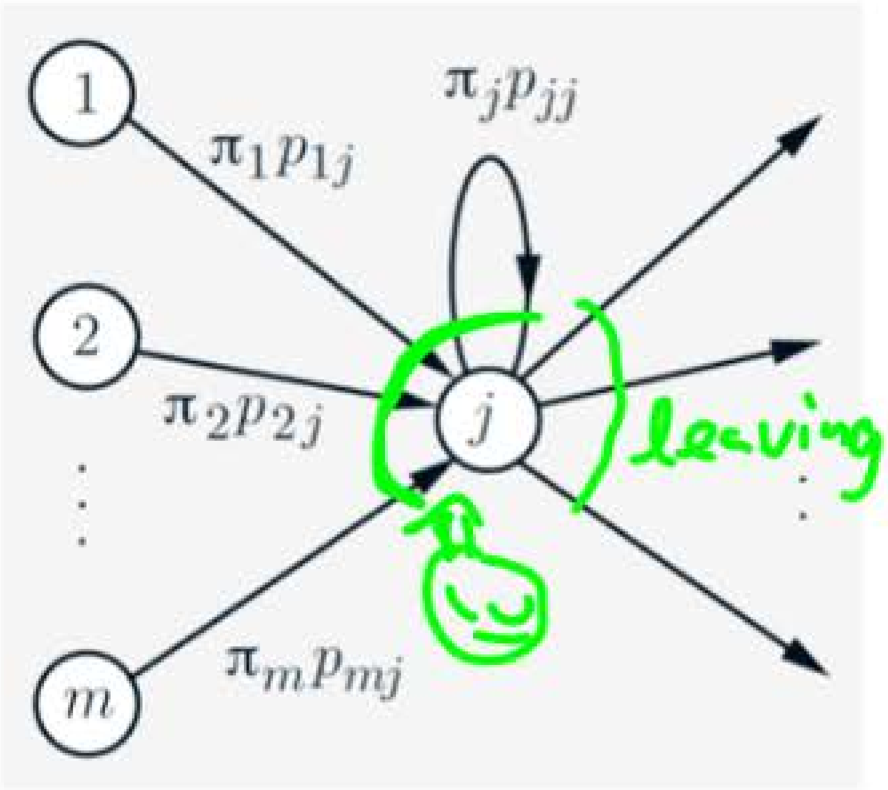
- frequency of transitions $1 \to j$: $\pi_1 p_{1j}$



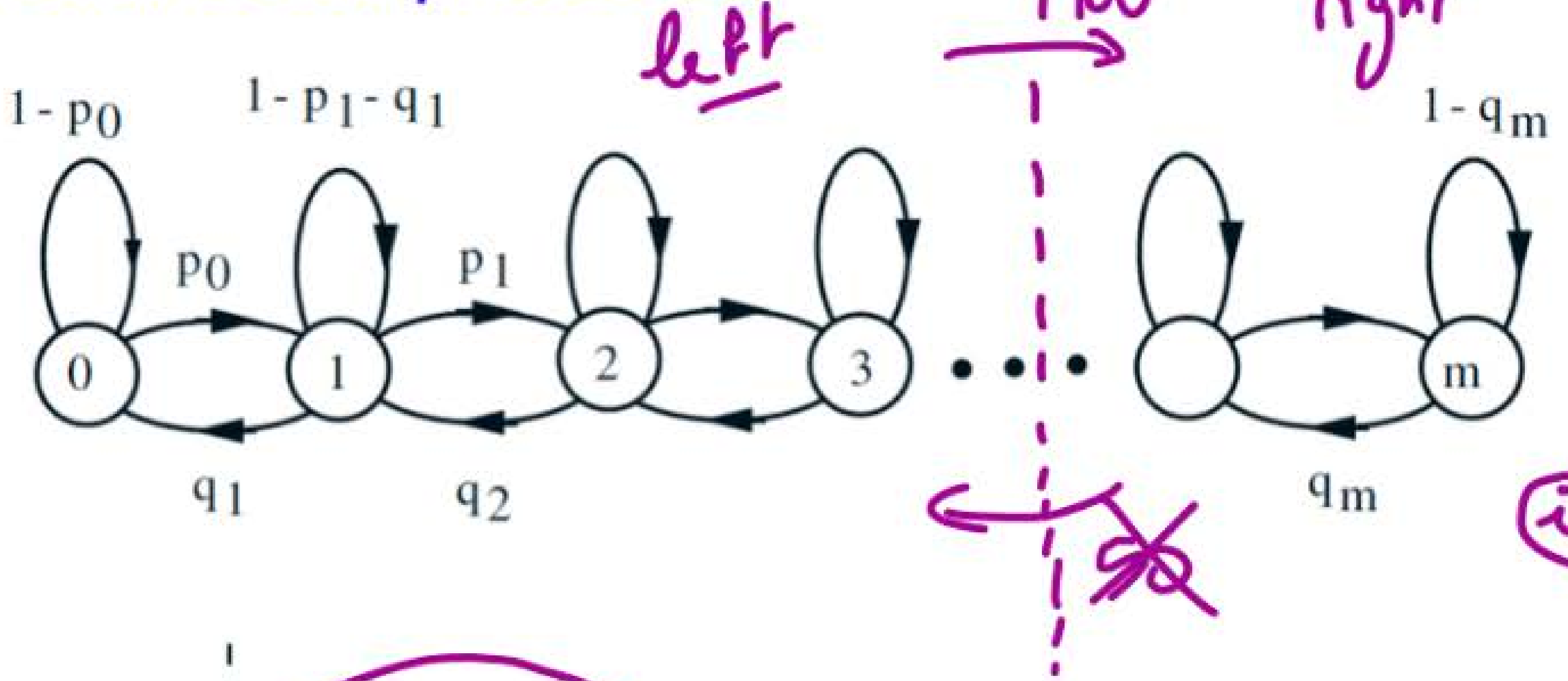
$\frac{Z_{1j}(n)}{n} \underset{n\ \text{large}}{\approx} \pi_1 \cdot p_{1j}$

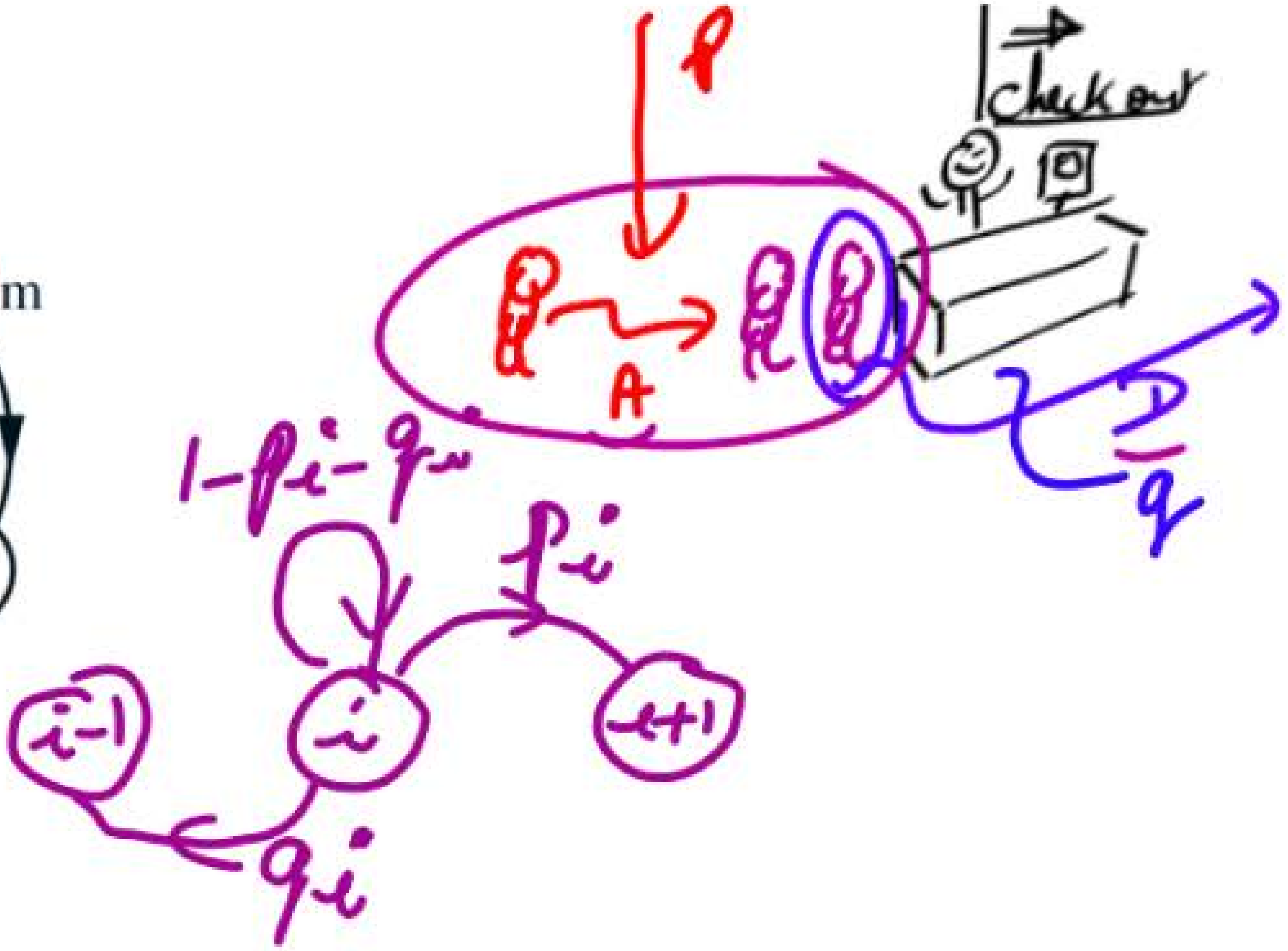
- frequency of transitions into j: $\sum_k \pi_k p_{kj}$

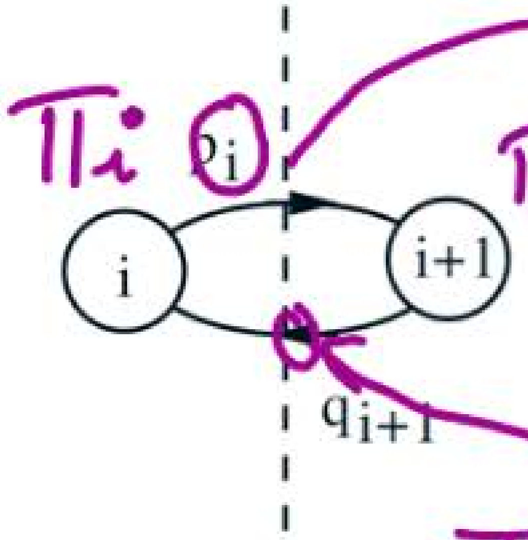




birth-death processes I







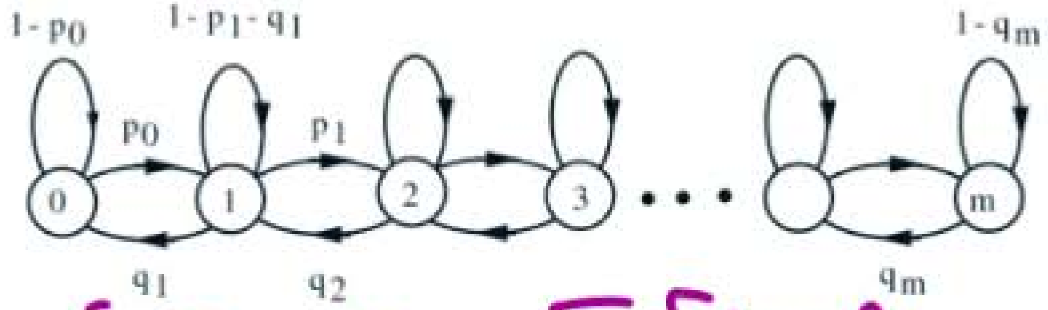
$$\boxed{\pi_i p_i = \pi_{i+1} q_{i+1}}$$

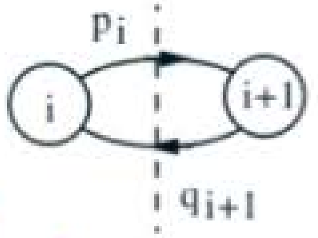
$$\pi_{i+1} = \pi_i \times \frac{p_i}{q_{i+1}} \quad i = 0, 1, \dots$$

$$\pi_0 \to \pi_1, \to \pi_2, \to \pi_3 \qquad / \qquad \pi_0 \; ?$$

$$\sum_j \pi_j = 1 \Rightarrow \pi_0 + \pi_0 \times \frac{p_0}{q_1} + \pi_0 \times \frac{p_0 \cdot p_1}{q_1 q_2} + \dots = 1$$

birth-death processes II





$\pi_i p_i = \pi_{i+1} q_{i+1}$

$\sum_j \pi_j = 1$

$\pi_{i+1} = \pi_i \times \frac{p_i}{q_{i+1}}$, $i = 0, \dots, m$ $\quad \pi_0 [1 + \frac{p_0}{q_1} + \dots] = 1$

special case: $p_i = p$ and $q_i = q$ for all i

$\rho = p/q \qquad \pi_{i+1} = \pi_i \frac{p}{q} = \pi_i \rho$ $\qquad \pi_1 = \pi_0 \rho, \; \pi_2 = \pi_1 \rho = \pi_0 \rho^2, \; \dots$

$\pi_i = \pi_0 \rho^i \quad i = 0, 1, \dots, m$ $\qquad \sum_j \pi_j = 1 \Rightarrow \pi_0 [1 + \rho + \rho^2 + \dots + \rho^m] = 1$

- assume $p = q$ $\quad \Rightarrow \pi_i = \pi_0 \quad i = 0, \dots, m \qquad \pi_0 [1 + m] = 1, \; \pi_0 = \frac{1}{1+m}$

- assume $p < q$ and $m \approx \infty$ $\quad \Rightarrow \sum_{i=1}^{\infty} \rho^i = \frac{1}{1-\rho}$ $\quad \frac{\rho}{1-\rho}$

$\boxed{\pi_0 = 1 - \rho}$ $\qquad \mathbf{E}[X_n] = \frac{\rho}{1-\rho}$ (in steady-state)

$\pi_i = \pi_0 \rho^i = (1-\rho)\rho^i, \; i = 0, \dots$