9. Car wash

Problem 8. Car wash

5.0/10.0 points (graded)

Starting at time $t = 0$, cars arrive at a car wash according to a Poisson process with rate of λ cars/hour. At any given moment, the car wash is either free or occupied. The car wash is initially free at time $t = 0$. If a car arrives at the car wash when it is free, the car is serviced immediately. Service lasts for $1/4$ hours, during which the car wash is occupied. If a car arrives at the car wash when it is occupied, the car is denied service and it leaves the car wash.

Whenever necessary, write down your answers using the standard notation. Type **fact(n)** for $n!$.

1. Write down the PMF $p_N(k)$ of N, the number of cars arriving at the car wash between times 0 and 3, in terms of λ and k.

   For $k = 0, 1, \ldots$,

   $p_N(k) =$

   (3*lambda)^k*exp(-3*lambda)/fact(k) ✔

   Answer: exp(-3*lambda)*(3*lambda)^k/(fact(k))

   $\frac{(3\cdot\lambda)^k\cdot\exp(-3\cdot\lambda)}{\text{fact}(k)}$

2. Find the probability that a car is accepted for service given that it arrives at time $t = 1/6$. Write down your answer in terms of λ, using the standard notation.

   exp(-1/6*lambda) ✔ **Answer:** exp(-lambda/6)

   $\exp\left(-\frac{1}{6}\cdot\lambda\right)$

3. Find the PDF $f_{T_2}(t)$ of T_2, the time until the second serviced car leaves the car wash.

   For $t \geq 0.5$,

   $f_{T_2}(t) =$

   (t*lambda)^2*t*exp(-t*lambda) ✖

   Answer: (lambda^2)*(t-0.5)*exp(-lambda*(t-0.5))

   $(t\cdot\lambda)^2\cdot t\cdot\exp(-t\cdot\lambda)$

4. Find the probability that exactly one car is accepted for service between $t = 0$ and $t = 1$.

   (3/4*lambda)*exp(-3/4*lambda)*exp(-3/4*lambda)+(1/4*lambda)*exp(-1/4*lai ✖

   Answer: 0.75*lambda*exp(-0.75*lambda) - exp(-lambda) + exp(-0.75*lambda)

   $\left(\frac{3}{4}\cdot\lambda\right)\cdot\exp\left(-\frac{3}{4}\cdot\lambda\right)\cdot\exp\left(-\frac{3}{4}\cdot\lambda\right)+\left(\frac{1}{4}\cdot\lambda\right)\cdot\exp\left(-\frac{1}{4}\cdot\lambda\right)$

STANDARD NOTATION

Solution:

Throughout, we take hours to be the unit of time.

1. Here N is a Poisson random variable with parameter 3λ. Its PMF is given by

$$p_N(k) = \frac{(3\lambda)^k e^{-3\lambda}}{k!} \qquad k = 0, 1, 2, \ldots$$

2. This is the probability that the car wash is free at time $t = 1/6$ hours. Let $N_{1/6}$ be the number of cars that arrive between 0 and $1/6$. The car wash is free at $t = 1/6$ min if $N_{1/6} = 0$. $N_{1/6}$ is a Poisson random variable with a parameter $\frac{1}{6}\lambda$:

$$p_{N_{1/6}}(n) = \frac{(\lambda/6)^n e^{-\lambda/6}}{n!}, \qquad \text{for } n = 0, 1, 2, \ldots$$

So, $p_{N_{1/6}}(0) = e^{-\lambda/6}$

3. Let X_1 be the time until the first car arrives at the car wash. Then, X_1 is an exponential random variable with parameter λ. The first car leaves at time $X_1 + 0.25$. By the fresh start property of the Poisson process, X_2, the additional time from $X_1 + 0.25$ until the arrival of the next accepted car is an exponential random variable with parameter λ, independent of X_1. The time until the second car leaves is:

$$T_2 = X_1 + 0.25 + X_2 + 0.25,$$

where X_1 and X_2 are i.i.d. exponential random variables. Their sum is an Erlang random variable of order 2:

$$f_{X_1+X_2}(x) = \frac{\lambda^2 x^{2-1} e^{\lambda x}}{(2-1)!}, \qquad x \geq 0.$$

It follows that T_2 is a shifted Erlang random variable with PDF:

$$f_{T_2}(t) = f_{X_1+X_2}(t - 0.5) = \lambda^2 (t - 0.5) e^{-\lambda(t-0.5)}, \qquad t \geq 0.5$$

4. Let T be the time of the first car arrival. Let A be the event that exactly one car is accepted between times 0 and 1. Then, note that, if $T > 1$, then event A cannot happen, and $\mathbf{P}(A|T = t) = 0$. Using the total probability theorem,

$$\begin{aligned}
\mathbf{P}(A) &= \int_0^\infty f_T(t)\mathbf{P}(A \mid T = t)\, dt \\
&= \int_0^{0.75} f_T(t)\mathbf{P}(A|T = t)dt + \int_{0.75}^1 f_T(t)\mathbf{P}(A|T = t)dt \\
&= \int_0^{0.75} f_T(t)\mathbf{P}[\text{No arrivals in } [t + 0.25, 1]]dt + \int_{0.75}^1 f_T(t)1dt \\
&= \int_0^{0.75} \lambda e^{-\lambda t} \underbrace{e^{-\lambda(1-t-0.25)}}_{\text{泊松pdf，k=0}} dt + \int_{0.75}^1 \lambda e^{-\lambda t} 1dt \\
&= 0.75\lambda e^{-0.75\lambda} - e^{-\lambda} + e^{-0.75\lambda}.
\end{aligned}$$

提交 你已经尝试了1次（总共可以尝试2次）

Answers are displayed within the problem

Error and Bug Reports/Technical Issues

显示讨论

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