3. Hypothesis test with a continuous observation

Problem 3. Hypothesis test with a continuous observation

0/3 points (graded)

Let Θ be a Bernoulli random variable that indicates which one of two hypotheses is true, and let $\mathbf{P}(\Theta = 1) = p$. Under the hypothesis $\Theta = 0$, the random variable X has a normal distribution with mean 0, and variance 1. Under the alternative hypothesis $\Theta = 1$, X has a normal distribution with mean 2 and variance 1.

Consider the MAP rule for deciding between the two hypotheses, given that $X = x$.

1. Suppose for this part of the problem that $p = 2/3$. The MAP rule can choose in favor of the hypothesis $\Theta = 1$ if and only if $x \geq c_1$. Find the value of c_1.

   $c_1 =$ (1/2)*ln(2) + 1 ✖ **Answer:** 0.6534

2. For this part, assume again that $p = 2/3$. Find the conditional probability of error for the MAP decision rule, given that the hypothesis $\Theta = 0$ is true.

   $\mathbf{P}(\text{error}|\Theta = 0) =$ ✖ **Answer:** 0.2578

3. Find the overall (unconditional) probability of error associated with the MAP rule for $p = 1/2$.

   ✖ **Answer:** 0.1587

You may want to consult to standard normal table.

Normal Table

Show

Solution:

1. For $0 < p < 1$, we can choose in favor of the hypothesis $\Theta = 1$ if and only if

$$f_{X|\Theta}(x \mid 1) p_\Theta(1) \geq f_{X|\Theta}(x|0) p_\Theta(0)$$

$$\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x-2)^2\right) \cdot p \geq \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \cdot (1-p)$$

$$
\begin{aligned}
\frac{x^2}{2} - \frac{(x-2)^2}{2} &\geq \ln \frac{1-p}{p} \\
x &\geq 1 + \frac{1}{2} \ln \frac{1-p}{p}.
\end{aligned}
$$

For $p = 2/3$, this threshold corresponds to $c_1 = 1 - (\ln 2)/2 \approx 0.6534$.

2. Under the hypothesis $\Theta = 0$, an error occurs if we decide $\Theta = 1$. Therefore,

$$
\begin{aligned}
\mathbf{P}(\text{error} \mid \Theta = 0) &= \mathbf{P}(X \geq c_1 \mid \Theta = 0) \\
&= 1 - \mathbf{P}(X < c_1 \mid \Theta = 0) \\
&\approx 1 - \Phi(0.65) \\
&\approx 0.2578,
\end{aligned}
$$

since under $\Theta = 0$, X is a standard normal random variable.

3. With $p = 1/2$, the threshold becomes 1. Therefore, we decide $\Theta = 1$, whenever $x \geq 1$, and decide $\Theta = 0$, whenever $x < 1$. f

$$
\begin{aligned}
\mathbf{P}(\text{error}) &= \mathbf{P}(\text{error} \mid \Theta = 0) p_\Theta(0) + \mathbf{P}(\text{error} \mid \Theta = 1) p_\Theta(1) \\
&= \mathbf{P}(X \geq 1 \mid \Theta = 0) \frac{1}{2} + \mathbf{P}(X < 1 \mid \Theta = 1) \frac{1}{2} \\
&= \frac{1 - \Phi(1)}{2} + \frac{\mathbf{P}(X - 2 < -1 \mid \Theta = 1)}{2} \\
&= \frac{1 - \Phi(1)}{2} + \frac{1 - \Phi(1)}{2} \\
&= 1 - \Phi(1) \\
&\approx 1 - 0.8413 = 0.1587.
\end{aligned}
$$

提交 You have used 2 of 3 attempts

Answers are displayed within the problem

讨论

显示讨论

Topic: Unit 7 / Problem Set / 3. Hypothesis test with a continuous observation