10. Asymptotic Normality of the Method of Moments Estimator - Example

Let $(E, \{\mathbf{P}_\theta\}_{\theta\in\Theta})$ denote a statistical model associated to a statistical experiment $X_1, \ldots, X_n \stackrel{iid}{\sim} \mathbf{P}_{\theta^*}$ for some unknown parameter θ^*. Under some technical conditions, the method of moments estimator $\hat{\theta}_n^{\mathrm{MM}}$ is **asymptotically normal** , which means that

$$\sqrt{n}\,(\hat{\theta}_n^{\mathrm{MM}} - \theta^*) \xrightarrow[n\to\infty]{(d)} \mathcal{N}\,(0, \sigma^2)\,.$$

The quantity σ^2 above is referred to as the **asymptotic variance** .

In the next few problems, we will demonstrate the asymptotic normality for the method of moments estimator for an exponential statistical model. To do so, first we will construct the method of moments estimator and then use the delta method to compute the asymptotic variance explicitly.

Step 1: Moments Map for an Exponential Statistical Model

2/2 points (graded)

Let $X_1, \ldots, X_n \sim \mathbf{Exp}\,(\lambda^*)$ denote a statistical experiment where λ^* is the true, unknown parameter. You construct the associated statistical model $((0,\infty)\,, \{\mathbf{Exp}\,(\lambda)\}_{\lambda\in(0,\infty)})$. Since the parameter λ is one-dimensional, we only consider the first moment with moment map:

$$\begin{aligned}\psi : \mathbb{R}\ &\to \mathbb{R}\\ \lambda\ &\mapsto m_1\,(\lambda) := \mathbb{E}\,[X]\,,\ \ (X \sim \mathbf{Exp}\,(\lambda))\,.\end{aligned}$$

是个函数

What is $\psi\,(\lambda)$?

$\psi\,(\lambda) =$ 1/lambda

$\frac{1}{\lambda}$

Answer: 1/lambda

What is $\psi^{-1}\,(m_1)$?

Type **m_1** for m_1.

$\psi^{-1}\,(m_1) =$ 1/m_1

$\frac{1}{m_1}$

Answer: 1/m_1

Solution:

$$\begin{aligned}m_1\,(\lambda)\ &= \mathbb{E}\,[X]\\ &= \int_0^\infty \lambda x e^{-\lambda x}\,dx\\ &= -xe^{-\lambda x}\Big|_\infty^0 - \int_0^\infty -e^{-\lambda x}\,dx = 1/\lambda.\end{aligned}$$

Hence, $\psi\,(\lambda) = 1/\lambda$. Since ψ is its own inverse, we also have $\psi^{-1}\,(m_1) = 1/m_1$.

提交 你已经尝试了1次（总共可以尝试2次）

Answers are displayed within the problem

Step 2a: Deriving the Method of Moments Estimator

2/2 points (graded)
We use the same set-up from the previous problem. Recall that $X_1, \ldots, X_n \sim \mathbf{Exp}(\lambda^*)$ where λ^* is the true, unknown parameter. Also recall the moments map

$$\begin{aligned} \psi : \mathbb{R} &\to \mathbb{R} \\ \lambda &\mapsto m_1(\lambda), \end{aligned}$$

where $m_1(\lambda) := \mathbb{E}[X]$ with $X \sim \mathbf{Exp}(\lambda)$.

What is the method of moments estimator $\widehat{\lambda}_n^{\text{MM}}$ for λ^*?

Type **X_i** for X_i. The following two answer boxes together represent a fraction with the "/" symbol representing division.

$\widehat{\lambda}_n^{\text{MM}} =$ n n **Answer:** n / $\sum_{i=1}^{n}$ X_i X_i **Answer:** X_i

Solution:

Recall that $\psi(\lambda) = 1/\lambda$ from the previous question. Hence,

$$\widehat{\lambda}_n^{\text{MM}} = \psi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{n}{\sum_{i=1}^{n} X_i}.$$

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Answers are displayed within the problem

Review: Central Limit Theorem

1/1 point (graded)
The **Central Limit Theorem** states that if

- $X_1, \ldots, X_n$ are iid,
- $\mathbb{E}[X_1] = \mu < \infty$, and
- Var $(X_1) = \sigma^2 < \infty$

then

$$\sqrt{n}\left(\overline{X}_n - \mu\right) = \sqrt{n}\left(\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) - \mu\right) \xrightarrow[n\to\infty]{(d)} Z,$$

where Z is a normal random variable with mean 0.

What is the variance of Z?

Var $(Z) =$ sigma^2 σ^2 **Answer:** sigma^2

Solution:

The variance of Z is given by the variance of X_i, which is σ^2 for all $1 \leq i \leq n$. You are encouraged to review the central limit theorem in Lecture 2.

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Answers are displayed within the problem

Step 2b: Deriving the Method of Moments Estimator

1/1 point (graded)
We use the same set-up from the previous problems. Recall that $X_1, \ldots, X_n \sim \mathbf{Exp}(\lambda^*)$ where λ^* is the true, unknown parameter.

By the central limit theorem,

$$\sqrt{n}\left(\widehat{m}_1(\lambda^*) - \frac{1}{\lambda^*}\right) = \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{\lambda^*}\right)$$

converges to a normal random variable $\mathcal{N}(0, \sigma^2)$, where σ^2 can be written in terms of λ^*. What is σ^2?

Type **L** for λ^*.

$\sigma^2 =$ 1/L^2 Answer: 1/L^2

$\frac{1}{L^2}$

Solution:

By the central limit theorem, $\sigma^2 = \mathsf{Var}(X)$ where $X \sim \mathbf{Exp}(\lambda^*)$. Hence, we need to compute the variance of $\mathbf{Exp}(\lambda^*)$. The second moment is

$$\begin{aligned}
\mathbb{E}[X^2] &= \int_0^\infty x^2 \lambda^* e^{-\lambda^* x}\, dx \\
&= -x^2 e^{-\lambda^* x}\Big|_\infty^0 - \int_0^\infty -2xe^{-\lambda^* x}\, dx \\
&= 2/(\lambda^*)^2
\end{aligned}$$

since we showed in the solution of the last question that $\int_0^\infty \lambda^* x e^{-\lambda^* x}\, dx = 1/\lambda^*$. Thus,

$$\mathsf{Var}(X) = 2/(\lambda^*)^2 - 1/(\lambda^*)^2 = 1/(\lambda^*)^2.$$

Therefore, $\sigma^2 = 1/(\lambda^*)^2$.

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Answers are displayed within the problem

Step 3: Computing the Asymptotic Variance of the Method of Moments Estimator

1/1 point (graded)
Suppose that

$$\sqrt{n}\left(\widehat{m}_1 - m_1(\theta)\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0, \sigma^2).$$

(Think of $\widehat{m}_1$ and m_1 as the first sample moment and first moment, respectively.)

Recall that the **delta method** states that if the above holds, then for any $g : \mathbb{R} \to \mathbb{R}$ that has a continuous first derivative,

$$\sqrt{n}\left(g\left(\widehat{m}_1\right) - g\left(m_1\left(\theta\right)\right)\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}\left(0, g'\left(m_1\left(\theta\right)\right)^2 \sigma^2\right)$$

We use the same set-up from the previous problem. Recall that $X_1, \ldots, X_n \sim \mathbf{Exp}\left(\lambda^*\right)$ where λ^* is the true, unknown parameter. Also recall the moments map

$$\begin{aligned} \psi : \mathbb{R} &\to \mathbb{R} \\ \lambda &\mapsto m_1\left(\lambda\right), \end{aligned}$$

where $m_1\left(\lambda\right) := \mathbb{E}\left[X\right]$ with $X \sim \mathbf{Exp}\left(\lambda\right)$.

By the central limit theorem for the method of moments estimator, $\widehat{\lambda}_n^{\text{MM}}$ is asymptotically normal, meaning that

$$\sqrt{n}\left(\widehat{\lambda}_n^{\text{MM}} - \lambda^*\right) \xrightarrow{(d)} \mathcal{N}\left(0, \tau^2\right)$$

where τ^2 is the asymptotic variance and can be expressed in terms of λ^*.

Applying the last problem and the delta method, what is the asymptotic variance τ^2 in terms of λ^*?

Use the letter **L** to stand for λ^*.

$\tau^2 =$ L^2 **Answer:** L^2

L^2

Solution:

By the previous problem, we have

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - 1/\lambda^*\right) \xrightarrow{(d)} \mathcal{N}\left(0, 1/(\lambda^*)^2\right).$$

Letting $g = \psi^{-1}$ in the statement of the delta method, and noting that $\left(\psi^{-1}\right)'\left(m_1\left(\lambda\right)\right) = -\lambda^2$, we see that

$$\sqrt{n}\left(\widehat{\lambda}_n^{\text{MM}} - \lambda^*\right) \xrightarrow{(d)} \mathcal{N}\left(0, (\lambda^*)^2\right).$$

提交 你已经尝试了1次（总共可以尝试3次）

Answers are displayed within the problem

讨论

显示讨论

主题： Unit 3 Methods of Estimation:Lecture 11: Fisher Information, Asymptotic Normality of MLE; Method of Moments / 10. Asymptotic Normality of the Method of Moments Estimator - Example