

Beyond Vectors

Hung-yi Lee

Introduction

- Many things can be considered as “vectors”.
 - E.g. a function can be regarded as a vector
- We can apply the concept we learned on those “vectors”.
 - Linear combination
 - Span
 - Basis
 - Orthogonal
- Reference: Chapter 6

Are they vectors?

Are they vectors?

- A matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

- A linear transform
- A polynomial

$$p(x) = a_0 + a_1x + \cdots + a_nx^n \Rightarrow \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Are they vectors?

What is the zero vector?

- Any function is a vector?

$$f(t) = e^t \quad \Rightarrow \quad v = \begin{bmatrix} \vdots \\ ? \\ \vdots \end{bmatrix}$$

我们在 e^t 的函数上取无穷的sample
每个sample就是一维
一共有无限维

$$g(t) = t^2 - 1 \quad \Rightarrow \quad g = \begin{bmatrix} \vdots \\ ? \\ \vdots \end{bmatrix}$$

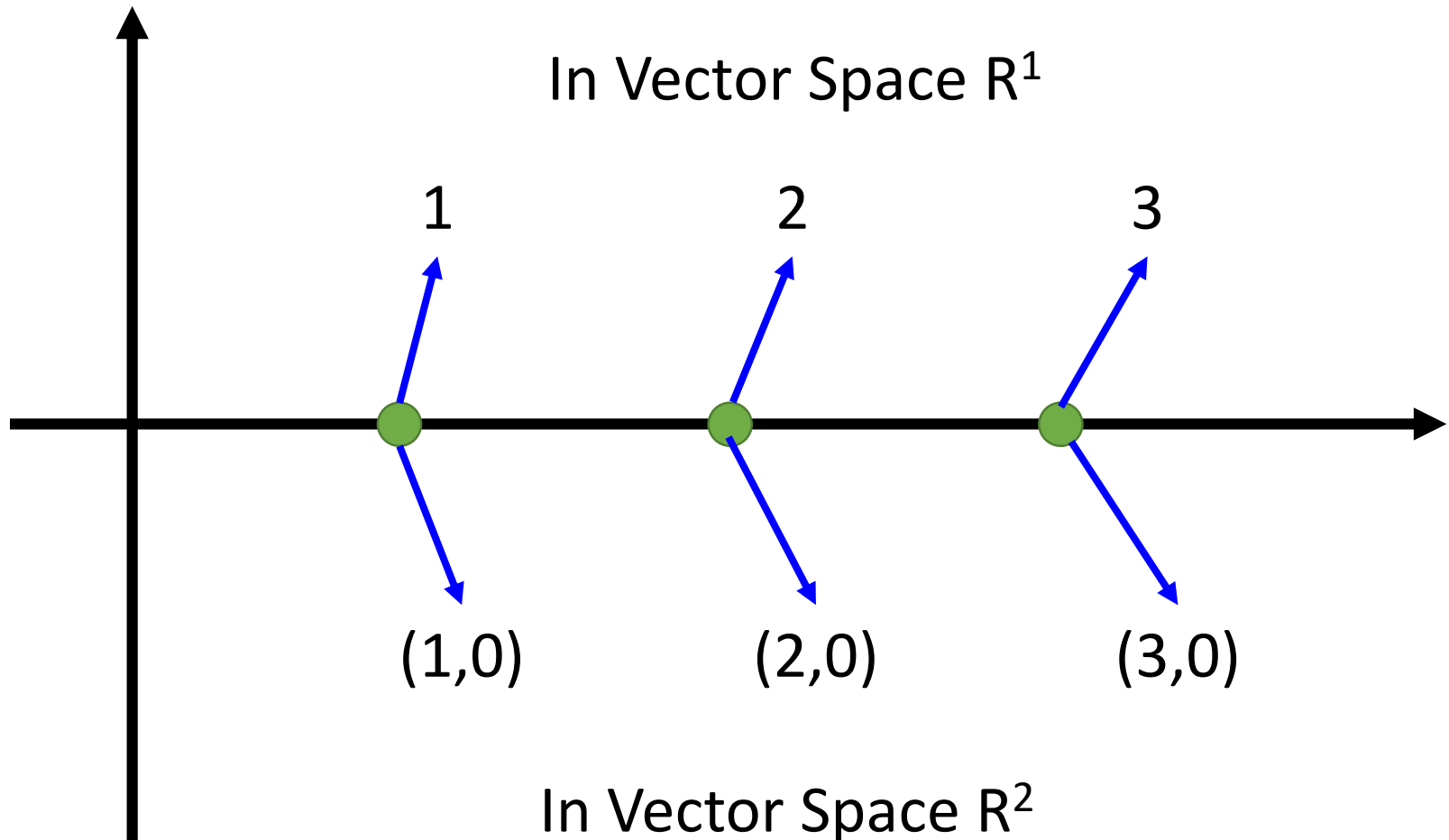
$$h(t) = e^t + t^2 - 1 \quad \Rightarrow \quad v + g$$

What is a vector?

\mathbb{R}^n is a
vector space

- If a set of objects V is a **vector space**, then the objects are “vectors”.
- Vector space:
 - There are operations called “***addition***” and “***scalar multiplication***”.
 - u, v and w are in V , and a and b are scalars. $u+v$ and au are unique elements of V
- The following axioms hold:
 - $u + v = v + u, (u + v) + w = u + (v + w)$
 - There is a “zero vector” 0 in V such that $u + 0 = u$ unique
 - There is $-u$ in V such that $u + (-u) = 0$
 - $1u = u, (ab)u = a(bu), a(u+v) = au + av, (a+b)u = au + bu$

Objects in Different Vector Spaces



Objects in Different Vector Spaces

All the polynomials with degree less than or equal to 2 as a vector space

$$\begin{array}{ccc} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ \uparrow & \uparrow & \uparrow \\ f(t) = 1 & g(t) = t + 1 & h(t) = t^2 + t + 1 \end{array}$$

Vectors with infinite dimensions

All functions as a vector space

Subspaces

Review: Subspace

- A vector set V is called a subspace if it has the following three properties:
- 1. The zero vector $\mathbf{0}$ belongs to V
- 2. If \mathbf{u} and \mathbf{w} belong to V , then $\mathbf{u} + \mathbf{w}$ belongs to V

Closed under (vector) addition

- 3. If \mathbf{u} belongs to V , and c is a scalar, then $c\mathbf{u}$ belongs to V

Closed under scalar multiplication

Are they subspaces?

- ✓ • All the functions pass 0 at t_0
- ✓ • All the matrices whose trace equal to zero
- ✓ • All the matrices of the form

$$\begin{bmatrix} a & a + b \\ b & 0 \end{bmatrix}$$

- ✓ • All the continuous functions
- ✗ • All the polynomials with degree n
- ✓ • All the polynomials with degree less than or equal to n

比如说，不符合closed addition，因为两个 n 方假如加起来等于0的话，0的degree不是 n ，所以不满足这个条件，掉出去了。

P: all polynomials, P_n : all polynomials with degree less than or equal to n

Linear Combination and Span

Linear Combination and Span

- Matrices

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

Linear combination with coefficient a, b, c

$$a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

What is Span S?

All 2x2 matrices whose trace equal to zero

Linear Combination and Span

- Polynomials

$$S = \{1, x, x^2, x^3\}$$

Is $f(x) = 2 + 3x - x^2$ linear combination of the “vectors” in S ?

$$f(x) = 2 \cdot 1 + 3 \cdot x + (-1) \cdot x^2$$

$$\text{Span}\{1, x, x^2, x^3\} = P_3$$

$$\text{Span}\{1, x, \dots, x^n, \dots\} = P$$

Linear Transformation

Linear transformation

- A mapping (function) T is called linear if for all “vectors” u, v and scalars c :
- Preserving vector addition:

$$T(u + v) = T(u) + T(v)$$

- Preserving vector multiplication: $T(cu) = cT(u)$

Is matrix transpose linear?

Input: $m \times n$ matrices, output: $n \times m$ matrices

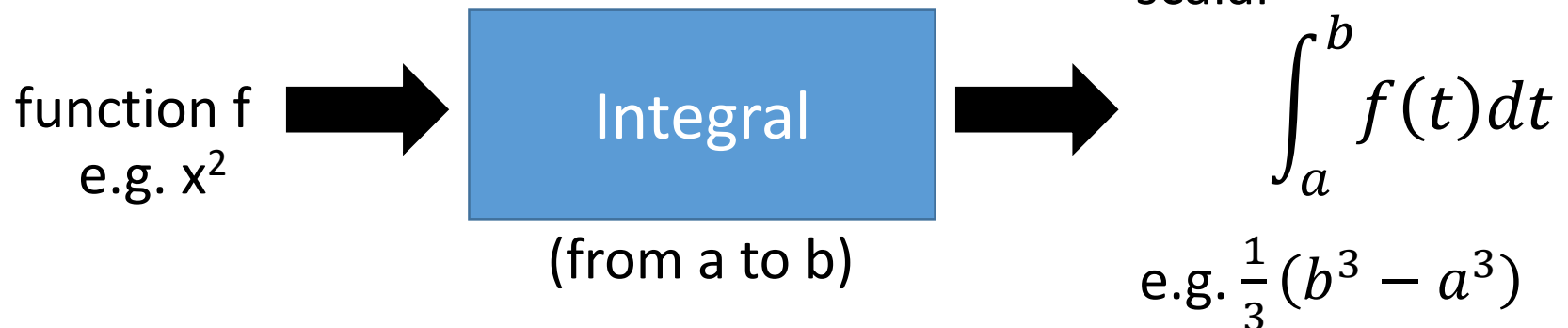
Linear transformation

- Derivative: linear?



- Integral from a to b

linear?



Null Space and Range

- Null Space
 - The null space of T is the set of all vectors such that $T(v)=0$
 - What is the null space of matrix transpose?
- Range
 - The range of T is the set of all images of T .
 - That is, the set of all vectors $T(v)$ for all v in the domain
 - What is the range of matrix transpose?

One-to-one and Onto

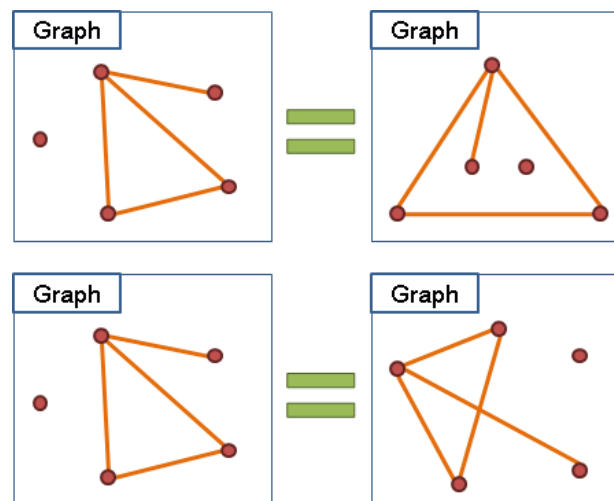
- $U: \mathcal{M}_{m \times n} \rightarrow \mathcal{M}_{n \times m}$ defined by $U(A) = A^T$.
 - Is U one-to-one? ☒ yes
 - Is U onto? ☒ yes
- $D: \mathcal{P}_3 \rightarrow \mathcal{P}_3$ defined by $D(f) = f' \in \mathcal{P}_2$
 - Is D one-to-one? ☐ no
 - Is D onto? ☐ no

Isomorphism (同構)

Biology 趋同演化



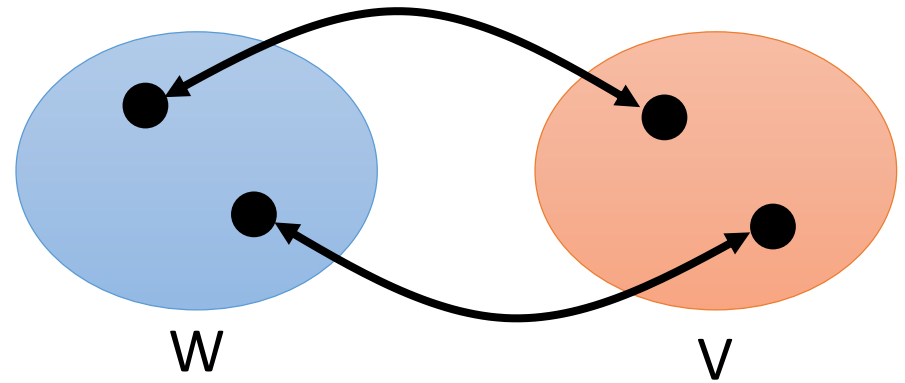
Graph



Chemistry



Isomorphism



- Let V and W be vector space.
- A linear transformation $T: V \rightarrow W$ is called an isomorphism if it is one-to-one and onto
 - **Invertible linear transform**
- W and V are isomorphic. 结构是一样的，比如身份证和学号

Example 1: $U: \mathcal{M}_{m \times n} \rightarrow \mathcal{M}_{n \times m}$ defined by $U(A) = A^T$.

Example 2: $T: \mathcal{P}_2 \rightarrow \mathcal{R}^3$

$$T\left(a + bx + \frac{c}{2}x^2\right) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Basis

A **basis** for subspace V is a **linearly independent** generation set of V .

Independent

- Example

$S = \{x^2 - 3x + 2, 3x^2 - 5x, 2x - 3\}$ is a subset of \mathcal{P}_2 .

Is it linearly independent?

$$3(x^2 - 3x + 2) + (-1)(3x^2 - 5x) + 2(2x - 3) = 0$$

No

- Example

$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ is a subset of 2x2 matrices.

Is it linearly independent?

$$a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

implies that $a = b = c = 0$

Yes

Independent

If $\{v_1, v_2, \dots, v_k\}$ are L.I., and T is an isomorphism, $\{T(v_1), T(v_2), \dots, T(v_k)\}$ are L.I.

- Example

The infinite vector set $\{1, x, x^2, \dots, x^n, \dots\}$

Is it linearly independent?

$$\sum_i c_i x^i = 0 \text{ implies } c_i = 0 \text{ for all } i.$$

Yes

- Example

$S = \{e^t, e^{2t}, e^{3t}\}$ Is it linearly independent?

Yes

$$ae^t + be^{2t} + ce^{3t} = 0$$

$$a + b + c = 0$$

$$ae^t + 2be^{2t} + 3ce^{3t} = 0$$

$$a + 2b + 3c = 0$$

$$ae^t + 4be^{2t} + 9ce^{3t} = 0$$

$$a + 4b + 9c = 0$$

Basis

- Example

For the subspace of all 2 x 2 matrices,

The basis is

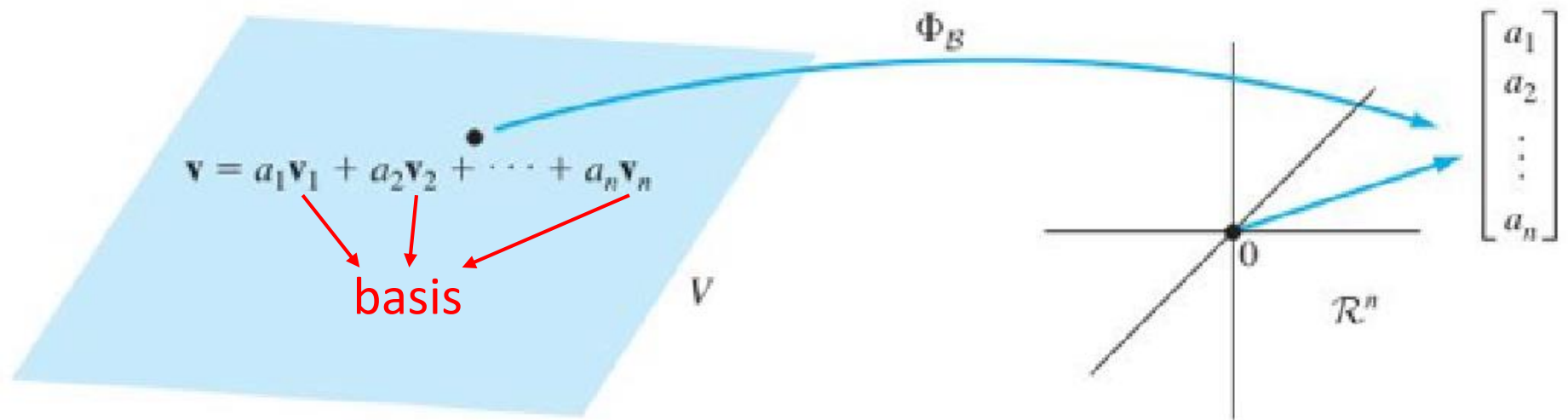
$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \quad \text{Dim} = 4$$

- Example

$$S = \{1, x, x^2, \dots, x^n, \dots\} \text{ is a basis of } \mathcal{P}. \quad \text{Dim} = \text{inf}$$

Vector Representation of Object

- Coordinate Transformation



P_n : Basis: $\{1, x, x^2, \dots, x^n\}$

$$p(x) = a_0 + a_1 x + \dots + a_n x^n$$



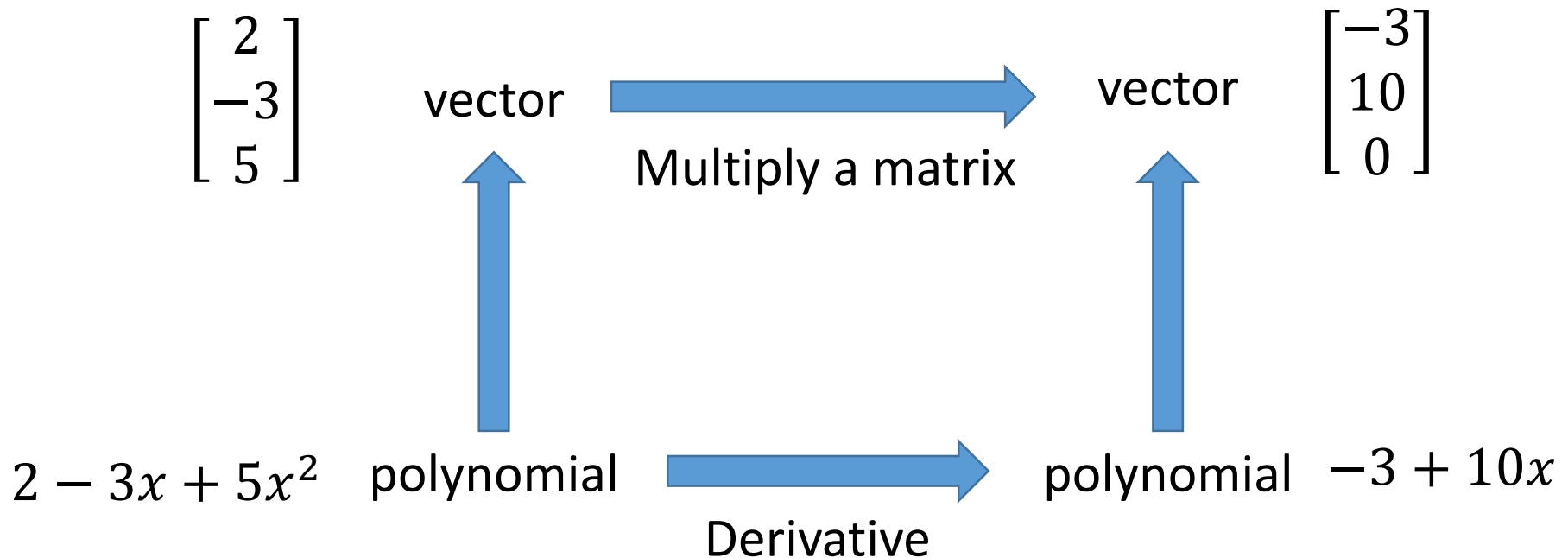
$$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Matrix Representation of Linear Operator

- Example:

- D (derivative): $P_2 \rightarrow P_2$

Represent it as a matrix

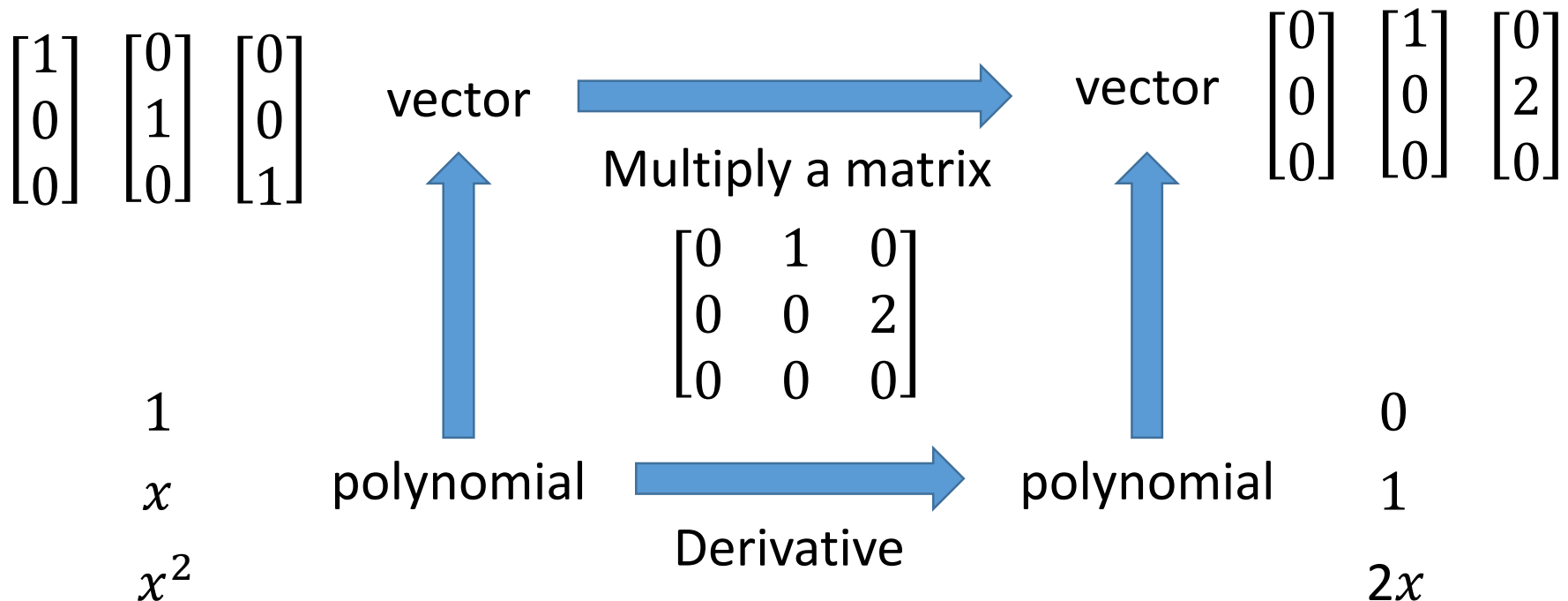


Matrix Representation of Linear Operator

- Example:

- D (derivative): $P_2 \rightarrow P_2$

Represent it as a matrix



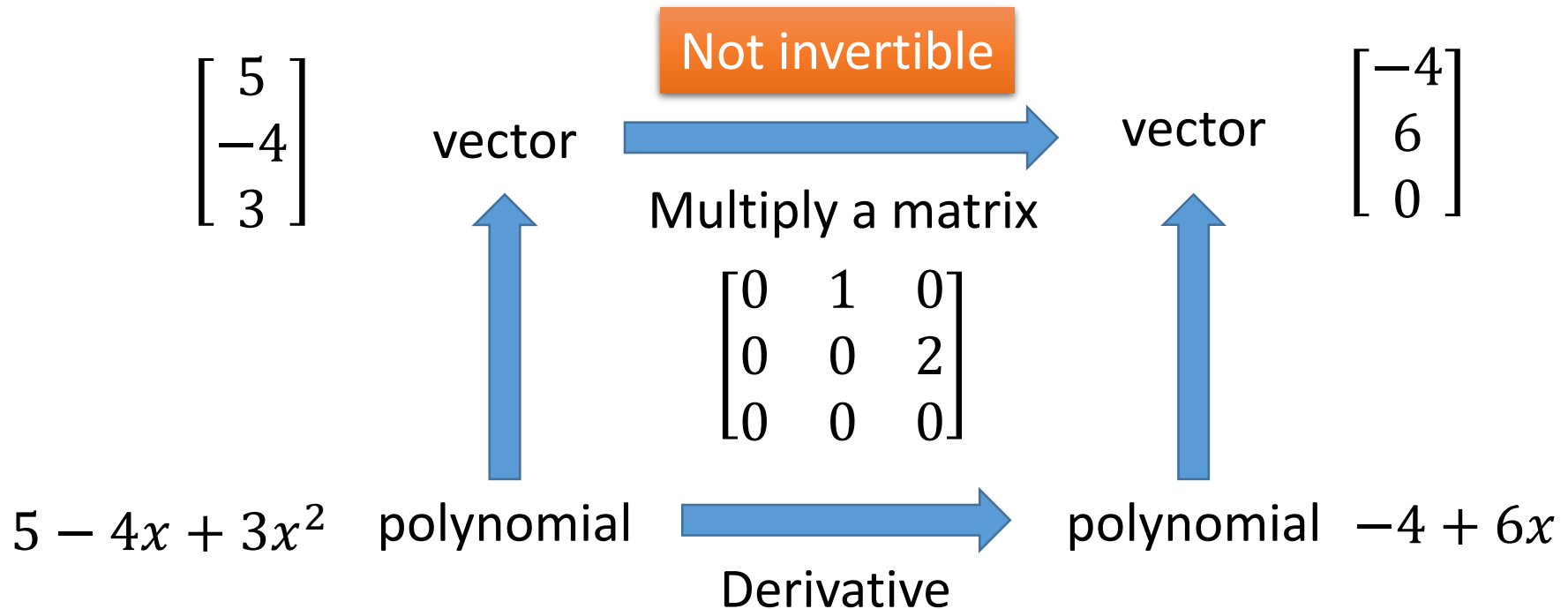
Matrix Representation of Linear Operator

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

- Example:

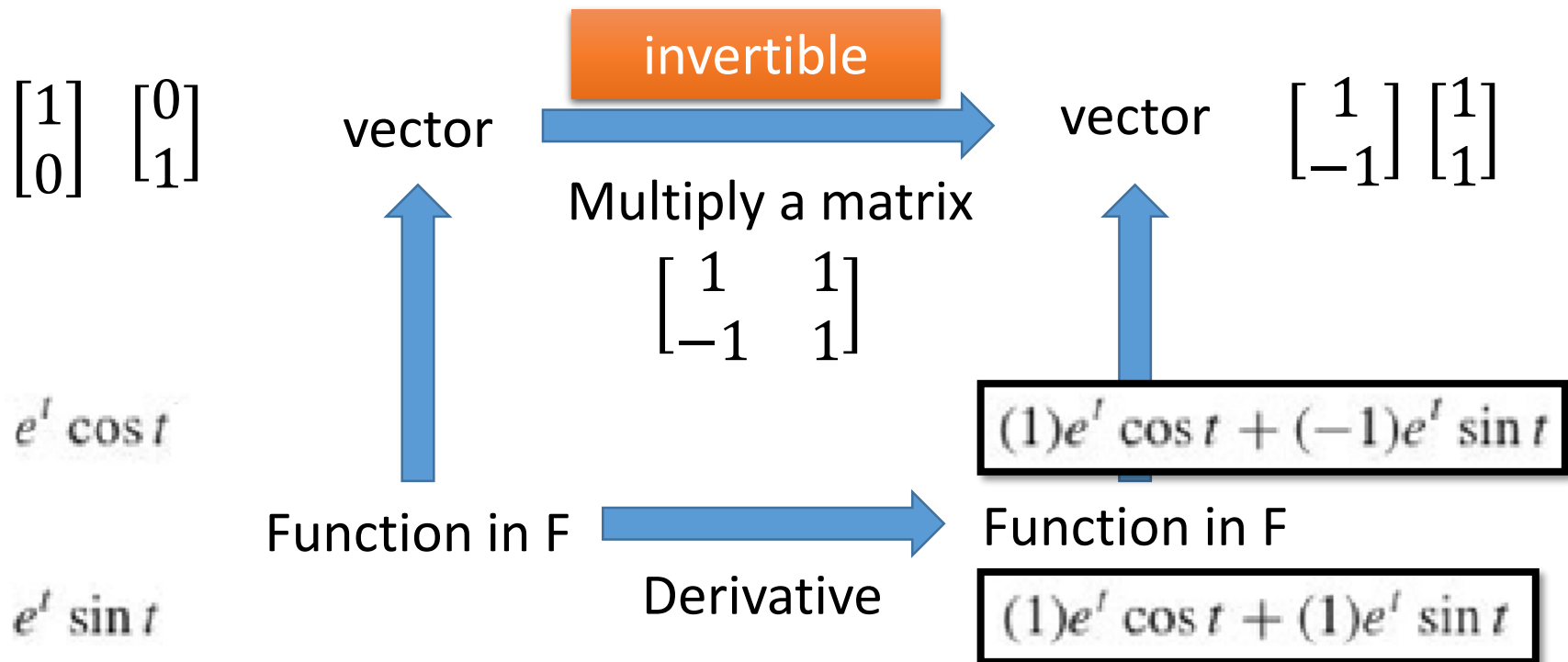
- D (derivative): $P_2 \rightarrow P_2$

Represent it as a matrix



Matrix Representation of Linear Operator

- Example:
 - D (derivative): Function set $F \rightarrow$ Function set F
 - Basis of F is $\{e^t \cos t, e^t \sin t\}$



Matrix Representation of Linear Operator

Basis of F is
 $\{e^t \cos t, e^t \sin t\}$

$$\begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$$

vector

$$\left(-\frac{1}{2}\right)e^t \cos t + \left(\frac{1}{2}\right)e^t \sin t$$

Function in F

$$\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

Multiply a matrix

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Antiderivative

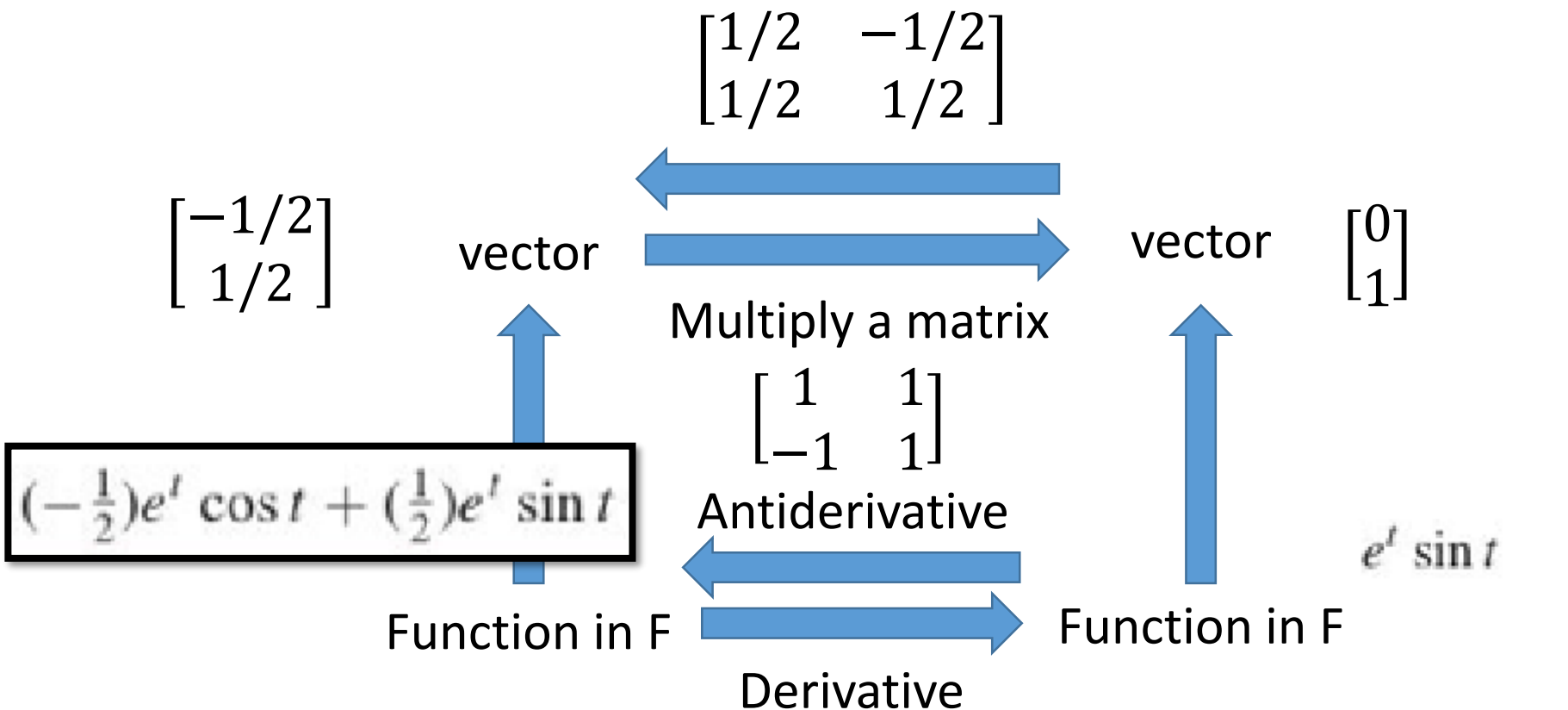
Derivative

vector

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$e^t \sin t$$

Function in F



Eigenvalue and Eigenvector

$$T(v) = \lambda v, v \neq 0, v \text{ is eigenvector}, \lambda \text{ is eigenvalue}$$

Eigenvalue and Eigenvector

- Consider derivative (linear transformation, input & output are functions)

Is $f(t) = e^{at}$ an “eigenvector”? What is the “eigenvalue”? a

Every scalar is an eigenvalue of derivative.

- Consider Transpose (also linear transformation, input & output are functions)

Is $\lambda = 1$ an eigenvalue?

Symmetric matrices form the eigenspace

Symmetric:

$$A^T = A$$

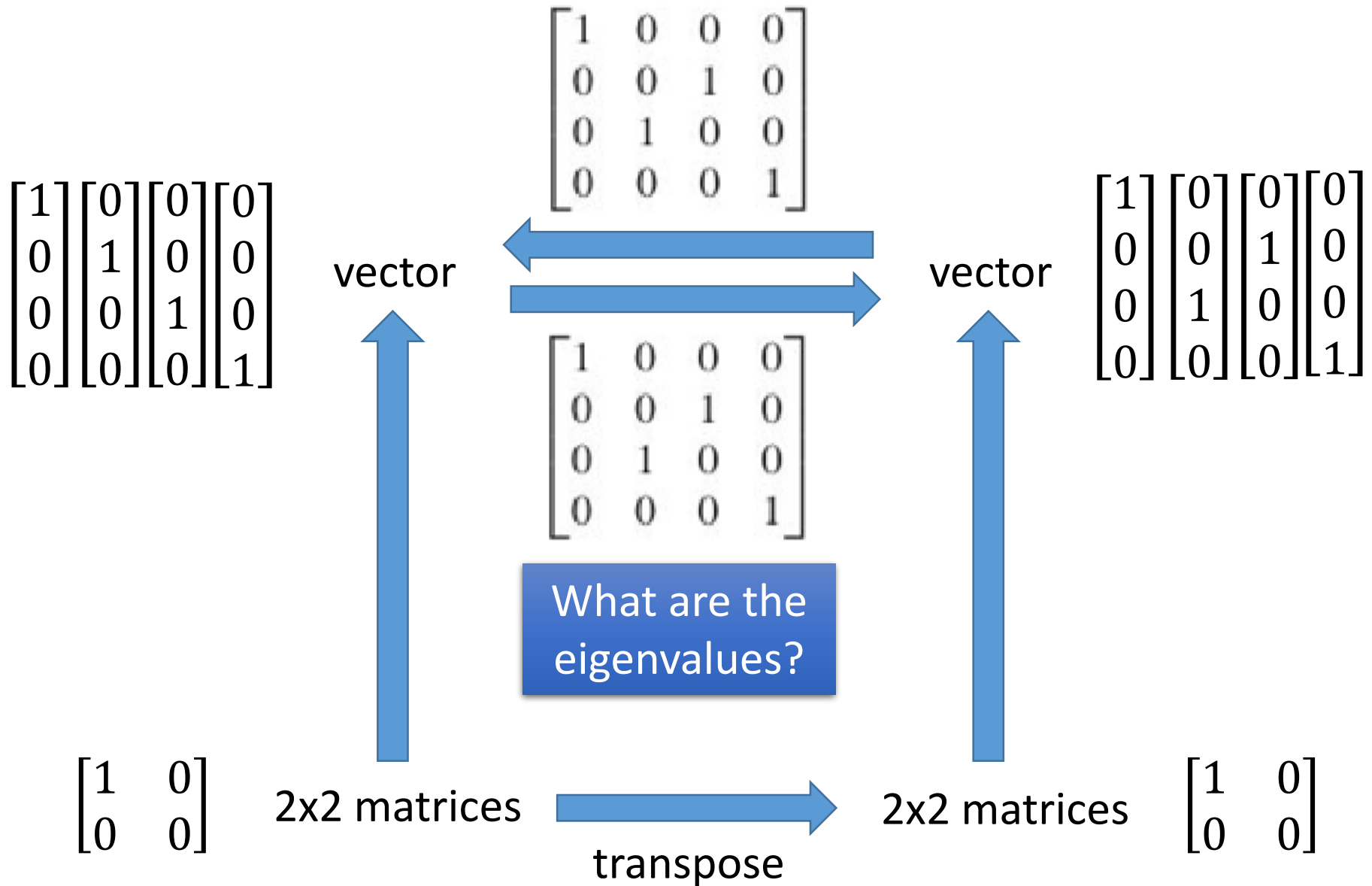
Is $\lambda = -1$ an eigenvalue?

Skew-symmetric:

Skew-symmetric matrices form the eigenspace.

$$A^T = -A$$

Consider Transpose of 2x2 matrices



Eigenvalue and Eigenvector

- **Consider Transpose of 2x2 matrices**

Matrix
representation
of transpose

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Characteristic polynomial

$$(t - 1)^3(t + 1)$$

$$\lambda = 1$$

Symmetric matrices

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Dim=3

$$\lambda = -1$$

Skew-symmetric matrices

$$\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$$

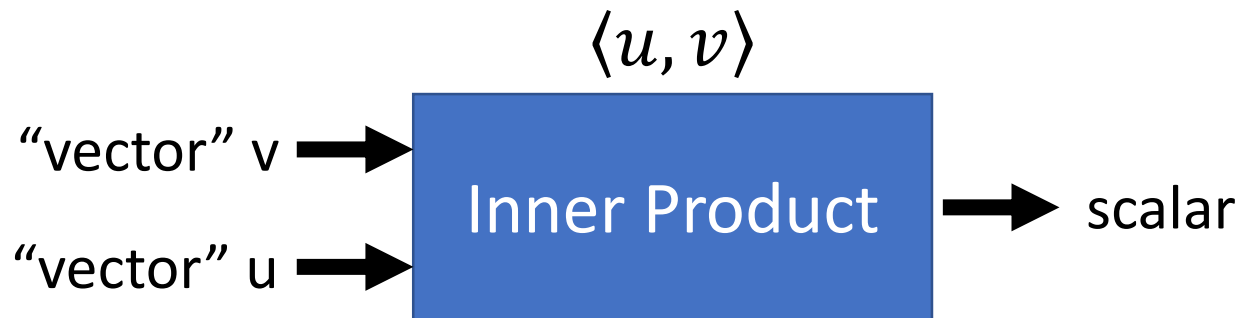
Dim=1

Inner Product

Inner Product

Norm (length): $\|v\| = \sqrt{\langle v, v \rangle}$

Orthogonal: Inner product is zero



For any vectors u, v and w , and any scalar a , the following axioms hold:

1. $\langle u, u \rangle > 0$ if $u \neq 0$
2. $\langle u, v \rangle = \langle v, u \rangle$
3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
4. $\langle au, v \rangle = a\langle u, v \rangle$

Dot product is a special case of inner product

Can you define other inner product for normal vectors?

Inner Product

- Inner Product of Matrix

Frobenius
inner product

$$\begin{aligned}\langle A, B \rangle &= \text{trace}(AB^T) \\ &= \text{trace}(BA^T)\end{aligned}$$

$$\left\langle \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \right\rangle = 1 \cdot 5 + 2 \cdot 6 + 3 \cdot 7 + 4 \cdot 8 = 70$$

trace在对角，也就是同样位置的相乘

Element-wise multiplication

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \|A\| = \sqrt{1^2 + 2^2 + 3^2 + 4^2}$$

Inner Product

1. $\langle u, u \rangle > 0$ if $u \neq 0$
2. $\langle u, v \rangle = \langle v, u \rangle$
3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
4. $\langle au, v \rangle = a\langle u, v \rangle$

- Inner product for general functions

$$\langle g, h \rangle = \int_{-1}^1 g(x)h(x) dx$$


Is $g(x) = 1$ and
 $h(x) = x$ orthogonal?

$$\langle g, h \rangle = \sum_{i=-10}^{10} g(i)h(i)$$


Can it be inner product for
general functions?

Orthogonal/Orthonormal Basis

- Let u be any vector, and w is the orthogonal projection of u on subspace W .
- Let $S = \{v_1, v_2, \dots, v_k\}$ be an orthogonal basis of W .

$$w = c_1 v_1 + c_2 v_2 + \dots + c_k v_k$$

$$\frac{u \cdot v_1}{\|v_1\|^2} \quad \frac{u \cdot v_2}{\|v_2\|^2} \quad \frac{u \cdot v_k}{\|v_k\|^2}$$

- Let $S = \{v_1, v_2, \dots, v_k\}$ be an orthonormal basis of W .

$$w = c_1 v_1 + c_2 v_2 + \dots + c_k v_k$$

$$u \cdot v_1 \quad u \cdot v_2 \quad u \cdot v_k$$

Orthogonal Basis

Let $\{u_1, u_2, \dots, u_k\}$ be a basis of a subspace V . How to transform $\{u_1, u_2, \dots, u_k\}$ into an orthogonal basis $\{v_1, v_2, \dots, v_k\}$?

**Gram-Schmidt
Process**

Then $\{v_1, v_2, \dots, v_k\}$ is an orthogonal basis for W

After normalization, you can
get orthonormal basis.

Orthogonal/Orthonormal Basis

- Find orthogonal/orthonormal basis for P_2
 - Define an inner product of P_2 by

$$\langle f(x), g(x) \rangle = \int_{-1}^1 f(t)g(t) dt$$

- Find a basis $\{1, x, x^2\}$ \longrightarrow $u_1, u_2, u_3 \longrightarrow v_1, v_2, v_3$

$$v_1 = u_1$$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2 = x^2 - \frac{1}{3}$$

Orthogonal/Orthonormal Basis

- Find orthogonal/orthonormal basis for P_2
 - Define an inner product of P_2 by

$$\langle f(x), g(x) \rangle = \int_{-1}^1 f(t)g(t) dt$$

- Get an orthogonal basis $\{1, x, x^2 - 1/3\}$

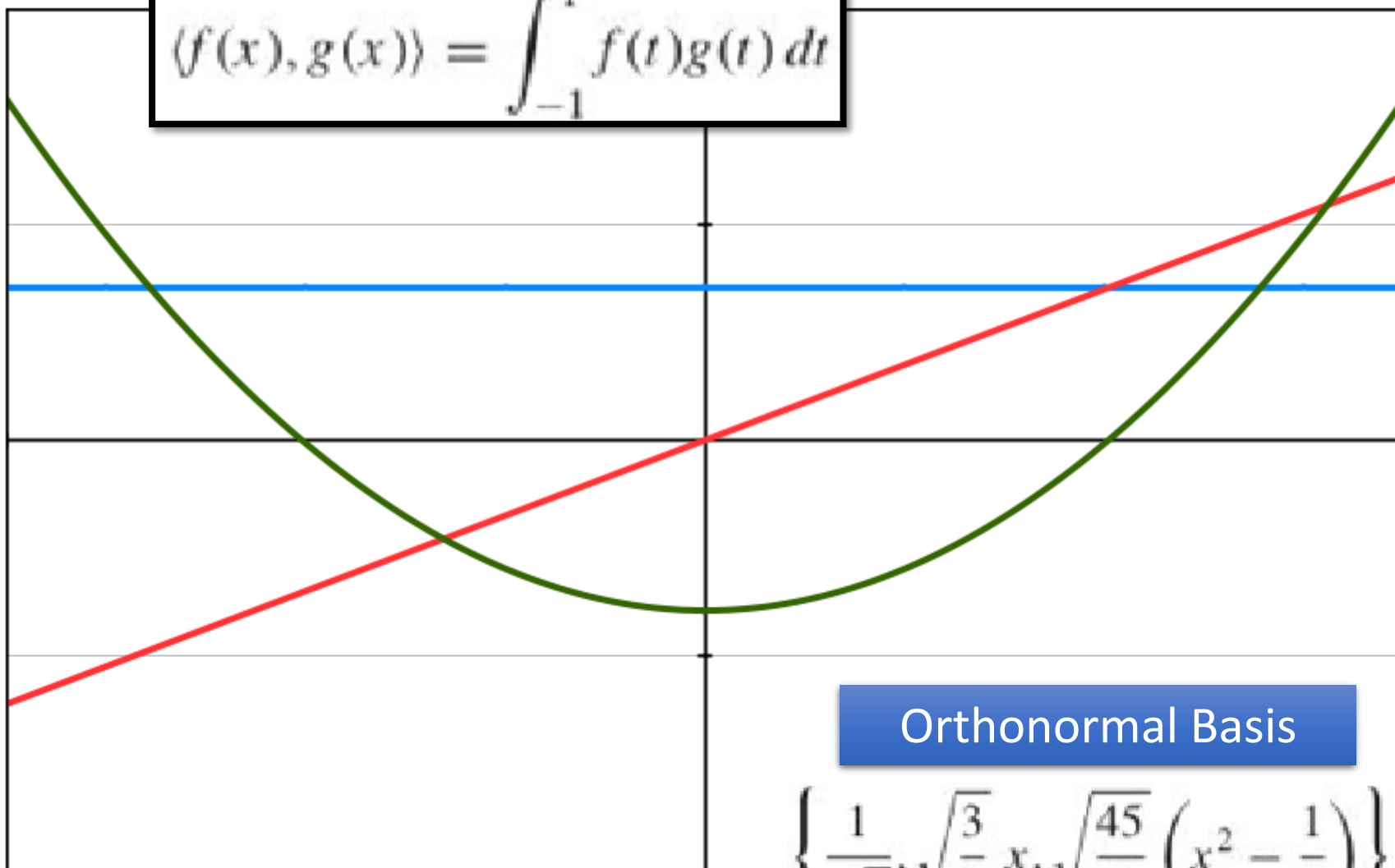
$$\|\mathbf{v}_1\| = \sqrt{\int_{-1}^1 1^2 dx} = \sqrt{2} \quad \|\mathbf{v}_2\| = \sqrt{\int_{-1}^1 x^2 dx} = \sqrt{\frac{2}{3}}$$

$$\|\mathbf{v}_3\| = \sqrt{\int_{-1}^1 \left(x^2 - \frac{1}{3}\right)^2 dx} = \sqrt{\frac{8}{45}}$$

Orthonormal Basis

$$\left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}} x, \sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3}\right) \right\}$$

$$\langle f(x), g(x) \rangle = \int_{-1}^1 f(t)g(t) dt$$



Orthonormal Basis

$$\left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}} x, \sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3} \right) \right\}$$