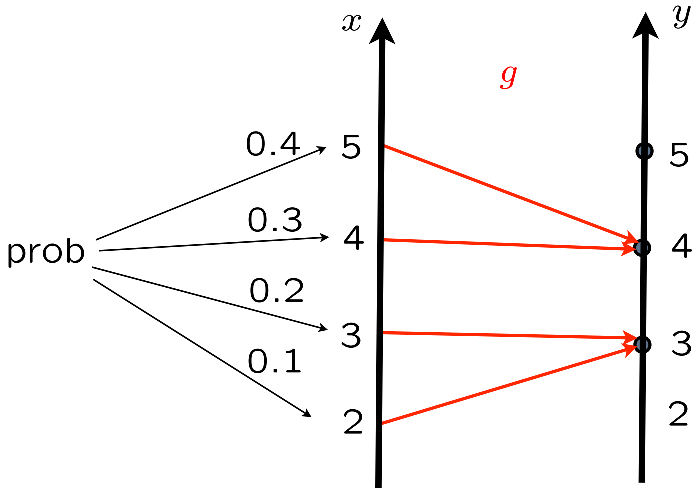
LECTURE 11: Derived distributions

- Given the distribution of X,
  find the distribution of $Y = g(X)$
  - the discrete case
  - the continuous case
  - general approach, using CDFs
  - the linear case: $Y = aX + b$
  - general formula when g is monotonic

- Given the (joint) distribution of X and Y,
  find the distribution of $Z = g(X, Y)$

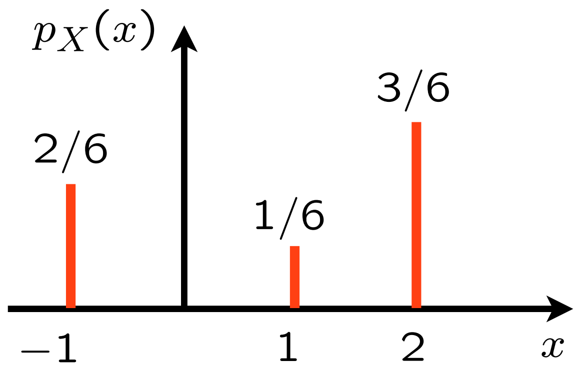
Derived distributions — the discrete case

$Y = g(X)$

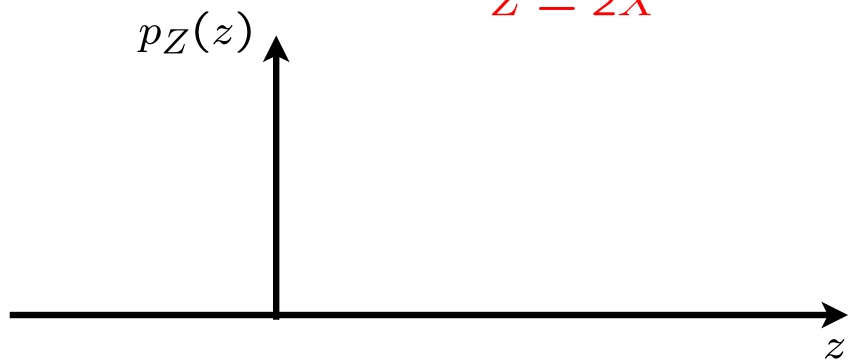


$$
\begin{aligned}
p_Y(y) &= \mathbf{P}(g(X) = y) \\
&= \sum_{x:\, g(x)=y} p_X(x)
\end{aligned}
$$

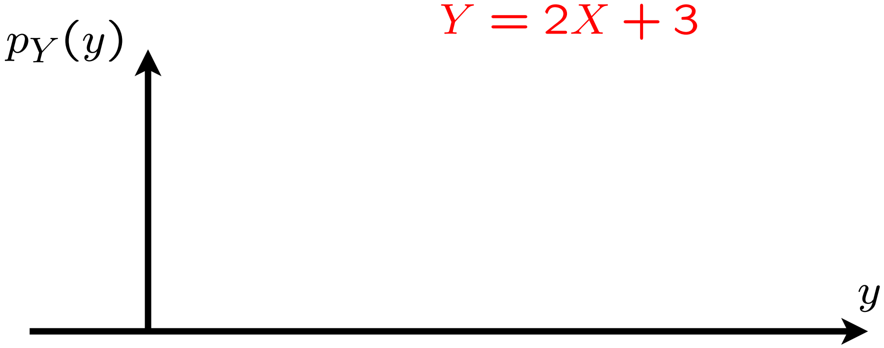
A linear function of a discrete r.v.



$Z = 2X$

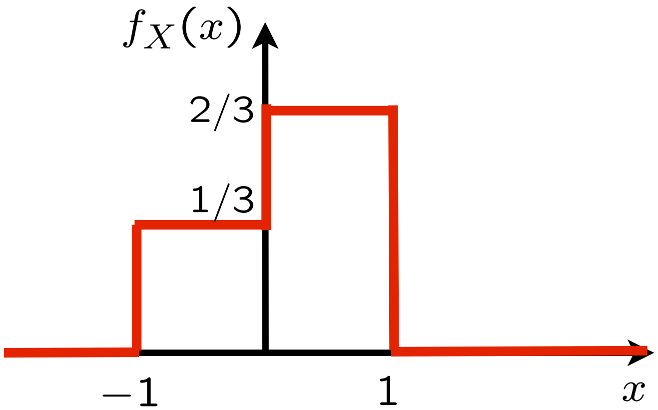


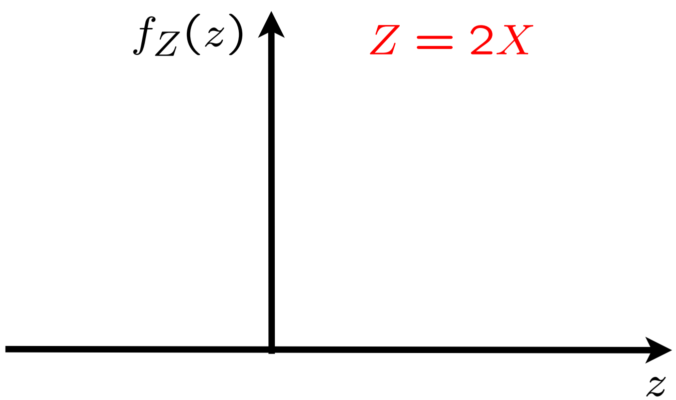
$Y = 2X + 3$

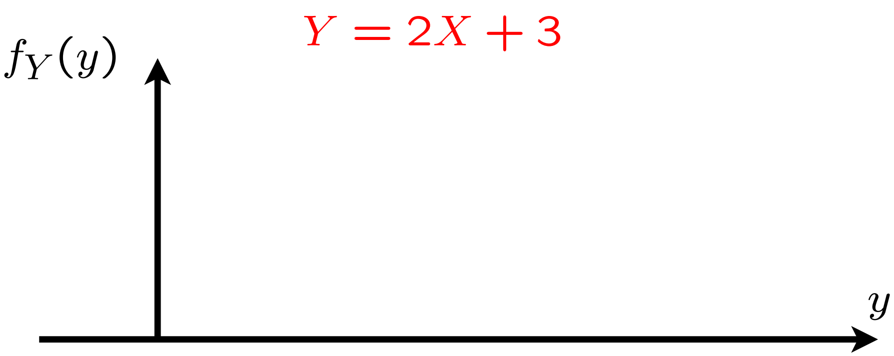


$$Y = aX + b: \quad p_Y(y) = p_X\left(\frac{y-b}{a}\right)$$

A linear function of a continuous r.v.







A linear function of a continuous r.v.

$Y = aX + b$

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

$$p_Y(y) = p_X\left(\frac{y-b}{a}\right)$$

A linear function of a normal r.v. is normal

$X \sim \mathcal{N}(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

$Y = aX + b, \qquad a \neq 0$

$$\boxed{f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)}$$

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

A general function $g(X)$ of a continuous r.v.

- **Two-step procedure:**
  - Find the CDF of Y: $F_Y(y) = \mathbf{P}(Y \le y)$
  - Differentiate: $f_Y(y) = \dfrac{dF_Y}{dy}(y)$

Example: $Y = X^3$; $\quad X$ **uniform on** $[0, 2]$

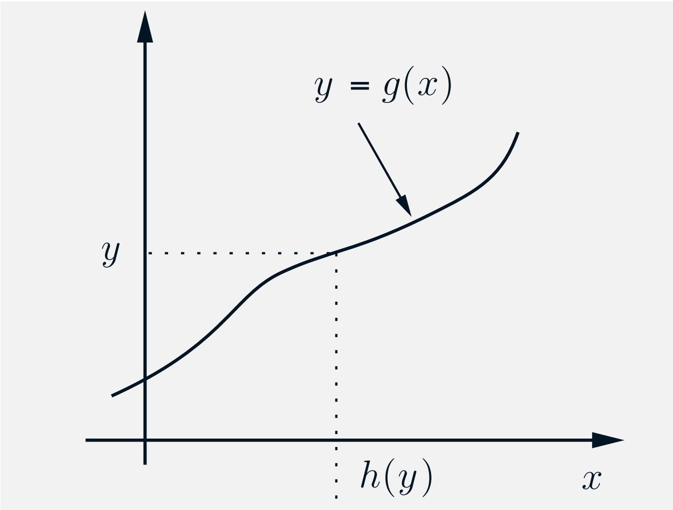
Example: $Y = a/X$

- You go to the gym and set the speed X of the treadmill to a number between 5 and 10 km/hr (with a uniform distribution). Find the PDF of the time it takes to run 10km.

A general formula for the PDF of $Y = g(X)$ when g is monotonic

Assume g strictly increasing

and differentiable



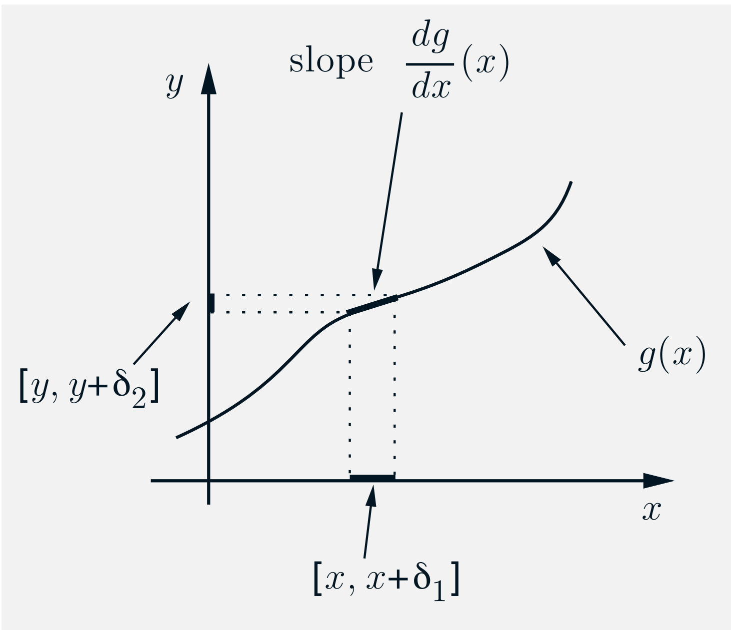
inverse function h

$$\boxed{f_Y(y) = f_X\big(h(y)\big)\left|\frac{dh}{dy}(y)\right|}$$

Example: $Y = X^2$; X uniform on $[0, 1]$

$$\boxed{f_Y(y) = f_X\big(h(y)\big)\left|\frac{dh}{dy}(y)\right|}$$

An intuitive explanation for the monotonic case



A nonmonotonic example: $Y = X^2$

- The discrete case:

  $p_Y(9) =$

  $p_Y(y) =$

- The continuous case:

A function of multiple r.v.'s: $Z = g(X, Y)$

- Same methodology: find CDF of Z

- Let $Z = Y/X$; X, Y independent, uniform on $[0, 1]$