2. Independent uniform random variables

Problem 1. Independent uniform random variables

5.0/5.0 points (graded)

Let X, Y, Z be three independent (i.e. mutually independent) random variables, each uniformly distributed on the interval $[0,1]$.

1. Find the mean and variance of $1/(Z+1)$.

   $\mathbf{E}[1/(Z+1)] =$ ln(2) ✔ **Answer:** 0.693

   $\text{var}(1/(Z+1)) =$ 1/2 - (ln(2))^2 ✔ **Answer:** 0.019

2. Find the mean of $XY/(Z+1)$.
   Hint: Use your answer to the previous part, together with the independence assumption.

   $\mathbf{E}[XY/(Z+1)] =$ 1/4 * ln(2) ✔ **Answer:** 0.173

3. Find the probability that $XY/Z \le 1$. Enter a numerical answer.

   $\mathbf{P}(XY/Z \le 1) =$ 3/4 ✔ **Answer:** 0.75

Solution:

1. Since Z is uniform on $[0,1]$, we can compute the expected value of $1/(Z+1)$ as follows:

$$\begin{aligned}\mathbf{E}\left[\frac{1}{Z+1}\right] &= \int_0^1 \frac{1}{z+1} f_Z(z)\, dz \\ &= \int_0^1 \frac{1}{z+1}\, dz \\ &= \ln(z+1)\Big|_0^1 = \ln(2) - \ln(1) \\ &= \ln(2) \\ &\approx 0.693.\end{aligned}$$

For the variance, we start by computing $\mathbf{E}\left[1/(Z+1)^2\right]$.

$$\begin{aligned}\mathbf{E}\left[\frac{1}{(Z+1)^2}\right] &= \int_0^1 \frac{1}{(z+1)^2} f_Z(z)\, dz \\ &= \int_0^1 \frac{1}{(z+1)^2}\, dz \\ &= -\frac{1}{z+1}\Big|_0^1 = 1 - \frac{1}{2} \\ &= \frac{1}{2}.\end{aligned}$$

Hence,

$$\text{var}(1/(Z+1)) = \mathbf{E}\left[\left(\frac{1}{(Z+1)^2}\right)\right] - \left(\mathbf{E}\left[\frac{1}{Z+1}\right]\right)^2 = \frac{1}{2} - (\ln(2))^2 \approx 0.019.$$

2. Using independence,

$$\mathbf{E}\left[\frac{XY}{Z+1}\right] = \mathbf{E}[X]\mathbf{E}[Y]\mathbf{E}\left[\frac{1}{Z+1}\right] = \frac{\ln(2)}{4} \approx 0.173.$$

3. $\mathbf{P}\left(\frac{XY}{Z} \le 1\right)$ is the same as $\mathbf{P}(XY \le Z) = 1 - \mathbf{P}(XY \ge Z)$. But since Z is a uniform r.v. in $[0,1]$, $\mathbf{P}(XY \ge Z)$ is simply the expected value of XY, which is 0.25, so the answer is 0.75. X或Y大于Z的概率是0.5，XY大于Z的概率就是0.25

Alternate longer solution: We first write down the joint density of X, Y, and Z:

$$f_{X,Y,Z}(x,y,z) = \begin{cases} 1, & \text{if } 0 \le x, y, z \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

To calculate the probability of interest, we find the region over which we should integrate this joint PDF. The region we are considering consists of the points (x,y,z) such that $0 \le x, y \le 1$, and $xy \le z \le 1$ (also notice that $xy \le 1$ always holds).

Using this,

$$\begin{aligned} \mathbf{P}\left(\frac{XY}{Z} \le 1\right) &= \mathbf{P}\left(XY \le Z\right) \\ &= \int_0^1 \int_0^1 \int_{xy}^1 dz\, dy\, dx \\ &= \int_0^1 \int_0^1 (1 - xy)\, dy\, dx \\ &= \int_0^1 \left(y - \frac{xy^2}{2}\right)\Big|_{y=0}^1 dx \\ &= \int_0^1 \left(1 - \frac{x}{2}\right) dx \\ &= \left(x - \frac{x^2}{4}\right)\Big|_0^1 \\ &= \frac{3}{4}. \end{aligned}$$

X, Y, Z are independent random variables, uniformly distributed on the interval $[0,1]$. So,

$$\begin{aligned} P(XY \le Z) &= P\left(Y \le \frac{Z}{X}\right) \\ &= \int_0^1 \int_0^1 \int_0^{min(1,\frac{z}{x})} f(x,y,z) dy\, dx\, dz \\ &= \int_0^1 \int_0^1 \int_0^{min(1,\frac{z}{x})} f(y|x,z) * f(x|z) * f(z) dy\, dx\, dz \\ &= \int_0^1 \int_0^1 \int_0^{min(1,\frac{z}{x})} f(y) * f(x) * f(z) dy\, dx\, dz \\ &= \int_0^1 \int_0^1 \int_0^{min(1,\frac{z}{x})} 1 * 1 * 1 * dy\, dx\, dz = \int_0^1 \int_0^1 \int_0^{min(1,\frac{z}{x})} dy\, dx\, dz \end{aligned}$$

提交 你已经尝试了1次（总共可以尝试2次）

A Poisson process-like interpretation of the last question.

discussion posted 4 days ago by an7777777

Consider 3 new random variables: $X' = -ln(X), Y' = -ln(Y), Z' = -ln(Z)$

It can be shown that: X', Y', Z' are mutually independent and distributed identically by $Exp(1)$.

As $X, Y, Z \ge 0$, $\mathbf{P}\left(XY/Z \le 1\right) = \mathbf{P}\left(-ln(XY/Z) \ge -ln(1)\right) = \mathbf{P}\left(X' + Y' - Z' \ge 0\right) = \mathbf{P}\left(X' + Y' \ge Z'\right)$.

The Poisson process-like interpretation of $\mathbf{P}\left(X' + Y' \ge Z'\right)$:

Image that you come to a bank in a late afternoon. The bank has only 2 bank tellers, one busy serving a customer and one is still free. Assume that the service times for you and for each of the customers being served are independent identically distributed by $Exp(1)$. You go to the free one and notice that there is one guy is coming right after you. You and 2 these guys are the last services. So, $\mathbf{P}\left(X' + Y' \ge Z'\right)$ is the probability that you will **NOT** be the last guy to leave.

显示讨论

$$1 - \boxed{\underbrace{\frac{1}{2}}_{\text{第一个不是你}} \cdot \underbrace{\frac{1}{2}}_{\text{第二个也不是你}}} = \frac{3}{4}$$

你是最后一个离开的