1. Asymptotic Variance of MLE for Curved Gaussian

(a)

3/3 points (graded)

Let $X_1, \ldots, X_n$ be n i.i.d. random variables with distribution $\mathcal{N}(\theta, \theta)$ for some unknown $\theta > 0$.

In the last homework, you have computed the maximum likelihood estimator $\hat{\theta}$ for θ in terms of the sample averages of the linear and quadratic means, i.e. $\overline{X}_n$ and $\overline{X_n^2}$, and applied the CLT and delta method to find its asymptotic variance.

In this problem, you will compute the asymptotic variance of $\hat{\theta}$ via the Fisher Information.

Denoting the log likelihood for one sample by $\ell(\theta, x)$, compute the second derivative $\frac{d^2}{d\theta^2}\ell(\theta, x)$.

$\frac{d^2}{d\theta^2}\ell(\theta, x) =$ (theta - 2*x^2)/(2*theta^

$\frac{\theta - 2\cdot x^2}{2\cdot\theta^3}$

Answer: -x^2/theta^3+1/(2*theta^2)

Then, compute the Fisher information $I(\theta)$.

as

$$I(\theta) = -\mathbb{E}\left[\frac{d^2}{d\theta^2}\ell(\theta, X)\right].$$

$I(\theta) =$ 1/theta + 1/(2*theta^2)

$\frac{1}{\theta} + \frac{1}{2\cdot\theta^2}$

Answer: (2 * theta + 1)/(2 * theta^2)

Finally, what does this tell us about the asymptotic variance of $\hat{\theta}$?

$V(\hat{\theta}) =$ 1/(1/theta + 1/(2*theta^2

$\frac{1}{\frac{1}{\theta} + \frac{1}{2\cdot\theta^2}}$

Answer: 2 * theta^2 /(2*theta + 1)

STANDARD NOTATION

Solution:

Let $\ell(\theta, x)$ denote the log likelihood for one sample. Recall its first derivative:

$$\begin{aligned}
\ell(\theta, x) &= -\frac{1}{2}(\log(2) + \log(\pi) + \log(\theta)) - \left(\frac{1}{2\theta}X^2 - X + \frac{1}{2}\theta\right) \\
\implies \frac{d}{d\theta}\ell(\theta, x) &= -\frac{1}{2\theta} + \frac{1}{2\theta^2}X^2 - \frac{1}{2}.
\end{aligned}$$

Differentiating one more time yields

$$\frac{d^2}{d\theta^2}\ell(\theta, x) = \frac{1}{2\theta^2} - \frac{x^2}{\theta^3}.$$

Since

$$\mathbb{E}[X^2] = \theta + \theta^2,$$

we obtain

$$I(\theta) = -\mathbb{E}\left[\frac{d^2}{d\theta^2}\ell(\theta, X)\right] = \frac{2\theta+1}{2\theta^2}.$$

By the theorem about the asymptotic variance of the MLE from class, we finally have

$$V(\hat{\theta}) = I(\theta)^{-1} = \frac{2\theta^2}{(2\theta+1)},$$

which coincides with the result from part (b).

提交 你已经尝试了3次（总共可以尝试3次）

Answers are displayed within the problem

讨论

显示讨论

主题： Unit 3 Methods of Estimation:Homework 6 Maximum Likelihood Estimation and Method of Moments / 1. Asymptotic Variance of MLE for Curved Gaussian