1. Implicit hypothesis testing

Given n i.i.d. samples $X_1, \ldots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu \in \mathbb{R}$ and $\sigma^2 > 0$, we want to find a test with asymptotic level 5% for the hypotheses

$$H_0 : \mu > \sigma \quad \text{vs} \quad H_1 : \mu \leq \sigma. \tag{7.1}$$

(a)

1/1得分 (计入成绩)

As a first step, define the maximum likelihood estimators

$$\hat{\mu} = \bar{X}_n, \quad \widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2.$$

Give a function $g(x, y)$ such that

$$g\left(\hat{\mu}, \widehat{\sigma^2}\right) \xrightarrow[n\to\infty]{\mathbf{P}} \mu - \sigma.$$

$g(x, y) =$ x - sqrt(y) Answer: x - sqrt(y)

$x - \sqrt{y}$

Solution:

Simply set

$$g(x, y) = x - \sqrt{y}.$$

By the consistency of the maximum likelihood estimators and continuity of g, we get

$$g\left(\hat{\mu}, \widehat{\sigma^2}\right) \xrightarrow[n\to\infty]{} g\left(\mu, \sigma^2\right) = \mu - \sigma.$$

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Answers are displayed within the problem

(b)

1/1得分 (计入成绩)

Note: To avoid too much double jeopardy, you will be able to see the solution to this part once you answered it correctly, and used all your attempts.

What is the asymptotic variance of $g\left(\hat{\mu}, \widehat{\sigma^2}\right)$ that you found in part (a)?

$V\left(g\left(\hat{\mu}, \widehat{\sigma^2}\right)\right) =$ sigma^2 + 1/2*sigma^2 **Answer:** 3/2*sigma^2

STANDARD NOTATION

Solution:

First, by the Theorem giving asymptotic normality for maximum likelihood estimators, we have

$$\sqrt{n}\begin{pmatrix} \hat{\mu} \\ \widehat{\sigma^2} \end{pmatrix} - \sqrt{n}\begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix} \xrightarrow[n\to\infty]{(D)} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, I(\mu,\sigma^2)^{-1}\right),$$

where $I\left(\mu, \sigma^2\right)$ denotes the Fisher information that we computed earlier to be

$$I\left(\mu,\sigma^2\right) = \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^4} \end{pmatrix}.$$

Hence,

$$\sqrt{n}\begin{pmatrix} \hat{\mu} \\ \widehat{\sigma^2} \end{pmatrix} - \sqrt{n}\begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix} \xrightarrow[n\to\infty]{(D)} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^4} \end{pmatrix}^{-1}\right).$$

Now, defining

$$g : \mathbb{R}\times(0,\infty) \to \mathbb{R}, \quad (x,y) \mapsto x - \sqrt{y},$$

we can compute

$$\nabla g\left(x,y\right) = \begin{pmatrix} 1 \\ -\frac{1}{2\sqrt{y}}. \end{pmatrix}$$

Then, apply the multivariate Delta method to obtain

$$\sqrt{n}\left(\hat{\mu} - \sqrt{\widehat{\sigma^2}} - (\mu - \sigma)\right) \xrightarrow[n\to\infty]{(D)} \mathcal{N}\left(0, \nabla g(\mu,\sigma^2)^T I(\mu,\sigma^2)^{-1} \nabla g\left(\mu,\sigma^2\right)\right) = \mathcal{N}\left(0, \frac{3}{2}\sigma^2\right).$$

That means

$$V\left(g\left(\hat{\mu}, \widehat{\sigma^2}\right)\right) = \frac{3}{2}\sigma^2.$$

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Answers are displayed within the problem

(c)

1.0/1得分 (计入成绩)

Using your result from part (b) together with a plug-in estimator for the asymptotic variance, give a test for

$$H_0 : \mu > \sigma \quad \text{vs} \quad H_1 : \mu \leq \sigma. \tag{7.2}$$

that is with asymptotic level 5% and of the form

$$\psi = \mathbf{1}\{f\left(\hat{\mu}, \widehat{\sigma^2}\right) > 0\},$$

where

$$f\left(\hat{\mu}, \widehat{\sigma^2}\right) = -T\left(\hat{\mu}, \widehat{\sigma^2}\right) - q$$

for some function T and $q > 0$.

(Enter **hatmu**, **hat(sigma^2)** for $\hat{\mu}$, $\widehat{\sigma^2}$, respectively. Use the quantile function q for best results. E.g: enter $q\left(0.01\right)$ for the 0.99-quantile.)

$f\left(\hat{\mu}, \widehat{\sigma^2}\right) =$ - sqrt(n/(3/2*hat(sigma^2))) * (hatmu - sqrt(hat(sigma^2))) - q(0.05)

Answer: -sqrt(n)*(hatmu - sqrt(hat(sigma^2)))/sqrt(3/2*hat(sigma^2))-q(0.05)

STANDARD NOTATION

Solution:

By part (b) and Slutsky's Theorem, we know that

$$\frac{\sqrt{n}}{\sqrt{\frac{3}{2}\widehat{\sigma^2}}}\left(\hat{\mu} - \sqrt{\widehat{\sigma^2}} - (\mu - \sigma)\right) \xrightarrow[n\to\infty]{(D)} \mathcal{N}\left(0, 1\right).$$

Therefore, set

$$T = \sqrt{n}\frac{\hat{\mu} - \sqrt{\widehat{\sigma^2}}}{\sqrt{\frac{3}{2}\widehat{\sigma^2}}},$$

and

$$\psi = \mathbf{1}\{-T - q > 0\},$$

for some $q > 0$. If $\mu = \sigma$, the above means that we reject H_0 if $T < -q$, which has probability

$$\mathbf{P}_{\mu,\sigma^2}\left(T < -q\right) \xrightarrow[n\to\infty]{} 1 - \Phi\left(q\right).$$

To achieve asymptotic level 5%, we therefore must set q to be at least $q_{5\%}$. It turns out that this value is sufficient. Indeed, if $\mu > \sigma$ (or equivalently, $\mu - \sigma > 0$), then by the Law of Large Numbers T has mean approaching ∞ due to the extra $\sqrt{n}$ scaling:

$$T = T_n\left(\hat{\mu}, \widehat{\sigma^2}\right) \to +\infty, \text{ as } \quad n \to \infty, \tag{7.3}$$

so in that case

$$\mathbf{P}_{\mu,\sigma^2}\left(T < -1.65\right) \xrightarrow[n\to\infty]{} 0.$$

Hence the (asymptotic) supremum of rejecting H_0 over $(\mu, \sigma^2) \in \Theta_0$ is exactly 0.05 if we set the threshold to be $q_{5\%}$. All in all, we have found a hypothesis test with asymptotic level 5% of the form

$$\psi\{f\left(\hat{\mu}, \widehat{\sigma^2}\right) > 0\},$$

where

$$f\left(\hat{\mu}, \widehat{\sigma^2}\right) = -\sqrt{n}\frac{\hat{\mu} - \sqrt{\widehat{\sigma^2}}}{\sqrt{\frac{3}{2}\widehat{\sigma^2}}} - 1.65.$$

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Answers are displayed within the problem

(d)

1.0/1得分 (计入成绩)

Using the same test as in part (c), give the (asymptotic) p-value of the test given observations $\hat{\mu}$ and $\widehat{\sigma^2}$.

(Enter **Phi(x)** for the cdf $\Phi\left(x\right)$ of a Normal distribution. Enter **hatmu**, **hat(sigma^2)** for $\hat{\mu}$, $\widehat{\sigma^2}$, respectively.)

p-value = 1 - Phi(- sqrt(n/(3/2*hat(sigma^2))) * (hatmu - sqrt(hat(sigma^2))))

Answer: 1-Phi(-sqrt(n)*(hatmu - sqrt(hat(sigma^2)))/sqrt(3/2*hat(sigma^2)))

STANDARD NOTATION

Solution:

By the same considerations as in part (c), we only have to control the level under the assumption $\mu = \sigma$. Looking for a p-value means that we are looking for the smallest α such that the test with level α would have rejected the null-hypothesis. Note that in our notation, this is often an infimum, so we might not be rejecting the null-hypothesis at the p-value α, but at any $\alpha + \varepsilon$, $\varepsilon > 0$.

Starting from the asymptotic normality result in part (c), if $\mu = \sigma$, we have

$$\mathbf{P}_{\mu,\sigma^2}\left(T < -q\right) \xrightarrow[n\to\infty]{} 1 - \Phi\left(q\right).$$

That means if we are looking for

$$\alpha\left(T\right) = \inf\{\alpha : \exists q > 0 \text{ such that } T < -q \text{ and } \alpha = 1 - \Phi\left(q\right)\}.$$

By the monotonicity of Φ, this comes down to

$$\alpha\left(T\right) = 1 - \Phi\left(-T\right).$$

Plugging in the form we found for T in part (c) yields

$$\text{p-value} = 1 - \Phi\left(-\sqrt{n}\frac{\hat{\mu} - \sqrt{\widehat{\sigma^2}}}{\sqrt{\frac{3}{2}\widehat{\sigma^2}}}\right).$$

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Answers are displayed within the problem

(e)

4/4得分 (计入成绩)

What is the (asymptotic) p-value if the sample size is $n = 100$, $\hat{\mu} = 2.41$, and $\widehat{\sigma^2} = 5.20$?

p-value = 0.679 **Answer:** 0.68

What if $n = 100$, $\hat{\mu} = 3.28$, and $\widehat{\sigma^2} = 15.95$?

p-value = 0.072 **Answer:** 0.07

In the second case, at level 10%, do you reject H_0?

- (•) Yes
- ( ) No

At 5%, do you reject H_0?

- ( ) Yes
- (•) No

Solution:

For the first example, we compute

$$T = \sqrt{n}\frac{\hat{\mu} - \sqrt{\widehat{\sigma^2}}}{\sqrt{\frac{3}{2}\widehat{\sigma^2}}} \approx 0.46,$$

which leads to a p-value of roughly 0.68, which is very high.

In the second example, we compute

$$T \approx -1.46,$$

which leads to a p-value of roughly 0.07, much lower. That means that at 10%, we can reject H_0, while at 5%, we cannot.

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