Fast Closest Pair Theorem

One of the keys to the described method for computing fast closest pairs is that fact that only the three points following a given point (sorted vertically) needs to be test for being the closest pair. The proof of this fact is very subtle so we won't give a full proof. However, here is a sketch of the reasoning behind this claim if you are interested.

Consider a sequence of points $p_k = (x_k, y_k)$ that satisfy the following three properties:

- The points are vertically ordered, that is $y_{k-1} \geq y_k$,
- The points lie within distance δ of the line $x = 0$, that is $|x_k| \leq \delta$,
- For all pairs p_i and p_j that lie on the same side of the line $x = 0$, $|p_i - p_j| \geq \delta$.

Theorem: Let (p_i, p_j) be the closest pair of points that lie on opposite sides of $x = 0$. If $|p_i - p_j| < \delta$, then $|i - j| < 4$.

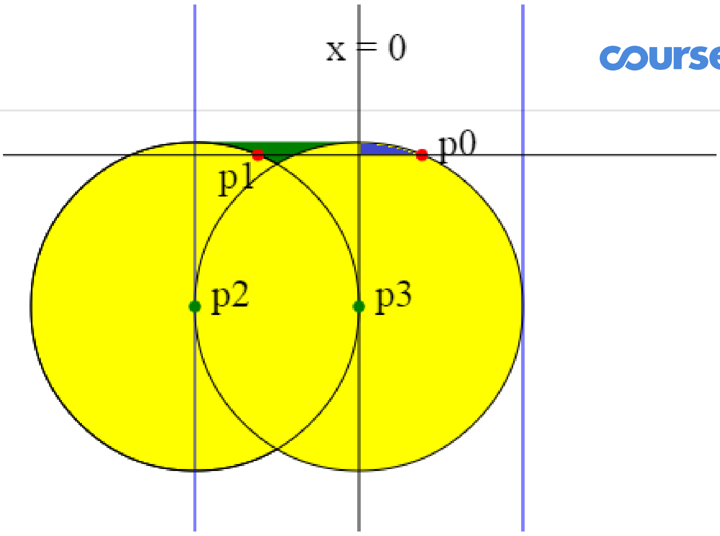
To prove this theorem, we will first prove the following lemma.

Lemma: Let p_0, p_1, p_2, p_3 be four consecutive points with $x_0 \geq 0$ and $x_1, x_2, x_3 \leq 0$. If $|p_3 - p_0| \leq \delta$, then $|p_1 - p_0| \leq |p_3 - p_0|$.

Proof: Consider the extremal case when $p_3 = (0, 0)$ and $p_2 = (-\delta, 0)$. The yellow circles in the diagram below cover all points within distance δ of p_2 and p_3. In this configuration, p_1 must lie somewhere in the green region on the left of $x = 0$ in the diagram since $y_1 - y_3 \leq y_0 - y_3 \leq \delta$. Given the position of p_1 in this region, p_0 must lie in the corresponding blue region on the right of $x = 0$ since $|p_3 - p_0| \leq \delta$ and $y_0 \geq y_1$.

Now, consider the perpendicular bisector of p_1 and p_3. We claim that the blue region on the right must lie entirely on the same side of this bisector as p_1 and, therefore, $|p_1 - p_0| \leq |p_3 - p_0|$. To confirm this observation, we note that the extremal case for this argument occurs exactly when $|p_2 - p_1| = \delta$ and $p_0 = (\delta + x_1, y_1)$ as shown. In this case, the p_i form a parallelogram and the perpendicular bisector between p_1 and p_3 passes through p_0. In any other configuration, the perpendicular bisector passes below p_0. **QED**

With this lemma in hand, we can now prove the main theorem. Assume that the closest pair of points (p_i, p_j) spanning $x = 0$ have $|p_i - p_j| \leq \delta$ and $|i - j| \geq 4$. Then, there must exist two points between p_i and p_j which lie on the same side as one of p_i and p_j. However, by the lemma, one of these points must also form a second closest pair that spans $x = 0$ with either p_i or p_j. This argument can repeated until we reach the situation where $|i - j| < 4$.


x = 0
p0
p1
p2
p3