LECTURE 7: Conditioning on a random variable; Independence of r.v.'s

- Conditional PMFs
  - Conditional expectations
  - Total expectation theorem
- Independence of r.v.'s
  - Expectation properties
  - Variance properties
- The variance of the binomial
- The hat problem: mean and variance

Conditional PMFs

$$p_{X|A}(x \mid A) = \mathbf{P}(X = x \mid A) \qquad p_{X|Y}(x \mid y) = \mathbf{P}(X = x \mid Y = y)$$

$$p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$ defined for y such that $p_Y(y) > 0$

$$\sum_x p_{X|Y}(x \mid y) = 1$$

y \ x	1	2	3	4
4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
1		1/20		

$$p_{X,Y}(x,y) = p_Y(y)\, p_{X|Y}(x \mid y)$$

$$p_{X,Y}(x,y) = p_X(x)\, p_{Y|X}(y \mid x)$$

Conditional PMFs involving more than two r.v.'s

- Self-explanatory notation

$$p_{X|Y,Z}(x \mid y, z)$$

$$p_{X,Y|Z}(x, y \mid z)$$

- Multiplication rule

$$\mathbf{P}(A \cap B \cap C) = \mathbf{P}(A)\,\mathbf{P}(B \mid A)\,\mathbf{P}(C \mid A \cap B)$$

$$p_{X,Y,Z}(x, y, z) = p_X(x)\,p_{Y|X}(y \mid x)\,p_{Z|X,Y}(z \mid x, y)$$

Conditional expectation

$$\mathbf{E}[X] = \sum_x x p_X(x) \qquad \mathbf{E}[X \mid A] = \sum_x x\, p_{X|A}(x) \qquad \mathbf{E}[X \mid Y = y] = \sum_x x p_{X|Y}(x \mid y)$$

- Expected value rule

$$\mathbf{E}[g(X)] = \sum_x g(x)\, p_X(x) \qquad \mathbf{E}[g(X) \mid A] = \sum_x g(x)\, p_{X|A}(x)$$

$$\mathbf{E}[g(X) \mid Y = y] = \sum_x g(x)\, p_{X|Y}(x \mid y)$$

Total probability and expectation theorems

- $A_1, \ldots, A_n$: partition of Ω

- $p_X(x) = \mathbf{P}(A_1)\, p_{X|A_1}(x) + \cdots + \mathbf{P}(A_n)\, p_{X|A_n}(x)$

$$p_X(x) = \sum_y p_Y(y)\, p_{X|Y}(x \mid y)$$

- $\mathbf{E}[X] = \mathbf{P}(A_1)\, \mathbf{E}[X \mid A_1] + \cdots + \mathbf{P}(A_n)\, \mathbf{E}[X \mid A_n]$

$$\boxed{\mathbf{E}[X] = \sum_y p_Y(y)\, \mathbf{E}[X \mid Y = y]}$$

- Fine print:
  Also valid when Y is a discrete r.v. that ranges over an infinite set,
  as long as $\mathbf{E}[|X|] < \infty$

Independence

- of two events: $\mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B)$ $\qquad$ $\mathbf{P}(A \mid B) = \mathbf{P}(A)$

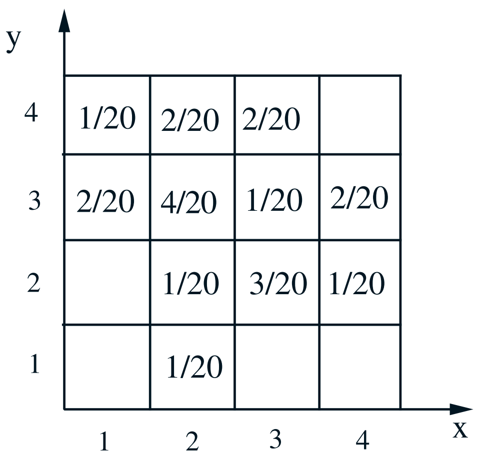
- of a r.v. and an event: $\mathbf{P}(X = x \text{ and } A) = \mathbf{P}(X = x) \cdot \mathbf{P}(A)$, for all x

- of two r.v.'s: $\mathbf{P}(X = x \text{ and } Y = y) = \mathbf{P}(X = x) \cdot \mathbf{P}(Y = y)$, for all x, y

  $p_{X,Y}(x, y) = p_X(x)\, p_Y(y)$, for all x, y

X, Y, Z are **independent** if:
$p_{X,Y,Z}(x, y, z) = p_X(x)\, p_Y(y)\, p_Z(z)$, for all x, y, z

Example: independence and conditional independence



- Independent?

- What if we condition on $X \leq 2$ and $Y \geq 3$?

Independence and expectations

- In general: $\mathbf{E}[g(X,Y)] \neq g\Big(\mathbf{E}[X], \mathbf{E}[Y]\Big)$

- Exceptions: $\mathbf{E}[aX+b] = a\mathbf{E}[X]+b$ $\quad\quad$ $\mathbf{E}[X+Y+Z] = \mathbf{E}[X]+\mathbf{E}[Y]+\mathbf{E}[Z]$

If X, Y are **independent:** $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$

$g(X)$ and $h(Y)$ are also independent: $\mathbf{E}[g(X)h(Y)] = \mathbf{E}[g(X)] \cdot \mathbf{E}[h(Y)]$

Independence and variances

- Always true: $\text{var}(aX) = a^2\text{var}(X)$ $\text{var}(X + a) = \text{var}(X)$

- In general: $\text{var}(X + Y) \neq \text{var}(X) + \text{var}(Y)$

If X, Y are independent: $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)$

- Examples:
  - If $X = Y$: $\text{var}(X + Y) =$
  - If $X = -Y$: $\text{var}(X + Y) =$
  - If X, Y independent: $\text{var}(X - 3Y) =$

Variance of the binomial

- X: binomial with parameters n, p
  - number of successes in n independent trials

$X_i = 1$ if ith trial is a success;
$X_i = 0$ otherwise

(indicator variable)

$X = X_1 + \cdots + X_n$

The hat problem

- n people throw their hats in a box and then pick one at random
  - All permutations equally likely
  - Equivalent to picking one hat at a time
- X: number of people who get their own hat
  - Find $\mathbf{E}[X]$

$$X_i = \begin{cases} 1, & \text{if } i \text{ selects own hat} \\ 0, & \text{otherwise.} \end{cases}$$

$$X = X_1 + X_2 + \cdots + X_n$$

- $\mathbf{E}[X_i] =$

The variance in the hat problem

- X: number of people who get their own hat
  - Find var(X)

$$X_i = \begin{cases} 1, & \text{if } i \text{ selects own hat} \\ 0, & \text{otherwise.} \end{cases}$$

$$X = X_1 + X_2 + \cdots + X_n$$

- var(X) $= \mathbf{E}[X^2] - (\mathbf{E}[X])^2$

$$X^2 = \sum_i X_i^2 + \sum_{i,j:i \neq j} X_i X_j$$

- $\mathbf{E}[X_i^2] =$

- For $i \neq j$: $\mathbf{E}[X_i X_j] =$

- $\mathbf{E}[X^2] =$