4. One-sided Test vs Wald's Test

In the problems on this page, $X_1, \ldots, X_n \stackrel{iid}{\sim} \mathrm{Exp}\,(\lambda)$, where $\lambda > 0$ is an unknown parameter. In this series of problems, we will compare two tests for the following null and alternative hypotheses:

$$H_0 \quad : \lambda \leq 1$$
$$H_1 \quad : \lambda > 1.$$

MLE and Fisher Information for an Exponential Statistical Model

2/2得分 (计入成绩)

What is the MLE $\hat{\lambda}$ for an exponential statistical model?

(Enter **barX_n** for $\overline{X}_n$.)

$\hat{\lambda} =$ 1/barX_n **Answer:** 1/barX_n

What is the Fisher information $I\,(\lambda)$ for an exponential statistical model?

$I\,(\lambda) =$ 1/lambda^2 **Answer:** 1/lambda^2

STANDARD NOTATION

Solution:

We computed the MLE in this homework for an exponential statistical model in the problem "Likelihood Ratio Test" on the page "MLE for a Shifted Exponential". This is precisely

$$\hat{\lambda}_n^{MLE} = \frac{1}{\overline{X}_n}.$$

To compute the Fisher information, the log-likelihood for a single observation is

$$\ell\,(\lambda) = \ln\left(\lambda e^{-x\lambda}\right) = \ln\,(\lambda) - \lambda x.$$

Therefore,

$$\ell''\,(\lambda) = -\frac{1}{\lambda^2},$$

and the Fisher information is given by

$$\mathcal{I}\,(\lambda) = -\mathbb{E}_\lambda\left[\ell''\,(\lambda)\right] = \frac{1}{\lambda^2}.$$

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☐ Answers are displayed within the problem

Test Statistic Based on the MLE for an Exponential Statistical Model

1/1得分 (计入成绩)

Assume that the technical conditions hold so that the MLE $\hat{\lambda}_n^{MLE}$ of an exponential statistical model is asymptotically normal. Then it follows that

$$\frac{\sqrt{n}\,(\hat{\lambda}_n^{MLE} - \lambda)}{g\,(\hat{\lambda}_n^{MLE})} \xrightarrow[n\to\infty]{(d)} N\,(0,1)$$

where $g\,(\hat{\lambda}_n^{MLE})$ is an expression that depends on $\hat{\lambda}_n^{MLE}$.

What is $g\,(\hat{\lambda}_n^{MLE})$?

(Enter **hatlambda** for $\hat{\lambda}_n^{MLE}$.)

$g\,(\hat{\lambda}_n^{MLE}) =$ hatlambda ☐ **Answer:** hatlambda

STANDARD NOTATION

Solution:

The asymptotic variance of the statistic

$$\sqrt{n}\,(\hat{\lambda}_n^{MLE} - \lambda)$$

is given by $\mathcal{I}(\lambda)^{-1} = \lambda^2$. Therefore,

$$\frac{\sqrt{n}\,(\hat{\lambda}_n^{MLE} - \lambda)}{\lambda} \xrightarrow[n\to\infty]{(d)} N\,(0,1)$$

Moreover, by Slutsky's theorem

$$\frac{\sqrt{n}\,(\hat{\lambda}_n^{MLE} - \lambda)}{\hat{\lambda}_n^{MLE}} \xrightarrow[n\to\infty]{(d)} N\,(0,1)\,.$$

Therefore $\mathbf{g\,(\hat{\lambda}_n^{MLE}) = \hat{\lambda}_n^{MLE}}$.

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☐ Answers are displayed within the problem

Evaluating the Test Based on the MLE

0/1得分 (计入成绩)

Let us define the test statistic

$$T_n = \frac{\sqrt{n}\,(\hat{\lambda}_n^{MLE} - 1)}{g\,(\hat{\lambda}_n^{MLE})}$$

where $g\left(\hat{\lambda}_n^{MLE}\right)$ is the expression from the previous problem.

We define the test $\psi = \mathbf{1}\left(T_n > \tau\right)$, where τ is a chosen so that ψ is a test at asymptotic level $\alpha = 0.05$. Suppose we observe $\overline{X}_n = 0.83$.

Does the test ψ **reject** or **fail to reject** H_0 on this data set? **Use** $n = 100$**.**

- (selected) Fail to reject
- Reject

Solution:

Recall that the 5% quantile of $N\left(0,1\right)$ is approximately 1.65. If $\overline{X}_n = 0.83$, then

$$T_n = \frac{\sqrt{n}\left(\frac{1}{0.83} - 1\right)}{1/0.83} \approx 1.7.$$

Therefore, the test as designed above will **reject** H_0.

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Answers are displayed within the problem

Wald's Test

2/2得分 (计入成绩)

Recall the test-statistic T_n from the previous problem, and let T_n^{Wald} denote the test-statistic associated to Wald's test for the hypotheses H_0 and H_1.

Express T_n^{Wald} in terms of T_n.

(Enter **T_n** for T_n.)

$T_n^{Wald} =$ (T_n)^2 **Answer:** T_n^2

$\left(T_n\right)^2$

Which of the following is true about T_n^{Wald} if we assume that $\lambda = 1$?

- T_n^{Wald} is distributed as $\mathcal{N}\left(0,1\right)$.
- T_n^{Wald} is asymptotically distributed as χ_2^2.
- T_n^{Wald} is distributed as χ_1^2.
- (selected) T_n^{Wald} is asymptotically distributed as χ_1^2.

STANDARD NOTATION

Solution:

By definition, we have that

$$T_n^{Wald} = n(\hat{\lambda}_n^{MLE} - 1)^T I(\hat{\lambda}_n^{MLE})(\hat{\lambda}_n^{MLE} - 1).$$

Since the MLE is 1-dimensional,

$$T_n^{Wald} = n(\hat{\lambda}_n^{MLE} - 1)^2 \cdot \frac{1}{(\hat{\lambda}_n^{MLE})^2}.$$

Now, for the first question, observe that $T_n^{Wald} = T_n^2$.

For the second question, since

$$T_n \xrightarrow[n\to\infty]{(d)} N(0,1),$$

we have that

$$T_n^{Wald} = T_n^2 \xrightarrow[n\to\infty]{(d)} \chi_1^2.$$

提交 你已经尝试了1次（总共可以尝试4次）

Answers are displayed within the problem

Evaluating Wald's Test on a Sample Data Set

1/1得分 (计入成绩)

Consider the test $\psi^{Wald} = \mathbf{1}(T_n^{Wald} > \tau)$ where τ is set so that the test ψ^{Wald} has asymptotic level 0.05. Suppose you observe $\overline{X}_n = 0.83$.

Does the test ψ^{Wald} **reject** or **fail to reject** on the given data set? **Use** $n = 100$.

- (●) Fail to reject
- ( ) Reject

Solution:

Consulting a table of values, we see that the 0.05-quantile of χ_1^2 is 3.84. Now observe that

$$T_n^{Wald} = (T_n^2) \approx (1.7)^2 \approx 2.89$$

as was computed in a previous problem. Therefore, using Wald's test, we would **fail to reject** H_0 on observing $\overline{X}_n = 0.83$.

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Answers are displayed within the problem

讨论

显示讨论