Let $X_1, \ldots, X_\alpha, Z, Y_1, \ldots Y_\beta$ be independent random variables, uniformly distributed over the interval $[0,1]$, and let A be the event

$$A = \{X_1 < \cdots < X_\alpha < Z < Y_1 < \cdots < Y_\beta\}.$$

Then,

$$\mathbf{P}(A) = \frac{1}{(\alpha+\beta+1)!},$$

because there are $(\alpha+\beta+1)!$ ways of ordering these $\alpha+\beta+1$ random variables, each order being equally likely, and exactly 1 of them is the order corresponding to event A.

Consider the following two events:

$$B = \big\{ \max\{X_1, \ldots, X_\alpha\} < Z \big\}, \qquad C = \big\{ Z < \min\{Y_1, \ldots, Y_\beta\} \big\}.$$

We have, using the total probability theorem,

$$\begin{aligned} \mathbf{P}(B \cap C) &= \int_0^1 \mathbf{P}(B \cap C \mid Z = \theta) f_Z(\theta)\, d\theta \\ &= \int_0^1 \mathbf{P}\big(\max\{X_1, \ldots, X_\alpha\} < \theta < \min\{Y_1, \ldots, Y_\beta\} \big)\, d\theta \\ &= \int_0^1 \mathbf{P}\big(\max\{X_1, \ldots, X_\alpha\} < \theta \big) \cdot \mathbf{P}\big(\theta < \min\{Y_1, \ldots, Y_\beta\} \big)\, d\theta \\ &= \int_0^1 \mathbf{P}(X_1 < \theta) \cdot \cdots \cdot \mathbf{P}(X_\alpha < \theta) \cdot \mathbf{P}(\theta < Y_1) \cdot \cdots \cdot \mathbf{P}(\theta < Y_\beta)\, d\theta \\ &= \int_0^1 \theta^\alpha (1-\theta)^\beta\, d\theta. \end{aligned}$$

We also have

$$\mathbf{P}(A \mid B \cap C) = \frac{1}{\alpha!\, \beta!},$$

because given the events B and C, all $\alpha!$ possible orderings of $X_1, \ldots, X_\alpha$ are equally likely, and all $\beta!$ possible orderings of $Y_1, \ldots, Y_\beta$ are equally likely.

By writing the equality

$$\mathbf{P}(A) = \mathbf{P}(B \cap C)\, \mathbf{P}(A \mid B \cap C)$$

in terms of the preceding relations, we finally obtain

$$\frac{1}{(\alpha+\beta+1)!} = \frac{1}{\alpha!\, \beta!} \int_0^1 \theta^\alpha (1-\theta)^\beta\, d\theta,$$

or

$$\int_0^1 \theta^\alpha (1-\theta)^\beta\, d\theta = \frac{\alpha!\, \beta!}{(\alpha+\beta+1)!}.$$