

## LECTURE 9: Conditioning on an event; Multiple continuous r.v.'s

- Conditioning a r.v. on an event
  - Conditional PDF
  - Conditional expectation and the expected value rule
  - Exponential PDF: memorylessness
  - Total probability and expectation theorems
  - Mixed distributions
- Jointly continuous r.v.'s and joint PDFs
  - From the joints to the marginals
  - Uniform joint PDF example
  - The expected value rule and linearity of expectations
  - The joint CDF

## Conditional PDF, given an event

$$P(A) > 0$$

$$p_X(x) = P(X = x)$$

$$f_X(x) \cdot \delta \approx P(x \leq X \leq x + \delta)$$

$$p_{X|A}(x) = P(X = x | A)$$

$$\underline{f_{X|A}(x)} \cdot \delta \approx P(x \leq X \leq x + \delta | A)$$

$$P(X \in B) = \sum_{x \in B} p_X(x)$$

$$P(X \in B) = \int_B f_X(x) dx$$

$$P(X \in B | A) = \sum_{x \in B} p_{X|A}(x)$$

$$P(X \in B | A) = \int_B f_{X|A}(x) dx \quad \text{Def}$$

$$\sum_x p_{X|A}(x) = 1$$

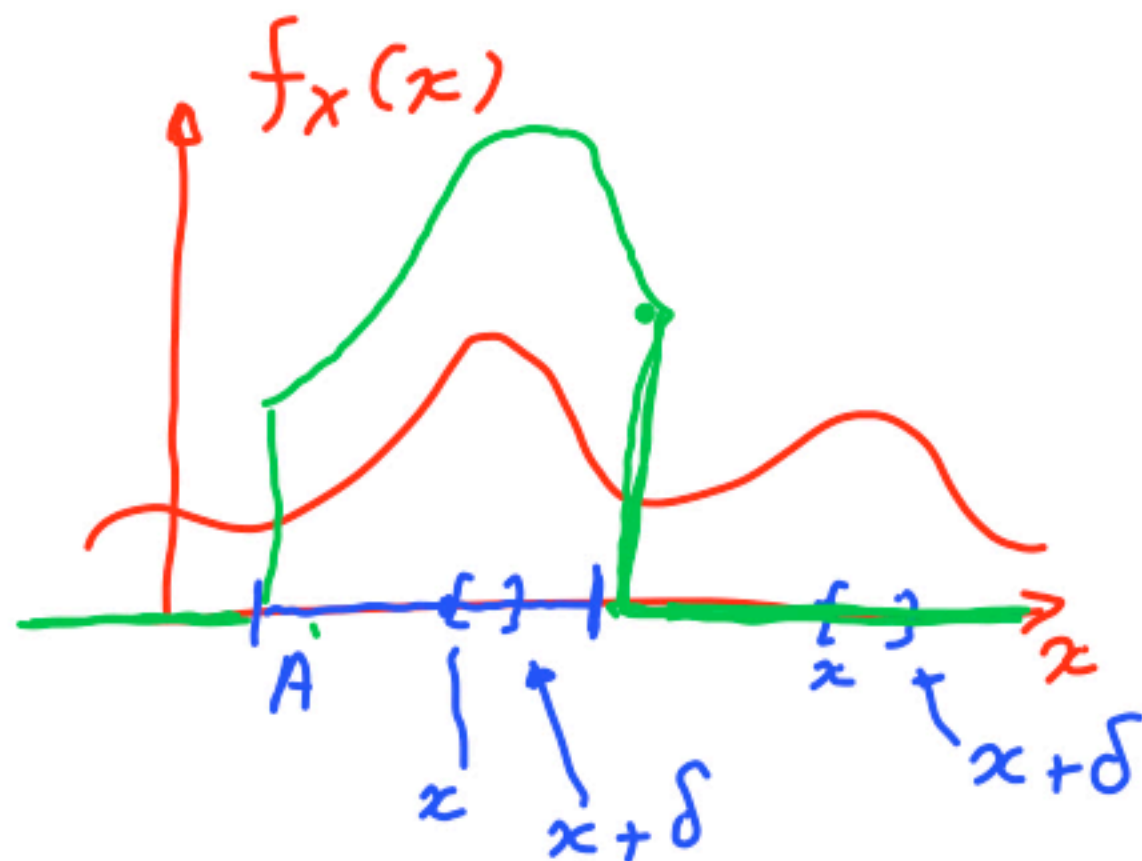
$$\int \underset{\bullet}{f_{X|A}(x)} dx = 1$$

Conditional PDF of  $X$ , given that  $X \in A$

$$P(x \leq X \leq x + \delta \mid X \in A) \approx f_{X|X \in A}(x) \cdot \delta$$

$$= \frac{P(x \leq X \leq x + \delta, X \in A)}{P(A)}$$

$$= \frac{P(x \leq X \leq x + \delta)}{P(A)} \approx \frac{f_X(x) \delta}{P(A)}$$



$$f_{X|X \in A}(x) = \begin{cases} 0, & \text{if } x \notin A \\ \frac{f_X(x)}{P(A)}, & \text{if } x \in A \end{cases}$$

## Conditional expectation of $X$ , given an event

$$\mathbf{E}[X] = \sum_x x p_X(x)$$

$$\mathbf{E}[X] = \int x f_X(x) dx$$

$$\mathbf{E}[X | A] = \sum_x x p_{X|A}(x)$$

$$\mathbf{E}[X | A] = \int x f_{X|A}(x) dx \quad \text{Def}$$

## Expected value rule:

$$\mathbf{E}[g(X)] = \sum_x g(x) p_X(x)$$

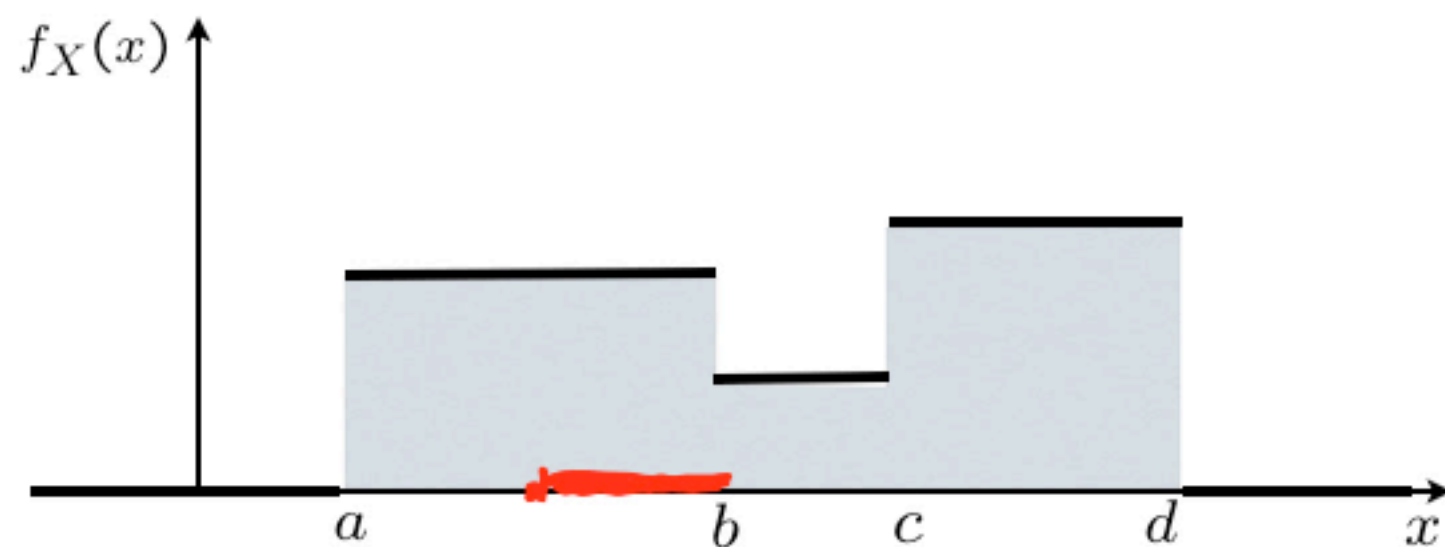
$$\mathbf{E}[g(X)] = \int g(x) f_X(x) dx$$

$$\mathbf{E}[g(X) | A] = \sum_x g(x) p_{X|A}(x)$$

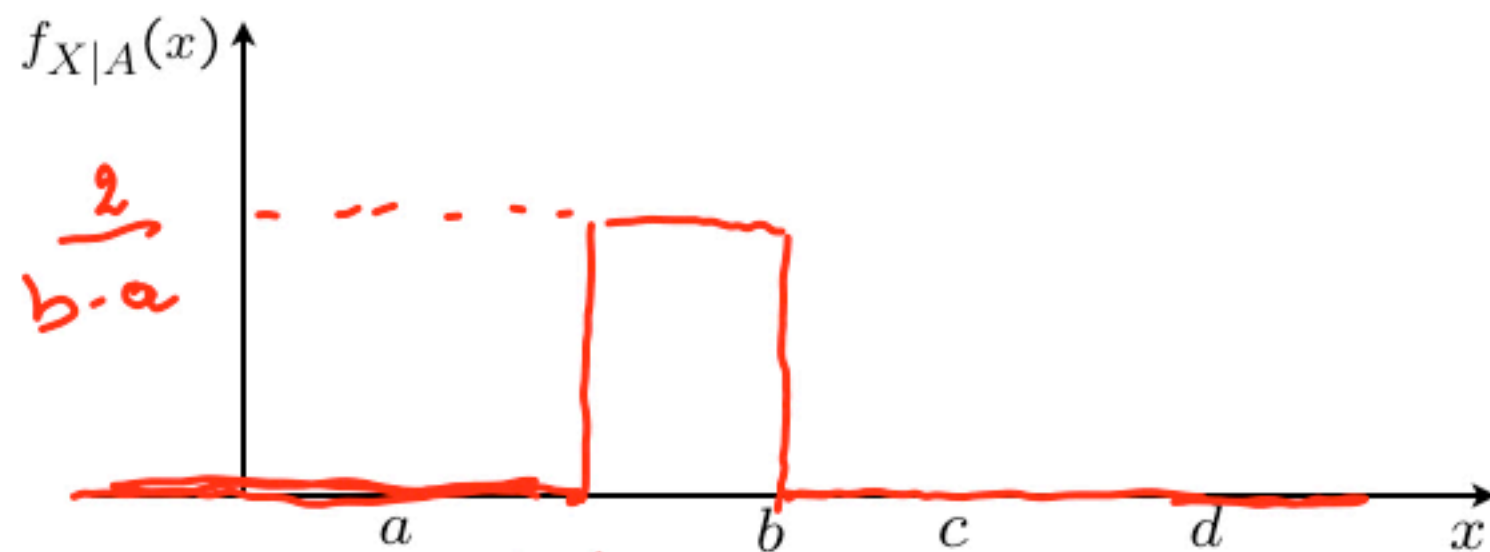
$$\mathbf{E}[g(X) | A] = \int g(x) f_{X|A}(x) dx$$

## Example

$$A: \frac{a+b}{2} \leq X \leq b$$



$$\frac{a+b}{2}$$



$$\frac{2}{b-a}$$

$$\frac{a+b}{2}$$

$$\mathbf{E}[X \mid A] = \frac{1}{2} \cdot \frac{a+b}{2} + \frac{1}{2} b$$

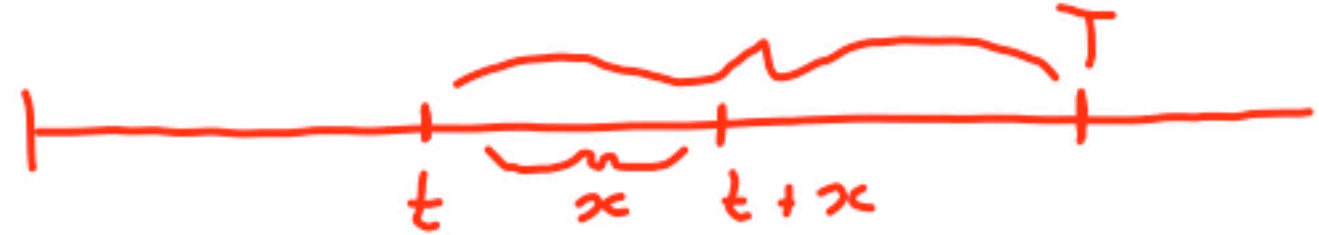
$$= \frac{1}{4} a + \frac{3}{4} b$$

$$\mathbf{E}[X^2 \mid A] =$$

$$\int_{\frac{a+b}{2}}^b \frac{2}{b-a} \cdot x^2 dx$$

## Memorylessness of the exponential PDF

- Do you prefer a used or a new “exponential” light bulb? **Probabilistically identical!**
- Bulb lifetime  $T$ :  $\text{exponential}(\lambda)$



$$P(T > x) = e^{-\lambda x}, \text{ for } x \geq 0$$

– we are told that  $T > t$

– r.v.  $X$ : remaining lifetime  $= T - t$

$$P(X > x | T > t) = e^{-\lambda x}, \text{ for } x \geq 0$$

$$= \frac{P(T - t > x, T > t)}{P(T > t)} = \frac{P(T > t + x, T > t)}{P(T > t)} = \frac{P(T > t + x)}{P(T > t)}$$

$$= \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} = e^{-\lambda x}$$



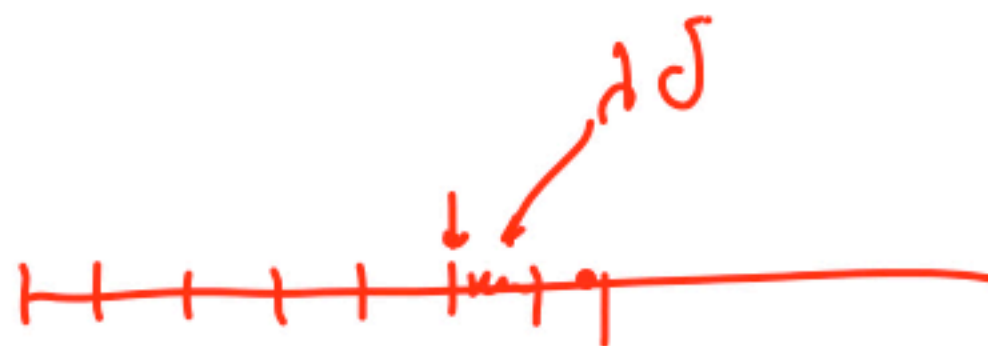
## Memorylessness of the exponential PDF

$$f_T(x) = \lambda e^{-\lambda x}, \quad \text{for } x \geq 0$$

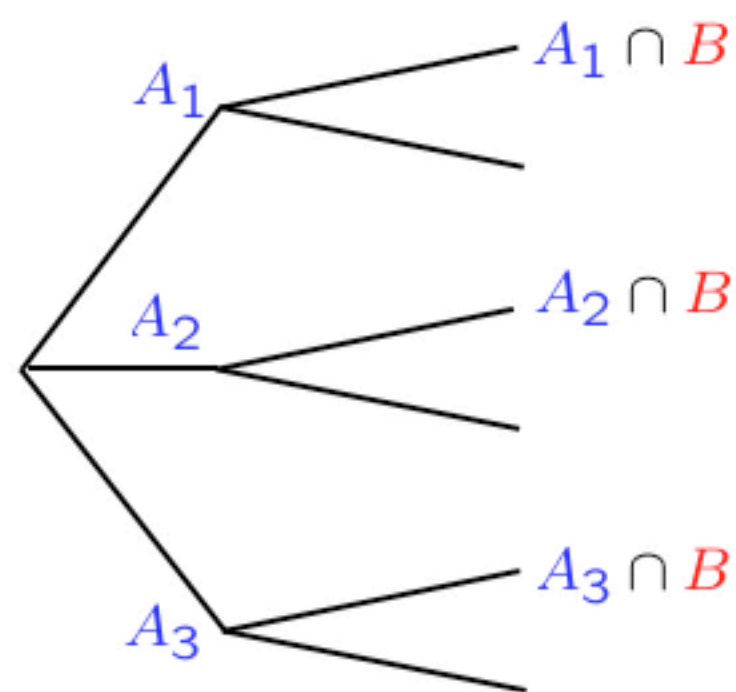
$$\mathbf{P}(0 \leq T \leq \delta) \approx f_T(0) \cdot \delta = \lambda \delta$$

$$\mathbf{P}(t \leq T \leq t + \delta \mid T > t) = \lambda \delta$$

similar to an independent coin flip,  
every  $\delta$  time steps,  
with  $\mathbf{P}(\text{success}) \approx \lambda \delta$



## Total probability and expectation theorems



$$\mathbf{P}(B) = \mathbf{P}(A_1)\mathbf{P}(B | A_1) + \cdots + \mathbf{P}(A_n)\mathbf{P}(B | A_n)$$

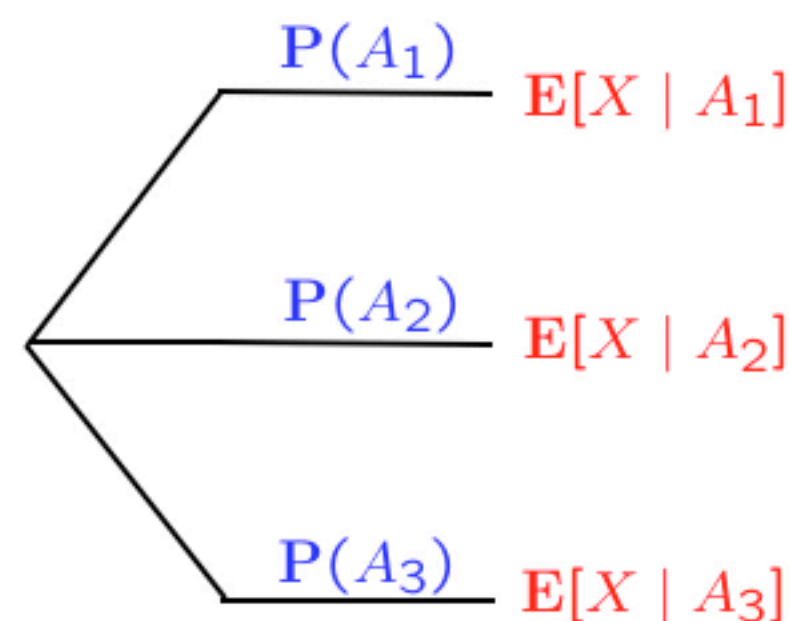
$$p_X(x) = \mathbf{P}(A_1) p_{X|A_1}(x) + \cdots + \mathbf{P}(A_n) p_{X|A_n}(x)$$

$$\begin{aligned} F_X(x) &= \mathbf{P}(X \leq x) = \mathbf{P}(A_1) \mathbf{P}(X \leq x | A_1) + \cdots \\ &= \mathbf{P}(A_1) F_{X|A_1}(x) + \cdots \end{aligned}$$

$$f_X(x) = \mathbf{P}(A_1) f_{X|A_1}(x) + \cdots + \mathbf{P}(A_n) f_{X|A_n}(x)$$

$$\int x f_X(x) dx = \mathbf{P}(A_1) \int x f_{X|A_1}(x) dx + \cdots$$

$$\mathbf{E}[X] = \mathbf{P}(A_1)\mathbf{E}[X | A_1] + \cdots + \mathbf{P}(A_n)\mathbf{E}[X | A_n]$$

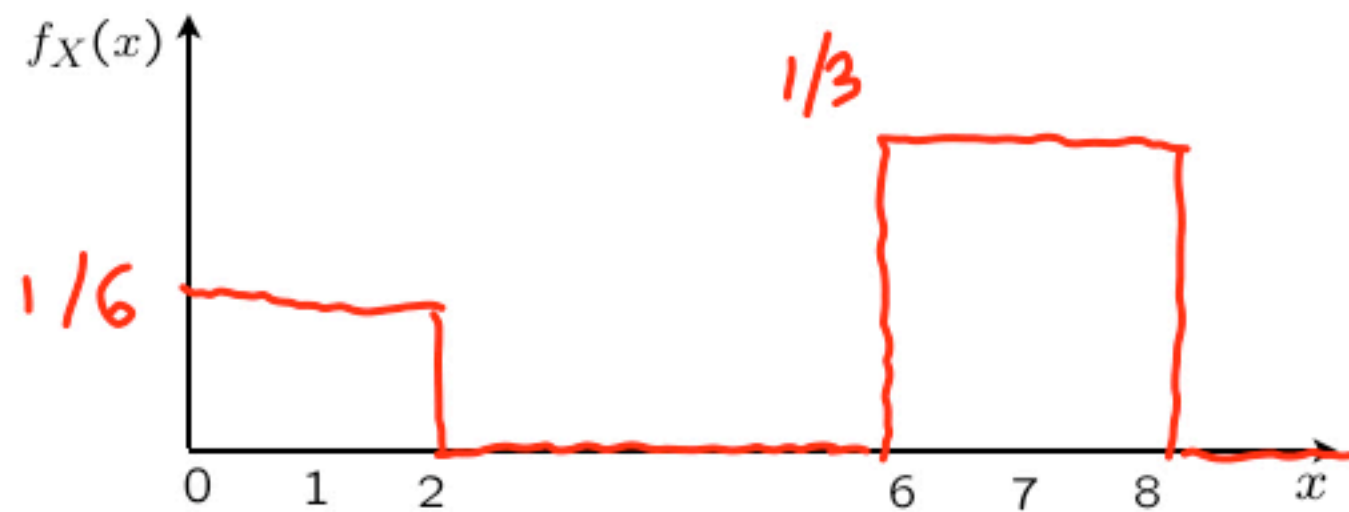




## Example

- Bill goes to the supermarket shortly, with probability  $1/3$ , at a time uniformly distributed between 0 and 2 hours from now; or with probability  $2/3$ , later in the day at a time uniformly distributed between 6 and 8 hours from now

$$P(A_1) = \frac{1}{3} \quad f_{X|A_1} \sim \text{unif}[0, 2] \quad P(A_2) = \frac{2}{3} \quad f_{X|A_2} \sim U[6, 8]$$



$$f_X(x) = P(A_1)f_{X|A_1}(x) + \cdots + P(A_n)f_{X|A_n}(x)$$

$$\bullet \quad E[X] = P(A_1)E[X|A_1] + \cdots + P(A_n)E[X|A_n]$$

$$\frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 7$$

## Mixed distributions

$$X = \begin{cases} \text{uniform on } [0, 2], & \text{with probability } 1/2 \\ 1, & \text{with probability } 1/2 \end{cases}$$

Is  $X$  discrete? *No*

Is  $X$  continuous? *No*

$$P(X=1) = 1/2$$

*X is mixed*

$Y$  discrete

$Z$  continuous

$$X = \begin{cases} Y, & \text{with probability } p \\ Z, & \text{with probability } 1 - p \end{cases}$$

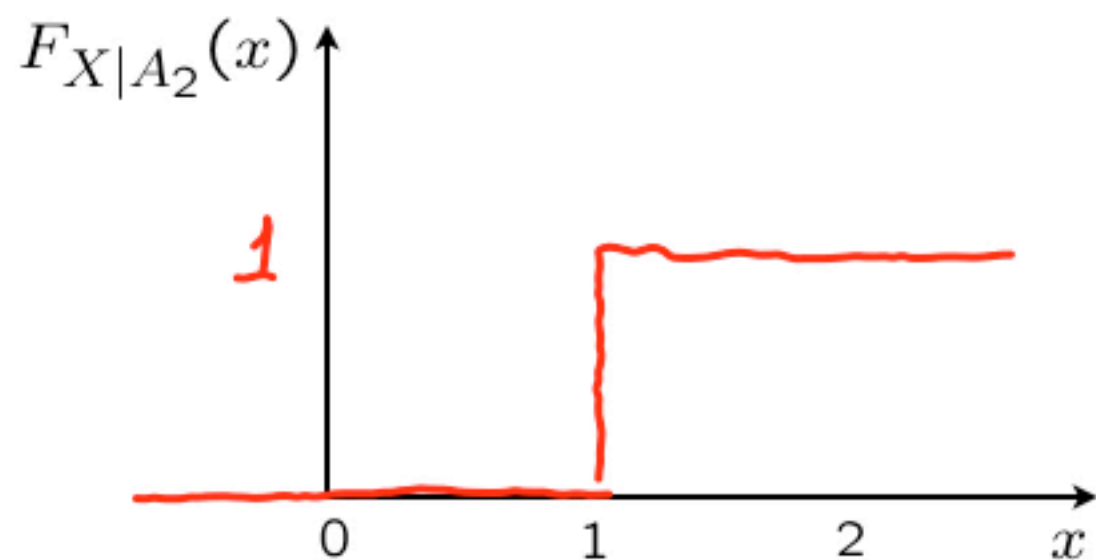
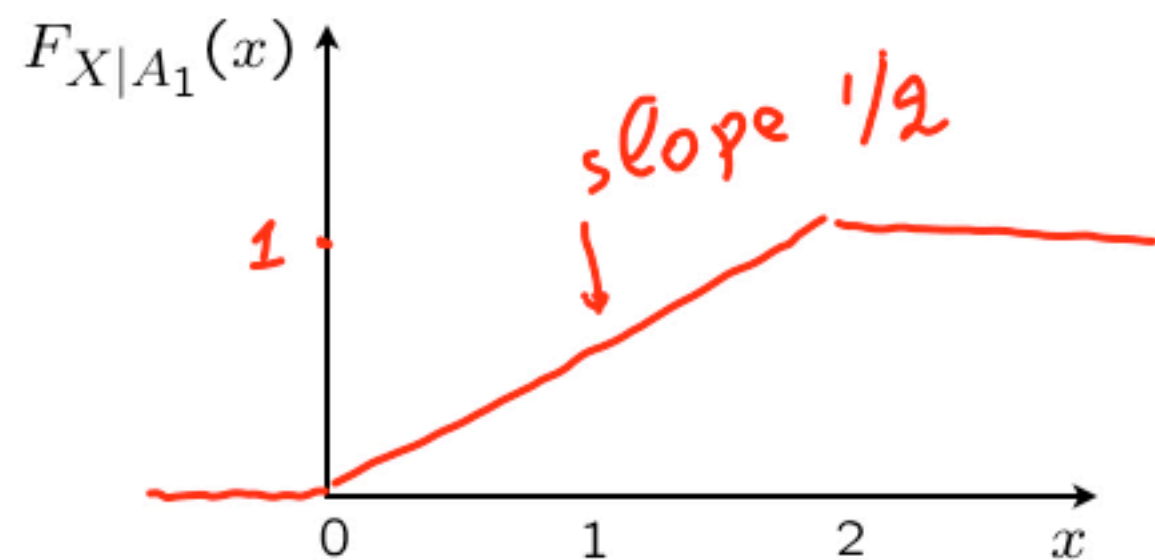
$$F_X(x) = p \cdot P(Y \leq x) + (1-p) P(Z \leq x)$$

$$= p F_Y(x) + (1-p) F_Z(x)$$

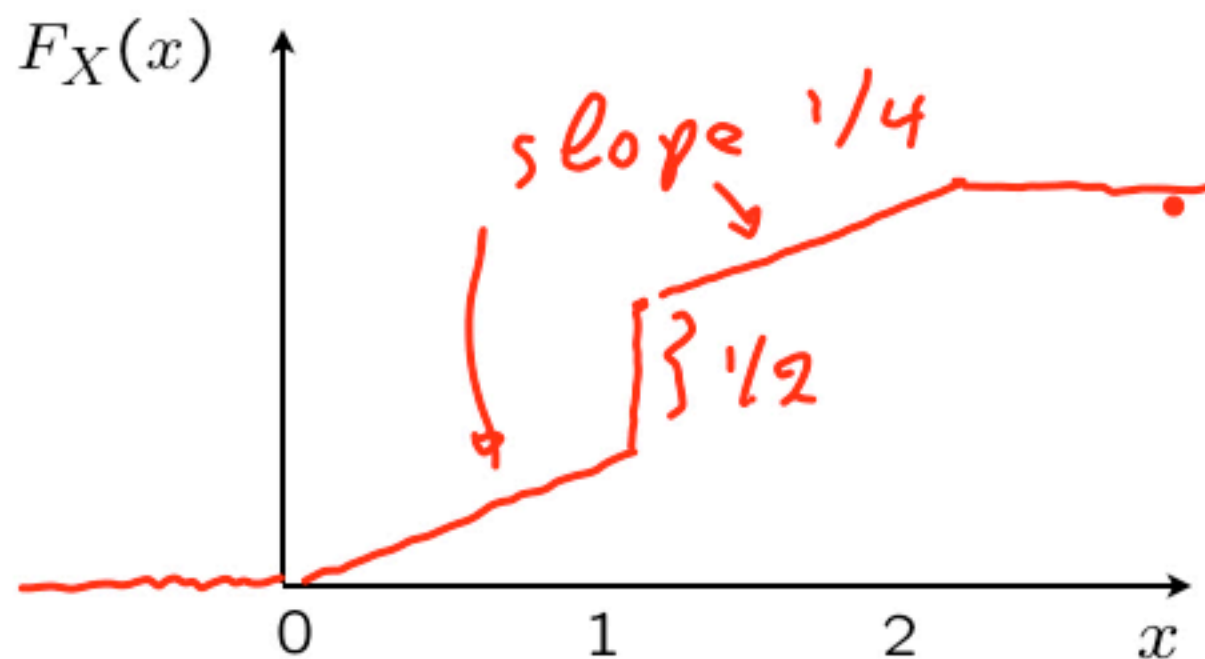
$$E[X] = p E[Y] + (1-p) E[Z]$$

## Mixed distributions

$$X = \begin{cases} \text{uniform on } [0, 2], & \text{with probability } 1/2 \quad A_1 \\ 1, & \text{with probability } 1/2 \quad A_2 \end{cases}$$



$$F_X(x) = P(A_1)F_{X|A_1}(x) + P(A_2)F_{X|A_2}(x)$$



## Jointly continuous r.v.'s and joint PDFs

$$p_X(x) \quad f_X(x)$$

$$p_{X,Y}(x, y) \quad f_{X,Y}(x, y)$$

$$p_{X,Y}(x, y) = \mathbf{P}(X = x \text{ and } Y = y) \geq 0$$

$$f_{X,Y}(x, y) \geq 0$$

$$\mathbf{P}((X, Y) \in B) = \sum_{(x,y) \in B} p_{X,Y}(x, y)$$

$$\mathbf{P}((X, Y) \in B) = \int \int_{(x,y) \in B} f_{X,Y}(x, y) dx dy \quad \bullet$$

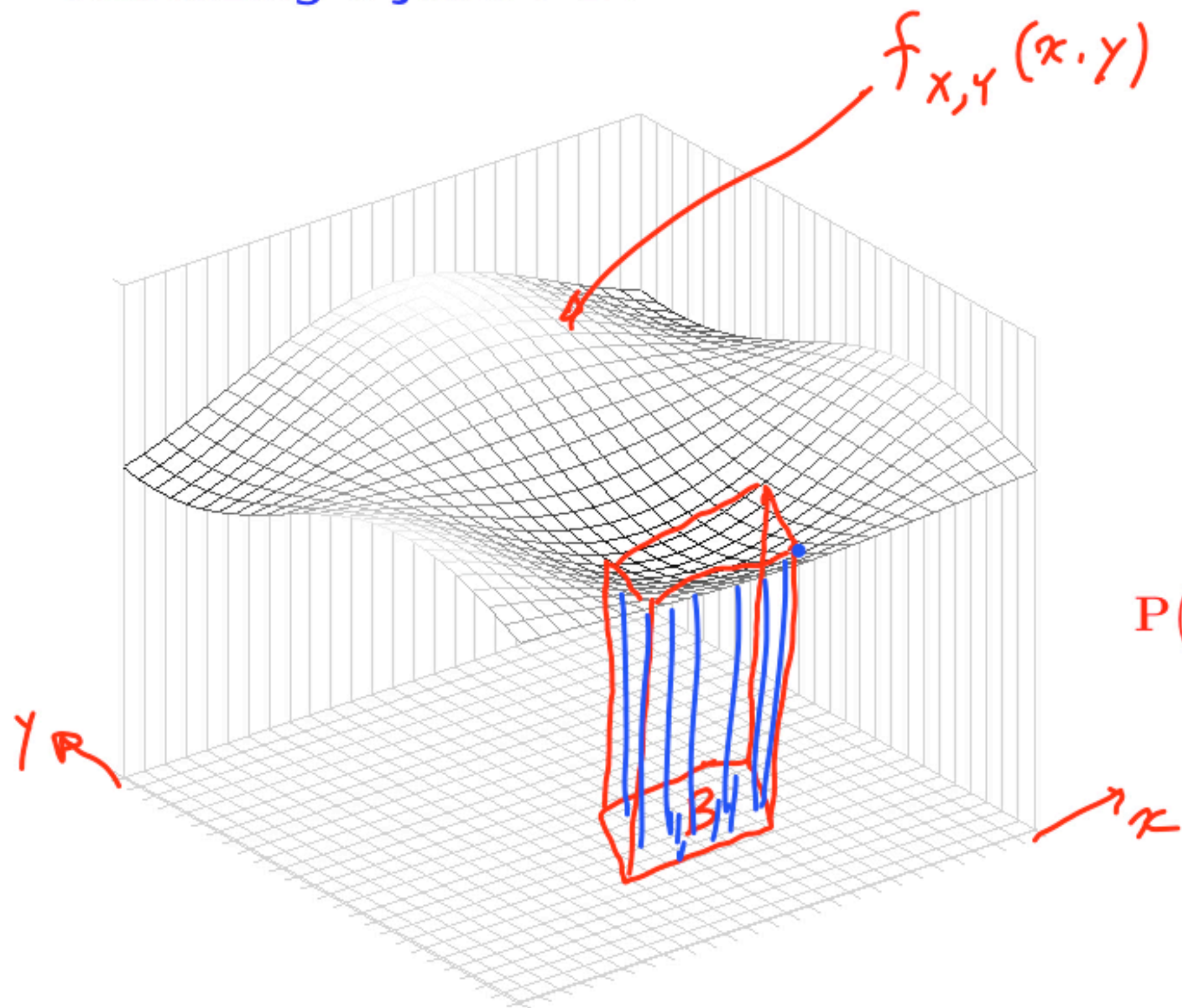
$$\sum_x \sum_y p_{X,Y}(x, y) = 1$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy = 1$$

**Definition:** Two random variables are **jointly continuous** if they can be described by a joint PDF



## Visualizing a joint PDF



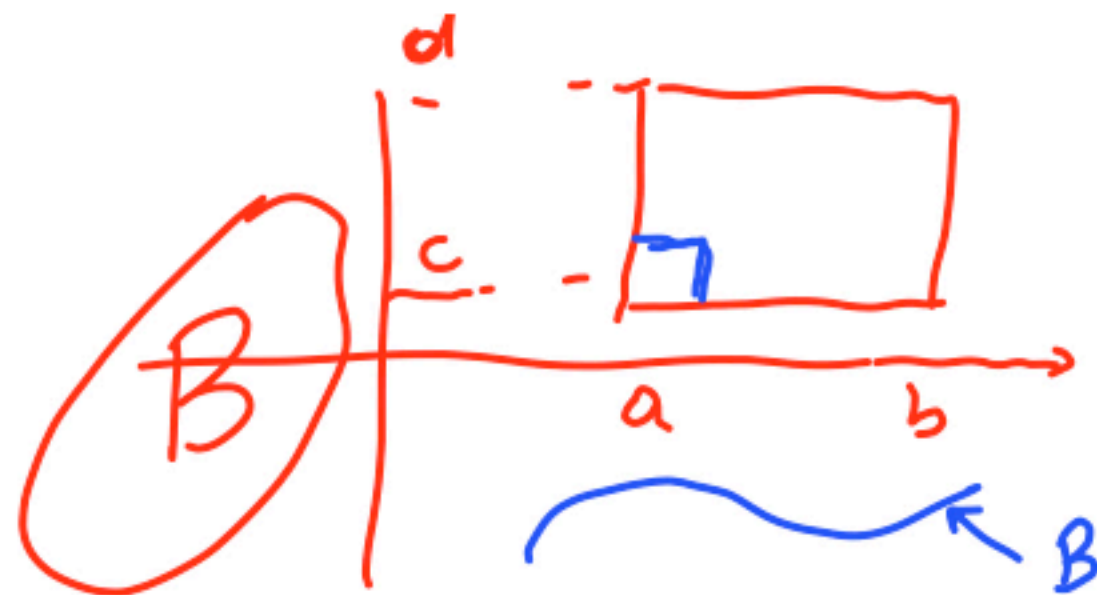
$$\mathbf{P}((X,Y) \in B) = \int \int_{(x,y) \in B} f_{X,Y}(x,y) \, dx \, dy$$



## On joint PDFs

$$\mathbf{P}((X, Y) \in B) = \int \int_{(x,y) \in B} f_{X,Y}(x, y) dx dy$$

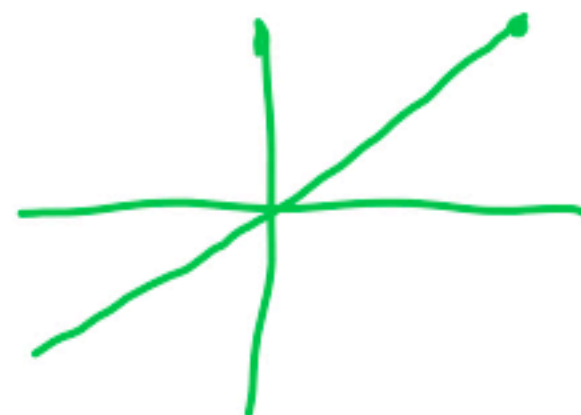
$$\mathbf{P}(a \leq X \leq b, c \leq Y \leq d) = \int_c^d \int_a^b f_{X,Y}(x, y) dx dy$$



$$\mathbf{P}(a \leq X \leq a + \delta, c \leq Y \leq c + \delta) \approx f_{X,Y}(a, c) \cdot \delta^2$$

$f_{X,Y}(x, y)$ : probability per unit area

$$\text{area}(B) = 0 \Rightarrow \mathbf{P}((X, Y) \in B) = 0$$



$$Y = X$$

## From the joint to the marginals

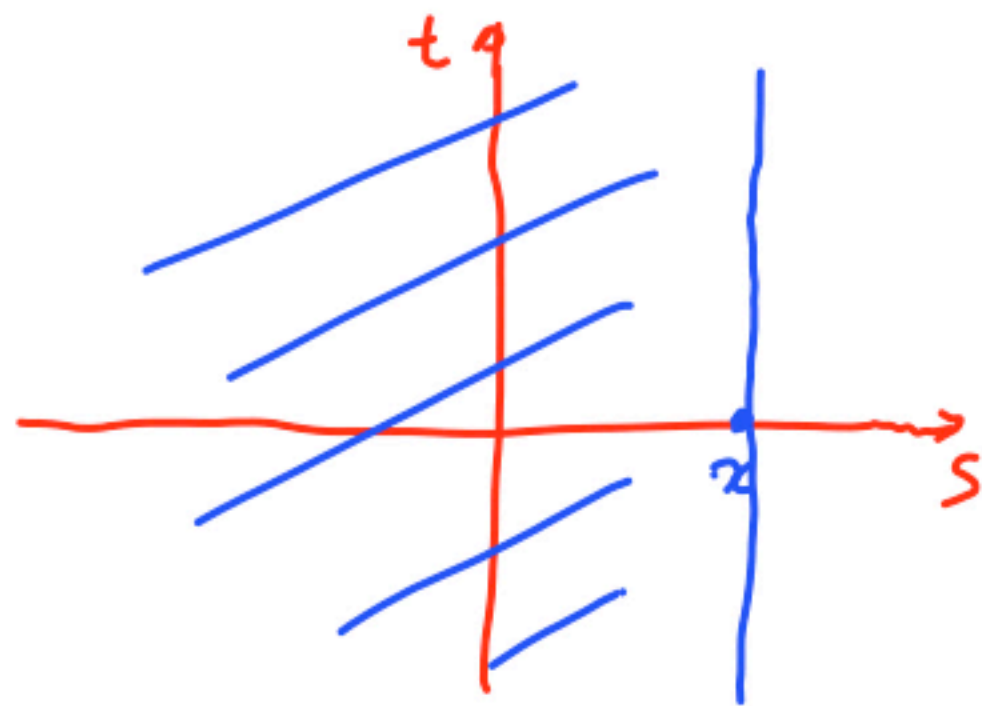
$$p_X(x) = \sum_y p_{X,Y}(x, y)$$

$$p_Y(y) = \sum_x p_{X,Y}(x, y)$$

$$f_X(x) = \int f_{X,Y}(x, y) dy$$

$$f_Y(y) = \int f_{X,Y}(x, y) dx$$

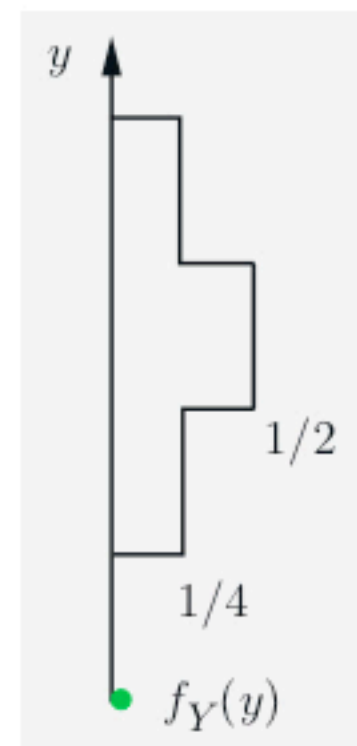
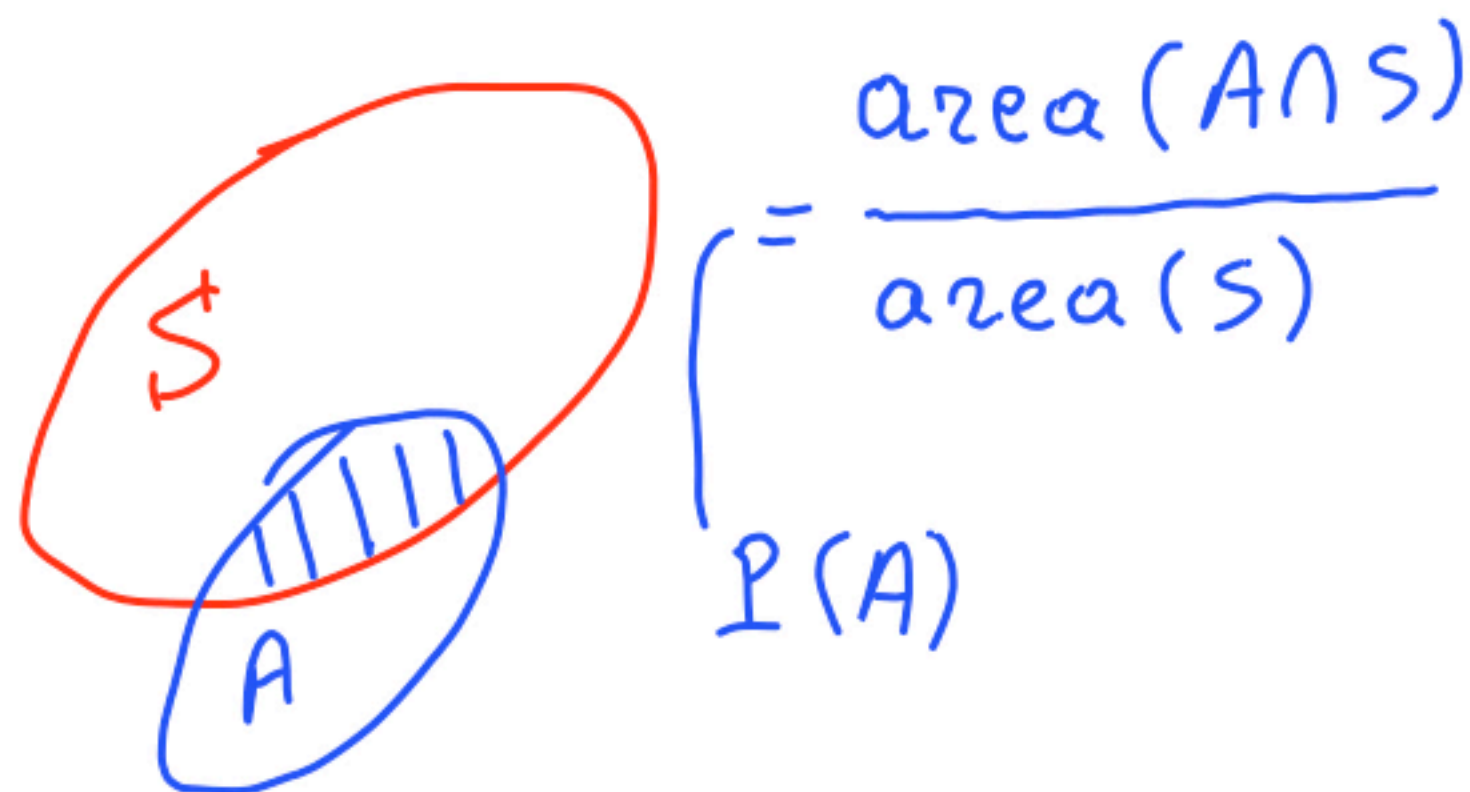
$$F_X(x) = P(X \leq x) = \int_{-\infty}^x \left[ \int_{-\infty}^{\infty} f_{X,Y}(s, t) dt \right] ds$$



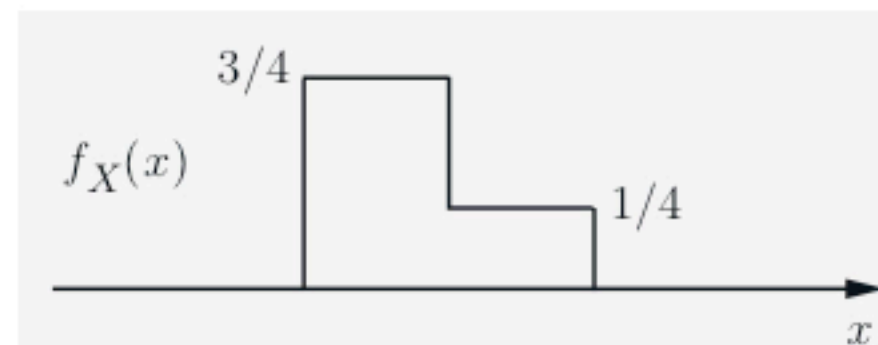
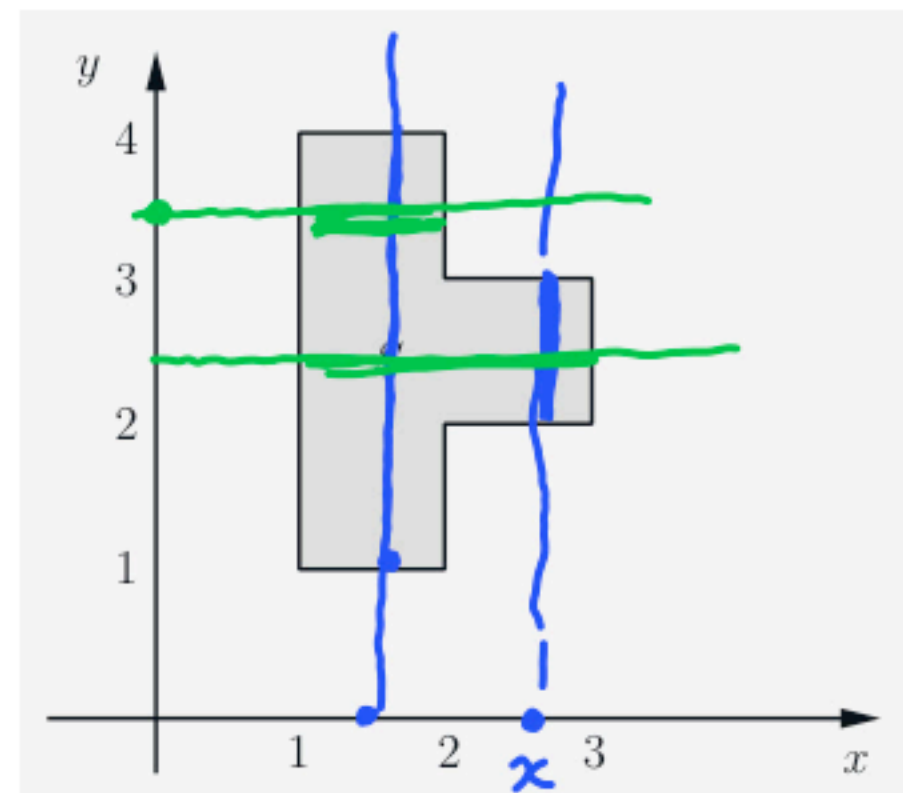
$$f_X(x) = \frac{dF_X}{dx}(x) = [ \quad ]$$

## Uniform joint PDF on a set $S$

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\text{area of } S}, & \text{if } (x,y) \in S, \\ 0, & \text{otherwise.} \end{cases}$$



$$f_{X,Y} = \frac{1}{4}$$



## More than two random variables

$$p_{X,Y,Z}(x, y, z)$$

$$f_{X,Y,Z}(x, y, z)$$

$$\sum_x \sum_y \sum_z p_{X,Y,Z}(x, y, z) = 1$$

$$p_X(x) = \sum_y \sum_z p_{X,Y,Z}(x, y, z)$$

$$p_{X,Y}(x, y) = \sum_z p_{X,Y,Z}(x, y, z)$$

## Functions of multiple random variables

$$Z = g(X, Y)$$

Expected value rule:

$$\mathbf{E}[g(X, Y)] = \sum_x \sum_y g(x, y) p_{X,Y}(x, y)$$

$$\mathbf{E}[g(X, Y)] = \int \int g(x, y) f_{X,Y}(x, y) dx dy$$

## Linearity of expectations

$$\mathbf{E}[aX + b] = a\mathbf{E}[X] + b$$

•

$$\mathbf{E}[X + Y] = \mathbf{E}[X] + \mathbf{E}[Y]$$

$$\mathbf{E}[X_1 + \cdots + X_n] = \mathbf{E}[X_1] + \cdots + \mathbf{E}[X_n]$$



## The joint CDF

$$F_X(x) = \mathbf{P}(X \leq x) = \int_{-\infty}^x f_X(t) dt$$

$$f_X(x) = \frac{dF_X}{dx}(x)$$

$$F_{X,Y}(x,y) = \mathbf{P}(X \leq x, Y \leq y) = \int_{-\infty}^y \int_{-\infty}^x \underbrace{f_{X,Y}(s,t)}_{\text{joint PDF}} ds dt$$

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}}{\partial x \partial y}(x,y)$$

$$F_{X,Y}(x,y) = xy$$

$$f_{X,Y}(x,y) = 1$$

