5. Hypothesis test between two normals

Problem 5. Hypothesis test between two normals

2 points possible (graded)

Conditioned on the result of an unbiased coin flip, the random variables $T_1, T_2, \ldots, T_n$ are independent and identically distributed, each drawn from a common normal distribution with mean zero. If the result of the coin flip is Heads, this normal distribution has variance 1; otherwise, it has variance 4. Based on the observed values $t_1, t_2, \ldots, t_n$, we use the MAP rule to decide whether the normal distribution from which they were drawn has variance 1 or variance 4. The MAP rule decides that the underlying normal distribution has variance 1 if and only if

$$\left| c_1 \sum_{i=1}^{n} t_i^2 + c_2 \sum_{i=1}^{n} t_i \right| < 1.$$

Find the values of $c_1 \geq 0$ and $c_2 \geq 0$ such that this is true. Express your answer in terms of n, and use "ln" to denote the natural logarithm function, as in "ln(3)".

$c_1 =$

Answer: 3/(8*n*ln(2))

$c_2 =$

Answer: 0

STANDARD NOTATION

Solution:

Let $\Theta = 0$ denote that the observations $t_1, t_2, \ldots, t_n$ were generated from a normal distribution with variance 1, and let $\Theta = 1$ denote that they were generated from a normal distribution with variance 4. For simplicity, let us use the notation $N(t_1, \ldots, t_n; 0, \sigma^2)$ to denote the joint PDF of n i.i.d. normal random variables with mean 0 and variance σ^2, evaluated at $t_1, \ldots, t_n$.

Therefore, given the observations $t_1, \ldots, t_n$, the posterior probability that the samples are generated from a normal distribution with variance 1 is

$$\mathbf{P}(\Theta = 0 \mid T_1 = t_1, \ldots, T_n = t_n) = \frac{(1/2) \cdot N(t_1, \ldots, t_n; 0, 1)}{(1/2) \cdot N(t_1, \ldots, t_n; 0, 1) + (1/2) \cdot N(t_1, \ldots, t_n; 0, 4)}.$$

Similarly, the probability that the samples are generated from a normal distribution with variance 4 is given by

$$\mathbf{P}(\Theta = 1 \mid T_1 = t_1, \ldots, T_n = t_n) = \frac{(1/2) \cdot N(t_1, \ldots, t_n; 0, 4)}{(1/2) \cdot N(t_1, \ldots, t_n; 0, 1) + (1/2) \cdot N(t_1, \ldots, t_n; 0, 4)}.$$

The MAP rule favors $\Theta = 0$ if the following inequality holds:

$$\mathbf{P}(\Theta = 0 \mid T_1 = t_1, \ldots, T_n = t_n) > \mathbf{P}(\Theta = 1 \mid T_1 = t_1, \ldots, T_n = t_n)$$

We notice that the denominators in the expressions for $\mathbf{P}(\Theta = 0 \mid \ldots)$ and $\mathbf{P}(\Theta = 1 \mid \ldots)$ are the same, so it suffices to compare the numerators. Therefore, the MAP rule favors $\Theta = 0$ if the following inequality holds:

$$\begin{aligned} N(t_1, \ldots, t_n; 0, 1) &> N(t_1, \ldots, t_n; 0, 4) \\ \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi \cdot 1}} e^{-\frac{t_i^2}{2 \cdot 1}} &> \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi \cdot 4}} e^{-\frac{t_i^2}{2 \cdot 4}}. \end{aligned}$$

With a little bit of algebra, we obtain

$$\left| \frac{3}{8} \sum_{i=1}^{n} t_i^2 \right| < n \cdot \ln(2).$$

Note: If the means under the two hypotheses were different, a similar answer would be obtained but with a nonzero coefficient c_2.

提交 You have used 0 of 3 attempts

Answers are displayed within the problem

讨论

显示讨论

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