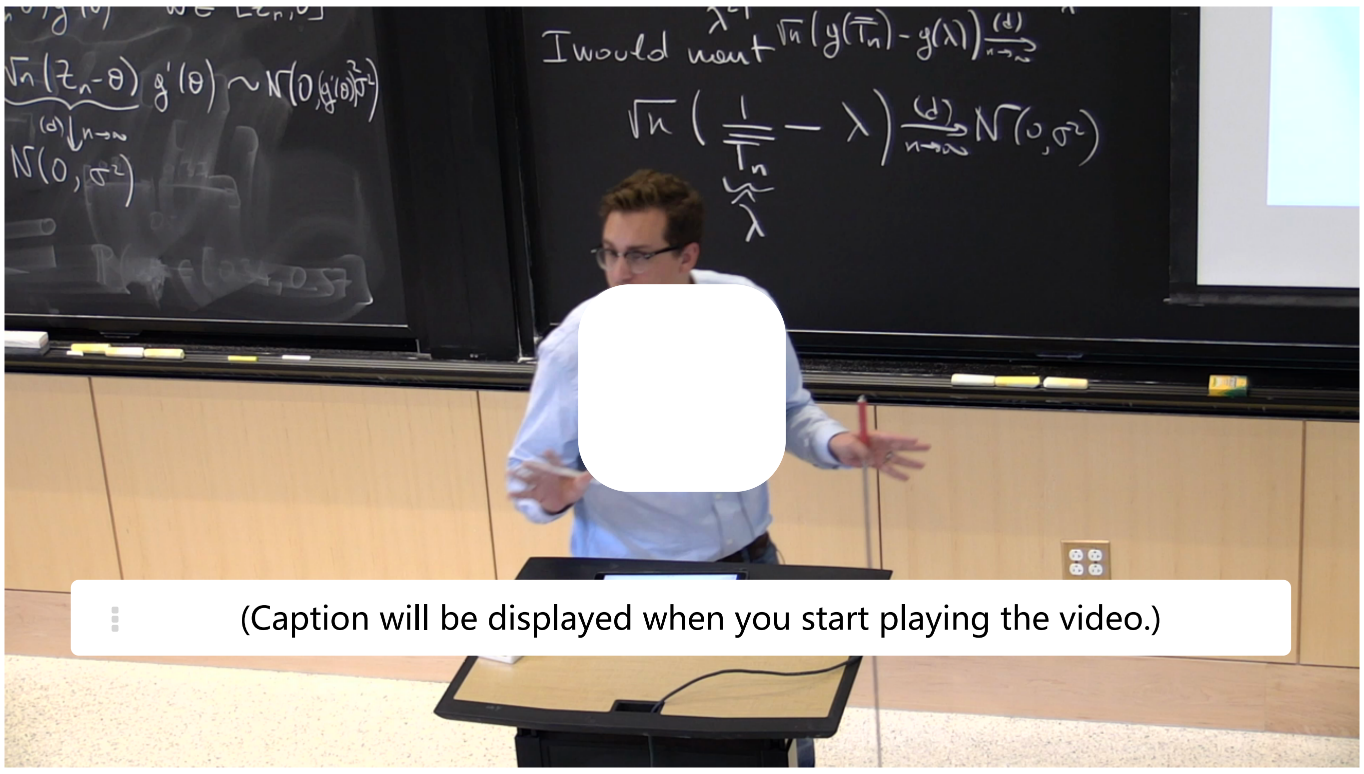
9. Applying the Delta Method

Applying the Delta Method



Start of transcript. Skip to the end.

Everybody knows where this is coming from?

I didn't drop it on you like unprepared?

So this is where the correction comes from.

When you apply the delta method, be very careful.

In our example, in the T example, what was theta?

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An Estimator for the Mean of an Exponential Random Variable

1/1 point (graded)

In the next two problems, we will repeat the computation in lecture.

Let $X_1, \ldots, X_n \sim \exp(\lambda)$ where $\lambda > 0$.

Since $\mathbb{E}[X] = \dfrac{1}{\lambda}$, by the central limit theorem,

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{\lambda}\right) \xrightarrow[n\to\infty]{(d)} N(0, \sigma^2).$$

What is σ^2 in terms of λ?

$\sigma^2 =$ 1/lambda^2

$\frac{1}{\lambda^2}$

✔ **Answer:** 1/lambda^2

STANDARD NOTATION

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Answers are displayed within the problem

Applying the Delta Method to an Exponential Random Variable

1/1 point (graded)

As above, let $X_1, \ldots, X_n \sim \exp(\lambda)$ where $\lambda > 0$. Let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ denote the sample mean. By the CLT, we know that

$$\sqrt{n}\left(\overline{X}_n - \frac{1}{\lambda}\right) \xrightarrow[n\to\infty]{(d)} N(0, \sigma^2)$$

for some value of σ^2 that depends on λ, which you computed in the problem above.

If we set g to be

$$\begin{aligned} g:\ & \mathbb{R} \to \mathbb{R} \\ & x \mapsto 1/x, \end{aligned}$$

then by the Delta method,

$$\sqrt{n}\left(g(\overline{X}_n) - g\left(\frac{1}{\lambda}\right)\right) \xrightarrow[n\to\infty]{(d)} N(0, \tau^2).$$

where τ^2 is the asymptotic variance and can be expressed in terms of λ.

What is the asymptotic variance τ^2 in terms of λ?
(Choose all that apply.)

- [ ] $g'(\lambda)\,\mathsf{Var}X$
- [ ] $g'(\lambda)\,\frac{1}{\lambda^2}$
- [x] $g'(E[X])^2\mathsf{Var}X$ ✔
- [x] $g'\left(\frac{1}{\lambda}\right)^2\frac{1}{\lambda^2}$ ✔
- [ ] $\frac{1}{\lambda^2}$
- [x] λ^2 ✔

✔

Solution:

By the previous problem, we have

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - 1/\lambda\right) \xrightarrow{(d)} \mathcal{N}\left(0, \frac{1}{\lambda^2}\right).$$

To apply the Delta method, first observe that $g'(x) = -1/x^2$ and that g is continuously differentiable for $x > 0$. By the Delta method,

$$\begin{aligned} \sqrt{n}\left(g(\overline{X}_n) - g(\mathbb{E}[X])\right) = \sqrt{n}\left(\frac{1}{\overline{X}_n} - \frac{1}{1/\lambda}\right) &\xrightarrow[n\to\infty]{(d)} \mathcal{N}\left(0, g'(\mathbb{E}[X])^2 var(X)\right) \\ &= \mathcal{N}\left(0, \left(g'\left(\frac{1}{\lambda}\right)\right)^2\left(\frac{1}{\lambda^2}\right)\right) \\ &= \mathcal{N}(0, \lambda^2). \end{aligned}$$

Since $g'(x) = -1/x^2$, $g'\left(\frac{1}{\lambda}\right) = -\lambda^2$, and hence the asymptotic variance of $g(\overline{X}_n)$ evaluates to λ^2.

Warning: It's very important that we apply g' to the value $1/\lambda$, and not λ. We start with a consistent estimator, namely $\overline{X}_n$, whose limit is $\mathbb{E}[X] = 1/\lambda$, and the Delta method asks us to apply g' to the limit of that consistent estimator. Be careful about this, as it is a common source of errors.

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Answers are displayed within the problem

When does the delta method apply?

0/1 point (graded)

Let $X_1, X_2, \ldots \stackrel{\text{i.i.d.}}{\sim} X$. The distribution of X depends on a **positive** parameter θ, which is a function of the mean μ, i.e $\theta = g(\mu)$. You estimate θ by the estimator $\hat{\theta} = g(\overline{X}_n)$.

For which function g can the delta method be applied? Remember that $\theta > 0$.
(Choose all that apply.)

- [x] $g(x) = x^3$ ✔
- [x] $g(x) = \sqrt{x}$ ✔
- [x] $g(x) = \ln(x)$ ✔
- [x] $g(x) = \begin{cases} x & \text{if } x \leq 1 \\ 2x - 1 & \text{if } x > 1 \end{cases}$
- [ ] $g(x) = \frac{1}{x-1}$ ✔

✖

Solution:

For the Delta method to apply, g' exists and is continuous at $\mathbb{E}[X] = g^{-1}(\theta)$. Since θ and $\mu = \mathbb{E}[X]$ are unknown, for the Delta method to apply, we need to make sure g is continuously differentiable at all possible values of $\mathbb{E}[X]$ given that $\theta > 0$. Let us first go through the correct choices:

1. $g(x) = x^3$ is continuously differentiable everywhere.

2. $g(x) = \sqrt{x}$ is continously differentiable for all $x > 0$. Given any $\theta > 0$, $\mu = g^{-1}(\theta) = \theta^2 > 0$. So for all possible values of $\mathbb{E}[X]$, g satisfies the requirement; hence Delta method applies.

3. Similarly, $g(x) = \ln x$ is continously differentiable for all $x > 0$. Given any $\theta > 0$, $\mu = g^{-1}(\theta) = e^{\theta} > 0$. Again, Delta method applies.

4. $g(x) = \frac{1}{x-1}$ is continously differentiable everywhere except at $x = 1$. However, inverting $\theta = g(\mu) = \frac{1}{\mu-1}$ gives $\mu = \frac{1}{\theta} + 1$, so $\mu \neq 1$ for all $\theta > 0$. Hence the Delta method applies.

Here is the incorrect choice: $g(x) = \begin{cases} x & \text{if } x \leq 1 \\ 2x - 1 & \text{if } x \geq 1 \end{cases}$ " is a 1-to-1 piecewise linear function and is continuously differentiable everywhere except at $x = 1$. Observe that $g(1) = 1$, hence when $\theta = 1$, $\mu = 1$. There is a possible value of μ when $g'(\mu)$ does not exist, so the Delta method does not apply.

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Answers are displayed within the problem