THE THEORETICAL SENSITIVITY OF THE DICKE RADIOMETER

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Summary

Following a brief review of the theory of operation, characteristics, and applications of the modern microwave radiometer, the Dicke radiometer circuit is analyzed by the autocorrelation function-power spectrum method. The treatment achieves a greater generality than prior investigations although the method is, in general, similar to that used by Goldstein.

The major improvement of this treatment is a result of replacing the detector characteristic assumed by Dicke and Goldstein, $V_o = kV_i^2$, with a general representation. The final result shows that the sensitivity of the radiometer is independent of the detector characteristic. In addition, the results of Dicke and Goldstein were corrected and found to be the same as the sensitivity obtained in the present work.

This paper illustrates the method and describes the results, but omits the main body of routine calculation. The theoretical results were experimentally tested.

Microwave Radiometer

The microwave radiometer is rapidly becoming one of the basic instruments of electronics. In a sense, any sensitive receiver is a radiometer. However, the term is usually restricted to instruments using special circuits and techniques to obtain improved sensitivity.

The basic scheme used in nearly all modern radiometers was developed by R. H. Dicke[1]. A number of modifications of this basic scheme have been devised for special problems. For instance, where the Dicke scheme is an autocorrelation process, a somewhat different approach is described by S. J. Goldstein[2] who cross-correlates two signals. This paper, after a brief description of the radiometer's application, characteristics, and principles of operation, will develop a more general analysis of the Dicke circuit than has been available previously.

The modern microwave radiometer is the foundation for the new field of radio astronomy. The instrument has literally added a new dimension to one of the most fundamental sciences. While this application alone would warrant the attention of the profession, the instrument has more mundane uses. One of the first applications of the instrument, by Dicke and others[3], was in the measurement of atmospheric absorption of microwave radiation. The attenuation was deduced from the varying intensity of the solar radio-frequency emission as the sun traversed the zenith. Other applications follow from the characteristics of the instrument.

Qualitative Description

Figure 1 is a block diagram of the Dicke circuit. In qualitative terms, the operation of the circuit can be described as follows:

The modulated signal and the internal noise are amplified and detected. The output of the detector consists of d-c terms which are filtered out, a noise continuum, and a concentration of signal power at the modulation frequency, ω_q. The primary purpose of the band-pass filter is to reject extraneous line frequency hum and spurious signals. For this reason, there is no advantage in making the band-pass stage particularly narrow. If the output of the band-pass filter is multiplied by cos $\omega_q t$, then the signal term appears as dc superimposed on a noise continuum. Since a low-pass filter can easily be made arbitrarily narrow whereas a band-pass filter cannot, the extra multiplier-detector stage is preferred to the alternative of a very narrow band-pass filter. Another advantage of the extra detection stage is its stability (i.e., drift of the modulation frequency relative to the center frequency of the band-pass filter becomes less important).

The radiometer can readily detect a thermal noise power 30 db less than the equivalent noise power input to the receiver. This thousand-fold gain in sensitivity is obtained by a sacrifice in response time, and to achieve a 30 db improvement requires a reduction in response time from the order of microseconds to seconds.

For the receiver bandwidths normally used, the signal level is in the neighborhood of -150 dbw. It is frequently convenient to measure the radiometer's sensitivity in terms of its minimum temperature resolution. This refers to a black body termination and amounts to the order of 1°C.

Circuit Analysis

The input to the radiometer of Figure 1 is s(t). While s(t) is a thermal noise voltage, the statistics of s(t) contain information. For this reason, s(t) is referred to in the following as the signal, despite its character. It may be generated by a resistor (waveguide termination), or by a remote black body whose emission is picked up by an antenna. In the first case, the noise power is kTB. In the second case, the emission from a black body is described in the radio-frequency spectrum by the Rayleigh-Jeans radiation law.* Since Burgess, and in a simpler fashion, Dicke, have demonstrated the equivalent of Rayleigh-Jeans radiation and Nyquist thermal noise, it is not necessary to specialize the problem for the purpose of analysis. The signal, s(t), is a spectrally uniform random voltage and distributed in amplitude according to a Gaussian probability density function. The noise power of this signal is denoted by $\sigma_s^2/2\omega_\alpha$ (watts/radian) where ω_α is the bandwidth of the receiver.

To achieve improved sensitivity, it is necessary to distinguish in some manner between s(t) and n(t), since both quantities are identical except in magnitude. The modulation of s(t) provides a means of separation if the process of modulation does not react on the noise term, n(t). This modulation is periodic, therefore its form is unimportant for the purpose of analysis since any periodic modulating waveform can be synthesized from sinusoidal Fourier components. The absolute sensitivity is, of course, slightly different since some schemes of modulation pass a greater signal power than others. For example, to convert the results of this work (which uses sinusoidal modulation) to square wave modulation, the sensitivity should be multiplied by the ratio of power in the fundamentals, $\pi/4$.

The modulation of the signal, s(t), converts this random waveform into a non-stationary random waveform. Fortunately, the present problem contains adequate regularity for the needs of analysis. It would be better, perhaps, to use the phrase "almost stationary" to suggest this regularity.

This non-stationary character of the problem, in common with the non-linear stages within the circuit, has led prior authors to represent the diode detector by the relation $V_o = kV_i^2$. As will become evident, the other non-linear devices in the circuit (the modulator and the multiplier) present no problem. The representation of a biased diode as a parabola has not been justified previous to this treatment. A much more satisfying characteristic for the diode with B volts of bias is:

$$\begin{aligned} V_o(t) &= k(V_i(t) - B)^N, \quad V_i(t) > B \\ &= 0 \qquad\qquad\qquad\;\, , \quad V_i(t) < B. \end{aligned} \tag{1}$$

The Van Vleck - North treatment described by Rice[5] is capable of dealing with functions such as (1) which are discontinuous in some derivative. The method is based upon the convolution of a Laplace step function with a continuous function. A double-contour integration completes the solution. Unfortunately, the process frequently yields integrals that are difficult or impossible to integrate. The integration that results for N non-integer appears to be impossible.

If the Van Vleck-North treatment is applied to a non-linear characteristic which is continuous in all derivatives, then the convolution with the Laplace step function is no longer required and the problem is simplified. However, an alternate approach is also available with this type of representation. Since both methods appeared laborious, the alternate approach was adopted because it yielded greater insight into the problem and permitted generalization of the final results.

It is rather easy to justify the use of a representation for a diode which is continuous in all derivatives and valid over only a finite range. First of all, the diode itself has this type of continuity due to the effect of contact potential. In most applications, the finite range of validity is assured since the input to the diode is limited by an amplifier somewhere in the preceding circuitry. If there is no limiting action on the input, the finite range of representation is still an excellent approximation. While it is true that s(t) is a Gaussian process and that there is a finite probability that some noise peak will exceed any arbitrary level, the number of such noise peaks is exceedingly small. Rice[6] has calculated that less than 0.05 percent of the noise peaks exceed four times the standard deviation of the random waveform. Thus, the

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* The Rayleigh-Jeans law is applicable when the parameter λT (centimeters-degrees Kelvin) exceeds about 100. For smaller values, the Planck radiation law must be used. According to Forsythe[4], there have been no direct tests of the Planck law in the range $10 < \lambda T < 100$ cm-deg. With a cooled termination attached to the input of the radiometer, the microwave radiometer would be an excellent instrument to use in making this last confirmation.

TABLE I

Coefficients of the Polynomial $x = \sum_{i=0}^{5} a_i y^i$ to Approximate the Function $\left\{ \begin{array}{l} x = y^n, \ 0 < y \\ \ \ = 0, \ \ 0 > y \end{array} \right\}$ in the Interval $-1 < y < 1$

N	a_0	a_1	a_2	a_3	a_4	a_5	N
1.0	0.05859	0.50000	0.82031	0.00000	-0.41016	0.00000	1.0
1.2	0.03666	0.37466	0.76979	0.24977	-0.32991	-0.13015	1.2
1.4	0.02164	0.27822	0.70690	0.41733	-0.24470	-0.20403	1.4
1.6	0.01140	0.20385	0.63849	0.52426	-0.15962	-0.23746	1.6
1.8	0.00451	0.14659	0.56870	0.58635	-0.07755	-0.24187	1.8
2.0	0.00000	0.10254	0.50000	0.61523	0.00000	-0.22559	2.0
2.2	-0.00282	0.06884	0.43382	0.61959	0.07230	-0.19473	2.2
2.4	-0.00442	0.04328	0.37096	0.60591	0.13911	-0.15381	2.4
2.6	-0.00514	0.02413	0.31181	0.57910	0.20045	-0.10617	2.6
2.8	-0.00524	0.01005	0.25653	0.54285	0.25653	-0.05429	2.8
3.0	-0.00488	0.00000	0.20508	0.50000	0.30762	0.00000	3.0

range of validity of the representation need not be great before the energy in the waveform that falls outside the interval becomes negligible.

With properly chosen coefficients, a number of possible representations for a non-linear element can be used. The simplest representation, a polynomial,

$$x(t) = \sum_{i=0}^{I} a_i \, y^i(t) \tag{2}$$

was used. It was found that with a fifth order polynomial, the curve of (2) agreed quite well with the curve of equation (1) over the interval of $-1 < y(t) < 1$. For $y(t) = 0$, the curve of (2) was positive and about 6 percent of the full scale value, thereby approximating very nicely the behavior of a diode without externally applied voltages. In the range of $0.2 < |y(t)| < 0.8$, the departure of (2) from (1) amounted to about 1 percent. By the least squares method, the coefficients of (2) have been evaluated using equation (1) as a base for values of N between 1.0 and 3.0. They are tabulated in Table I.

Neglecting the irrelevant gain factor of the amplifier, the input to the diode is

$$y(t) = s(t)\left(\tfrac{1}{2} + \tfrac{1}{2}\cos \omega_q t\right) + n(t) - B \tag{3}$$

where the multiplier of s(t) is due to modulation and the diode bias voltage is B. The spectral distribution of y(t) is defined by the amplifier that precedes the diode. With this general formulation of the problem, it was believed that, if a solution could be obtained, it would also be possible to state the optimum diode characteristic. Dicke had suggested that linear detection would provide improved sensitivity relative to square law detection although he did not imply that linear detection was necessarily optimum. (The author was not aware until recently that Selove[7] had corrected Dicke's work. A discussion of these errors would be repetitious and is omitted from this paper.)

The complete details of the treatment are tedious and are not included here since they are available elsewhere.[8] Instead, an outline of the method is given.

The output of the detector, x(t), is applied to the band-pass filter. The action of the filter on the signal is readily computed if the power spectrum of x(t) is used. To obtain the power spectrum, the autocorrelation function is calculated and the power spectrum obtained from the autocorrelation function by a Fourier transformation. The autocorrelation function of x(t) is

$$R_x(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)\, x(t+\tau)\, dt \tag{4}$$

and the power spectrum

$$W_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_x(\tau) e^{-i\omega\tau}\, d\tau \tag{5}$$

A direct substitution of (3) into (2) produces x(t) with the resulting expression containing 56 terms. The direct substitution of x(t) into equation (4) would produce about 3100 terms before simplication. To avoid such nonsense, it is possible to identify the terms of the integrand which will have zero means in the limit. Forms of the integrand which have zero means are:

$$A\left\{\begin{matrix}\sin m\omega t\\ \cos m\omega t\end{matrix}\right\} \quad \text{and}$$

$$s^a(t)\ s^b(t+\tau)\ n^c(t)\ n^d(t+\tau)\ A\left\{\begin{matrix}\sin m\omega t\\ \cos m\omega t\end{matrix}\right\}$$

where A is a constant, m is a positive integer, and a, b, c, and d are non-negative integers where either a + b, c + d, or both are odd. With this preliminary simplification, the autocorrelation function can be written in 140 terms involving higher-order compound correlation functions. The evaluation of these compound correlation functions is not difficult since s(t) is independent of n(t). If a typical term of equation (5) is

$$R(\tau) = \overline{s^a(t)\ s^b(t+\tau)\ n^c(t)\ n^d(t+\tau)}$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_0^T s^a(t)\ s^b(t+\tau)\,n^c(t)\,n^d(t+\tau)dt$$

and is denoted for simplicity by $\overline{a\ b\ c\ d}$, the associative and commutation rules, $\overline{a\ b\ c\ d} = \overline{a\ b}\ \ \overline{cd} = \overline{b\ a}\ \ \overline{d\ c}$, can easily be demonstrated. Since the effects of modulating s(t) have been explicitly accounted for in the simplification of equation (4), the higher-order, compound correlation functions can be computed from the autocorrelation functions of s(t) and n(t) whose spectral distributions are now defined by the pass band of the amplifier.

The analysis proceeds smoothly by transforming to the power spectrum domain when a filtering operation is involved, and back to the autocorrelation domain when the signal is applied to a non-linear device. As has been illustrated, the non-linear device produces no difficulty when a continuous representation is available. A simple treatment of the multiplier can be made. It is assumed that the autocorrelation of the input to the multiplier has been calculated. In accordance with the nomenclature of Figure 1, this autocorrelation function is designated $R_v(\tau)$. The problem is to find the output autocorrelation function $R_w(\tau)$ when

$$w(t) = v(t)\ \cos\ \omega_q\ t. \tag{6}$$

According to equation (4)

$$R_w(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_0^T w(t)w(t+\tau)dt \tag{7}$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_0^T v(t)\cos\omega_q t\, v(t+\tau)\cos\omega_q(t+\tau)dt$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_0^T v(t)v(t+\tau)\frac{\cos\omega_q\tau}{2}\ dt$$

$$+\lim_{T\to\infty}\frac{1}{T}\int_0^T v(t)v(t+\tau)\frac{\cos\omega_q(2t+\tau)}{2}\ dt$$

Since the second integral on the right vanishes in the limit,

$$R_w(\tau) = R_v(\tau)\ \frac{\cos\ \omega_q\tau}{2} \tag{8}$$

The completion of the analysis is now routine and for small signal input, the final power output is found to be

$$P_o = \frac{A^2}{32}\sigma_s{}^4 + A^2\frac{\omega_\gamma}{\omega_a}\sigma_n{}^4 \tag{9}$$

where

$$A = a_2 - 3a_3\ B + 6a_4B^2 - 10a_5B^3 \tag{10}$$

Defining the sensitivity of the instrument by a unity signal-to-noise ratio and recognizing the first term of (9) as signal, the second as noise,

$$\sigma_m{}^2 = \sigma_n{}^2\ \sqrt{\frac{32\ \omega_\gamma}{\omega_a}} \tag{11}$$

where $\sigma_m{}^2$ is the minimum detectable signal power. The cancellation of coefficient A is of major importance. Since A is determined by the diode characteristic, but does not appear in the sensitivity relation, the radiometer's sensitivity is independent of the detector characteristic. While it is not evident from this abbreviated treatment, this same result would be obtained if I in equation (2) were arbitrarily large and fitted any desired characteristic with precision over an arbitrarily large interval. In other words, the sensitivity expression is not based upon an approximation. The signal and noise powers are also given by

$$\sigma_m{}^2 = \frac{k\Delta T\omega_a}{2\pi} \tag{12}$$

and

$$\sigma_n{}^2 = \frac{FkT_o\,\omega_a}{2\pi} \tag{13}$$

where ΔT is the temperature difference of the source relative to the temperature of the resistive element of the modulator, F is noise figure of the radiometer measured with minimum modulator attentuation, T_0 is the reference temperature for the measurement of F (usually 300°K) and k is Stephan-Boltzmann's constant. The relation (9) may be expressed in terms of the minimum detectable temperature difference,

$$\Delta T = F\,T_0 \sqrt{\frac{32 \ \omega_\gamma}{\omega_a}} \tag{14}$$

The theoretical results were experimentally tested and verified within reasonable limits. The conclusion that sensitivity is independent of detector characteristic was checked with a number of diodes and with varying bias. No significant change was measured although the nature of the measurement precludes a precise verification. The absolute measured sensitivity was about one-half as good as that computed by equation (14). In view of the uncertainty of this type of measurement as well as the almost inevitable large experimental errors, this agreement was considered quite satisfactory.

In conclusion, the author would like to combine an acknowledgement of his debt to prior investigators, along with a comment or two to place this work in perspective relative to their work. One cannot help but be pleased by the clever circuit that Dicke devised to obtain measurements 30 db and more below the noise level. On the other hand, Dicke's analysis is difficult to follow and is dangerous; the latter point is emphasized by the fact that his analysis contains two errors. The author is indebted to Goldstein for the method of analysis that has allowed the generalizations described here. In addition to the justification for using a simple detector characteristic, the present treatment consistently follows the autocorrelation-power spectrum method. When Goldstein departed from the method he fell into error. Goldstein's original work[2] has a correct statement of the instrument's sensitivity, but is based on an incorrect autocorrelation function (the error cancelled out when the signal-to-noise ratio was computed). Recently, Goldstein corrected a portion of his work (with the result that the common error no longer cancelled out) and thereby obtained an answer in error by a factor of $\sqrt{2}$.

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List of References

1. R. H. Dicke, "The measurement of thermal radiation at microwave frequencies," Rev. Sci. Instr., vol. 17, pp 268-275; July, 1946

2. S. J. Goldstein, Jr., "A comparison of two radiometer circuits," Proc. IRE, vol. 43, pp 1663-1666; November, 1955

3. R. H. Dicke, R. Beringer, R. L. Kyhl, and A. B. Vane, "Atmospheric absorption measurements," Phys. Rev., vol. 70, pp 340-348; September, 1946

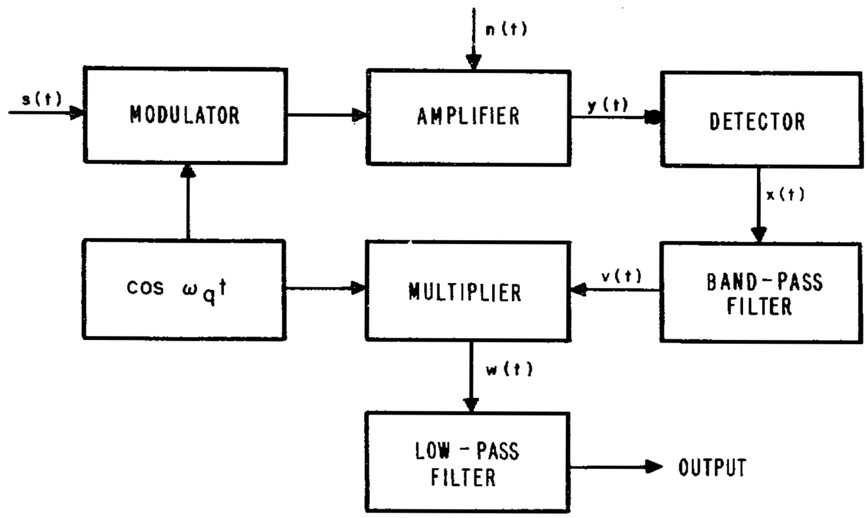
4. W. E. Forsythe, Measurement of Radiant Energy, New York, McGraw-Hill, 1937, p 8.

5. S. O. Rice, "Mathematical analysis of random noise," B.S.T.J., vol. 24, Part IV; January, 1945

6. Ibid, p 92

7. Walter Selove, "A dc comparison radiometer," Rev. Sci. Instr., vol. 25, pp 120-124; February, 1954

8. L. D. Strom, "The theoretical sensitivity of the microwave radiometer," a dissertation, the University of Texas; February, 1957, (available from University Microfilms, Ann Arbor, Michigan)



NOTES:

1. s(t) and n(t) are thermal noise voltages with average powers of $\sigma_s{}^2/2\omega_\alpha$ and $\sigma_n{}^2/2\omega_\alpha$ watts/radian.

2. The amplifier pass band is rectangular and of width ω_α centered at $\pm\ \omega_1$.

3. The band-pass filter is rectangular and of width ω_β centered at $\pm\ \omega_q$.

4. The low-pass filter has a rectangular pass band extending from $-\omega_\gamma$ to $+\omega_\gamma$.

Fig. 1