Algebra Tutorial-III

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1. Let H be a normal subgroup of a group G and K be any subgroup of G. Then prove that
$$HK = \{hk \mid h \in H,\, k \in K\}$$
is a subgroup of G.

2. Let G be a group and let $Z(G)$ denote the center of G. If $G/Z(G)$ is cyclic, then show that G is Abelian.

3. Let $f : G \to G'$ be a group homomorphism. Then prove the following:
   (a) $f(H) = \{f(h) \mid h \in H\}$ is a subgroup of G'. If $K \leq G'$, then $f^{-1}(K) = \{g \in G \mid f(g) \in K\}$ is a subgroup of G.
   (b) If H is cyclic, then $f(H)$ is cyclic, and if H is Abelian, then $f(H)$ is Abelian.
   (c) If $H \trianglelefteq G$, then $f(H) \trianglelefteq f(G)$. Also, if $K \trianglelefteq G'$, then $f^{-1}(K) \trianglelefteq G$.
   (d) If $|H| = n$, then $|f(H)| \mid n$.

4. Let $G = (\mathbb{R}, +)$ and $G' = (\mathbb{R}^+, \cdot)$. Prove that $G \cong G'$ under the mapping $f(x) = 2^x$.

5. Show that any infinite cyclic group is isomorphic to $\mathbb{Z}$.

6. Show that every group is isomorphic to a group of permutations.

7. Compute $\mathrm{Aut}(\mathbb{Z}/10\mathbb{Z})$. Show that it is isomorphic to $(\mathbb{Z}/10\mathbb{Z})^*$.

8. Let f be an automorphism of $\mathbb{C}^*$, the group of nonzero complex numbers under multiplication. Determine $f(-1)$ and $f(i)$.

9. Let H be a subgroup of a group G. Show that $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$.

10. Prove that every group of order 35 is cyclic.

11. Suppose that $f : \mathbb{Z}_{50} \to \mathbb{Z}_{15}$ is a group homomorphism with $f(7) = 6$. Determine the map, its kernel and image, i.e., determine $f(x)$, $\mathrm{Im}(f)$ and $\mathrm{Ker}(f)$.