Algebra Tutorial-II

Dr. Abhash Kumar Jha / MA-111

Groups

1. Determine which of the following are groups under the corresponding binary operations:

   (a) $\mathbb{Z}$ with the operation $a \cdot b = a - b$.

   (b) $\mathbb{Z} \times \mathbb{Z}$ with the operation $(a, b) \cdot (x, y) = (ad + bc, bd)$.

   (c) The set of complex numbers with absolute value 1 under multiplication.

   (d) $\{a + b\sqrt{2} \in \mathbb{R} \ : \ a, b \in \mathbb{Q}\}$ with respect to addition.

   (e) $G = \{a \in \mathbb{R} \ : \ 0 \leq a < 1\}$ with the operation defined as $a \cdot b = a + b - [a + b]$, where $[a]$ is the greatest integer less or equal to a.

   (f) Upper triangular, unipotent, orthogonal matrices under multiplication.

2. Define
$$\mathbb{H} := \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \mid z, w \in \mathbb{C} \right\}.$$
Show that $\mathbb{H}$ is a group under multiplication. It is called the 'skew-field' of quaternions. Later we will show that $\mathbb{H}$ is *identical* to the set
$$G = \{a + ib + jc + kd \ : \ a, b, c, d \in \mathbb{R}\} \text{ (with some group operation)}$$
as groups.

3. Let Q be an $n \times n$ real symmetric matrix. Then there is an analogue of the orthogonal group, defined as
$$O_Q = \left\{ A \in \mathrm{GL}_n(\mathbb{R}) : A^T Q A = Q \right\}.$$
Show that this is a subgroup of $\mathrm{GL}_n(\mathbb{R})$.

4. Is
$$G = \left\{ X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ \mid \ X^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$
a group with respect to multiplication? Is it a familiar group? What about
$$H = \left\{ X = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \ \mid \ a \in \mathbb{R} \setminus \{0\} \right\}?$$

Subgroups

1. Determine whether the following are subgroups of the given group $(\mathbb{C}, +)$ (Complex numbers under addition)

   (a) $\mathbb{R}$ (real numbers)

   (b) $\mathbb{Q}^+$ (positive rationals)

   (c) $7\mathbb{Z}$ (integer multiples of 7)

   (d) $i\mathbb{R}$ (purely imaginary numbers)

   (e) $\pi\mathbb{Q}$ (rational multiples of π)

   (f) $\{\pi^n \mid n \in \mathbb{Z}\}$ (powers of π)

2. Determine whether the following are subgroups of $(\mathbb{C}^*, \cdot)$ (Nonzero complex numbers under multiplication)

   (a) $\mathbb{R}$

   (b) $\mathbb{Q}^+$

   (c) $7\mathbb{Z}$

   (d) $i\mathbb{R}$

   (e) $\pi\mathbb{Q}$

   (f) $\{\pi^n \mid n \in \mathbb{Z}\}$

3. Let G be an abelian group under multiplication. Then $H = \{x^2 \mid x \in G\}$ is a subgroup of G. What about $H' = \{x \in G \mid x^2 = e\}$, where e is the identity of G?

4. Show that $H = \{x \in G \mid x^n = e \text{ for some integer } n\}$ is a subgroup of G.

5. The center of a group G is the subset $Z(G)$ of elements in G that commute with every element of G, i.e., $Z(G) = \{g \in G \mid ga = ag \text{ for all } a \in G\}$. Show that $Z(G)$ is a subgroup of G.

6. Let a be a fixed element of a group G. The centralizer $C(a)$ of a in G is the set of all elements in G that commute with a. Write the set notation for $C(a)$ and show that it is a subgroup of G.

7. Show that $Z(G)$ is a subgroup of $C(a)$ for any $a \in G$. What happens when G is abelian?

8. Let G be an abelian group and H and K be subgroups of G. Then show that $HK = \{hk \mid h \in H, k \in K\}$ is a subgroup of G.

9. Consider the group $G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, a, b, c, d \in \mathbb{Z} \right\}$ under addition and

$$H = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G \,\middle|\, a + b + c + d = 0 \right\}.$$

   Prove that H is a subgroup of G.