Algebra Tutorial-III

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1 Symmetric groups

1. Write the cycle decompositions and orders of the following permutations.

$1 \mapsto 13$	$2 \mapsto 2$	$3 \mapsto 15$	$4 \mapsto 14$	$5 \mapsto 10$
$6 \mapsto 6$	$7 \mapsto 12$	$8 \mapsto 3$	$9 \mapsto 4$	$10 \mapsto 1$
$11 \mapsto 7$	$12 \mapsto 9$	$13 \mapsto 5$	$14 \mapsto 11$	$15 \mapsto 8$

Figure 1: σ

$1 \mapsto 14$	$2 \mapsto 9$	$3 \mapsto 10$	$4 \mapsto 2$	$5 \mapsto 12$
$6 \mapsto 6$	$7 \mapsto 5$	$8 \mapsto 11$	$9 \mapsto 15$	$10 \mapsto 3$
$11 \mapsto 8$	$12 \mapsto 7$	$13 \mapsto 4$	$14 \mapsto 1$	$15 \mapsto 13$

Figure 2: τ

   Also, find the cycle decompositions and orders of $\sigma^2, \sigma\tau, \tau\sigma, \tau^2\sigma$.

2. Prove that an n-cycle has order n. Using this prove that the order of a permutation consisting of disjoint cycles is the *least common multiple* of the lengths of the cycles.

3. Determine the number of elements in S_7 of order 12. What about the number of elements of order 3.

4. Show that every permutation in S_n is a product of transpositions. Also, show that if a permutation σ can be expressed as a product of an even (odd) number of transpositions, then any other decomposition of σ is so.

5. Show that there are $\frac{1}{2}(n!)$ even permutations in S_n. Do they form a group?

2 Dihedral groups

For $n \geq 3$, the dihedral group D_n is defined as the symmetries of a regular n-gon, with the operation being composition.

1. Show that the number of elements in D_n is $2n$.
2. Show that $D_n = \{1, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}\}$ for some element r and s in D_n.
3. With the above notation show that $rs = sr^{-1}$.
4. Show that D_n can be considered as a subgroup of a S_n.
5. Find the centre of D_4. Also, find the centralizer of $A = \{1, r, r^2, r^3\}$ in D_4.

3 Cyclic groups

1. For $n = 6, 8$ and 20, find all generators of $\mathbb{Z}/n\mathbb{Z}$.
2. Prove that $\mathbb{Q} \times \mathbb{Q}$ is not cyclic.
3. Let G be a group and $a, b \in G$ with $|a| = 1000$, $|b| = 30$. Find the order of
$$a^{62}, a^{185}, a^{400}, b^{17}, b^{18}, \text{ and } b^{26}.$$
4. Prove that a group of order 3 must be cyclic.
5. Let G be a finite group. Show that there exists a fixed positive integer n such that $a^n = e$ for all a in G.
6. Prove that a finite group is the union of proper subgroups if and only if the group is not cyclic.
7. Prove that no group can have exactly two elements of order 2.
8. Let $\phi(n)$ denote the number of positive integers less than n and relatively prime to n. Show that for relatively prime m and n,
$$\phi(mn) = \phi(m)\phi(n).$$
Also, show that for a prime p,
$$\phi(p^n) = p^n - p^{n-1}.$$
9. If d is a positive divisor of n, show that the number of elements of order d in a cyclic group of order n is $\phi(d)$.