Equivalence relation

An *equivalence relation* on a set S is a subset $R \subseteq S \times S$ satisfying the *reflexivity*, *symmetry* and *transitivity* properties. We write $a \sim b$ to indicate that $(a, b) \in R$, and say that $\sim$ is the equivalence relation.

Problems

1. Let $A = \{0, 1, 2, 3\}$ and define a relation R on A by

$$R = \{(0,0),\ (0,1),\ (0,3),\ (1,0),\ (1,1),\ (2,2),\ (3,0),\ (3,3)\}.$$

   Is it an equivalence relation? If so, what are the equivalence classes?

2. For $a, b \in \mathbb{Z}$, let $a \sim b$ iff $a - b$ is even. Show that $\sim$ is an equivalence relation. What are the equivalence classes?

3. Let $\mathbb{N}$ be the set of natural numbers, and define the relation $R \subseteq \mathbb{N} \times \mathbb{N}$ by $R = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid 2x + y = 10\}$. Show that the relation is neither reflexive, nor symmetric, nor transitive.

4. Define a partition of a set.

   Show that every equivalence relation on a set S yields a partition of the set, and vice versa.

5. Let n be a fixed positive integer. Define a relation on $\mathbb{Z}$ by

$$a \sim b \quad \text{iff} \quad n|(b - a).$$

   Show that the relation is an equivalent one and determine the equivalence classes.

   The set of equivalence classes of this relation is denoted by $\mathbb{Z}/n\mathbb{Z}$.

6. Let us define a relation on $\mathbb{R}^n \setminus \{(0, 0, \cdots, 0)\}$ by

$$(a_1, a_2, \cdots, a_n) \sim (b_1, b_2, \cdots, b_n) \quad \text{iff} \quad a_i = \lambda b_i, \quad (i = 1, 2, \cdots, n)$$

   for some nonzero $\lambda \in \mathbb{R}$. The equivalence classes are the lines through the origin and the set of all equivalence classes is called the real projective space $\mathbb{RP}^n$.

7. The 'similarity' relation on $M_n(\mathbb{R})$ is defined by

$$A \sim B \quad \text{iff} \quad B = CAC^{-1}$$

   for some invertible $C \in M_n(\mathbb{R})$. Show that the relation is an equivalence relation.

   What about the 'congruence' relation defined by

$$A \sim B \quad \text{iff} \quad B = CAC^{\mathrm{T}}?$$