Solution 1

Let $a = 1$, then if $S = \{1, b, c, d\}$, $d = 6$. It then follows that $b = 2$ and $c = 4$, resulting in $a + b + c + d = \boxed{13}$.

Solution 2

$$k = \frac{5 + 9 + ... + 4n + 1}{n} = \frac{n + 4(\frac{n(n+1)}{2})}{n} = 2n + 3$$

It is given that $k = n^2 - 41n + 225$ as well, setting these equal to each other, we get:

$$\begin{aligned} 2n + 3 &= n^2 - 41n + 225 \\ 0 &= (n - 37)(n - 6) \\ n &= \boxed{37}. \end{aligned}$$

Solution 3

$\binom{30}{2} = 15 \cdot 29$, the amount of ways to pick two shapes, not counting permutations.

$$\begin{gathered} 3 + 30 \\ 4 + 29 \\ \vdots \\ 16 + 17 \end{gathered}$$

$= 14$ ways to have friends, not counting permutations. Since $\frac{14}{15 \cdot 29}$ cannot be simplified, $mn = 14 \cdot 15 \cdot 29 = \boxed{6090}$.

Solution 4

Using Vieta's:

$$\begin{aligned} abc &= -\frac{c}{a} \\ a + b + c &= 0 \end{aligned}$$

This gives $a = \pm 1, b = -1, c = 0$, so $\underline{a^2}\ \underline{b^2}\ \underline{c^2} = \boxed{110}$.

Solution 5 (Brute Force)

There exists $14 \cdot 13 = 182$ ways for C and D to be chosen. If $\overline{AC}$ is vertical or horizontal, there are 4 ways for it to intersect $\overline{BD}$. If C is directly above D, there are $11 + 9 + 8 = 28$ ways for the lines to intersect. Checking the rest of the points gives $5 + 7 + 6 + 1 + 3 + 3 = 25$. Finally $25 + 28 + 4 = 57$ giving us $\frac{57}{182}$. Since the fraction cannot simplify, we get that $m + n = \boxed{239}$.

Solution 6

$y = \binom{x-1}{x-3} \cdot \binom{x-2}{x-4} \cdot \binom{x-3}{x-5} \cdot \ldots \cdot \binom{x-2023}{x-2025} = \frac{1}{2^{2023}}(x-1)(x-2)^2 \ldots (x-2023)^2(x-2024)$. Since y is only positive for $x < 1$ and $x > 2024$ it will hit $-tanh(x)$ $2 + 2022 \cdot 2 = \boxed{4046}$ times.

Solution 7

Triangle area $A = rs$ where r is the in-circle radius and s is the semi perimeter. Since we have equilateral triangles, $A = 3 \cdot \frac{1}{2}R^2 \cdot \frac{\sqrt{3}}{2}$ where R is the outer radius and $s = \frac{3}{2}side = \frac{3R\sqrt{3}}{2}$. So we have

$$r\frac{3R\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}R^2$$
$$r = \frac{1}{2}R$$

which makes it clear that $f(n) = 2^{n-1}$. So $n = 10$ and $\frac{1}{f(n)} = \boxed{512}$.

Solution 8

Let $A(n)$ denote the area and $P(n)$ denote the perimeter of a *n-pinwheel.* $A(4) = A(3) \implies 85A_4 = 21A_3$. Let a and b be the shortest length of the triangle with area A_4 and A_3, respectively. This gives us $85 \cdot \frac{1}{2}a^2 = 21 \cdot \frac{1}{2}b^2 \cdot \frac{\sqrt{3}}{2} \implies a\sqrt{\frac{170}{21\sqrt{3}}} = b$ and from here we just get the perimeters:

$$P(4) = a + 2a + 4a + 7a + a\sqrt{2} + 2a\sqrt{2} + 4a\sqrt{2} + 8a\sqrt{2} = (14 + 15\sqrt{2})a$$
$$P(3) = b + 2b + 3b + b\sqrt{3} + 2b\sqrt{3} + 4b\sqrt{3} = (6 + 7\sqrt{3})b$$

Note: The sides still double in length between each triangle since a is proportional to b. Anyways, we get that

$$\begin{aligned}
\frac{P(3)}{P(4)} &= \frac{(6+7\sqrt{3})b}{(14+15\sqrt{2})a} \\
&= \frac{6+7\sqrt{3}}{14+15\sqrt{2}} \cdot \sqrt{\frac{170}{21\sqrt{3}}} \\
\frac{m}{n} &= \frac{6+7\sqrt{3}}{14+15\sqrt{2}} \\
m+n &= 20 + 15\sqrt{2} + 7\sqrt{3} \\
\lceil m+n \rceil &= \boxed{54}.
\end{aligned}$$