# RESOLUÇÃO · LISTA · REGRA DA CADEIA

01 Em cada caso, verifique se a função real $u$, satisfaz à equação à derivadas parciais indicada:

a) $u = f(x,y)$; $x(r,s) = r^3 - s^3$; $y(r,s) = s^3 - r^3$; $s^2 \dfrac{\partial u}{\partial r} + r^2 \dfrac{\partial u}{\partial s} = 0$;

Solução: usemos o quadro 01:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \therefore \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}(3r^2) + \frac{\partial u}{\partial y}(-3r^2) \therefore s^2\frac{\partial u}{\partial r} = 3s^2r^2\frac{\partial u}{\partial x} - 3s^2r^2\frac{\partial u}{\partial y};$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \therefore \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}(-3s^2) + \frac{\partial u}{\partial y}(3s^2) \therefore r^2\frac{\partial u}{\partial r} = -3r^2s^2\frac{\partial u}{\partial x} + 3r^2s^2\frac{\partial u}{\partial y};$$

Logo, $s^2\dfrac{\partial u}{\partial r} + r^2\dfrac{\partial u}{\partial r} = 3s^2r^2\dfrac{\partial u}{\partial x} - 3s^2r^2\dfrac{\partial u}{\partial y} - 3r^2s^2\dfrac{\partial u}{\partial x} + 3r^2s^2\dfrac{\partial u}{\partial y} = 0$;

Sim, satisfaz;

b) $u = f(x,y)$; $x(r,s) = r+s$; $y(r,s) = r-s$; $\left(\dfrac{\partial u}{\partial x}\right)^2 - \left(\dfrac{\partial u}{\partial y}\right)^2 = \dfrac{\partial u}{\partial r}\dfrac{\partial u}{\partial s}$;

Solução: novamente usaremos o quadro 01:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \therefore \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}.(1) + \frac{\partial u}{\partial y}.(1) = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y};$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \therefore \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}.(1) + \frac{\partial u}{\partial y}.(-1) = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y};$$

Logo, $\dfrac{\partial u}{\partial r}\dfrac{\partial u}{\partial s} = \left(\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}\right)\left(\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y}\right) = \left(\dfrac{\partial u}{\partial x}\right)^2 - \left(\dfrac{\partial u}{\partial y}\right)^2$; Sim, satisfaz;

c) $u = f(x,y,z)$; $x(r,s,t) = r-s$; $y(r,s,t) = s-t$; $z(r,s,t) = t-r$; $\dfrac{\partial u}{\partial r} + \dfrac{\partial u}{\partial s} + \dfrac{\partial u}{\partial t} = 0$;

Solução: usaremos o quadro 05:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r} = \frac{\partial u}{\partial x}.(1) + \frac{\partial u}{\partial y}.(0) + \frac{\partial u}{\partial z}.(-1) = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial z};$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s} = \frac{\partial u}{\partial x}.(-1) + \frac{\partial u}{\partial y}.(1) + \frac{\partial u}{\partial z}.(0) = -\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y};$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} = \frac{\partial u}{\partial x}.(0) + \frac{\partial u}{\partial y}.(-1) + \frac{\partial u}{\partial z}.(1) = -\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z};$$

Logo, $\dfrac{\partial u}{\partial r} + \dfrac{\partial u}{\partial s} + \dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial z} - \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} - \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$;

Sim, satisfaz;

(d) $u = f(x,y)$; $x(r,s) = \dfrac{s-r}{rs}$; $y(s,t) = \dfrac{t-s}{st}$; $r,s,t \neq 0$; $r^2\dfrac{\partial u}{\partial r} + s^2\dfrac{\partial u}{\partial s} + t^2\dfrac{\partial u}{\partial t} = 0$;

Solução: em primeiro lugar, como $r, s$ e $t$ são não-nulos, escrevamos:

$$x(r,s) = \frac{s}{rs} - \frac{r}{rs} = \frac{1}{r} - \frac{1}{s}; \quad y(s,t) = \frac{t-s}{st} = \frac{t}{st} - \frac{s}{st} = \frac{1}{s} - \frac{1}{t};$$

Em segundo lugar, observemos que nenhum dos quadros da teoria se encaixa, exatamente, em nossa situação.

Vamos então criar nosso quadro:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} = \frac{\partial u}{\partial x}\left(\frac{-1}{r^2}\right);$$
$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial u}{\partial x}\left(\frac{1}{s^2}\right) + \frac{\partial u}{\partial y}\left(\frac{-1}{s^2}\right);$$
$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial u}{\partial y}\cdot\left(\frac{1}{t^2}\right);$$

Logo, $r^2\dfrac{\partial u}{\partial r} + s^2\dfrac{\partial u}{\partial s} + t^2\dfrac{\partial u}{\partial t} = r^2\left(\dfrac{\partial u}{\partial x}\cdot\left(\dfrac{-1}{r^2}\right)\right) + s^2\left(\dfrac{\partial u}{\partial x}\cdot\left(\dfrac{1}{s^2}\right) - \dfrac{\partial u}{\partial y}\left(\dfrac{1}{s^2}\right)\right) + t^2\left(\dfrac{\partial u}{\partial y}\cdot\left(\dfrac{1}{t^2}\right)\right)$;

$$= -\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} + \frac{\partial u}{\partial y} = 0; \text{ Sim, satisfaz;}$$

(e) $u = f(ar+bs)$; $a$ e $b \in \mathbb{R}$; $b\dfrac{\partial u}{\partial r} - a\dfrac{\partial u}{\partial s} = 0$;

Solução: começamos introduzindo a variável $x(r,s) = ar+bs$;

Assim, ficamos com $u = f(x)$; $x(r,s) = ar+bs$; Usaremos o quadro 03;

$$\frac{\partial u}{\partial r} = \frac{du}{dx}\frac{\partial x}{\partial r} = \frac{du}{dx}\cdot a; \qquad \text{Logo, } b\frac{\partial u}{\partial r} - a\frac{\partial u}{\partial s} = ba\frac{du}{dx} - ab\frac{du}{dx} = 0;$$
$$\frac{\partial u}{\partial s} = \frac{du}{dx}\frac{\partial x}{\partial s} = \frac{du}{dx}\cdot b;$$

Sim, satisfaz;

(f) $u(r,s) = s + f(r^2-s^2)$; $s\dfrac{\partial u}{\partial r} + r\dfrac{\partial u}{\partial s} = r$;

Solução: vamos introduzir, $v = f(x)$ com $x(r,s) = r^2 - s^2$;

Então, $\dfrac{\partial u}{\partial r} = \dfrac{\partial(s)}{\partial r} + \dfrac{\partial v}{\partial r} = \dfrac{\partial v}{\partial r}$; e $\dfrac{\partial u}{\partial s} = \dfrac{\partial(s)}{\partial s} + \dfrac{\partial v}{\partial s} = 1 + \dfrac{\partial v}{\partial s}$;

Falta descobrirmos $\dfrac{\partial v}{\partial r}$ e $\dfrac{\partial v}{\partial s}$; Usaremos, novamente, o quadro 03 para $v = f(x)$;

$$\frac{\partial v}{\partial r} = \frac{dv}{dx}\cdot\frac{\partial x}{\partial r} = 2r\frac{dv}{dx};$$
$$\frac{\partial v}{\partial s} = \frac{dv}{dx}\cdot\frac{\partial x}{\partial s} = -2s\frac{dv}{dx};$$

$$s\frac{\partial u}{\partial r} + r\frac{\partial u}{\partial s} = s\frac{\partial v}{\partial r} + r\left(1 + \frac{\partial v}{\partial s}\right)$$
$$= s\left(2r\frac{dv}{dx}\right) + r + r\left(-2s\frac{dv}{dx}\right)$$
$$= 2rs\frac{dv}{dx} + r - 2rs\frac{dv}{dx} = r;$$

Sim, satisfaz;

(8) $u(r,s) = rs + f(r^2+s^2)$; $s\dfrac{\partial u}{\partial r} - r\dfrac{\partial u}{\partial s} = s^2 - r^2$;

<u>Solução:</u> vamos introduzir, $v = f(x)$ com $x(r,s) = r^2+s^2$;

Então, $\dfrac{\partial u}{\partial r} = \dfrac{\partial (rs)}{\partial r} + \dfrac{\partial v}{\partial r} = s + \dfrac{\partial v}{\partial r}$; e $\dfrac{\partial u}{\partial s} = \dfrac{\partial (rs)}{\partial s} + \dfrac{\partial v}{\partial s} = r + \dfrac{\partial v}{\partial s}$;

E, para encontrarmos $\dfrac{\partial v}{\partial r}$ e $\dfrac{\partial v}{\partial s}$ vamos usar o quadro (03):

$$\left.\begin{array}{l} \dfrac{\partial v}{\partial r} = \dfrac{dv}{dx}\dfrac{\partial x}{\partial r} = 2r\dfrac{dv}{dx}; \\ \dfrac{\partial v}{\partial s} = \dfrac{dv}{dx}\dfrac{\partial x}{\partial s} = 2s\dfrac{dv}{dx}; \end{array}\right\} s\dfrac{\partial u}{\partial r} - r\dfrac{\partial u}{\partial s} = s\left(s + \dfrac{\partial v}{\partial r}\right) - r\left(r + \dfrac{\partial v}{\partial s}\right)$$

$$= s^2 + 2rs\dfrac{dv}{dx} - r^2 - 2rs\dfrac{dv}{dx} = s^2 - r^2;$$

Sim, satisfaz;

(02) Sejam $u = f(x,y)$; $x(r,\theta) = r\cos\theta$; $y(r,\theta) = r\,\text{sen}\,\theta$; Mostre que:

$$\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2 = \left(\dfrac{\partial u}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial u}{\partial \theta}\right)^2;$$

<u>Solução:</u> usaremos o quadro (01):

$$\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r} = \dfrac{\partial u}{\partial x}\cos\theta + \dfrac{\partial u}{\partial y}\text{sen}\,\theta;$$

$$\dfrac{\partial u}{\partial \theta} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial \theta} = \dfrac{\partial u}{\partial x}(-r\,\text{sen}\,\theta) + \dfrac{\partial u}{\partial y}(r\cos\theta) = r\left[\dfrac{\partial u}{\partial y}\cos\theta - \dfrac{\partial u}{\partial x}\text{sen}\,\theta\right];$$

Então:

$$\left(\dfrac{\partial u}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial u}{\partial \theta}\right)^2 = \left(\dfrac{\partial u}{\partial x}\cos\theta + \dfrac{\partial u}{\partial y}\text{sen}\,\theta\right)^2 + \dfrac{1}{r^2}\left[r^2\left(\dfrac{\partial u}{\partial y}\cos\theta - \dfrac{\partial u}{\partial x}\text{sen}\,\theta\right)^2\right]$$

$$= \left(\dfrac{\partial u}{\partial x}\right)^2\cos^2\theta + \underbrace{2\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y}\cos\theta\,\text{sen}\,\theta} + \left(\dfrac{\partial u}{\partial y}\right)^2\text{sen}^2\theta + \left(\dfrac{\partial u}{\partial y}\right)\cos^2\theta \underbrace{- 2\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y}\cos\theta\,\text{sen}\,\theta} + \left(\dfrac{\partial u}{\partial x}\right)^2\text{sen}^2\theta$$

$$= \left(\dfrac{\partial u}{\partial x}\right)^2(\cos^2\theta + \text{sen}^2\theta) + \left(\dfrac{\partial u}{\partial y}\right)^2(\text{sen}^2\theta + \cos^2\theta)$$

$$= \left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2;$$

(03) [Equações de Cauchy-Riemann em coordenadas polares;]

Sejam $u(x,y)$ e $v(x,y)$ funções reais satisfazendo às equações de Cauchy (1789-1857) - Riemann (1826-1866). Mostre que se $x(r,\theta) = r\cos\theta$ e $y(r,\theta) = r \operatorname{sen}\theta$, então: $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$ e $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$;

Solução:

Pelo que acabamos de ver na Questão (02) a Regra da Cadeia nos diz, que:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\operatorname{sen}\theta;$$

$$\frac{\partial v}{\partial \theta} = r\left(\frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\operatorname{sen}\theta\right);$$

E Cauchy-Riemann nos diz, que: $\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}; \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}; \end{cases}$

Logo, $\dfrac{1}{r}\dfrac{\partial v}{\partial \theta} = \dfrac{\partial v}{\partial y}\cos\theta - \dfrac{\partial v}{\partial x}\operatorname{sen}\theta = \dfrac{\partial u}{\partial x}\cos\theta + \dfrac{\partial u}{\partial y}\operatorname{sen}\theta = \dfrac{\partial u}{\partial r}$;

Bem como,

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\operatorname{sen}\theta;$$

$$\frac{\partial u}{\partial \theta} = r\left(\frac{\partial u}{\partial y}\cos\theta - \frac{\partial u}{\partial x}\operatorname{sen}\theta\right);$$

Logo, $-\dfrac{1}{r}\dfrac{\partial u}{\partial \theta} = -\dfrac{\partial u}{\partial y}\cos\theta + \dfrac{\partial u}{\partial x}\operatorname{sen}\theta = \dfrac{\partial v}{\partial x}\cos\theta + \dfrac{\partial v}{\partial y}\operatorname{sen}\theta = \dfrac{\partial v}{\partial r}$;

(04) [Fórmula de Leibniz:]

Sejam $w = F(u,v) = \displaystyle\int_v^u p(t)\,dt$; $u = g(x)$; $v = h(x)$; Mostre a seguinte Fórmula devida à Leibniz (1646-1716): $\dfrac{dw}{dx} = p(g(x))\,g'(x) - p(h(x))\,h'(x)$;

Solução: Usaremos o quadro (07) e o Teorema Fundamental do Cálculo:

$$\frac{dw}{dx} = \frac{\partial w}{\partial u}\frac{du}{dx} + \frac{\partial w}{\partial v}\frac{dv}{dx} = \frac{\partial\left(\int_v^u p(t)\,dt\right)}{\partial u}\cdot g'(x) + \frac{\partial\left(\int_v^u p(t)\,dt\right)}{\partial v}\cdot h'(x)$$

$$= p(u)\cdot g'(x) + \frac{\partial\left(-\int_u^v p(t)\,dt\right)}{\partial v}\cdot h'(x)$$

$$= p(g(x))\,g'(x) - p(v)\,h'(x)$$

$$= p(g(x))\,g'(x) - p(h(x))\,h'(x);$$

(05) [Teorema de Euler para Funções homogêneas;]

Diz-se que $F(x,y)$ é homogênea de grau $n$, $n \geq 0$ um inteiro, quando $F(tx,ty) = t^n F(x,y)$. Mostre a seguinte igualdade, devida à Euler (1707-1783);

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = nF(x,y);$$

Solução: Vamos denotar $w = F(tx,ty) = t^n F(x,y)$;

• Então, de $w = t^n F(x,y)$ vemos que $\frac{\partial w}{\partial t} = n t^{n-1} F(x,y)$; (I)

• Agora para encontrarmos $\frac{\partial w}{\partial t}$ na expressão $w = F(tx,ty)$, introduzamos as variáveis $u(t,x) = tx$ e $v(t,y) = ty$. Então:

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial t} = \frac{\partial w}{\partial u}x + \frac{\partial w}{\partial v}y; \text{ (II)}$$

Logo, concluímos de (I) e (II) que para todos $t$, $x$ e $y$ reais, temos:

$$x\frac{\partial w}{\partial u} + y\frac{\partial w}{\partial v} = n t^{n-1} F(x,y);$$

O que Euler, genialmente, percebeu é que, em particular, esta igualdade vale para $t=1$ e para todos $x$ e $y$ reais, e que nesse caso $u = x$ e $v = y$. Logo ela se torna: $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = nF(x,y)$;

(06) Em cada caso, mostre que se a equação:

(a) $F(x,y) = 0$, definir $y$ como uma função diferenciável de $x$ e $F_y \neq 0$, então $\frac{dy}{dx} = -\frac{F_x}{F_y}$;

Solução: Como $y$ é função derivável de $x$, o quadro (07) nos informa:

De $F(x,y)=0$, temos: $\frac{d\,F(x,y)}{dx} = 0 \therefore \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \therefore F_x + F_y\frac{dy}{dx} = 0 \therefore$

E como $F_y \neq 0$, $\frac{dy}{dx} = -\frac{F_x}{F_y}$;

(b) $F(x,y,z) = 0$, definir $z$ como uma função diferenciável de $x$ e $y$ e $F_z \neq 0$, então: $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$; e $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$;

Solução: como $z$ é função diferenciável de $x$ e $y$, o quadro (02) nos informa:

• De $F(x,y,z)=0 \therefore \frac{\partial(F(x,y,z))}{\partial x} = 0 \therefore \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \therefore F_x + F_z\frac{\partial z}{\partial x} = 0$;

E como, $F_z \neq 0$, então: $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$;

• De $F(x,y,z)=0 \therefore \frac{\partial(F(x,y,z))}{\partial y} = 0 \therefore \frac{\partial F}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial F}{\partial y}\frac{\partial(y)}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0 \therefore F_y + F_z\frac{\partial z}{\partial y} = 0$;

E como, $F_z \neq 0$, temos: $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$;