Q05. Seja $X_1, X_2, \ldots, X_n$ uma amostra aleatória de uma variável aleatória $X \sim N(\theta, \theta)$, $\theta > 0$. Sugira uma quantidade pivotal para construir uma intervalo de confiança para θ com $\gamma = 1-\alpha$..

Solução:

Sabemos que

$$\bar{X} \sim N(\theta, \frac{\theta}{n}),$$

mas

$$Z = \frac{\bar{X} - \theta}{\sqrt{\theta/n}} \sim N(0, 1),$$

que é a nossa quantidade pivotal procurada.

Seja $z = z_{\alpha/2}$ de sorte que:

$$P\left(-z \leq Z \leq z\right) = \gamma = 1 - \alpha$$

$$P\left(-z \leq \frac{\bar{X} - \theta}{\sqrt{\theta/n}} \leq z\right) = \gamma$$

$$P\left(-z\sqrt{\theta/n} \leq \bar{X} - \theta \leq z\sqrt{\theta/n}\right) = \gamma$$

$$P\left(|\bar{X} - \theta| \leq z\sqrt{\theta/n}\right) = \gamma$$

Elevando ao quadrado temos:

$$\left(\bar{X} - \theta\right)^2 \leq z^2 \, \frac{\theta}{n}$$

$$\bar{X}^2 - 2\bar{X}\theta + \theta^2 - \frac{z^2}{n} \, \theta \leq 0$$

$$\theta^2 - (2\bar{X} + \frac{z^2}{n})\theta + \bar{X}^2 \leq 0.$$

O nosso intervalo é a solução da inequação.

Vamos fazer um exemplo no **R**:

Vamos gerar uma amostra de tamanho $n = 100$ de $X \sim N(16, 16)$ e construir um intervalo de confiança de 95% para θ.

```
 set.seed(32)
>
> X=rnorm(100,16,4);round(X,2)
[1] 16.06 19.49 11.89 18.74 17.80 17.63 17.14 13.50 19.36 17.25 17.90 15.60
[13] 16.81 15.62 16.40 14.93 21.38 15.41 16.20 19.33 14.84 11.66 19.76 17.44
[25] 18.96 19.53 18.11  7.80 19.93 17.89 19.28 18.36 12.75 20.08 22.21 19.94
[37] 15.53 11.15 18.64 14.78 16.32 16.33 12.29 15.52 12.50  9.59 11.40 13.54
```

```
[49] 13.43 11.01 17.50 15.22 14.11 15.75  9.89 13.05 19.08 15.25 19.53 13.60
[61] 11.93  7.68 14.81 16.87 16.50 12.28 12.86 17.63 17.57 15.85 11.52 17.27
[73] 17.41 13.93 12.49 14.57 21.56  5.21 17.35 14.37 21.44 11.19 11.23  8.29
[85] 17.06 20.39 19.75 16.67 19.30 12.97 13.89 16.45 21.24 21.36 18.37 16.95
[97] 12.94 13.94 13.23 15.96
> xb=mean(X);xb
[1] 15.74275
>
> s2=var(X);s2
[1] 11.89615
>
> alfa=0.05
> z_tab=qnorm(1-alfa/2);z_tab
[1] 1.959964
>
> z=1.96
>
> ### teta^2-(2xb+z^2/n) teta + xb^2=0
>
> a=1;b=-(2*xb+z^2/n);c=xb^2
>
> n=100
> a=1;b=-(2*xb+z^2/n);c=xb^2
>
> a;b;c
[1] 1
[1] -31.52392
[1] 247.8343
>
>
> polyroot(c(c,b,a))
[1] 14.98405-0i 16.53987+0i
>
> ICteta95=c(14.98,16.54);ICteta95
[1] 14.98 16.54
>
```

$$IC[\theta, 95\%] = [14,98; 16,54].$$

Usando a distribuição t temos:

$$V = \frac{(n_1)S^2}{\sigma^2} = \frac{(n-1)S^2}{\theta} \sim \chi^2(n-1).$$

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t(n-1).$$

```
 set.seed(32)
>
```

```
> X=rnorm(100,16,4);round(X,2)
[1] 16.06 19.49 11.89 18.74 17.80 17.63 17.14 13.50 19.36 17.25 17.90 15.60
[13] 16.81 15.62 16.40 14.93 21.38 15.41 16.20 19.33 14.84 11.66 19.76 17.44
[25] 18.96 19.53 18.11  7.80 19.93 17.89 19.28 18.36 12.75 20.08 22.21 19.94
[37] 15.53 11.15 18.64 14.78 16.32 16.33 12.29 15.52 12.50  9.59 11.40 13.54
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[73] 17.41 13.93 12.49 14.57 21.56  5.21 17.35 14.37 21.44 11.19 11.23  8.29
[85] 17.06 20.39 19.75 16.67 19.30 12.97 13.89 16.45 21.24 21.36 18.37 16.95
[97] 12.94 13.94 13.23 15.96
> xb=mean(X);xb
[1] 15.74275
>
> s2=var(X);s2
[1] 11.89615
> s=sd(X);s
[1] 3.44908
> alfa=0.05
> t_tab=qt(1-alfa/2,n-1);t_tab
[1] 1.984217
> xb;s
[1] 15.74275
[1] 3.44908
> e=t_tab*s/sqrt(n);e
[1] 0.6843723
> IC1=xb+c(-1,1)*e;IC1
[1] 15.05838 16.42713
> round(IC1,2)
[1] 15.06 16.43
> t.test(X)$conf.int
[1] 15.05838 16.42713
attr(,"conf.level")
[1] 0.95
```

Note que: