Análise de Dados Categorizados
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Para as distribuições de probabilidade Binomial e Poisson, encontre as estatísticas Wald, Score, Gradiente e a Razão de Verossimilhanças, considerando as hipóteses $H_0 : \pi = \pi_0 \quad vs \quad H_1 : \pi \neq \pi_0$ para a Binomial, e $H_0 : \mu = \mu_0 \quad \text{vs} \quad H_1 : \mu \neq \mu_0$ para a Poisson.

- **Binomial**

Propriedades da distribuição:

$$P(Y = y) = \binom{n}{y} \pi^y (1-\pi)^{n-y} I_A(y), A = \{0, 1, \ldots, n\};$$

$$\hat{\pi} = \frac{y}{n};$$

$$l(\pi; y) = \log \binom{n}{y} + y \log(\pi) + (n-y) \log(1-\pi);$$

$$\frac{\partial l(\pi; y)}{\partial \pi} = U(\pi) = \frac{y}{\pi} - \frac{(n-y)}{1-\pi}; \tag{1}$$

$$I_F(\pi) = -E\left[\frac{\partial^2 l(\pi; y)}{\partial \pi^2}\right] = \frac{n}{\pi(1-\pi)};$$

$$E[\hat{\pi}] = n\pi \quad \text{e} \quad Var[\hat{\pi}] = I_F^{-1}(\pi) = \frac{\pi(1-\pi)}{n}.$$

a) Estatística *Wald*

$$W^2 = \frac{(\hat{\pi} - \pi_0)^2}{\frac{\hat{\pi}(1-\hat{\pi})}{n}} \xrightarrow{D} \chi^2_{(1)} \quad \text{e} \quad \sqrt{W} = \frac{(\hat{\pi} - \pi_0)}{\sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}} \xrightarrow{D} N(0, 1).$$

b) Estatística *Score*

A equação (1) pode ser decomposta da seguinte forma:

$$U(\pi_0) = \frac{y}{\pi_0} - \frac{(n-y)}{1-\pi_0} = \frac{y(1-\pi_0) - \pi_0(n-y)}{\pi_0(1-\pi_0)} = \frac{y - n\pi_0}{\pi_0(1-\pi_0)}. \tag{2}$$

Assim,

$$U(\pi_0)^2 = \left[\frac{y - n\pi_0}{\pi_0(1-\pi_0)}\right]^2 = \frac{\left[n\left(\frac{y}{n} - \pi_0\right)\right]^2}{\pi_0^2(1-\pi_0)^2} = \frac{n^2(\hat{\pi} - \pi_0)^2}{\pi_0^2(1-\pi_0)^2}.$$

Logo,

$$S^2 = \frac{U(\pi_0)^2}{I_F(\pi_0)} = \frac{\dfrac{n^2(\hat{\pi} - \pi_0)^2}{\pi_0^2(1-\pi_0)^2}}{\dfrac{n}{\pi_0(1-\pi_0)}} = \frac{n(\hat{\pi} - \pi_0)^2}{\pi_0(1-\pi_0)} \xrightarrow{D} \chi^2_{(1)} \text{ e},$$

$$\sqrt{S^2} = \sqrt{\frac{n(\hat{\pi} - \pi_0)^2}{\pi_0(1-\pi_0)}} = \frac{\hat{\pi} - \pi_0}{\sqrt{\dfrac{\pi_0(1-\pi_0)}{n}}} \xrightarrow{D} N(0,1).$$

c) Estatística *Gradiente*

Vimos que a função Score pode ser posta na forma (2). Portanto, obtemos a estatística Gradiente por

$$G^2 = U(\pi_0)(\hat{\pi} - \pi_0) = \frac{n(\frac{y}{n} - \pi_0)}{\pi_0(1-\pi_0)}(\hat{\pi} - \pi_0) = \frac{n(\hat{\pi} - \pi_0)}{\pi_0(1-\pi_0)}(\hat{\pi} - \pi_0) = \frac{n(\hat{\pi} - \pi_0)^2}{\pi_0(1-\pi_0)}, \text{e}$$

$$G = \sqrt{\frac{n(\hat{\pi} - \pi_0)^2}{\pi_0(1-\pi_0)}} = \frac{\hat{\pi} - \pi_0}{\sqrt{\dfrac{\pi_0(1-\pi_0)}{n}}}.$$

$$G^2 \xrightarrow{D} \chi^2_{(1)} \quad \text{e} \quad G \xrightarrow{D} N(0,1).$$

c) *Razão de Verossimilhanças*

$$\begin{aligned}
RV &= -2\left[\log\binom{n}{y} + y\log\pi_0 + (n-y)\log(1-\pi_0) - \log\binom{n}{y} - y\log\hat{\pi} - (n-y)\log(1-\hat{\pi})\right] \\
&= -2\left[y\log\left(\frac{\pi_0}{\hat{\pi}}\right) + (n-y)\log\left(\frac{1-\pi_0}{1-\hat{\pi}}\right)\right] \\
&= -2\left[y\log\left(\frac{\pi_0}{\frac{y}{n}}\right) + (n-y)\log\left(\frac{1-\pi_0}{1-\frac{y}{n}}\right)\right] \\
&= -2\left[y\log\left(\frac{n\pi_0}{y}\right) + (n-y)\log\left(\frac{1-\pi_0}{\frac{n-y}{n}}\right)\right] \\
&= 2\left[y\log\left(\frac{y}{n\pi_0}\right) + (n-y)\log\left(\frac{n-y}{n(1-\pi_0)}\right)\right] \xrightarrow{D} \chi^2_{(1)}.
\end{aligned}$$

- **Poisson**

  Propriedades da distribuição:

$$P(Y = y) = \frac{e^{-\mu}\mu^y}{y!} I_A(y), A = \{0, 1, ...\};$$

$$\hat{\mu} = \bar{y};$$

$$L(\mu; \mathbf{y}) = \frac{e^{-n\mu}\mu^{n\bar{y}}}{\prod_{i=1}^n y_i!} \quad \text{e} \quad l(\mu; \mathbf{y}) = -n\mu + n\bar{y}\log\mu - \sum_{i=1}^n \log y_i!;$$

$$\frac{\partial l(\mu; y)}{\partial\mu} = U(\mu) = -n + \frac{n\bar{y}}{\mu} = \frac{n(\bar{y} - \mu)}{\mu};$$

$$I_F(\mu) = Var\left[U(\mu)\right] = \frac{n}{\mu};$$

$$E[\hat{\mu}] = \mu \quad \text{e} \quad Var[\hat{\mu}] = I_F^{-1}(\mu) = \frac{\mu}{n}.$$

a) Estatística *Wald*

$$W^2 = \frac{(\hat{\mu} - \mu_0)^2}{\frac{\hat{\mu}}{n}} \xrightarrow{D} \chi^2_{(1)} \quad \text{e} \quad \sqrt{W^2} = \frac{\hat{\mu} - \mu_0}{\sqrt{\frac{\hat{\mu}}{n}}} \xrightarrow{D} N(0, 1).$$

b) Estatística *Score*

$$S^2 = \frac{\frac{n^2(\bar{y} - \mu_0)^2}{\mu_0^2}}{\frac{n}{\mu_0}} = \frac{n(\bar{y} - \mu_0)^2}{\mu_0} = \frac{n(\hat{\mu} - \mu_0)^2}{\mu_0} \xrightarrow{D} \chi^2_{(1)} \text{ e,}$$

$$\sqrt{S^2} = \frac{\hat{\mu} - \mu_0}{\sqrt{\frac{\mu_0}{n}}} \xrightarrow{D} N(0, 1).$$

c) Estatística *Gradiente*

$$G^2 = \frac{n(\bar{y} - \mu_0)}{\mu_0}(\bar{y} - \mu_0) = \frac{n(\bar{y} - \mu_0)^2}{\mu_0} = \frac{n(\hat{\mu} - \mu_0)^2}{\mu_0} \xrightarrow{D} \chi^2_{(1)} \text{ e,}$$

$$\sqrt{G^2} = \frac{\hat{\mu} - \mu_0}{\sqrt{\frac{\mu_0}{n}}} \xrightarrow{D} N(0, 1).$$

d) *Razão de Verossimilhanças*

$$
\begin{aligned}
RV &= -2\left[-n\mu_0 + n\bar{y}\log\mu_0 - \sum_{i=1}^{n} y_i! + n\hat{\mu} - n\bar{y}\log\hat{\mu} + \sum_{i=1}^{n} y_i!\right] \\
&= -2\left[n\left(\hat{\mu} - \mu_0 + \bar{y}\log\left(\frac{\mu_0}{\hat{\mu}}\right)\right)\right] \\
&= 2n\left[(\mu_0 - \hat{\mu}) + \hat{\mu}\log\left(\frac{\hat{\mu}}{\mu_0}\right)\right] \xrightarrow{D} \chi^2_{(1)}.
\end{aligned}
$$