2.12. Seja $Y_1, Y_2, \ldots, Y_n$ variáveis aleatórias independentes em que

$$Y_i \sim N\left(\beta X_i, \sigma^2\right),$$

com $X_i, i = 1, 2, \ldots, n$ são conhecidos. Note que, neste caso, as variáveis Y_i não são identicamente distribuídas .

i. Encontre uma estatística conjuntamente suficiente para β e σ^2.
ii. Baseado nessa estatística, obtenha os **ENVVUM** para β e σ^2.

Solução: A f.d.p. de Y_i é dada por:

$$f_{Y_i}(y_i \mid \beta, \sigma^2) = (2\pi)^{-1/2} \, (\sigma^2)^{-1/2} \, \exp\left(-\frac{1}{2\sigma^2}\,(y_i - \beta X_i)^2\right)$$

A distribuição conjunta da amostra é dada por:

$$f(y_1, y_2, \ldots, y_n|\beta, \sigma^2) = \prod_{i=1}^{n} f_{Y_i}(y_i \mid \beta, \sigma^2)$$

$$f(y_1, y_2, \ldots, y_n|\beta, \sigma^2) = \prod_{i=1}^{n} (2\pi)^{-1/2} \, (\sigma^2)^{-1/2} \, \exp\left(-\frac{1}{2\sigma^2}\,(y_i - \beta X_i)^2\right)$$

$$= (2\pi)^{-n/2} \, (\sigma^2)^{-n/2} \, \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta X_i)^2\right)$$

$$= (2\pi)^{-n/2} \, (\sigma^2)^{-n/2} \, \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(y_i^2 - 2\beta X_i y_i + \beta^2 X_i^2\right)\right)$$

$$= (2\pi)^{-n/2} \, (\sigma^2)^{-n/2} \, \exp\left(-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n} y_i^2 - 2\beta \sum_{i=1}^{n} X_i y_i + \beta^2 \sum_{i=1}^{n} X_i^2\right)\right)$$

Fazendo

$$\theta = (\beta, \sigma^2)$$

$$h(y_1, y_2, y_n) = (2\pi)^{-n/2}$$

e

$$T_1 = \sum_{i=1}^{n} X_i Y_i \quad e \quad T_2 = \sum_{i=1}^{n} Y_i^2$$

$$f(y_1, y_2, \ldots, y_n|\beta, \sigma^2) = h(y_1, y_2, y_n) \times g_\theta(T_1, T_2)$$

Logo

$$(T_1, T_2) = \left(\sum_{i=1}^{n} X_i Y_i, \sum_{i=1}^{n} Y_i^2\right)$$

é uma estatística conjuntamente suficiente para β e σ^2.

Note que:

Devemos procurar $h(S)$ de sorte que

$$E[T_1] = \sum_{i=1}^{n} X_i E(Y_i) = \sum_{i=1}^{n} X_i \beta X_i = \beta \sum_{i=1}^{n} X_i^2.$$

O estimador não viciado de β é dado por:

$$b = \frac{T_1}{\sum_{i=1}^{n} X_i^2} = \frac{\sum_{i=1}^{n} X_i Y_i}{\sum_{i=1}^{n} X_i^2}$$

Por outro lado

$$\sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (Y_i - bX_i)^2 = \sum_{i=1}^{n} Y_i^2 - 2b \sum_{i=1}^{n} X_i Y_i + b^2 \sum_{i=1}^{n} X_i^2$$

$$\sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} Y_i^2 - 2b \times b \sum_{i=1}^{n} X_i^2 + b^2 \sum_{i=1}^{n} X_i^2$$

$$\sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} Y_i^2 - b^2 \sum_{i=1}^{n} X_i^2$$

Note que:

$$E(Y_i^2) = Var(Y_i) + E^2(Y_i) = \sigma^2 + \beta^2 X_i^2.$$

$$\sum_{i=1}^{n} E(Y_i^2) = \sum_{i=1}^{n} (\sigma^2 + \beta^2 X_i^2) = n\sigma^2 + \beta^2 \sum_{i=1}^{n} X_i^2$$

Note que:

$$Var(b) = \frac{1}{\left(\sum_{i=1}^{n} X_i^2\right)^2} \sum_{i=1}^{n} X_i^2 Var(Y_i) = \frac{\sigma^2}{\sum_{i=1}^{n} X_i^2}.$$

$$\sum_{i=1}^{n} X_i^2 Var(b) + \beta^2.$$

$$E(b^2) = Var(b) + \beta^2 = .$$

$$E(b^2) = \frac{\sigma^2}{\sum_{i=1}^{n} X_i^2} + \beta^2$$

$$\sum_{i=1}^{n} E(e_i^2) = \sum_{i=1}^{n} E(Y_i^2) - E(b^2)\ \sum_{i=1}^{n} X_i^2$$

$$\sum_{i=1}^{n} E(e_i^2) = n\sigma^2\ + \beta^2\ \sum_{i=1}^{n} X_i^2 - \left(\frac{\sigma^2}{\sum_{i=1}^{n} X_i^2} + \beta^2\right)\ \sum_{i=1}^{n} X_i^2$$

$$\sum_{i=1}^{n} E(e_i^2) = n\sigma^2\ - \sigma^2 = (n-1)\sigma^2.$$

Assim

$$\hat{\sigma^2} = \frac{\sum_{i=1}^{n} e_i^2}{n-1} = \frac{\sum_{i=1}^{n}(Y_i - bX_i)^2}{n-1} = \frac{\sum_{i=1}^{n} Y_i^2 - b^2\sum_{i=1}^{n} X_i^2}{n-1}$$

é o nosso estimador procurado.