Dimostrazione 1 ($x^n < y$).

$$x^n < y \iff \exists \epsilon \in \mathbb{R}, \epsilon > 0 \ : \ (x+\epsilon)^n < y \tag{1}$$

Sia $\epsilon \in]0,1[$

$$(x+\epsilon)^n = ((x+\epsilon)^n - x^n) + x^n = ((\cancel{x}+\epsilon)-\cancel{x})((x+\epsilon)^{n-1} + \ldots + x^{n-1}) + x^n \tag{2}$$

$$\Downarrow \tag{3}$$

$$\epsilon((x+\epsilon)^{n-1} + \cdots + x^{n-1}) + x^n \ \leq \ \epsilon \cdot n \cdot (x+1)^{n-1} + x^n \tag{4}$$

$$\Downarrow \tag{5}$$

$$\epsilon n(x+1)^{n-1} + x^n < y \iff \epsilon < \frac{y - x^n}{n(x+1)^{n-1}} :\overset{def.}{=} \epsilon > 0 \tag{6}$$

$$0 \ < \ \epsilon \ < \ \frac{y - x^n}{n(x+1)^{n-1}} \tag{7}$$

Questo è valido per: $\epsilon \in \mathbb{R}$