

Ripasso

Corso Informatica UNIPD 2021/2022

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DIMOSTRAZIONE 1 (Principio d'induzione e binomio di Newton).

IPOTESI: $P(n)$ è vera

$$P(n) = (a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} \quad (1)$$

Per il principio di induzione dovremmo essere capaci di dimostrare anche: $P(n+1)$ come vera.

$$P(n+1) = (a+b)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k} \quad (2)$$

$$(a+b)^{n+1} = (a+b)(a+b)^n \Rightarrow (a+b) \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} \quad (3)$$

Ricordiamo la proprietà distributiva: $(a+b)c = ac + bc$

$$\sum_{k=0}^n \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^n \binom{n}{k} a^k b^{n+1-k} \quad (4)$$

\Downarrow

$$\sum_{k=1}^{n+1} \binom{n}{k-1} a^k b^{n+1-k} + \sum_{k=0}^n \binom{n}{k} a^k b^{n+1-k} + \binom{n}{n} a^0 b^{n+1-0} \quad (5)$$

\Downarrow

$$\sum_{k=0}^{n+1} [\binom{n}{k-1} + \binom{n}{k}] a^k b^{n+1-k} \quad (6)$$

$$\Downarrow$$

$$\frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} = \frac{\underline{n!}}{(k-1)!(n-k+1)(n-k)!} + \frac{\underline{n!}}{k(k-1)!(n-k)!} \quad (7)$$

$$\Downarrow$$

$$\frac{n!}{(k-1)!(n-k)!} \left(\frac{1}{(n-k+1)} + \frac{1}{k} \right) \quad (8)$$

$$\Downarrow$$

$$\frac{n!}{(k-1)!(n-k)!} \cdot \frac{k+n-k+1}{k(n-k+1)} \quad (9)$$

$$\Downarrow$$

$$\overbrace{\frac{n!}{(k-1)!(n-k)!} \cdot \frac{n+1}{k(n-k+1)}} = \frac{(n+1)!}{k!((n+1)-k)!} \quad (10)$$

L'ultima semplificazione è conseguenza di: $(n+1)! = (n+1) \cdot n!$ e di $(n-1)! \cdot n = n!$

In conclusione abbiamo che:

$$\sum_{k=0}^{n+1} [(n^{k-1}) + (n^k)] a^k b^{n+1-k} = \sum_{k=0}^{n+1} (n^{k+1}) a^k b^{n+1-k} = (a+b)^{n+1} \quad (11)$$

DIMOSTRAZIONE 2 ($x^n < y$).

$$x^n < y \iff \exists \epsilon \in \mathbb{R}, \epsilon > 0 : (x+\epsilon)^n < y \quad (12)$$

Sia $\epsilon \in]0, 1[$

$$(x+\epsilon)^n = ((x+\epsilon)^n - x^n) + x^n = ((x+\epsilon) - x)((x+\epsilon)^{n-1} + \dots + x^{n-1}) + x^n \quad (13)$$

$$\Downarrow$$

$$(14)$$

$$\epsilon((x + \epsilon)^{n-1} + \dots + x^{n-1}) + x^n \leq \epsilon \cdot n \cdot (x + 1)^{n-1} + x^n \quad (15)$$

$$\Downarrow \quad (16)$$

$$\epsilon n(x + 1)^{n-1} + x^n < y \iff \epsilon < \frac{y - x^n}{n(x + 1)^{n-1}} \stackrel{def.}{=} \epsilon > 0 \quad (17)$$

$$0 < \epsilon < \frac{y - x^n}{n(x + 1)^{n-1}} \quad (18)$$

Questo è valido per: $\epsilon \in \mathbb{R}$

$$a > 1 \quad a^n > 1 \quad \forall n \in \mathbb{Q}, n > 0 \quad p, q \in \mathbb{N} \wedge q \neq 0 : n = \frac{p}{q} \quad (19)$$

$$a^n = \sqrt[q]{a^p} \geq \sqrt[q]{a} > \sqrt[q]{1} = 1 \quad (20)$$

Questo perchè $a^n > 1 \wedge a > 1 \Rightarrow 1,000000000001^n > 1$

Dalle proprietà delle potenze ricordiamo che:

$$x < y \Rightarrow a^x < a^y \Rightarrow \frac{1}{a^x} > \frac{1}{a^y} \quad (21)$$

$$x - y < 0 \xrightarrow{\cdot(-1)} y - x > 0 \quad (22)$$

$$a^x - a^y < 0 \xrightarrow{\cdot(-1)} a^y - a^x > 0 \quad (23)$$

$$\Downarrow \quad (24)$$

$$\underbrace{\underbrace{a^x}_{>0} \underbrace{(a^{y-x} - 1)}_{a^n > 1}}_{>0} > 0 \quad (25)$$

DIMOSTRAZIONE 3 (Il limite se esiste è unico).

$$\lim_{x \rightarrow \infty} a_n = l \quad \wedge \quad \lim_{x \rightarrow \infty} a_n = m \iff l = m \quad (26)$$

ESEMPIO 1.

Poniamo per assurdo che $l \neq m$ Fissiamo $\epsilon > 0$

$$\underbrace{|a_n - l| < \frac{\epsilon}{2}}_{n > \bar{n}_1} \quad \& \quad \underbrace{|a_n - m| < \frac{\epsilon}{2}}_{n > \bar{n}_2} \quad (27)$$

$$\underbrace{\hspace{10em}}_{n > \max\{\bar{n}_1, \bar{n}_2\}}$$

\Downarrow

Ricordiamo che $|a_n - m| = |m - a_n|$

$$|\cancel{a_n} - l - \cancel{a_n} + m| |a_n - l| + |m - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad (28)$$

\Downarrow

$$|m - l| < \epsilon \implies |m - l| = 0 \quad (29)$$

Ma questo è assurdo perchè: $\epsilon > 0, \forall \epsilon \in \mathbb{R}$

$$m = l \quad (30)$$

DIMOSTRAZIONE 4 (Limiti).

Se $\{a_n\}_{n \in \mathbb{N}}$ converge $l \in \mathbb{R} \implies \{a_{k_n}\}_{k_n \in \mathbb{N}}$ converge $l \in \mathbb{R}$

\Downarrow

Si ha che:

$$\forall \epsilon > 0 \exists \bar{n} \in \mathbb{N} : n > \bar{n} \implies |a_n - l| < \epsilon \quad (31)$$

$$\forall \epsilon > 0 \exists \bar{n} \in \mathbb{N} : n > \bar{n} \implies |a_{k_n} - l| < \epsilon \quad (32)$$

$$\lim_{n \rightarrow \infty} a_{k_n} = l \quad (33)$$

ESEMPIO 2.

$$\lim_{n \rightarrow +\infty} \frac{1}{n} = 0 \quad \& \quad k = 2, \lim_{k_n \rightarrow +\infty} \frac{1}{k_n} = 0 \quad (34)$$

DIMOSTRAZIONE 5.

$$\lim_{n \rightarrow +\infty} (a_n + b_n) = l + m \quad (35)$$

$$\lim_{n \rightarrow +\infty} a_n = l \quad \& \quad \lim_{n \rightarrow +\infty} b_n = m \quad (36)$$

\Downarrow

$$|a_n - l| < \frac{\epsilon}{2} \quad \text{se} \quad n > \overline{n}_1 \quad (37)$$

$$|b_n - m| < \frac{\epsilon}{2} \quad \text{se} \quad n > \overline{n}_2 \quad (38)$$

$$n > \max\{\overline{n}_1, \overline{n}_2\}$$

$$|a_n + b_n - l - m| \leq |a_n - l| + |b_n - m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad (39)$$

\Downarrow

$$\forall \epsilon > 0, \exists \overline{n} \equiv \max\{\overline{n}_1, \overline{n}_2\} : n > \overline{n} \Rightarrow \underbrace{|(a_n + b_n) - (l + m)|}_0 < \epsilon \quad (40)$$

$$(a_n + b_n) - (l + m) = 0 \quad (41)$$

$$a_n + b_n = l + m \quad (42)$$

DIMOSTRAZIONE 6.

$$\forall \epsilon > 0, \exists \overline{n} \in \mathbb{N} : n > \overline{n} \Rightarrow \underbrace{|a_n - l|}_{l - \epsilon < a_n < l + \epsilon \quad \forall n > \overline{n}} < \epsilon \quad (43)$$

$$\epsilon = |l|$$

Da questo otteniamo che

$$\underbrace{l - |l|}_0 < a_n < \underbrace{l + |l|}_{2l} \quad (44)$$

In conclusione avremo che:

$$\text{se } l > 0 \Rightarrow a_n > 0$$

$$\text{se } l < 0 \Rightarrow a_n < 0$$

Definizione 1 (Teorema dei 2 carabinieri):

Se $\underbrace{\{a_n\}, \{b_n\}}_{\text{convergono a } l}, \{c_n\}$

$$\text{è ovvio che: } a_n \leq c_n \leq b_n \implies c_n \text{ converge a } l \quad (45)$$

DIMOSTRAZIONE 7.

$$\forall \epsilon > 0, \exists \overline{n_1}, \overline{n_2} \in \mathbb{N} : \quad (46)$$

\Downarrow

$$l - \epsilon < a_n < l + \epsilon \quad \& \quad l - \epsilon < b_n < l + \epsilon \quad (47)$$

se $n > \max\{\overline{n_1}, \overline{n_2}\}$

\Downarrow

$$l - \epsilon < a_n \leq c_n \leq b_n < l + \epsilon \quad \forall n > \overline{n} \quad (48)$$

$$\underbrace{l - \epsilon < c_n < l + \epsilon}_{|c_n - l| < \epsilon} \implies \lim_{n \rightarrow +\infty} c_n = l \quad (49)$$