Dimostrazione 1 (Principio d'induzione e binomio di Newton).
IPOTESI: $P(n)$ è vera

$$P(n) \;=\; (a+b)^n \;=\; \sum_{k=0}^{n} \binom{k}{n}\, a^k\, b^{n-k} \tag{1}$$

Per il principio di induzione dovremmo essere capaci di dimostrare anche: $P(n+1)$ come vera.

$$P(n+1) \;=\; (a+b)^{n+1} \;=\; \sum_{k=0}^{n+1} \binom{k}{n+1}\, a^k\, b^{n+1-k} \tag{2}$$

$$(a+b)^{n+1} \;=\; (a+b)(a+b)^n \;\Rightarrow\; (a+b)\sum_{k=0}^{n} \binom{k}{n}\, a^k\, b^{n-k} \tag{3}$$

Ricordiamo la proprietà distributiva: $(a+b)c \;=\; ac+bc$

$$\sum_{k=0}^{n} \binom{k}{n}\, a^{k+1}\, b^{n-k} \;+\; \sum_{k=0}^{n} \binom{k}{n}\, a^k\, b^{n+1-k} \tag{4}$$

$$\Downarrow$$

$$\sum_{k=1}^{n+1} \binom{k-1}{n}\, a^k\, b^{n+1-k} \;+\; \sum_{k=0}^{n} \binom{k}{n}\, a^k\, b^{n+1-k} \;+\binom{0}{n}\, a^0\, b^{n+1-0} \tag{5}$$

$$\Downarrow$$

$$\sum_{k=0}^{n+1} \left[\binom{k-1}{n} + \binom{k}{n}\right] a^k\, b^{n+1-k} \tag{6}$$

$$\Downarrow$$

$$\frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \;=\; \frac{\underline{n!}}{(\underline{k-1)!}(n-k+1)(\underline{n-k)!}} + \frac{\underline{n!}}{k\underline{(k-1)!}\;\underline{(n-k)!}} \tag{7}$$

$$\Downarrow$$

$$\frac{n!}{(k-1)!(n-k)!}\left(\frac{1}{(n-k+1)} \;+\frac{1}{k}\right) \tag{8}$$

$$\Downarrow$$

$$\frac{n!}{(k-1)!(n-k)!} \cdot \frac{\cancel{k}+n-\cancel{k}+1}{k(n-k+1)} \tag{9}$$

$$\Downarrow$$

$$\overbrace{\frac{n!}{(k-1)!(n-k)!} \cdot \frac{n+1}{k(n-k+1)}} = \frac{(n+1!)}{k!((n+1)-k)!} \tag{10}$$

L'ultima semplificazione è conseguenza di: $(n+1)! = (n+1) \cdot n!$ e di $(n-1)! \cdot n = n!$

In conclusione abbiamo che:

$$\sum_{k=0}^{n+1} [\tbinom{k-1}{n} + \tbinom{k}{n}]\ a^k\ b^{n+1-k} = \sum_{k=0}^{n+1} \tbinom{k}{n+1}\ a^k\ b^{n+1-k} = (a+b)^{n+1} \tag{11}$$

Dimostrazione 2 ($x^n < y$).

$$x^n < y \iff \exists \epsilon \in \mathbb{R}, \epsilon > 0 \ : \ (x+\epsilon)^n < y \tag{12}$$

Sia $\epsilon \in]0,1[$

$$(x+\epsilon)^n = ((x+\epsilon)^n - x^n) + x^n = ((\cancel{x}+\epsilon)-\cancel{x})((x+\epsilon)^{n-1} + \ldots + x^{n-1}) + x^n \tag{13}$$

$$\Downarrow \tag{14}$$

$$\epsilon((x+\epsilon)^{n-1} + \cdots + x^{n-1}) + x^n \le \epsilon \cdot n \cdot (x+1)^{n-1} + x^n \tag{15}$$

$$\Downarrow \tag{16}$$

$$\epsilon n(x+1)^{n-1} + x^n < y \iff \epsilon < \frac{y-x^n}{n(x+1)^{n-1}} :\stackrel{def.}{=} \epsilon > 0 \tag{17}$$

$$0 < \epsilon < \frac{y-x^n}{n(x+1)^{n-1}} \tag{18}$$

Questo è valido per: $\epsilon \in \mathbb{R}$

$$a > 1 \quad a^n > 1 \quad \forall n \in \mathbb{Q}, n > 0 \qquad p, q \in \mathbb{N} \ \wedge \ q \neq 0 \ : n = \frac{p}{q} \tag{19}$$

$$a^n = \sqrt[q]{a^p} \geq \sqrt[q]{a} > \sqrt[q]{1} = 1 \tag{20}$$

Questo perchè $a^n > 1 \wedge a > 1 \ \Rightarrow \ 1,000000000001^n > 1$
Dalle proprietà delle potenze ricordiamo che:

$$x < y \ \Rightarrow \ a^x < a^y \ \Rightarrow \ \frac{1}{a^x} > \frac{1}{x^y} \tag{21}$$

$$x - y < 0 \ \overset{\cdot(-1)}{\Longrightarrow} \ y - x > 0 \tag{22}$$

$$a^x - a^y < 0 \ \overset{\cdot(-1)}{\Longrightarrow} \ a^y - a^x > 0 \tag{23}$$

$$\Downarrow \tag{24}$$

$$\underbrace{\underbrace{a^x}_{>0} \underbrace{(a^{y-x} - 1)}_{a^n > 1}}_{>0} > 0 \tag{25}$$

Dimostrazione 3 (Il limite se esiste è unico).

$$\lim_{x \to \infty} a_n = l \quad \wedge \quad \lim_{x \to \infty} a_n = m \qquad \Longleftrightarrow \qquad l \ = \ m \tag{26}$$

Esempio 1.
Poniamo per assurdo che $l \neq m$ Fissiamo $\epsilon > 0$

$$\underbrace{\underbrace{|a_n - l| < \frac{\epsilon}{2}}_{n > \overline{n_1}} \quad \& \quad \underbrace{|a_n - m| < \frac{\epsilon}{2}}_{n > \overline{n_2}}}_{n \ > \ \max\{\overline{n_1}, \overline{n_2}\}} \tag{27}$$

$$\Downarrow$$

Ricordiamo che $|a_n - m| = |m - a_n|$

$$|\cancel{-a_n} - l - \cancel{-a_n} + m||a_n - l| \ + \ |m - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{28}$$

$$\Downarrow$$

$$|m-l|<\epsilon \quad \Longrightarrow \quad |m-l|=0 \tag{29}$$

Ma questo è assurdo perchè: $\epsilon > 0, \forall \epsilon \in \mathbb{R}$

$$m = l \tag{30}$$

Dimostrazione 4 (Limiti).
Se $\{a_n\}_{n\in\mathbb{N}}$ converge $l \in \mathbb{R} \quad \Longrightarrow \quad \{a_{k_n}\}_{k_{n\in\mathbb{N}}}$ converge $l \in \mathbb{R}$

$$\Downarrow$$

Si ha che:

$$\forall \epsilon > 0 \ \exists \overline{n} \in \mathbb{N} \ : \ n > \overline{n} \implies |a_n - l| < \epsilon \tag{31}$$

$$\forall \epsilon > 0 \ \exists \overline{n} \in \mathbb{N} \ : \ n > \overline{n} \implies |a_{k_n} - l| < \epsilon \tag{32}$$

$$\lim_{n\to\infty} a_{k_n} \ = \ l \tag{33}$$

Esempio 2.

$$\lim_{n\to+\infty} \frac{1}{n} = 0 \qquad \& \qquad k = 2, \lim_{k_n\to+\infty} \frac{1}{k_n} = 0 \tag{34}$$

Dimostrazione 5.

$$\lim_{n\to+\infty} \underset{(a_n+b_n)}{=} l + m \tag{35}$$

$$\lim_{n\to+\infty} a_n = l \quad \& \quad \lim_{n\to+\infty} b_n = m \tag{36}$$

$$\Downarrow$$

$$|a_n - l| < \frac{\epsilon}{2} \quad \text{se} \quad n > \overline{n_1} \tag{37}$$

$$|b_n - m| < \frac{\epsilon}{2} \quad \text{se} \quad n > \overline{n_2} \tag{38}$$

$n > \max\{\overline{n_1}, \overline{n_2}\}$

$$|a_n + b_n - l - m| \leq |a_n - l| + |b_n - m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{39}$$

$$\Downarrow$$

$$\forall\epsilon > 0, \exists\overline{n} \equiv \max\{\overline{n_1}, \overline{n_2}\} \ : \ n > \overline{n} \ \Rightarrow \ \underbrace{|(a_n + b_n) - (l + m)|}_{0} < \epsilon \tag{40}$$

$$(a_n + b_n) - (l + m) = 0 \tag{41}$$

$$a_n + b_n = l + m \tag{42}$$

Dimostrazione 6.

$$\forall\epsilon > 0, \exists\overline{n} \in \mathbb{N} \ : \ n > \overline{n} \ \Rightarrow \ \underbrace{|a_n - l < \epsilon|}_{l-\epsilon<a_n<l+\epsilon \quad \forall n>\overline{n}} \tag{43}$$

$$\epsilon = |l|$$

Da questo otteniamo che

$$\underbrace{l - |l|}_{0} < a_n < \underbrace{l + |l|}_{2l} \tag{44}$$

In conclusione avremo che:
se $l > 0 \Rightarrow a_n > 0$
se $l < 0 \Rightarrow a_n < 0$

Definizione 1 (Teorema dei 2 carabinieri)**:**
Se $\underbrace{\{a_n\}\ , \{b_n\}}_{convergono\ a\ \ l}\ , \{c_n\}$

$$\textit{è ovvio che: } a_n \leq c_n \leq b_n \ \Longrightarrow \ c_n \textit{converge a } l \tag{45}$$

Dimostrazione 7.

$$\forall\epsilon > 0, \exists\overline{n_1}, \overline{n_2} \in \mathbb{N} : \tag{46}$$

$$\Downarrow$$

$$l - \epsilon < a_n < l + \epsilon \qquad \& \qquad l - \epsilon < b_n < l + \epsilon \tag{47}$$

se $n > \max\{\overline{n_1}, \overline{n_2}\}$

$$\Downarrow$$

$$l - \epsilon < a_n \leq c_n \leq b_n < l + \epsilon \qquad \forall n > \overline{n} \tag{48}$$

$$\underbrace{l - \epsilon < c_n < l + \epsilon}_{|c_n - l| < \epsilon} \Longrightarrow \lim_{n \to +\infty} c_n = l \tag{49}$$