$$f(x) = |e^{-3x} - e^{-x}|$$

① Trovare dominio e segno.

$$e^{\cdot} : \mathbb{R} \to ]0, +\infty[$$

$$|\cdot| : \mathbb{R} \to [0, +\infty[$$

$$\Rightarrow Dom(f) = \mathbb{R}$$

Segno: $f(x) \geq 0 \iff |e^{-3x} - e^{-x}| \geq 0 \iff \forall x \in Dom(f)$

$\iff \forall x \in \mathbb{R}$

$$f(x) = 0 \iff |e^{-3x} - e^{-x}| = 0$$

$$\iff e^{-3x} - e^{-x} = 0 \iff e^{-3x} = e^{-x}$$

$$\iff -3x = -x \iff -2x = 0 \iff x = 0$$

$x = 0$ è uno ZERO di $f(x)$

• Calcoliamo i limiti agli estremi del dominio

$$\lim_{x \to +\infty} |\underbrace{e^{\overbrace{-3x}^{\to -\infty}}}_{\to 0} - \underbrace{e^{\overbrace{-x}^{\to -\infty}}}_{\to 0}| = 0 \Rightarrow$$ $y = 0$ è asintoto orizzontale per $x \to +\infty$

$$\lim_{x \to -\infty} |\underbrace{e^{-3x}}_{(+\infty)} - \underbrace{e^{-x}}_{(+\infty)}| = \lim_{x \to -\infty} \left|e^{-3x}\left(1 - \frac{e^{-x}}{e^{-3x}}\right)\right| =$$

$(+\infty) - (+\infty)$

$$= \lim_{x \to -\infty} |\underbrace{e^{-3x}}_{\to +\infty} (1 - \underbrace{e^{2x}}_{\to 0})| = +\infty$$

Cerchiamo se c'è un asintoto obliquo a $-\infty$.

$$\lim_{x \to -\infty} \frac{|e^{-3x} - e^{-x}|}{x} = \lim_{x \to -\infty} \frac{|e^{-3x}(1-e^{2x})|}{x} = -\infty$$

per la gerarchia degli infiniti.

$$\log(x) << x^{\alpha} << e^{x} << x! << x^{x}$$

$\Rightarrow$ Non c'è un asintoto obliquo a $-\infty$

Oss

$\lim_{x \to 0^+} x \log(x) = 0$ ed è una conseguenza della gerarchia degli infiniti.

Infatti: $\lim_{x \to 0^+} x \log(x) \underset{x = \frac{1}{y}}{=} \lim_{y \to +\infty} \frac{1}{y} \cdot \log\left(\frac{1}{y}\right) = \lim_{y \to +\infty} \frac{-\log(y)}{y} = 0$

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- Studiare $f'$ e $f''$

$$\operatorname{Sgn}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

$$f(x) = |e^{-3x} - e^{-x}|$$

$$f'(x) = \left(-3e^{-3x} - (-e^{-x})\right) \cdot \operatorname{sgn}(e^{-3x} - e^{-x})$$

$$e^{-3x}-e^{-x} \geq 0 \iff e^{-3x} \geq e^{-x} \iff -3x \geq -x$$

$$\iff 2x \leq 0 \iff x \leq 0$$

$$\Rightarrow f'(x) = \begin{cases} -3e^{-3x}+e^{-x} & \text{se } x<0 \\ -(-3e^{-3x}+e^{-x}) & \text{se } x>0 \end{cases}$$

$x=0$ è un punto di non derivabilità:

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{|e^{-3x}-e^{-x}|}{x} =$$

$$= \lim_{x\to 0^+} \frac{|1-3x+o(x)-(1-x+o(x))|}{x} = \lim_{x\to 0^+} \frac{|-2x+o(x)|}{x} =$$

$$= \lim_{x\to 0^+} \frac{\left|-2x\left(1+\frac{o(x)}{2x}\right)\right|}{x} = \lim_{x\to 0^+} \frac{|-2x|\cdot|1+o(1)|}{x} =$$

$$= \lim_{x\to 0^+} \frac{2x\cdot|1+o(1)|}{x} = 2$$

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \dots = \lim_{x\to 0^-} \frac{|-2x+o(x)|}{x} = \dots$$

$$= \lim_{x\to 0^-} \frac{|-2x|\cdot|1+o(1)|}{x} = \lim_{x\to 0^-} \frac{-2x\cdot|1+o(1)|}{x} =$$

$$= -2$$

$\Rightarrow x=0$ è un punto angoloso!

$$f'(x) = \begin{cases} -3e^{-3x}+e^{-x} & \text{se } x<0 \\ -(-3e^{-3x}+e^{-x}) & \text{se } x>0 \end{cases}$$

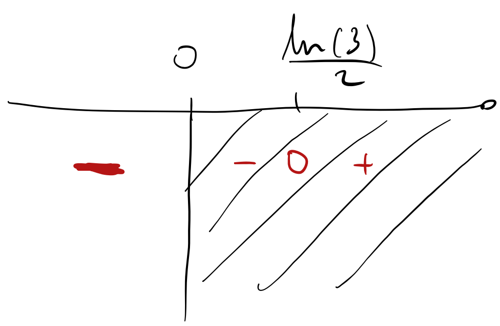
$f'(x) \geq 0$

• $x<0$: $\quad -3e^{-3x}+e^{-x} \geq 0$

$\iff e^{3x}\cdot e^{-x} \geq 3e^{-3x}\cdot e^{3x} \iff e^{2x} \geq 3$

$\iff 2x \geq \ln(3) \iff x \geq \frac{\ln(3)}{2} > 0$

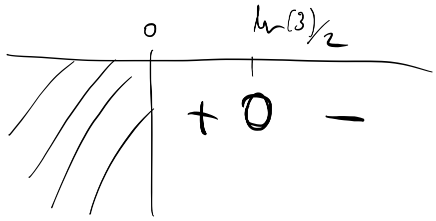
$\Rightarrow$ Se $x<0$, allora $f'(x) < 0$



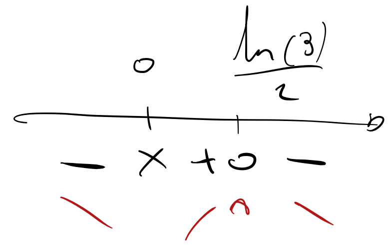

• $x>0$: $\quad f'(x) \geq 0$

$-(-3e^{-3x}+e^{-x}) \geq 0 \iff 3e^{-3x} \geq e^{-x} \iff$

$\iff 3 \geq e^{2x} \iff 2x \leq \ln(3) \iff x \leq \frac{\ln(3)}{2}$




Il segno della derivata prima è




$$f'(x) = \begin{cases} -3e^{-3x} + e^{-x} & \text{se } x < 0 \\ -(-3e^{-3x} + e^{-x}) & \text{se } x > 0 \end{cases}$$

$$f''(x) = \begin{cases} 9e^{-3x} - e^{-x} & \text{se } x < 0 \\ -9e^{-3x} + e^{-x} & \text{se } x > 0 \end{cases}$$

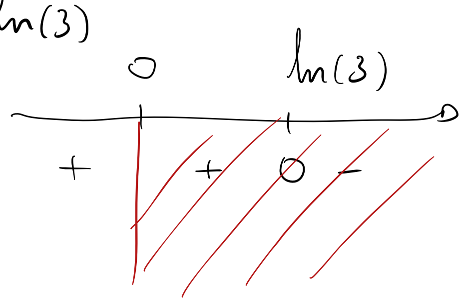
- $\underline{x < 0}$

$$f''(x) \geq 0 \iff 9e^{-3x} - e^{-x} \geq 0$$

$$\iff 9e^{-3x} \geq e^{-x} \iff e^{2x} \leq 9 \iff x \leq \frac{\ln(9)}{2}$$

$\ln(9) = \ln(3^2)$ $\iff x \leq \ln(3)$

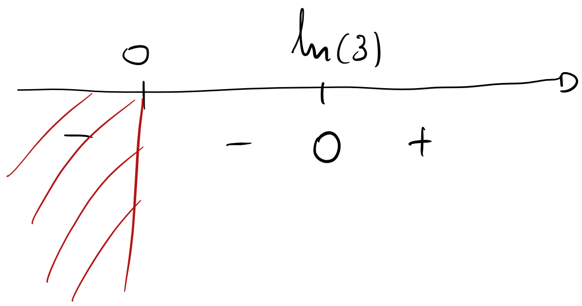
$= 2\ln(3)$



• $\underline{x>0}$, $f''(x) \geq 0 \iff -9e^{-3x} + e^{-x} \geq 0$

$\iff e^{-x} \geq 9e^{-3x} \iff e^{2x} \geq 9 \iff x \geq \frac{\ln(9)}{2}$

$\iff x \geq \ln(3)$




$\Rightarrow$ La concavità/convessità di $f$ è