$$\lim_{n\to+\infty} \frac{2^n\cdot n^\alpha - 2^n\cdot n^3 + 2^{-n}\cos(n^n)}{2^n\cdot n^4 - 2^{-n} n^8 + n^{10}} \quad , \quad \alpha>0$$

$$\lim_{n\to+\infty} \frac{n^2+3n}{5n^2+1} = \lim_{n\to+\infty} \frac{\cancel{n^2}\left(1+\underbrace{\frac{3}{n}}_{\to 0}\right)}{\cancel{n^2}\left(5+\underbrace{\frac{1}{n^2}}_{\to 0}\right)} = \frac{1}{5}$$

$$\lim_{n\to+\infty} \frac{2^n+n^4}{2^n+n^2} = \lim_{n\to+\infty} \frac{\cancel{2^n}\left(1+\underbrace{\frac{n^4}{2^n}}_{\to 0}\right)}{\cancel{2^n}\left(1+\underbrace{\frac{n^2}{2^n}}_{\to 0}\right)} = \frac{1}{1} = 1$$

$\alpha>0$ $\quad$ $a>1$

Ordini tra infiniti: $\quad c_1\cdot n^\alpha \ll c_2\cdot a^n \ll c_3\cdot n! \ll c_4\cdot n^n$

$c_1, c_2, c_3, c_4 > 0$

$$\lim_{n\to+\infty} \frac{n!}{n^n} = 0$$

$$\lim_{n\to+\infty} \frac{\cos(n)}{n} =$$

$$0 \leftarrow \left[-\frac{1}{n} < \frac{\cos(n)}{n} < \frac{1}{n}\right] \rightarrow 0$$

$\Rightarrow$ Per il teo dei due carabinieri:

$$\lim_{n\to+\infty} \frac{\cos(n)}{n} = 0$$

$$\nexists \lim_{n\to+\infty} \cos(n)$$

$$\lim_{n\to+\infty} \frac{2^n \cdot n^4 - 2^n \cdot n^3 + n^{10} + 2^n}{2^n \cdot n^2 - 2^n \cdot n^4}$$

$$= \lim_{n\to+\infty} \frac{\cancel{2^n \cdot n^4}\left(1 - \frac{\cancel{2^n} \cdot \cancel{n^3}}{\cancel{2^n} \cdot \cancel{n^4}} + \frac{n^{10}}{2^n \cdot n^4} + \frac{2^n}{2^n \cdot n^4}\right)}{\cancel{2^n \cdot n^4}\left(\frac{\cancel{2^n} n^2}{\cancel{2^n} \cdot n^4} - 1\right)} =$$

$$= \lim_{n\to+\infty} \frac{1 - \underbrace{\frac{1}{n}}_{\to 0} + \underbrace{\frac{n^6}{2^n}}_{\to 0} + \underbrace{\frac{1}{n^4}}_{\to 0}}{\underbrace{\frac{1}{n^2}}_{\to 0} - 1} = \frac{1}{-1} = -1$$

$$n^6 \in o(2^n) \iff \lim_{n\to+\infty} \frac{n^6}{2^n} = 0$$

$$\lim_{n\to+\infty} \frac{2^n \cdot n^3 + 3^n}{5 \cdot 3^n + n^5} = \lim_{n\to+\infty} \frac{\cancel{3^n}\left(\underbrace{\frac{2^n \cdot n^3}{3^n}}_{\to 0} + 1\right)}{\cancel{3^n}\left(5 + \underbrace{\frac{n^5}{3^n}}_{\to 0}\right)} = \frac{1}{5}$$

$2^n \cdot n^3 \in o(3^n)$!

gerarchia degli infiniti

$$\lim_{n\to+\infty} \frac{2^n \cdot n^3}{3^n} = \lim_{n\to+\infty} \underbrace{\left(\frac{2}{3}\right)^n}_{\to 0} \cdot \underbrace{n^3}_{\to +\infty} = \lim_{n\to+\infty} \frac{n^3}{\left(\frac{3}{2}\right)^n} = 0$$

$$\Rightarrow 2^n \cdot n^3 \in o(3^n)$$

$$\lim_{n\to+\infty} \frac{2^n \cdot n^\alpha - 2^n \cdot n^3 + 2^{-n} \cos(n^n)}{\underbrace{2^n \cdot n^4 - 2^{-n} n^8 + n^{10}}_{a_n,\ n \geq 1}} = , \quad \alpha > 0$$

$$\lim_{n\to+\infty} \frac{N\,?}{2^n\cdot n^4\left(1-\frac{2^{-n}n^8}{2^n\cdot n^4}+\frac{n^{10}}{2^n\cdot n^4}\right)}$$

$$\frac{n^8}{2^n} \xrightarrow{n\to+\infty} 0$$

$$N = 2^n\cdot n^\alpha - 2^n\cdot n^3 + 2^{-n}\cos(n^n)$$

$\alpha < 4$:
- $0<\alpha<3$
- $\alpha=3$
- $3<\alpha<4$

$\alpha = 4$

$\alpha > 4$

$\alpha = 3$: $N = 2^n\cdot n^3 - 2^n\cdot n^3 + 2^{-n}\cos(n^n)$
$= 2^{-n}\cos(n^n)$

• $0<\alpha<3$:

$$\lim_{n\to+\infty}\frac{2^n\cdot n^\alpha - 2^n\cdot n^3 + 2^{-n}\cos(n^n)}{2^n\cdot n^4 - 2^{-n}n^8 + n^{10}} =$$

$$= \lim_{n\to+\infty}\frac{\cancel{2^n}\cdot\cancel{n^3}\left(\underbrace{\frac{2^n n^\alpha}{2^n\cdot n^3}}_{\to 0} - 1 + \underbrace{\frac{\cos(n^n)}{2^{2n}\cdot n^3}}_{\to 0}\right)}{\underbrace{\cancel{2^n}\,n^{\cancel{4}}}_{= \frac{1}{n}\to 0}\left(1-\underbrace{\frac{n^4}{2^{2n}}}_{\to 0}+\underbrace{\frac{n^6}{2^n}}_{\to 0}\right)} = 0$$

• $\alpha = 3$

$$\lim_{n\to+\infty}\frac{\cancel{2^n\cdot n^3} - \cancel{2^n\cdot n^3} + \underbrace{2^{-n}\cos(n^n)}_{\to 0}}{\underbrace{2^n n^4}_{\to+\infty}\left(1-\frac{n^4}{2^{2n}}+\frac{n^6}{2^n}\right)} = \left[\frac{0}{+\infty}\right] = 0$$

- $\underline{\alpha > 3}$

$$\lim_{n\to+\infty} \frac{2^n \cdot n^\alpha \left(1 - \underbrace{\frac{2^n n^3}{2^n n^\alpha}}_{\to 0} + \underbrace{\frac{2^{-n}\cos(n^n)}{2^n \cdot n^\alpha}}_{\to 0}\right)}{2^n \cdot n^4 \left(1 - \underbrace{\frac{n^4}{2^{2n}}}_{\to 0} + \underbrace{\frac{n^6}{2^n}}_{\to 0}\right)} =$$

$$= \begin{cases} 0 & 3 < \alpha < 4 \\ 1 & \alpha = 4 \\ +\infty & \alpha > 4 \end{cases}$$

$$\Rightarrow \lim_{n\to+\infty} \frac{2^n \cdot n^\alpha - 2^n \cdot n^3 + 2^{-n}\cos(n^n)}{2^n \cdot n^4 - 2^{-n} n^8 + n^{10}} =$$

$$= \begin{cases} 0 & \text{se} \quad 0 < \alpha < 4 \\ 1 & \text{se} \quad \alpha = 4 \\ +\infty & \text{se} \quad \alpha > 4 \end{cases}$$

$$\lim_{n\to+\infty} 2^{-n}\cos(n^n) = \lim_{n\to+\infty} \frac{\cos(n^n)}{2^n} = 0$$

$$-1 \leq \cos(n) \leq 1 \quad \Rightarrow \quad -1 \leq \cos(n^n) \leq 1$$

$$\Rightarrow \quad -\frac{1}{\underbrace{2^n}_{\to 0}} \leq \frac{\cos(n^n)}{2^n} \leq \frac{1}{\underbrace{2^n}_{\to 0}}$$

$$\lim_{n\to+\infty} \frac{n^\alpha - n^4 + n^2}{n^3 - 3n^4}, \quad \alpha > 0$$

$$\cancel{=} \lim_{n\to+\infty} \frac{n^\alpha\left(1 - \frac{n^4}{n^\alpha} + \frac{n^2}{n^\alpha}\right)}{n^4\left(\frac{1}{n} - 3\right)} \; \overset{???}{\underset{???}{\cancel{=}}} \; \begin{cases} 0 & 0<\alpha<4 \\ -\frac{1}{3} & \alpha = 4 \\ -\infty & \alpha > 4 \end{cases}$$

No!

- $0 < \alpha < 4$

$$n^\alpha - n^4 + n^2 = n^4\left(\frac{n^\alpha}{n^4} - 1 + \frac{n^2}{n^4}\right)$$

$$\lim_{n\to+\infty} \frac{n^\alpha - n^4 + n^2}{n^3 - 3n^4} = \lim_{n\to+\infty} \frac{\cancel{n^4}\left(\underbrace{\frac{n^\alpha}{n^4}}_{\to 0} - 1 + \underbrace{\frac{n^2}{n^4}}_{\to 0}\right)}{\cancel{n^4}\left(\underbrace{\frac{n^3}{n^4}}_{\to 0} - 3\right)} =$$

$$= \frac{-1}{-3} = \frac{1}{3}$$

- $\alpha = 4$

$$\lim_{n\to+\infty} \frac{\cancel{n^4} - \cancel{n^4} + n^2}{n^3 - 3n^4} = \lim_{n\to+\infty} \frac{n^2}{n^4\left(\underbrace{\frac{n^3}{n^4}}_{\to 0} - 3\right)} = 0$$

- $\alpha > 4$

$$\lim_{n\to+\infty} \frac{n^\alpha - n^4 + n^2}{n^3 - 3n^4} = \lim_{n\to+\infty} \frac{\boxed{\frac{n^\alpha}{n^4}}^{\to +\infty}\left(1 - \underbrace{\frac{n^4}{n^\alpha}}_{\to 0} + \underbrace{\frac{n^2}{n^\alpha}}_{\to 0}\right)}{\left(\underbrace{\frac{n^3}{n^4}}_{\to 0} - 3\right)} = -\infty$$

$$\Rightarrow \lim_{n\to+\infty} \frac{n^\alpha - n^4 + n^2}{n^3 - 3n^4} = \begin{cases} +\frac{1}{3} & \text{se } 0<\alpha<4 \\ 0 & \text{se } \alpha = 4 \\ -\infty & \text{se } \alpha > 4 \end{cases}$$

Esercizio

$$\lim_{n\to+\infty} \frac{2^n \cdot n^\alpha - 2^n \cdot n^4 + 2^{-n}\cos(n^n)}{2^n \cdot n^4 - 2^{-n} n^8 + n^{10}} \quad , \quad \alpha > 0$$

• Utilizzando la definizione di limite, provare che:

$$\lim_{n\to+\infty} \underbrace{\frac{n^2+2}{3n^2+4}}_{=a_n} = \frac{1}{3}$$

$$\forall \varepsilon > 0 \quad \exists \bar{n} \in \mathbb{N} : \forall n \left(n > \bar{n} \Rightarrow \left|a_n - \tfrac{1}{3}\right| < \varepsilon \right)$$

Prendiamo $\varepsilon > 0$:

$$\left| \frac{n^2+2}{3n^2+4} - \frac{1}{3} \right| = \left| \frac{3n^2+6-(3n^2+4)}{3(3n^2+4)} \right| =$$

$$\left|a_n - \frac{1}{3}\right| = \left| \frac{2}{3(3n^2+4)} \right| < \varepsilon$$

$$\Leftrightarrow \frac{2}{3(3n^2+4)} < \varepsilon \Leftrightarrow \frac{2}{3\cdot\varepsilon} < 3n^2+4$$

$$\Leftrightarrow 3n^2 > \frac{2}{3\cdot\varepsilon} - 4 \Leftrightarrow n^2 > \frac{1}{3}\left(\frac{2}{3\cdot\varepsilon} - 4\right)$$

• Se $\underbrace{\boxed{\frac{1}{3}\left(\frac{2}{3\cdot\varepsilon} - 4\right)}}_{=d_\varepsilon} < 0$ (*) allora scegliamo $\bar{n} = 0$

$n^2 > a_\varepsilon$ ma $a_\varepsilon < 0 \Rightarrow n^2 > a_\varepsilon \ \forall n \in \mathbb{N}$

$\Leftrightarrow n \geq \boxed{0 = \bar{n}}$

(*) $\frac{1}{3}\left(\frac{2}{3\cdot\varepsilon} - 4\right) < 0 \Leftrightarrow \frac{2}{3\cdot\varepsilon} - 4 < 0$

$\Leftrightarrow \frac{2}{3\cdot\varepsilon} < 4 \Leftrightarrow \frac{2}{3\cdot 4} < \varepsilon \Leftrightarrow \varepsilon > \frac{1}{6}$

- Se $\frac{1}{3}\left(\frac{2}{3\cdot\varepsilon} - 4\right) \geq 0 \Leftrightarrow 0 < \varepsilon \leq \frac{1}{6}$

Allora: $n > \sqrt{\frac{1}{3}\left(\frac{2}{3\cdot\varepsilon} - 4\right)}$

$\Rightarrow \bar{n} \geq \sqrt{\frac{1}{3}\left(\frac{2}{3\cdot\varepsilon} - 4\right)}$

$$\boxed{\bar{n} \begin{cases} = 0 & \text{se } \varepsilon > \frac{1}{6} \\ \geq \sqrt{\frac{1}{3}\left(\frac{2}{3\cdot\varepsilon} - 4\right)} & \text{se } 0 < \varepsilon < \frac{1}{6} \end{cases}}$$

$\Rightarrow$ Questo conclude la dimostrazione.

$$\lim_{n\to+\infty} \underbrace{\frac{n^2+2}{3n^2+4}}_{a_n} \overset{?}{=} \frac{1}{4}$$

$\forall \varepsilon > 0 \ \exists \bar{n} \in \mathbb{N}$ t.c. $\forall n \left(n > \bar{n} \text{ e } \left|a_n - \frac{1}{4}\right| < \varepsilon\right)$

$$\left|a_n - \frac{1}{4}\right| = \left|\frac{n^2+2}{3n^2+4} - \frac{1}{4}\right| = \left|\frac{4n^2+8-3n^2-4}{4(3n^2+4)}\right| =$$

$$= \left|\frac{n^2+4}{4(3n^2+4)}\right| = \frac{n^2+4}{4(3n^2+4)} < \varepsilon \Leftrightarrow$$

$$\Leftrightarrow \quad n^2+4 < \varepsilon \cdot 4\,(3n^2+4)$$

$$\Leftrightarrow \quad n^2+4 < 12\,\varepsilon \cdot n^2 + 16\varepsilon$$

$$\Leftrightarrow \quad (12\varepsilon-1)\,n^2 + 16\varepsilon - 4 > 0$$

$$\Leftrightarrow \quad (12\varepsilon-1)\,n^2 \underset{\star}{>} 4 - 16\varepsilon \qquad \left[12\varepsilon-1<0 \Leftrightarrow \varepsilon < \tfrac{1}{12}\right]$$

Se $0 < \varepsilon < \frac{1}{12}$, allora $(12\varepsilon-1) < 0$

$$\star \Leftrightarrow \quad n^2 < \frac{4-16\varepsilon}{12\varepsilon-1} \qquad \cancel{\Rightarrow n > \bar{n}}$$

Se $0 < \varepsilon < \frac{1}{12}$, allora $\nexists\, \bar{n} \in \mathbb{N}$ t.c.

$$\forall n \left(n > \bar{n} \Rightarrow \left| a_n - \tfrac{1}{4} \right| < \varepsilon \right)$$
