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CS161 Homework 8

1. *Prove:*

Generalized product rule: $Pr(A,B|K) = Pr(A|B,K)Pr(B|K)$.

$$Pr(A,B|K) = Pr(A|B,K)Pr(B|K)$$
$$Pr(A,B,K) / Pr(K) = Pr(A,B,K) / Pr(B,K) * Pr(B,K) / Pr(K)$$
$$Pr(A,B,K) / Pr(K) = Pr(A,B,K) / Pr(K)$$

Therefore, the generalized product rule is shown to be true.

Generalized Bayes' rule: $Pr(A|B,K) = Pr(B|A,K)\ Pr(A|K)/Pr(B|K)$.

$$Pr(A|B,K) = Pr(B|A,K)\ Pr(A|K)/Pr(B|K)$$
$$Pr(A,B|K) / Pr(B|K) = Pr(B,A,K) / Pr(A,K) * Pr(A,K) / Pr(K) * Pr(B,K) / Pr(K)$$

*With left side using the generalized product rule from above

$$Pr(A,B|K) / Pr(B|K) = Pr(B,A,K) / Pr(K) * Pr(B,K) / Pr(K)$$
$$Pr(A,B,K) / Pr(K) * Pr(B,K) / Pr(K) = Pr(B,A,K) / Pr(K) * Pr(B,K) / Pr(K)$$
$$Pr(B,A,K) / Pr(K) * Pr(B,K) / Pr(K) = Pr(B,A,K) / Pr(K) * Pr(B,K) / Pr(K)$$

Therefore, the generalized Bayes' rule is shown to be true.

2. *Given the information below:*

Pr(oil)=0.5
Pr(natural gas only)=0.2
Pr(neither)=0.3
Pr(positive | oil)=0.9
Pr(positive | natural gas only)=0.3
Pr(positive | neither)=0.1

Suppose the test comes back positive. What's the probability that oil is present? That is, find Pr(oil | positive).

Pr(oil | positive) = Pr(positive | oil) * Pr(oil) / Pr(positive)

where Pr(positive) =
Pr(positive | oil)*Pr(oil) + Pr(positive | natural gas only)*Pr(natural gas only) + Pr(positive | neither)*Pr(neither)
= 0.9(0.5) + 0.3(0.2) + 0.1(0.3)
= 0.54

Pr(oil | positive) = 0.9(0.5)/0.54
Pr(oil | positive) = 0.8333

Therefore, knowing that the test shows positive, the probability that oil is present is 0.83.

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3. *Compute the probabilities of the following events:*
   α_1: the object is black;
   α_2: the object is square;
   α_3: if the object is one or black, then it is also square

   Since there are 9 black objects out of 13 total objects, we get:
   $Pr(\alpha_1)$ = 9/13
   Since there are 8 square objects out of 13 total objects, we get:
   $Pr(\alpha_2)$ = 8/13

   $Pr(\alpha_3)$ = Pr(square | one v black)
   = Pr(square ^ (one v black)) / Pr(one v black)
   = 7/13 *13/11
   = **7/11**

   Therefore (as shown above), since out of the 11 objects that are one or black, 7 of them are squares, we find that **$Pr(\alpha_3)$ = 7/11**

World	Black?	Square?	One?	Pr()
1	True	True	True	2/13
2	True	True	False	4/13
3	True	False	True	1/13
4	True	False	False	2/13
5	False	True	True	1/13
6	False	True	False	1/13
7	False	False	True	1/13
8	False	False	False	1/13

Since the question asks us to compute α_1, α_2, and α_3 using the probability of the worlds above, we recalculate them below:

$Pr(\alpha_1) = Pr(\omega_1) + Pr(\omega_2) + Pr(\omega_3) + Pr(\omega_4)$ = **9/13**

$Pr(\alpha_2) = Pr(\omega_1) + Pr(\omega_2) + Pr(\omega_5) + Pr(\omega_6)$ = **8/13**

$Pr(\alpha_3) = (Pr(\omega_1) + Pr(\omega_2) + Pr(\omega_5)) / (Pr(\omega_1) + Pr(\omega_2) + Pr(\omega_3) + Pr(\omega_4) + Pr(\omega_5) + Pr(\omega_7))$ = (7/13) / (11/13) = **7/11**

The following two sentences show α independent from β given that γ is true:

- Given that γ=not black, α=square and β=one are independent.
- Given that γ=not black, α=not one and β=square are independent.

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4. *Construct the joint probability distribution of this problem and use it to compute probabilities. Identify two sets of sentences α, β, γ such that α is independent of β given γ with respect to the constructed distribution.*

   For example, C is independent of G given F. Similarly, B is independent of G given D.

   a. *List the Markovian assumptions asserted by the DAG.*

      I (A, ∅, {B, E})
      I (B, ∅, {A, C})
      I (C, A, {B, D, E})
      I (D, {A, B}, {C, E})
      I (E, B, {A, C, D, F, G})
      I (F, {C, D}, {A, B, E})
      I (G, F, {A, B, C, D, E, H})
      I (H, {E, F}, {A, B, C, D, G})

   b. *True of false? Why?*
      i. *d_separated(A, BH, E)*
         →**False; H is known, but F (which stems from A) and E converge on H, so A and E are not independent.**

      ii. *d_separated(G, D, E)*
         →**True; G and E are separated and blocked by D from a common divergent path through B.**

      iii. *d_separated(AB, F, GH)*
         →**False; knowing F blocks G from A, but B can still connect to H (B→E→H) and so they are dependent and connected.**

   c. *Express Pr(a,b,c,d,e,f,g,h) in factored form using the chain rule of Bayesian networks.*

      Pr(a,b,c,d,e,f,g,h) = Pr(a | b,c,d,e,f,g,h) * Pr(b,c,d,e,f,g,h)
         = Pr(a | b,c,d,e,f,g,h) * Pr(b | c,d,e,f,g,h) * Pr(c,d,e,f,g,h)
         = …
         = **Pr(a | b,c,d,e,f,g,h) * Pr(b | c,d,e,f,g,h) * Pr(c | d,e,f,g,h) * Pr(d | e,f,g,h) * Pr(e | f,g,h) * Pr(f | g,h) * Pr(g | h) * Pr(h)**

   d. *Compute Pr(A=0, B=0) and Pr(E=1 | A=1). Justify your answers.*

      Pr(A=0, B=0) = P(A=0) * P(B=0)
         = 0.8 * 0.3
         = **0.24**

      Pr(A=0, B=0) is calculated to be 0.24, since A and B are independent.

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$$\begin{aligned}
Pr(E=1 \mid A=1) &= Pr(E=1, A=1) / Pr(A=1) \\
&= [Pr(A=1) * Pr(E=1)] / Pr(A=1) \\
&= Pr(E=1) \\
&= Pr(E=1, B=0) + Pr(E=1, B=1) \\
&= Pr(E=1 \mid B=0) * Pr(B=0) + Pr(E=1 \mid B=1) * Pr(B=1) \\
&= 0.9 * 0.3 + 0.1 * 0.7 \\
&= \mathbf{0.34}
\end{aligned}$$

Therefore, Pr(E=1 | A=1) is calculated to be 0.34. Since A and E are independent, this is the same as Pr(E).