Advanced Data Analysis
Homework Week 10

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1 Question

We are given the following squared energy distance:

$$\begin{aligned}\mathbf{D_E^2} &= 2\,\mathbb{E}_{x'\sim p_{test},x\sim q_\pi}\,||x'-x|| - \mathbb{E}_{x',\tilde{x}'\sim p_{test}}\,||x'-\tilde{x}'|| \\ &\quad - \mathbb{E}_{x,\tilde{x}\sim q_\pi}\,||x-\tilde{x}|| \qquad (1)\\ q_\pi(x) &= \pi p_{train}(x|y=+1) + (1-\pi)p_{train}(x|y=-1) \qquad (2)\end{aligned}$$

We need to derive the following for $D_E(p_{test,q_\pi})$ w.r.t π:

$$\begin{aligned}J(\pi) = (2A_{+1,-1} - A_{+1,+1} - A_{-1,-1})\pi^2 - 2(A_{+1,-1} - A_{-1,-1} - b_{+1} + b_{-1})\pi \\ + Const \qquad (3)\end{aligned}$$

where,

$$\begin{aligned}A_{y,\tilde{y}} &= \mathbb{E}_{x\sim p_{train},\tilde{x}\sim p_{train}(x|\tilde{y})}\,||x-\tilde{x}|| \qquad (4)\\ b_y &= \mathbb{E}_{x'\sim p_{test},x\sim p_{train}(x|y)}\,||x'-x|| \qquad (5)\end{aligned}$$

we are also given that,

$$\mathbb{E}_{\tilde{x}\sim q_\pi}[f(\tilde{x})] = \pi\,\mathbb{E}_{\tilde{x}\sim p_{train}(\tilde{x}|+1)}[f(\tilde{x})] + (1-\pi)\,\mathbb{E}_{\tilde{x}\sim p_{train}(\tilde{x}|-1)}[f(\tilde{x})] \qquad (6)$$

Derivation $(x', \tilde{x}')$ and $(x, \tilde{x})$:

To derive the expression for $J(\pi)$, substitute (6) into (1) :

$$\begin{aligned}\mathbf{D_E^2} &= 2\,\mathbb{E}_{x'\sim p_{test},x'\sim q_\pi}\,||x'-x|| - \mathbb{E}_{x',\tilde{x}'\sim p_{test}}\,||x'-\tilde{x}'|| - \mathbb{E}_{x,\tilde{x}\sim q_\pi}\,||x-\tilde{x}|| \\ &= 2\,\mathbb{E}_{x'\sim p_{test},x'\sim q_\pi}\,||x'-x|| - \mathbb{E}_{x',\tilde{x}'\sim p_{test}}\,||x'-\tilde{x}'|| - \mathbb{E}_{x\sim q_\pi}[\pi\,\mathbb{E}_{\tilde{x}\sim p_{train}(\tilde{x}|y=+1)}\,||x-\tilde{x}|| \\ &\quad + (1-\pi)\,\mathbb{E}_{\tilde{x}\sim p_{train}(\tilde{x}|y=-1)}\,||x-\tilde{x}||]\end{aligned}$$

Rearranging we get:

$$\begin{aligned}\mathbf{D_E^2} &= 2\,\mathbb{E}_{x'\sim p_{test}}\,[\mathbb{E}_{x\sim q_\pi}\,||x'-x||] - \mathbb{E}_{x',\tilde{x}'\sim p_{test}}\,||x'-\tilde{x}'|| \\ &\quad - \pi\,\mathbb{E}_{x\sim q_\pi,\tilde{x}\sim p_{train}(\tilde{x}|y=+1)}\,||x-\tilde{x}|| - (1-\pi)\,\mathbb{E}_{x\sim q_\pi,\tilde{x}\sim p_{train}(\tilde{x}|y=-1)}\,||x-\tilde{x}||\end{aligned}$$

For the first term we get:

$$
\begin{aligned}
2\,\mathbb{E}_{x'\sim p_{test}}\left[\mathbb{E}_{x\sim q_\pi}||x'-x||\right] &= 2\,\mathbb{E}_{x'\sim p_{test}}\left[\,\pi\,\mathbb{E}_{x\sim p_{train}(x|+1)}[||x'-x||]+(1-\pi)\,\mathbb{E}_{x\sim p_{train}(x|-1)}[||x'-x||]\right] \\
&= 2\pi\left(\mathbb{E}_{x'\sim p_{test},x\sim p_{train}(x|y=+1)}||x'-x||\right)+2(1-\pi)\left(\mathbb{E}_{x'\sim p_{test},x\sim p_{train}(x|y=-1)}||x'-x||\right) \\
&= 2\pi b_{+1}+2(1-\pi)b_{-1}\ using\ (5)
\end{aligned}
$$

For the second term:

$$
\mathbb{E}_{x'\sim p_{test},\tilde{x}'\sim p_{test}}||x'-\tilde{x}'|| = const\ (independent\ of\ \pi)
$$

For the third term:

$$
\begin{aligned}
&\pi\,\mathbb{E}_{x\sim q_\pi,\tilde{x}\sim p_{train}(x|y=+1)}||x-\tilde{x}|| = \pi\,\mathbb{E}_{x\sim q_\pi}||x-\tilde{x}||\cdot\mathbb{E}_{\tilde{x}\sim p_{train}(\tilde{x}|y=+1)}||x-\tilde{x}|| \\
&= \pi(\pi\,\mathbb{E}_{x\sim p_{train}(x|y=+1)}||x-\tilde{x}||+(1-\pi)\,\mathbb{E}_{x\sim p_{train}(x|y=-1)}||x-\tilde{x}||)\,\mathbb{E}_{\tilde{x}\sim p_{train}(\tilde{x}|y=+1)}||x-\tilde{x}|| \\
&= \pi^2(\mathbb{E}_{x\sim p_{train}(x|y=+1),\tilde{x}\sim p_{train}(\tilde{x}|y=+1)}||x-\tilde{x}||)+(\pi-\pi^2)(\mathbb{E}_{x\sim p_{train}(x|y=-1),\tilde{x}\sim p_{train}(\tilde{x}|y=+1)}||x-\tilde{x}||) \\
&= \pi^2 A_{+1,+1}+(\pi-\pi^2)A_{-1,+1}\ using\ (4)
\end{aligned}
$$

Similarly the fourth term can be written as:

$$
\begin{aligned}
(1-\pi)\,\mathbb{E}_{x\sim q_\pi,\tilde{x}\sim p_{train}(x|y=-1)}||x-\tilde{x}|| &= \pi\,\mathbb{E}_{x\sim q_\pi}||x-\tilde{x}||\cdot\mathbb{E}_{\tilde{x}\sim p_{train}(\tilde{x}|y=-1)}||x-\tilde{x}|| \\
&= (\pi-\pi^2)A_{+1,-1}+(1-\pi)^2A_{-1,-1}\ using\ (4)
\end{aligned}
$$

Substituting these results back into $\mathbf{D^2_E}$ we get:

$$
\begin{aligned}
\mathbf{D^2_E} &= J(\pi) = 2\pi b_{+1}+2(1-\pi)b_{-1}-\pi^2A_{+1,+1}-(\pi-\pi^2)A_{-1,+1}-(\pi-\pi^2)A_{+1,-1}-(1-\pi)^2A_{-1,-1} \\
&= (2A_{+1,-1}-A_{+1,+1}-A_{-1,-1})\pi^2-2(A_{+1,-1}-A_{-1,-1}-b_{+1}+b_{-1})\pi+const \\
&\quad (A_{+1,-1}=A_{-1,+1})
\end{aligned}
$$

Thus, we have derived the expression for $J(\pi)$. The constant terms independent of π has been absorbed into the const term in $J(\pi)$.