Advanced Data Analysis
Homework Week 3

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Given the linear model $f_\theta(x) = \sum_{j=1}^{b} \theta_j \phi_j(x)$, where $\{\phi_j(x)\}_{j=1}^{b}$ are the basis functions, the weighted least squares objective is given by,

$$\hat{\theta} = \min_\theta \frac{1}{2} \sum_{i=1}^{n} \tilde{w}_i \Big(f_\theta(x_i) - y_i \Big)^2$$

To derive the analytical solution we rewrite the above objective using the weighting matrix $\tilde{W} = diag(\tilde{w}_1, \ldots, \tilde{w}_n)$ in matrix format as shown below.

$$\hat{\theta} = \min_\theta \left[\frac{1}{2} (\Phi\theta - \mathbf{y})^T \tilde{W} (\Phi\theta - \mathbf{y}) \right]$$

where Φ is the design matrix and $\mathbf{y} = (y_1, \ldots, y_n)^T$. Expanding the brackets and then taking the derivative w.r.t θ we get,

$$\begin{aligned}
\hat{\theta} &= \min_\theta \left[\frac{1}{2} (\Phi\theta - \mathbf{y})^T \tilde{W} (\Phi\theta - \mathbf{y}) \right] \\
&= \min_\theta \left[\frac{1}{2} (\theta^T \Phi^T - \mathbf{y}^T) \tilde{W} (\Phi\theta - \mathbf{y}) \right] \\
&= \min_\theta \left[\frac{1}{2} (\theta^T \Phi^T - \mathbf{y}^T) \tilde{W} (\Phi\theta - \mathbf{y}) \right] \\
&= \min_\theta \frac{1}{2} \left[\theta^T \Phi^T \tilde{W} \Phi \theta - \theta^T \Phi^T \tilde{W} \mathbf{y} - \mathbf{y}^T \tilde{W} \Phi \theta + \mathbf{y}^T \tilde{W} \mathbf{y} \right] \quad \textit{Taking derivative w.r.t } \theta \\
&= \left[\Phi^T \tilde{W} \Phi \theta - \Phi^T \tilde{W} \mathbf{y} \right] = 0
\end{aligned}$$

Thus we get the required result,

$$\hat{\theta} = (\Phi^T \tilde{W} \Phi)^{-1} \Phi^T \tilde{W} \mathbf{y}$$