Advanced Data Analysis
Homework Week - 11

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1. Question

We are asked to prove,

$$\sum_{i,i'=1}^{n} W_{i,i'} \|Tx_i - Tx_{i'}\|^2 = 2 \operatorname{tr} \left(TXLX^T T^T\right) \tag{1}$$

given,

$$
\begin{aligned}
X &= (x_1, ..., x_n) \\
L &= D - W \\
D &= \operatorname{diag} \left(\sum_{i,i'=1}^{n} W_{i,i'}\right)
\end{aligned}
\tag{2}
$$

also we use the properties,

$$
\begin{aligned}
a^T b &= \operatorname{tr}\left(ba^T\right) \\
TT^T &= I_m \\
W_{i,i'} &= W_{i',i}
\end{aligned}
\tag{3}
$$

Starting with LHS of (1),

$$
\begin{aligned}
&\sum_{i,i'=1}^{n} W_{i,i'} \|Tx_i - Tx_{i'}\|^2 \\
&= \sum_{i,i'=1}^{n} W_{i,i'} (Tx_i - Tx_{i'})^T \cdot (Tx_i - Tx_{i'}) \\
&= \sum_{i,i'=1}^{n} W_{i,i'} (x_i^T T^T T x_i - x_i^T T^T T x_{i'} - x_{i'}^T T^T T x_i + x_{i'}^T T^T T x_{i'}) \\
&= \sum_{i=1}^{n}\sum_{i'=1}^{n} W_{i,i'} (\operatorname{tr}\left(Tx_i x_i^T T^T\right) + \operatorname{tr}\left(Tx_{i'} x_{i'}^T T^T\right) - \operatorname{tr}\left(Tx_i x_{i'}^T T^T\right) - \operatorname{tr}\left(Tx_{i'} x_i^T T^T\right)) \\
&= 2\sum_{i=1}^{n}\sum_{i=1}^{n} W_{i,i} (\operatorname{tr}\left(Tx_i x_i^T T^T\right)) - 2\sum_{i=1}^{n}\sum_{i'=1}^{n} W_{i,i'} \operatorname{tr}\left(Tx_i x_{i'}^T T^T\right), \text{using property of trace, (3)} \\
&= 2(\operatorname{tr}\left(TXDX^T T^T\right) - \operatorname{tr}\left(TXWX^T T^T\right)) \text{ using (2)} \\
&= 2 \operatorname{tr}\left(TXLX^T T^T\right)
\end{aligned}
\tag{4}
$$