Notes For Bachelor of Science (TU)

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Section 10.6 Problems:

1) Find the steady state solution of the heat conduction equation $\alpha^2 u_{xx} = u_t$ that satisfies the given boundary conditions.

$u(0,t) = 10, \quad u(50,t) = 40.$

Soln

Here the given boundary conditions are

$u(0,t) = 10$ — (i), $u(50,t) = 40$ — (ii)

for the heat condition equation

$$\alpha^2 u_{xx} = u_t \quad \text{— (iii)}$$

Let us suppose after a long time (i.e $t \to \infty$), a steady temperature distribution $v(x)$ will be reached which is independent of time t and initial conditions.

∴ The eqn (iii) can be written as

$v_{xx} = 0$ — (iv) with boundary condition

$v(0) = 10$ — (v)

and $v(50) = 40$ — (vi)

on integrating (iv) w.r.t x we get

$v_x = A$ which on further int. gives

$V(x) = Ax + B$ — (vii)

from (v) and (viii)

$10 = A \cdot 0 + B$

$\therefore B = 10$

and from (vi) and (vii)

$40 = 50A + 10$

$\therefore A = \frac{3}{5}$

substituting the value of A and B in (vii) we get

$$V(x) = 10 + \frac{3}{5}x \quad \underline{\underline{Ans}}$$

hich is the required steady-state solution.

ind the steady state solution of the heat conduction quation $\alpha^2 u_{xx} = u_t$ that satisfies the set of boundary nditions

$u(0,t) = 30$, $u(40,t) = -20$

re the given heat conduction equation is

$$\alpha^2 u_{xx} = u_t \quad \text{--- (i)}$$

the given boundary conditions are

$u(0,t) = 30$ —— (ii) and

$u(40,t) = -20$ —— (iii)

us suppose after a long time (i.e $t \to \infty$) a steady temperature ibution $v(x)$ will be reached which is independent of time t the initial conditions.

u the eqn (i) can be written as

$v_{xx} = 0$ —— (iv) with boundary conditions $v(0) = 30$ —— (v) &

ich on integration with respect to x gives $\quad v(40) = -20$ —— (vi)

$$v_x = A$$

ich on further integration gives

$$v(x) = Ax + B \quad \text{—— (vii)}$$

from (v) and (vii) we get

$30 = A.0 + B$

$\therefore B = 30$

and from (vi) and (vii) we get

$-20 = 40.A + 30$

or $A = -\frac{5}{4}$

On substituting the value of A and B in (vii) we get

$v(x) = -\frac{5x}{4} + 30$

$\therefore v(x) = 30 - \frac{5x}{4}$ is the required steady state solution of the heat conduction eqn. Ans

4) Find the steady state solution of the heat conduction equation $\alpha^2 u_{xx} = u_t$ that satisfies the set of boundary conditions

$u_x(0,t) = 0, \quad u(L,t) = T$

Soln.

Here the given heat conduction equation is

$\alpha^2 u_{xx} = u_t$ —(i)

and the given boundary conditions is are

$u_x(0,t) = 30$ — (ii) and

$u(L,t) = T$ — (iii)

Let us suppose after a long time (i.e $t \to \infty$) a steady temperature

4

ibution $v(x)$ will be reached which is independent of time
nd initial conditions.

eqn (i) can be written as
$V_{xx}=0$ — (iv) with boundary conditions

$V_x(0)=0$ — (v)
nd $V(L)=T$ — (vi)
Int. eqn (iv) w.r.t x we get

$$V_x(x)=A \quad \text{— (vii)}$$

which on further integrating w.r.t x gives

$$V(x)=Ax+B \quad \text{— (viii)}$$

om eqn (v) and (vii) we get
$A=0$

and from eqn (vi) and (viii) we get

$T=0\cdot L+B$
$\therefore B=T$
substituting these values of A and B in (viii) we get

$V(x)=T$ which is the required steady-state solutions Ans

5) Find the steady-state solution of the heat conduction equation $\alpha^2 u_{xx} = u_t$ that satisfies the given boundary conditions
$u(0,t) = 0$, $u_x(L,t) = 0$

Solⁿ

Here the give heat conduction eqⁿ is
$$\alpha^2 u_{xx} = u_t \quad \text{--- (i)}$$
and the given boundary conditions are

$u(0,t) = 0$ --- (ii) and
$u_x(L,t) = 0$ --- (iii)

Now we suppose at large interval of time (i.e $t \to \infty$), steady temperature distribution $v(x)$ will be reached which is independent of time t and the initial condition.
Now eqⁿ (i) can be written as

$v_{xx} = 0$ --- (iv) with the boundary conditions
$v(0) = 0$ --- (v)
and $v_x(L) = 0$ --- (vi)

Int. (iv) w.r.t x we get
$v_x = A$ --- (vii)
which on further integration gives

$v(x) = Ax + B$ --- (viii)

Using (vii) and (vi) we get
$A = 0$
and on using (v) and (viii) we get
$0 = 0 + B$
$\therefore B = 0$
On substituting the values of A and B in (viii) we get
$v(x) = 0$ which is required steady-state solution Ans

Find the steady-state solution of the heat conduction equation $\alpha^2 u_{xx} = u_t$ that satisfies the given set of boundary conditions.

$u(0,t) = T$; $u_x(L,t) = 0$

The given heat conduction eqn is

$\alpha^2 u_{xx} = u_t$ — (i)

The given boundary conditions are

$u(0,t) = T$ — (ii) and

$u_x(L,t) = 0$ — (iii)

So we suppose that at large interval of time (i.e $t \to \infty$) the steady temperature will be reached which is independent of time 't and initial condition.

eqn (i) can be written as

$v_{xx} = 0$ — (iv) ~~with~~ with initial boundary condition

$v(0) = T$ ~~an~~ — (v) and

$v'(L) = 0$ — (vi)

Integrating eqn (iv) we get

$v_x = A$ — (vii)

on further integration it gives

$v(x) = Ax + B$ — (viii)

Using (vi) and (vii) we get

$A = 0$

and on using (v) and (viii) we get $B = T$

On substituting these values of A and B in (viii) we get

$$V(x) = T$$

which is the required steady state solution. Ans

8) Find the steady state solution of heat conduction equation $\alpha^2 u_{xx} = u_t$ that satisfies the given boundary conditions:

$u(0,t) = T$, $u_x(L,t) + u(L,t) = 0$.

Soln,

Here the given heat conduction is

$$\alpha^2 u_{xx} = u_t \quad \text{—(i)}$$

and the given boundary conditions are

$u(0,t) = T$ —(ii) and

$u_x(L,t) + u(L,t) = 0$ —(iii)

Now, we suppose that after long time interval of time (i.e. $t \to \infty$), steady temperature distribution will be reached which is independent of time (t) and initial condition.

Now eq^n (i) can be written as

$V_{xx} = 0$ —(iv) with the boundary conditions

$V(0) = T$ —(v) and

$V_x(L) + V(L) = 0$ —(vi)

Now on int. of eq^n (iv) we get

$V_x(x) = A$ —(vii) $\therefore V_x(L) = A$

which on further int gives $V(x) = Ax + B$ —(viii)

$\therefore V(L) = AL + B$ —(ix).

$v_x(L) + v(L) = AL + B + A$

$v_x(L) + v(L) = A(L+1) + B$ —— (x)

ing (v) and (viii) we get

$T = A \cdot 0 + B$

$\therefore B = T$

using (x) and (vi)

$0 = A(L+1) + B$

$\therefore A = \dfrac{-T}{L+1}$

ubstituting these values of A and B in (viii) we get

$v(x) = Ax + B$

$v(x) = \dfrac{-T}{L+1} x + T$

$v(x) = \dfrac{(L+1)T - Tx}{(L+1)}$ Ans

$v(x) = \dfrac{T(L+1-x)}{(1+L)}$ is the require steady-state solution.

Ans.

The Wave Equation: Vibration of an Elastic Strings:

Suppose that an elastic string of length L is tightly stretched between two supports at the same horizontal level, so that the x-axis lies along the stree. Suppose that the string is set in motion so that it vibrates in vertical plane and let $u(x,t)$ denote the vertical displacement experienced by the string at the point x at time t. Damping such as air resistance are neglected, if the amplitude of the motion is not too large, then $u(x,t)$ satisfies the partial differential equation

$$\alpha^2 u_{xx} = u_{tt} \quad \text{—(i)} \quad 0 < x < L, \quad t > 0$$

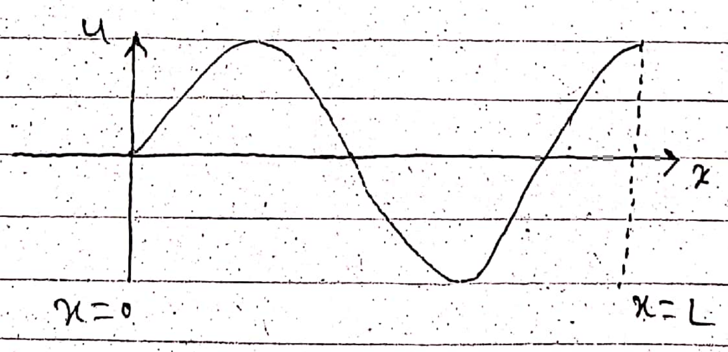
This equation is known as one dimensional wave equation.

where

$\alpha^2 = T/\rho$ — T is tension in string, ρ is mass per unit length.

For the motion of string i.e for the the displacement $u(x,t)$, we need to specify the suitable initial and boundary conditions. For this we assume that the ends of the string are fixed so, the boundary conditions are

$$u(0,t) = 0 \quad , \quad u(L,t) = 0 \quad t > 0 \quad \text{—(ii)}$$



ve the diff. eqⁿ (i) of second order w.r.t to t, we need to
scribe two initial conditions

$u(x,0) = f(x) \quad 0 \le x \le L$ — (iii)

id its ~~the~~ initial velocity

$u_t(x,0) = g(x) \quad 0 \le x \le L$ — (iv)

here $f(x)$ and $g(x)$ are the given functions

and since $u(0,t) = 0$

$\therefore u(0,0) = f(0)$

$\therefore f(0) = 0$ — (v)

and $u(L,t) = 0$

or $u(L,0) = f(L)$

$\therefore 0 = f(L)$

$\therefore f(L) = 0$ — (vi)

and same is for $g(x)$

i.e $g(0) = 0$ — (vii)

$g(L) = 0$ — (viii)

Section 10.7

Elastic String with non-zero Initial Displacement

Suppose that the string is disturbed from its equilibrium position and then released at time $t=0$ with zero velocity to vibrate freely. Then the vertical displacement $u(x,t)$ must satisfy the wave equation

$$\alpha^2 u_{xx} = u_{tt} \quad 0<x<L, \; t>0 \quad \text{— (i)}$$

the boundary conditions

$$u(0,t)=0 \text{ — (ii)}, \quad u(L,t)=0 \text{ — (iii)}, \quad t \geq 0;$$

and the initial conditions

$$u(x,0)=f(x) \text{ — (iv)}; \quad u_t(x,0)=0 \quad 0 \leq x \leq L \text{ — (v)}$$

where $f(x)$ is the given function.

Now,

Let $u(x,t) = X(x)\,T(t)$ — (vi) be the solution of eq^n (i) then

$$u_{xx} = X''(x)\,T(t)$$

and $u_{tt} = X(x)\,T''(t)$

On substituting these values in (i) we get

$$\alpha^2 X''(x)\,T(t) = X(x)\,T''(t)$$

or $$\frac{X''(x)}{X(x)} = \frac{T''(t)}{\alpha^2 T(t)} = -\lambda \text{ (say)} \quad \text{— (vii)}$$

Otherwise $u=0$, which is out of interest.

from (vii)

$$X''(x) + \lambda X(x) = 0 \quad \text{— (viii)}$$

and $$T''(t) + \alpha^2 \lambda T(t) = 0 \quad \text{— (ix)}$$

stly we solve for (viii)

$X''(x) + \lambda X(x) = 0$ is a second order linear homogeneous equation so its characteristic eqn is

$r^2 + \lambda = 0$

for eigen values we put $\lambda = \mu^2$

$\therefore r^2 + \mu^2 = 0$

or $r^2 = -\mu^2$

$\therefore r = \pm i\mu$

The solution of (viii) is

$X(x) = A_1 \cos \mu x + B_1 \sin \mu x$ — (x)

& we have,

$u(0,t) = 0$

$\therefore$ and $u(x,t) = X(x)\, T(t)$

or $u(0,t) = X(0)\, T(t)$

or $0 = X(0)\, T(t)$

$\therefore X(0) = 0$ — (xi)

and $u(L,t) = 0$

and $u(x,t) = X(x)\, T(t)$

or $u(L,t) = X(L)\, T(t)$

or $0 = X(L)\, T(t)$

$\therefore \cancel{T(t) =}\ X(L) = 0$ — (xii)

n using (x) and (xi) we get

$0 = A_1$

n using (x) and (xii) we get

$0 = B_1 \sin \mu L$

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$\therefore B \neq 0$, otherwise the solution is trival,

$\therefore \sin \mu L = 0$

$\therefore \sin \mu L = \sin n\pi$

$\therefore \mu L = n\pi$

$\therefore \mu = \frac{n\pi}{L}$

$\therefore \lambda_n = \mu_n^2 = \frac{n^2\pi^2}{L^2}$ —

$\therefore$ The required ~~solution is~~ eigen function is

$\cancel{X_n(x) = B_1 \sin n\pi}$

$X_n(x) = \sin \frac{n\pi x}{L}$ — (xiii)

Now, substituting the value of λ in (ix) we get

$T''(t) + \frac{\alpha^2 n^2\pi^2}{L^2} T(t) = 0$ — (xiv)

or $\frac{T''(t)}{T(t)} = -\frac{\alpha^2 n^2\pi^2}{L^2}$ characteristic eq^n for (xiv) is

$\therefore r^2 + \frac{\alpha^2 n^2\pi^2}{L^2} = 0$

or $r^2 = -\frac{\alpha^2\pi^2 n^2}{L^2}$

$\therefore r = \pm i \frac{\alpha n\pi}{L}$

$\therefore$ The solution of (xiv) is

$T(t) = A_2 \cos \frac{\alpha n\pi t}{L} + B_2 \sin \frac{\alpha n\pi t}{L}$ — (xv)

and we have $u_t(x, 0) = 0$

and $u_t(x,t) = X(x)\, T'(t)$

$\therefore u_t(x,0) = X(x)\,T'(0)$

$\therefore 0 = X(x)\,T'(0)$

$\therefore T'(0) = 0$ — (xvi)

~~from (xv) and (xvi) we get~~

$T'(t) = -A_2 \frac{\alpha n\pi}{L} \sin \frac{\alpha n\pi t}{L} + B_2 \frac{\alpha n\pi}{L} \cos \frac{\alpha n\pi t}{L}$ — (xvii)

from (xvi) and (xvii) we get

$0 = B_2 \frac{\alpha n\pi}{L} \cos \frac{\alpha n\pi}{L} t$

$\therefore \cancel{\cos \frac{\alpha n\pi}{L} t}$ $\therefore B_2 = 0$.

Now,

$\therefore T(t) = A_2 \cancel{\cos} \cos \frac{\alpha n\pi t}{L}$

Required solution of wave eq^n is

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \cos \frac{\alpha n\pi t}{L}$$

and we have $u(x,0) = f(x)$

or $\sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} \cos \frac{0}{L} = f(x)$

$\therefore f(x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L}$

which is fourier sine series.

$\therefore c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$

Thus the required solution of vibrating string is

$u(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} \cos \frac{\alpha n\pi t}{L}$ where

$c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$

→ Here, the quantity $\frac{\alpha n\pi}{L} = \lambda_n$ is said to be the natural frequency i.e the frequency at which the string is free to vibrate

and the factor $\sin \frac{n\pi x}{L}$ represents the displacement patterns occuring in the string when it is executing vibration of the given frequency.

→ Each displacement pattern is said to be natural mode of vibration whose time period is $\frac{2L}{n}$ and it is said to be the wavelength mode with frequency $\frac{n\pi\alpha}{L}$.

ion 10.7 Problems:

onsider an elastic string of length L whose ends are held ixed. The string is set in motion with no initial velocity rom an initial position $u(x,0) = f(x)$. Let $L = 10$, find the isplacement $u(x,t)$ for the given initial position $f(x)$.

$$f(x) = \begin{cases} 4x/L & ; 0 \le x \le L/4 \\ 1 & ; L/4 < x < 3L/4 \\ 4(L-x)/L & ; 3L/4 \le x \le L \end{cases}$$

Soln

une the given initial position is

$$u(x,0) = f(x) = \begin{cases} 4x/L & ; 0 \le x \le L/4 \\ 1 & ; L/4 < x < 3L/4 \\ 4(L-x)/L & ; 3L/4 \le x \le L \end{cases}$$

where the length of the rod $(L) = 10$

or one-dimensional wave eqn we have,

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} \cos\frac{\alpha n\pi t}{L} \quad \text{---} (i)$$

When

$$\therefore u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{10} \cos\frac{\alpha n\pi t}{10}$$

where $c_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}$

or $c_n = \frac{2}{10}\int_0^{10} f(x) \sin\frac{n\pi x}{L}$

or $C_n = \frac{1}{5}\left\{\int_0^{10/4}\frac{4x}{10}\sin\frac{n\pi x}{10}\,dx + \int_{10/4}^{30/4} 1\cdot\sin\frac{n\pi x}{10}\,dx + \int_{30/4}^{10}\frac{4}{10}(10-x)\sin\frac{n\pi x}{10}\,dx\right\}$

or $C_n = \frac{1}{5}\left\{\frac{4}{10}\left[-\frac{10x\cos}{n\pi}\frac{n\pi x}{10} + \frac{100}{n^2\pi^2}\sin\frac{n\pi x}{10}\right]_0^{10/4} - \frac{10}{n\pi}\left[\cos\frac{n\pi x}{10}\right]_{10/4}^{30/4}\right.$

$-\frac{4\cdot 10}{n\pi}\left[\cos\frac{n\pi x}{10}\right]_{30/4}^{10} - \frac{4}{10}\left[-\frac{10x\cos}{n\pi}\frac{n\pi x}{10} + \frac{100}{n^2\pi^2}\sin\frac{n\pi x}{10}\right]_{30/4}^{10}$

or $C_n = \frac{1}{5}\left\{\frac{4}{10}\left\{-\frac{10\times 10\cos}{n\pi\, 4}\frac{n\pi}{4} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{4}\right\} - \frac{10\cos}{n\pi}\frac{3n\pi}{4} + \frac{10}{n\pi}\cos\frac{n\pi}{4}\right.$

$-\frac{40}{n\pi}\cos n\pi + \frac{40}{n\pi}\cos\frac{3n\pi}{4} - \frac{4}{10}\left\{-\frac{10\times 10\cos n\pi}{n\pi} + \frac{10\times 30\cos 3n\pi}{n\pi\, 4\ \ 4} - \frac{100}{n^2\pi^2}\sin\frac{3n\pi}{4}\right\}\Bigg\}$

or $C_n = \frac{1}{5}\left[-\frac{40\cos}{n\pi}\cancel{\frac{n\pi}{4}} + \frac{40}{n^2\pi^2}\sin\frac{n\pi}{4} - \cancel{\frac{10}{n\pi}\cos\frac{3n\pi}{4}} + \cancel{\frac{10}{n\pi}\cos\frac{n\pi}{4}}\right.$

$\cancel{-\frac{40}{n\pi}}\cos n\pi + \frac{40}{n\pi}\cancel{\cos\frac{3n\pi}{4}} + \cancel{\frac{40}{n\pi}}\cos n\pi - \frac{30}{n\pi}\cancel{\cos\frac{3n\pi}{4}} +$

$\left.\frac{40}{n^2\pi^2}\sin\frac{3n\pi}{4}\right]$

or $C_n = \frac{1}{8}\left(\frac{16}{n^2\pi^2}\sin\frac{n\pi}{4}\right.\ \frac{1}{5}\left\{\frac{40}{n^2\pi^2}\sin\frac{3n\pi}{4} + \frac{40}{n^2\pi^2}\sin\frac{n\pi}{4}\right\}$

or $C_n = \left(\frac{4}{n\pi}\right)^2\sin\frac{n\pi}{4}\ \ \frac{1}{5}\times\frac{40}{n^2\pi^2}\left[\sin\frac{n\pi}{4} + \sin\frac{3n\pi}{4}\right]$

$\therefore C_n = \frac{8}{n^2\pi^2}\left[\sin\frac{n\pi}{4} + \sin\frac{3n\pi}{4}\right]$

Substituting the value of C_n in (i) we get

$$u(x,t) = \sum_{n=1}^{\infty} \frac{16}{n^2\pi^2} \sin\frac{3n\pi}{4} \sin\frac{n\pi x}{L} \cos\frac{\alpha n\pi t}{L}$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2}\left[\sin\frac{n\pi}{4} + \sin\frac{3n\pi}{4}\right] \sin\frac{n\pi x}{L} \cos\frac{\alpha n\pi t}{L}$$

$$\therefore u(x,t) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\sin n\pi/4 + \sin 3n\pi/4\right) \sin\frac{n\pi x}{L} \cos\frac{\alpha n\pi t}{L}$$

which is the required solⁿ for wave equation Ans.

3) Consider an elastic string of length L whose ends are held fixed. The string is set in motion with no initial velocity from an initial position $u(x,0) = f(x)$. Let $L = 10$, and find the displacement $u(x,t)$ for the given initial position $f(x)$,

$$f(x) = 8x(L-x)^2/L^3$$

Sol^n

Here the given initial position is

$$u(x,0) = f(x) = \frac{8x(L-x)^2}{L^3}$$

where $L = 10$

$$\therefore u(x,0) = f(x) = \frac{8x(10-x)^2}{1000}$$

Now for one dimensional wave equation we have

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} \cos\frac{n\pi \alpha t}{L}$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{10} \cos\frac{n\pi \lambda t}{10} \quad \text{—— (1)}$$

where for c_n, we have

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$\text{or } c_n = \frac{2}{10}\int_0^{10} \frac{8x(10-x)^2}{1000}\sin\frac{n\pi x}{10}\,dx$$

$$\text{or } c_n = \frac{1}{5}\times\frac{8}{1000}\left[\int_0^{10}\left(100x\sin\frac{n\pi x}{10} - 20x^2\sin\frac{n\pi x}{10} + x^3\sin\frac{n\pi x}{10}\right)dx\right.$$

$$c_n = \frac{1}{625}\left\{100\int_0^{10} x \sin\frac{n\pi x}{10}\,dx - 20\int_0^{10} x^2 \sin\frac{n\pi x}{10}\,dx + \int_0^{10} x^3 \sin\frac{n\pi x}{10}\,dx\right\}$$

$$c_n = \frac{1}{625}\left\{100\left[\frac{-10x}{n\pi}\cos\frac{n\pi x}{10} + \frac{100}{n^2\pi^2}\sin\frac{n\pi x}{10}\right]_0^{10} - 20\left[\frac{10}{n\pi}\cos\frac{n\pi x}{10}\left\{\frac{200}{(n\pi)^2} - x^2\right\} + \frac{200x}{(n\pi)^2}\sin\frac{n\pi x}{10}\right]_0^{10}\right.$$

$$\left. + \left[\frac{-10x^3}{n\pi}\cos\frac{n\pi x}{10} + \frac{30}{n\pi}\left[\frac{10x^2}{n\pi}\sin\frac{n\pi x}{10} + \frac{200x}{(n\pi)^2}\cos\frac{n\pi x}{10} - \frac{2000}{(n\pi)^3}\sin\frac{n\pi x}{10}\right]\right]_0^{10}\right\}$$

$$= \frac{1}{625}\left[\left\{100 \times \left(-\frac{100}{n\pi}\cos n\pi\right) - 20\left(\frac{10}{n\pi}\cos n\pi\left(\frac{200}{(n\pi)^2} - 100\right\}\right) - \frac{10}{n\pi}\left(\frac{200}{(n\pi)^2}\right)\right\}\right.$$

$$\left. + \left\{\frac{-10000\cos n\pi}{(n\pi)} + \frac{60000}{(n\pi)^2}\cos n\pi - 0\right\}\right.$$

$$= \frac{1}{625}\left[\frac{-10000}{n\pi}\cos n\pi - \frac{40000}{(n\pi)^3}\cos n\pi + \frac{2000}{n\pi}\cos n\pi - \frac{2000}{(n\pi)^3} - \frac{10000}{n\pi}\cos n\pi + \frac{60000}{(n\pi)^2}\cos n\pi\right]$$

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21

$$\text{or } C_n = \frac{1}{625}\left\{100\int_0^{10} x\sin\frac{n\pi x}{10}\,dx - 20\int_0^{10} x^2\sin\frac{n\pi x}{10}\,dx + \int_0^{10} x^3\sin\frac{n\pi x}{10}\,dx\right\}$$

$$\text{or } C_n = \frac{1}{625}\left\{100\left[\frac{-10x}{(n\pi)}\cos\frac{n\pi x}{10} + \frac{100}{(n\pi)^2}\sin\frac{n\pi x}{10}\right]_0^{10} - 20\left[\frac{-10x^2}{(n\pi)}\cos\frac{n\pi x}{10} + \frac{200x}{(n\pi)^2}\sin\frac{n\pi x}{10}\right.\right.$$

$$\left. + \frac{2000}{(n\pi)^3}\cos\frac{n\pi x}{10}\right]_0^{10} + \left[\frac{-10x^3}{(n\pi)}\cos\frac{n\pi x}{10} + \frac{300x^2}{(n\pi)^2}\sin\frac{n\pi x}{10} + \frac{6000x}{(n\pi)^3}\cos\frac{n\pi x}{10}\right.$$

$$\left.\left. - \frac{60000}{(n\pi)^4}\sin\frac{n\pi x}{10}\right]_0^{10}\right\}$$

$$\text{or } C_n = \frac{1}{625}\left\{100\left\{\frac{-100}{(n\pi)}\cos n\pi + \frac{100}{(n\pi)^2}\sin n\pi\right\} - 20\left\{\frac{-1000}{(n\pi)}\cos n\pi + \frac{2000}{(n\pi)^2}\sin n\pi\right.\right.$$

$$\left. + \frac{2000}{(n\pi)^3}\cos n\pi - \frac{2000}{(n\pi)^3}\right\} + \left\{\frac{-10000}{(n\pi)}\cos n\pi + \frac{30000}{(n\pi)^2}\sin n\pi\right.$$

$$\left.\left. + \frac{60000}{(\pi n)^3}\cos n\pi - \frac{60000}{(n\pi)^4}\sin n\pi\right\}\right\}$$

$$\text{or } C_n = \frac{1}{625}\left\{\frac{-10000}{(n\pi)}\cos n\pi + \frac{10000}{(n\pi)^2}\sin n\pi + \frac{20000}{n\pi}\cos n\pi - \frac{40000}{(n\pi)^2}\sin n\pi\right.$$

$$- \frac{40000}{(n\pi)^3}\cos n\pi + \frac{40000}{(n\pi)^3} - \frac{10000}{(n\pi)}\cos n\pi + \frac{30000}{(n\pi)^2}\sin n\pi + \frac{60000}{(n\pi)^3}\cos n\pi$$

$$\left. - 60000\sin n\pi/(n\pi)^4\right\}$$

22

$$-\left\{-\frac{\cancel{10000}}{(n\pi)}\cos n\pi + \frac{\cancel{20000}}{(n\pi)}\cos n\pi - \frac{40000}{(n\pi)^3}\cos n\pi + \frac{40000}{(n\pi)^3}\right.$$

$$\left. - \frac{\cancel{10000}}{n\pi}\cos n\pi + \frac{60000}{(n\pi)^3}\cos n\pi\right\}\frac{1}{625}$$

$$= \left\{\frac{20000}{(n\pi)^3}\cos n\pi + \frac{40000}{(n\pi)^3}\right\}\frac{1}{625}$$

$$= \frac{20000}{(n\pi)^3}(\cos n\pi + 2) \times \frac{1}{625} = \frac{32}{n^3\pi^3}(\cos n\pi + 2)$$

Substituting the value of C_n in (i) we get

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{10}\cos\frac{n\pi \alpha t}{10}$$

$$u(x,t) = \sum_{n=1}^{\infty}\frac{32}{n^3\pi^3}(2+\cos n\pi)\sin\frac{n\pi x}{10}\cos\frac{n\pi\alpha t}{10}$$

$$u(x,t) = \frac{32}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}(2+\cos n\pi)\sin\frac{n\pi x}{10}\cos\frac{n\pi\alpha t}{10}$$

This is the required solution for displacement.

5) Consider an elastic string of length L whose ends are held fixed. The string is set in motion from its equilibrium position with an initial velocity $u_t(x,0) = g(x)$. Let $L = 10$, find the displacement $u(x,t)$ for the given $g(x)$.

$$f(x) = \begin{cases} 2x/L & 0 \le x \le L/2 \\ 2(L-x)/L & L/2 < x \le L \end{cases}$$

Soln

Here the given initial velocity of the string is

$$u_t(x,0) = g(x) = \begin{cases} 2x/L & 0 \le x \le L/2 \\ 2(L-x)/L & L/2 < x \le L \end{cases}$$

where $L = 0$.

Now for one dimensional wave eqn we have

$$u(x,t) = \sum_{n=1}^{\infty} K_n \sin\frac{n\pi x}{L} \sin\frac{n\pi \alpha t}{L}$$

or $$u(x,t) = \sum_{n=1}^{\infty} K_n \sin\frac{n\pi x}{10} \sin\frac{n\pi \alpha t}{10} \quad \text{---(i)}$$

and $$u_t(x,t) = \sum_{n=1}^{\infty} K_n \frac{n\pi\alpha}{10} \sin\frac{\pi x n}{10} \cos\frac{n\pi\alpha t}{10}$$

and we have, $u_t(x,0) = g(x)$

or $$\sum_{n=1}^{\infty} K_n \frac{n\alpha\pi}{10} \sin\frac{\pi x n}{10} = g(x)$$

which is sine fourier series

$K_n \frac{n\alpha\pi}{10}$ is the fourier coefficient of fourier sine series and is given by

$$K_n \frac{n\alpha\pi}{10} = \frac{2}{10}\int_0^{10} g(x)\sin\frac{n\pi x}{10}\,dx$$

$$K_n \frac{n\alpha\pi}{10} = \frac{2}{10}\left[\int_0^{5} g(x)\sin\frac{n\pi x}{10}\,dx + \int_5^{10} g(x)\sin\frac{n\pi x}{10}\,dx\right]$$

$$K_n \frac{n\alpha\pi}{10} = \frac{2}{10}\left[\int_0^{5} \frac{2x}{10}\sin\frac{n\pi x}{10}\,dx + \int_5^{10} \frac{2(10-x)}{10}\sin\frac{n\pi x}{10}\,dx\right]$$

$$K_n \frac{n\alpha\pi}{10} = \frac{2}{10}\left[\frac{1}{5}\left[\frac{-10x}{n\pi}\cos\frac{n\pi x}{10} + \frac{100}{(n\pi)^2}\sin\frac{n\pi x}{10}\right]_0^5 + \frac{1}{5}\left\{\int_5^{10} 10\sin\frac{n\pi x}{10}\,dx - \int_5^{10} x\sin\frac{n\pi x}{10}\,dx\right\}\right]$$

$$K_n \frac{n\alpha\pi}{10} = \frac{2}{10}\left[\frac{1}{5}\left[\frac{-10x}{(n\pi)}\cos\frac{n\pi x}{10} + \frac{100}{(n\pi)^2}\sin\frac{n\pi x}{10}\right]_0^5 + \frac{1}{5}\left\{-10\,\frac{10}{n\pi}\left[\cos\frac{n\pi x}{10}\right]_5^{10} - \left[\frac{-10x}{n\pi}\cos\frac{n\pi x}{10} + \frac{100}{(n\pi)^2}\sin\frac{n\pi x}{10}\right]_5^{10}\right\}\right]$$

$$K_n \frac{n\alpha\pi}{10} = \frac{2}{10}\left[\frac{1}{5}\left[\frac{-50}{n\pi}\cos 2n\pi + \frac{100}{(n\pi)^2}\sin 2n\pi - 0\right] + \frac{1}{5}\left\{\frac{-100}{n\pi}\cos n\pi + \frac{100}{(n\pi)}\cos 2n\pi + \frac{100}{n\pi}\cos n\pi + \frac{100}{(n\pi)^2}\sin n\pi - \frac{50}{n\pi}\cos 2n\pi + \frac{100}{(n\pi)^2}\sin 2n\pi\right\}\right]$$

25

or $K_n \frac{n\pi a}{10} = \frac{2}{10}\left[\frac{-10}{n\pi}\cancel{\cos 2n\pi} - \frac{20}{n\pi}\cancel{\cos n\pi} + \frac{20}{n\pi}\cancel{\cos 2n\pi} + \frac{20}{\cancel{n\pi}}\cancel{\cos n\pi}\right.$

$\left. + \frac{20-10}{n\pi}\cancel{\cos 2n\pi}\right]$

or $K_n \frac{n\pi a}{10} = 0$

$\therefore K_n = 0$

Consider an elastic string of length L whose ends are held fixed. The string is set in motion from its equillibrium position with an initial velocity $u_t(x,0) = g(x)$. Let us suppose that $L=10$ and find the displacement $u(x,t)$ for given $g(x)$:

Q[n].

$$g(x) = \begin{cases} 2x/L & 0 \le x \le L/2 \\ 2(L-x)/L & L/2 < x \le L \end{cases}$$

S[n].

We the given initial velocity is

$$u_t(x,0) = g(x) = \begin{cases} 2x/L & 0 \le x \le L/2 \\ 2(L-x)/L & L/2 < x \le L \end{cases}$$

and $L=10$

$$\therefore u_t(x,0) = g(x) = \begin{cases} 2x/10 & 0 \le x \le 5 \\ 2(10-x)/10 & 5 < x \le 10 \end{cases}$$

for one dimensional wave equation we have

$$u(x,t) = \sum_{n=1}^{\infty} k_n \sin\frac{n\pi x}{L} \sin\frac{n\pi\alpha t}{L} \quad \text{---(i)}$$

$$u(x,t) = \sum_{n=1}^{\infty} k_n \sin\frac{n\pi x}{10} \sin\frac{n\pi\alpha t}{10}$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{n\pi\alpha}{10} k_n \sin\frac{n\pi x}{10} \cos\frac{n\pi\alpha t}{10}$$

and we have,

$$u_t(x,0) = g(x)$$

$$\therefore g(x) = \sum_{n=1}^{\infty} \frac{n\pi\alpha}{10} k_n \sin\frac{n\pi x}{10}$$

which is fourier sine representation.
and $\frac{n\alpha\pi}{10} k_n$ is the sine fourier coefficient which is given by

$$\frac{n\alpha\pi}{10} k_n = \frac{2}{L}\int_0^L g(x) \sin\frac{n\pi x}{L}\, dx$$

$$\text{or } \frac{n\pi\alpha}{10} k_n = \frac{2}{10}\int_0^{10} g(x) \sin\frac{n\pi x}{10}\, dx$$

$$\text{or } n\pi\alpha k_n = 2\left\{\int_0^5 g(x) \sin\frac{n\pi x}{10} dx + \int_5^{10} g(x) \sin\frac{n\pi x}{L} dx\right\}$$

$$\text{or } n\pi\alpha k_n = 2\left\{\int_0^5 \frac{2x}{10} \sin\frac{n\pi x}{10} dx + \int_5^{10} \frac{2}{10}(10-x) \sin\frac{n\pi x}{10} dx\right\}$$

$$\text{or } n\pi\alpha k_n = 2\left\{\frac{2}{10}\int_0^5 x \sin\frac{n\pi x}{10} dx + \frac{2\times 10}{10}\int_5^{10} \sin\frac{n\pi x}{10} dx - \frac{2}{10}\int_5^{10} x \sin\frac{n\pi x}{10} dx\right\}$$

$$\text{or } n\pi\alpha k_n = 2\left\{\frac{2}{10}\left[\frac{-10x}{n\pi}\cos\frac{n\pi x}{10} + \frac{100}{(n\pi)^2}\sin\frac{n\pi x}{10}\right]_0^5 - \frac{2\times 10}{n\pi}\left[\cos\frac{n\pi x}{10}\right]_5^{10}\right.$$
$$\left. - \frac{2}{10}\left[\frac{-10x}{n\pi}\cos\frac{n\pi x}{10} + \frac{100}{(n\pi)^2}\sin\frac{n\pi x}{10}\right]_5^{10}\right\}$$

$$\text{or } n\pi\alpha k_n = 2\left\{\frac{2}{10}\left[\frac{-50}{n\pi}\cos\frac{n\pi}{2} + \frac{100}{(n\pi)^2}\sin\frac{n\pi}{2}\right] - \frac{20}{n\pi}\left[\cos n\pi - \cos\frac{n\pi}{2}\right]\right.$$
$$- \frac{2}{10}\left[\frac{-100}{n\pi}\cos n\pi + \frac{100}{(n\pi)^2}\sin n\pi + \frac{50}{n\pi}\cos\frac{n\pi}{2} - \frac{100}{(n\pi)^2}\sin\frac{n\pi}{2}\right]$$

$$\text{or } n\pi\alpha k_n = 2\left[\cancel{\frac{-10}{n\pi}\cos\frac{n\pi}{2}} + \frac{20}{(n\pi)^2}\sin\frac{n\pi}{2} - \cancel{\frac{20}{n\pi}\cos n\pi} + \frac{20}{n\pi}\cos\frac{n\pi}{2} + \cancel{\frac{20}{n\pi}\cos n\pi} - \right.$$
$$\left. 20/(n\pi)^2 \sin n\pi - 10\cos\frac{n\pi}{2}/n\pi + 20\sin n\pi/2/(n\pi)^2\right]$$

$\therefore \alpha K_n = 2\left\{\frac{40}{(n\pi)^2}\sin\frac{n\pi}{2}\right\}$

$K_n = \frac{80}{\pi^3 n^3 \alpha}\sin\frac{n\pi}{2}$

Substituting the value of K_n in (i) we get

$$u(x,t) = \sum_{n=1}^{\infty} \frac{80}{\pi^3 n^3 \alpha}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10}\sin\frac{n\pi\alpha t}{10}$$

$$\therefore u(x,t) = \frac{80}{\pi^3 \alpha}\sum_{n=1}^{\infty}\frac{\sin n\pi/2}{n^3}\sin\frac{n\pi x}{10}\sin\frac{n\pi\alpha t}{10} \quad \text{Ans}$$

29

12) Dimensionless variable can be introduced into the wave equation $\alpha^2 u_{xx} = u_{tt}$ in the following manner. Let $s = x/L$ and show that the wave equation becomes

$$\alpha^2 u_{ss} = L^2 u_{tt}$$

Then show that L/α has the dimensions of time and thus can be used as the unit on the time scale. Finally let $\tau = \alpha t/L$ and show that the wave equation then reduces to

$$u_{ss} = u_{\tau\tau}$$

Sol^n

Section 10.8 Laplace's Equation:

The equation of the form $u_{xx} + u_{yy} = 0$ (in two dimensions) and $u_{xx} + u_{yy} + u_{zz} = 0$ (in three dimension) are Laplace equation.

Dirichlet Problem for a Rectangle:

Find the function $u(x,y)$ which satisfy the laplace equation

$$u_{xx} + u_{yy} = 0$$

in rectangle $0 < x < a$, $0 < y < b$ and also satisfying the boundary conditions

$$u(x,0) = 0, \quad u(x,b) = 0 \qquad 0 < x < a,$$
$$u(0,y) = 0, \quad u(a,y) = f(y) \qquad 0 \leq y \leq b,$$

where $f(y)$ is a given function on $0 \leq y \leq b$

Sol:

Here the given laplace equation is

$$u_{xx} + u_{yy} = 0 \quad \text{---(i)}$$

in a rectangle $0 < x < a$, $0 < y < b$ and with the boundary conditions

$$u(x,0) = 0, \quad 0 < x < a \quad \text{---(ii)}$$
$$u(x,b) = 0, \quad 0 < x < a \quad \text{---(iii)}$$
$$u(0,y) = 0, \quad 0 \leq y \leq b \quad \text{---(iv)}$$

and $u(a,y) = f(y)$, $\quad 0 \leq y \leq b$ ---(v)

where $f(y)$ is the given function on $0 \leq y \leq b$.

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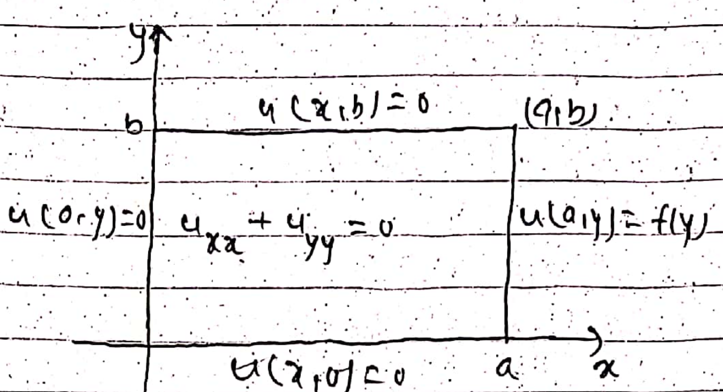


Let $u(x,y)$ be the solution of eq^n (i) such that

$$u(x,y) = X(x)\,Y(y) \quad \text{—— (vi)}$$

or $u_x(x,y) = X'(x)\,Y(y)$

or $u_{xx}(x,y) = X''(x)\,Y(y)$

or $u_y(x,y) = X(x)\,Y'(y)$

or $u_{yy}(x,y) = X(x)\,Y''(y)$

On substituting the value of u_{xx} and u_{yy} in (i) we get

$$X''(x)\,Y(y) + X(x)\,Y''(y) = 0$$

or $\dfrac{X''(x)}{X(x)} = -\dfrac{Y''(y)}{Y(y)} = \lambda$ (say)

$\therefore$ $X''(x) - \lambda X(x) = 0$ —— (vii)

and $Y''(y) + \lambda Y(y) = 0$ —— (viii)

Now, for (vi)

we have, $u(0,y) = 0$ {from (iv)}

or $X(0)\,Y(y) = 0$

$\therefore$ $X(0) = 0$ —— (ix)

and we have, $u(x,0) = 0$

or $X(x)\,Y(0) = 0$

i.e. $Y(0) = 0$ —— (x)

and $u(x,b) = 0$

or $X(x)\,Y(b) = 0$ $\therefore$ $Y(b) = 0$ —— (xi)

For (vii)

$X''(x) - \lambda X(x) = 0$

for eigen values we put $\lambda = \mu^2$

$\therefore X''(x) - \mu^2 X(x) = 0$

characteristic eq^n is

$r^2 - \mu^2 = 0$

$\therefore r = \pm \mu$

For (viii)

$Y''(y) + \lambda Y(y) = 0$

for the eigenvalues we put $\lambda = \mu^2$

or $Y''(y) + \mu^2 Y(y) = 0$

characteristic eq^n is

$r^2 + \mu^2 = 0$

$\therefore r = \pm i\mu$

$\therefore$ Solution of (viii) is

$Y(y) = c_1 \cos \mu y + c_2 \sin \mu y$ —— (xii)

from (xii) and (x)

$Y(y=0) = 0$

or $c_1 = 0$

again from (xii) and (xi)

$Y(y=b) = 0$

or $c_2 \sin \mu b = 0$

$\because c_2 \neq 0$ otherwise the solution is trival so

$\sin \mu b = 0$

or $\sin \mu_1 b = \sin n\pi$

$\therefore \mu_n = \frac{n\pi}{b}$

$\therefore \lambda_n = \mu_n^2 = \left(\frac{n\pi}{b}\right)^2$

$\therefore$ The eigen function of (viii) is

$$Y(y) = \sin\left(\frac{n\pi y}{b}\right) \quad \text{—— (xiii)}$$

Now substituting the value of λ_n in (vii) we get

$$X''(x) - \frac{n^2\pi^2}{b^2} X(x) = 0$$

Now the characteristics eqn is

$$r^2 - \left(\frac{n\pi}{b}\right)^2 = 0$$

$$\therefore r = \pm \frac{n\pi}{b}$$

$\therefore$ The solution of (vii) is

$$X(x) = A_1 \cosh \frac{n\pi x}{b} + B_1 \sinh \frac{n\pi x}{b} \quad \text{—— (xiv)}$$

from (ix) and (xiv) we get

$$X(x=0) = A_1 = 0$$

So $X(x)$ must be proportional to $\sinh \frac{n\pi x}{b}$

$\therefore$ The eigen function of (viii) is

$$X_n(x) = \sinh \frac{n\pi x}{b} \quad \text{—— (xv)}$$

∴ The solution for homogeneous boundary condition is

$u_n(x,y) = X_n(x) Y_n(y)$

or $u_n(x,y) = \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}$

Now for the non-homogeneous boundary condition

$u(a,y) = f(y)$ ie at $x = a$

we have solution

$u(x,y) = \sum_{n=1}^{\infty} c_n u_n(x,y)$

or $u(x,y) = \sum_{n=1}^{\infty} c_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}$ —— (XVI)

which is the fourier sine series and

$c_n \sinh \frac{n\pi x}{b}$ is the coefficient of fourier sine series.

and $c_n \sinh \frac{n\pi x}{b} = \frac{2}{b} \int_0^b f(y) \sin \frac{n\pi x}{b} \, dy$

This is the required solution.

and for large value of n we have

$\sinh \alpha = \frac{e^{\alpha}}{2}$

$\therefore \frac{\sinh(n\pi x/b)}{\sinh(n\pi a/b)} = \frac{e^{n\pi x/b}}{2} \times \frac{2}{e^{n\pi a/b}} = e^{-\frac{n\pi}{b}(a-x)}$

~~# Drit~~

Dirichlet Problem for a circle:

Find the function $u(r,\theta)$ which satify the laplace's equation of polar form

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

In a circular region $r < a$ subjected to the boundary condition

$$u(a,\theta) = f(\theta)$$

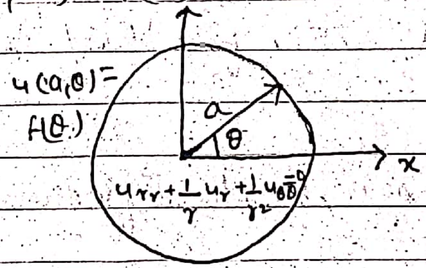
where $f(\theta)$ is the given function on $0 \le \theta < 2\pi$

Soln

Here the given laplace equation in the polar form is

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad \text{—(i)}$$ In a circular region $r < a$ subjected to boundary condition

$$u(a,\theta) = f(\theta) \quad \text{—(i)} \qquad 0 \le \theta < 2\pi$$



Now we suppose that $u(r,\theta)$ is the solution of (i) such that

$$u(r,\theta) = R(r) \cdot F(\theta) \quad \text{— (ii)}$$

$$\therefore u_r(r,\theta) = R'(r) F(\theta)$$

$$\therefore u_{rr}(r,\theta) = R''(r) F(\theta)$$

and $u_\theta(r,\theta) = R(r) F'(\theta)$

$$u_{\theta\theta}(r,\theta) = R(r) F''(\theta)$$

In ~~un~~ substituting the value of $u_{rr}, u_r, u_{\theta\theta}$ in (i) we get

$$R''(r)F(\theta) + \frac{1}{r}R'(r)F(\theta) + \frac{1}{r^2}R(r)F''(\theta) = 0$$

$$\text{or } r^2R''(r)F(\theta) + rR'(r)F(\theta) + R(r)F''(\theta) = 0$$

$$\text{or } r^2R''(r)F(\theta) + rR'(r)F(\theta) = -R(r)F''(\theta)$$

$$\text{or } \frac{r^2R''(r)F(\theta)}{R(r)F(\theta)} + \frac{rR'(r)F(\theta)}{R(r)F(\theta)} = -\frac{R(r)F''(\theta)}{R(r)F(\theta)}$$

$$\text{or } r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = -\frac{F''(\theta)}{F(\theta)} = \lambda$$

This gives,

$$r^2R''(r) + rR'(r) - \lambda R(r) = 0 \quad \text{— (iii)}$$

$$\text{and } F''(\theta) + \lambda F(\theta) = 0 \quad \text{— (iv)}$$

Now it is necessary that $u(r,\theta)$ be bounded and periodic in θ with period 2π, for this λ must be real
~~So we put~~ and $\lambda > 0$.
So we put

$\lambda = \mu^2$ where $\mu > 0$ then from (iii) and (iv)

$$r^2R''(r) + rR'(r) - \mu^2R(r) = 0 \quad \text{— (v)}$$

$$\text{and } F''(\theta) + \mu^2F(\theta) = 0 \quad \text{— (vi)}$$

eq^n (v) is Euler's equation and has solution

$$R(r) = k_1r^{\mu} + k_2r^{-\mu} \quad \text{— (vii)}$$

and eq^n (vi) has solution

$$F(\theta) = c_1\sin\mu\theta + c_2\cos\mu\theta \quad \text{— (viii)}$$

In order that $F(\theta)$ be periodic with period 2π it is necessary that μ be positive integer n.
with $\mu = n$, when $\mu = n$, the solution $r^{-\mu}$ becomes unbounded as $r \to 0$, so $r^{-\mu}$ must be neglected. which gives $K_2 = 0$.

$\therefore$ eq^n (vii) ~~becomes~~ and (viii) becomes

$R(r) = K_1 r^n$

and $F(\theta) = c_1 \sin n\theta + c_2 \cos n\theta$

$\therefore$ ~~the~~ putting the value of $R(r)$ and $F(\theta)$ in eq^n (ii) we get

~~ϕ~~ ~~$u + R$~~, $u_n(r, \theta) = r^n (c_n \sin n\theta + c_2 \cos n\theta)$

Now

Section 18.8 problem:

2) Find the solution $u(x,y)$ of Laplace's equation in rectangle $0<x<a$, $0<y<b$ that satisfies the boundary conditions:

$$u(0,y)=0, \quad u(a,y)=0, \quad 0<y<b$$
$$u(x,0)=h(x) \quad u(x,b)=0. \quad 0\le x\le a.$$

Solⁿ:

We have laplace's equation is

$$u_{xx}+u_{yy}=0 \text{ --- (i)}$$

and the given rectangle is

$$0<x<a, \quad 0<y<b$$

and the given boundary conditions are

$$u(0,y)=0 \qquad 0<y<b \text{ --- (ii)}$$
$$u(a,y)=0 \qquad 0<y<b \text{ --- (iii)}$$
$$u(x,0)=h(x) \qquad 0\le x\le a \text{ --- (iv)}$$
$$u(x,b)=0 \qquad 0\le x\le a \text{ --- (v)}$$

Now, we have, for, dirichilet problem for rectangle

$$u\cancel{(x,y)} = \sum_{n=1}^{\infty} c_n \cancel{\sinh \frac{n\pi x}{b}} \cancel{\sin \frac{n\pi y}{b}} \text{ --- (vi)}$$

We assume that $u(x,y)$ be the solution of (i) such that

$$u(x,y)=X(x)\,Y(y) \text{ --- (vi)}$$

$$\therefore u_x(x,y)=X'(x)\,Y(y)$$
$$u_{xx}(x,y)=X''(x)\,Y(y)$$
$$u_y(x,y)=X(x)\,Y'(y)$$
$$u_{yy}(x,y)=X(x)=Y''(y)$$

Substituting the value of u_{xx} and u_{yy} in (i) we get

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

or $X''(x)Y(y) = -X(x)Y''(y)$

or $\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$ (say)

$\therefore X''(x) - \lambda X(x) = 0$ —— (vii)

and $Y''(y) + \lambda Y(y) = 0$ —— (viii)

again,
we have $u(x,y) = X(x)Y(y)$

and also we have, $u(0,y)$

or $X(0)Y(y) = 0$

$\therefore X(0) = 0$ —— (ix)

again, $u(a,y) = 0$

or $X(a)Y(y) = 0$

$\therefore X(a) = 0$ —— (x)

and $u(x,b) = 0$

or $X(x)Y(b) = 0$

$\therefore Y(b) = 0$ —— (xi)

For (vii)

$$X''(x) - \lambda X(x) = 0$$

for eigen values we put $\lambda = \mu^2$

$$X''(x) - \mu^2 X(x) = 0$$

its characteristic equation

$$r^2 - \mu^2 = 0$$

$\therefore r = \pm\mu$

∴ The solution of (vii) is

$$X(x) = c_1 \cosh \mu x + c_2 \sinh \mu x \quad \text{— (xii)}$$

using (xii) and (ix) we get

$$0 = c_1$$

using (x) and (xii) we get

$$0 = c_2 \sinh 4a$$