Notes For Bachelor of Science (TU)

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Partial differential equation of 1st order

Tuesday

Chapter 7

→ An equation involving partial derivative of dependent variable with respect to two or more independent variables is said to be partial differential equation.

If the eqⁿ of the form

$$F(x,y,z,p,q) = 0 \quad \text{where } p = \frac{\partial z}{\partial x}, \; q = \frac{\partial z}{\partial y}$$

and z is dependent variable, x and y are independent variables.

→ Origin of first order partial differential equation:

Let us consider an equation of sphere with centre (0,0,c) and radius a i.e

$$x^2+y^2+(z-c)^2 = a^2 \quad \text{—— (i)}$$

diff. partially with respect to x and y we get

w.r.t x

$$2x + 2(z-c)p = 0$$

$$\therefore x + (z-c)p = 0 \quad \text{—— (ii)}$$

w.r.t y

$$2y + 2(z-c)q = 0$$

$$\therefore y + (z-c)q = 0 \quad \text{—— (iii)}$$

eliminating $(z-c)$ from eqⁿ (ii) and (iii) we get

$$-\frac{x}{p} = -\frac{y}{q}$$

$$\boxed{\therefore \ xq = yp}$$

which is the required first order partial differential equation

<u>note</u>: The set of all spheres with centre on z-axis is represented by the partial differential equation:

$$\boxed{xq = yp}$$

example:
Let us consider an equation of cone whose axis of symmetry is z axis and represented by $x^2+y^2=(z-c)^2\tan^2\alpha$ where c and α are arbitary.

Solⁿ
$$x^2+y^2 = (z-c)^2\tan^2\alpha \quad \text{--- (i)}$$

diff. partially with respect to x and y we get
w.r.t. x

$$2x = 2(z-c)p\tan^2\alpha$$
$$\therefore x = (z-c)p\tan^2\alpha \quad \text{--- (ii)}$$

w.r.t y

$$2y = 2(z-c)q\tan^2\alpha$$
$$\therefore y = (z-c)q\tan^2\alpha \quad \text{--- (iii)}$$

eliminating $(z-c)$ from eqⁿ (ii) and (iii) we get

$$\frac{x}{y} = \frac{p}{q}.$$

$$\boxed{\therefore xq = yp}$$

Which is the required partial differential equation

Note!

From the above two example, the sphere and the cone have in common is that they are surface of revolution which the line z-axis (OZ) as an axis of symmetry.

eg.3:

All surfaces of revolution which have the line OZ (z-axis) as the axis of symmetry be generated by an equation of the form

$$z = f(x^2+y^2) \text{ — (i) where } f \text{ is arbitrary}$$

sol^n

put $x^2+y^2 = u$

then $z = f(u)$

Now, diff. partially w.r.t x and y we get

w.r.t x

$$\frac{\partial z}{\partial x} = f'(u)\,2x$$

$$p = f'(2x) \text{ — (ii)}$$

w.r.t y

$$\frac{\partial z}{\partial y} = q = f'(2y) \text{ — (iii)}$$

From (ii) and (iii) we get

$$\frac{p}{2x} = \frac{q}{2y}$$

$$\boxed{\therefore xq = yp}$$

Note:
if the eq^n of the form
$F(x, y, z, a, b) = 0$ —— (i)
diff. partially w.r.t x and y we get
w.r.t x

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\therefore \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} p = 0 \quad \text{—— (ii)}$$

~~an~~ w.r.t y

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} q = 0 \quad \text{—— (iii)}$$

From eq^n (i), (ii) and (iii) which has two arbitrary constants a and b and it will be possible to eliminate these constants from these eq^n and get a relation

$$f(x, y, z, p, q) = 0$$

which is 1st order partial differential eq^n.

Note:

If the equation of the form,

$f(u,v)=0$ — (i) where u and v are function of x, y and z

and F is arbitary function of u and v.

diff. partially w.r.t x and y we get

w.r.t x

$$\boxed{\begin{array}{l}\frac{\partial F}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial x}=0 \\ \because u \text{ is function of } x, y, z\end{array}}$$

$$\therefore \frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right]+\frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right]=0$$

w.r.t y

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=0$$

eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ we get

$$p\frac{\partial(u,v)}{\partial(y,z)}+q\frac{\partial(u,v)}{\partial(z,x)}=\frac{\partial(u,v)}{\partial(x,y)} \quad \text{—— (ii)}$$

which is a partial differential equation of the form

$$f(x, y, z, p, q) = 0 \quad \text{---(iii)}$$

Note:

It should be observed that the partial differential eqⁿ (ii) $f(x, y, z, p, q) = 0$ is a linear eqⁿ {i.e the powers of p and q are both unity}; it is not necessary that [illegible] (iii) $f(x, y, z, p, q) = 0$ is 1st order linear partial differential equation

eg: Let us consider a sphere of radius 1 and centre at XOY plane $(a, b, 0)$ i.e

$$(x-a)^2 + (y-b)^2 + z^2 = 1 \quad \text{---(i)}$$

diff. partially w.r.t x and y we get

w.r.t x

$$2(x-a) + 2zp = 0 \quad \text{---(ii)}$$

w.r.t y

$$2(y-b) + 2zq = 0 \quad \text{---(iii)}$$

substituting the values of $(x-a)$ and $(y-b)$ in eqⁿ (i) we get

$$(-zp)^2 + (-zq)^2 + z^2 = 1$$

or $z^2(p^2+q^2+1) = 1$ which is reqᵈ partial diff eqⁿ but it is not linear.

Origin of first order partial differential equation

Problems

eliminate the constant 'a' and 'b' from the following equations:

a) $z = (x+a)(y+b)$

b) $2z = (ax+y)^2 + b$

Eliminate the arbitrary function f from the eqn.

a) $z = xy + f(x^2+y^2)$

b) $z = x+y + f(xy)$

c) $z = f\{(xy)/z\}$

e) $f(x^2+y^2+z^2, z^2 - 2xy) = 0$

Soln.

a) Here the given eqn is

$$z = (x-a)(y+b) \quad \text{—(i)}$$

diff. partially w.r.t x and y we get

w.r.t x

$$\frac{\partial z}{\partial x} = y+b$$

i.e. $p = y+b$ —(ii) $\left\{\because \frac{\partial z}{\partial x} = p\right\}$

w.r.t y

$$\frac{\partial z}{\partial y} = x-a$$

$$q = x-a \quad \text{—(iii)}$$

multiplying eqn (ii) and (iii) we get

$pq = (x-a)(y-b)$

from (i) we get

$pq = z$

which is the required first order differential eqⁿ.

sol^n,

b) the given eq^n is

$2z = (ax+y)^2 + b$

or $z = \frac{(ax+y)^2}{2} + \frac{b}{2}$ —— (i)

differentiating partially w.r.t x and y we get

w.r.t x

$\frac{\partial z}{\partial x} = \frac{1}{2} 2(ax+y)\cdot a = a(ax+y)$.

$\therefore p = a(ax+y)$ —— (ii)

w.r.t y $\frac{\partial z}{\partial y} = (ax+y)$

$q = (ax+y)$ —— (iii)

or $a = \frac{q-y}{x}$

$\therefore$ eq^n ~~(iii)~~ (ii) becomes

$p = \left[\left(\frac{q-y}{x}\right)x + y\right]\left(\frac{q-y}{x}\right)$

$\therefore$ $\boxed{px = q(q-y)}$

which is required diff equation.

2 (a) $z = xy + f(x^2+y^2)$

Sol^n

Here given eq^n is:

$$z = xy + f(x^2+y^2) \quad \text{--- (i)}$$

we put $u = x^2+y^2$

$$\therefore z = xy + f(u) \quad \text{--- (ii)}$$

Now partially diff w.r.t x we get

$$\frac{\partial z}{\partial x} = y + 2x f'(u)$$

$$\therefore p = y + 2x f'(u)$$

$$\therefore p - y = \{f'(u)\}(2x) \quad \text{--- (iii)}$$

Now, partially diff w.r.t y we get

$$\frac{\partial z}{\partial y} = x + 2y f'(u)$$

$$\text{or } q = x + 2y f'(u)$$

$$\therefore q - x = 2y f'(u) \quad \text{--- (iv)}$$

dividing eq^n (iii) by (iv) we get

$$\frac{p-y}{q-x} = \frac{\cancel{2}x \cancel{f'(u)}}{\cancel{2}y \cancel{f'(u)}}$$

$$\therefore y(p-y) = x(q-x) \quad \text{Ans}$$

2(b) $z = x + y + f(xy)$

Sol^n

Here the given diff eq^n is

$$z = x + y + f(xy)$$

we put $u = xy$

$$\therefore z = x + y + f(u) \text{ ——— (i)}$$

Now partially diff w.r.t. x we get

$$\frac{\partial z}{\partial x} = 1 + y f'(u)$$

or $p = 1 + y f'(u)$

$\therefore p - 1 = y f'(u)$ ——— (ii)

again diff w.r.t. y we get

$$\frac{\partial z}{\partial y} = 1 + x f'(u)$$

or $q = 1 + x f'(u)$

$\therefore q - 1 = x f'(u)$ ——— (iii)

dividing eq^n (ii) by (iii) we get

$$\frac{p-1}{q-1} = \frac{y \cancel{f'(u)}}{x \cancel{f'(u)}}$$

$$\therefore x(p-1) = y(q-1) \quad \text{Ans}$$

Q(C) $z = f\left(\frac{xy}{z}\right)$

Solⁿ
Here the given diff eqⁿ is

$$z = f\left(\frac{xy}{z}\right) \text{ --- (i)}$$

we put $u = \left(\frac{xy}{z}\right)$

$\therefore z = f(u)$ --- (ii)

Now diff eqⁿ (ii) w.r.t x we get

$$\frac{\partial z}{\partial x} = \frac{\partial f(u)}{\partial x} = f'(u)\left\{\frac{\partial\, xy/z}{\partial x}\right\} = \frac{\left\{z\frac{\partial xy}{\partial x} - xy\frac{\partial z}{\partial x}\right\}}{z^2} f'(u)$$

or $p = \frac{(zy - xyp)}{z^2} f'(u)$

$\therefore p = \frac{y(z - xp)}{z^2} f'(u)$ --- (iii)

Now, diff eqⁿ (ii) w.r.t y we get

$$\frac{\partial z}{\partial y} = \frac{\partial f(u)}{\partial y} = f'(u)\frac{\partial (xy/z)}{\partial y} = \frac{\left\{z\frac{\partial xy}{\partial y} - xy\frac{\partial z}{\partial y}\right\}}{z^2} f'(u)$$

or $q = \frac{zx - xyq}{z^2} f'(u)$

or $q = \frac{x(z - yq)}{z^2} f'(u)$ --- (iv) dividing eq^n (iii) by (iv) we get

$$\frac{p}{q} = \frac{y(z-xp)\,\cancel{f'(u)}/\cancel{z^2}}{x(z-yq)\,\cancel{f'(u)}/\cancel{z^2}}$$

or $x(z-yq)p = y(z-xp)q$ Ans

#2.3 Cauchy's problem for first order partial differential equation:

i) If the $x_0(\mu)$, $y_0(\mu)$ and $z_0(\mu)$ are the function which together with their derivatives, are continuous in the interval M where $M: \mu_1 < \mu < \mu_2$ and

ii) if $F(x, y, z, p, q)$ is a continuous function of x, y, z, p and in a certain region U of the xyzpq space, then it is required to establish the existence of a function $\phi(x, y)$ such that

a) $\phi(x, y)$ and its partial derivative with respect to x and y are continuous function of x and y in Region R of xy plane.

b) For all values of x and y lying in R, the point $\{x, y, \phi(x,y), \phi_x(x,y), \phi_y(x,y)\}$ lies in U and

$F[x, y, \phi(x,y), \phi_x(x,y), \phi_y(x,y)] = 0$ is the required equation.

c) For all μ belonging to the interval M, the point $\{x_0(\mu), y_0(\mu)\}$ belongs to the region R and

$$\phi\{x_0(\mu), y_0(\mu)\} = z_0$$

Theorem (Without Proof):

If $g(y)$ and all its derivative are continuous for $|y-y_0|<\delta$, if x_0 is a given number and $z_0 = g(y_0)$, $q_0 = g'(y_0)$ and if $f(x,y,z,q)$ and all its derivative are continuous in a region S defined by

$$|x-x_0|<\delta, \quad |y-y_0|<\delta, \quad |q-q_0|<\delta$$

then there exist a unique function $\phi(x,y)$ such that

a) $\phi(x,y)$ and all its partial derivative are continuous in a Region R defined by $|x-x_0|<\delta_1$, $|y-y_0|<\delta_2$

b) For all (x,y) in R, $z=\phi(x,y)$ is a solution of the equation

$$\frac{\partial z}{\partial x} = f\left(x, y, z, \frac{\partial z}{\partial y}\right)$$

c) For all values of y in the interval $|y-y_0|<\delta_1$, $\phi(x_0,y) = g(y)$.

#2.4 Linear Equation of the First Order:

The partial differential equation of the form

$$Pp + Qq = R \quad \text{—— (i)}$$

where P, Q, R are the functions of x, y, z respectively (which do not dependent on p or q)

$$p = \frac{\partial z}{\partial x} \qquad q = \frac{\partial z}{\partial y}$$

Equation (i) is also known as Lagrange's equation. Its generalization to n independent variables is obviously the equation

$$X_1 P_1 + X_2 P_2 + \cdots + X_n P_n = Y$$

where $X_1, X_2, \ldots, X_n$ are the function of n independent variable $x_1, x_2, \ldots, x_n$ and $P_i = \frac{\partial f}{\partial x_i}$ where

$i = 1, 2, \ldots, n$

eg: $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z^2 + x^2$ is a linear equation.

But

$x\frac{\partial z}{\partial x} = z^2 + x^2$ is not linear equation.

Theorem:

The general solution of the linear partial differential equation

$$Pp + Qq = R \quad \text{is} \quad F(u, v) = 0$$

where F is an arbitary function and $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are obtained ~~by~~ from the solution of the equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Proof:

The given diff. eq^n is

$$Pp + Qq = R \quad \text{———} \; (i)$$

Let $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are two functions of x, y, z.

Now, diff $u(x, y, z) = c_1$ partially w.r.t x, y

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0$$

or $u_x dx + u_y dy + u_z dz = 0$ ——— (ii)

and diff $v(x, y, z) = c_2$ partially w.r.t x, y we get

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = 0$$

or $v_x dx + v_y dy + v_z dz = 0$ ——— (iii)

Let us now suppose that it is possible to eq (ii) and (iii) are comparable with

$$\frac{dx}{P} = \frac{dy}{dQ} = \frac{dz}{R}$$

Then we get

$$Pu_x + Qu_y + Rv_z = 0 \quad \text{(iv)}$$

and

$$Pv_x + Qv_y + Rv_z = 0 \quad \text{(v)}$$

From (iv) and (v)

$$\frac{P}{u_y v_z - u_z v_y} = \frac{Q}{u_z v_x - u_x v_z} = \frac{R}{u_x v_y - u_y v_x}$$

or

$$\frac{P}{\frac{\partial(u,v)}{\partial(y,z)}} = \frac{Q}{\frac{\partial(u,v)}{\partial(x,z)}} = \frac{R}{\frac{\partial(u,v)}{\partial(x,y)}} \quad \text{(vi)}$$

while

$$\frac{\partial(u,v)}{\partial(y,z)} = u_y v_z - u_z v_y = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}$$

$$\frac{\partial(u,v)}{\partial(x,z)} = u_z v_x - u_x v_z = \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = u_x v_y - u_y v_x = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

We know the relation

$f(u,v)=0$ leads to partial diff. equation

$$p\frac{\partial(u,v)}{\partial(y,z)}+q\frac{\partial(u,v)}{\partial(z,x)}=\frac{\partial(u,v)}{\partial(x,y)} \quad \text{---(vii)}$$

Substituting the values from (vi) and (vii)

we see that $f(u,v)=0$ is a solution of
$Pp+Qq=R$

This completes the proof.

Example V.)
Find the general solution of the diff. eqn

$$x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=(x+y)z$$

Soln
Here the given partial diff. eqn is

$$x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=(x+y)z \quad \text{---(i)}$$

The integral surface of this equation are generated by integral curve of the equation

$$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{(x+y)z} \quad \text{---(ii)}$$

Now from (ii)

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

On integrating we get

$$\frac{-1}{x} + \frac{1}{y} = c$$

or $\cancel{\frac{1}{x} - \frac{1}{y} = c}$

$\therefore$ or $\frac{x-y}{xy} = c_1$ —— (iii)

again from (ii)

$$\frac{dx - dy}{x^2 - y^2} = \frac{dz}{z(x+y)}$$

or $$\frac{dx - dy}{(x+y)(x-y)} = \frac{dz}{z(x+y)}$$

or $$\frac{dx - dy}{x-y} = \frac{dz}{z}$$

On integrating we get

$$\log(x-y) = \log z + \log c_2$$

or $\log(x-y) - \log z = \log c_2$

or $\log\left(\frac{x-y}{z}\right) = \log c_2$

$\therefore$ $\cancel{\frac{x-y}{z}}$ $\frac{x-y}{z} = c_2$ —— (iv)

Thus the required solution is given by

$$F\left(\frac{x-y}{xy}, \frac{x-y}{z}\right)$$

On combining eqn (iii) and (iv) we get

$$xy c_1 = z c_2$$

or $$\frac{xy}{z} = \frac{c_2}{c_1}$$

$$\therefore \frac{xy}{z} = c_3 \quad \text{—— (v)}$$

Hence the solution of the given eqn is

$$F\left(\frac{xy}{z}, \frac{x-y}{z}\right) = 0 \quad \text{Ans}$$

Theorem: (Without Proof).

If $u_i(x_1, x_2, \cdots x_n, z) = c_i$ $(i = 1, 2, \cdots n)$ are the independent solution of the equations

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \cdots\cdots = \frac{dx_n}{P_n} = \frac{dz}{R}$$

then the relation

$\phi(u_1, u_2, \cdots, u_n) = 0$, in which the function ϕ is arbitrary, is a general solution of the linear partial diff. eqn

$$P_1 \frac{\partial z}{\partial x_1} + P_2 \frac{\partial z}{\partial x_2} + \cdots\cdots + P_n \frac{\partial z}{\partial x_n} = R$$

Example:

If u is a function of x, y, and z which satisfies the partial differential eqn

$$(y-z)\frac{\partial u}{\partial x} + (z-x)\frac{\partial u}{\partial y} + (x-y)\frac{\partial u}{\partial z} = 0$$

show that u contains x, y, z only in the combination $x+y+z$ and $x^2+y^2+z^2$

Soln:

Here the given partial diff. eqn is

$$(y-z)\frac{\partial u}{\partial x} + (z-x)\frac{\partial u}{\partial y} + (x-y)\frac{\partial u}{\partial z} = 0 \quad \text{--- (1)}$$

its auxilliary eqns are

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} = \frac{du}{0} \text{ —— (ii)}$$

from (ii)

$du = 0$

On intigrating we get

$u = c_1$ —— (iii)

again from (ii)

$dx + dy + dz = 0$

On integrating we get

$x + y + z = c_2$ —— (iv)

again from (ii)

$x\,dx + y\,dy + z\,dz = 0$

On integrating we get

$x^2 + y^2 + z^2 = c_3$

Hence the general solution of given diff eq^n is in the form of

$u = f(c_2, c_3)$

$\therefore u = f(x+y+z,\ x^2+y^2+z^2)$ Ans

2.5. Integral surface passing through a given curve

Let us suppose $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ be two ~~solutions~~ ~~general~~ particular solutions of ~~a linear partial diff eqn~~ auxilliary eqn

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{--- (i)}$$

Then any solution of the corresponding linear eqn ($Pp + Qq = R$) is in the form of

$$F(u, v) = 0 \quad \text{--- (ii)}$$

$\Rightarrow F(c_1, c_2) = 0$ (iii), where c_1 and c_2 are integrating constant

If we wish to find the integral surface which passes through the curve C whose parametric eqn's are

$x = x(t)$; $y = y(t)$, $z = z(t)$

where t is a parameter, then the particular solution (i) must be such that

$$u\{x(t), y(t), z(t)\} = c_1$$

and $v\{x(t), y(t), z(t)\} = c_2$

We therefore have two eqn's from which we ~~many~~ may eliminate the single variable t to obtain the relation of type

$$F(c_1, c_2) = 0$$

Now, the required solution is $F(u, v) = 0$

T.U.S

Example:-

Find the integral surface of the linear partial diff. equation

$$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$$

which contains the straight line

$$x+y=0, \quad z=1$$

Solⁿ

Here the given partial diff eqⁿ is

$$x(y^2+z)p - y.(x^2+z)q = (x^2-y^2)z \quad \text{—— (i)}$$

eqⁿ (i) is first order partial linear diff eqⁿ.
So its auxilliary equation is

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z} \quad \text{—— (ii)}$$

Now, from (ii)

$$\frac{x\,dx - z\,dz}{x^2y^2 + \cancel{x^2z} - \cancel{z^2x} + z^2y^2} = \cancel{dy}$$

$$\frac{yz\,dx + xz\,dy + xy\,dz}{\cancel{xy^3z} + \cancel{xyz^2} - \cancel{x^3yz} - \cancel{xyz^2} + \cancel{x^3yz} - \cancel{xy^3z}} = \frac{dx}{x(y^2+z)}$$

or $$\frac{yz\,dx + xz\,dy + xy\,dz}{0} = \frac{dx}{x(y^2+z)}$$

or $yz\,dx + xz\,dy + xy\,dz = 0$

or $d(xyz) = 0$

On integrating we get

$$\int d(xyz) = \int 0$$

$$xyz = c_1 \quad \text{—— (iii)}$$

Again from (ii)

$$\frac{x\,dx + y\,dy}{x^2y + x^2z - x^2y - y^2z} = \frac{dz}{(x^2-y^2)z}$$

or $\dfrac{x\,dx + y\,dy}{z(x^2-y^2)} = \dfrac{dz}{(x^2-y^2)z}$

or $x\,dx + y\,dy = dz$

On integrating we get

$$\frac{x^2}{2} + \frac{y^2}{2} = z + c$$

or $x^2 + y^2 - 2z = 2c$

$\therefore \quad x^2 + y^2 - 2z = c_2 \quad \text{—— (iv)}$

we have the given

$x + y = 0 \quad z = 1$

we put $x = t \quad \therefore y = -t$

Substituting the value of x and y in (iii) and (iv) we get
in (iii)

$$tx - t \times 1 = c_1$$

$$\therefore c_1 = -t^2 \quad \text{--- (v)}$$

and in (iv)

$$t^2 + t^2 - 2 = c_2$$

or, $\therefore c_2 = 2t^2 - 2 \quad \text{--- (vi)}$

from (v) and (vi)

$$c_2 = -2c_1^2 - 2$$

~~or $2c_1^2 - c_2 - 2 = 0$~~

or $2c_1 + c_2 + 2 = 0 \quad \text{--- (vii)}$

Again from (iii) & (iv) and (vii) we get

$2xyz + x^2 + y^2 - 2z + 2 = 0$ This is the required integral surface.

Integral surface passing through a given curve

PROBLEM

1) Find the equation of the integral surface of the diff. eqⁿ

$$2y(z-3)p + (2x-z)q = y(2x-3)$$

which passes through the circle $z=0,\ x^2+y^2=2x$

Solⁿ

Here the given diff eqⁿ is

$$2y(z-3)p + (2x-z)q = y(2x-3) \quad \text{---(i)}$$

This is first order linear partial diff. eqⁿ. So its auxiliary eqⁿs are

$$\frac{dx}{2y(z-3)} = \frac{dy}{(2x-z)} = \frac{dz}{y(2x-3)} \quad \text{---(ii)}$$

from (ii)

$$\frac{x\,dx - z\,dz}{2xyz - 6xy - 2xyz + 3yz} = \frac{dy}{2x-z}$$

or $$\frac{x\,dx - z\,dz}{-3y(2x-z)} = \frac{dy}{(2x-z)}$$

or $x\,dx - z\,dz = -3y\,dy$

On integrating we get

$$\frac{x^2}{2} - \frac{z^2}{2} + \frac{3y^2}{2} = c_0$$

or $x^2 - z^2 + 3y^2 = 2c_0$

$\therefore x^2 - z^2 + 3y^2 = c_1$ ---(iii)

Again from (ii)

$$\frac{dx - 2dz}{2yz - 6y - 4xy + 6y} = \frac{dy}{(2x-z)}$$

or $$\frac{dx - 2dz}{-2y(2x-z)} = \frac{dy}{(2x-z)}$$

or $dx - 2dz = -2y\,dy$

On integrating we get

$x - 2z + y^2 = c_2$ —— (iv)

Now we have given circle is $z=0,\ x^2+y^2=2x$

we put $x=t$ $\therefore y^2 = 2t-t^2$

Now, substituting the values of x and y in (iii) and (iv)

in (iii)

$$t^2 - 0^2 + 3(2t-t^2) = c_1$$

or $t^2 + 6t - 3t^2 = c_1$

$\therefore c_1 = 6t - 2t^2 \quad \therefore c_1 = 2(3t - t^2)$ —— (v)

in (iv)

$$t - 2\times 0 + (2t - t^2) = c_2$$

$$t + 2t - t^2 = c_2$$

$\therefore c_2 = 3t - t^2$ —— (vi)

Again from (v) and (vi)

$$c_2 = \frac{c_1}{2}$$

$\therefore 2c_2 - c_1 = 0$ —— (vii)

Again from (iii) (iv) and (vii)

$$2x - 4z + 2y^2 - x^2 + z^2 - 3y^2 = 0$$

or, $x^2 + y^2 - z^2 - 2x + 4z = 0$

This is the required integral surface

2) Find the integral surface of partial diff. eqn

$$(2xy - 1)p + (z - 2x^2)q = 2(x - yz)$$

and also the particular integral which passes through the line $x = 1,\ y = 0$.

Soln

Here the given partial diff. eqn is

$$(2xy - 1)p + (z - 2x^2)q = 2(x - yz) \text{ ---(i)}$$

which is 1st order Linear partial differential So auxiliary eqns are

$$\frac{dx}{(2xy - 1)} = \frac{dy}{(z - 2x^2)} = \frac{dz}{2(x - yz)} \text{ ---(ii)}$$

Now from (ii)

$$\frac{x\,dx + y\,dy}{2x^2y - x + yz - 2x^2y} = \frac{dz}{2(x - yz)}$$

or $$\frac{x\,dx + y\,dy}{-(x - yz)} = \frac{dz}{2(x - yz)}$$

or $$x\,dx + y\,dy = -\frac{dz}{2}$$

on integrating we get

$$\frac{x^2}{2}+\frac{y^2}{2}=-\frac{z^2}{2}+c_1$$

or $x^2+y^2+z=c_1$ —— (iii)

again from (ii)

$$\frac{2z\,dx+x\,dz}{4xy-2x+2x-2yz}=\frac{dy}{z-2x^2} \qquad \frac{z\,dx+x\,dz}{2xyz-z+2x^2-2xyz}=\frac{dy}{(z-2x^2)}$$

or $\frac{2x\,dx+dz}{-2y(z-2x^2)}=\frac{dy}{(z-2x^2)}$ $\qquad \frac{z\,dx+x\,dz}{-(z-2x^2)}=\frac{dy}{(z-2x^2)}$

or $2x\,dx+dz=-2y\,dy$. $\qquad$ or $z\,dx+x\,dz=-dy$

On integrating we get $\qquad$ or $d(zx)=-dy$

x^2+y^2 $\qquad$ On integrating we get

$xz+y=c_2$ —— (iv)

we have given the line

$x=1, y=0$ So we put $z=1$ in (iii) and (iv)

in (iii)

$1+0+1=c_1$

$\therefore c_1=1+1$ —— (v)

In (iv)

$1\times1+0=c_2$

$c_2=1$ —— (vi)

from (v) and (vi)

$c_1=1+c_2$

i.e $c_1 - c_2 = 1$ —— (vii)

Now, from (iii) (iv) and (vii) we have

$x^2 + y^2 + z - xz + y = 1$

$\therefore x^2 + y^2 - xz + y + z - 1 = 0$ is the required solution.

3) Find the integral surface of the equation $(x-y)y^2 p + (y-x)x^2 q = (x^2+y^2)z$ through the curve $xz = a^3,\ y = 0$.

Solⁿ

Here the given eqⁿ is

$(x-y)y^2 p + (y-x)x^2 q = (x^2+y^2)z$ —— (i)

which is first order linear partial diff eqⁿ. So its auxilliary eqⁿs are

$$\frac{dx}{(x-y)y^2} = \frac{dy}{(y-x)x^2} = \frac{dz}{(x^2+y^2)z} \text{ —— (ii)}$$

Now, from (ii)

$$\frac{x\,dx + y\,dy}{x^3y^2 - xy^3 + x^2y^2 + x^3y} = \frac{dz}{(x^2+y^2)z}$$

or $$\frac{x\,dx - y\,dy}{xy(y^2-x^2)} = \frac{dz}{(x^2+y^2)z}$$

$$\frac{dx}{(x-y)y^2} = \frac{dy}{(y-x)x^2}$$

or $$\frac{dx}{-(y-x)y^2} = \frac{dy}{(y-x)x^2}$$

or $x^2\,dx = -y^2\,dy$

On integrating we get

$x^3 + y^3 = c_1$ —— (iii)

again from (ii)

$$\frac{\frac{dx}{(x-y)} + \frac{dy}{(y-x)}}{(x^2+y^2)} = \frac{dz}{(x^2+y^2)z}$$

or $\frac{dx - dy}{(x-y)} = \frac{dz}{z}$

or $\frac{d(x-y)}{(x-y)} = \frac{dz}{z}$

On integrating we get

$\log(x-y) = \log z + \log c_2$

or $\log\left(\frac{x-y}{z}\right) = \log c_2$

$\therefore \frac{x-y}{z} = c_2$ —— (iv)

And we have the given curve

$xz = a^3$, ~~and~~ $y = 0$

So we put $x = t^3$ and

$\therefore z = \frac{a^3}{t^3}$ in (iii) and (iv)

In (iii)

$\therefore (t^3)^2 + 0^3 = c_1^6$

$\therefore c_1 = t$ —— (v)

and in (iv) $\frac{t^3 - 0}{\frac{a^3}{t^3}} = c_2$

$\therefore C_2 = \frac{1}{a^3}$ ——— (vi)

Now, from (v) and (vi)

or $\frac{C_1}{C_2} = a^3$

$\therefore C_1 = a^3 C_2$ ——— (vii)

Now, from (iii), (iv) and (vii) we have

$$x^3 + y^3 = a^3\left(\frac{x-y}{z}\right)$$

or $x^3 z + y^3 z - a^3 x - a^3 y = 0$

$\therefore x(x^2 z - a^3) + y(y^2 z - a^3) = 0$ is the required soln

1) Find the general solution of the eq^n

$$2x(y+z^2)p + y(2y+z^2)q = z^3$$

Sol^n

Here the given diff. eq^n is

$$2x(y+z^2)p + y(2y+z^2)q = z^3 \quad \text{---- (i)}$$

The integral surface of eq^n (i) are generated by the integral curves of equations

$$\frac{dx}{2x(y+z^2)} = \frac{dy}{y(2y+z^2)} = \frac{dz}{z^3} \quad \text{---- (ii)}$$

from (ii)

$$\frac{y\,dx - 2x\,dy}{\cancel{2xy^2} + \cancel{2xyz^2} - 4xy^2 - \cancel{2xyz^2}} = \frac{dz}{z^3}$$

or $\dfrac{y\,dx - 2x\,dy = 0}{-4xy^2} = \dfrac{dz}{z^3}$

or $\dfrac{dx}{2x} = \dfrac{dy}{y}$

On integrating we get

$\therefore \frac{1}{2}\log x = \log y + \log c_1$

or $\log\left(\frac{\sqrt{x}}{y}\right) = \log c_1$

$\therefore \frac{\sqrt{x}}{y} = c_1 \quad \text{---- (iii)}$

again from (ii) $\frac{dx}{x-y} = \frac{dy}{y-x-z} = \frac{dz}{z}$

$\frac{z\,dy - y\,dz}{2y^2z + \cancel{yz^3} - \cancel{yz^3}} = \frac{dz}{z^3}$

$\frac{dx + dy}{x - \cancel{y} + \cancel{y} - \cancel{x} - z} = \frac{-dz}{z}$

or $\frac{z\,dy - y\,dz}{2y^2\cancel{z}} = \frac{dz}{z^{\cancel{3}2}}$

$dx + dy = -dz$

$x + y + z = c_1$

or $\frac{1}{2}\,d\left(\frac{y}{z}\right) =$

or $-\frac{1}{2}\,d\left(\frac{z}{y}\right) = \frac{dz}{z^2}$

$\frac{dz}{x-y} = \frac{dy}{y-x}$

$\frac{dy + dz}{y - x - \cancel{z} + \cancel{z}} = \frac{dx}{x-y}$

On integrating we get

$\frac{dx + dz}{x - y + z} = \frac{dy}{-(x-y+z)}$

or $-\frac{1}{2}\frac{z}{y} = \frac{-1}{z} + c_2$

or $-\frac{1}{2}\frac{z}{y} + \frac{1}{z} = c_2$

or $c_2 = \frac{2y - z^2}{2yz}$

$\frac{\partial(x-y)}{\partial x} + \frac{\partial(y-x-z)}{\partial y} = z$

$(x-y)\frac{dz}{x-}$

$\frac{1}{}+\frac{1}{z^2} = z^2$

$z\frac{dx}{(x-y)}$

$\frac{dx}{x-y}$

The req. orthogonal surface is a solution of

$$p\frac{\partial f}{\partial x}+q\frac{\partial f}{\partial y}=\frac{\partial f}{\partial z} \text{ or } p\frac{3z}{3z+1}+q\frac{z}{3z+1}=\frac{x+y}{(3z+1)^2}$$

$$\therefore z(3z+1)p+z(3z+1)q=x+y \longrightarrow (3)$$

The Lagrange's auxiliary eq^n corresponding to the given eq^n are

$$\frac{dx}{z(3z+1)}=\frac{dy}{z(3z+1)}=\frac{dz}{x+y} \longrightarrow (4)$$

From first two fractions of (4), by integration, we get $x-y=c_1$, c_1 being an arbitrary constant $\longrightarrow (5)$

Taking $x, y, -z(3z+1)$ as multipliers, each fraction of (4) $=\frac{x\,dx+y\,dy-z(3z+1)dz}{0}$

and so $x^2+y^2-2z^3-z^2=c_2$, c_2 being an arbitrary constant $\longrightarrow (6)$

Hence any surface which is orthogonal to the given surfaces (1) has eq^n of the form

$x^2+y^2-2z^3-z^2=f(x-y)$, f being any arbitrary function. $\longrightarrow (7)$

Also,

for the particular surface passing through circle $x^2+y^2=1,\ z=1$

$$\therefore f(x-y)=1-2\times1^3-1^2=-2$$

$\therefore$ The req. surface is

$$x^2+y^2-2z^3-z^2=-2$$

$$\therefore x^2+y^2=2z^3+z^2-2$$

2.6 Surface orthogonal to given system of surfaces:

Suppose we are given a one-parameter family of surfaces characterized by the equation

$$f(x, y, z) = c \quad \text{--- (i)}$$

and we wish to find a system of surfaces which cut each of these given surfaces at right angle.

The normal at the point (x, y, z), to the surface of the system (i) which passes through that point is the direction given by the direction ratios

$$(P, Q, R) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \quad \text{--- (ii)}$$

If the surface $z = \phi(x, y)$ --- (iii) cuts each surface of the given system orthogonally, then its normal at the point (x, y, z) which is in the direction

$$\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$$

is perpendicular to the direction (P, Q, R) of the normal to the surface of the set (i).

We therefore have the linear partial diff eq^n

$$P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R \quad \text{--- (iv)}$$

Now from (ii) $\frac{\partial f}{\partial x}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial z}{\partial y} = \frac{\partial f}{\partial z}$

Conversely,

any solution of the linear partial diff eq^n (iv) is orthogonal to every surface of the system characteri

(5)

by equation (i). The linear equation A(iv) is therefore the general partial diff. equation determining the surfaces orthogonal to members of the system.
Therefore the surfaces orthogonal to the system (i) are the surfaces generated by the integral curves of the equation

$$\frac{dx}{\partial f/\partial x} = \frac{dy}{\partial f/\partial y} = \frac{dz}{\partial f/\partial z}$$

Example:

Find the surface which intersect the surface of the $z(x+y) = c(3z+1)$ orthogonally and passes through the circle $x^2+y^2=1$ and $z=1$.

Soln

Here the given family of surface is

$$z(x+y) = c(3z+1)$$

$$\therefore \frac{z(x+y)}{(3z+1)} = c = f(x,y,z) \quad \text{---(i)}$$

we know that,
the surface orthogonal to the system (i) are the surface generated by the integral curves of the eqn

$$\frac{dx}{\partial f/\partial x} = \frac{dy}{\partial f/\partial y} = \frac{dz}{\partial f/\partial z}$$

From (i)

$$\frac{dx}{z/(3z+1)} = \frac{dy}{z/(3z+1)} = \frac{dz}{(x+y)/(3z+1)^2} \quad \text{---(ii)}$$

Now, from (ii)

$$\frac{dx}{z/(3z+1)} = \frac{dy}{z/(3z+1)}$$

or $dx = dy$

On integrating we get

$$x - y = c_1 \quad \text{--- (iii)}$$

again from (ii)

$$\frac{dx+dy}{\frac{2z}{(3z+1)}} = \frac{dz}{(x+y)/(3z+1)^2}$$

or $\frac{(3z+1)(dx+dy)}{2z} = \frac{dz}{(x+y)}$

or $\frac{(3z+1)}{2z}(dx+dy) = \frac{(3z+1)^2\, dz}{(x+y)}$

or $(x+y)\{dx+dy\} = 2z(3z+1)\, dz$

or $(x+y)\, d(x+y) = 2z(3z+1)\, dz$

On integrating we get

$$\frac{(x+y)^2}{2} = 2z^3 + z^2 + c_0$$

or $x^2+y^2+2xy-2z^3-2z^3 = c_2$

Soln:- ~~The given system of surfaces is~~

$$z(x+y) = c(3z+1)$$

$$\therefore \frac{z(x+y)}{3z+1} = c = f(x,y,z) \longrightarrow (1)$$

Now, $\frac{\partial f}{\partial x} = \frac{z}{3z+1}$, $\frac{\partial f}{\partial y} = \frac{z}{3z+1}$ and $\frac{\partial f}{\partial z} = (x+y)\frac{(3z+1)-3z}{(3z+1)^2} = \frac{x+y}{(3z+1)^2} \rightarrow (2)$

Thus any surface which is orthogonal to the given surface has eqn

$$x^2+y^2-2z^3-z^2 = f(x-y)$$

also,

for the particular surface passing through circle

$$x^2+y^2=1, \quad z=1$$

$$\therefore f(x-y) = 1-2\times1^3-1^2 = -2$$

$\therefore$ The required surface is

$$x^2+y^2-2z^3-z^2=-2$$

$$\therefore x^2+y^2 = 2z^3+z^2-2 \quad \text{Ans}$$

Surfaces orthogonal to given system of surfaces

Problems:

1) Find the surface which is orthogonal to the one-parameter system

$$z = cxy(x^2+y^2)$$

and which passes through the hyperbola $x^2-y^2=a^2, z=0$

soln

Here the given surface is

$$z = cxy(x^2+y^2)$$

$$\therefore \frac{z}{xy(x^2+y^2)} = c = f(x,y,z) \quad \text{--- (1)}$$

we know that the surface orthogonal to the system (i) are the surface generated by the integral curve of the eq^n

$$\frac{dx}{\partial f/\partial x} = \frac{dy}{\partial f/\partial y} = \frac{dz}{\partial f/\partial z} \quad \text{---- (ii)}$$

Now, from (i) and (ii)

$$\frac{dx}{\quad}$$

:Lo Charpit's Methods

A method of solving the partial diff. eq^n

$$f(x, y, z, p, q) = 0 \quad \text{—— (i)}$$

is based on the considerations of the last section. The fundamental idea in Charpit's method is the introduction of a second partial differential equation of first order

$$g(x, y, z, p, q, a) = 0 \quad \text{—— (ii)}$$

which contains arbitary constant a and which is such that

$p = p(x, y, z, a)$, $q = q(x, y, z, a)$ {by solving (i) & (ii)}

and ~~dz =~~ The equation

$$dz = p(x, y, z, a)\,dx + q(x, y, z, a)\,dy \text{ is integrable.} \quad \text{(iii)}$$

When such function g has been found, the solution of equation (iii)

$$F(x, y, z, a, b) = 0 \quad \text{—— (iv)}$$ containing two arbitary constant a, b; will be the solution of eq^n(i).

Note:

The auxilliary equation of the differential eq^n(i) is

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

Example:

Find the complete integral of the equation.

$$p^2x + q^2y = z$$

Sol^n

Here the given diff eq^n can be written as

$$f = p^2x + q^2y - z = 0 \quad \text{---} (i)$$

Now, its auxilliary eq^{n}s are

$$\frac{dx}{2px} = \frac{dy}{2qy} = \frac{dz}{2p^2x + 2q^2y} = \frac{dp}{-(p^2-p)} = \frac{dq}{-(q^2-q)}$$

or $$\frac{dx}{2px} = \frac{dy}{2qy} = \frac{dz}{2(p^2x+q^2y)} = \frac{dp}{p-p^2} = \frac{dq}{q-q^2} \quad \text{---} (ii)$$

Now, from (ii)

$$\frac{p^2dx + 2pxdp}{2p^3\cancel{x} + 2p^2x - 2p^3\cancel{x}} = \frac{q^2dy + 2qydq}{2q^3\cancel{y} + 2q^2y - 2q^3\cancel{y}}$$

or $$\frac{p^2dx + 2pxdp}{2p^2x} = \frac{q^2dy + 2qydq}{2q^2y}$$

On integrating we get

$$\log(p^2x) = \log(q^2y) + \log a$$

or $$p^2x = aq^2y \quad \text{---} (iii)$$

also, we have,

$$p^2x + q^2y = z$$

or $aq^2y + q^2y = z$

or $q^2 = \frac{z}{(a+1)y}$

$\therefore q = \sqrt{\frac{z}{(1+a)y}}$ ——— (iv)

and $p^2 = \frac{ay}{x}\left[\frac{z}{(1+a)y}\right]$

or $p = \frac{ayz}{xy(1+a)}$

$\therefore p = \sqrt{\frac{az}{x(1+a)}}$

again we have,

$$dz = pdx + qdy$$

$$dz = \sqrt{\frac{az}{x(1+a)}}\,dx + \sqrt{\frac{z}{y(1+a)}}\,dy$$

or $dz = \sqrt{\frac{a}{1+a}}\sqrt{\frac{z}{x}}\,dx + \sqrt{\frac{1}{1+a}}\sqrt{\frac{z}{y}}\,dy$

or $\dfrac{dz}{} = \dfrac{\sqrt{z}}{\sqrt{1+a}} \left\{ \sqrt{a}\dfrac{dx}{\sqrt{x}} + \dfrac{dy}{\sqrt{y}} \right\}$

or $\sqrt{1+a}\dfrac{dz}{\sqrt{z}} = \sqrt{a}\dfrac{dx}{\sqrt{x}} + \dfrac{dy}{\sqrt{y}}$

On integrating we get

$\therefore \sqrt{1+a}\dfrac{1}{2\sqrt{z}} = \sqrt{a}\left(\dfrac{1}{2\sqrt{x}} + \dfrac{1}{2\sqrt{y}}\right) + \dfrac{b}{2}$

$\therefore \sqrt{1+a}\, 2\sqrt{z} = \sqrt{a}\, 2\sqrt{x} + 2\sqrt{y} + 2b$

or $\sqrt{(1+a)z} = \sqrt{ax} + \sqrt{y} + b$

is the required complete Integral.

Charpit's Method

Problem:

1) Find the complete integrals of the equation

a) $(p^2+q^2)y = qz$

Here the given diff. eq^n can be written as

$$f = (p^2+q^2)y - qz = 0 \quad \text{---(i)}$$

$\therefore$ The auxilliary eq^n of (i) are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p+qf_q} = \frac{dp}{-(f_x+pf_z)} = \frac{dq}{-(f_y+qf_z)}$$

$$\therefore \frac{dx}{2py} = \frac{dy}{2qy-z} = \frac{dz}{2p^2y+2q^2y-qz} = \frac{dp}{-(-pq)} = \frac{dq}{-(p^2+q^2-q^2)}$$

$$\text{or } \frac{dx}{2py} = \frac{dy}{2qy-z} = \frac{dz}{2p^2y+2q^2y-qz} = \frac{dp}{pq} = \frac{dq}{-p^2} \quad \text{---(ii)}$$

Now, from (ii)

$$\frac{dp}{pq} = \frac{dq}{-p^2}$$

or $p\,dp + q\,dq = 0$

On integrating we get

$$p^2+q^2 = a^2 \quad \text{---(iii)}$$

Now, from (i) and (iii)

$$a^2y - qz = 0$$

$$\therefore q = \frac{a^2y}{z} \quad \text{---(iv)}$$

Now, from (iii) and (iv)

$$\cancel{p^2 + \frac{a^4y^2}{z^2} \cdot y = \frac{a^2y}{z} \times z}$$

$$p^2 + \frac{a^4y^2}{z^2} = a^2$$

or $p^2 = a^2 - \dfrac{a^4y^2}{z^2}$

or $p^2 = \dfrac{a^2z^2 - a^4y^2}{z^2}$

or $p^2 = \dfrac{a^2}{z^2}(z^2 - a^2y^2)$

or $p = \sqrt{\dfrac{a^2}{z^2} z^2 - a^2y^2}$

$\therefore p = \dfrac{a}{z}\sqrt{z^2 - a^2y^2}$ — (v)

Now, we have,

$dz = p\,dx + q\,dy$

or $dz = \dfrac{a}{z}\sqrt{z^2 - a^2y^2}\,dx + \dfrac{a^2y}{z}\,dy$

or $z\,dz = a\sqrt{z^2 - a^2y^2}\,dx + a^2y\,dy$

or $a\sqrt{z^2 - a^2y^2}\,dx = z\,dz - a^2y\,dy$

or $a\,dx = \dfrac{z\,dz - a^2y\,dy}{\sqrt{z^2 - a^2y^2}}$

or $2a\,dx = \dfrac{2z\,dz - 2a^2 y\,dy}{\sqrt{z^2 - a^2 y}}$

On integrating we get

$2ax + b = 2\sqrt{z^2 - a^2 y}$

$\therefore 2\sqrt{z^2 - a^2 y} = 2ax + b$ is the complete integral of the eqn (i).

b) $p = (z + qy)^2$

soln

Here the given diff. eqn can be written as

$f = p - (z + qy)^2 = 0$ ——— (i)

Now, the auxilliary eqn of eqn (i) are

$\dfrac{dx}{f_p} = \dfrac{dy}{f_q} = \dfrac{dz}{pf_p + qf_q} = \dfrac{dp}{-(f_x + pf_z)} = \dfrac{dq}{-(f_y + qf_z)}$

or $\dfrac{dx}{1} = \dfrac{dy}{-2y(z + qy)} = \dfrac{dz}{p - 2qy(z+qy)} = \dfrac{dp}{-(-2p(z+qy))} = \dfrac{dq}{-[-2q(z+qy) - qy(z+qy)]}$ ——— (ii)

Now, from (ii)

$\dfrac{dy}{-2y} = \dfrac{dp}{2p}$

or $\dfrac{dy}{y} + \dfrac{dp}{p} = 0$

On integrating we get

$\log y + \log p = \log a$

or $\log(yp) = \log a$

$\therefore yp = a$ ——— (iii)

$\therefore p = \frac{a}{y}$

from (iii) and (i)

$$\frac{a}{y} = (z+qy)^2$$

or $z + qy = \sqrt{\frac{a}{y}}$

or $q = \frac{\sqrt{a/y} - z}{y}$

we have,

$dz = p\,dx + q\,dy$

or $dz = \frac{a}{y}dx + \frac{\sqrt{a/y} - z}{y}dy$

or $dz = \frac{a}{y}dx + \frac{1}{y}\sqrt{a/y}\,dy - \frac{z}{y}dy$

or $dz + \frac{z}{y}dy = \frac{a}{y}dx + \frac{1}{y}\sqrt{\frac{a}{y}}\,dy$

or $y\,dz + z\,dy = a\,dx + \sqrt{\frac{a}{y}}\,dy$

or $d(yz) = a\,dx + \sqrt{a}\frac{dy}{\sqrt{y}}$

On integrating we get

$yz = ax + \sqrt{a}\, 2\sqrt{y} + b$ —— (iv) which is a required solution.

) $2(z + xp + yq) = yp^2$

Sol

Here the given diff. eq can be written as

$$f = 2(z + xp + yq) - yp^2 = 0 \quad \text{—— (i)}$$

Now the auxilliary eqns for (i) are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_p + pf_z)} = \frac{dq}{-(f_y + qf_z)} \quad \text{—— (ii)}$$

~~from (ii)~~

or $$\frac{dx}{2x - 2py} = \frac{dy}{2y} = \frac{dz}{p(2x - 2py) + 2qy} = \frac{dp}{-(2p + 2p)} = \frac{dq}{-(2q - q^2) + 2}$$ —— (ii)

Now, from (ii)

$$\frac{dy}{2y} = \frac{dp}{-4p}$$

or $$\frac{dy}{y} = -\frac{1}{2}\frac{dp}{p}$$

On integrating we get

$$\log y + \frac{1}{2}\log p = \log a$$

$$y p^{1/2} = a$$

$$\therefore y\sqrt{p} = a$$

$$\text{or } \sqrt{p} = a/y$$

$$\therefore p = \frac{a^2}{y^2} \quad \text{—— (iii)}$$

from (i) and (iii)

$$2\left(z+x\cdot\frac{a^2}{y^2}+yq\right)=y\left(a^2/y^2\right)^2$$

or $$z+\frac{xa^2}{y^2}+yq=\frac{a^4}{2y^3}$$

or $$yq=\frac{a^4}{2y^3}-z-\frac{xa^2}{y^2}$$

$$\therefore q=\frac{a^4}{2y^4}-\frac{z}{y}-\frac{xa^2}{y^3}$$

Now, we have

$$dz=p\,dx+q\,dy$$

or $$dz=\frac{a^2}{y^2}dx+\left\{\frac{a^4}{2y^4}-\frac{z}{y}-\frac{xa^2}{y^3}\right\}dy$$

or $$dz=\frac{a^2}{y^2}dx+\frac{a^4}{2y^4}dy-\frac{z}{y}dy-\frac{xa^2}{y^3}dy$$

or $$dz+\frac{z}{y}dy=\frac{a^2}{y^2}dx+\frac{a^4}{2y^4}dy-\frac{xa^2}{y^3}dy$$

or $$y\,dz+z\,dy=\frac{a^2}{y}dx+\frac{a^4dy}{2y^3}-x\frac{a^2}{y^2}dy$$

or $$d(yz)=\frac{a^4dy}{2y^3}+a^2\left\{\frac{y\,dx-x\,dy}{y^2}\right\}$$

or $$dyz=\frac{a^4}{2y^3}dy+a^2d\left(\frac{x}{y}\right)$$

On integrating we get

$$yz = -\frac{a^4}{2} \cdot \frac{1}{2yz} + \frac{a^2 x}{y} + b$$

$\therefore$ $yz = \frac{a^2 x}{y} - \frac{a^4}{4yz} + b$ is the required sol^n

) $2(y+zq) = q(xp+yq)$

Sol^n

Here the given eq^n can be written as

$$f = 2(y+zq) - q(xp+yq) = 0 \quad \text{—— (i)}$$

Its auxilliary eq^{n}s are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p+qf_q} = \frac{dp}{-(f_x+pf_z)} = \frac{dq}{-(f_y+qf_z)}$$

or $$\frac{dx}{-qx} = \frac{dy}{2z-xp-2yq} = \frac{dz}{-pqx+2zq-xpq-2yq^2} = \frac{dp}{-(-pq+2pq)} = \frac{dq}{-(2-q^2+2q}$$ —— (ii)

Now, from (ii)

$$\frac{dx}{-qx} = \frac{dq}{-(2-q^2+2q^2)}$$

or $$\frac{dx}{qx} = \frac{dq}{2+q^2}$$

or $$\frac{dx}{x} = \frac{q\,dq}{q^2+2}$$

On integrating we get

$$\log x - \frac{1}{2}\log(q^2+2) = \log a$$

or $\log\left\{\frac{x}{\sqrt{q^2+2}}\right\} = \log a$

or $\frac{x}{\sqrt{q^2+2}} = a$

or $\sqrt{q^2+2} = \frac{x}{a}$

or $q^2+2 = \frac{x^2}{a^2}$

or $q = \sqrt{\frac{x^2-2a^2}{a^2}}$

$\therefore q = \frac{\sqrt{x^2-2a^2}}{a}$ ——— (iii)

from (i) and (iii)

$2(y+zq) = q(xp+yq)$

or $\frac{2}{q}(y+zq) - yq = xp$

or $xp = \frac{2y}{\frac{\sqrt{x^2-2a^2}}{a}} + 2z - y\frac{\sqrt{x^2-2a^2}}{a}$

or $xp = \frac{2ya}{\sqrt{x^2-2a^2}} + 2z - \frac{y}{a}\sqrt{x^2-2a^2}$

$\therefore p = \frac{2ay}{x\sqrt{x^2-2a^2}} + \frac{2z}{x} - \frac{y}{ax}\sqrt{x^2-2a^2}$

We have

$dz = p\,dx + q\,dy$

or. $dz = \left\{\frac{2ay}{x\sqrt{x^2-2a^2}} + \frac{2z}{x} - \frac{y}{ax}\sqrt{x^2-2a^2}\right\}dx + \left\{\frac{\sqrt{x^2-2a^2}}{a}\right\}dy$

or. $dz - \frac{2z}{x}dx = \frac{2ay}{x\sqrt{x^2-2a^2}}dx - \frac{y}{ax}\sqrt{x^2-2a^2}\,dx + \frac{1}{a}\sqrt{x^2-2a^2}\,dy$

or $x\,dz - 2z\,dx = \frac{2ay}{\sqrt{x^2-2a^2}}dx - \frac{y}{a}\sqrt{x^2-2a^2}\,dx + \frac{x}{a}\sqrt{x^2-2a^2}\,dy$

or $x\sqrt{x^2-2a^2}\,dz - 2z\sqrt{x^2-2a^2}\,dx = 2ay\,dx - y$

c) $2(z+xp+yq)=yp^2$

Here the given differential equations ~~are~~ can be written

$$f = 2(z+xp+yq) - yp^2 = 0 \quad \text{---(i)}$$

Charpit's auxilliary eq^ns are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p+qf_q} = \frac{dp}{-(f_x+pf_z)} = \frac{dq}{-(f_y+qf_z)}$$

$$\text{or } \frac{dx}{2x-2yp} = \frac{dy}{2y} = \frac{dz}{2xp-2yp^2+2yq} = \frac{dp}{-(2p+2p)} = \frac{dq}{-(2q-p^2+2q)}$$

$$\text{or } \frac{dx}{2x-2yp} = \frac{dy}{2y} = \frac{dz}{2xp-2yp^2+2yq} = \frac{dp}{-4p} = \frac{dq}{p^2-4q} \quad \text{---(ii)}$$

Now, from (ii)

$$\frac{dy}{2y} = \frac{dp}{-4p}$$

$$\text{or } \frac{dy}{y} = \frac{dp}{-2p}$$

On integrating we get

$$\log y = -\frac{1}{2}\log p + \log a$$

$$\text{or } \log y + \log\sqrt{p} = \log a$$

$$\text{or } \log(y\sqrt{p}) = \log a$$

$$a = y\sqrt{p}$$

$$\therefore p = \frac{a^2}{y^2} \quad \text{---(iii)}$$

Now,

$$2\left(z + x\left(\frac{a^2}{y^2}\right) + yq\right) - y\,\frac{a^4}{y^4} = 0$$

$$\text{or } 2z + 2x\frac{a^2}{y^2} + 2yq = \frac{a^4}{y^3}$$

$$\text{or } q = \frac{a^4}{2y^4} - \frac{z}{y} - \frac{2xa^2}{y^3}$$

we have,

$$dz = p\,dx + q\,dy$$

$$\text{or } dz = \frac{a^2}{y^2}dx + \left(\frac{a^4}{2y^4} - \frac{z}{y} - \frac{2xa^2}{y^3}\right)dy$$

$$\text{or } dz = \frac{a^2}{y^2}dx + \frac{a^4}{2y^4}dy - \frac{z}{y}dy - \frac{xa^2}{y^3}dy$$

$$\text{or } dz + \frac{z}{y}dy = \frac{a^2}{y^2}dx + \frac{a^4}{2y^4}dy - \frac{xa^2}{y^3}dy$$

$$\text{or } y\,dz + z\,dy = \frac{a^2}{y}dx + \frac{a^4}{2y^3}dy - \frac{xa^2}{y^2}dy$$

$$\text{or } d(yz) = \frac{a^4}{2y^3}dy + a^2\left\{\frac{dx}{y} - \frac{x\,dy}{y^2}\right\}$$

$$\text{or } d(yz) = \frac{a^4}{2y^3}dy + a^2\left\{\frac{y\,dx - x\,dy}{y^2}\right\} = \frac{a^4}{2y^3}dy + a^2\,d(x/y)$$

(12)

On integrating we get

$$yz = \frac{a^4}{2(-2)} y^{-2} + a^2 \frac{x}{y}$$

or $$yz = \frac{a^4}{-4y^2} + \frac{a^2 x}{y}$$

or $$yz + \frac{a^4}{4y^2} - \frac{a^2 x}{y} = 0$$

or $$\frac{y^3 z + a^4 - a^2 x y}{y^2} = 0$$

This is the required condition

2.12 Special type of First order diff. eqn.

1) Equations Involving Only p and q

Let a differential equation of the form $f(p,q)=0$ —— (1)

From Charpit's method we have auxilliary eqns

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p+qf_q} = \frac{dp}{-(f_x+pf_z)} = \frac{dq}{-(f_y+qf_z)}$$

here,

$$\frac{dx}{f_p} = \frac{dy}{f_p} = \frac{dz}{pf_p+qf_q} = \frac{dp}{[0]} = \frac{dz}{[0]}$$

Now,

$\therefore dp=0$

$\therefore p=a$ —— (ii)

Now, from (i) and (ii)

$\therefore f(a,q)=0$

$\therefore q = Q(a)$

we have,

$dz = p\,dx + q\,dy$

or $dz = a\,dx + Q(a)\,dy$

On integrating we get

$z = ax + Q(a)y + b$

which is the required solution of the given diff. eqn.

Examples 8

Find the complete solution of the eq^n $pq=1$

Sol^n

Here the given eq^n can be written as

$$f = pq-1=0 \quad \text{——(i)}$$

Now the auxilliary eq^{n}s are

$$\frac{dx}{q} = \frac{dy}{p} = \frac{dz}{pq+pq} = \frac{dp}{0} = \frac{dq}{0} \quad \text{——(ii)}$$

∴ from (ii) $\cancel{\frac{dy}{p}}$ $\frac{dp}{0}$

or $\cancel{dy=p}$ ∴ $dp=0$

On integrating we get

$$p=a \quad \text{——(iii)}$$

from (i) and (iii)

$aq-1=0$

or $aq=1$

∴ $q=1/a$

again we have,

$dz=pdx+qdy$

or $dz=adx+\frac{1}{a}dy$

or $adz=a^2dx+dy$

On integrating we get

$$az = a^2x+y+b$$

which is the required complete solution.

xample 9.

Soln $p+q=pq$

Here the given diff. eqn can be written as

$\therefore f = p+q-pq=0$ —— (i)

$\therefore$ From auxilliary eqn we have,

$$\frac{dx}{1-q} = \frac{dy}{1-p} = \frac{dz}{p+q-2pq} = \frac{dp}{0} = \frac{dq}{0} \quad \text{—— (ii)}$$

from (ii)

$$\frac{dp}{0} = 0$$

$\therefore dp = 0.$

On integrating we get

$$p = a \quad \text{—— (iii)}$$

from (i) and (iii)

$$a+q-aq=0$$

or $-q(a-1) = -a$

$$\therefore q = \frac{a}{a-1}$$

we have

$$dz = p\,dx + q\,dy$$

or $dz = a\,dx + \frac{a}{a-1}dy$

On integrating we get

$$(a-1)z = (a^2-a)x + \frac{a}{a-1}dy.$$

which is the required soln.

Equation not involving the independent variables:

Let a diff eq^n of the form

$$f(z, p, q) = 0 \quad \text{—— (i)}$$

Let its auxilliary equation is

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-pf_z} = \frac{dq}{-qf_z}$$

Taking last two ratios

$$\frac{dp}{p} = \frac{dq}{q}$$

Integrating we get

$$\log p = \log q + \log a$$

or $\log (p/q) = \log a$

or $\frac{p}{q} = a$

$\therefore p = aq$ —— (ii)

Solving eq^n (i) and (ii) we get the expression for p and q and substituting these values in $dz = pdx + qdy$ and again integrating we get the required solution of given diff. eq^n.

Example 9:

Find the complete integral of the equation $p^2z^2+q^2=1$

Soln

Let $f=p^2z^2+q^2-1=0$ —— (i)

i.e $f(p,q,z)=0$

from charpit's auxilliary eqn we get

$$\frac{dp}{p}=\frac{dq}{q}$$

on integrating we get

$$\log p=\log q+\log a$$

or $\log p-\log q=\log a$

or $\log(p/q)=\log a$

$$p/q=a$$

$\therefore p=aq$ ← (ii)

from (i) and (ii) we have

$$a^2q^2z^2+q^2-1=0$$

or $q^2=\dfrac{1}{a^2z^2+1}$

$$\therefore q=\frac{1}{\sqrt{a^2z^2+1}}$$

$$\therefore p=\frac{a}{\sqrt{a^2z^2+1}}$$

Also we have

$$dz = p\,dx + q\,dy$$

$$\text{or } dz = \frac{a}{\sqrt{a^2z^2+1}}\,dx + \frac{1}{\sqrt{a^2z^2+1}}\,dy$$

$$\text{or } \sqrt{a^2z^2+1}\,dz = a\,dx + dy$$

On integrating we get

$$\frac{1}{2a}\left\{az\sqrt{a^2z^2+1} + \log\left(\sqrt{a^2z^2+1} + az\right)\right\} = ax + y + b$$

$\therefore\ az\sqrt{a^2z^2+1} + \log\left(az + \sqrt{a^2z^2+1}\right) = 2a(ax + y + b)$ is the required general solution.

Special types of first order equation:

Problems:

$p+q=pq$

Soln

Here the given diff equation is

$$p+q=pq \quad \text{—— (i)}$$

eqn (i) can be written as $p+q-pq=0$

$\therefore f(p,q)=0$

From charpit's auxilliary eqn we have

$$\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{dp}{-(f_x+pf_z)}=\frac{dq}{-(f_y+qf_z)}$$

or $$\frac{dx}{1-q}=\frac{dy}{1-p}=\frac{dz}{p+q-2pq}=\frac{dp}{0}=\frac{dq}{0} \quad \text{—— (ii)}$$

From (ii) we have,

$$\frac{dp}{0}=0$$

or $dp=0$

On integrating we get

$$p=a \longrightarrow \text{(iii)}$$

from (i) and (iii) we have

$$a+q=aq$$

or $a=q(a-1)$

$$\therefore q=\frac{a}{a-1}$$

we have,

$$dz = p\,dx + q\,dy$$

$$\text{or } dz = a\,dx + \frac{a}{a-1}\,dy$$

On integrating we get

$$z = ax + \frac{a}{a-1}y + b$$ is the required condition.

2.) $z = p^2 - q^2$

Here the given differential equation is

$$z = p^2 - q^2 \quad \text{---(i)}$$

eqn(i) can be written as

$$f = z - p^2 + q^2 = 0 \quad \text{---(ii)}$$

By using charpit's auxilliary equation

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

$$\therefore \frac{dx}{-2p} = \frac{dy}{2q} = \frac{dz}{-2p^2 + 2q^2} = \frac{dp}{-p} = \frac{dq}{-q} \quad \text{---(iii)}$$

From (iii) we have,

$$\frac{dp}{-p} = \frac{dq}{-q}$$

$$\text{or } \frac{dp}{p} = \frac{dq}{q}$$

On integrating we get

$$\log p = \log q + \log a$$

or $\log p = \log aq$

$\therefore p = aq \quad \text{--- (iv)}$

from (ii) and (iv)

$f = z - a^2 q^2 + q^2 = 0$

or $q^2(1-a^2) = -z$

or $q^2 = \frac{-z}{1-a^2}$

$\therefore q = \sqrt{\frac{z}{a^2-1}}$

and $p = a\sqrt{\frac{z}{a^2-1}}$

we have

$dz = p\,dx + q\,dy$

or $dz = \frac{a\sqrt{z}}{\sqrt{a^2-1}}dx + \frac{\sqrt{z}}{\sqrt{a^2-1}}dy$

or $\frac{dz}{\sqrt{z}} = \frac{a}{\sqrt{a^2-1}}dx + \frac{dy}{\sqrt{a^2-1}}$

On integrating we get

$2\sqrt{z} = \frac{ax}{\sqrt{a^2-1}} + \frac{y}{\sqrt{a^2-1}}$

$\therefore 2\sqrt{z} = \frac{ax+y}{\sqrt{a^2-1}}$ is the required solⁿ

3) $zpq = p+q$

Here the given diff. eqn. is

$$zpq = p+q \quad \text{—— (i)}$$

eqn (i) can be written as

$$f = zpq - p - q = 0 \quad \text{—— (ii)}$$

Now using charpit's auxilliary eqn.

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p+qf_q} = \frac{dp}{-(f_x+pf_z)} = \frac{dq}{-(f_y+qf_z)}$$

$$\frac{dx}{zq-1} = \frac{dy}{zp-1} = \frac{dz}{zpq-p+zpq-q} = \frac{dp}{-p^2q} = \frac{dq}{-q^2p} \quad \text{—— (iii)}$$

from (iii) we have

$$\frac{dp}{-p^2\cancel{q}} = \frac{dq}{-q^{\cancel{2}}q\cancel{p}}$$

$$\therefore \frac{dp}{p} = \frac{dq}{q}$$

On integrating we get

$$\log p = \log q + \log a$$

or $\log p = \log (q \cdot a)$

$$\therefore p = aq \quad \text{—— (iv)}$$

from (i) and (iv) we have

$$zaq^2 = aq+q$$

or $zaq^2 = q(1+a)$

$$\therefore q = \frac{1+a}{az}$$

$\therefore p = \frac{1+a}{z}$

Now, we have,

$dz = p\,dx + q\,dy$

or $dz = \frac{1+a}{z}dx + \frac{1+a}{az}dy$

or $z\,dz = (1+a)dx + \frac{(1+a)}{a}dy$

On integration we get

$$\frac{z^2}{2} = (1+a)x + \frac{(1+a)}{a}y + b$$

$\therefore z^2 = \frac{a(1+a)x + (1+a)y + ab}{a}$ is the required complete integrals of the eqns.

(18)

Seperable Equation:

If a differential equation is of the form

~~F = f(x)~~

$F = f(x,p) - g(y,q)$ — (i) then its Charpit's equation

$$\frac{dx}{pf_p} = \frac{dy}{g_q} = \frac{dz}{pf_q - qg_q} = \frac{dp}{-f_x} = \frac{dq}{-g_y} \quad \text{— (ii)}$$

from (ii) $\frac{dx}{f_p} = \frac{dp}{-f_x}$

or $\frac{dx}{f_p} + \frac{dp}{f_x} = 0$

or $\frac{dp}{dx} + \frac{f_x}{f_p} = 0$ — (iii)

This eqn in x and p may be solved to give p as a function of x and an arbitary constant a, writing this eqn in the form

$f_p\,dp + f_x\,dx = 0$

and we see that the solution of the eqn is

$f(x,p) = a$.

Hence we determine p, q from the relation

$f(x,p) = a$ and $g(y,q) = a$

and substitute value in $dz = p\,dx + q\,dy$ and on integration we ~~the~~ get the required complete integral of the given differential equation.

Find the complete integral of the equation
$p^2 y(1+x^2) = qx^2$

Sol^n

Here the given differential equation is
$p^2 y(1+x^2) = qx^2$ —— (i)

eq^n (i) can be written as

$$p^2\left(1+\frac{1}{x^2}\right) = \frac{q}{y}$$

i.e $f(x,p) = g(y,q)$

By charpit's auxilliary equation we get

$$\frac{dx}{f_p} = \frac{dy}{g_q} = \frac{dz}{pf_q - qg_p} = \frac{dp}{-f_x} = \frac{dq}{-g_y} \quad \text{—— (ii)}$$

from (ii) we have,

$$\frac{dx}{f_p} = -\frac{dp}{f_x}$$

or $f_x dx + f_p dp = 0$

or $\frac{-2p^2}{x^3} dx + 2p\left(\frac{1}{x^2}+1\right) dp = 0$

or $\cancel{2}p\left(\frac{1}{x^2}+1\right) dp = \frac{\cancel{2}p^2}{x^3} dx$

or $$\frac{dp}{p} = \frac{dx}{x^3\left(\frac{1}{x^2}+1\right)}$$

or $$\frac{dp}{p} = \frac{dx}{x+x^3}$$

Ans-or $\frac{dP}{P} = \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx$

On integrating we get

$\log p = \log x - \frac{1}{2} \log(x^2+1) + \log a$

or $\log p = \log x - \log \sqrt{x^2+1} + \log a$

or $\log p = \log\left(\frac{x}{\sqrt{x^2+1}}\right) + \log a$

or $\log p = \log \left\{\frac{ax}{\sqrt{x^2+1}}\right\}$

$\therefore \quad p = \frac{ax}{\sqrt{x^2+1}}$

$a = \frac{p\sqrt{x^2+1}}{x}$

also we have,

$qx^2 = p^2 y (1+x^2)$

or $qx^2 = \frac{a^2 x^2 y}{x^2+1} \; x^2+1$

or $q = a^2 y$

Now,

$dz = p\,dx + q\,dy$

or $dz = \frac{ax}{\sqrt{x^2+1}} dx + a^2 y\, dy$

$dz =$

On integrating we get

$$z = a\sqrt{x^2+1} + \frac{a^2y^2}{2} + b$$

which is the required equation.

Clairaut Equations:

A first-order partial differential differential equation is said to be Clairaut equation if it can be written in the form

$$z = px + qy + f(p,q) \quad \text{---(i)}$$

The corresponding Charpit Equations are

$$\frac{dx}{x+f_p} = \frac{dy}{y+f_q} = \frac{dz}{px+qy+pf_p+qf_q} = \frac{dp}{0} = \frac{dq}{0} \quad \text{---(ii)}$$

From (ii) we can take

$$\frac{dp}{0} = 0 \quad \text{and} \quad \frac{dq}{0} = 0$$

or $dp = 0$ or $dq = 0$

On integrating we get On integrating we get

$p = a$ $q = b$

Then $z = ax + by + f(a,b)$ which is the required solution of given differential equation.

(2C)

Q// Find a Complete integral of the equation $(p+q)(z-xp-yq)=1$

Sol^n

The given diff. eq^n is

$$(p+q)(z-xp-yq)=1$$

$$\therefore z-xp-yq=\frac{1}{p+q}$$

$$\therefore z=\frac{1}{p+q}+xp+yq \quad \text{--- (i)}$$ This is a Clairaut equation

By Charpit's equation

$$\frac{dx}{x+f_p}=\frac{dy}{y+f_q}=\frac{dz}{px+yq+pf_p+qf_q}=\frac{dp}{0}=\frac{dq}{0} \quad \text{--- (ii)}$$

from (ii) we have,

$$\cancel{\frac{dp}{0}=0} \quad \text{and} \quad \cancel{\frac{dq}{0}=0}$$

or $dp=0$ and $dq=0$

On integrating we get $p=a$ On integrating we get $q=b$

Substituting p and q in (i) we get

$$z=\frac{1}{a+b}+ax+by$$ where a and b are int. constants

Special types of 1st order equations

Problems:

1) $pqz = p^2(xq+q^2)+q^2(yp+q^2)$

Here the given diff. eqn is

$$pqz = p^2(xq+p^2)+q^2(yp+q^2)$$

or $z = \frac{p}{q}(xq+p^2)+\frac{q}{p}(yp+q^2)$

or $z = px+\frac{p^3}{q}+qy+\frac{q^3}{p}$

$\therefore f = z-(px+qy+\frac{p^3}{q}+\frac{q^3}{p})$ —— (i) which is in clairaut form

So the charpit's eqns are

$$\frac{dx}{x+f_p} = \frac{dy}{y+f_q} = \frac{dz}{px+qy+pf_p+qf_q} = \frac{dp}{0} = \frac{dq}{0} \quad \text{—— (ii)}$$

or $\frac{dx}{\frac{q^3}{p^2}-\frac{3p^2}{q}} =$ From (ii) we have

$$\frac{dp}{0} = 0 = \frac{dq}{0}$$

or $\frac{dp}{0} = 0$ and $\frac{dq}{0} = 0$

or $dp = 0$ and $dq = 0$

on integrating we get $\quad$ on integrating we get

$p = a \quad q = b$

Substituting p and q in (i) we get

$z = ax+by+\frac{a^3}{b}+\frac{b^3}{a}$ which is the required solution