

Graph Theory Facts and Propositions

General:

1. Handshake Theorem: $\sum_{v \in V(G)} \deg(v) = 2|E|$
2. Proposition: Every graph with ≥ 2 vertices has two vertices of the same degree
3. Proposition: the n-cube has 2^n vertices and $n * 2^{n-1}$ edges
4. Theorem: if there is a walk from vertex x to vertex y in G then there is a path from x to y in G.
5. Corollary: if there is a path from x to y in G and a path from y to z in G then there is a path from x to z in G.
6. Theorem: let G be a graph and let v be a vertex in G. If for each w in G there is a path from v to w, then G is connected. *For any vertex, you can get to any other vertex.*
7. Theorem: a graph G is **not connected** iff there exists a property subset of x of V(G) such that the **cut** induced by x is empty.
8. Proposition: if every vertex has degree ≥ 2 then G has a cycle.
9. Theorem (Dirac): if G is a graph on $n > 3$ vertices where every vertex has degree $\geq \frac{n}{2}$, then G has a cycle containing every vertex. G is a **Hamiltonian Graph**.
10. Theorem (Chvatal '72): if G is a graph on n vertices with degree $d_1 \leq d_2 \leq d_3 \dots \leq d_n$ then if $d_i \geq i$ or $d_{n-i} \geq n - i$ for all $i \leq \frac{n}{2}$, then G is Hamiltonian.
11. Theorem (Tutte): Every 4-connected graph that can be drawn in the plane without crossings is Hamiltonian.
12. Theorem: Every connected graph in which every vertex has even degree is *Eulerian*. An Eulerian graph has an Euler tour, which is a closed walk that contains every edge once.
13. Lemma: if $e = \{x,y\}$ is a bridge of a connected graph G, then $G-e$ has precisely two components. Furthermore, x and y are in different components.
14. Theorem: An edge is a bridge of a graph G iff it is not contained in a cycle of G
15. Corollary: If there are two distinct paths from u to v in G then G contains a cycle.

16. Lemma: There is a unique path between every pair of vertices u and v in a tree.
17. Lemma: Every edge of a tree T is a bridge.
18. Theorem: A tree with at least 2 vertices has at least two vertices of degree 1.
19. Theorem: if T is a tree, then $|E(T)| = |V(T) - 1|$.
20. Proposition: If x, y are vertices of a tree T , then there is a unique path of T from x to y .
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22. Proposition: Every edge of a tree is a bridge.
23. Proposition: If x, y are vertices of a tree T , then there is a unique path of T from x to y .
24. Proposition: A graph G has a spanning tree iff it is connected.
25. Corollary: Every connected graph on n vertices has $\geq n - 1$ edges.
26. Corollary: Every connected graph on n vertices, $n-1$ edges is a tree.
27. Proposition: Every tree is bipartite.
28. Proposition: If G is a bipartite graph and $u, v \in V(G)$ then if u and v are in the same part of a bipartition, then every walk from u to v has even length. If u, v are in different parts, then every walk from u to v has odd length.
29. Proposition: If G is a graph with no odd cycles, then G is bipartite.
30. Theorem: Prim's algorithm outputs a min-weight spanning tree.
31. Proposition: A graph is planar iff it has a spherical embedding.
32. Theorem: if there is a planar embedding of 2-connected graph G with faces f_1, f_2, \dots then $\sum_{i=1} \deg(f_i) = 2|E(G)|$
33. Corollary: If the connected graph G has a planar embedding with f faces, then average degree of a face is $\frac{2|E(G)|}{f}$.
34. Theorem: let G be a connected graph with $|V|$ vertices and $|E|$ edges. If G has a planar embedding with $|F|$ faces, then $|V| - |E| + |F| = 2$.
35. Theorem: There are exactly five non-isomorphic platonic solids.
36. Lemma: Let G be a planar embedding with $|V|$ vertices, $|E|$ edges and $|F|$ faces. Then $\{d, k\}$ is one of the five pairs of faces and vertices: $\{3, 3\}$, $\{3, 4\}$, $\{4, 3\}$, $\{5, 3\}$, $\{3, 5\}$

37. Lemma: If G is connected and not a tree then in a planar embedding of G , the boundary of each face contains a cycle.
38. Lemma: Let G be a planar embedding with $|V|$ vertices and $|E|$ edges. If each face has degree at least d , then $(d-2)|E| \leq d(|V|-2)$.
39. Corollary: In any planar embedding of a graph with at least 2 faces, each face has degree ≥ 3 .
40. Lemma: In any planar embedding of a graph with ≥ 1 cycle, the boundary of every face contains a cycle.
41. Lemma (Test 1): If $G = (V, E)$ is a planar graph and $|E| \geq 2$, then $|E| \leq 3|V| - 6$.
42. Corollary: K_5 is non-planar $|V| = 5$, $|E| = 10$.
43. Corollary: A planar graph has a vertex of degree at most 5.
44. Lemma (Test 2): If $G = (V, E)$ is a planar graph and every cycle has length $\geq g$, where g is the girth, the length of the smallest cycle, and $|E| \geq \frac{1}{2}g$, then $|E| \leq \frac{g}{g-2}(|V| - 2)$.
45. Corollary: $K_{3,3}$ is non-planar because it has no triangles, so $g = 4$ and it fails Test 2.
46. Kuratowski's Theorem: A graph is planar iff it has no subdivision of $K_{3,3}$ or K_5 as a subgraph.
47. Theorem: A graph is 2-colourable iff it is bipartite.
48. Theorem: K_n is n -colourable and not k -colourable for $k < n$.
49. Five-Colour-Theorem: Every planar graph is 5-colourable.
50. Theorem: Every planar graph is 4-colourable.
51. Lemma: M is not a maximum matching iff there exists an M -augmenting path.
52. Lemma: If M is a matching of G and C is a cover of G then $|M| \leq |C|$.
53. Lemma: If M is matching and C is a cover and $|M| = |C|$ then M is a maximum matching and C is a minimum cover.
54. Theorem (Konig's Theorem): If G is bipartite, then the size of the maximum matching is equal to the size of the minimum cover.
55. Lemma: Let G be a bipartite graph with bipartition A, B where $|A| = |B| = n$. If G has $|E|$ edges then G has a matching of at least size $\frac{|E|}{n}$.
56. Theorem (Hall's): An (A, B) -bigraph G has a matching that saturates A iff for every S subset of A , $|S| \leq |N(S)|$.

- 57. Corollary: An (A,B) bigraph G has a perfect matching iff $|A|=|B|$ and for S is a subset of A , $|S| \leq |N(S)|$.
- 58. Proposition: If $k \geq 1$ and G is a k -regular bipartite graph, then G has a perfect matching.
- 59. Corollary: If G is a k -regular bipartite graph, then $E(G)$ has a partition into k perfect matches of G .
- 60. Corollary: Following from right above, every k -regular bipartite graph is k -edge colourable.