Graph Theory Facts and Propositions

General:

1. Handshake Theorem: $\sum_{v \in V(G)} deg(v) = 2|E|$
2. Proposition: Every graph with ≥ 2 vertices has two vertices of the same degree
3. Proposition: the n-cube has 2^n vertices and $n * 2^{n-1}$ edges
4. Theorem: if there is a walk from vertex x to vertex y in G then there is a path from x to y in G.
5. Corollary: if there is a path from x to y in G and a path from y to z in G then there is a path from x to z in G.
6. Theorem: let G be a graph and let v be a vertex in G. If for each w in G there is a path from v to w, then G is connected. *For any vertex, you can get to any other vertex.*
7. Theorem: a graph G is **not connected** iff there exists a property subset of x of V(G) such that the **cut** induced by x is empty.
8. Proposition: if every very has degree ≥ 2 then G has a cycle.
9. Theorem (Dirac): if G is a graph on n $>$ 3 vertices where every vertex has degree $\geq \frac{n}{2}$, then G has a cycle containing every vertex. G is a **Hamiltonian Graph**.
10. Theorem (Chvtal '72): if G is a graph on n vertices with degree $d_1 \leq d_2 \leq d_3 ... \leq d_n$ then if $d_i \geq i$ or $d_{n-i} \geq n - i$ for all $i \leq \frac{n}{2}$, then G is Hamiltonian.
11. Theorem (Tutte): Every 4-connected graph that can be drawn in the plane without crossings is Hamiltonian.
12. Theorem: Every connected graph in which every vertex has even degree is *Eulerian.* An Eulerian graph has an Euler tour, which is a closed walk that contains every edge once.
13. Lemma: if e = {x,y} is a bridge of a connected graph G, then G-e has precisely two components. Furthermore, x and y are in different components.
14. Theorem: An edge is a bridge of a graph G iff it is not contained in a cycle of G
15. Corollary: If there are two distinct paths from u to v in G then G contains a cycle.

16. Lemma: There is a unique path between every pair of vertices u and v in a tree.
17. Lemma: Every edge of a tree T is a bridge.
18. Theorem: A tree with at least 2 vertices has at least two vertices of degree 1.
19. Theorem: if T is a tree, then |E(T)| = |V(T) - 1|.
20. Proposition: If x,y are vertices of a tree T, then there is a unique path of T from x to y.
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24. Proposition: A graph G has a spanning tree iff it is connected.
25. Corollary: Every connected graph on n vertices has $\geq n-1$ edges.
26. Corollary: Every connected graph on n vertices, n-1 edges is a tree.
27. Proposition: Every tree is bipartite.
28. Proposition: If G is a bipartite graph and u,v $\in V(G)$ then if u and v are in the same part of a bipartition, then every walk from u to v has even length. If u,v are in different parts, then every walk from u to v has odd length.
29. Proposition: If G is a graph with no odd cycles, then G is bipartite.
30. Theorem: Prim's algorithm outputs a min-weight spanning tree.
31. Proposition: A graph is planar iff it has a spherical embedding.
32. Theorem: if there is a planar embedding of 2-connected graph G with faces $f_1, f_2, ...$ then $\sum_{i=1} deg(f_i) = 2|E(G)|$
33. Corollary: If the connected graph G has a planar embedding with f faces, then average degree of a face is $\frac{2|E(G)|}{f}$.
34. Theorem: let G be a connected graph with |V| vertices and |E| edges. If G has a planar embedding with |F| faces, then |V| - |E| + |F| = 2.
35. Theorem: There are exactly five non-isomorphic platonic solids.
36. Lemma: Let G be a planar embedding with |V| vertices, |E| edges and |F| faces. Then {d,k} is one of the five pairs of faces and vertices: {3,3}, {3,4}, {4,3}, {5,3}, {3,5}

37. Lemma: If G is connected and not a tree then in a planar embedding of G, the boundary of each face contains a cycle.
38. Lemma: Let G be a planar embedding with |V| vertices and |E| edges. If each face has degree at least d, then (d-2)|E| $\leq$ d(|V|-2)$.
39. Corollary: In any planar embedding of a graph with at least 2 edges, each face has degree ≥ 3.
40. Lemma: In any planar embedding of a graph with ≥ 1 cycle, the boundary of every face contains a cycle.
41. Lemma (Test 1): If G = (V,E) is a planar graph and |E| $\geq$ 2, then |E| $\leq$ 3|V|-6.
42. Corollary: K_5 is non-planar |V| = 5, |E| = 10.
43. Corollary: A planar graph has a vertex of degree at most 5.
44. Lemma (Test 2): If G = (V,E) is a planar graph and every cycle has length $\geq$ g, where g is the girth, the length of the smallest cycle, and |E| $\geq \frac{1}{2}g$, then $|E| \leq \frac{g}{g-2}(|V|-2)$
45. Corollary: $K_{3,3}$ is non-planar because it has no triangles, so g = 4 and it fails Test 2.
46. Kuratowski's Theorem: A graph is planar iff it has no subdivision of $K_{3,3}$ or K_5 as a subgraph.
47. Theorem: A graph is 2-colourable iff it is bipartite.
48. Theorem: K_n is n-colourable and not k-colourable for k < n.
49. Five-Colour-Theorem: Every planar graph is 5-colourable.
50. Theorem: Every planar graph is 4-colourable.
51. Lemma: M is not a maximum matching iff there exists an M-augmenting path.
52. Lemma: If M is a matching of G and C is a cover of G then $|M| \leq |C|$.
53. Lemma: If M is matching and C is a cover and |M| = |C| then M is a maximum matching and C is a minimum cover.
54. Theorem (Konig's Theorem): If G is bipartite, then the size of the maximum matching is equal to the size of the minimum cover.
55. Lemma: Let G be a bipartite graph with bipartition A,B where |A| = |B| = n. If G has |E| edges then G has a matching of at least size $\frac{q}{n}$.
56. Theorem (Hall's): An (A,B)-bigraph G has a matching that saturates A iff for every S subset of A, |S| $\leq$ |N(S)|.

57. Corollary: An (A,B) bigraph G has a perfect matching iff $|A|=|B|$ and for S is a subset of A, $|S| \leq |N(S)|$.

58. Proposition: If $k \geq 1$ and G is a k-regular bipartite graph, then G has a perfect matching.

59. Corollary: If G is a k-regular bipartite graph, then E(G) has a partition into k perfect matches of G.

60. Corollary: Following from right above, every k-regular bipartite graph is k-edge colourable.