Recursion

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Outline

- Call Stack
- Recursion
- Time and Space complexity of Recursion
- Mergesort
- Multiple recursive

Section 1

Call Stack

How a Call Stack Works

- Your computer internally uses a call stack (stack ADT) to execute functions
- When you run your Python file, the `main` functions is called. `main` is pushed onto the stack
  - Sounds familiar? `if __name__ == "__main__":`
- As the main function executes, it may call other functions, each functions is pushed to the top of the stack
  - The currently executing function is at the top of the stack
- When each function is executed, it is popped from the stack
- The function may return a value, which is passes to the calling function (the function below in the stack).
- The calling function can use the return value and continue execution until the stack is empty

Basic Example

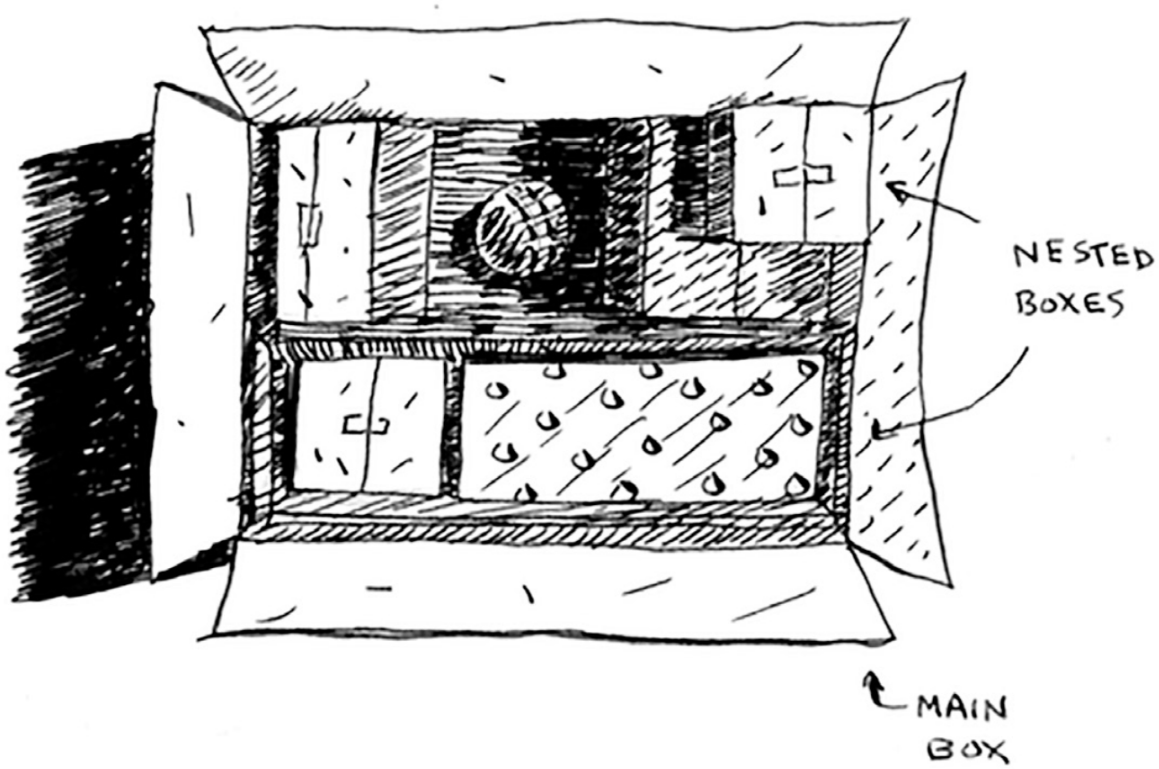
- If I run `round(float("20.24"))`, I expect 20
  - The `round` function is first to be called, it is pushed on the call stack
  - Then, `float("20.24")` is called and pushed on the call stack
- Now, we pop each function off the call stack.
  - `float("20.24")` returns `20.24`
  - `round` uses the return value of the previous function, `20.24`. It executes `round(20.24)`, which returns 20
  - The stack is empty, so the program finishes

Section 2

Recursion

Motivating Example

- Suppose you are looking for a key in a box, but the box contains more boxes!



- 2 minutes: write down the steps of the algorithm you would take to search for the key

Algorithm 1: Loop

1. Make a pile of all the boxes
2. Grab a box and open it
3. If it contains a box, append it to your pile of boxes
4. If it contains the key, you're done!
5. Repeat

Algorithm 2: Recursion

1. Grab a box and open it
2. If it contains a box, repeat step 1
3. If it contains the key, you're done!

Which algorithm do you like more?

- Notice the function is recursive because it calls itself
- Both algorithms achieve the same thing, but recursion is clearer (to me)

Formula to write a recursive function

- Since recursive functions call themselves, its easy to write an infinite loop
- Let's write a function that does a countdown

```
def countdown(i):
  print(i)
  countdown(i - 1)
```

- This runs forever, so we need a *base case* to tell the code when to stop

```
def countdown(i):
  print(i)
  if i <= 0:
    return
  else:
    countdown(i - 1)
```

Factorial

- The *factorial* is the product of all positive integers less than or equal to the given integer
  - $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
  - We define $1! = 1$
- Let's use recursion to calculate factorials

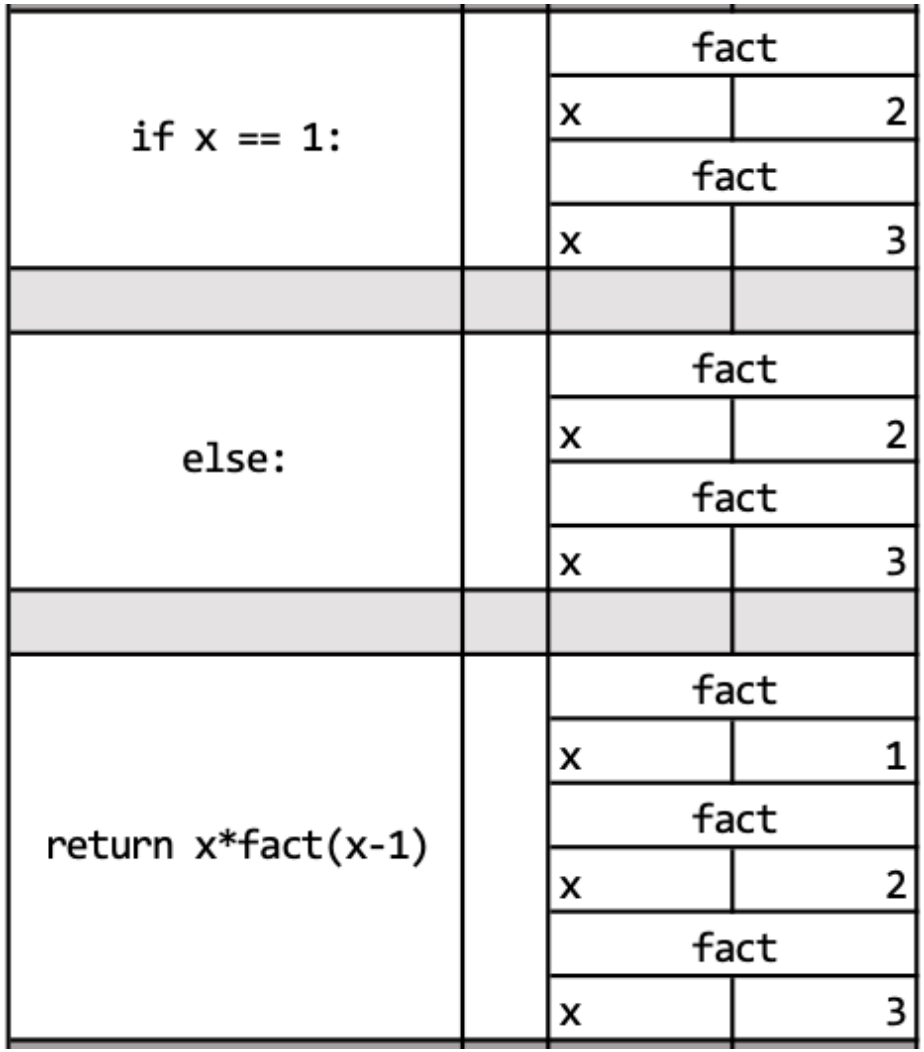
```
def factorial(n):
  if x == 1:
    return 1
  else:
    return x * factorial(x - 1)
```

- Let's examine the call stack when we call `factorial(3)`

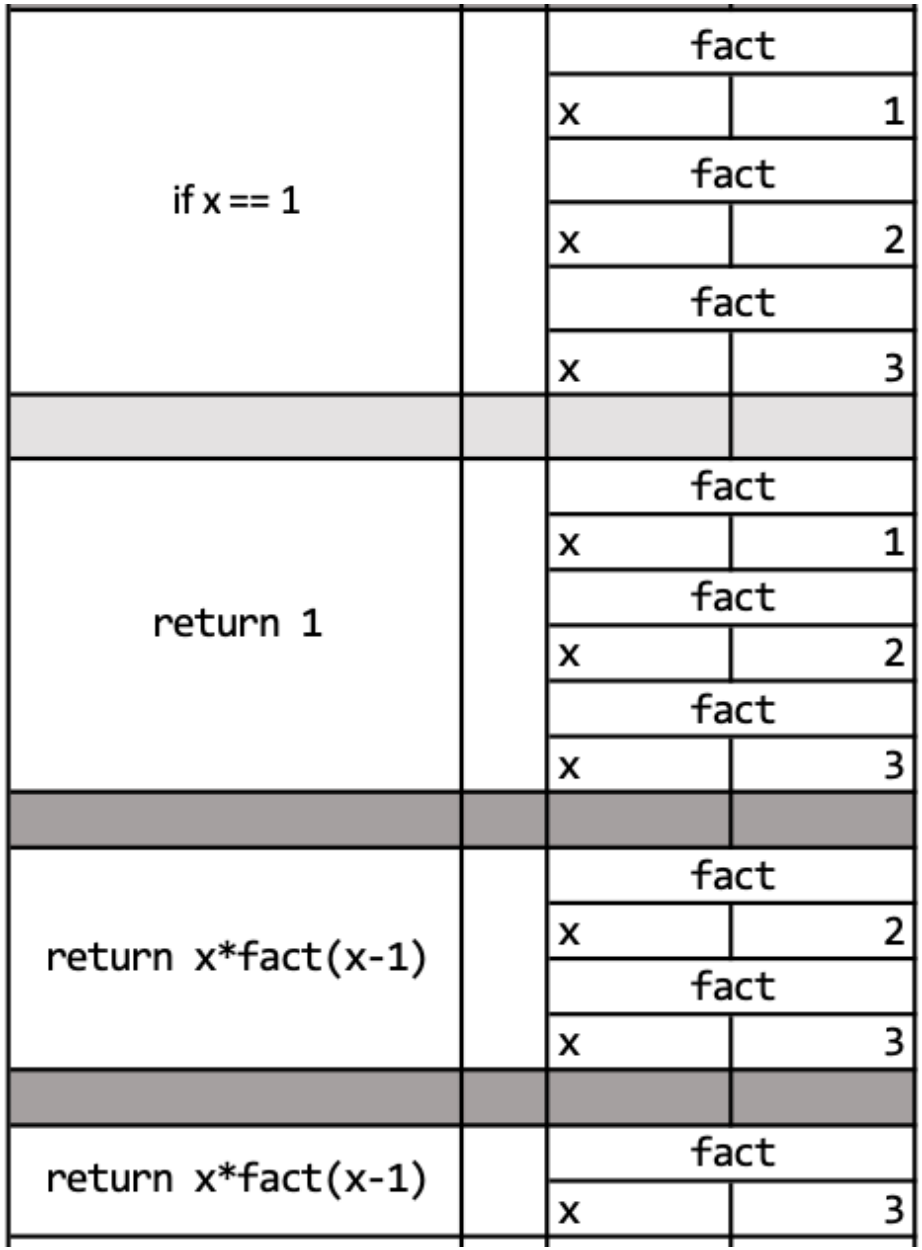
Recursion and the Call Stack

Code		Call Stack	
fact(3)		fact	
		x	3
if x == 1:		fact	
		x	3
else:		fact	
		x	3
return x*fact(x-1)		fact	
		x	2
		fact	
		x	3

Recursion and the Call Stack



Recursion and the Call Stack



Multiple Recursive Calls: Fibonacci Sequence

- In calculating the factorial, each recursion only calls itself once. This doesn't have to be the case
- The Fibonacci Sequence is a sequence of numbers where the first two numbers are 0 and 1, with each subsequent number being being the sum of the previous two numbers in the sequence.
  - Notice how the problem is defined recursively

```
def fib(n):
  if n <= 1:
    return n
  else:
    return fib(n - 1) + fib(n - 2)
```

2 minutes: what is its time and space complexity?

Section 3

Time and Space Complexity of Recursion

Time Complexity of Recursion

- Generally, recursion doesn't have performance benefits compared to loops (in problems like finding a key in nested boxes)
  - However, it is simpler to understand
- The time complexity of recursion depends on the number of time the function calls itself (branches)
  - Factorial: the `fact` is called n times before reaching the base case so its $O(1^n) = O(n)$
  - If a recursive function called itself twice, then its (2^n)
- When a recursive function makes multiple calls, the run time will often be $O(branches^{depth})$

Tricky Example

```
def recursive(n):
  for i in range(n):
    # Something happens
    i += 2
  if n <= 0:
    return 1
  else:
    return 1 + recursive(n - 3)
```

- Loop takes $n/2$ steps, because we increase `i` by 2
- Recursion takes $n/3$ steps **and** the loop is called recursively.
  - In other words, for each recursion, run the loop.
- The time complexity is $n/2 \times n/3 = \frac{n^2}{6} = O(n^2)$

Space complexity of recursion

- Notice the call stack takes up space in memory. How much depends on the depth of the recursion
- Think about the maximum amount of space the call stack will need
  - Factorial: $O(n)$, when recursion reaches the base case
- Even when you have multiple branches, it's possible only 1 branch at depth n is in memory at a time
- 2 minutes: to find the key in nested boxes, what is the memory complexity of the recursive approach versus the loop approach?

Live Coding

Given an list of positive integers and an integer x, we want to find all unique combinations in the list where the sum is equal to x. A number in the list can be used multiple times.

Example

```
# INPUT
lst = [1,2,5,6]
x = 6
# OUTPUT
[1, 5]
[6]
```

Section 4

Mergesort

Divide and Conquer Algorithms

- Divide and Conquer (D&C) is a general method to solve problems utilizing recursion.
  - Figure out the simplest case and use it as the base case
  - Figure out how to reduce your problem to the base case
- Let's start with a trivial example: how would you sum a list of integers?
  - Solution is obvious with a loop
  - Let's do it recursively

Divide and Conquer Algorithms

Step 1

- What is the simplest array to sum?
- Arrays with no elements or 1 element
  - sum of `[]` is 0, sum of `[8]` is 8

Step 2

- How can we reduce all arrays to empty array?
- Notice `sum[2, 4, 5]` $= 2 +$ `sum[4, 5]`, but the second version reduced the problem

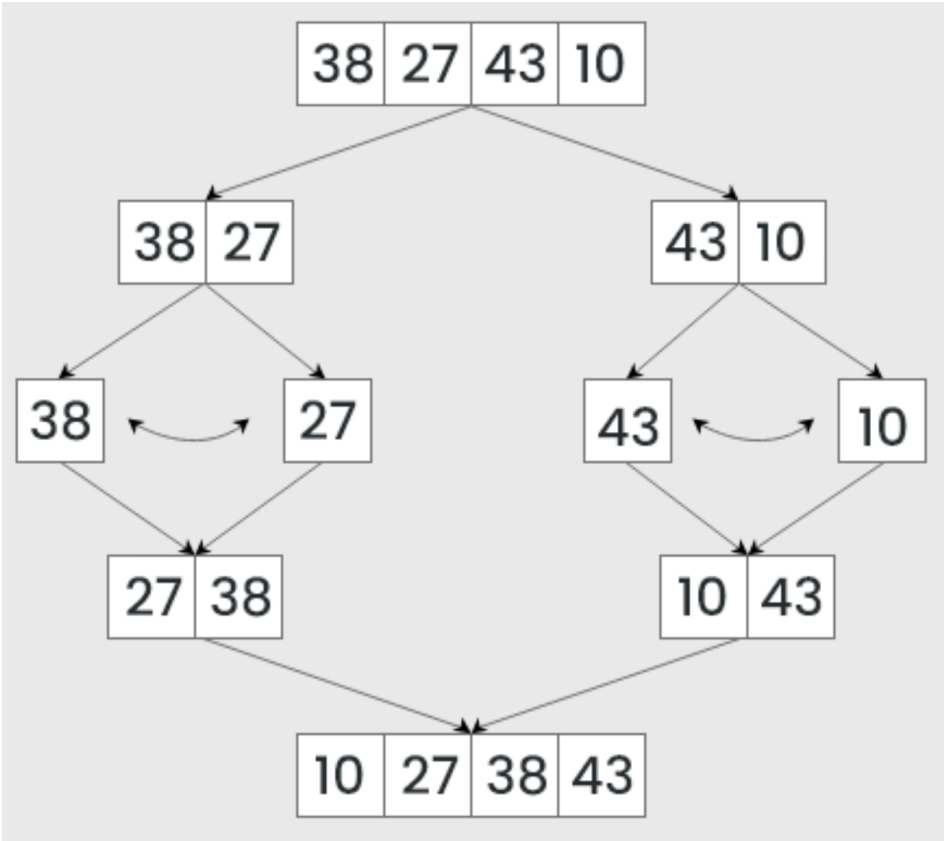
Divide and Conquer Algorithms

```
def rec_sum(lst):
  if not lst:
    return 0
  else:
    return lst[0] + rec_sum(lst[1:])
```

- Let's work on a real problem next!

Mergesort

- Some lists don't need to be sorted
  - Lists of size 1! This is our base case
- We can split lists in half until they contain 1 element, then merge all of the sub-lists
- Python's sort function uses a hybrid of merge and insertion sort, both of which you've learned!



Big-O of Merge Sort

First consider the non-recursive part of the code

- The "divide" step takes linear time, since slicing operations take roughly $n/2$ steps to make a left and right copy respectively.
- The merge operation also takes n steps approximately
- All other operations are constants
- Together, the non-recursive part of this algorithm is $O(n)$

Next consider the recursive calls

- Recall the big-O of recursion depends on the recursion depth and number of calls. $O(branches^{depth})$
- The depth in Merge Sort is the number of times you need to divide to get to a list of length 1.
- Mathematically, $2^{\text{depth}} = n$, then depth $= \log n$. So there are approximately log n levels

Big-O of Mergesort

- Since the $O(n)$ steps must be performed each recursion, the total run time is $O(n\log n)$. Our analysis only depended on the size of the list, so the best and worst case of mergesort is the same
- This is much faster than insertion sort!
- 2 minutes: does it have less space complexity than insertion sort?

Section 5

Recommended Problems and References

Recommended Problems

- Bhargava: Chapter 4 exercises
  - 4.1 to 4.8
- Write a recursive function that produces the `RecursionError: maximum recursion depth exceeded` error.
- Write a iterative function to calculate the *n*th Fibonacci number. What is its time and space complexity?
- Write a recursive function to determine if a string is a palindrome. What is its time and space complexity?
- Write a recursive function to check if a given positive integer is a prime number. What is its time and space complexity?

Recommended Problems

- Suppose you have a plot of land and want to divide the land into even square plots, while keeping the plots as big as possible. How would you do this using D&C? See Bhargava pg. 52.
- Explain why the "merge" step in mergesort is $O(n)$
- Implement mergesort. You might find using helper functions useful.
- Write a recursive function to perform binary search on a sorted list

Bonus Readings

- You may be interested in learning more about quicksort in Bhargava chapter 4 or here. Quicksort is another recursive sorting method

References

- Bhargava, A. Y. (2016). *Grokking algorithms: An illustrated guide for programmers and other curious people.* Manning. Chapter 3 and 4.
- Cormen, T. H. (Ed.). (2009). *Introduction to algorithms* (3rd ed). MIT Press. Chapter 4.