# TWISTED CHARACTERISTIC CLASSES AND GENERA: COMPUTATIONAL TOOLKIT

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ABSTRACT. This paper provides continuation of a previous paper, where we now consider various twisted expressions appearing in index theory and expression from quantum field theory, string theory, and M-theory. This is based on the undergraduate project of the second and third authors advised by the first author.

CONTENTS



## 1. INTRODUCTION

## 2. TWISTED GENERA

In the final results section of this paper we will consider the twisted genera, which arise as combinations of a genus with the Chern character. The twisted structures have applications in physics, where the genera represent *spinors* (the matter), while the Chern character represents *charge*. Hence, by combining the two structures one can study "matter that is charged".

One will notice that the twisted structures throughout the section are only in degrees divisible by 4. Firstly, the indexing in this case is different from the previous expansions, since here the notation stands for the *total degree*. The Â-genus $A_k$ and L-genus $L_k$ actually have the total degree of $4k$. Meanwhile, the Chern character $c_k$ has the total degree $2k$. Since only elements with the same degree can be combined, then one can only include the even Chern character degrees within the twisted structure (see *Expansion* **??**). Additionally, one can only combine Chern classes with Chern classes and Pontrjagin classes with Pontrjagin classes. Hence, once again we must use the idea encapsulated in the equation **??** in section **??** to write Chern character in terms of the Pontrjagin classes which causes the even degree classes to vanish. This two way reasoning demonstrates how the theory aligns with the practical computations. However, we can also invert this reasoning and write the twisted structures in terms of the Chern classes, which we do.

In the following section all the results adopt the notation that $c_i$ or $p_i$ are the characteristic classes of the bundle $E$, while $c_i'$ and $p_i'$ are the characteristic classes of the bundle $F$. The notation $F_r$ indicates that $r$ is the rank of the bundle $F$.

### 2.1. Twisted Â-genus.

The function on the next page is designed to compute the twisted structures of the Â-genus and L-genus. (A separate function for computing twisted Todd is stated later on.) The given function can compute the twisted structures either in terms of the Chern classes or Pontrjagin classes by combining the appropriate expansions of the genera and Chern character. For clarity we show a brief example of how the computation mechanism works.

**Example 1.** *Let us compute $\hat{A}$-genus combined with the Chern character of complexification written in terms of the Pontrjagin classes $p_i$ for the total degree 4. We need to retrieve Expansions* **??** *and* **??** *and write them* *out as shown below. Recall that both the Â-genus and the Chern character in terms of the Pontrjagin classes have total degrees $4k$. Finally, we multiply out the polynomials and keep only terms with the total degree 4.*

$$\begin{aligned}
[\hat{A}\cdot \mathrm{ch}(E_{\mathbb{C}})]_4 &= [(1-\tfrac{1}{24}p_1+\ldots)(r+\tfrac{1}{2}p_1'+\ldots)]_4\\
&= [\tfrac{1}{2}p_1' - \tfrac{1}{24}p_1 r - \tfrac{1}{48}p_1p_1' + \ldots]_4\\
&= \tfrac{1}{2}p_1' - \tfrac{1}{24}p_1 r.
\end{aligned}$$

As in the brief example above we use *Expansions* **??** and **??** to obtain the below expansion.

**Expansion 1.** *Â-genus combined with the Chern character of complexification written in terms of the Pontrjagin classes $p_i$.*

**Â-genus combined with the Chern character in terms of the Pontrjagin classes**

$$\begin{aligned}
[A(E)\mathrm{ch}(F_r)]_0 =& r\\
[A(E)\mathrm{ch}(F_r)]_4 =& \tfrac{1}{2^1}p_1' - \tfrac{r}{2^3.3^1}p_1\\
[A(E)\mathrm{ch}(F_r)]_8 =& \tfrac{r}{2^7.3^2.5^1}[7p_1^2-4p_2] - \tfrac{1}{2^4.3^1}p_1p_1' + \tfrac{1}{2^3.3^1}[p'^2_1 - 2p'_2]\\
[A(E)\mathrm{ch}(F_r)]_{12} =& \tfrac{r}{2^{10}.3^3.5^1.7^1}[-31p_1^3+44p_1p_2-16p_3] + \tfrac{1}{2^8.3^2.5^1}(7p_1^2-4p_2)p'_1 - \tfrac{1}{2^6.3^2}p_1(p'^2_1-2p'_2) + \tfrac{1}{2^4.3^2.5^1}(p'^3_1 - 3p'_1p'_2\\
&+ 3p'_3)\\
[A(E)\mathrm{ch}(F_r)]_{16} =& \tfrac{r}{2^{15}.3^4.5^2.7^1}[381p_1^4+208p_2^2-904p_1^2p_2+512p_1p_3-192p_4] + \tfrac{1}{2^{11}.3^3.5^1.7^1}[-31p_1^3+44p_1p_2-16p_3]p'_1\\
&+ \tfrac{1}{2^{10}.3^3.5^1}(7p_1^2-4p_2)(p'^2_1-2p'_2) - \tfrac{1}{2^7.3^3.5^1}p_1(p'^3_1-3p'_1p'_2+3p'_3) + \tfrac{1}{2^7.3^2.5^1.7^1}[p'^4_1+2p'^2_2-4p'^2_1p'_2\\
&+ 4p'_1p'_3 - 4p'_4]
\end{aligned}$$

Next we repeat the same combination but in terms of the Chern classes via *Expansions* **??** and **??**. Observe how the expansion in terms of the Chern classes on the next page is so much more complex than the expansion in terms of the Pontrjagin classes above.

**Expansion 2.** *Â-genus combined with the Chern character written in terms of the Chern classes $c_i$ of complexification.*

**Twisted Â-genus in terms of Chern classes**

$$\begin{aligned}
[A(E)\mathrm{ch}(F_r)]_0 =& r\\
[A(E)\mathrm{ch}(F_r)]_4 =& \tfrac{r}{2^3.3^1}c_2 + \tfrac{1}{2^1}[c'^2_1 - 2c'_2]\\
[A(E)\mathrm{ch}(F_r)]_8 =& \tfrac{r}{2^7.3^2.5^1}[7c_2^2-4c_4] + \tfrac{1}{2^4.3^1}c_2[c'^2_1-2c'_2] + \tfrac{1}{2^3.3^1}[c'^4_1+2c'^2_2-4c'^2_1c'_2+4c'_1c'_3-4c'_4]\\
[A(E)\mathrm{ch}(F_r)]_{12} =& \tfrac{r}{2^{10}.3^3.5^1.7^1}[31c_2^3-44c_2c_4+16c_6] + \tfrac{1}{2^8.3^2.5^1}(7c_2^2-4c_4)(c'^2_1-2c'_2) + \tfrac{1}{2^6.3^2}c_2[c'^4_1+2c'^2_2-4c'^2_1c'_2+4c'_1c'_3\\
&- 4c'_4] + \tfrac{1}{2^4.3^2.5^1}[c'^6_1 - 2c'^3_2 + 9c'^2_1c'^2_2 + 3c'^2_3 - 6c'^4_1c'_2 + 6c'^3_1c'_3 - 12c'_1c'_2c'_3 - 6c'^2_1c'_4 + 6c'_2c'_4 + 6c'_1c'_5\\
&- 6c'_6]\\
[A(E)\mathrm{ch}(F_r)]_{16} =& \tfrac{r}{2^{15}.3^4.5^2.7^1}[381c_2^4+208c_4^2-904c_2^2c_4+512c_2c_6-192c_8] + \tfrac{1}{2^{11}.3^3.5^1.7^1}\left(31c_2^3-44c_2c_4+16c_6\right)\left(c'^2_1-2c'_2\right)\\
&+ \tfrac{1}{2^{10}.3^3.5^1}\left(7c_2^2-4c_4\right)\left(c'^4_1+2c'^2_2-4c'^2_1c'_2+4c'_1c'_3-4c'_4\right)\\
&+ \tfrac{1}{2^7.3^3.5^1}c_2[c'^6_1-2c'^3_2+9c'^2_1c'^2_2+3c'^2_3-6c'^4_1c'_2+6c'^3_1c'_3-12c'_1c'_2c'_3-6c'^2_1c'_4+6c'_2c'_4+6'_1c'_5-6c'_6]\\
&+ \tfrac{1}{2^7.3^2.5^1.7^1}[c'^8_1+2c'^4_2-16c'^2_1c'^3_2+20c'^4_1c'^2_2+12c'^2_1c'^2_3-8c'_2c'^2_3+4c'^2_4-8c'^6_1c'_2+8c'^5_1c'_3-32c'^3_1c'_2c'_3\\
&+ 24c'_1c'^2_2c'_3 - 8c'^4_1c'_4 + 24c'^2_1c'_2c'_4 - 8c'^2_2c'_4 - 16c'_1c'_3c'_4 + 8c'^3_1c'_5 - 16c'_1c'_2c'_5 + 8c'_3c'_5 - 8c'^2_1c'_6 + 8c'_2c'_6\\
&+ 8c'_1c'_7 - 8c'_8]
\end{aligned}$$

**Expansion 3.** *Twisted Â-genus combined with the Chern character of a $SU(r)$ bundle ($c'_1=0$).*

**Â-genus twisted with $SU(r)$ bundle in terms of Chern classes**

$$
\begin{aligned}
[A(E)\mathrm{ch}(F_r)]_0 =& r \\
[A(E)\mathrm{ch}(F_r)]_4 =& \tfrac{r}{2^3.3^1} c_2 - {c'}_2 \\
[A(E)\mathrm{ch}(F_r)]_8 =& \tfrac{r}{2^7.3^2.5^1}[7c_2^2 - 4c_4] - \tfrac{1}{2^3.3^1} c_2 {c'}_2 + \tfrac{1}{2^2.3^1}[{c'}_2^2 - 2{c'}_4] \\
[A(E)\mathrm{ch}(F_r)]_{12} =& \tfrac{r}{2^{10}.3^3.5^1.7^1}[31c_2^3 - 44c_2c_4 + 16c_6] - \tfrac{1}{2^7.3^2.5^1}(7c_2^2 - 4c_4){c'}_2 + \tfrac{1}{2^5.3^2} c_2[{c'}_2^2 - 2{c'}_4] \\
&+ \tfrac{1}{2^4.3^2.5^1}[-2{c'}_2^3 + 3{c'}_3^2 + 6{c'}_2{c'}_4 - 6{c'}_6] \\
[A(E)\mathrm{ch}(F_r)]_{16} =& \tfrac{r}{2^{15}.3^4.5^2.7^1}[381c_2^4 + 208c_4^2 - 904c_2^2c_4 + 512c_2c_6 - 192c_8] - \tfrac{1}{2^{10}.3^3.5^1.7^1}\left(31c_2^3 - 44c_2c_4 + 16c_6\right)c'_2 \\
&+ \tfrac{1}{2^9.3^3.5^1}\left(7c_2^2 - 4c_4\right)\left({c'}_2^2 - 2c'_4\right) + \tfrac{1}{2^7.3^3.5^1} c_2[-2{c'}_2^3 + 3{c'_3}^2 + 6c'_2c'_4 - 6c'_6] \\
&+ \tfrac{1}{2^7.3^2.5^1.7^1}[2{c'_2}^4 - 8c'_2{c'_3}^2 + 4{c'_4}^2 - 8{c'_2}^2c'_4 + 8c'_3c'_5 + 8c'_2c'_6 - 8c'_8]
\end{aligned}
$$

**Twisted Â-genus with simplifications.** Sometimes it is useful to consider twisted structures with vanishing classes both in terms of the Pontrjagin and Chern classes. However, one must not think that the vector bundles are equivalent, i.e. that setting $p_1 = 0$ would produce the same effect as $p'_1 = 0$. The twisted structure is a combination of two distinct elements.

The first simplification is when we take the String structure on the natural bundles itself.

**Expansion 4.** *Twisted Â-genus of a String bundle ($p_1 = 0$) combined with the Chern character of complexification (in terms of $p_i$'s).*

**Twisted A-genus with a String structure**

$$
\begin{aligned}
[A(E)\mathrm{ch}(F_r)]_0 =& r \\
[A(E)\mathrm{ch}(F_r)]_4 =& \tfrac{1}{2^1} p'_1 \\
[A(E)\mathrm{ch}(F_r)]_8 =& -\tfrac{r}{2^5.3^2.5^1} p_2 + \tfrac{1}{2^3.3^1}[{p'}_1^2 - 2{p'}_2] \\
[A(E)\mathrm{ch}(F_r)]_{12} =& -\tfrac{r}{2^6.3^3.5^1.7^1} p_3 - \tfrac{1}{2^6.3^2.5^1} p_2{p'}_1 + \tfrac{1}{2^4.3^2.5^1}[{p'}_1^3 - 3{p'}_1{p'}_2 + 3{p'}_3] \\
[A(E)\mathrm{ch}(F_r)]_{16} =& \tfrac{r}{2^{15}.3^4.5^2.7^1}[208p_2^2 - 192p_4] - \tfrac{1}{2^7.3^3.5^1.7^1} p_3{p'}_1 - \tfrac{1}{2^8.3^3.5^1} p_2[{p'}_1^2 - 2{p'}_2] \\
&+ \tfrac{1}{2^7.3^2.5^1.7^1}[{p'}_1^4 + 2{p'}_2^2 - 4{p'}_1^2{p'}_2 + 4{p'}_1{p'}_3 - 4{p'}_4]
\end{aligned}
$$

**Expansion 5.** *Twisted Â-genus combined with the Chern character of complexification for a String bundle ($p'_1 = 0$).*

**Â-genus twisted with a String bundle**

$$
\begin{aligned}
[A(E)\mathrm{ch}(F_r)]_0 =& r \\
[A(E)\mathrm{ch}(F_r)]_4 =& -\tfrac{r}{2^3.3^1} p_1 \\
[A(E)\mathrm{ch}(F_r)]_8 =& \tfrac{r}{2^7.3^2.5^1}[7p_1^2 - 4p_2] - \tfrac{1}{2^2.3^1}{p'}_2 \\
[A(E)\mathrm{ch}(F_r)]_{12} =& \tfrac{r}{2^{10}.3^3.5^1.7^1}[-31p_1^3 + 44p_1p_2 - 16p_3] + \tfrac{1}{2^5.3^2} p_1{p'}_2 + \tfrac{1}{2^4.3^1.5^1}{p'}_3 \\
[A(E)\mathrm{ch}(F_r)]_{16} =& \tfrac{r}{2^{15}.3^4.5^2.7^1}[381p_1^4 + 208p_2^2 - 904p_1^2p_2 + 512p_1p_3 - 192p_4] \\
&- \tfrac{1}{2^9.3^3.5^1}(7p_1^2 - 4p_2){p'}_2 - \tfrac{1}{2^7.3^2.5^1} p_1{p'}_3 + \tfrac{1}{2^6.3^2.5^1.7^1}[{p'}_2^2 - 2{p'}_4]
\end{aligned}
$$

We can now combine the above two function for when both the natural bundle and the twisting bundle have String structures.

**Expansion 6.** *Twisted Â-genus of $p_1$-structure or String bundles - the Â-genus of a String bundle ($p_1 = 0$) combined with the Chern character of complexification for a String bundle ($p'_1 = 0$).*

**Twisted A-genus with both bundles having String structures**

$$
\begin{aligned}
[A(E)\mathrm{ch}(F_r)]_0 =& r \\
[A(E)\mathrm{ch}(F_r)]_4 =& 0 \\
[A(E)\mathrm{ch}(F_r)]_8 =& -\tfrac{r}{2^5.3^2.5^1} p_2 - \tfrac{1}{2^2.3^1} {p'}_2 \\
[A(E)\mathrm{ch}(F_r)]_{12} =& -\tfrac{r}{2^6.3^3.5^1.7^1} p_3 + \tfrac{1}{2^4.3^1.5^1} {p'}_3 \\
[A(E)\mathrm{ch}(F_r)]_{16} =& \tfrac{r}{2^{15}.3^4.5^2.7^1} [208p_2^2 - 192p_4] + \tfrac{1}{2^7.3^3.5^1} p_2 {p'}_2 + \tfrac{1}{2^6.3^2.5^1.7^1} [{p'}_2^2 - 2{p'}_4]
\end{aligned}
$$

2.2. **Twisted L-genus.** The twisted L-genus was computed using exactly the same method as for the twisted Â-genus, which was described at the beginning of the section. The first expansion of the twisted L-genus combines *Expansions* **??** and **??**.

**Expansion 7.** *Twisted L-genus combined with the Chern character of complexification written in terms of the Pontrjagin classes.*

**Twisted L-genus in terms of the Pontrjagin classes**

$$
\begin{aligned}
[L(E)\mathrm{ch}(F_r)]_0 =& r \\
[L(E)\mathrm{ch}(F_r)]_4 =& \tfrac{r}{3^1} p_1 + \tfrac{1}{2^1} p'_1 \\
[L(E)\mathrm{ch}(F_r)]_8 =& \tfrac{r}{3^2.5^1} [7p_2 - p_1^2] + \tfrac{1}{2^1.3^1} p_1 {p'}_1 + \tfrac{1}{2^3.3^1} [{p'}_1^2 - 2p'_2] \\
[L(E)\mathrm{ch}(F_r)]_{12} =& \tfrac{r}{3^3.5^1.7^1} [2p_1^3 - 13p_1p_2 + 62p_3] + \tfrac{1}{2^1.3^2.5^1} [7p_2 - p_1^2] p'_1 + \tfrac{1}{2^3.3^2} p_1 [{p'}_1^2 - 2p'_2] + \tfrac{1}{2^4.3^2.5^1} [{p'}_1^3 - 3p'_1p'_2 + 3p'_3] \\
[L(E)\mathrm{ch}(F_r)]_{16} =& \tfrac{r}{3^4.5^2.7^1} [-3p_1^4 - 19p_2^2 + 22p_1^2p_2 - 71p_1p_3 + 381p_4] + \tfrac{1}{2^1.3^3.5^1.7^1} [2p_1^3 - 13p_1p_2 + 62p_3] p'_1 \\
&+ \tfrac{1}{2^3.3^3.5^1} (7p_2 - p_1^2)({p'}_1^2 - 2p'_2) + \tfrac{1}{2^4.3^3.5^1} p_1 [{p'}_1^3 - 3p'_1p'_2 + 3p'_3] + \tfrac{1}{2^7.3^2.5^1.7^1} [{p'}_1^4 + 2{p'}_2^2 - 4{p'}_1^2 p'_2 + 4p'_1p'_3 \\
&- 4{p'}_4]
\end{aligned}
$$

For the twisted L-genus expansion in terms of the Chern classes one must use *Expansions* **??** and **??**.

**Expansion 8.** *Twisted L-genus combined with the Chern character written in terms of the Chern classes of complexification.*

**Twisted L-genus in terms of the Chern classes**

$$
\begin{aligned}
[L(E)\mathrm{ch}(F_r)]_0 =& r \\
[L(E)\mathrm{ch}(F_r)]_4 =& -\tfrac{r}{3^1} c_2 + \tfrac{1}{2^1} [{c'}_1^2 - 2c'_2] \\
[L(E)\mathrm{ch}(F_r)]_8 =& \tfrac{1}{3^2.5^1} r [7c_4 - c_2^2] - \tfrac{1}{2^1.3^1} c_2 [{c'}_1^2 - 2c'_2] + \tfrac{1}{2^3.3^1} [{c'}_1^4 + 2{c'}_2^2 - 4{c'}_1^2 c'_2 + 4c'_1c'_3 - 4c'_4] \\
[L(E)\mathrm{ch}(F_r)]_{12} =& \tfrac{1}{3^3.5^1.7^1} r [-2c_2^3 + 13c_2c_4 - 62c_6] + \tfrac{1}{2^1.3^2.5^1} (7c_4 - c_2^2)({c'}_1^2 - 2c'_2) - \tfrac{1}{2^3.3^2} c_2 [{c'}_1^4 + 2{c'}_2^2 - 4{c'}_1^2 {c'}_2 + 4c'_1c'_3 - 4c'_4] \\
&+ \tfrac{1}{2^4.3^2.5^1} [{c'}_1^6 - 2{c'}_2^3 + 9{c'}_1^2 {c'}_2^2 + 3{c'}_3^2 - 6{c'}_1^4 c'_2 + 6{c'}_1^3 c'_3 - 12c'_1c'_2c'_3 - 6{c'}_1^2 c'_4 + 6c'_2c'_4 + 6c'_1c'_5 - 6c'_6] \\
[L(E)\mathrm{ch}(F_r)]_{16} =& \tfrac{1}{3^4.5^2.7^1} r \left(-3c_2^4 - 19c_4^2 + 22c_2^2c_4 - 71c_2c_6 + 381c_8\right) + \tfrac{1}{2^1.3^3.5^1.7^1} \left(-2c_2^3 + 13c_2c_4 - 62c_6\right) \left({c'}_1^2 - 2c'_2\right) \\
&+ \tfrac{1}{2^3.3^3.5^1} \left(7c_4 - c_2^2\right) \left({c'}_1^4 + 2{c'}_2^2 - 4{c'}_1^2 c'_2 + 4c'_1c'_3 - 4c'_4\right) \\
&- \tfrac{1}{2^4.3^3.5^1} [c_2({c'}_1^6 - 2{c'}_2^3 + 9{c'}_1^2 {c'}_2^2 + 3{c'}_3^2 - 6{c'}_1^4 c'_2 + 6{c'}_1^3 c'_3 - 12c'_1c'_2c'_3 - 6{c'}_1^2 c'_4 + 6c'_2c'_4 + 6c'_1c'_5 - 6c'_6)] \\
&+ \tfrac{1}{2^7.3^2.5^1.7^1} [{c'}_1^8 + 2{c'}_2^4 - 16{c'}_1^2 {c'}_2^3 + 20{c'}_1^4 {c'}_2^2 + 12{c'}_1^2 {c'}_3^2 - 8c'_2 {c'}_3^2 + 4{c'}_4^2 - 8{c'}_1^6 c'_2 + 8{c'}_1^5 c'_3 - 32{c'}_1^3 c'_2c'_3 \\
&+ 24c'_1 {c'}_2^2 c'_3 - 8{c'}_1^4 c'_4 + 24{c'}_1^2 c'_2c'_4 - 8{c'}_2^2 c'_4 - 16c'_1c'_3c'_4 + 8{c'}_1^3 c'_5 - 16c'_1c'_2c'_5 + 8c'_3c'_5 - 8{c'}_1^2 c'_6 + 8c'_2c'_6 \\
&+ 8c'_1c'_7 - 8c'_8]
\end{aligned}
$$

**Twisted L-genus with simplifications.** All the simplifications were done using `ReplaceAll` operator on the two expansion of the twisted L-genus above.

**Expansion 9.** *Twisted L-genus of a String bundle ($p_1 = 0$) combined with the Chern character of complexification (in terms of $p_i$'s).*

**Twisted L-genus of String bundle**

$$\begin{aligned}
[L(E)\mathrm{ch}(F_r)]_0 =& r \\
[L(E)\mathrm{ch}(F_r)]_4 =& \tfrac{1}{2^1} {p'}_1 \\
[L(E)\mathrm{ch}(F_r)]_8 =& \tfrac{7r}{3^2.5^1} p_2 + \tfrac{1}{2^3.3^1} [{p'}_1^2 - 2p'_2] \\
[L(E)\mathrm{ch}(F_r)]_{12} =& \tfrac{62r}{3^3.5^1.7^1} p_3 + \tfrac{7}{2^1.3^2.5^1} p_2 p'_1 + \tfrac{1}{2^4.3^2.5^1} [{p'}_1^3 - 3p'_1 p'_2 + 3p'_3] \\
[L(E)\mathrm{ch}(F_r)]_{16} =& \tfrac{r}{3^4.5^2.7^1} [-19p_2^2 + 381p_4] + \tfrac{31}{3^3.5^1.7^1} p_3 p'_1 + \tfrac{7}{2^3.3^3.5^1} p_2 [{p'}_1^2 - 2p'_2] \\
&+ \tfrac{1}{2^7.3^2.5^1.7^1} [{p'}_1^4 + 2{p'}_2^2 - 4{p'}_1^2 p'_2 + 4p'_1 p'_3 - 4p'_4]
\end{aligned}$$

**Expansion 10.** *Twisted L-genus combined with the Chern character of complexification for a String bundle ($p'_1 = 0$).*

**Twisted L-genus twisted by a String bundle**

$$\begin{aligned}
[L(E)\mathrm{ch}(F_r)]_0 =& r \\
[L(E)\mathrm{ch}(F_r)]_4 =& \tfrac{r}{3^1} p_1 \\
[L(E)\mathrm{ch}(F_r)]_8 =& \tfrac{r}{3^2.5^1} [7p_2 - p_1^2] - \tfrac{1}{2^2.3^1} p'_2 \\
[L(E)\mathrm{ch}(F_r)]_{12} =& \tfrac{r}{3^3.5^1.7^1} [2p_1^3 - 13p_1 p_2 + 62p_3] - \tfrac{1}{2^2.3^2} p_1 p'_2 + \tfrac{1}{2^4.3^1.5^1} p'_3 \\
[L(E)\mathrm{ch}(F_r)]_{16} =& \tfrac{r}{3^4.5^2.7^1} [-3p_1^4 - 19p_2^2 + 22p_1^2 p_2 - 71p_1 p_3 + 381p_4] \\
&- \tfrac{1}{2^2.3^3.5^1} (7p_2 - p_1^2) p'_2 + \tfrac{1}{2^4.3^2.5^1} p_1 p'_3 + \tfrac{1}{2^6.3^2.5^1.7^1} [{p'}_2^2 - 2p'_4]
\end{aligned}$$

Observe how significantly simpler the expansions below are as compared to their initial form presented above.

**Expansion 11.** *Twisted L-genus of $p_1$-structure or String bundles - the L-genus of a String bundle ($p_1 = 0$) combined with the Chern character of complexification for a String bundle ($p'_1 = 0$).*

**Twisted L-genus of String bundles twisted by String bundles**

$$\begin{aligned}
[L(E)\mathrm{ch}(F_r)]_0 =& r \\
[L(E)\mathrm{ch}(F_r)]_4 =& 0 \\
[L(E)\mathrm{ch}(F_r)]_8 =& \tfrac{7r}{3^2.5^1} p_2 - \tfrac{1}{2^2.3^1} p'_2 \\
[L(E)\mathrm{ch}(F_r)]_{12} =& \tfrac{62r}{3^3.5^1.7^1} p_3 + \tfrac{1}{2^4.3^1.5^1} p'_3 \\
[L(E)\mathrm{ch}(F_r)]_{16} =& \tfrac{r}{3^4.5^2.7^1} [-19p_2^2 + 381p_4] - \tfrac{7}{2^2.3^3.5^1} p_2 p'_2 + \tfrac{1}{2^6.3^2.5^1.7^1} [{p'}_2^2 - 2p'_4]
\end{aligned}$$

**Expansion 12.** *Twisted L-genus combined with the Chern character of a $SU(r)$ bundle ($c'_1 = 0$).*

**L-genus twisted by $SU(r)$ bundle**

$$\begin{aligned}
[L(E)\mathrm{ch}(F_r)]_0 =& r \\
[L(E)\mathrm{ch}(F_r)]_4 =& -\tfrac{r}{3^1} c_2 - c'_2 \\
[L(E)\mathrm{ch}(F_r)]_8 =& \tfrac{1}{3^2.5^1} r[7c_4 - c_2^2] + \tfrac{1}{3^1} c_2 c'_2 + \tfrac{1}{2^2.3^1} [{c'}_2^2 - 2c'_4] \\
[L(E)\mathrm{ch}(F_r)]_{12} =& \tfrac{r}{3^3.5^1.7^1} [-2c_2^3 + 13c_2 c_4 - 62c_6] - \tfrac{1}{3^2.5^1} [7c_4 - c_2^2] c'_2 - \tfrac{1}{2^3.3^2} c_2 [2{c'}_2^2 - 4c'_4] \\
&+ \tfrac{1}{2^4.3^2.5^1} [-2{c'}_2^3 + 3{c'}_3^2 + 6c'_2 c'_4 - 6c'_6] \\
[L(E)\mathrm{ch}(F_r)]_{16} =& \tfrac{1}{3^4.5^2.7^1} r[-3c_2^4 - 19c_4^2 + 22c_2^2 c_4 - 71c_2 c_6 + 381c_8] - \tfrac{1}{3^3.5^1.7^1} [-2c_2^3 + 13c_2 c_4 - 62c_6] c'_2 \\
&+ \tfrac{1}{2^2.3^3.5^1} \left(7c_4 - c_2^2\right) \left({c'}_2^2 - 2c'_4\right) - \tfrac{1}{2^4.3^3.5^1} [-2{c'}_2^3 + 3{c'}_3^2 + 6c'_2 c'_4 - 6c'_6] \\
&+ \tfrac{1}{2^7.3^2.5^1.7^1} [2{c'}_2^4 - 8c'_2 {c'}_3^2 + 4{c'}_4^2 - 8{c'}_2^2 c'_4 + 8c'_3 c'_5 + 8c'_2 c'_6 - 8c'_8]
\end{aligned}$$

2.3. **Twisted Todd genus.** The method for computing the twisted Todd genus is the same in the previous two cases with the Â-genus and L-genus. However, we wrote a separate Mathematica function for computing the twisted Todd, due to the fact that originally the Todd genus is written in terms of the Chern classes as opposed to the Â and L genera that are written in terms of Pontrjagin classes. Therefore, it was technically a better solution to introduce separate formulas for each of the two cases. Since both Todd genus and Chern character are in terms of Chern classes then no further conversions are necessary.

One should use *Expansions* **??** and **??** to obtain the following expansion.

**Expansion 13.** *Todd genus combined with the Chern character in terms of the Chern classes.*

**Twisted Todd genus in terms of the Chern classes**

$$
\begin{aligned}
[\mathrm{Td}(E)\mathrm{ch}(F_r)]_0 =& r \\
[\mathrm{Td}(E)\mathrm{ch}(F_r)]_4 =& \tfrac{1}{2^2.3^1} r \left(c_1^2 + c_2\right) + \tfrac{1}{2^1} c_1 c_1' + \tfrac{1}{2^1}\left({c'}_1^2 - 2c_2'\right) \\
[\mathrm{Td}(E)\mathrm{ch}(F_r)]_8 =& \tfrac{1}{2^4.3^2.5^1} r \left(-c_1^4 + c_3 c_1 + 3c_2^2 + 4c_1^2 c_2 - c_4\right) + \tfrac{1}{2^3.3^1} c_1 c_2 c_1' + \tfrac{1}{2^3.3^1}\left(c_1^2 + c_2\right)\left({c'}_1^2 - 2c_2'\right) \\
&+ \tfrac{1}{2^2.3^1} c_1 \left({c'}_1^3 - 3c_1' c_2' + 3c_3'\right) + \tfrac{1}{2^3.3^1}\left({c'}_1^4 + 2{c'}_2^2 - 4{c'}_1^2 c_2' + 4c_1' c_3' - 4c_4'\right) \\
[\mathrm{Td}(E)\mathrm{ch}(F_r)]_{12} =& \tfrac{r}{2^6.3^3.5^1.7^1}[2c_1^6 + 10c_2^3 + 11c_1^2 c_2^2 - c_3^2 - 12c_1^4 c_2 + 5c_1^3 c_3 + 11c_1 c_2 c_3 - 5c_1^2 c_4 - 9c_2 c_4 - 2c_1 c_5 + 2c_6] \\
&+ \tfrac{1}{2^5.3^2.5^1}[c_3 c_1^2 - c_4 c_1 + 3c_1 c_2^2 - c_1^3 c_2] c_1' + \tfrac{1}{2^5.3^2.5^1}[-c_1^4 + c_3 c_1 + 3c_2^2 + 4c_1^2 c_2 - c_4]\left({c'}_1^2 - 2c_2'\right) \\
&+ \tfrac{1}{2^4.3^2} c_1 c_2 [{c'}_1^3 - 3c_1' c_2' + 3c_3'] + \tfrac{1}{2^5.3^2}\left(c_1^2 + c_2\right)[{c'}_1^4 + 2{c'}_2^2 - 4{c'}_1^2 c_2' + 4c_1' c_3' - 4c_4'] \\
&+ \tfrac{1}{2^4.3^1.5^1} c_1 [{c'}_1^5 + 5c_1' {c'}_2^2 - 5{c'}_1^3 c_2' + 5{c'}_1^2 c_3' - 5c_2' c_3' - 5c_1' c_4' + 5c_5'] \\
&+ \tfrac{1}{2^4.3^2.5^1}[{c'}_1^6 - 2{c'}_2^3 + 9{c'}_1^2 {c'}_2^2 + 3{c'}_3^2 - 6{c'}_1^4 c_2' + 6{c'}_1^3 c_3' - 12c_1' c_2' c_3' - 6{c'}_1^2 c_4' + 6c_2' c_4' + 6c_1' c_5' - 6c_6']
\end{aligned}
$$

**Twisted Todd genus with simplifications.** The simplifications were implemented via the `ReplaceAll` function in Mathematica. Although one could argue that the simplified expansions are still quite elaborate, it is definitely a drastic reduction from the standard twisted Todd listed above.

**Expansion 14.** *Todd genus of a complex bundle ($c_1 = 0$) combined with the Chern character.*

**Twisted Todd genus of a complex bundle $c_1$ zero**

$$
\begin{aligned}
[\mathrm{Td}(E)\mathrm{ch}(F_r)]_0 =& r \\
[\mathrm{Td}(E)\mathrm{ch}(F_r)]_4 =& \tfrac{1}{2^2.3^1} r c_2 + \tfrac{1}{2^1}[{c'}_1^2 - 2c_2'] \\
[\mathrm{Td}(E)\mathrm{ch}(F_r)]_8 =& \tfrac{1}{2^4.3^2.5^1} r[3c_2^2 - c_4] + \tfrac{1}{2^3.3^1} c_2[{c'}_1^2 - 2c_2'] + \tfrac{1}{2^3.3^1}[{c'}_1^4 + 2{c'}_2^2 - 4{c'}_1^2 c_2' + 4c_1' c_3' - 4c_4'] \\
[\mathrm{Td}(E)\mathrm{ch}(F_r)]_{12} =& \tfrac{r}{2^6.3^3.5^1.7^1}[10c_2^3 - c_3^2 - 9c_2 c_4 + 2c_6] + \tfrac{1}{2^5.3^2.5^1}(3c_2^2 - c_4)({c'}_1^2 - 2c_2') + \tfrac{1}{2^5.3^2} c_2[{c'}_1^4 + 2{c'}_2^2 - 4{c'}_1^2 c_2' + 4c_1' c_3' \\
&- 4c_4'] + \tfrac{1}{2^4.3^2.5^1}[{c'}_1^6 - 2{c'}_2^3 + 9{c'}_1^2 {c'}_2^2 + 3{c'}_3^2 - 6{c'}_1^4 c_2' + 6{c'}_1^3 c_3' - 12c_1' c_2' c_3' - 6{c'}_1^2 c_4' + 6c_2' c_4' + 6c_1' c_5' \\
&- 6c_6']
\end{aligned}
$$

One might notice that vanishing $c_1 = 0$ simplified the twisted Todd more than vanishing of $c_1' = 0$, which emphasizes the earlier point on how the twisted structure is created out of two indeed distinct elements.

**Expansion 15.** *Todd genus combined with the Chern character of a $SU(r)$-bundle ($c_1' = 0$).*

**Twisted Todd genus with a twist by an SU bundle $c_1'$ zero**

$$
\begin{aligned}
[\mathrm{Td}(E)\mathrm{ch}(F_r)]_0 =& r \\
[\mathrm{Td}(E)\mathrm{ch}(F_r)]_4 =& \tfrac{1}{2^2.3^1} r \left(c_1^2 + c_2\right) - c_2' \\
[\mathrm{Td}(E)\mathrm{ch}(F_r)]_8 =& \tfrac{1}{2^4.3^2.5^1} r \left(-c_1^4 + c_3 c_1 + 3c_2^2 + 4c_1^2 c_2 - c_4\right) - \tfrac{1}{2^2.3^1}\left(c_1^2 + c_2\right) c_2' + \tfrac{1}{2^2} c_1 c_3' + \tfrac{1}{2^2.3^1}\left({c'}_2^2 - 2c_4'\right) \\
[\mathrm{Td}(E)\mathrm{ch}(F_r)]_{12} =& \tfrac{r}{2^6.3^3.5^1.7^1}[2c_1^6 + 10c_2^3 + 11c_1^2 c_2^2 - c_3^2 - 12c_1^4 c_2 + 5c_1^3 c_3 + 11c_1 c_2 c_3 - 5c_1^2 c_4 - 9c_2 c_4 - 2c_1 c_5 + 2c_6] \\
&- \tfrac{1}{2^4.3^2.5^1}[-c_1^4 + c_3 c_1 + 3c_2^2 + 4c_1^2 c_2 - c_4] c_2' + \tfrac{1}{2^4.3^1} c_1 c_2 c_3' + \tfrac{1}{2^4.3^2}\left(c_1^2 + c_2\right)[{c'}_2^2 - 2c_4'] \\
&+ \tfrac{1}{2^4.3^1} c_1[-c_2' c_3' + c_5'] + \tfrac{1}{2^4.3^2.5^1}[-2{c'}_2^3 + 3{c'}_3^2 + 6c_2' c_4' - 6c_6']
\end{aligned}
$$

**Expansion 16.** *Twisted Todd genus of complex bundles - the Todd genus of a $SU$-bundle ($c_1 = 0$) combined with the Chern character of a $SU(r)$-bundle ($c_1' = 0$).*

**Twisted Todd genus of complex bundles $c_1$ $c_1'$ zero**

$$
\begin{aligned}
[\mathrm{Td}(E)\mathrm{ch}(F_r)]_0 =& r\\
[\mathrm{Td}(E)\mathrm{ch}(F_r)]_4 =& \tfrac{1}{2^2.3^1} r c_2 - c_2'\\
[\mathrm{Td}(E)\mathrm{ch}(F_r)]_8 =& \tfrac{1}{2^4.3^2.5^1} r\left(3c_2^2 - c_4\right) - \tfrac{1}{2^2.3^1} c_2 c_2' + \tfrac{1}{2^2.3^1}\left({c'}_2^2 - 2c_4'\right)\\
[\mathrm{Td}(E)\mathrm{ch}(F_r)]_{12} =& \tfrac{r}{2^6.3^3.5^1.7^1}[10c_2^3 - c_3^2 - 9c_2c_4 + 2c_6] - \tfrac{1}{2^4.3^2.5^1}[3c_2^2 - c_4]c_2' + \tfrac{1}{2^4.3^2} c_2[{c'}_2^2 - 2c_4']\\
&+ \tfrac{1}{2^4.3^2.5^1}[-2{c'}_2^3 + 3{c'}_3^2 + 6c_2'c_4' - 6c_6']
\end{aligned}
$$

With this final result we conclude the twisted genera section as well as the result part of the paper more broadly.

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