COMPUTATIONS OF CHARACTERISTIC CLASSES AND GENERA: A PRACTICAL TOOLKIT FOR BEGINNERS AND PRACTITIONERS

HISHAM SATI, SILVIU-MARIAN UDRESCU, AND ERIKA ZOGLA

Abstract. This paper provides explicit combinatorial expansions of the Chern classes, Pontrjagin classes, Chern character, as well as the Chern character and Chern classes of a tensor product of two vector bundles. One will also find expansions of the Â-genus, L-genus, Todd genus, their twisted forms, as well as relations among the three genera. The computational methods are elementary, but we hope that the clearly displayed and ready-to-use results – which do not seem to be otherwise available – will be useful. This is the first part of research based on the undergraduate project of the second and third authors advised by the first author.

Contents

1. Introduction 1
2. Mathematica implementation and computational tools 3
2.1. Chern classes and Chern character 4
2.2. The Splitting Principle 4
3. Expansions of the Characteristic Classes 5
3.1. Chern character & Chern classes 5
Chern character 6
3.2. Pontrjagin classes 8
Pontrjagin classes 8
3.3. Tensor product 12
4. Expansions of the Genera 16
4.1. Genera 16
1. The Â-genus 16
2. The L-genus 16
3. The Todd genus 17
4.2. Â-genus 18
4.3. The L-genus 24
L-genus with simplifications 24
L-genus and complexification 27
4.4. The Todd genus 30
5. Relations among genera 32
5.1. **Â and L genera** 32
5.2. **Todd and L genera** 34
5.3. **Todd and Â genera** 34
References 35

1. Introduction

The goal of this paper is to perform computations of characteristic classes and genera of vector bundles which would yield explicit handy results that would be readily utilizable by mathematicians and physicists not wishing to perform such straightforward but cumbersome computations themselves. The main focus has been on explicitly expand existing combinatorial formulas that are often lengthy and complicated to compute by hand and that cannot be easily looked up in the literature. The results are tabulated throughout the paper which some readers may find very useful. While this work by no means exhausts the subject, every attempt was made to provide comprehensive listings of computations with clear notation and descriptions in the few narrow lines of study tackled by this paper. Additionally, Mathematica codes for every expansion are available upon request.

The calculations are elementary but systematic. Why would this be useful? Someone who is willing to go through the task can certainly reproduce them. However, for someone who uses characteristic classes extensively, having to do such elementary but often laborious calculations repeatedly might not be the most efficient task. So we provide the list at least for those students and researchers who use such formulas extensively. We believe that being able to reproduce the formulas and having these displayed in front of a reader are two different things. One byproduct of the latter might be allowing for known patterns to be seen visually and perhaps for new ones to be discovered. Indeed, the patterns do reveal specific interrelations that might otherwise not be obvious from the general formulas.

We also give an overview of previous work done in this direction, a description of the computational approach theoretically, and the resulting expansions themselves. As for the organization, some overarching theoretical background and explanation of the approach is presented in section 2, yet in most cases the reader will find further details at the beginning of the corresponding sections and commentaries directly next to the expansions. We will be mainly interested in the Chern classes and the Pontrjagin classes, and the Chern character.

A bundle consists of a base, which may be a space or a manifold, and of a fiber which is attached to every point of the base [16] [26]. The fiber may be a vector space, a Lie group or another manifold (or topological space). Yet for the purposes of the current paper we will specifically consider a *vector bundle* $V \to E \to M$ which has a manifold M as its base, a vector space V as its fiber, and E denotes the total space. Nevertheless, one can deduce results for other types of bundles by associating them to vector bundles, although we will not do this explicitly here (see, e.g., [27]).

Characteristic classes [3] are polynomials that determine whether a bundle is trivial or not, as well as give some idea about the extent of its non-triviality. A non-trivial bundle is "twisted" and it would be indicated by a non-zero result for its characteristic class. The characteristic class of a given bundle depends on the dimension, whether the vector space fiber is real or complex, as well as on the nontrivial topology (naively, the (non)linearity and presence or absence of holes) in the manifold base. There are groups of transformations that act on V. If the fiber (isomorphic to) V is a real vector space then the group of transformations is $O(n)$, the rotation group. On the other hand, if V is complex then the group of transformations is $U(n)$, the unitary group.

We will consider characteristic classes from a combinatorial point of view, inevitably hiding the rich geometric nature (via Chern-Weil theory) and only touching briefly on the rich algebraic-topological nature (via cohomology of classifying spaces). Extensive theoretical background on characteristic classes and genera can be found in [3][4][5][6][12][26][8][7][21][24], with useful summaries and applications, for instance, in [10][28][2][34][9]. We will only provide enough background to be able to state the results meaningfully.

The expansions obtained are grouped into two large sections called *Characteristic classes* and *Genera*. Section 3, *Characteristic classes*, tabulates expansions of combinatorial formulas of characteristic classes, particularly, expansions that express Chern classes, Chern character and Pontrjagin classes in terms of each other. It also features a few simplifications of these expressions where relevant. Tensor products $E \otimes F$ of two vector bundles E and F are considered in the final subsection of *Characteristic classes*, where expansions of Chern character and Chern classes of a $E \otimes F$ are given.

Section 4 deals with the three main classical genera, the Â-genus, the L-genus, and the Todd genus. The first three subsections of the *Genera* section tabulate the standard expansions of the three genera as well as their various simplifications, for instance, in the presence of String and Fivebrane structures, in the sense of [30] [31]. Additionally, these three subsections also include complexification expansions. The fourth subsection of the *Genera* directly builds upon the results listed above by finding relations among the three genera.

	Real bundles	**Complex bundles**
Classes	Pontrjagin p_i	Chern c_i
Characters	Pontrjagin Ph	Chern ch
Genera	A-genus $\widehat{A}$, L-genus L	Todd Td

Finally, the fifth subsection features computations of the twisted genera which are combinations of each of the three genera with the Chern character. Like in the previous cases this section also provides various types of simplifications in the presence of extra structures (complex, Calabi-Yau [11], String [1], Fivebrane [30][31], and Ninebrane [29] structures). [1]

[1]We emphasize that we will be working rationally, so that issues about congruence and divisibility in the corresponding cohomology rings can be avoided.

Name of structure	Calabi-Yau	complex String	String	Fivebrane	Ninebrane
Condition	$c_1 = 0$	$c_1 = 0 = c_2$	$p_1 = 0$	$p_1 = 0 = p_2$	$p_1 = p_2 = p_3 = 0$

The genera are typically computed directly from their defining series expansions, which were stated in Section 4. The first few expansions of all three genera are readily available - one may even find accurate formulae up to rank 4 on [33], the Todd genus up to rank 5 in [28], rank 4-6 in the notes [9], and for the Â-genus up to rank 6 in [14], which also explains in detail how to compute these expansions in Mathematica. Now, for the L-genus up to an impressive rank 14 one should visit Carl McTague's blog [25]. As a matter of fact, Carl McTague wrote a universal Mathematica formula that allows one to compute a genus expansion by taking its characteristic power series and the desired degree as input. The formula is presented below and it was indeed crucial, for this paper for obtaining the results in Section 4 by modifications throughout Section 4.

Although one may find the standard expansions of the three genera in question up to certain degrees, there is no literature that would explicitly state the various simplifications applicable to different types of bundles, genera of complexification, relations among them, or the twisted structures. All of these are tackled later in the paper. To help with patterns that might arise, we have chosen to write down (large) numerical factors in terms of their prime factorizations. As indicated in Section 3.1 such computations are very important for anomaly cancellations in field theory [2], as well as in string theory in the presence of extra higher structures [30].

We have chosen to use Mathematica as our tool of choice for programming the functions, as we believe this would enable a wider audience to benefit from this work, due to the program's popularity. Although the computations for high degrees in Mathematica may take longer than a low-level highly specialized programming language like SINGULAR [17], there is nevertheless a clear trade-off with the ease of use. Naturally, the speed of computation depends also on the machine used.

The need for such computations to be carried out is illustrated well by the fact that Carl McTague wrote the L-genus expansion based on a request by Andrew Ranicki [25]. Besides mathematicians working on Topology or Algebraic Geometry, the results should be also useful to physicists and applied mathematicians working on physics problems. This is due to the fact that all forces of nature, namely, gravity, electromagnetism, strong nuclear force, and weak nuclear force, can be modelled using fiber bundles [2][28]. Furthermore, the numerical quantities obtained as a result of computing characteristic classes, which measure the (non)triviality of a bundle, also provide some measurement of a physical quantity, such as the charge, in the respective theories that interpret each of the forces.

This paper grew from an undergraduate project of the second and third authors under the guidance of the first author.

2. Mathematica implementation and computational tools

In this section we will present a brief overview of previous work done in finding explicit expressions of the characteristic classes and genera. This discussion will be intertwined with explanations of various approaches of computation which were developed and/or used in the literature.

We have kept in mind that the goal is to provide ready-made formulae and code that would be convenient for others to use towards their own ends. Therefore, we prioritized coherence, completeness, accuracy, and usability. Some of results (e.g. L-genus, Chern classes of tensor bundles) reproduce existing in other sources (but mostly in lower degrees), but we include for coherence and completeness. Other times we thought it worthwhile to rewrite code that would allow one to make such computations, because previous ones were done in a non-standard programming language. This is the case with the work O. Iena, who developed a library called `chern.lib`, but in a very specialized programming language called SINGULAR[2] in the context of algebraic geometry. While that work is indeed very comprehensive, the specificity of the SINGULAR environment may be prohibitive for many. We chose Mathematica as the programming tool this paper exactly because it seems to be the most widely used programming environment within the field. Finally, in many cases the seemingly duplicated lists were actually shorter elsewhere and have been been extended here (e.g. the Todd genus). In other instances, combinatorial formulas do exist, but we found them to be still quite abstract and not readily utilizable (see [22]).

Most of the expressions are truncated around degree 10 but not always so. Some lists of expansions were truncated at a certain degree because their next iteration would have been incredibly long, for instance, the Twisted Todd expansion. In other instances Mathematica was taking a very long time in producing the next iteration, as was the case with the genera expansions. Consider that to obtain only the 12-th degree for the

[2]See https://www.singular.uni-kl.de/.

Todd genus in terms of Pontrjagin classes we had to wait for over 2 hours, whereas the computation of degrees 0 to 10 took around 15 minutes in total. The rest were arbitrarily truncated around degree 9 or 10, trusting that one would be able to make use of the Mathematica function to obtain further results if needed.

2.1. **Chern classes and Chern character.** In [17], via symbolic computations with Chern classes the author works out expansions with the existing formulas, and also demonstrates how alternative ways of computation compare in terms of the computer time taken to solve the problems. That paper is among the fundamental sources upon which the current paper builds. Let us expand on some of the conclusions presented in that paper which are relevant for our case.

For instance, let us consider the observations made in [17] regarding the Chern character computations. The sum of the Chern roots raised to the k-th power yields a symmetric polynomial in $a_1, ..., a_r$ of degree k, which can be written as a polynomial in $c_1, ..., c_k$. There exists an approach, called the *elimination method*, which allows to work out the computation of $\mathrm{ch}_k(c_1, ..., c_r)$. However, there also exists an alternative and, in fact, faster method which uses *Newton's identities*

$$P_{k+1} = c_1 P_k - c_2 P_{k-1} + ... + (-1)^k (k+1) c_{k+1}, \quad k \in \mathbb{Z}_{\geqslant 0} \tag{2.1}$$

to compute the polynomials $\mathrm{ch}_k(c_1, ..., c_r) = \frac{1}{k!} P_k(c_1, ..., c_k)$. [3] Explicitly, the first few terms are

$$\begin{aligned}
P_1 &= c_1, \\
P_2 &= c_1 P_1 - 2c_2 = c_1^2 - 2c_2, \\
P_3 &= c_1 P_2 - c_2 P_1 + 3c_3 = c_1^3 - 3c_1 c_2 + 3c_3, \\
P_4 &= c_1 P_3 - c_2 P_2 + 3c_3 P_1 = c_1^4 - 4c_1^2 c_2 + 4c_1 c_3 + 2c_2^2 - 4c_4.
\end{aligned}$$

Since c_k's are integral $\in \mathbb{Z}$, then it means all P_k take values in $\mathbb{Z}$ as well, since they are expressed in terms of the Chern classes without division. It follows that $\mathrm{ch}_k(c_1, ..., c_k) \in \mathbb{Q}[c_1, ..., c_k]$ due to the formula that expresses the $\mathrm{ch}_k(c_1, ..., c_k)$ in terms of the Newton's identities. Also the Newton's identities can be rewritten to define the Chern classes in terms of the Chern character [4]

$$c_{k+1} = \tfrac{1}{k+1}(c_k \cdot \mathrm{ch}_1 - 2! \cdot c_{k-1} \cdot \mathrm{ch}_2 + ... + (-1)^k (k+1)! \cdot \mathrm{ch}_{k+1}). \tag{2.2}$$

According to the conclusions in [17], starting from $k = 8$ the time it takes to compute the components ch_k of the Chern character exponentially increases for the elimination method, whereas there is only slight increase in computer time for the Newton's method. Therefore, Newton's identities prove to be significantly more efficient for higher powers of k [17]. Taking this into account, we used the approach with Newton's identities when expressing jth Chern character in terms of the Chern classes (see *Expansion 1* in Section 3), which we then inverted to write jth Chern class in terms of the Chern character (see *Expansion 2*).

2.2. **The Splitting Principle.** The *splitting principle* is one of the most important methods of computation for our case. It is based on the notion that a vector bundle can 'pretend' to be decomposable into a sum of line bundles. In other words, the vector bundle E is said to be split into a direct (Whitney) sum of line bundles L_i, for which we know the characteristic classes are easily computed, leading to a result for the initial vector bundle.

$$E = L_1 \oplus L_2 \oplus \cdots \oplus L_r \,, \quad r = \mathrm{rank}(E).$$

Moreover, the principle is especially strong, because it is applicable in cases when the vector bundle is not actually decomposable or split, since the computational result holds true regardless [12] [7]. The splitting principle is widely used in characteristic class computations (see [32] and [20]).

Particularly, for computations with Chern classes, this method enables one to obtain results for tensor product of two vector bundles. In the context of this paper the splitting principle was most notably used to compute the jth Chern character in terms of Chern classes (see *Expansion 2*) as well as to expand the jth Chern class and jth Chern character of a tensor product of two vector bundles (see Section 3.3). Next follows a demonstration on the essential mechanism of the splitting principle and how it works in computations of Chern character (see [7] [28]).

Example: Take L_j to be complex line bundles with rank $1 \leqslant j \leqslant r$. We deduce from the Whitney sum property of the Chern character (section 3.1) that for $E = L_1 \oplus L_2 \oplus ... \oplus L_r$,

$$\mathrm{ch}(E) = \mathrm{ch}(L_1) \oplus \mathrm{ch}(L_2) \oplus ... \oplus \mathrm{ch}(L_r). \tag{2.3}$$

[3] The notation of P indicating the recursive steps should not be confused with the Pontrjagin classes denoted by lower case p.
[4] Note that the above equation had a small error as stated in [17] – the factorial coefficients had been omitted.

Yet, since $\mathrm{ch}(L_j) = \exp(x_j)$, then

$$\mathrm{ch}(E) = \prod_{j=1}^{k} \exp(x_j) \tag{2.4}$$

which is Chern character in terms elementary symmetric polynomials. Therefore, the Chern character of a vector bundle is expressed by the Whitney sum of r complex line bundles. Note that the x_i are called *Chern roots*. The Chern classes are given in terms of Chern roots by

$$c_i(E) = \sigma_i(x_1, \dots, x_r) \tag{2.5}$$

where σ_i is the i-th elementary symmetric polynomial, defined as [23] [12] [20]

$$\sigma_i(x_1, x_2, ..., x_r) = \sum_{1 \leqslant j_1 < j_2 ... < j_i \leqslant r} x_{j_1} ... x_{j_i}. \tag{2.6}$$

The first few elementary symmetric polynomials are given as follows

$$\begin{aligned} \sigma_0(x_1, x_2, ..., x_r) &= 1\ , \\ \sigma_1(x_1, x_2, ..., x_r) &= \sum_{1 \leqslant j \leqslant r} x_j\ , \\ \sigma_2(x_1, x_2, ..., x_r) &= \sum_{1 \leqslant j < k \leqslant r} x_j x_k\ , \\ \sigma_3(x_1, x_2, ..., x_r) &= \sum_{1 \leqslant j < k < l \leqslant r} x_j x_k x_l\ , \end{aligned}$$

while the n-th polynomial is $\sigma_n(x_1, x_2, ..., x_r) = x_1 x_2 ... x_r$.

3. Expansions of the Characteristic Classes

The next two sections will present the actual expansions obtained in the course of this paper as well as explanations of the programming approach and at times additional theoretical background.

Both Chern and Pontrjagin classes are cohomology classes $C_k(\xi) \in H^k(X; R)$ (with complex $\mathbb{C}$ or real $\mathbb{R}$ coefficients, respectively) assigned to each vector bundle $\xi : E \to X$. Note that the Chern classes turn out to be integral classes, i.e. taking values in cohomology with $\mathbb{Z}$ coefficients, while the Pontrjagin classes can be taken over $\mathbb{Z}$ or the rational numbers $\mathbb{Q}$. The two main characteristic classes are further defined below, as those will be the ones manipulated and combined in different formulas throughout the paper.

Note that throughout this section we denote the jth Chern class by c_j, the jth Chern character by ch_j, and the jth Pontrjagin class by p_j.

Characteristic classes we consider arise from the cohomology of classifying spaces $BO(n)$ and $BU(n)$ for the orthogonal group $\mathrm{O}(n)$ and the unitary group $\mathrm{U}(n)$, respectively. One of the main results that allows for a systematic characterization of characteristic classes is the following classical result (see [21, Theorem 17])

Theorem 1. *The family of all characteristic classes of n-dimensional real (complex) vector bundles is in one-to-one correspondence with the cohomology ring $H^*(BO(n))$ (respectively, with $H^*(BU(n))$).*

These are called the universal characteristic classes. Characteristic classes of a (vector bundles over a) manifold M are obtained by pulling back the universal ones via a classifying map $M \to BG$, where G is an orthogonal group or a unitray group depending on whether the bundle is real or complex, respectively.

3.1. **Chern character & Chern classes.**

Chern classes. Chern classes are used to characterize complex vector bundles. From an algebraic point of view, these classes are generators of certain ring related to the complex unitary group. More precisely, one has the classical result (see [21, Theorem 18]):

Theorem 2. *The ring $H^*(BU(n); \mathbb{Z})$ of integer cohomology classes is isomorphic to the polynomial ring $\mathbb{Z}[c_1, c_2, \cdots, c_n]$, where*

$$c_k \in H^{2k}(BU(n); \mathbb{Z}).$$

The generators c_k can be chosen so that one has certain axioms. One of the properties of Chern classes is that given another vector bundle F with fiber $\mathbb{C}^k$ the total Chern class of a tensor product bundle $E \otimes F$ is given by

$$c(E \otimes F) = c(E) \cdot c(F).$$

In fact, we can also define the total Chern class $c(E)$ using the following axioms

(1) Naturality: $c(f^*E) = f^*c(E)$.
(2) Finiteness: For a rank k complex bundle E, $c(E) = c_0(E) + c_1(E) + ... + c_k(E)$, $c_i(E) \in H^{2i}(M)$; $c_j(E) = 0$, $j > k$.
(3) Whitney sum: $c(E \oplus F) = c(E)c(F)$.
(4) Normalization: $c(L) = 1 + x \qquad (x = c_1(L)$, L is canonical line bundle over the complex projective space $\mathbb{C}P^n)$.

Chern character. The idea of the Chern character is that one would like to get a multiplicative combinations of the Chern classes (think exponential vs. log).

The Chern character has the following properties:

(1) Naturality: Let $\pi : E \to M$ be a vector bundle with fiber $F = \mathbb{C}^k$ and let $f : N \to M$ be a smooth map, then $\mathrm{ch}(f^*M) = f^*\mathrm{ch}(M)$.
(2) Tensor product: $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \wedge \mathrm{ch}(F)$.
(3) Whitney sum: $\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) \oplus \mathrm{ch}(F)$.

Proofs of these properties are provided in [28] and [34] as well as further discussion of Chern character overall. Note that Chern character has one particularly important application, as they appear in the Atiyah-Singer index theorem [28]. Indeed, this will be demonstrated in explicit calculations in the second paper to appear. Such computations are very important for anomalies in quantum field theory and string theory (see [2][28][31]).

Expansion 1. *Computing the ith Chern character in terms of the Chern classes.*

Chern character in terms of Chern classes

$$
\begin{aligned}
\mathrm{ch}_0 =& r \\
\mathrm{ch}_1 =& c_1 \\
\mathrm{ch}_2 =& \tfrac{1}{2}\left[c_1^2 - 2c_2\right] \\
\mathrm{ch}_3 =& \tfrac{1}{2^1}\Big[+ \tfrac{1^1}{3^1} c_1^3 - 1^1 c_1c_2 + 1^1 c_3\Big] \\
\mathrm{ch}_4 =& \tfrac{1}{2^1.3^1}\Big[+ \tfrac{1^1}{2^2} c_1^4 - 1^1 c_1^2c_2 + 1^1 c_1c_3 + \tfrac{1^1}{2^1} c_2^2 - 1^1 c_4\Big] \\
\mathrm{ch}_5 =& \tfrac{1}{2^3.3^1}\Big[+ \tfrac{1^1}{5^1} c_1^5 - 1^1 c_1^3c_2 + 1^1 c_1^2c_3 + 1^1 c_1c_2^2 - 1^1 c_1c_4 - 1^1 c_2c_3 + 1^1 c_5\Big] \\
\mathrm{ch}_6 =& \tfrac{1}{2^2.5^1}\Big[+ \tfrac{1^1}{2^2.3^2} c_1^6 - \tfrac{1^1}{2^1.3^1} c_1^4c_2 + \tfrac{1^1}{2^1.3^1} c_1^3c_3 + \tfrac{1^1}{2^2} c_1^2c_2^2 - \tfrac{1^1}{2^1.3^1} c_1^2c_4 - \tfrac{1^1}{3^1} c_1c_2c_3 + \tfrac{1^1}{2^1.3^1} c_1c_5 - \tfrac{1^1}{2^1.3^2} c_2^3 \\
&+ \tfrac{1^1}{2^1.3^1} c_2c_4 + \tfrac{1^1}{2^2.3^1} c_3^2 - \tfrac{1^1}{2^1.3^1} c_6\Big] \\
\mathrm{ch}_7 =& \tfrac{1}{2^3.3^1.5^1}\Big[+ \tfrac{1^1}{2^1.3^1.7^1} c_1^7 - \tfrac{1^1}{2^1.3^1} c_1^5c_2 + \tfrac{1^1}{2^1.3^1} c_1^4c_3 + \tfrac{1^1}{3^1} c_1^3c_2^2 - \tfrac{1^1}{2^1.3^1} c_1^3c_4 - \tfrac{1^1}{2^1} c_1^2c_2c_3 + \tfrac{1^1}{2^1.3^1} c_1^2c_5 - \tfrac{1^1}{2^1.3^1} c_1c_2^3 \\
&+ \tfrac{1^1}{3^1} c_1c_2c_4 + \tfrac{1^1}{2^1.3^1} c_1c_3^2 - \tfrac{1^1}{2^1.3^1} c_1c_6 + \tfrac{1^1}{2^1.3^1} c_2^2c_3 - \tfrac{1^1}{2^1.3^1} c_2c_5 - \tfrac{1^1}{2^1.3^1} c_3c_4 + \tfrac{1^1}{2^1.3^1} c_7\Big] \\
\mathrm{ch}_8 =& \tfrac{1}{2^2.3^1.7^1}\Big[+ \tfrac{1^1}{2^5.3^1.5^1} c_1^8 - \tfrac{1^1}{2^2.3^1.5^1} c_1^6c_2 + \tfrac{1^1}{2^2.3^1.5^1} c_1^5c_3 + \tfrac{1^1}{2^3.3^1} c_1^4c_2^2 - \tfrac{1^1}{2^2.3^1.5^1} c_1^4c_4 - \tfrac{1^1}{3^1.5^1} c_1^3c_2c_3 \\
&+ \tfrac{1^1}{2^2.3^1.5^1} c_1^3c_5 - \tfrac{1^1}{2^1.3^1.5^1} c_1^2c_2^3 + \tfrac{1^1}{2^2.5^1} c_1^2c_2c_4 + \tfrac{1^1}{2^3.5^1} c_1^2c_3^2 - \tfrac{1^1}{2^2.3^1.5^1} c_1^2c_6 + \tfrac{1^1}{2^2.5^1} c_1c_2^2c_3 - \\
&\tfrac{1^1}{2^1.3^1.5^1} c_1c_2c_5 - \tfrac{1^1}{2^1.3^1.5^1} c_1c_3c_4 + \tfrac{1^1}{2^2.3^1.5^1} c_1c_7 + \tfrac{1^1}{2^4.3^1.5^1} c_2^4 - \tfrac{1^1}{2^2.3^1.5^1} c_2^2c_4 - \tfrac{1^1}{2^2.3^1.5^1} c_2c_3^2 + \\
&\tfrac{1^1}{2^2.3^1.5^1} c_2c_6 + \tfrac{1^1}{2^2.3^1.5^1} c_3c_5 + \tfrac{1^1}{2^3.3^1.5^1} c_4^2 - \tfrac{1^1}{2^2.3^1.5^1} c_8\Big] \\
\mathrm{ch}_9 =& \tfrac{1}{2^5.3^1.7^1}\Big[+ \tfrac{1^1}{2^2.3^3.5^1} c_1^9 - \tfrac{1^1}{2^2.3^1.5^1} c_1^7c_2 + \tfrac{1^1}{2^2.3^1.5^1} c_1^6c_3 + \tfrac{1^1}{2^2.5^1} c_1^5c_2^2 - \tfrac{1^1}{2^2.3^1.5^1} c_1^5c_4 - \tfrac{1^1}{2^2.3^1} c_1^4c_2c_3 \\
&+ \tfrac{1^1}{2^2.3^1.5^1} c_1^4c_5 - \tfrac{1^1}{2^1.3^2} c_1^3c_2^3 + \tfrac{1^1}{3^1.5^1} c_1^3c_2c_4 + \tfrac{1^1}{2^1.3^1.5^1} c_1^3c_3^2 - \tfrac{1^1}{2^2.3^1.5^1} c_1^3c_6 + \tfrac{1^1}{2^1.5^1} c_1^2c_2^2c_3 - \tfrac{1^1}{2^2.5^1} c_1^2c_2c_5 \\
&- \tfrac{1^1}{2^2.5^1} c_1^2c_3c_4 + \tfrac{1^1}{2^2.3^1.5^1} c_1^2c_7 + \tfrac{1^1}{2^2.3^1.5^1} c_1c_2^4 - \tfrac{1^1}{2^2.5^1} c_1c_2^2c_4 - \tfrac{1^1}{2^2.5^1} c_1c_2c_3^2 + \tfrac{1^1}{2^1.3^1.5^1} c_1c_2c_6 \\
&+ \tfrac{1^1}{2^1.3^1.5^1} c_1c_3c_5 + \tfrac{1^1}{2^2.3^1.5^1} c_1c_4^2 - \tfrac{1^1}{2^2.3^1.5^1} c_1c_8 - \tfrac{1^1}{2^2.3^1.5^1} c_2^3c_3 + \tfrac{1^1}{2^2.3^1.5^1} c_2^2c_5 + \tfrac{1^1}{2^1.3^1.5^1} c_2c_3c_4 \\
&- \tfrac{1^1}{2^2.3^1.5^1} c_2c_7 + \tfrac{1^1}{2^2.3^2.5^1} c_3^3 - \tfrac{1^1}{2^2.3^1.5^1} c_3c_6 - \tfrac{1^1}{2^2.3^1.5^1} c_4c_5 + \tfrac{1^1}{2^2.3^1.5^1} c_9\Big]
\end{aligned}
$$

Now we can invert the above relations $\mathrm{ch}_i(c)$ to get $c_j(\mathrm{ch})$ in the following.

Expansion 2. *Computing the ith Chern class in terms of Chern character.*

Chern classes in terms of the Chern character

$$
\begin{aligned}
c_0 =& 1\\
c_1 =& \tfrac{1}{1^1}\mathrm{ch}_1\\
c_2 =& \tfrac{1}{2^1}\left[\mathrm{ch}_1^2 - 2^1\mathrm{ch}_2\right]\\
c_3 =& \tfrac{1}{2^1 \cdot 3^1}\left[\mathrm{ch}_1^3 - 2^1 \cdot 3^1\mathrm{ch}_1\mathrm{ch}_2 + 2^2 \cdot 3^1\mathrm{ch}_3\right]\\
c_4 =& \tfrac{1}{2^3 \cdot 3^1}\left[\mathrm{ch}_1^4 - 2^2 \cdot 3^1\mathrm{ch}_1^2\mathrm{ch}_2 + 2^2 \cdot 3^1\mathrm{ch}_2^2 + 2^4 \cdot 3^1\mathrm{ch}_1\mathrm{ch}_3 - 2^4 \cdot 3^2\mathrm{ch}_4\right]\\
c_5 =& \tfrac{1}{2^3 \cdot 3^1 \cdot 5^1}\Big[\mathrm{ch}_1^5 - 2^2 \cdot 5^1\mathrm{ch}_1^3\mathrm{ch}_2 + 2^2 \cdot 3^1 \cdot 5^1\mathrm{ch}_1\mathrm{ch}_2^2 + 2^3 \cdot 3^1 \cdot 5^1\mathrm{ch}_1^2\mathrm{ch}_3 - 2^4 \cdot 3^1 \cdot 5^1\mathrm{ch}_2\mathrm{ch}_3 - 2^4 \cdot 3^2 \cdot\\
& 5^1\mathrm{ch}_1\mathrm{ch}_4 + 2^6 \cdot 3^2 \cdot 5^1\mathrm{ch}_5\Big]\\
c_6 =& \tfrac{1}{2^4 \cdot 3^2 \cdot 5^1}\Big[\mathrm{ch}_1^6 - 2^1 \cdot 3^1 \cdot 5^1\mathrm{ch}_1^4\mathrm{ch}_2 + 2^2 \cdot 3^2 \cdot 5^1\mathrm{ch}_1^2\mathrm{ch}_2^2 - 2^3 \cdot 3^1 \cdot 5^1\mathrm{ch}_2^3 + 2^4 \cdot 3^1 \cdot 5^1\mathrm{ch}_1^3\mathrm{ch}_3 - 2^5 \cdot 3^2 \cdot 5^1\mathrm{ch}_1\mathrm{ch}_2\mathrm{ch}_3\\
& + 2^5 \cdot 3^2 \cdot 5^1\mathrm{ch}_3^2 - 2^4 \cdot 3^3 \cdot 5^1\mathrm{ch}_1^2\mathrm{ch}_4 + 2^5 \cdot 3^3 \cdot 5^1\mathrm{ch}_2\mathrm{ch}_4 + 2^7 \cdot 3^3 \cdot 5^1\mathrm{ch}_1\mathrm{ch}_5 - 2^7 \cdot 3^3 \cdot 5^2\mathrm{ch}_6\Big]\\
c_7 =& \tfrac{1}{2^4 \cdot 3^2 \cdot 5^1 \cdot 7^1}\Big[\mathrm{ch}_1^7 - 2^1 \cdot 3^1 \cdot 7^1\mathrm{ch}_1^5\mathrm{ch}_2 + 2^2 \cdot 3^1 \cdot 5^1 \cdot 7^1\mathrm{ch}_1^3\mathrm{ch}_2^2 - 2^3 \cdot 3^1 \cdot 5^1 \cdot 7^1\mathrm{ch}_1\mathrm{ch}_2^3 + 2^2 \cdot 3^1 \cdot 5^1 \cdot 7^1\mathrm{ch}_1^4\mathrm{ch}_3\\
& - 2^4 \cdot 3^2 \cdot 5^1 \cdot 7^1\mathrm{ch}_1^2\mathrm{ch}_2\mathrm{ch}_3 + 2^4 \cdot 3^2 \cdot 5^1 \cdot 7^1\mathrm{ch}_2^2\mathrm{ch}_3 + 2^5 \cdot 3^2 \cdot 5^1 \cdot 7^1\mathrm{ch}_1\mathrm{ch}_3^2 - 2^4 \cdot 3^2 \cdot 5^1 \cdot 7^1\mathrm{ch}_1^3\mathrm{ch}_4\\
& + 2^5 \cdot 3^3 \cdot 5^1 \cdot 7^1\mathrm{ch}_1\mathrm{ch}_2\mathrm{ch}_4 - 2^6 \cdot 3^3 \cdot 5^1 \cdot 7^1\mathrm{ch}_3\mathrm{ch}_4 + 2^6 \cdot 3^3 \cdot 5^1 \cdot 7^1\mathrm{ch}_1^2\mathrm{ch}_5 - 2^7 \cdot 3^3 \cdot 5^1 \cdot 7^1\mathrm{ch}_2\mathrm{ch}_5\\
& - 2^7 \cdot 3^3 \cdot 5^2 \cdot 7^1\mathrm{ch}_1\mathrm{ch}_6 + 2^8 \cdot 3^4 \cdot 5^2 \cdot 7^1\mathrm{ch}_7\Big]\\
c_8 =& \tfrac{1}{2^7 \cdot 3^2 \cdot 5^1 \cdot 7^1}\Big[\mathrm{ch}_1^8 - 2^3 \cdot 7^1\mathrm{ch}_1^6\mathrm{ch}_2 + 2^3 \cdot 3^1 \cdot 5^1 \cdot 7^1\mathrm{ch}_1^4\mathrm{ch}_2^2 - 2^5 \cdot 3^1 \cdot 5^1 \cdot 7^1\mathrm{ch}_1^2\mathrm{ch}_2^3 + 2^4 \cdot 3^1 \cdot 5^1 \cdot 7^1\mathrm{ch}_2^4 + 2^5 \cdot 3^1 \cdot 7^1\mathrm{ch}_1^5\mathrm{ch}_3\\
& - 2^7 \cdot 3^1 \cdot 5^1 \cdot 7^1\mathrm{ch}_1^3\mathrm{ch}_2\mathrm{ch}_3 + 2^7 \cdot 3^2 \cdot 5^1 \cdot 7^1\mathrm{ch}_1\mathrm{ch}_2^2\mathrm{ch}_3 + 2^7 \cdot 3^2 \cdot 5^1 \cdot 7^1\mathrm{ch}_1^2\mathrm{ch}_3^2 - 2^8 \cdot 3^2 \cdot 5^1 \cdot 7^1\mathrm{ch}_2\mathrm{ch}_3^2\\
& - 2^5 \cdot 3^2 \cdot 5^1 \cdot 7^1\mathrm{ch}_1^4\mathrm{ch}_4 + 2^7 \cdot 3^3 \cdot 5^1 \cdot 7^1\mathrm{ch}_1^2\mathrm{ch}_2\mathrm{ch}_4 - 2^7 \cdot 3^3 \cdot 5^1 \cdot 7^1\mathrm{ch}_2^2\mathrm{ch}_4 - 2^9 \cdot 3^3 \cdot 5^1 \cdot 7^1\mathrm{ch}_1\mathrm{ch}_3\mathrm{ch}_4\\
& + 2^8 \cdot 3^4 \cdot 5^1 \cdot 7^1\mathrm{ch}_4^2 + 2^9 \cdot 3^2 \cdot 5^1 \cdot 7^1\mathrm{ch}_1^3\mathrm{ch}_5 - 2^{10} \cdot 3^3 \cdot 5^1 \cdot 7^1\mathrm{ch}_1\mathrm{ch}_2\mathrm{ch}_5 + 2^{11} \cdot 3^3 \cdot 5^1 \cdot 7^1\mathrm{ch}_3\mathrm{ch}_5 - 2^9 \cdot 3^3 \cdot 5^2 \cdot 7^1\mathrm{ch}_1^2\mathrm{ch}_6\\
& + 2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^1\mathrm{ch}_2\mathrm{ch}_6 + 2^{11} \cdot 3^4 \cdot 5^2 \cdot 7^1\mathrm{ch}_1\mathrm{ch}_7 - 2^{11} \cdot 3^4 \cdot 5^2 \cdot 7^2\mathrm{ch}_8\Big]\\
c_9 =& \tfrac{1}{2^7 \cdot 3^4 \cdot 5^1 \cdot 7^1}\Big[\mathrm{ch}_1^9 - 2^3 \cdot 3^2\mathrm{ch}_1^7\mathrm{ch}_2 + 2^3 \cdot 3^3 \cdot 7^1\mathrm{ch}_1^5\mathrm{ch}_2^2 - 2^5 \cdot 3^2 \cdot 5^1 \cdot 7^1\mathrm{ch}_1^3\mathrm{ch}_2^3 + 2^4 \cdot 3^3 \cdot 5^1 \cdot 7^1\mathrm{ch}_1\mathrm{ch}_2^4 + 2^4 \cdot 3^2 \cdot 7^1\mathrm{ch}_1^6\mathrm{ch}_3\\
& - 2^5 \cdot 3^3 \cdot 5^1 \cdot 7^1\mathrm{ch}_1^4\mathrm{ch}_2\mathrm{ch}_3 + 2^6 \cdot 3^4 \cdot 5^1 \cdot 7^1\mathrm{ch}_1^2\mathrm{ch}_2^2\mathrm{ch}_3 - 2^7 \cdot 3^3 \cdot 5^1 \cdot 7^1\mathrm{ch}_2^3\mathrm{ch}_3 + 2^7 \cdot 3^3 \cdot 5^1 \cdot 7^1\mathrm{ch}_1^3\mathrm{ch}_3^2\\
& - 2^8 \cdot 3^4 \cdot 5^1 \cdot 7^1\mathrm{ch}_1\mathrm{ch}_2\mathrm{ch}_3^2 + 2^9 \cdot 3^3 \cdot 5^1 \cdot 7^1\mathrm{ch}_3^3 - 2^5 \cdot 3^4 \cdot 7^1\mathrm{ch}_1^5\mathrm{ch}_4 + 2^7 \cdot 3^4 \cdot 5^1 \cdot 7^1\mathrm{ch}_1^3\mathrm{ch}_2\mathrm{ch}_4\\
& - 2^7 \cdot 3^5 \cdot 5^1 \cdot 7^1\mathrm{ch}_1\mathrm{ch}_2^2\mathrm{ch}_4 - 2^8 \cdot 3^5 \cdot 5^1 \cdot 7^1\mathrm{ch}_1^2\mathrm{ch}_3\mathrm{ch}_4 + 2^9 \cdot 3^5 \cdot 5^1 \cdot 7^1\mathrm{ch}_2\mathrm{ch}_3\mathrm{ch}_4 + 2^8 \cdot 3^6 \cdot 5^1 \cdot 7^1\mathrm{ch}_1\mathrm{ch}_4^2\\
& + 2^7 \cdot 3^4 \cdot 5^1 \cdot 7^1\mathrm{ch}_1^4\mathrm{ch}_5 - 2^9 \cdot 3^5 \cdot 5^1 \cdot 7^1\mathrm{ch}_1^2\mathrm{ch}_2\mathrm{ch}_5 + 2^9 \cdot 3^5 \cdot 5^1 \cdot 7^1\mathrm{ch}_2^2\mathrm{ch}_5 + 2^{11} \cdot 3^5 \cdot 5^1 \cdot 7^1\mathrm{ch}_1\mathrm{ch}_3\mathrm{ch}_5\\
& - 2^{11} \cdot 3^6 \cdot 5^1 \cdot 7^1\mathrm{ch}_4\mathrm{ch}_5 - 2^9 \cdot 3^4 \cdot 5^2 \cdot 7^1\mathrm{ch}_1^3\mathrm{ch}_6 + 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^1\mathrm{ch}_1\mathrm{ch}_2\mathrm{ch}_6 - 2^{11} \cdot 3^5 \cdot 5^2 \cdot 7^1\mathrm{ch}_3\mathrm{ch}_6\\
& + 2^{10} \cdot 3^6 \cdot 5^2 \cdot 7^1\mathrm{ch}_1^2\mathrm{ch}_7 - 2^{11} \cdot 3^6 \cdot 5^2 \cdot 7^1\mathrm{ch}_2\mathrm{ch}_7 - 2^{11} \cdot 3^6 \cdot 5^2 \cdot 7^2\mathrm{ch}_1\mathrm{ch}_8 + 2^{14} \cdot 3^6 \cdot 5^2 \cdot 7^2\mathrm{ch}_9\Big]
\end{aligned}
$$

Note that for all further expansions we denote a rank r bundle by E_r.

Expansion 3. *The total Chern character expansion for various ranks of a bundle.*

Total Chern character for a rank r bundle E_r in terms of Chern classes

$$
\begin{aligned}
\mathrm{ch}(E_1) =& r + c_1\\
\mathrm{ch}(E_2) =& \mathrm{ch}(E_1) + \tfrac{1}{2^1}\left[c_1^2 - 2c_2\right]\\
\mathrm{ch}(E_3) =& \mathrm{ch}(E_2) + \tfrac{1}{2^1 \cdot 3^1}\left[c_1^3 - 3c_1c_2 + 3c_3\right]\\
\mathrm{ch}(E_4) =& \mathrm{ch}(E_3) + \tfrac{1}{2^3 \cdot 3^1}\left[c_1^4 + 2c_2^2 - 4c_1^2c_2 + 4c_1c_3 - 4c_4\right]\\
\mathrm{ch}(E_5) =& \mathrm{ch}(E_4) + \tfrac{1}{2^3 \cdot 3^1 \cdot 5^1}\left[c_1^5 + 5c_1c_2^2 - 5c_1^3c_2 + 5c_1^2c_3 - 5c_2c_3 - 5c_1c_4 + 5c_5\right]\\
\mathrm{ch}(E_6) =& \mathrm{ch}(E_5) + \tfrac{1}{2^4 \cdot 3^2 \cdot 5^1}\left[c_1^6 - 2c_2^3 + 9c_1^2c_2^2 + 3c_3^2 - 6c_1^4c_2 + 6c_1^3c_3 - 12c_1c_2c_3 - 6c_1^2c_4 + 6c_2c_4 + 6c_1c_5 - 6c_6\right]\\
\mathrm{ch}(E_7) =& \mathrm{ch}(E_6) + \tfrac{1}{2^4 \cdot 3^2 \cdot 5^1 \cdot 7^1}\Big[c_1^7 - 7c_1c_2^3 + 14c_1^3c_2^2 + 7c_1c_3^2 - 7c_1^5c_2 + 7c_1^4c_3 - 21c_1^2c_2c_3 + 7c_2^2c_3 - 7c_1^3c_4 + 14c_1c_2c_4 -\\
& - 7c_3c_4 + 7c_1^2c_5 - 7c_2c_5 - 7c_1c_6 + 7c_7\Big]
\end{aligned}
$$

Expansion 4. *Chern classes for Calabi-Yau or SU-bundles:* $c_1 = 0$

We have the following nice simplification.

Total Chern character for a rank r SU-bundle E_r in terms of Chern classes

$$
\begin{aligned}
\mathrm{ch}(E_1) =& r \\
\mathrm{ch}(E_2) =& \mathrm{ch}(E_1) - c_2 \\
\mathrm{ch}(E_3) =& \mathrm{ch}(E_2) + \tfrac{1}{2^1} c_3 \\
\mathrm{ch}(E_4) =& \mathrm{ch}(E_3) + \tfrac{1}{2^2.3^1}\left[c_2^2 - 2c_4\right] \\
\mathrm{ch}(E_5) =& \mathrm{ch}(E_4) + \tfrac{1}{2^3.3^1}\left[-c_2c_3 + c_5\right] \\
\mathrm{ch}(E_6) =& \mathrm{ch}(E_5) + \tfrac{1}{2^4.3^2.5^1}\left[-2c_2^3 + 3c_3^2 + 6c_2c_4 - 6c_6\right] \\
\mathrm{ch}(E_7) =& \mathrm{ch}(E_6) + \tfrac{1}{2^4.3^2.5^1}\left[c_2^2c_3 - c_3c_4 - c_2c_5 + c_7\right]
\end{aligned}
$$

Now, sometimes it is desirable to consider situations when only even Chern classes are nonzero. For instance, when realifying a complex bundle, i.e., going from $E_{\mathbb{C}}$ to $E_{\mathbb{R}}$, or when the underlying space has cohomology of degree $4k$. With this in mind the above expansion is simplified by vanishing all odd Chern classes to obtain the result below.

Expansion 5. *Total Chern character expansion in terms of only the even Chern classes ($c_{2i+1} = 0$).*

Total Chern character expansion in terms of only the even Chern classes

$$
\begin{aligned}
\mathrm{ch}(E_1) = \mathrm{ch}(E_0) =& r \\
\mathrm{ch}(E_3) = \mathrm{ch}(E_2) =& \mathrm{ch}(E_1) - c_2 \\
\mathrm{ch}(E_5) = \mathrm{ch}(E_4) =& \mathrm{ch}(E_3) + \tfrac{1}{2^2.3^1}\left[c_2^2 - 2c_4\right] \\
\mathrm{ch}(E_7) = \mathrm{ch}(E_6) =& \mathrm{ch}(E_5) + \tfrac{1}{2^3.3^2.5^1}\left[-c_2^3 + 3c_2c_4 - 3c_6\right] \\
\mathrm{ch}(E_9) = \mathrm{ch}(E_8) =& \mathrm{ch}(E_7) + \tfrac{1}{2^6.3^2.5^1.7^1}\left[c_2^4 + 2c_4^2 - 4c_2^2c_4 + 4c_2c_6 - 4c_8\right]
\end{aligned}
$$

3.2. **Pontrjagin classes.**

Pontrjagin classes. In the real case, there are two complications: The first is the presence of torsion in cohomology, which forces one to study the generators with $\mathbb{Z}_2$ coefficients and with rational $\mathbb{Q}$ coefficients. For our purposes the latter is enough. The second complication is that the ring for $BO(n)$ will take a slightly different form depending on whether n is even or odd. We will provide the precise statement although we will not be highlighting such differences. Again, the following fundamental statement is classical (see[21, Theorem 19]).

Theorem 3. *The ring $H^*(BSO(2m);\mathbb{Q})$ is isomorphic to the ring of polynomials $\mathbb{Q}[p_1, p_2, \cdots, p_{m-1}, \chi]$, where*

$$p_k \in H^{4k}(BSO(2m);\mathbb{Q}), \chi \in H^{2m}(BSO(2m);\mathbb{Q}) \;,$$

while the ring $H^*(BSO(2m+1);\mathbb{Q})$ is isomorphic to the ring of polynomials $\mathbb{Q}[p_1, p_2, \cdots, p_m]$, where

$$p_k \in H^{4k}(BSO(2m);\mathbb{Q}) \;.$$

We will not consider the Euler characteristic χ separately as in many cases it can be determined by the corresponding Pontrjagin classe. The generators can be chosen so that similar axioms to those of the Chern class are satisfied.

Pontrjagin classes have the property (again rationally)

$$p(E \oplus F) = p(E) \cdot p(F). \tag{3.1}$$

Chern and Pontrjagin classes are directly related. As we stated above Chern classes are defined only for complex vector bundles; consequently in order to compare it is necessary to *complexify* (convert from $\mathbb{R} \to \mathbb{C}$) the fiber of the vector bundle E. We will denote such a bundle with $E_{\mathbb{C}}$ throughout the paper. The corresponding relation is

$$p_j(E) = (-1)^j c_{2j}(E_{\mathbb{C}}). \tag{3.2}$$

Complexification and realification. The complexification of a real vector space V is defined to be $V_{\mathbb{C}} = V \oplus V$, in which a pair (v_1, v_2) can be thought of as a formal sum $v_1 + iv_2$, with multiplication law $(a + bi)(v_1, v_2) = (av_1 - bv_2, bv_1 + av_2)$, where a and b are real.

For E a real vector bundle, its complexification is $E_{\mathbb{C}} := E \otimes \mathbb{C}$ is isomorphic to $E \oplus \overline{E}$ as complex vector bundles. Here $\overline{E}$ is the complex conjugate bundle of E with the opposite complex structure. This is the operation of complexification c of a real vector bundle.

The Chern classes of the conjugate bundle are given as $c_i(\overline{E}) = (-1)^k c_i(E)$. Then the first Chern class is given by $c_1(E \otimes \mathbb{C}) = c_1(E) + c_1(\overline{E}) = c_1(E) - c_1(E) = 0$, while the second Chern class $c_2(E \otimes \mathbb{C}) = c_1(E)c_1(\overline{E}) = -c_1(E)^2 = -p_1(E)$, the first Pontrjagin class of E. See [16, Remark 6.2].

Likewise, forgetting the complex structure on a complex vector bundle E turns it into a real vector bundle $E_{\mathbb{R}}$. This operation is called the realification of a complex vector bundle E. Note that the composition of realification and complexification is (see [21, Sec. 1.4.1])

$$rcE = E \oplus E \qquad \text{while} \qquad crE = E \oplus \overline{E}\,.$$

In explicit terms the reader will find Pontrjagin classes in terms of Chern classes presented in section 3 in *Expansion 6*. The above equation will be very important for the practical calculations later on and we will refer back to it frequently.

The next expansions employs the concept of realification, when an initially complex bundle is converted to the real space $E \to E_{\mathbb{R}}$ [26]. The function is implemented according to Corollary 15.5 in [26], which states that for any complex vector bundle E, the Chern classes of $c_i(E)$ determine the Pontrjagin classes $p_k(E_{\mathbb{R}})$ by the formula

$$1 - p_1 + p_2 - ... \pm p_n = (1 - c_1 + c_2 - ... \pm c_n)(1 + c_1 + c_2 + ... + c_n). \tag{3.3}$$

Therefore,

$$p_k(E_{\mathbb{R}}) = c_k(E)^2 - 2c_{k-1}(E)c_{k+1}(E) + ... \pm 2c_1(E)c_{2k-1}(E) \pm 2c_{2k}(E). \tag{3.4}$$

Expansion 6. *Pontrjagin classes of a realified bundle $E_{\mathbb{R}}$ in terms of Chern classes of a complex bundle E.*

Pontrjagin classes of a realified bundle in terms of the Chern classes of a complex bundle

$$\begin{aligned}
p_0 =&1\\
p_1 =&c_1^2 - 2c_2\\
p_2 =&c_2^2 - 2c_1c_3 + 2c_4\\
p_3 =&c_3^2 - 2c_2c_4 + 2c_1c_5 - 2c_6\\
p_4 =&c_4^2 - 2c_3c_5 + 2c_2c_6 - 2c_1c_7 + 2c_8\\
p_5 =&c_5^2 - 2c_4c_6 + 2c_3c_7 - 2c_2c_8 + 2c_1c_9 - 2c_{10}\\
p_6 =&c_6^2 - 2c_5c_7 + 2c_4c_8 - 2c_3c_9 + 2c_2c_{10} - 2c_1c_{11} + 2c_{12}\\
p_7 =&c_7^2 - 2c_6c_8 + 2c_5c_9 - 2c_4c_{10} + 2c_3c_{11} - 2c_2c_{12} + 2c_1c_{13} - 2c_{14}\\
p_8 =&c_8^2 - 2c_7c_9 + 2c_6c_{10} - 2c_5c_{11} + 2c_4c_{12} - 2c_3c_{13} + 2c_2c_{14} - 2c_1c_{15} + 2c_{16}\\
p_9 =&c_9^2 - 2c_8c_{10} + 2c_7c_{11} - 2c_6c_{12} + 2c_5c_{13} - 2c_4c_{14} + 2c_3c_{15} - 2c_2c_{16} + 2c_1c_{17} - 2c_{18}\\
p_{10} =&c_{10}^2 - 2c_9c_{11} + 2c_8c_{12} - 2c_7c_{13} + 2c_6c_{14} - 2c_5c_{15} + 2c_4c_{16} - 2c_3c_{17} + 2c_2c_{18} - 2c_1c_{19} + 2c_{20}\\
p_{11} =&c_{11}^2 - 2c_{10}c_{12} + 2c_9c_{13} - 2c_8c_{14} + 2c_7c_{15} - 2c_6c_{16} + 2c_5c_{17} - 2c_4c_{18} + 2c_3c_{19} - 2c_2c_{20} + 2c_1c_{21} - 2c_{22}\\
p_{12} =&c_{12}^2 - 2c_{11}c_{13} + 2c_{10}c_{14} - 2c_9c_{15} + 2c_8c_{16} - 2c_7c_{17} + 2c_6c_{18} - 2c_5c_{19} + 2c_4c_{20} - 2c_3c_{21} + 2c_2c_{22} - 2c_1c_{23} + 2c_{24}\\
p_{13} =&c_{13}^2 - 2c_{12}c_{14} + 2c_{11}c_{15} - 2c_{10}c_{16} + 2c_9c_{17} - 2c_8c_{18} + 2c_7c_{19} - 2c_6c_{20} + 2c_5c_{21} - 2c_4c_{22} + 2c_3c_{23} - 2c_2c_{24} + 2c_1c_{25}\\
&- 2c_{26}\\
p_{14} =&c_{14}^2 - 2c_{13}c_{15} + 2c_{12}c_{16} - 2c_{11}c_{17} + 2c_{10}c_{18} - 2c_9c_{19} + 2c_8c_{20} - 2c_7c_{21} + 2c_6c_{22} - 2c_5c_{23} + 2c_4c_{24} - 2c_3c_{25} + 2c_2c_{26}\\
&- 2c_1c_{27} + 2c_{28}
\end{aligned}$$

We now compute the Pontrjagin classes of a realified bundle in terms of the Chern character of a complex bundle.

Expansion 7. *Pontrjagin classes of realified bundle $E_{\mathbb{R}}$ expressed in terms of Chern character of a bundle E.*

Pontrjagin classes of realified bundle $E_{\mathbb{R}}$ expressed in terms of Chern character of a bundle E

$$
\begin{aligned}
p_1 =& 2\mathrm{ch}_2\\
p_2 =& 2(\mathrm{ch}_2^2 - 6\mathrm{ch}_4)\\
p_3 =& \tfrac{4}{3}\left(\mathrm{ch}_2^3 - 18\mathrm{ch}_2\mathrm{ch}_4 + 180\mathrm{ch}_6\right)\\
p_4 =& \tfrac{2}{3}\mathrm{ch}_2^4 - 24\mathrm{ch}_2^2\mathrm{ch}_4 + 480\mathrm{ch}_2\mathrm{ch}_6 + 72\left(\mathrm{ch}_4^2 - 140\mathrm{ch}_8\right)\\
p_5 =& \tfrac{4}{15}\mathrm{ch}_2^5 - 16\mathrm{ch}_2^3\mathrm{ch}_4 + 480\mathrm{ch}_2^2\mathrm{ch}_6 + 144\mathrm{ch}_2\left(\mathrm{ch}_4^2 - 140\mathrm{ch}_8\right) - 2880\left(\mathrm{ch}_4\mathrm{ch}_6 - 252\mathrm{ch}_{10}\right)\\
p_6 =& \tfrac{4}{45}\mathrm{ch}_2^6 - 8\mathrm{ch}_2^4\mathrm{ch}_4 + 320\mathrm{ch}_2^3\mathrm{ch}_6 + 144\mathrm{ch}_2^2\left(\mathrm{ch}_4^2 - 140\mathrm{ch}_8\right) - 5760\mathrm{ch}_2\left(\mathrm{ch}_4\mathrm{ch}_6 - 252\mathrm{ch}_{10}\right) -\\
&- 288\left[\mathrm{ch}_4^3 - 420\mathrm{ch}_4\mathrm{ch}_8 - 100\left(\mathrm{ch}_6^2 - 2772\mathrm{ch}_{12}\right)\right]\\
p_7 =& \tfrac{8}{315}\mathrm{ch}_2^7 - \tfrac{16}{5}\mathrm{ch}_2^5\mathrm{ch}_4 + 160\mathrm{ch}_2^4\mathrm{ch}_6 + 96\mathrm{ch}_2^3\left(\mathrm{ch}_4^2 - 140\mathrm{ch}_8\right) - 5760\mathrm{ch}_2^2\left(\mathrm{ch}_4\mathrm{ch}_6 - 252\mathrm{ch}_{10}\right) -\\
&- 576\mathrm{ch}_2\left[\mathrm{ch}_4^3 - 420\mathrm{ch}_4\mathrm{ch}_8 - 100\left(\mathrm{ch}_6^2 - 2772\mathrm{ch}_{12}\right)\right] + 17280\left(\mathrm{ch}_4^2\mathrm{ch}_6 - 140\mathrm{ch}_6\mathrm{ch}_8 - 504\mathrm{ch}_4\mathrm{ch}_{10} + 720720\mathrm{ch}_{14}\right)\\
p_8 =& \tfrac{2}{315}\mathrm{ch}_2^8 - \tfrac{16}{15}\mathrm{ch}_2^6\mathrm{ch}_4 + 64\mathrm{ch}_2^5\mathrm{ch}_6 + 48\mathrm{ch}_2^4\left(\mathrm{ch}_4^2 - 140\mathrm{ch}_8\right) - 3840\mathrm{ch}_2^3\left(\mathrm{ch}_4\mathrm{ch}_6 - 252\mathrm{ch}_{10}\right) -\\
&- 576\mathrm{ch}_2^2\left[\mathrm{ch}_4^3 - 420\mathrm{ch}_4\mathrm{ch}_8 - 100\left(\mathrm{ch}_6^2 - 2772\mathrm{ch}_{12}\right)\right] + 34560\mathrm{ch}_2\left(\mathrm{ch}_6\mathrm{ch}_4^2 - 140\mathrm{ch}_6\mathrm{ch}_8 - 504\mathrm{ch}_4\mathrm{ch}_{10} + 720720\mathrm{ch}_{14}\right) +\\
&+ 864\left[\mathrm{ch}_4^4 - 840\mathrm{ch}_4^2\mathrm{ch}_8 - 400\mathrm{ch}_4\left(\mathrm{ch}_6^2 - 2772\mathrm{ch}_{12}\right) + 8400\left(7\mathrm{ch}_8^2 + 24\mathrm{ch}_6\mathrm{ch}_{10} - 360360\mathrm{ch}_{16}\right)\right]\\
p_9 =& \tfrac{4}{2835}\mathrm{ch}_2^9 - \tfrac{32}{105}\mathrm{ch}_2^7\mathrm{ch}_4 + \tfrac{64}{3}\mathrm{ch}_2^6\mathrm{ch}_6 + \tfrac{96}{5}\mathrm{ch}_2^5\left(\mathrm{ch}_4^2 - 140\mathrm{ch}_8\right) - 1920\mathrm{ch}_2^4\left(\mathrm{ch}_4\mathrm{ch}_6 - 252\mathrm{ch}_{10}\right) -\\
&- 384\mathrm{ch}_2^3\left[\mathrm{ch}_4^3 - 420\mathrm{ch}_4\mathrm{ch}_8 - 100\left(\mathrm{ch}_6^2 - 2772\mathrm{ch}_{12}\right)\right] + 34560\mathrm{ch}_2^2\left(\mathrm{ch}_4^2\mathrm{ch}_6 - 140\mathrm{ch}_6\mathrm{ch}_8 - 504\mathrm{ch}_4\mathrm{ch}_{10} + 720720\mathrm{ch}_{14}\right) +\\
&+ 1728\mathrm{ch}_2\left[\mathrm{ch}_4^4 - 840\mathrm{ch}_4^2\mathrm{ch}_8 - 400\mathrm{ch}_4\left(\mathrm{ch}_6^2 - 2772\mathrm{ch}_{12}\right) + 8400\left(7\mathrm{ch}_8^2 + 24\mathrm{ch}_6\mathrm{ch}_{10} - 360360\mathrm{ch}_{16}\right)\right] -\\
&- 23040\Big[3\mathrm{ch}_4^3\mathrm{ch}_6 - 2268\mathrm{ch}_4^2\mathrm{ch}_{10} - 1260\mathrm{ch}_4\left(\mathrm{ch}_6\mathrm{ch}_8 - 5148\mathrm{ch}_{14}\right) - 20\Big(5\mathrm{ch}_6^3 - 41580\mathrm{ch}_6\mathrm{ch}_{12} - 15876\big(\mathrm{ch}_8\mathrm{ch}_{10}\\
&\quad - 97240\mathrm{ch}_{18}\big)\Big)\Big]
\end{aligned}
$$

Sometimes one encounters complex bundles with vanishing c_1 and c_2, or real bundles with vanishing p_1, such as the String structures presented later on in Section 4. Hence, we compute a simplification of the above expansion by vanishing Chern character ch_2. Note that vanishing of ch_1 is pointless, since all odd Chern characters have dimension $4k+2$, whereas the Pontrjagin classes p_k have dimension $4k$ which means that the odd Chern classes are already absent.

Expansion 8. *Pontrjagin classes expressed in terms of Chern character with simplification ($\mathrm{ch}_2 = 0$).*

Note that ch_1 does not appear in the above expressions, but ch_2, so we set the latter to zero.

Pontrjagin classes expressed in terms of Chern character with zero second Chern character

$$
\begin{aligned}
p_1 =& 0\\
p_2 =& -12\mathrm{ch}_4\\
p_3 =& 240\mathrm{ch}_6\\
p_4 =& 72\left(\mathrm{ch}_4^2 - 140\mathrm{ch}_8\right)\\
p_5 =& -2880\left(\mathrm{ch}_4\mathrm{ch}_6 - 252\mathrm{ch}_{10}\right)\\
p_6 =& -288\left[\mathrm{ch}_4^3 - 420\mathrm{ch}_4\mathrm{ch}_8 - 100\left(\mathrm{ch}_6^2 - 2772\mathrm{ch}_{12}\right)\right]\\
p_7 =& 17280\left(\mathrm{ch}_4^2\mathrm{ch}_6 - 140\mathrm{ch}_6\mathrm{ch}_8 - 504\mathrm{ch}_4\mathrm{ch}_{10} + 720720\mathrm{ch}_{14}\right)\\
p_8 =& 864\left[\mathrm{ch}_4^4 - 840\mathrm{ch}_4^2\mathrm{ch}_8 - 400\mathrm{ch}_4\left(\mathrm{ch}_6^2 - 2772\mathrm{ch}_{12}\right) + 8400\left(7\mathrm{ch}_8^2 + 24\mathrm{ch}_6\mathrm{ch}_{10} - 360360\mathrm{ch}_{16}\right)\right]\\
p_9 =& -23040[3\mathrm{ch}_4^3\mathrm{ch}_6 - 2268\mathrm{ch}_4^2\mathrm{ch}_{10} - 1260\mathrm{ch}_4\left(\mathrm{ch}_6\mathrm{ch}_8 - 5148\mathrm{ch}_{14}\right) -\\
&\qquad - 20\big(5\mathrm{ch}_6^3 - 41580\mathrm{ch}_6\mathrm{ch}_{12} - 15876\left(\mathrm{ch}_8\mathrm{ch}_{10} - 97240\mathrm{ch}_{18}\right)\big)]
\end{aligned}
$$

Now consider the restriction to $\mathrm{ch}_2 = \mathrm{ch}_4 = 0$:

Expansion 9. *Pontrjagin classes expressed in terms of Chern character with zero ch_2 and ch_4.*

Pontrjagin classes expressed in terms of Chern character with zero ch_2 and ch_4

$$
\begin{aligned}
p_1 =&0\\
p_2 =&0\\
p_3 =&240\mathrm{ch}_6\\
p_4 =&72(140\mathrm{ch}_8)\\
p_5 =&-2880\left(-252\mathrm{ch}_{10}\right)\\
p_6 =&-288\left[-100\left(\mathrm{ch}_6^2-2772\mathrm{ch}_{12}\right)\right]\\
p_7 =&17280\left(-140\mathrm{ch}_6\mathrm{ch}_8+720720\mathrm{ch}_{14}\right)\\
p_8 =&864\left[8400\left(7\mathrm{ch}_8^2+24\mathrm{ch}_6\mathrm{ch}_{10}-360360\mathrm{ch}_{16}\right)\right]\\
p_9 =&-23040[-20\big(5\mathrm{ch}_6^3-41580\mathrm{ch}_6\mathrm{ch}_{12}-15876\left(\mathrm{ch}_8\mathrm{ch}_{10}-97240\mathrm{ch}_{18}\right)\big)]
\end{aligned}
$$

Observations: We do get the Ninebrane class as ch_6. For p_4 the factor is $2.7!$. For p_5 it is $2.9!$ etc. This pattern then gives

$$p_i = 2(2i+1)!\mathrm{ch}_{2i} \qquad \text{when } \mathrm{ch}_{2j}=0\ ,\ j<i\ .$$

This allows one to read off the obstructions in the Whitehead tower up to a factor of two appearing in alternating degrees (see [30][31][29]).

Now we invert the above expressions.

Expansion 10. *Chern character in terms of Pontrjagin classes.*

Chern character in terms of Pontrjagin classes

$$
\begin{aligned}
\mathrm{ch}_0 =&r\\
\mathrm{ch}_2 =&\tfrac{1}{2}p_1\\
\mathrm{ch}_4 =&\tfrac{1}{2^3.3^1}[p_1^2-2p_2]\\
\mathrm{ch}_6 =&\tfrac{1}{2^4.3^2.5^1}[p_1^3-3p_1p_2+3p_3]\\
\mathrm{ch}_8 =&\tfrac{1}{2^7.3^2.5^1.7^1}[p_1^4+2p_2^2-4p_1^2p_2+4p_1p_3-4p_4]\\
\mathrm{ch}_{10} =&\tfrac{1}{2^8.3^4.5^2.7^1}[p_1^5+5p_1p_2^2-5p_1^3p_2+5p_1^2p_3-5p_2p_3-5p_1p_4+5p_5]\\
\mathrm{ch}_{12} =&\tfrac{1}{2^{10}.3^5.5^2.7^1.11^1}[p_1^6-2p_2^3+9p_1^2p_2^2+3p_3^2-6p_1^4p_2+6p_1^3p_3-12p_1p_2p_3-6p_1^2p_4+6p_2p_4+6p_1p_5-6p_6]\\
\mathrm{ch}_{14} =&\tfrac{1}{2^{11}.3^5.5^2.7^2.11^1.13^1}[p_1^7-7p_1p_2^3+14p_1^3p_2^2+7p_1p_3^2-7p_1^5p_2+7p_1^4p_3-21p_1^2p_2p_3+7p_2^2p_3-7p_1^3p_4+14p_1p_2p_4\\
&-7p_3p_4+7p_1^2p_5-7p_2p_5-7p_1p_6+7p_7]
\end{aligned}
$$

Sometimes it is desirable to study vector bundles with a specific rank. The following sums ch_j's from the previous expansion up to the rank of the vector bundle E_r.

Expansion 11. *Total Chern character in terms of Pontrjagin classes for a rank r bundle E_r.*

Total Chern character for a rank r bundle E_r in terms of Pontrjagin classes

$$
\begin{aligned}
\mathrm{ch}(E_0) =&r\\
\mathrm{ch}(E_2) =&\mathrm{ch}(E_0)+\tfrac{1}{2^1}p_1\\
\mathrm{ch}(E_4) =&\mathrm{ch}(E_2)+\tfrac{1}{2^3.3^1}\left[p_1^2-2p_2\right]\\
\mathrm{ch}(E_6) =&\mathrm{ch}(E_4)+\tfrac{1}{2^4.3^2.5^1}\left[p_1^3-3p_1p_2+3p_3\right]\\
\mathrm{ch}(E_8) =&\mathrm{ch}(E_6)+\tfrac{1}{2^7.3^2.5^1.7^1}[p_1^4+2p_2^2-4p_1^2p_2+4p_1p_3-4p_4]\\
\mathrm{ch}(E_{10}) =&\mathrm{ch}(E_8)+\tfrac{1}{2^8.3^4.5^2.7^1}[p_1^5+5p_1p_2^2-5p_1^3p_2+5p_1^2p_3-5p_2p_3-5p_1p_4+5p_5]
\end{aligned}
$$

3.3. **Tensor product.**

Tensor product. In [17] computations of Chern classes of a tensor product of two vector bundles are also performed. From the theoretical point of view these computations are possible due to the splitting principle (see Section 2.2). Let us observe an example (see [32]) on how to compute Chern classes of a tensor product of two vector bundles.

Example: Let *E* be a vector bundle and *L* a line bundle. How to obtain Chern classes of $E \otimes L$ in terms of *E* and *L*? Consider, first case when E is a line bundle, then

$$c(E \otimes L) = 1 + c_1(E \otimes L) = 1 + c_1(E) + c_1(L).$$

If E is decomposable into line bundles, $E = L_1 \otimes ... \otimes L_r$ where r is the rank of the vector bundle, then by linearity it follows that $c(E) = (1+c_1(L_1))...(1+c_1(L_r))$. Further, we have $E \otimes L = (L_1 \otimes L) \otimes ... \otimes (L_r \otimes L)$, hence

$$\begin{aligned} c(E) &= (1 + c_1(L_1) + c_1(L))...(1 + c_1(L_r) + c_1(L)) \\ &= 1 + (c_1(E) + rc_1(L)) + \left(c_2(E) + (r-1)c_1(E)c_1(L) + \binom{r}{2} c_1(L)^2\right) + ... \, . \end{aligned} \tag{3.5}$$

In fact, the result of the above equation holds even without the assumption that the vector bundle is decomposable or split. In general, for any vector bundle E and line bundle L the expression just stated shows the breakdown of the Chern class of a tensor product of two bundles into known Chern classes of the two elements by the splitting principle. The general formula for the tensor product of two vector bundles is

$$c(E \otimes F) = \prod_{a \leqslant i \leqslant r, 1 \leqslant j \leqslant s} (1 + a_i + b_j),$$

where a_i and b_j are the Chern roots of vector bundles E and F, respectively.

Regarding computerized calculations, there are three methods for computing $c(E \otimes F)$: Elimination method, Lascoux formula [19], and multiplicity of the Chern character method [17]. The latter uses the tensor product property $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \cdot \mathrm{ch}(F)$, together with Newton's identities. It appears to be the most efficient approach as demonstrated in [17]. Hence, it is quite natural that we followed this method in reproducing the expansions in Section 3.3. In [17] the reader will find explicit formulas of Chern classes of a tensor product of two vector bundles up to degree four. [5]

We also would like mention L. Manivel's paper [22], which describes formulae that may assist one in obtaining the same results. However, the paper itself does not contain explicit expansions of Chern classes of tensor bundles and more work is needed to extract them. One could even argue that such expressions are still rather abstract and not very efficient [18].

Throughout this section r denotes the rank of the bundle E_r and its Chern classes are c_i, whereas R denotes the rank of the bundle F_R and its Chern classes are C_i.

Chern character of a tensor product. Essentially, the way we perform the computations below is by first expanding the Chern character as

$$\mathrm{ch}_0(E \otimes F) + \mathrm{ch}_0(E \otimes F) + ... = [\mathrm{ch}_0(E) + \mathrm{ch}_1(E) + \mathrm{ch}_2(E) + ...][\mathrm{ch}_0(F) + \mathrm{ch}_1(F) + \mathrm{ch}_2(F) + ...] \,, \tag{3.6}$$

and then match the total degrees on the right hand side with the degree j of $\mathrm{ch}_j(E \otimes F)$ on the left hand side and write them out sequentially.

Expansion 12. *Chern character of a tensor product of two complex bundles E_r and F_R.*

[5]However, we have to remark that the expansion of $c_3(E \otimes F)$ has an error on the third line - ranks of F and E are swapped in two of the parenthesis.

Chern character of tensor product of two vector bundles

$$
\begin{aligned}
\mathrm{ch}_0(E_r \otimes F_R) =& rR \\
\mathrm{ch}_1(E_r \otimes F_R) =& Rc_1 + rC_1 \\
\mathrm{ch}_2(E_r \otimes F_R) =& \tfrac{1}{2^1}\left[R(c_1^2 - 2c_2) + 2c_1C_1 + r(C_1^2 - 2C_2)\right] \\
\mathrm{ch}_3(E_r \otimes F_R) =& \tfrac{1}{2^1.3^1}\left[R(c_1^3 - 3c_1c_2 + 3c_3) + 3\left(c_1^2 - 2c_2\right)C_1 + 3c_1\left(C_1^2 - 2C_2\right) + r\left(C_1^3 - 3C_1C_2 + 3C_3\right)\right] \\
\mathrm{ch}_4(E_r \otimes F_R) =& \tfrac{1}{2^3.3^1}[R(c_1^4 + 2c_2^2 - 4c_1^2c_2 + 4c_1c_3 - 4c_4) + r(C_1^4 + 2C_2^2 - 4C_1^2C_2 + 4C_1C_3 - 4C_4) \\
& + 4C_1(c_1^3 - 3c_1c_2 + 3c_3) + 4c_1\left(C_1^3 - 3C_1C_2 + 3C_3\right) + 6c_1^2C_1^2 - 12(c_1^2C_2 + c_2C_1^2) + 24c_2C_2] \\
\mathrm{ch}_5(E_r \otimes F_R) =& \tfrac{1}{2^3.3^1.5^1}\Big[R[c_1^5 - 5c_1^3c_2 + 5c_1^2c_3 - 5c_2c_3 + 5c_1(c_2^2 - c_4) + 5c_5] + r[C_1^5 - 5C_1^3C_2 + 5C_1^2C_3 - 5C_2C_3 + 5C_1(C_2^2 \\
& - C_4) + 5C_5] + 10(c_1^3C_1^2 + c_1^2C_1^3) + 30(c_3C_1^2 + c_1^2C_3) + 5(c_1^4C_1 + c_1C_1^4) + 10(c_2^2C_1 + c_1C_2^2) - 20(c_1^3C_2 \\
& + c_2C_1^3) - 20(c_4C_1 + c_1C_4) - 30(c_1^2C_1C_2 + c_1c_2C_1^2) - 20(c_1^2c_2C_1 + c_1C_1^2C_2) + 20(c_1c_3C_1 + c_1C_1C_3) \\
& - 60(c_3C_2 + c_2C_3) + 60(c_2C_1C_2 + c_1c_2C_2)\Big]
\end{aligned}
$$

Simplifications. We now provide the simplifications, imposing conditions appropriate for the tangential structure desired. These include instances when the structured bundle can be viewed as the lift of the tangent bundle, hence tensoring the tangent bundle with extra structure (e.g. Calabi-Yau, complex version of String structure, etc.) with an auxiliary bundle.

Expansion 13. *Chern character of a tensor product of two bundles E_r and F_R, where E_r is a special unitary bundle $\mathrm{SU}(r)$ with $c_1 = 0$.*

Chern character of tensor product of a vector bundle with an SU-bundle

$$
\begin{aligned}
\mathrm{ch}_0(E_r \otimes F_R) =& rR \\
\mathrm{ch}_1(E_r \otimes F_R) =& rC_1 \\
\mathrm{ch}_2(E_r \otimes F_R) =& -Rc_2 + \tfrac{1}{2^1}r\left(C_1^2 - 2C_2\right) \\
\mathrm{ch}_3(E_r \otimes F_R) =& \tfrac{1}{2^1.3^1}[3Rc_3 - 6c_2C_1 + r\left(C_1^3 - 3C_1C_2 + 3C_3\right)] \\
\mathrm{ch}_4(E_r \otimes F_R) =& \tfrac{1}{2^3.3^1}[2R(c_2^2 - 2c_4) + r(C_1^4 + 2C_2^2 - 4C_1^2C_2 + 4C_1C_3 - 4C_4) + 12C_1c_3 - 12c_2C_1^2 + 24c_2C_2] \\
\mathrm{ch}_5(E_r \otimes F_R) =& \tfrac{1}{2^3.3^1.5^1}\Big[5R(-c_2c_3 + c_5) + r[C_1^5 - 5C_1^3C_2 + 5C_1^2C_3 - 5C_2C_3 + 5C_1(C_2^2 - C_4) + 5C_5] \\
& + 10C_1(3c_3C_1 + c_2^2 - 2c_2C_1^2 - 2c_4) - 60(c_3C_2 + c_2C_3) + 60c_2C_1C_2\Big] \\
\mathrm{ch}_6(E_r \otimes F_R) =& \tfrac{1}{2^4.3^2.5^1}\Big[r(C_1^6 - 2C_2^3 + 9C_1^2C_2^2 + 3C_3^2 - 6C_1^4C_2 + 6C_1^3C_3 - 12C_1C_2C_3 + 6C_2C_4 - 6C_1^2C_4 + 6C_1C_5 - 6C_6) \\
& + R(-2c_2^3 + 6c_2c_4 + 3c_3^2 - 6c_6) + 30\left(-2c_4C_1^2 + c_5C_1 + 4c_4C_2 + c_2^2(C_1^2 - 2C_2)\right) \\
& + 60c_3(C_1^3 - 3C_1C_2 + 3C_3) - 30c_2(C_1^4 + c_3C_1 + 2C_2^2 - 4C_1^2C_2 + 4C_1C_3 - 4C_4)\Big]
\end{aligned}
$$

Expansion 14. *Chern character of a tensor product of two bundles E_r and F_R, where $c_1 = C_1 = 0$, i.e. the bundles are of type $SU(r)$ and $SU(R)$, respectively.*

Chern character of tensor product of two SU-bundles

$$
\begin{aligned}
\mathrm{ch}_0(E_r\otimes F_R) =& rR\\
\mathrm{ch}_1(E_r\otimes F_R) =& 0\\
\mathrm{ch}_2(E_r\otimes F_R) =& -Rc_2-rC_2\\
\mathrm{ch}_3(E_r\otimes F_R) =& \tfrac{1}{2^1}\left[Rc_3+rC_3\right]\\
\mathrm{ch}_4(E_r\otimes F_R) =& \tfrac{1}{2^2.3^1}\left[R(c_2^2-2c_4)+12c_2C_2+r\left(C_2^2-2C_4\right)\right]\\
\mathrm{ch}_5(E_r\otimes F_R) =& \tfrac{1}{2^3.3^1}\left[R(c_5-c_2c_3)-12(c_3C_2+c_2C_3)+r(C_5-C_2C_3)\right]\\
\mathrm{ch}_6(E_r\otimes F_R) =& \tfrac{1}{2^4.3^2.5^1}[R(-2c_2^3+3c_3^2+6c_2c_4-6c_6)+r(-2C_2^3+3C_3^2+6C_2C_4-6C_6)\\
& +60(-c_2^2C_2+2c_4C_2+3c_3C_3+2c_2C_4-c_2C_2^2)]\\
\mathrm{ch}_7(E_r\otimes F_R) =& \tfrac{1}{2^4.3^2.5^1}\Big[R(c_2^2c_3-c_2c_5-c_3c_4+c_7)+r(C_2^2C_3-C_2C_5-C_3C_4+C_7)\\
& +30\left\{C_3c_2^2+c_3C_2^2+c_3C_2c_2+C_3C_2c_2-(C_5c_2+c_5C_2)-2(c_4C_3+c_3C_4)\right\}\Big]\\
\mathrm{ch}_8(E_r\otimes F_R) =& \tfrac{1}{2^6.3^2.5^1.7^1}[R(c_2^4-4c_2c_3^2-4c_2^2c_4+4c_2c_6+4c_3c_5+2c_4^2-4c_8)\\
& +r(C_2^4-4C_2C_3^2-4C_2^2C_4+4C_2C_6+4C_3C_5+2C_4^2-4C_8)\\
& +140c_2^2C_2^2+56(c_2^3C_2+c_2C_2^3)-84(c_3^2C_2+c_2C_3^2)-420(c_3C_2C_3+c_2c_3C_3)\\
& -280(c_2^2C_4+c_4C_2^2)-168(c_2c_4C_2+c_2C_2C_4)+168(c_6C_2+C_6c_2)+420(c_5C_3+c_3C_5)+560c_4C_4]
\end{aligned}
$$

Now we take E_r and F_R to both be complex String bundles, i.e., $c_1=c_2=0$ and $C_1=C_2=0$.

Expansion 15. *Chern character of tensor product of complex String bundles.*

Chern character of tensor product of complex String bundles

$$
\begin{aligned}
\mathrm{ch}_0(E_r\otimes F_R) =& rR\\
\mathrm{ch}_1(E_r\otimes F_R) =& 0\\
\mathrm{ch}_2(E_r\otimes F_R) =& 0\\
\mathrm{ch}_3(E_r\otimes F_R) =& \tfrac{1}{2^1}\left[Rc_3+rC_3\right]\\
\mathrm{ch}_4(E_r\otimes F_R) =& -\tfrac{1}{2^1.3^1}[Rc_4+rC_4]\\
\mathrm{ch}_5(E_r\otimes F_R) =& \tfrac{1}{2^3.3^1}\left[Rc_5+rC_5\right]\\
\mathrm{ch}_6(E_r\otimes F_R) =& \tfrac{1}{2^4.3^2.5^1}[R(3c_3^2-6c_6)+r(3C_3^2-6C_6)+180c_3C_3]\\
\mathrm{ch}_7(E_r\otimes F_R) =& \tfrac{1}{2^4.3^2.5^1}[R(-c_3c_4+c_7)+r(-C_3C_4+C_7)-60(c_4C_3+c_3C_4)]\\
\mathrm{ch}_8(E_r\otimes F_R) =& \tfrac{1}{2^6.3^2.5^1.7^1}[R(4c_3c_5+2c_4^2-4c_8)+r(4C_3C_5+2C_4^2-4C_8)+420(c_5C_3+c_3C_5)+560c_4C_4]
\end{aligned}
$$

Chern classes of a tensor product. The Chern classes, unlike the Chern character, are not multiplicative. Thus, we combine Expansion 12 with Expansion 2 to write Chern classes of a tensor product.

Expansion 16. *Chern classes of a tensor product of two bundles E_r and F_R.*

Chern classes of tensor product of two vector bundles

$$
\begin{aligned}
c_0(E_r \otimes F_R) =& rR \\
c_1(E_r \otimes F_R) =& Rc_1 + rC_1 \\
c_2(E_r \otimes F_R) =& \tfrac{1}{2^1}[R(R-1)c_1^2 + r(r-1)C_1^2 + 2(Rc_2 + rC_2) + 2(rR-1)c_1C_1] \\
c_3(E_r \otimes F_R) =& \tfrac{1}{2^1 \cdot 3^1}[(Rc_1 + rC_1)^3 - 3\left(Rc_1 + rC_1\right)\left(Rc_1^2 - 2Rc_2 + 2c_1C_1 + rC_1^2 - 2rC_2\right) \\
& + 2R\left(c_1^3 - 3c_1c_2 + 3c_3\right) + 2r(C_1^3 - 3C_1C_2 + 3C_3) + 6\left(c_1^2 - 2c_2\right)C_1 + 6c_1\left(C_1^2 - 2C_2\right)] \\
c_4(E_r \otimes F_R) =& \tfrac{1}{2^3 \cdot 3^1}\Big[(Rc_1 + rC_1)^4 + 3\left(R(c_1^2 - 2c_2) + 2c_1C_1 + r\left(C_1^2 - 2C_2\right)\right)^2 - 6\left(Rc_1 + rC_1\right)^2 \Big(R(c_1^2 - 2c_2) + 2c_1C_1 \\
& + r\left(C_1^2 - 2C_2\right)\Big) + 8\left(Rc_1 + rC_1\right)\Big[R\left(c_1^3 - 3c_1c_2 + 3c_3\right) + 3(c_1^2 - 2c_2)C_1 + 3c_1\left(C_1^2 - 2C_2\right) \\
& + r\left(C_1^3 - 3C_1C_2 + 3C_3\right)\Big] - 6(Rc_1^4 + rC_1^4) - 12(2c_1C_1^3 + 3c_1^2C_1^2 + 2c_1^3C_1) - 12(Rc_2^2 + 12c_2C_2 + rC_2^2) \\
& + 24(rC_1^2C_2 + Rc_1^2c_2 + 3c_1^2C_2 + 3c_2C_1^2 + 3c_1c_2C_1 + 3c_1C_1C_2) + 24(-rC_1C_3 - 3c_3C_1 - Rc_1c_3 - 3c_1C_3) \\
& + 24(rC_4 + Rc_4)\Big]
\end{aligned}
$$

Simplifications.

Expansion 17. *Chern classes of a tensor product of two bundles E_r and F_R, where $c_1 = C_1 = 0$, i.e. the bundles are of type $SU(r)$ and $SU(R)$, respectively.*

Chern classes of tensor product of two SU-bundles

$$
\begin{aligned}
c_0(E_r \otimes F_R) =& rR \\
c_1(E_r \otimes F_R) =& 0 \\
c_2(E_r \otimes F_R) =& Rc_2 + rC_2 \\
c_3(E_r \otimes F_R) =& Rc_3 + rC_3 \\
c_4(E_r \otimes F_R) =& \tfrac{1}{2^1}[R(R-1)c_2^2 + 2(Rc_4 + rC_4) + 2(rR-6)c_2C_2 + r(r-1)C_2^2] \\
c_5(E_r \otimes F_R) =& [C_5r + Rc_5 + (Rr-12)(c_3C_2 + C_3c_2) + r(r-1)C_2C_3 + R(R-1)c_2c_3] \\
c_6(E_r \otimes F_R) =& \tfrac{1}{2^1 \cdot 3^1}\Big[R(R^2 - 3R + 2)c_2^3 + r(r^2 - 3r + 2)C_2^3 + 3\left(rR^2 - (r+12)R + 20\right)c_2^2C_2 + 3\left(Rr^2 - (R+12)r + 20\right)c_2C_2^2 \\
& + 6R(R-1)c_2c_4 + 6r(r-1)C_2C_4 + 6(rR-20)(c_4C_2 + C_4c_2) \\
& + 6(rR-30)c_3C_3 + 3R(R-1)c_3^2 + 3r(r-1)C_3^2 + 6(Rc_6 + rC_6)\Big]
\end{aligned}
$$

Observation: It is now possible to group the terms in such as way as to make the symmetry: $r \leftrightarrow R$ and $c_i \leftrightarrow C_i$ manifest.

Now we take both E_r and F_R to be complex String bundles.

Expansion 18. *Chern classes of tensor product of complex String bundles, i.e., $c_1 = c_2 = 0$ and $C_1 = C_2 = 0$.*

Chern classes of tensor product of complex String bundles

$$
\begin{aligned}
c_0(E_r \otimes F_R) =& rR \\
c_1(E_r \otimes F_R) =& 0 \\
c_2(E_r \otimes F_R) =& 0 \\
c_3(E_r \otimes F_R) =& Rc_3 + rC_3 \\
c_4(E_r \otimes F_R) =& Rc_4 + rC_4 \\
c_5(E_r \otimes F_R) =& Rc_5 + rC_5 \\
c_6(E_r \otimes F_R) =& (rR-30)c_3C_3 + \tfrac{1}{2}R(R-1)c_3^2 + \tfrac{1}{2}r(r-1)C_3^2 + (Rc_6 + rC_6) \\
c_7(E_r \otimes F_R) =& R(R-1)c_3c_4 + (rR-60)(c_3C_4 + c_4C_3) + r(r-1)C_3C_4 + (Rc_7 + rC_7)
\end{aligned}
$$

4. Expansions of the Genera

4.1. **Genera.** We will provide some basic theoretical background on genera (see [12][26][13]). A genus (singular of "genera") is a certain combinatorial formula that involve characteristic classes. It is a functor on cobordism, i.e. is related to whether a manifold can be a boundary of another and also how two manifolds combine to get a third of the same or different dimension. A genus g assigns a number $g(X)$ to each manifold X so that the following holds true (see [12] [26])

(1) **Additivity**: $g(X \coprod Y) = g(X) + g(Y)$ where $\coprod$ is disjoint union,
(2) **Multiplicativity**: $g(X \times Y) = g(X)g(Y)$,
(3) **Triviality**: $g(X) = 0$ if X is a boundary of some other manifold.

We will find the following useful for the computations. One calls a sequence of polynomials $P_1, P_2, ...$ in variables $s_1, s_2, ..$ *multiplicative* if there exists a factorization (see Hirzebruch)

$$\begin{aligned} 1 + s_1 t + s_2 t^2 + ... &= (1 + y_1 t + y_2 t^2 + ...)(1 + z_1 t + z_2 t^2 + ...) \\ \Rightarrow \sum_j P_j(s_1, s_2, ...)t^j &= \sum_j P_j(y_1, y_2, ...)t^j \sum_k P_k(z_1, z_2, ...)t^k. \end{aligned}$$

A multiplicative sequence $P = 1 + P_1 + P_2 + ...$ is then given by

$$P(s_1, s_2, s_3...) = Q(y_1)Q(y_2)Q(y_3)...$$

where s_i is the i-th elementary symmetric function of y_j and $Q(u)$ is a formal power series in u which starts with 1.

The above combinatorial formulas will be in terms of the Pontrjagin and Chern classes once the manifold under investigation admits the corresponding structure (i.e. real or oriented for Pontrjagin and complex for Chern). Indeed, the genus g of oriented manifolds corresponding to Q is written as $g(X) = P(s_1, s_2, s_3, ...)$ where s_i are Pontrjagin classes p_i of X. The power series Q are the characteristic power series of the genus g. A similar description holds for the Chern classes.

Next we will combinatorially define the three genera of interest.

1. The Â-genus. The *Â-genus* is defined as

$$\begin{aligned} \hat{A}(x) &= \prod_{j=1}^{k} \frac{x_j/2}{\sinh(x_j/2)} \\ &= \prod_{j=1}^{k} \left(1 + \sum_{n \geqslant 1} (-1)^n \tfrac{2^{2n}-1}{(2n)!} B_n x_j^{2n}\right) \end{aligned}$$

where B_n are the Bernoulli numbers given as

$$B_1 = \tfrac{1}{6},\ B_2 = \tfrac{1}{30},\ B_3 = \tfrac{1}{42},\ B_4 = \tfrac{1}{30},\ B_5 = \tfrac{5}{66},\ \cdots.$$

The Â-genus is an even function of x_j which can be expanded in the Pontrjagin classes p_i. One of the properties of the Â-genus is that it satisfies the following

$$\hat{A}(E \oplus F) = \hat{A}(E) \cdot \hat{A}(F).$$

The next section has more on how to expand Â-genus, while *Expansion 19* in Section 4 states the actual expressions.

2. The L-genus. The *Hirzebruch L-polynomial* or *L-genus* is defined by

$$L(x) = \prod_{j=1}^{k} \frac{\sqrt{x_j}}{\tanh\sqrt{x_j}} \tag{4.1}$$

$$= \prod_{j=1}^{k} \left(1 + \sum_{n \geqslant 1} (-1)^{n-1} \tfrac{2^{2n}}{(2n)!} B_n x_j^{2n}\right). \tag{4.2}$$

$L(x)$ is also an even function of x_j which can be expressed in terms of the Pontrjagin classes and enjoys the same property as the Â-genus

$$L(E \oplus F) = L(E) \cdot L(F).$$

The L-genus expressions are listed in *Expression 28* in Section 4.

3. The Todd genus. The Todd genus is associated with complex vector bundles and is defined by the series

$$T(x) = \prod_j \frac{x_j}{1 - e^{-x_j}}. \tag{4.3}$$

If the *Todd genus* is expanded in powers of x_j, then the expression looks like

$$\begin{aligned} \mathrm{Td}(E) &= \prod_j \Big(1 + \tfrac{1}{2}x_j + \sum_{k\geqslant 1}(-1)^{k-1}\tfrac{1}{(2k)!}B_k x_j^{2k}\Big) \\ &= 1 + \tfrac{1}{2}\sum_j x_j + \tfrac{1}{12}\sum_j x_j^2 + \tfrac{1}{4}\sum_{j\ k} x_j x_k + \ldots \\ &= 1 + \tfrac{1}{2}c_1(E) + \tfrac{1}{12}[c_1(E)^2 + c_2(E)] + \ldots , \end{aligned}$$

where we use the identification between the Chern classes and the Chern roots. As with the other two genera, the Todd genus also satisfies

$$\mathrm{Td}(E \oplus F) = \mathrm{Td}(E) \cdot \mathrm{Td}(F).$$

The reader will find the Todd genus written out explicitly in *Expansion 37*.

The genera expansions are important because they capture information about the vector bundles. The Â-genus and L-genus are relevant for real spaces, while the Todd genus is relevant for complex spaces. Details can be found in [9][12], while geometric perspectives are provided, e.g. in [28][27][34].

4.2. **Â-genus.** The Â-genus is computed by plugging the characteristic power series of Â-genus $\frac{x_j/2}{\sinh(x_j/2)}$ (see Section 4) into the function below originally developed by C. McTague [25].

Expansion 19. *Â-genus in terms of Pontrjagin classes.*

Â-genus in terms of Pontrjagin classes

$$
\begin{aligned}
A_0 =& 1\\
A_1 =& \tfrac{1}{2^3.3^1}\Big[-1^1p_1\Big]\\
A_2 =& \tfrac{1}{2^5.3^2.5^1}\Big[\tfrac{7^1}{2^2}p_1^2-1^1p_2\Big]\\
A_3 =& \tfrac{1}{2^6.3^3.5^1.7^1}\Big[-\tfrac{31^1}{2^4}p_1^3+\tfrac{11^1}{2^2}p_1p_2-1^1p_3\Big]\\
A_4 =& \tfrac{1}{2^6.3^3.5^2.7^1}\Big[\tfrac{127^1}{2^9}p_1^4-\tfrac{113^1}{2^6.3^1}p_1^2p_2+\tfrac{1^1}{3^1}p_1p_3+\tfrac{13^1}{2^5.3^1}p_2^2-\tfrac{1^1}{2^3}p_4\Big]\\
A_5 =& \tfrac{1}{2^{10}.3^4.5^1.11^1}\Big[-\tfrac{73^1}{2^8.3^1}p_1^5+\tfrac{29^1\cdot 37^1}{2^5.3^1.5^1.7^1}p_1^3p_2-\tfrac{61^1}{2^3.5^1.7^1}p_1^2p_3-\tfrac{311^1}{2^4.3^1.5^1.7^1}p_1p_2^2+\tfrac{53^1}{2^2.3^1.5^1.7^1}p_1p_4+\tfrac{1^1}{2^1.5^1}p_2p_3\\
&-\tfrac{1^1}{3^1.7^1}p_5\Big]\\
A_6 =& \tfrac{1}{2^{12}.3^5.5^2.7^2.11^1.13^1}\Big[\tfrac{23^1\cdot 89^1\cdot 691^1}{2^{10}.3^1.5^1}p_1^6-\tfrac{1540453^1}{2^8.3^1.5^1}p_1^4p_2+\tfrac{29^1\cdot 1249^1}{2^3.3^1.5^1}p_1^3p_3+\tfrac{19^1\cdot 4013^1}{2^6.3^1}p_1^2p_2^2-\tfrac{16759^1}{2^4.5^1}p_1^2p_4-\tfrac{3491^1}{2^1.5^1}p_1p_2p_3\\
&+\tfrac{23^1\cdot 53^1}{2^1.5^1}p_1p_5-\tfrac{19^1\cdot 211^1}{2^4.5^1}p_2^3+\tfrac{73^1\cdot 79^1}{2^2.3^1.5^1}p_2p_4+\tfrac{19^1\cdot 37^1}{3^1.5^1}p_3^2-\tfrac{691^1}{3^1.5^1}p_6\Big]\\
A_7 =& \tfrac{1}{2^{11}.3^4.7^1.13^1}\Big[-\tfrac{8191^1}{2^{14}.3^2.5^2.11^1}p_1^7+\tfrac{37^1\cdot 31121^1}{2^{12}.3^3.5^3.7^1.11^1}p_1^5p_2-\tfrac{67^1\cdot 127^1}{2^{10}.5^3.7^1.11^1}p_1^4p_3-\tfrac{9161^1}{2^{10}.3^1.5^2.7^1.11^1}p_1^3p_2^2+\tfrac{23^1\cdot 127^1}{2^8.3^2.5^2.7^1.11^1}p_1^3p_4\\
&+\tfrac{179^1\cdot 317^1}{2^7.3^3.5^3.7^1.11^1}p_1^2p_2p_3-\tfrac{2543^1}{2^6.3^2.5^3.7^1.11^1}p_1^2p_5+\tfrac{109^1\cdot 307^1}{2^8.3^3.5^3.7^1.11^1}p_1p_2^3-\tfrac{67^1}{2^6.5^3.11^1}p_1p_2p_4-\tfrac{97^1}{3^3.5^3.7^1.11^1}p_1p_3^2\\
&+\tfrac{101^1}{2^4.3^3.5^3.7^1}p_1p_6-\tfrac{23^1\cdot 233^1}{2^6.3^3.5^3.7^1.11^1}p_2^2p_3+\tfrac{1^1}{2^4.3^3.11^1}p_2p_5+\tfrac{283^1}{2^4.3^2.5^3.7^1.11^1}p_3p_4-\tfrac{1^1}{2^2.3^2.5^2.11^1}p_7\Big]\\
A_8 =& \tfrac{1}{2^{15}.3^5.5^2.7^1.17^1}\Big[\tfrac{31^1\cdot 151^1\cdot 3617^1}{2^{16}.3^2.5^2.11^1.13^1}p_1^8-\tfrac{2241667^1}{2^{12}.3^1.5^2.11^1.13^1}p_1^6p_2+\tfrac{3661841^1}{2^7.3^3.5^2.7^1.11^1.13^1}p_1^5p_3+\tfrac{71^1\cdot 268823^1}{2^{11}.3^3.5^2.11^1.13^1}p_1^4p_2^2\\
&-\tfrac{3941363^1}{2^9.3^2.5^2.7^1.11^1.13^1}p_1^4p_4-\tfrac{317^1\cdot 4129^1}{2^4.3^3.5^2.7^1.11^1.13^1}p_1^3p_2p_3+\tfrac{26921^1}{2^5.3^2.5^2.11^1.13^1}p_1^3p_5-\tfrac{29^1\cdot 41^1\cdot 227^1}{2^8.3^3.5^1.11^1.13^1}p_1^2p_2^3\\
&+\tfrac{38923^1}{2^6.3^2.7^1.11^1.13^1}p_1^2p_2p_4+\tfrac{907^1}{2^2.3^1.5^2.7^1.13^1}p_1^2p_3^2-\tfrac{197033^1}{2^4.3^3.5^2.7^1.11^1.13^1}p_1^2p_6+\tfrac{9091^1}{2^3.3^3.5^1.11^1.13^1}p_1p_2^2p_3\\
&-\tfrac{23339^1}{2^3.3^3.5^2.11^1.13^1}p_1p_2p_5-\tfrac{39887^1}{2^1.3^3.5^2.7^1.11^1.13^1}p_1p_3p_4+\tfrac{1063^1}{2^2.5^2.7^1.11^1.13^1}p_1p_7+\tfrac{11^1\cdot 1249^1}{2^8.3^3.5^1.7^1.13^1}p_2^4-\tfrac{275593^1}{2^5.3^3.5^2.7^1.11^1.13^1}p_2^2p_4\\
&-\tfrac{31^1}{3^2.5^2.11^1}p_2p_3^2+\tfrac{11299^1}{2^1.3^3.5^2.7^1.11^1.13^1}p_2p_6+\tfrac{11^1\cdot 181^1}{2^2.3^3.5^2.7^1.13^1}p_3p_5+\tfrac{73^1\cdot 199^1}{2^4.3^2.5^2.7^1.11^1.13^1}p_4^2-\tfrac{3617^1}{2^2.3^2.5^2.7^1.11^1.13^1}p_8\Big]\\
A_9 =& \tfrac{1}{2^{13}.3^7.5^2.7^2.19^1}\Big[-\tfrac{43867^1\cdot 131071^1}{2^{21}.3^2.5^1.7^1.11^1.13^1.17^1}p_1^9+\tfrac{47^1\cdot 907^1\cdot 7949^1}{2^{17}.3^1.5^2.7^1.11^1.17^1}p_1^7p_2-\tfrac{9397^1\cdot 33317^1}{2^{15}.5^2.7^1.11^1.13^1.17^1}p_1^6p_3\\
&-\tfrac{83^1\cdot 3688543^1}{2^{16}.3^1.5^2.7^1.11^1.13^1}p_1^5p_2^2+\tfrac{5303^1\cdot 44519^1}{2^{14}.3^2.5^1.7^1.11^1.13^1.17^1}p_1^5p_4+\tfrac{601^1\cdot 4429813^1}{2^{13}.3^2.5^2.7^1.11^1.13^1.17^1}p_1^4p_2p_3+\tfrac{601^1\cdot 4429813^1}{2^{13}.3^2.5^2.7^1.11^1.13^1.17^1}p_1^4p_2p_3\\
&-\tfrac{47756197^1}{2^{12}.3^2.5^1.7^1.11^1.13^1.17^1}p_1^4p_5+\tfrac{1625000107^1}{2^{13}.3^2.5^2.7^1.11^1.13^1.17^1}p_1^3p_2^3-\tfrac{613^1\cdot 688087^1}{2^{11}.3^2.5^2.7^1.11^1.13^1.17^1}p_1^3p_2p_4-\tfrac{23^1\cdot 653^1}{2^9.5^1.13^1.17^1}p_1^3p_3^2\\
&+\tfrac{4569683^1}{2^9.3^2.5^1.7^1.11^1.13^1.17^1}p_1^3p_6-\tfrac{59^1\cdot 163^1\cdot 8377^1}{2^{11}.3^2.5^2.11^1.13^1.17^1}p_1^2p_2^2p_3+\tfrac{1511459^1}{2^9.3^1.5^2.7^1.11^1.17^1}p_1^2p_2p_5+\tfrac{23^1\cdot 263^1\cdot 3187^1}{2^9.3^1.5^2.7^1.11^1.13^1.17^1}p_1^2p_3p_4\\
&-\tfrac{23^1\cdot 86573^1}{2^6.3^2.5^2.7^1.11^1.13^1.17^1}p_1^2p_7-\tfrac{37857689^1}{2^{13}.5^2.7^1.11^1.13^1.17^1}p_1p_2^4+\tfrac{11389153^1}{2^{10}.5^2.7^1.11^1.13^1.17^1}p_1p_2^2p_4+\tfrac{199^1\cdot 4231^1}{2^7.3^1.5^1.7^1.11^1.13^1.17^1}p_1p_2p_3^2\\
&-\tfrac{3301303^1}{2^6.3^2.5^2.7^1.11^1.13^1.17^1}p_1p_2p_6-\tfrac{127^1\cdot 2029^1}{2^7.3^2.7^1.11^1.13^1.17^1}p_1p_3p_5-\tfrac{311^1\cdot 41263^1}{2^9.3^2.5^2.7^1.11^1.13^1.17^1}p_1p_4^2+\tfrac{467^1\cdot 4969^1}{2^7.3^2.5^2.7^1.11^1.13^1.17^1}p_1p_8\\
&+\tfrac{15013651^1}{2^9.3^2.5^2.7^1.11^1.13^1.17^1}p_2^3p_3-\tfrac{991^1\cdot 1123^1}{2^8.3^2.5^1.7^1.11^1.13^1.17^1}p_2^2p_5-\tfrac{43^1\cdot 42239^1}{2^7.3^1.5^2.7^1.11^1.13^1.17^1}p_2p_3p_4+\tfrac{41^1\cdot 557^1}{2^3.3^2.5^1.7^1.11^1.13^1.17^1}p_2p_7\\
&-\tfrac{61^1\cdot 3659^1}{2^5.3^2.5^2.7^1.11^1.13^1.17^1}p_3^3+\tfrac{13829^1}{3^2.5^2.7^1.11^1.13^1.17^1}p_3p_6+\tfrac{43^1\cdot 4091^1}{2^6.3^2.5^1.7^1.11^1.13^1.17^1}p_4p_5-\tfrac{43867^1}{2^5.3^2.5^1.7^1.11^1.13^1.17^1}p_9\Big]
\end{aligned}
$$

Â-genus with simplifications. One might be interested in bundles with higher notions of symmetries beyond the usual SO or SU structures; for example, String or p_1-structures ($p_1 = 0$), Fivebrane or p_2-structures ($p_1 = p_2 = 0$) [30][31], and Ninebrane or p_3-structures ($p_1 = p_2 = p_3 = 0$) [29].

The expansions in this subsection were obtained by using two Mathematica formulas, both of which are modifications of C. McTague's code [25]. The two new functions deal with genera simplification and only differ in how they set which characteristic classes would vanish based on user's input.

Expansion 20. *Â-genus with String or p_1-structure ($p_1 = 0$).*

Â-genus of String or p_1-structure manifolds

$$
\begin{aligned}
A_0 =& 1\\
A_1 =& 0\\
A_2 =& \tfrac{1}{2^5.3^2.5^1}\Big[-1^1 p_2\Big]\\
A_3 =& \tfrac{1}{2^6.3^3.5^1.7^1}\Big[-1^1 p_3\Big]\\
A_4 =& \tfrac{1}{2^9.3^3.5^2.7^1}\Big[\tfrac{13^1}{2^2.3^1}p_2^2 - 1^1 p_4\Big]\\
A_5 =& \tfrac{1}{2^{10}.3^4.5^1.11^1}\Big[\tfrac{1^1}{2^1.5^1}p_2p_3 - \tfrac{1^1}{3^1.7^1}p_5\Big]\\
A_6 =& \tfrac{1}{2^{12}.3^5.5^3.7^2.11^1.13^1}\Big[-\tfrac{19^1\cdot 211^1}{2^4}p_2^3 + \tfrac{73^1\cdot 79^1}{2^2.3^1}p_2p_4 + \tfrac{19^1\cdot 37^1}{3^1}p_3^2 - \tfrac{691^1}{3^1}p_6\Big]\\
A_7 =& \tfrac{1}{2^{13}.3^6.7^1.11^1.13^1}\Big[-\tfrac{23^1\cdot 233^1}{2^4.3^1.5^3.7^1}p_2^2p_3 + \tfrac{1^1}{2^2.3^1}p_2p_5 + \tfrac{283^1}{2^2.5^3.7^1}p_3p_4 - \tfrac{1^1}{5^2}p_7\Big]\\
A_8 =& \tfrac{1}{2^{15}.3^7.5^3.7^1.17^1}\Big[\tfrac{11^1\cdot 1249^1}{2^8.3^1.7^1.13^1}p_2^4 - \tfrac{275593^1}{2^5.3^1.5^1.7^1.11^1.13^1}p_2^2p_4 - \tfrac{31^1}{5^1.11^1}p_2p_3^2 + \tfrac{11299^1}{2^1.3^1.5^1.7^1.11^1.13^1}p_2p_6\\
&+ \tfrac{11^1\cdot 181^1}{2^2.3^1.5^1.7^1.13^1}p_3p_5 + \tfrac{73^1\cdot 199^1}{2^4.5^1.7^1.11^1.13^1}p_4^2 - \tfrac{3617^1}{2^2.5^1.7^1.11^1.13^1}p_8\Big]\\
A_9 =& \tfrac{1}{2^{13}.3^8.5^3.7^3.11^1.13^1.17^1.19^1}\Big[\tfrac{15013651^1}{2^9.3^1.5^1}p_2^3p_3 - \tfrac{991^1\cdot 1123^1}{2^8.3^1}p_2^2p_5 - \tfrac{43^1\cdot 42239^1}{2^7.5^1}p_2p_3p_4 + \tfrac{41^1\cdot 557^1}{2^3.3^1}p_2p_7 - \tfrac{61^1\cdot 3659^1}{2^5.3^1.5^1}p_3^3\\
&+ \tfrac{13829^1}{3^1.5^1}p_3p_6 + \tfrac{43^1\cdot 4091^1}{2^6.3^1}p_4p_5 - \tfrac{43867^1}{2^5.3^1}p_9\Big]
\end{aligned}
$$

Expansion 21. *Â-genus of p_2-structure manifolds ($p_2 = 0$).*

Â-genus with p_2-structure

$$
\begin{aligned}
A_0 =& 1\\
A_1 =& \tfrac{1}{2^3.3^1}\Big[-1^1 p_1\Big]\\
A_2 =& \tfrac{1}{2^7.3^2.5^1}\Big[7^1 p_1^2\Big]\\
A_3 =& \tfrac{1}{2^6.3^3.5^1.7^1}\Big[-\tfrac{31^1}{2^4}p_1^3 - 1^1 p_3\Big]\\
A_4 =& \tfrac{1}{2^6.3^3.5^2.7^1}\Big[\tfrac{127^1}{2^9}p_1^4 + \tfrac{1^1}{3^1}p_1p_3 - \tfrac{1^1}{2^3}p_4\Big]\\
A_5 =& \tfrac{1}{2^{10}.3^4.5^1.11^1}\Big[-\tfrac{73^1}{2^8.3^1}p_1^5 - \tfrac{61^1}{2^3.5^1.7^1}p_1^2p_3 + \tfrac{53^1}{2^2.3^1.5^1.7^1}p_1p_4 - \tfrac{1^1}{3^1.7^1}p_5\Big]\\
A_6 =& \tfrac{1}{2^{12}.3^5.5^3.7^2.11^1.13^1}\Big[\tfrac{23^1\cdot 89^1\cdot 691^1}{2^{10}.3^1}p_1^6 + \tfrac{29^1\cdot 1249^1}{2^3.3^1}p_1^3p_3 - \tfrac{16759^1}{2^4}p_1^2p_4 + \tfrac{23^1\cdot 53^1}{2^1}p_1p_5 + \tfrac{19^1\cdot 37^1}{3^1}p_3^2 - \tfrac{691^1}{3^1}p_6\Big]\\
A_7 =& \tfrac{1}{2^{11}.3^4.5^2.7^1.13^1}\Big[-\tfrac{8191^1}{2^{14}.3^2.11^1}p_1^7 - \tfrac{67^1\cdot 127^1}{2^{10}.5^1.7^1.11^1}p_1^4p_3 + \tfrac{23^1\cdot 127^1}{2^8.3^2.7^1.11^1}p_1^3p_4 - \tfrac{2543^1}{2^6.3^2.5^1.7^1.11^1}p_1^2p_5 - \tfrac{97^1}{3^3.5^1.7^1.11^1}p_1p_3^2\\
&+ \tfrac{101^1}{2^4.3^3.5^1.7^1}p_1p_6 + \tfrac{283^1}{2^4.3^2.5^1.7^1.11^1}p_3p_4 - \tfrac{1^1}{2^2.3^2.11^1}p_7\Big]\\
A_8 =& \tfrac{1}{2^{16}.3^5.5^4.7^1.13^1.17^1}\Big[\tfrac{31^1\cdot 151^1\cdot 3617^1}{2^{15}.3^2.11^1}p_1^8 + \tfrac{3661841^1}{2^6.3^3.7^1.11^1}p_1^5p_3 - \tfrac{3941363^1}{2^8.3^2.7^1.11^1}p_1^4p_4 + \tfrac{26921^1}{2^4.3^2.11^1}p_1^3p_5 + \tfrac{907^1}{2^1.3^1.7^1}p_1^2p_3^2\\
&- \tfrac{197033^1}{2^3.3^3.7^1.11^1}p_1^2p_6 - \tfrac{39887^1}{3^3.7^1.11^1}p_1p_3p_4 + \tfrac{1063^1}{2^1.7^1.11^1}p_1p_7 + \tfrac{11^1\cdot 181^1}{2^1.3^3.7^1}p_3p_5 + \tfrac{73^1\cdot 199^1}{2^3.3^2.7^1.11^1}p_4^2 - \tfrac{3617^1}{2^1.3^2.7^1.11^1}p_8\Big]\\
A_9 =& \tfrac{1}{2^{13}.3^7.5^2.7^2.13^1.17^1.19^1}\Big[-\tfrac{43867^1\cdot 131071^1}{2^{21}.3^2.5^1.7^1.11^1}p_1^9 - \tfrac{9397^1\cdot 33317^1}{2^{15}.5^2.7^1.11^1}p_1^6p_3 + \tfrac{5303^1\cdot 44519^1}{2^{14}.3^2.5^1.7^1.11^1}p_1^5p_4 - \tfrac{47756197^1}{2^{12}.3^2.5^1.7^1.11^1}p_1^4p_5\\
&- \tfrac{23^1\cdot 653^1}{2^9.5^1}p_1^3p_3^2 + \tfrac{4569683^1}{2^9.3^2.5^1.7^1.11^1}p_1^3p_6 + \tfrac{23^1\cdot 263^1\cdot 3187^1}{2^9.3^1.5^2.7^1.11^1}p_1^2p_3p_4 - \tfrac{23^1\cdot 86573^1}{2^6.3^2.5^2.7^1.11^1}p_1^2p_7 - \tfrac{127^1\cdot 2029^1}{2^7.3^2.7^1.11^1}p_1p_3p_5\\
&- \tfrac{311^1\cdot 41263^1}{2^9.3^2.5^2.7^1.11^1}p_1p_4^2 + \tfrac{467^1\cdot 4969^1}{2^7.3^2.5^2.7^1.11^1}p_1p_8 - \tfrac{61^1\cdot 3659^1}{2^5.3^2.5^2.7^1.11^1}p_3^3 + \tfrac{13829^1}{3^2.5^2.7^1.11^1}p_3p_6 + \tfrac{43^1\cdot 4091^1}{2^6.3^2.5^1.7^1.11^1}p_4p_5\\
&- \tfrac{43867^1}{2^5.3^2.5^1.7^1.11^1}p_9\Big]
\end{aligned}
$$

Expansion 22. *Â-genus of p_3-structure manifolds ($p_3 = 0$).*

Â-genus with p_3-structure

$$
\begin{aligned}
A_0 =& 1\\
A_1 =& \tfrac{1}{2^3.3^1}\Big[-1^1 p_1\Big]\\
A_2 =& \tfrac{1}{2^5.3^2.5^1}\Big[\tfrac{7^1}{2^2}p_1^2 - 1^1 p_2\Big]\\
A_3 =& \tfrac{1}{2^8.3^3.5^1.7^1}\Big[-\tfrac{31^1}{2^2}p_1^3 + 11^1 p_1p_2\Big]\\
A_4 =& \tfrac{1}{2^9.3^3.5^2.7^1}\Big[\tfrac{127^1}{2^6}p_1^4 - \tfrac{113^1}{2^3.3^1}p_1^2p_2 + \tfrac{13^1}{2^2.3^1}p_2^2 - 1^1 p_4\Big]\\
A_5 =& \tfrac{1}{2^{10}.3^5.5^1.11^1}\Big[-\tfrac{73^1}{2^8}p_1^5 + \tfrac{29^1\cdot 37^1}{2^5.5^1.7^1}p_1^3p_2 - \tfrac{311^1}{2^4.5^1.7^1}p_1p_2^2 + \tfrac{53^1}{2^2.5^1.7^1}p_1p_4 - \tfrac{1^1}{7^1}p_5\Big]\\
A_6 =& \tfrac{1}{2^{12}.3^5.5^2.7^2.11^1.13^1}\Big[\tfrac{23^1\cdot 89^1\cdot 691^1}{2^{10}.3^1.5^1}p_1^6 - \tfrac{1540453^1}{2^8.3^1.5^1}p_1^4p_2 + \tfrac{19^1\cdot 4013^1}{2^6.3^1}p_1^2p_2^2 - \tfrac{16759^1}{2^4.5^1}p_1^2p_4 + \tfrac{23^1\cdot 53^1}{2^1.5^1}p_1p_5 - \tfrac{19^1\cdot 211^1}{2^4.5^1}p_2^3\\
&+ \tfrac{73^1\cdot 79^1}{2^2.3^1.5^1}p_2p_4 - \tfrac{691^1}{3^1.5^1}p_6\Big]\\
A_7 =& \tfrac{1}{2^{13}.3^4.7^1.13^1}\Big[-\tfrac{8191^1}{2^{12}.3^2.5^2.11^1}p_1^7 + \tfrac{37^1\cdot 31121^1}{2^{10}.3^3.5^3.7^1.11^1}p_1^5p_2 - \tfrac{9161^1}{2^8.3^1.5^2.7^1.11^1}p_1^3p_2^2 + \tfrac{23^1\cdot 127^1}{2^6.3^2.5^2.7^1.11^1}p_1^3p_4 - \tfrac{2543^1}{2^4.3^2.5^3.7^1.11^1}p_1^2p_5\\
&+ \tfrac{109^1\cdot 307^1}{2^6.3^3.5^3.7^1.11^1}p_1p_2^3 - \tfrac{67^1}{2^4.5^3.11^1}p_1p_2p_4 + \tfrac{101^1}{2^2.3^3.5^3.7^1}p_1p_6 + \tfrac{1^1}{2^2.3^3.11^1}p_2p_5 - \tfrac{1^1}{3^2.5^2.11^1}p_7\Big]\\
A_8 =& \tfrac{1}{2^{16}.3^5.5^2.7^1.13^1.17^1}\Big[\tfrac{31^1\cdot 151^1\cdot 3617^1}{2^{15}.3^2.5^2.11^1}p_1^8 - \tfrac{2241667^1}{2^{11}.3^1.5^2.11^1}p_1^6p_2 + \tfrac{71^1\cdot 268823^1}{2^{10}.3^3.5^2.11^1}p_1^4p_2^2 - \tfrac{3941363^1}{2^8.3^2.5^2.7^1.11^1}p_1^4p_4 + \tfrac{26921^1}{2^4.3^2.5^2.11^1}p_1^3p_5\\
&- \tfrac{29^1\cdot 41^1\cdot 227^1}{2^7.3^3.5^1.11^1}p_1^2p_2^3 + \tfrac{38923^1}{2^5.3^2.7^1.11^1}p_1^2p_2p_4 - \tfrac{197033^1}{2^3.3^3.5^2.7^1.11^1}p_1^2p_6 - \tfrac{23339^1}{2^2.3^3.5^2.11^1}p_1p_2p_5 + \tfrac{1063^1}{2^1.5^2.7^1.11^1}p_1p_7 + \tfrac{11^1\cdot 1249^1}{2^7.3^3.5^1.7^1}p_2^4\\
&- \tfrac{275593^1}{2^4.3^3.5^2.7^1.11^1}p_2^2p_4 + \tfrac{11299^1}{3^3.5^2.7^1.11^1}p_2p_6 + \tfrac{73^1\cdot 199^1}{2^3.3^2.5^2.7^1.11^1}p_4^2 - \tfrac{3617^1}{2^1.3^2.5^2.7^1.11^1}p_8\Big]\\
A_9 =& \tfrac{1}{2^{16}.3^7.5^3.7^3.11^1.19^1}\Big[-\tfrac{43867^1\cdot 131071^1}{2^{18}.3^2.13^1.17^1}p_1^9 + \tfrac{47^1\cdot 907^1\cdot 7949^1}{2^{14}.3^1.5^1.17^1}p_1^7p_2 - \tfrac{83^1\cdot 3688543^1}{2^{13}.3^1.5^1.13^1}p_1^5p_2^2 + \tfrac{5303^1\cdot 44519^1}{2^{11}.3^2.13^1.17^1}p_1^5p_4 - \tfrac{47756197^1}{2^9.3^2.13^1.17^1}p_1^4p_5\\
&+ \tfrac{1625000107^1}{2^{10}.3^2.5^1.13^1.17^1}p_1^3p_2^3 - \tfrac{613^1\cdot 688087^1}{2^8.3^2.5^1.13^1.17^1}p_1^3p_2p_4 + \tfrac{4569683^1}{2^6.3^2.13^1.17^1}p_1^3p_6 + \tfrac{1511459^1}{2^6.3^1.5^1.17^1}p_1^2p_2p_5 - \tfrac{23^1\cdot 86573^1}{2^3.3^2.5^1.13^1.17^1}p_1^2p_7\\
&- \tfrac{37857689^1}{2^{10}.5^1.13^1.17^1}p_1p_2^4 + \tfrac{11389153^1}{2^7.5^1.13^1.17^1}p_1p_2^2p_4 - \tfrac{3301303^1}{2^3.3^2.5^1.13^1.17^1}p_1p_2p_6 - \tfrac{311^1\cdot 41263^1}{2^6.3^2.5^1.13^1.17^1}p_1p_4^2 + \tfrac{467^1\cdot 4969^1}{2^4.3^2.5^1.13^1.17^1}p_1p_8\\
&- \tfrac{991^1\cdot 1123^1}{2^5.3^2.13^1.17^1}p_2^2p_5 + \tfrac{41^1\cdot 557^1}{3^2.13^1.17^1}p_2p_7 + \tfrac{43^1\cdot 4091^1}{2^3.3^2.13^1.17^1}p_4p_5 - \tfrac{43867^1}{2^2.3^2.13^1.17^1}p_9\Big]
\end{aligned}
$$

One may go further and impose vanishing of multiple Pontrjagin classes at once. This is demonstrated in a few of the next expansions.

Expansion 23. *Â-genus of Fivebrane manifolds ($p_1 = p_2 = 0$).*

Â-genus with Fivebrane structure

$$
\begin{aligned}
A_3 =& \tfrac{1}{2^6.3^3.5^1.7^1}\Big[-1^1 p_3\Big]\\
A_4 =& \tfrac{1}{2^9.3^3.5^2.7^1}\Big[-1^1 p_4\Big]\\
A_5 =& \tfrac{1}{2^{10}.3^5.5^1.7^1.11^1}\Big[-1^1 p_5\Big]\\
A_6 =& \tfrac{1}{2^{12}.3^6.5^3.7^2.11^1.13^1}\Big[19^1\cdot 37^1 p_3^2 - 691^1 p_6\Big]\\
A_7 =& \tfrac{1}{2^{13}.3^6.5^2.7^1.11^1.13^1}\Big[\tfrac{283^1}{2^2.5^1.7^1}p_3p_4 - 1^1 p_7\Big]\\
A_8 =& \tfrac{1}{2^{17}.3^7.5^4.7^2.13^1.17^1}\Big[\tfrac{11^1\cdot 181^1}{3^1}p_3p_5 + \tfrac{73^1\cdot 199^1}{2^2.11^1}p_4^2 - \tfrac{3617^1}{11^1}p_8\Big]\\
A_9 =& \tfrac{1}{2^{13}.3^9.5^3.7^3.11^1.13^1.17^1.19^1}\Big[-\tfrac{61^1\cdot 3659^1}{2^5.5^1}p_3^3 + \tfrac{13829^1}{5^1}p_3p_6 + \tfrac{43^1\cdot 4091^1}{2^6}p_4p_5 - \tfrac{43867^1}{2^5}p_9\Big]
\end{aligned}
$$

Expansion 24. *Â-genus of Ninebrane manifolds ($p_1 = p_2 = p_3 = 0$).*

Â-genus with Ninebrane structure

$$
\begin{aligned}
A_4 &= \frac{1}{2^9 \cdot 3^3 \cdot 5^2 \cdot 7^1}\Big[-1^1 p_4\Big] \\
A_5 &= \frac{1}{2^{10} \cdot 3^5 \cdot 5^1 \cdot 7^1 \cdot 11^1}\Big[-1^1 p_5\Big] \\
A_6 &= \frac{1}{2^{12} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11^1 \cdot 13^1}\Big[-691^1 p_6\Big] \\
A_7 &= \frac{1}{2^{13} \cdot 3^6 \cdot 5^2 \cdot 7^1 \cdot 11^1 \cdot 13^1}\Big[-1^1 p_7\Big] \\
A_8 &= \frac{1}{2^{17} \cdot 3^7 \cdot 5^4 \cdot 7^2 \cdot 11^1 \cdot 13^1 \cdot 17^1}\Big[\frac{73^1 \cdot 199^1}{2^2} p_4^2 - 3617^1 p_8\Big] \\
A_9 &= \frac{1}{2^{18} \cdot 3^9 \cdot 5^3 \cdot 7^3 \cdot 11^1 \cdot 13^1 \cdot 17^1 \cdot 19^1}\Big[\frac{43^1 \cdot 4091^1}{2^1} p_4 p_5 - 43867^1 p_9\Big]
\end{aligned}
$$

As in the case of Chern classes, situations might arise where one only has cohomology classes of degree $8k$, as all those of degree $8k+4$ vanish; hence, the next expansion. Observe that if further odd Pontrjagin classes would be set to zero, then all odd Â-genus would equal zero too.

Expansion 25. *Â-genus simplification - vanishing of all odd Pontrjagin classes ($p_{2i+1} = 0$).*

This necessarily makes the odd degree A-genus vanish.

Â-genus with only even Pontrjagin classes

$$
\begin{aligned}
A_0 &= 1 \\
A_2 &= \frac{1}{2^5 \cdot 3^2 \cdot 5^1}\Big[-1^1 p_2\Big] \\
A_4 &= \frac{1}{2^9 \cdot 3^3 \cdot 5^2 \cdot 7^1}\Big[\frac{13^1}{2^2 \cdot 3^1} p_2^2 - 1^1 p_4\Big] \\
A_6 &= \frac{1}{2^{12} \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 11^1 \cdot 13^1}\Big[-\frac{19^1 \cdot 211^1}{2^4} p_2^3 + \frac{73^1 \cdot 79^1}{2^2 \cdot 3^1} p_2 p_4 - \frac{691^1}{3^1} p_6\Big] \\
A_8 &= \frac{1}{2^{16} \cdot 3^7 \cdot 5^3 \cdot 7^2 \cdot 13^1 \cdot 17^1}\Big[\frac{11^1 \cdot 1249^1}{2^7 \cdot 3^1} p_2^4 - \frac{275593^1}{2^4 \cdot 3^1 \cdot 5^1 \cdot 11^1} p_2^2 p_4 + \frac{11299^1}{3^1 \cdot 5^1 \cdot 11^1} p_2 p_6 + \frac{73^1 \cdot 199^1}{2^3 \cdot 5^1 \cdot 11^1} p_4^2 - \frac{3617^1}{2^1 \cdot 5^1 \cdot 11^1} p_8\Big]
\end{aligned}
$$

Note that if an ith expansion of a genus is zero, then it is often omitted in this paper.

4.2.1. **Â-genus and complexification.** Here we are using Equation 3.2, which relates Pontrjagin and Chern class, to convert the L-genus or Â-genus from Pontrjagin to Chern classes.

Expansion 26. *Â-genus expansion in terms of Chern classes of complexification.*

Â-genus expansion in terms of Chern classes of complexification

$$
\begin{aligned}
A_0 =& 1\\
A_1 =& \tfrac{1}{2^{3}.3^{1}}\left[c_2\right]\\
A_2 =& \tfrac{1}{2^{5}.3^{2}.5^{1}}\left[\tfrac{7^{1}}{2^{2}}c_2^2 - 1^{1}c_4\right]\\
A_3 =& \tfrac{1}{2^{6}.3^{3}.5^{1}.7^{1}}\left[\tfrac{31^{1}}{2^{4}}c_2^3 - \tfrac{11^{1}}{2^{2}}c_2c_4 + 1^{1}c_6\right]\\
A_4 =& \tfrac{1}{2^{6}.3^{3}.5^{2}.7^{1}}\left[\tfrac{127^{1}}{2^{9}}c_2^4 - \tfrac{113^{1}}{2^{6}.3^{1}}c_2^2c_4 + \tfrac{1^{1}}{3^{1}}c_2c_6 + \tfrac{13^{1}}{2^{5}.3^{1}}c_4^2 - \tfrac{1^{1}}{2^{3}}c_8\right]\\
A_5 =& \tfrac{1}{2^{10}.3^{4}.5^{1}.11^{1}}\left[\tfrac{73^{1}}{2^{8}.3^{1}}c_2^5 - \tfrac{29^{1}\cdot 37^{1}}{2^{5}.3^{1}.5^{1}.7^{1}}c_2^3c_4 + \tfrac{61^{1}}{2^{3}.5^{1}.7^{1}}c_2^2c_6 + \tfrac{311^{1}}{2^{4}.3^{1}.5^{1}.7^{1}}c_2c_4^2 - \tfrac{53^{1}}{2^{2}.3^{1}.5^{1}.7^{1}}c_2c_8 - \tfrac{1^{1}}{2^{1}.5^{1}}c_4c_6 + \tfrac{1^{1}}{3^{1}.7^{1}}c_{10}\right]\\
A_6 =& \tfrac{1}{2^{12}.3^{5}.5^{2}.7^{2}.11^{1}.13^{1}}\Big[\tfrac{23^{1}\cdot 89^{1}\cdot 691^{1}}{2^{10}.3^{1}.5^{1}}c_2^6 - \tfrac{1540453^{1}}{2^{8}.3^{1}.5^{1}}c_2^4c_4 + \tfrac{29^{1}\cdot 1249^{1}}{2^{3}.3^{1}.5^{1}}c_2^3c_6 + \tfrac{19^{1}\cdot 4013^{1}}{2^{6}.3^{1}}c_2^2c_4^2 - \tfrac{16759^{1}}{2^{4}.5^{1}}c_2^2c_8 - \tfrac{3491^{1}}{2^{1}.5^{1}}c_2c_4c_6\\
&+ \tfrac{23^{1}\cdot 53^{1}}{2^{1}.5^{1}}c_2c_{10} - \tfrac{19^{1}\cdot 211^{1}}{2^{4}.5^{1}}c_4^3 + \tfrac{73^{1}\cdot 79^{1}}{2^{2}.3^{1}.5^{1}}c_4c_8 + \tfrac{19^{1}\cdot 37^{1}}{3^{1}.5^{1}}c_6^2 - \tfrac{691^{1}}{3^{1}.5^{1}}c_{12}\Big]\\
A_7 =& \tfrac{1}{2^{11}.3^{4}.7^{1}.13^{1}}\Big[\tfrac{8191^{1}}{2^{14}.3^{2}.5^{2}.11^{1}}c_2^7 - \tfrac{37^{1}\cdot 31121^{1}}{2^{12}.3^{3}.5^{3}.7^{1}.11^{1}}c_2^5c_4 + \tfrac{67^{1}\cdot 127^{1}}{2^{10}.5^{3}.7^{1}.11^{1}}c_2^4c_6 + \tfrac{9161^{1}}{2^{10}.3^{1}.5^{2}.7^{1}.11^{1}}c_2^3c_4^2 - \tfrac{23^{1}\cdot 127^{1}}{2^{8}.3^{2}.5^{2}.7^{1}.11^{1}}c_2^3c_8\\
&- \tfrac{179^{1}\cdot 317^{1}}{2^{7}.3^{3}.5^{3}.7^{1}.11^{1}}c_2^2c_4c_6 + \tfrac{2543^{1}}{2^{6}.3^{2}.5^{3}.7^{1}.11^{1}}c_2^2c_{10} - \tfrac{109^{1}\cdot 307^{1}}{2^{8}.3^{3}.5^{3}.7^{1}.11^{1}}c_2c_4^3 + \tfrac{67^{1}}{2^{6}.5^{3}.11^{1}}c_2c_4c_8 + \tfrac{97^{1}}{3^{3}.5^{3}.7^{1}.11^{1}}c_2c_6^2\\
&- \tfrac{101^{1}}{2^{4}.3^{3}.5^{3}.7^{1}}c_2c_{12} + \tfrac{23^{1}\cdot 233^{1}}{2^{6}.3^{3}.5^{3}.7^{1}.11^{1}}c_4^2c_6 - \tfrac{1^{1}}{2^{4}.3^{3}.11^{1}}c_4c_{10}\\
&- \tfrac{283^{1}}{2^{4}.3^{2}.5^{3}.7^{1}.11^{1}}c_6c_8 + \tfrac{1^{1}}{2^{2}.3^{2}.5^{2}.11^{1}}c_{14}\Big]\\
A_8 =& \tfrac{1}{2^{15}.3^{5}.5^{2}.7^{1}.17^{1}}\Big[\tfrac{31^{1}\cdot 151^{1}\cdot 3617^{1}}{2^{16}.3^{2}.5^{2}.11^{1}.13^{1}}c_2^8 - \tfrac{2241667^{1}}{2^{12}.3^{1}.5^{2}.11^{1}.13^{1}}c_2^6c_4 + \tfrac{3661841^{1}}{2^{7}.3^{3}.5^{2}.7^{1}.11^{1}.13^{1}}c_2^5c_6 + \tfrac{71^{1}\cdot 268823^{1}}{2^{11}.3^{3}.5^{2}.11^{1}.13^{1}}c_2^4c_4^2\\
&- \tfrac{3941363^{1}}{2^{9}.3^{2}.5^{2}.7^{1}.11^{1}.13^{1}}c_2^4c_8 - \tfrac{317^{1}\cdot 4129^{1}}{2^{4}.3^{3}.5^{2}.7^{1}.11^{1}.13^{1}}c_2^3c_4c_6 + \tfrac{26921^{1}}{2^{5}.3^{2}.5^{2}.11^{1}.13^{1}}c_2^3c_{10} - \tfrac{29^{1}\cdot 41^{1}\cdot 227^{1}}{2^{8}.3^{3}.5^{1}.11^{1}.13^{1}}c_2^2c_4^3\\
&+ \tfrac{38923^{1}}{2^{6}.3^{2}.7^{1}.11^{1}.13^{1}}c_2^2c_4c_8 + \tfrac{907^{1}}{2^{2}.3^{1}.5^{2}.7^{1}.13^{1}}c_2^2c_6^2 - \tfrac{197033^{1}}{2^{4}.3^{3}.5^{2}.7^{1}.11^{1}.13^{1}}c_2^2c_{12} + \tfrac{9091^{1}}{2^{3}.3^{3}.5^{1}.11^{1}.13^{1}}c_2c_4^2c_6\\
&- \tfrac{23339^{1}}{2^{3}.3^{3}.5^{2}.11^{1}.13^{1}}c_2c_4c_{10} - \tfrac{39887^{1}}{2^{1}.3^{3}.5^{2}.7^{1}.11^{1}.13^{1}}c_2c_6c_8 + \tfrac{1063^{1}}{2^{2}.5^{2}.7^{1}.11^{1}.13^{1}}c_2c_{14} + \tfrac{11^{1}\cdot 1249^{1}}{2^{8}.3^{3}.5^{1}.7^{1}.13^{1}}c_4^4 - \tfrac{275593^{1}}{2^{5}.3^{3}.5^{2}.7^{1}.11^{1}.13^{1}}c_4^2c_8\\
&- \tfrac{31^{1}}{3^{2}.5^{2}.11^{1}}c_4c_6^2 + \tfrac{11299^{1}}{2^{1}.3^{3}.5^{2}.7^{1}.11^{1}.13^{1}}c_4c_{12} + \tfrac{11^{1}\cdot 181^{1}}{2^{2}.3^{3}.5^{2}.7^{1}.13^{1}}c_6c_{10} + \tfrac{73^{1}\cdot 199^{1}}{2^{4}.3^{2}.5^{2}.7^{1}.11^{1}.13^{1}}c_8^2 - \tfrac{3617^{1}}{2^{2}.3^{2}.5^{2}.7^{1}.11^{1}.13^{1}}c_{16}\Big]\\
A_9 =& \tfrac{1}{2^{13}.3^{7}.5^{2}.7^{2}.19^{1}}\Big[\tfrac{43867^{1}\cdot 131071^{1}}{2^{21}.3^{2}.5^{1}.7^{1}.11^{1}.13^{1}.17^{1}}c_2^9 - \tfrac{47^{1}\cdot 907^{1}\cdot 7949^{1}}{2^{17}.3^{1}.5^{2}.7^{1}.11^{1}.17^{1}}c_2^7c_4 + \tfrac{9397^{1}\cdot 33317^{1}}{2^{15}.5^{2}.7^{1}.11^{1}.13^{1}.17^{1}}c_2^6c_6 + \tfrac{83^{1}\cdot 3688543^{1}}{2^{16}.3^{1}.5^{2}.7^{1}.11^{1}.13^{1}}c_2^5c_4^2\\
&- \tfrac{5303^{1}\cdot 44519^{1}}{2^{14}.3^{2}.5^{1}.7^{1}.11^{1}.13^{1}.17^{1}}c_2^5c_8 - \tfrac{601^{1}\cdot 4429813^{1}}{2^{13}.3^{2}.5^{2}.7^{1}.11^{1}.13^{1}.17^{1}}c_2^4c_4c_6 + \tfrac{47756197^{1}}{2^{12}.3^{2}.5^{1}.7^{1}.11^{1}.13^{1}.17^{1}}c_2^4c_{10}\\
&- \tfrac{1625000107^{1}}{2^{13}.3^{2}.5^{2}.7^{1}.11^{1}.13^{1}.17^{1}}c_2^3c_4^3 + \tfrac{613^{1}\cdot 688087^{1}}{2^{11}.3^{2}.5^{2}.7^{1}.11^{1}.13^{1}.17^{1}}c_2^3c_4c_8 + \tfrac{23^{1}\cdot 653^{1}}{2^{9}.5^{1}.13^{1}.17^{1}}c_2^3c_6^2 - \tfrac{4569683^{1}}{2^{9}.3^{2}.5^{1}.7^{1}.11^{1}.13^{1}.17^{1}}c_2^3c_{12}\\
&+ \tfrac{59^{1}\cdot 163^{1}\cdot 8377^{1}}{2^{11}.3^{2}.5^{2}.11^{1}.13^{1}.17^{1}}c_2^2c_4^2c_6 - \tfrac{1511459^{1}}{2^{9}.3^{1}.5^{2}.7^{1}.11^{1}.17^{1}}c_2^2c_4c_{10} - \tfrac{23^{1}\cdot 263^{1}\cdot 3187^{1}}{2^{9}.3^{1}.5^{2}.7^{1}.11^{1}.13^{1}.17^{1}}c_2^2c_6c_8 + \tfrac{23^{1}\cdot 86573^{1}}{2^{6}.3^{2}.5^{2}.7^{1}.11^{1}.13^{1}.17^{1}}c_2^2c_{14}\\
&+ \tfrac{37857689^{1}}{2^{13}.5^{2}.7^{1}.11^{1}.13^{1}.17^{1}}c_2c_4^4 - \tfrac{11389153^{1}}{2^{10}.5^{2}.7^{1}.11^{1}.13^{1}.17^{1}}c_2c_4^2c_8 - \tfrac{199^{1}\cdot 4231^{1}}{2^{7}.3^{1}.5^{1}.7^{1}.11^{1}.13^{1}.17^{1}}c_2c_4c_6^2 + \tfrac{3301303^{1}}{2^{6}.3^{2}.5^{2}.7^{1}.11^{1}.13^{1}.17^{1}}c_2c_4c_{12}\\
&+ \tfrac{127^{1}\cdot 2029^{1}}{2^{7}.3^{2}.7^{1}.11^{1}.13^{1}.17^{1}}c_2c_6c_{10} + \tfrac{311^{1}\cdot 41263^{1}}{2^{9}.3^{2}.5^{2}.7^{1}.11^{1}.13^{1}.17^{1}}c_2c_8^2 - \tfrac{467^{1}\cdot 4969^{1}}{2^{7}.3^{2}.5^{2}.7^{1}.11^{1}.13^{1}.17^{1}}c_2c_{16} - \tfrac{15013651^{1}}{2^{9}.3^{2}.5^{2}.7^{1}.11^{1}.13^{1}.17^{1}}c_4^3c_6\\
&+ \tfrac{991^{1}\cdot 1123^{1}}{2^{8}.3^{2}.5^{1}.7^{1}.11^{1}.13^{1}.17^{1}}c_4^2c_{10} + \tfrac{43^{1}\cdot 42239^{1}}{2^{7}.3^{1}.5^{2}.7^{1}.11^{1}.13^{1}.17^{1}}c_4c_6c_8 - \tfrac{41^{1}\cdot 557^{1}}{2^{3}.3^{2}.5^{1}.7^{1}.11^{1}.13^{1}.17^{1}}c_4c_{14} + \tfrac{61^{1}\cdot 3659^{1}}{2^{5}.3^{2}.5^{2}.7^{1}.11^{1}.13^{1}.17^{1}}c_6^3\\
&- \tfrac{13829^{1}}{3^{2}.5^{2}.7^{1}.11^{1}.13^{1}.17^{1}}c_6c_{12} - \tfrac{43^{1}\cdot 4091^{1}}{2^{6}.3^{2}.5^{1}.7^{1}.11^{1}.13^{1}.17^{1}}c_8c_{10} + \tfrac{43867^{1}}{2^{5}.3^{2}.5^{1}.7^{1}.11^{1}.13^{1}.17^{1}}c_{18}\Big]
\end{aligned}
$$

Note that the denominators in the above expressions are exactly those showing up in the corresponding term in the expansion via Pontrjagin classes, Expansion 19, since the relation between the p's and the c's only involves a minus sign and no numerical factors.

Expansion 27. *Â-genus in terms of Chern classes with complex String structure ($c_2 = 0$).*

In this case, the first Chern class is necessarily trivial, so indeed $c_2 = 0$ defines a complex String structure.

Â-genus in terms of Chern classes with complex String structure

$$
\begin{aligned}
A_0 =&1\\
A_1 =&0\\
A_2 =&\tfrac{1}{2^5.3^2.5^1}\Big[-1^1 c_4\Big]\\
A_3 =&\tfrac{1}{2^6.3^3.5^1.7^1}\Big[c_6\Big]\\
A_4 =&\tfrac{1}{2^9.3^3.5^2.7^1}\Big[\tfrac{13^1}{2^2.3^1}c_4^2-1^1c_8\Big]\\
A_5 =&\tfrac{1}{2^{10}.3^4.5^1.11^1}\Big[-\tfrac{1^1}{2^1.5^1}c_4c_6+\tfrac{1^1}{3^1.7^1}c_{10}\Big]\\
A_6 =&\tfrac{1}{2^{12}.3^5.5^3.7^2.11^1.13^1}\Big[-\tfrac{19^1\cdot 211^1}{2^4}c_4^3+\tfrac{73^1\cdot 79^1}{2^2.3^1}c_4c_8+\tfrac{19^1\cdot 37^1}{3^1}c_6^2-\tfrac{691^1}{3^1}c_{12}\Big]\\
A_7 =&\tfrac{1}{2^{13}.3^6.7^1.11^1.13^1}\Big[\tfrac{23^1\cdot 233^1}{2^4.3^1.5^3.7^1}c_4^2c_6-\tfrac{1^1}{2^2.3^1}c_4c_{10}-\tfrac{283^1}{2^2.5^3.7^1}c_6c_8+\tfrac{1^1}{5^2}c_{14}\Big]\\
A_8 =&\tfrac{1}{2^{15}.3^7.5^3.7^1.17^1}\Big[\tfrac{11^1\cdot 1249^1}{2^8.3^1.7^1.13^1}c_4^4-\tfrac{275593^1}{2^5.3^1.5^1.7^1.11^1.13^1}c_4^2c_8-\tfrac{31^1}{5^1.11^1}c_4c_6^2+\tfrac{11299^1}{2^1.3^1.5^1.7^1.11^1.13^1}c_4c_{12}\\
&\quad+\tfrac{11^1\cdot 181^1}{2^2.3^1.5^1.7^1.13^1}c_6c_{10}+\tfrac{73^1\cdot 199^1}{2^4.5^1.7^1.11^1.13^1}c_8^2-\tfrac{3617^1}{2^2.5^1.7^1.11^1.13^1}c_{16}\Big]\\
A_9 =&\tfrac{1}{2^{13}.3^8.5^3.7^3.11^1.13^1.17^1.19^1}\Big[-\tfrac{15013651^1}{2^9.3^1.5^1}c_4^3c_6+\tfrac{991^1\cdot 1123^1}{2^8.3^1}c_4^2c_{10}+\tfrac{43^1\cdot 42239^1}{2^7.5^1}c_4c_6c_8-\tfrac{41^1\cdot 557^1}{2^3.3^1}c_4c_{14}+\tfrac{61^1\cdot 3659^1}{2^5.3^1.5^1}c_6^3\\
&\quad-\tfrac{13829^1}{3^1.5^1}c_6c_{12}-\tfrac{43^1\cdot 4091^1}{2^6.3^1}c_8c_{10}+\tfrac{43867^1}{2^5.3^1}c_{18}\Big]
\end{aligned}
$$

Note that, similarly to the previous case, the denominators in the above expressions are exactly those showing up in the corresponding term in the expansion with a p_1 or String structure, i.e., Expansion 20.

4.3. **The L-genus.** The strategy and implementation for the L-genus as well as for Todd genus below follow the same pattern as that of the Â-genus, since most of the Mathematica functions are shared. Here, we are presenting the results of the L-genus using the characteristic power series $\frac{\sqrt{x_j}}{\tanh\sqrt{x_j}}$, as defined in Section 4.1.

Expansion 28. *L-genus in terms of Pontrjagin classes.*

L-genus in terms of Pontrjagin classes

$$
\begin{aligned}
L_0 =& 1\\
L_1 =& \tfrac{1}{3^1}\big[p_1\big]\\
L_2 =& \tfrac{1}{3^2\cdot 5^1}\big[-1^1p_1^2+7^1p_2\big]\\
L_3 =& \tfrac{1}{3^3\cdot 5^1\cdot 7^1}\big[+2^1p_1^3-13^1p_1p_2+2^1\cdot 31^1p_3\big]\\
L_4 =& \tfrac{1}{3^3\cdot 5^2\cdot 7^1}\Big[-1^1p_1^4+\tfrac{2^1\cdot 11^1}{3^1}p_1^2p_2-\tfrac{71^1}{3^1}p_1p_3-\tfrac{19^1}{3^1}p_2^2+127^1p_4\Big]\\
L_5 =& \tfrac{1}{3^4\cdot 5^1\cdot 11^1}\Big[+\tfrac{2^1}{3^1\cdot 7^1}p_1^5-\tfrac{83^1}{3^1\cdot 5^1\cdot 7^1}p_1^3p_2+\tfrac{79^1}{5^1\cdot 7^1}p_1^2p_3+\tfrac{127^1}{3^1\cdot 5^1\cdot 7^1}p_1p_2^2-\tfrac{919^1}{3^1\cdot 5^1\cdot 7^1}p_1p_4-\tfrac{2^4}{5^1}p_2p_3+\tfrac{2^1\cdot 73^1}{3^1}p_5\Big]\\
L_6 =& \tfrac{1}{3^5\cdot 5^2\cdot 7^2\cdot 11^1\cdot 13^1}\Big[-\tfrac{2^1\cdot 691^1}{3^1\cdot 5^1}p_1^6+\tfrac{2^1\cdot 6421^1}{3^1\cdot 5^1}p_1^4p_2-\tfrac{33863^1}{3^1\cdot 5^1}p_1^3p_3-\tfrac{5527^1}{3^1}p_1^2p_2^2+\tfrac{40841^1}{5^1}p_1^2p_4+\tfrac{83^1\cdot 349^1}{5^1}p_1p_2p_3\\
&-\tfrac{2^5\cdot 29^1\cdot 181^1}{5^1}p_1p_5+\tfrac{2^1\cdot 1453^1}{5^1}p_2^3-\tfrac{159287^1}{3^1\cdot 5^1}p_2p_4-\tfrac{167^1\cdot 241^1}{3^1\cdot 5^1}p_3^2+\tfrac{2^1\cdot 23^1\cdot 89^1\cdot 691^1}{3^1\cdot 5^1}p_6\Big]\\
L_7 =& \tfrac{1}{3^2\cdot 5^1\cdot 7^1\cdot 13^1}\Big[+\tfrac{2^2}{3^4\cdot 5^1\cdot 11^1}p_1^7-\tfrac{2^1\cdot 2161^1}{3^5\cdot 5^2\cdot 7^1\cdot 11^1}p_1^5p_2+\tfrac{2^2}{5^2\cdot 7^1}p_1^4p_3+\tfrac{2^3}{3^2\cdot 5^1\cdot 7^1}p_1^3p_2^2-\tfrac{2^1\cdot 113^1}{3^4\cdot 5^1\cdot 7^1}p_1^3p_4-\tfrac{39341^1}{3^5\cdot 5^2\cdot 7^1\cdot 11^1}p_1^2p_2p_3\\
&+\tfrac{2^4\cdot 277^1}{3^4\cdot 5^2\cdot 7^1}p_1^2p_5-\tfrac{2^1\cdot 3989^1}{3^5\cdot 5^2\cdot 7^1\cdot 11^1}p_1p_2^3+\tfrac{1399^1}{3^3\cdot 5^2\cdot 11^1}p_1p_2p_4+\tfrac{22027^1}{3^5\cdot 5^2\cdot 7^1\cdot 11^1}p_1p_3^2-\tfrac{2^1\cdot 305633^1}{3^5\cdot 5^2\cdot 7^1\cdot 11^1}p_1p_6+\tfrac{2^3\cdot 2087^1}{3^5\cdot 5^2\cdot 7^1\cdot 11^1}p_2^2p_3\\
&-\tfrac{2^1\cdot 23^2}{3^5\cdot 11^1}p_2p_5-\tfrac{2^1\cdot 97^1\cdot 107^1}{3^4\cdot 5^2\cdot 7^1\cdot 11^1}p_3p_4+\tfrac{2^2\cdot 8191^1}{3^4\cdot 5^1\cdot 11^1}p_7\Big]\\
L_8 =& \tfrac{1}{3^3\cdot 5^3\cdot 7^1\cdot 17^1}\Big[-\tfrac{3617^1}{3^4\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1}p_1^8+\tfrac{2^2\cdot 41^1\cdot 83^1}{3^3\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1}p_1^6p_2-\tfrac{2^1\cdot 143483^1}{3^5\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1}p_1^5p_3-\tfrac{2^1\cdot 43^1\cdot 431^1}{3^5\cdot 5^1\cdot 7^1\cdot 13^1}p_1^4p_2^2+\tfrac{2^1\cdot 162011^1}{3^4\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1}p_1^4p_4\\
&+\tfrac{2^6\cdot 29^1\cdot 739^1}{3^5\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1}p_1^3p_2p_3-\tfrac{97^1\cdot 12889^1}{3^4\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1}p_1^3p_5+\tfrac{2^7\cdot 661^1}{3^5\cdot 7^1\cdot 11^1\cdot 13^1}p_1^2p_2^3-\tfrac{97^1\cdot 2273^1}{3^4\cdot 7^1\cdot 11^1\cdot 13^1}p_1^2p_2p_4-\tfrac{107^1\cdot 857^1}{3^3\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1}p_1^2p_3^2\\
&+\tfrac{1091^1\cdot 13789^1}{3^5\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1}p_1^2p_6-\tfrac{2^1\cdot 9619^1}{3^5\cdot 7^1\cdot 11^1}p_1p_2^2p_3+\tfrac{2^1\cdot 617147^1}{3^5\cdot 5^1\cdot 11^1\cdot 13^1}p_1p_2p_5+\tfrac{29^1\cdot 98057^1}{3^5\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1}p_1p_3p_4-\tfrac{37117^1}{5^1\cdot 11^1\cdot 13^1}p_1p_7\\
&-\tfrac{13687^1}{3^5\cdot 7^1\cdot 11^1\cdot 13^1}p_2^4+\tfrac{2^1\cdot 101^1\cdot 6491^1}{3^5\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1}p_2^2p_4+\tfrac{2^3\cdot 311^1}{3^4\cdot 5^1\cdot 11^1}p_2p_3^2-\tfrac{191^1\cdot 14143^1}{3^5\cdot 5^1\cdot 11^1\cdot 13^1}p_2p_6-\tfrac{29^1\cdot 31^1\cdot 79^1}{3^5\cdot 5^1\cdot 13^1}p_3p_5-\tfrac{167^1\cdot 2663^1}{3^4\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1}p_4^2\\
&+\tfrac{31^1\cdot 151^1\cdot 3617^1}{3^4\cdot 5^1\cdot 11^1\cdot 13^1}p_8\Big]\\
L_9 =& \tfrac{1}{3^7\cdot 5^3\cdot 7^1\cdot 19^1}\Big[+\tfrac{2^1\cdot 43867^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1}p_1^9-\tfrac{41^1\cdot 53^1\cdot 827^1}{3^1\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1}p_1^7p_2+\tfrac{1337617^1}{5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1}p_1^6p_3+\tfrac{541^1\cdot 761^1}{3^1\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1}p_1^5p_2^2\\
&-\tfrac{29^1\cdot 79^1\cdot 3467^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1}p_1^5p_4-\tfrac{3931^1\cdot 17981^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1}p_1^4p_2p_3+\tfrac{61^1\cdot 491^1\cdot 1013^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1}p_1^4p_5-\tfrac{2^1\cdot 31^1\cdot 475957^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1}p_1^3p_2^3\\
&+\tfrac{2^2\cdot 1697^1\cdot 25951^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1}p_1^3p_2p_4+\tfrac{2^1\cdot 70003^1}{7^1\cdot 11^1\cdot 13^1\cdot 17^1}p_1^3p_3^2-\tfrac{29^1\cdot 53^1\cdot 181^1\cdot 433^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1}p_1^3p_6+\tfrac{2^1\cdot 93133^1}{3^2\cdot 5^1\cdot 13^1\cdot 17^1}p_1^2p_2^2p_3\\
&-\tfrac{2^1\cdot 83892287^1}{3^1\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1}p_1^2p_2p_5-\tfrac{743^1\cdot 73597^1}{3^1\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1}p_1^2p_3p_4+\tfrac{887^1\cdot 210719^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 17^1}p_1^2p_7+\tfrac{13^1\cdot 7297^1}{5^1\cdot 7^2\cdot 11^1\cdot 17^1}p_1p_2^4\\
&-\tfrac{2^1\cdot 8359009^1}{5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1}p_1p_2^2p_4-\tfrac{2^2\cdot 83^1\cdot 15667^1}{3^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1}p_1p_2p_3^2+\tfrac{47^1\cdot 1291^1\cdot 22709^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1}p_1p_2p_6+\tfrac{641^1\cdot 121169^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1}p_1p_3p_5\\
&+\tfrac{67^1\cdot 101^1\cdot 13883^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1}p_1p_4^2-\tfrac{43^1\cdot 237398563^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1}p_1p_8-\tfrac{2^4\cdot 31^1\cdot 139^1\cdot 293^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1}p_2^3p_3+\tfrac{2^2\cdot 31^1\cdot 314173^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1}p_2^2p_5\\
&+\tfrac{2^2\cdot 89^1\cdot 11071^1}{3^1\cdot 5^1\cdot 7^2\cdot 13^1\cdot 17^1}p_2p_3p_4-\tfrac{2^1\cdot 23^1\cdot 1933^1\cdot 6857^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1}p_2p_7+\tfrac{2^3\cdot 89^1\cdot 17929^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1}p_3^3-\tfrac{2^1\cdot 367^1\cdot 1150249^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1}p_3p_6\\
&-\tfrac{2^2\cdot 13569497^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1}p_4p_5+\tfrac{2^1\cdot 43867^1\cdot 131071^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1}p_9\Big]
\end{aligned}
$$

L-genus with simplifications. As in the case of Â-genus one can get simplifications in the presence of the higher structures. These expressions were obtained by using the two genus simplification formulas stated in the Â-genus section above.

Expansion 29. *L-genus with String manifolds or p_1-structure ($p_1 = 0$).*

L-genus with String pr p_1 structure

$$
\begin{aligned}
L_0 =& 1 \\
L_1 =& 0 \\
L_2 =& \tfrac{1}{3^2 \cdot 5^1}\Big[+ 7^1 p_2\Big] \\
L_3 =& \tfrac{1}{3^3 \cdot 5^1 \cdot 7^1}\Big[+ 2^1 \cdot 31^1 p_3\Big] \\
L_4 =& \tfrac{1}{3^3 \cdot 5^2 \cdot 7^1}\Big[- \tfrac{19^1}{3^1} p_2^2 + 127^1 p_4\Big] \\
L_5 =& \tfrac{1}{3^4 \cdot 5^1 \cdot 11^1}\Big[- \tfrac{2^4}{5^1} p_2 p_3 + \tfrac{2^1 \cdot 73^1}{3^1} p_5\Big] \\
L_6 =& \tfrac{1}{3^5 \cdot 5^3 \cdot 7^2 \cdot 11^1 \cdot 13^1}\Big[+ 2^1 \cdot 1453^1 p_2^3 - \tfrac{159287^1}{3^1} p_2 p_4 - \tfrac{167^1 \cdot 241^1}{3^1} p_3^2 + \tfrac{2^1 \cdot 23^1 \cdot 89^1 \cdot 691^1}{3^1} p_6\Big] \\
L_7 =& \tfrac{1}{3^6 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1}\Big[+ \tfrac{2^3 \cdot 2087^1}{3^1 \cdot 5^2 \cdot 7^1} p_2^2 p_3 - \tfrac{2^1 \cdot 23^2}{3^1} p_2 p_5 - \tfrac{2^1 \cdot 97^1 \cdot 107^1}{5^2 \cdot 7^1} p_3 p_4 + \tfrac{2^2 \cdot 8191^1}{5^1} p_7\Big] \\
L_8 =& \tfrac{1}{3^7 \cdot 5^3 \cdot 7^1 \cdot 17^1}\Big[- \tfrac{13687^1}{3^1 \cdot 7^1 \cdot 11^1 \cdot 13^1} p_2^4 + \tfrac{2^1 \cdot 101^1 \cdot 6491^1}{3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1} p_2^2 p_4 + \tfrac{2^3 \cdot 311^1}{5^1 \cdot 11^1} p_2 p_3^2 - \tfrac{191^1 \cdot 14143^1}{3^1 \cdot 5^1 \cdot 11^1 \cdot 13^1} p_2 p_6 - \tfrac{29^1 \cdot 31^1 \cdot 79^1}{3^1 \cdot 5^1 \cdot 13^1} p_3 p_5 \\
& - \tfrac{167^1 \cdot 2663^1}{5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1} p_4^2 + \tfrac{31^1 \cdot 151^1 \cdot 3617^1}{5^1 \cdot 11^1 \cdot 13^1} p_8\Big] \\
L_9 =& \tfrac{1}{3^8 \cdot 5^3 \cdot 7^3 \cdot 13^1 \cdot 17^1 \cdot 19^1}\Big[- \tfrac{2^4 \cdot 31^1 \cdot 139^1 \cdot 293^1}{3^1 \cdot 5^1 \cdot 11^1} p_2^3 p_3 + \tfrac{2^2 \cdot 31^1 \cdot 314173^1}{3^1 \cdot 11^1} p_2^2 p_5 + \tfrac{2^2 \cdot 89^1 \cdot 11071^1}{5^1} p_2 p_3 p_4 - \tfrac{2^1 \cdot 23^1 \cdot 1933^1 \cdot 6857^1}{3^1 \cdot 11^1} p_2 p_7 \\
& + \tfrac{2^3 \cdot 89^1 \cdot 17929^1}{3^1 \cdot 5^1 \cdot 11^1} p_3^3 - \tfrac{2^1 \cdot 367^1 \cdot 1150249^1}{3^1 \cdot 5^1 \cdot 11^1} p_3 p_6 - \tfrac{2^2 \cdot 13569497^1}{3^1 \cdot 11^1} p_4 p_5 + \tfrac{2^1 \cdot 43867^1 \cdot 131071^1}{3^1 \cdot 11^1} p_9\Big]
\end{aligned}
$$

Expansion 30. *L-genus with p_2-structure manifolds ($p_2 = 0$).*

L-genus with p_2-structure

$$
\begin{aligned}
L_0 =& 1 \\
L_1 =& \tfrac{1}{3^1}\Big[p_1\Big] \\
L_2 =& \tfrac{1}{3^2 \cdot 5^1}\Big[- 1^1 p_1^2\Big] \\
L_3 =& \tfrac{1}{3^3 \cdot 5^1 \cdot 7^1}\Big[+ 2^1 p_1^3 + 2^1 \cdot 31^1 p_3\Big] \\
L_4 =& \tfrac{1}{3^3 \cdot 5^2 \cdot 7^1}\Big[- 1^1 p_1^4 - \tfrac{71^1}{3^1} p_1 p_3 + 127^1 p_4\Big] \\
L_5 =& \tfrac{1}{3^4 \cdot 5^1 \cdot 11^1}\Big[+ \tfrac{2^1}{3^1 \cdot 7^1} p_1^5 + \tfrac{79^1}{5^1 \cdot 7^1} p_1^2 p_3 - \tfrac{919^1}{3^1 \cdot 5^1 \cdot 7^1} p_1 p_4 + \tfrac{2^1 \cdot 73^1}{3^1} p_5\Big] \\
L_6 =& \tfrac{1}{3^5 \cdot 5^3 \cdot 7^2 \cdot 11^1 \cdot 13^1}\Big[- \tfrac{2^1 \cdot 691^1}{3^1} p_1^6 - \tfrac{33863^1}{3^1} p_1^3 p_3 + 40841^1 p_1^2 p_4 - 2^5 \cdot 29^1 \cdot 181^1 p_1 p_5 - \tfrac{167^1 \cdot 241^1}{3^1} p_3^2 + \tfrac{2^1 \cdot 23^1 \cdot 89^1 \cdot 691^1}{3^1} p_6\Big] \\
L_7 =& \tfrac{1}{3^2 \cdot 5^2 \cdot 7^1 \cdot 13^1}\Big[+ \tfrac{2^2}{3^4 \cdot 11^1} p_1^7 + \tfrac{2^2}{5^1 \cdot 7^1} p_1^4 p_3 - \tfrac{2^1 \cdot 113^1}{3^4 \cdot 7^1} p_1^3 p_4 + \tfrac{2^4 \cdot 277^1}{3^4 \cdot 5^1 \cdot 7^1} p_1^2 p_5 + \tfrac{22027^1}{3^5 \cdot 5^1 \cdot 7^1 \cdot 11^1} p_1 p_3^2 - \tfrac{2^1 \cdot 305633^1}{3^5 \cdot 5^1 \cdot 7^1 \cdot 11^1} p_1 p_6 \\
& - \tfrac{2^1 \cdot 97^1 \cdot 107^1}{3^4 \cdot 5^1 \cdot 7^1 \cdot 11^1} p_3 p_4 + \tfrac{2^2 \cdot 8191^1}{3^4 \cdot 11^1} p_7\Big] \\
L_8 =& \tfrac{1}{3^3 \cdot 5^4 \cdot 7^1 \cdot 13^1 \cdot 17^1}\Big[- \tfrac{3617^1}{3^4 \cdot 7^1 \cdot 11^1} p_1^8 - \tfrac{2^1 \cdot 143483^1}{3^5 \cdot 7^1 \cdot 11^1} p_1^5 p_3 + \tfrac{2^1 \cdot 162011^1}{3^4 \cdot 7^1 \cdot 11^1} p_1^4 p_4 - \tfrac{97^1 \cdot 12889^1}{3^4 \cdot 7^1 \cdot 11^1} p_1^3 p_5 - \tfrac{107^1 \cdot 857^1}{3^3 \cdot 7^1 \cdot 11^1} p_1^2 p_3^2 \\
& + \tfrac{1091^1 \cdot 13789^1}{3^5 \cdot 7^1 \cdot 11^1} p_1^2 p_6 + \tfrac{29^1 \cdot 98057^1}{3^5 \cdot 7^1 \cdot 11^1} p_1 p_3 p_4 - \tfrac{37117^1}{11^1} p_1 p_7 - \tfrac{29^1 \cdot 31^1 \cdot 79^1}{3^5} p_3 p_5 - \tfrac{167^1 \cdot 2663^1}{3^4 \cdot 7^1 \cdot 11^1} p_4^2 + \tfrac{31^1 \cdot 151^1 \cdot 3617^1}{3^4 \cdot 11^1} p_8\Big] \\
L_9 =& \tfrac{1}{3^7 \cdot 5^3 \cdot 7^2 \cdot 11^1 \cdot 17^1 \cdot 19^1}\Big[+ \tfrac{2^1 \cdot 43867^1}{3^2 \cdot 7^1 \cdot 13^1} p_1^9 + \tfrac{1337617^1}{5^1 \cdot 7^1 \cdot 13^1} p_1^6 p_3 - \tfrac{29^1 \cdot 79^1 \cdot 3467^1}{3^2 \cdot 7^1 \cdot 13^1} p_1^5 p_4 + \tfrac{61^1 \cdot 491^1 \cdot 1013^1}{3^2 \cdot 7^1 \cdot 13^1} p_1^4 p_5 + \tfrac{2^1 \cdot 70003^1}{13^1} p_1^3 p_3^2 \\
& - \tfrac{29^1 \cdot 53^1 \cdot 181^1 \cdot 433^1}{3^2 \cdot 7^1 \cdot 13^1} p_1^3 p_6 - \tfrac{743^1 \cdot 73597^1}{3^1 \cdot 5^1 \cdot 7^1 \cdot 13^1} p_1^2 p_3 p_4 + \tfrac{887^1 \cdot 210719^1}{3^2 \cdot 5^1 \cdot 7^1} p_1^2 p_7 + \tfrac{641^1 \cdot 121169^1}{3^2 \cdot 7^1 \cdot 13^1} p_1 p_3 p_5 + \tfrac{67^1 \cdot 101^1 \cdot 13883^1}{3^2 \cdot 5^1 \cdot 7^1 \cdot 13^1} p_1 p_4^2 \\
& - \tfrac{43^1 \cdot 237398563^1}{3^2 \cdot 5^1 \cdot 7^1 \cdot 13^1} p_1 p_8 + \tfrac{2^3 \cdot 89^1 \cdot 17929^1}{3^2 \cdot 5^1 \cdot 7^1 \cdot 13^1} p_3^3 - \tfrac{2^1 \cdot 367^1 \cdot 1150249^1}{3^2 \cdot 5^1 \cdot 7^1 \cdot 13^1} p_3 p_6 - \tfrac{2^2 \cdot 13569497^1}{3^2 \cdot 7^1 \cdot 13^1} p_4 p_5 + \tfrac{2^1 \cdot 43867^1 \cdot 131071^1}{3^2 \cdot 7^1 \cdot 13^1} p_9\Big]
\end{aligned}
$$

Expansion 31. *L-genus with p_3-structure manifolds ($p_3 = 0$).*

L-genus with p_3-structure

$$
\begin{aligned}
L_0 =&1\\
L_1 =&\tfrac{1}{3^1}\big[p_1\big]\\
L_2 =&\tfrac{1}{3^2.5^1}\big[-1^1p_1^2+7^1p_2\big]\\
L_3 =&\tfrac{1}{3^3.5^1.7^1}\big[+2^1p_1^3-13^1p_1p_2\big]\\
L_4 =&\tfrac{1}{3^3.5^2.7^1}\Big[-1^1p_1^4+\tfrac{2^1\cdot 11^1}{3^1}p_1^2p_2-\tfrac{19^1}{3^1}p_2^2+127^1p_4\Big]\\
L_5 =&\tfrac{1}{3^5.5^1.11^1}\Big[+\tfrac{2^1}{7^1}p_1^5-\tfrac{83^1}{5^1.7^1}p_1^3p_2+\tfrac{127^1}{5^1.7^1}p_1p_2^2-\tfrac{919^1}{5^1.7^1}p_1p_4+2^1\cdot 73^1p_5\Big]\\
L_6 =&\tfrac{1}{3^5.5^2.7^2.11^1.13^1}\Big[-\tfrac{2^1\cdot 691^1}{3^1.5^1}p_1^6+\tfrac{2^1\cdot 6421^1}{3^1.5^1}p_1^4p_2-\tfrac{5527^1}{3^1}p_1^2p_2^2+\tfrac{40841^1}{5^1}p_1^2p_4-\tfrac{2^5\cdot 29^1\cdot 181^1}{5^1}p_1p_5+\tfrac{2^1\cdot 1453^1}{5^1}p_2^3\\
&-\tfrac{159287^1}{3^1.5^1}p_2p_4+\tfrac{2^1\cdot 23^1\cdot 89^1\cdot 691^1}{3^1.5^1}p_6\Big]\\
L_7 =&\tfrac{1}{3^4.5^1.7^1.13^1}\Big[+\tfrac{2^2}{3^2.5^1.11^1}p_1^7-\tfrac{2^1\cdot 2161^1}{3^3.5^2.7^1.11^1}p_1^5p_2+\tfrac{2^3}{5^1.7^1}p_1^3p_2^2-\tfrac{2^1\cdot 113^1}{3^2.5^1.7^1}p_1^3p_4+\tfrac{2^4\cdot 277^1}{3^2.5^2.7^1}p_1^2p_5-\tfrac{2^1\cdot 3989^1}{3^3.5^2.7^1.11^1}p_1p_2^3\\
&+\tfrac{1399^1}{3^1.5^2.11^1}p_1p_2p_4-\tfrac{2^1\cdot 305633^1}{3^3.5^2.7^1.11^1}p_1p_6-\tfrac{2^1\cdot 23^2}{3^3.11^1}p_2p_5+\tfrac{2^2\cdot 8191^1}{3^2.5^1.11^1}p_7\Big]\\
L_8 =&\tfrac{1}{3^3.5^3.7^1.13^1.17^1}\Big[-\tfrac{3617^1}{3^4.5^1.7^1.11^1}p_1^8+\tfrac{2^2\cdot 41^1\cdot 83^1}{3^3.5^1.7^1.11^1}p_1^6p_2-\tfrac{2^1\cdot 43^1\cdot 431^1}{3^5.5^1.7^1}p_1^4p_2^2+\tfrac{2^1\cdot 162011^1}{3^4.5^1.7^1.11^1}p_1^4p_4-\tfrac{97^1\cdot 12889^1}{3^4.5^1.7^1.11^1}p_1^3p_5\\
&+\tfrac{2^7\cdot 661^1}{3^5.7^1.11^1}p_1^2p_2^3-\tfrac{97^1\cdot 2273^1}{3^4.7^1.11^1}p_1^2p_2p_4+\tfrac{1091^1\cdot 13789^1}{3^5.5^1.7^1.11^1}p_1^2p_6+\tfrac{2^1\cdot 617147^1}{3^5.5^1.11^1}p_1p_2p_5-\tfrac{37117^1}{5^1.11^1}p_1p_7\\
&-\tfrac{13687^1}{3^5.7^1.11^1}p_2^4+\tfrac{2^1\cdot 101^1\cdot 6491^1}{3^5.5^1.7^1.11^1}p_2^2p_4-\tfrac{191^1\cdot 14143^1}{3^5.5^1.11^1}p_2p_6-\tfrac{167^1\cdot 2663^1}{3^4.5^1.7^1.11^1}p_4^2+\tfrac{31^1\cdot 151^1\cdot 3617^1}{3^4.5^1.11^1}p_8\Big]\\
L_9 =&\tfrac{1}{3^7.5^3.7^3.11^1.19^1}\Big[+\tfrac{2^1\cdot 43867^1}{3^2.13^1.17^1}p_1^9-\tfrac{41^1\cdot 53^1\cdot 827^1}{3^1.5^1.13^1.17^1}p_1^7p_2+\tfrac{541^1\cdot 761^1}{3^1.5^1.13^1}p_1^5p_2^2-\tfrac{29^1\cdot 79^1\cdot 3467^1}{3^2.13^1.17^1}p_1^5p_4+\tfrac{61^1\cdot 491^1\cdot 1013^1}{3^2.13^1.17^1}p_1^4p_5\\
&-\tfrac{2^1\cdot 31^1\cdot 475957^1}{3^2.5^1.13^1.17^1}p_1^3p_2^3+\tfrac{2^2\cdot 1697^1\cdot 25951^1}{3^2.5^1.13^1.17^1}p_1^3p_2p_4-\tfrac{29^1\cdot 53^1\cdot 181^1\cdot 433^1}{3^2.13^1.17^1}p_1^3p_6-\tfrac{2^1\cdot 83892287^1}{3^1.5^1.13^1.17^1}p_1^2p_2p_5+\tfrac{887^1\cdot 210719^1}{3^2.5^1.17^1}p_1^2p_7\\
&+\tfrac{13^1\cdot 7297^1}{5^1.17^1}p_1p_2^4-\tfrac{2^1\cdot 8359009^1}{5^1.13^1.17^1}p_1p_2^2p_4+\tfrac{47^1\cdot 1291^1\cdot 22709^1}{3^2.5^1.13^1.17^1}p_1p_2p_6+\tfrac{67^1\cdot 101^1\cdot 13883^1}{3^2.5^1.13^1.17^1}p_1p_4^2-\tfrac{43^1\cdot 237398563^1}{3^2.5^1.13^1.17^1}p_1p_8\\
&+\tfrac{2^2\cdot 31^1\cdot 314173^1}{3^2.13^1.17^1}p_2^2p_5-\tfrac{2^1\cdot 23^1\cdot 1933^1\cdot 6857^1}{3^2.13^1.17^1}p_2p_7-\tfrac{2^2\cdot 13569497^1}{3^2.13^1.17^1}p_4p_5+\tfrac{2^1\cdot 43867^1\cdot 131071^1}{3^2.13^1.17^1}p_9\Big]
\end{aligned}
$$

Expansion 32. *L-genus for Fivebrane manifolds ($p_1 = p_2 = 0$).*

L-genus with Fivebrane structure

$$
\begin{aligned}
L_3 =&\tfrac{1}{3^3.5^1.7^1}\big[+2^1\cdot 31^1p_3\big]\\
L_4 =&\tfrac{1}{3^3.5^2.7^1}\big[+127^1p_4\big]\\
L_5 =&\tfrac{1}{3^5.5^1.11^1}\big[+2^1\cdot 73^1p_5\big]\\
L_6 =&\tfrac{1}{3^6.5^3.7^2.11^1.13^1}\big[-167^1\cdot 241^1p_3^2+2^1\cdot 23^1\cdot 89^1\cdot 691^1p_6\big]\\
L_7 =&\tfrac{1}{3^6.5^2.7^1.11^1.13^1}\Big[-\tfrac{2^1\cdot 97^1\cdot 107^1}{5^1.7^1}p_3p_4+2^2\cdot 8191^1p_7\Big]\\
L_8 =&\tfrac{1}{3^7.5^4.7^1.13^1.17^1}\Big[-\tfrac{29^1\cdot 31^1\cdot 79^1}{3^1}p_3p_5-\tfrac{167^1\cdot 2663^1}{7^1.11^1}p_4^2+\tfrac{31^1\cdot 151^1\cdot 3617^1}{11^1}p_8\Big]\\
L_9 =&\tfrac{1}{3^9.5^3.7^3.11^1.13^1.17^1.19^1}\Big[+\tfrac{2^3\cdot 89^1\cdot 17929^1}{5^1}p_3^3-\tfrac{2^1\cdot 367^1\cdot 1150249^1}{5^1}p_3p_6-2^2\cdot 13569497^1p_4p_5+2^1\cdot 43867^1\cdot 131071^1p_9\Big]
\end{aligned}
$$

Expansion 33. *L-genus with Ninebrane manifolds ($p_1 = p_2 = p_3 = 0$).*

L-genus with Ninebrane structure

$$
\begin{aligned}
L_3 =&0\\
L_4 =&\tfrac{1}{3^3.5^2.7^1}\big[+127^1p_4\big]\\
L_5 =&\tfrac{1}{3^5.5^1.11^1}\big[+2^1\cdot 73^1p_5\big]\\
L_6 =&\tfrac{1}{3^6.5^3.7^2.11^1.13^1}\big[+2^1\cdot 23^1\cdot 89^1\cdot 691^1p_6\big]\\
L_7 =&\tfrac{1}{3^6.5^2.7^1.11^1.13^1}\big[+2^2\cdot 8191^1p_7\big]\\
L_8 =&\tfrac{1}{3^7.5^4.7^1.11^1.13^1.17^1}\Big[-\tfrac{167^1\cdot 2663^1}{7^1}p_4^2+31^1\cdot 151^1\cdot 3617^1p_8\Big]\\
L_9 =&\tfrac{1}{3^9.5^3.7^3.11^1.13^1.17^1.19^1}\Big[-2^2\cdot 13569497^1p_4p_5+2^1\cdot 43867^1\cdot 131071^1p_9\Big]
\end{aligned}
$$

There is indeed no limit to how far one could go with the vanishing classes, as long as the machine is able to process the computations.

Observe that, as with the Â-genus, if all odd p_i's were set to zero, then all the odd L-genus expansions would vanish too.

Expansion 34. *L-genus simplification - vanishing of all odd Pontrjagin classes ($p_{2i+1} = 0$).*

Here all L_{2i+1} vanish.

L-genus with only even Pontrjagin classes

$$\begin{aligned}
L_0 =& 1 \\
L_2 =& \tfrac{1}{3^2 \cdot 5^1}\big[+ 7^1 p_2 \big] \\
L_4 =& \tfrac{1}{3^3 \cdot 5^2 \cdot 7^1}\big[- \tfrac{19^1}{3^1} p_2^2 + 127^1 p_4 \big] \\
L_6 =& \tfrac{1}{3^5 \cdot 5^3 \cdot 7^2 \cdot 11^1 \cdot 13^1}\Big[+ 2^1 \cdot 1453^1 p_2^3 - \tfrac{159287^1}{3^1} p_2 p_4 + \tfrac{2^1 \cdot 23^1 \cdot 89^1 \cdot 691^1}{3^1} p_6 \Big] \\
L_8 =& \tfrac{1}{3^7 \cdot 5^3 \cdot 7^1 \cdot 11^1 \cdot 13^1 \cdot 17^1}\Big[- \tfrac{13687^1}{3^1 \cdot 7^1} p_2^4 + \tfrac{2^1 \cdot 101^1 \cdot 6491^1}{3^1 \cdot 5^1 \cdot 7^1} p_2^2 p_4 - \tfrac{191^1 \cdot 14143^1}{3^1 \cdot 5^1} p_2 p_6 - \tfrac{167^1 \cdot 2663^1}{5^1 \cdot 7^1} p_4^2 + \tfrac{31^1 \cdot 151^1 \cdot 3617^1}{5^1} p_8 \Big]
\end{aligned}$$

L-genus and complexification. The L-genus is expressed in terms of the Chern classes similarly to way as it was described for the Â-genus.

Expansion 35. *L-genus in terms of the Chern classes of complexification.*

L-genus in terms of the Chern classes of complexification

$$
\begin{aligned}
L_0 =& 1\\
L_1 =& \tfrac{1}{3^1}\Big[-1^1 c_2\Big]\\
L_2 =& \tfrac{1}{3^2\cdot 5^1}\Big[-1^1 c_2^2 + 7^1 c_4\Big]\\
L_3 =& \tfrac{1}{3^3\cdot 5^1\cdot 7^1}\Big[-2^1 c_2^3 + 13^1 c_2 c_4 - 2^1\cdot 31^1 c_6\Big]\\
L_4 =& \tfrac{1}{3^3\cdot 5^2\cdot 7^1}\Big[-1^1 c_2^4 + \tfrac{2^1\cdot 11^1}{3^1} c_2^2 c_4 - \tfrac{71^1}{3^1} c_2 c_6 - \tfrac{19^1}{3^1} c_4^2 + 127^1 c_8\Big]\\
L_5 =& \tfrac{1}{3^4\cdot 5^1\cdot 11^1}\Big[-\tfrac{2^1}{3^1\cdot 7^1} c_2^5 + \tfrac{83^1}{3^1\cdot 5^1\cdot 7^1} c_2^3 c_4 - \tfrac{79^1}{5^1\cdot 7^1} c_2^2 c_6 - \tfrac{127^1}{3^1\cdot 5^1\cdot 7^1} c_2 c_4^2 + \tfrac{919^1}{3^1\cdot 5^1\cdot 7^1} c_2 c_8 + \tfrac{2^4}{5^1} c_4 c_6 - \tfrac{2^1\cdot 73^1}{3^1} c_{10}\Big]\\
L_6 =& \tfrac{1}{3^5\cdot 5^2\cdot 7^2\cdot 11^1\cdot 13^1}\Big[-\tfrac{2^1\cdot 691^1}{3^1\cdot 5^1} c_2^6 + \tfrac{2^1\cdot 6421^1}{3^1\cdot 5^1} c_2^4 c_4 - \tfrac{33863^1}{3^1\cdot 5^1} c_2^3 c_6 - \tfrac{5527^1}{3^1} c_2^2 c_4^2 + \tfrac{40841^1}{5^1} c_2^2 c_8 + \tfrac{83^1\cdot 349^1}{5^1} c_2 c_4 c_6\\
&- \tfrac{2^5\cdot 29^1\cdot 181^1}{5^1} c_2 c_{10} + \tfrac{2^1\cdot 1453^1}{5^1} c_4^3 - \tfrac{159287^1}{3^1\cdot 5^1} c_4 c_8 - \tfrac{167^1\cdot 241^1}{3^1\cdot 5^1} c_6^2 + \tfrac{2^1\cdot 23^1\cdot 89^1\cdot 691^1}{3^1\cdot 5^1} c_{12}\Big]\\
L_7 =& \tfrac{1}{3^2\cdot 5^1\cdot 7^1\cdot 13^1}\Big[-\tfrac{2^2}{3^4\cdot 5^1\cdot 11^1} c_2^7 + \tfrac{2^1\cdot 2161^1}{3^5\cdot 5^2\cdot 7^1\cdot 11^1} c_2^5 c_4 - \tfrac{2^2}{5^2\cdot 7^1} c_2^4 c_6 - \tfrac{2^3}{3^2\cdot 5^1\cdot 7^1} c_2^3 c_4^2 + \tfrac{2^1\cdot 113^1}{3^4\cdot 5^1\cdot 7^1} c_2^3 c_8 + \tfrac{39341^1}{3^5\cdot 5^2\cdot 7^1\cdot 11^1} c_2^2 c_4 c_6\\
&- \tfrac{2^4\cdot 277^1}{3^4\cdot 5^2\cdot 7^1} c_2^2 c_{10} + \tfrac{2^1\cdot 3989^1}{3^5\cdot 5^2\cdot 7^1\cdot 11^1} c_2 c_4^3 - \tfrac{1399^1}{3^3\cdot 5^2\cdot 11^1} c_2 c_4 c_8 - \tfrac{22027^1}{3^5\cdot 5^2\cdot 7^1\cdot 11^1} c_2 c_6^2 + \tfrac{2^1\cdot 305633^1}{3^5\cdot 5^2\cdot 7^1\cdot 11^1} c_2 c_{12} - \tfrac{2^3\cdot 2087^1}{3^5\cdot 5^2\cdot 7^1\cdot 11^1} c_4^2 c_6\\
&+ \tfrac{2^1\cdot 23^2}{3^5\cdot 11^1} c_4 c_{10} + \tfrac{2^1\cdot 97^1\cdot 107^1}{3^4\cdot 5^2\cdot 7^1\cdot 11^1} c_6 c_8 - \tfrac{2^2\cdot 8191^1}{3^4\cdot 5^1\cdot 11^1} c_{14}\Big]\\
L_8 =& \tfrac{1}{3^3\cdot 5^3\cdot 7^1\cdot 17^1}\Big[-\tfrac{3617^1}{3^4\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1} c_2^8 + \tfrac{2^2\cdot 41^1\cdot 83^1}{3^3\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1} c_2^6 c_4 - \tfrac{2^1\cdot 143483^1}{3^5\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1} c_2^5 c_6 - \tfrac{2^1\cdot 43^1\cdot 431^1}{3^5\cdot 5^1\cdot 7^1\cdot 13^1} c_2^4 c_4^2 + \tfrac{2^1\cdot 162011^1}{3^4\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1} c_2^4 c_8\\
&+ \tfrac{2^6\cdot 29^1\cdot 739^1}{3^5\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1} c_2^3 c_4 c_6 - \tfrac{97^1\cdot 12889^1}{3^4\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1} c_2^3 c_{10} + \tfrac{2^7\cdot 661^1}{3^5\cdot 7^1\cdot 11^1\cdot 13^1} c_2^2 c_4^3 - \tfrac{97^1\cdot 2273^1}{3^4\cdot 7^1\cdot 11^1\cdot 13^1} c_2^2 c_4 c_8 - \tfrac{107^1\cdot 857^1}{3^3\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1} c_2^2 c_6^2\\
&+ \tfrac{1091^1\cdot 13789^1}{3^5\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1} c_2^2 c_{12} - \tfrac{2^1\cdot 9619^1}{3^5\cdot 7^1\cdot 11^1} c_2 c_4^2 c_6 + \tfrac{2^1\cdot 617147^1}{3^5\cdot 5^1\cdot 11^1\cdot 13^1} c_2 c_4 c_{10} + \tfrac{29^1\cdot 98057^1}{3^5\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1} c_2 c_6 c_8 - \tfrac{37117^1}{5^1\cdot 11^1\cdot 13^1} c_2 c_{14}\\
&- \tfrac{13687^1}{3^5\cdot 7^1\cdot 11^1\cdot 13^1} c_4^4 + \tfrac{2^1\cdot 101^1\cdot 6491^1}{3^5\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1} c_4^2 c_8 + \tfrac{2^3\cdot 311^1}{3^4\cdot 5^1\cdot 11^1} c_4 c_6^2 - \tfrac{191^1\cdot 14143^1}{3^5\cdot 5^1\cdot 11^1\cdot 13^1} c_4 c_{12} - \tfrac{29^1\cdot 31^1\cdot 79^1}{3^5\cdot 5^1\cdot 13^1} c_6 c_{10}\\
&- \tfrac{167^1\cdot 2663^1}{3^4\cdot 5^1\cdot 7^1\cdot 11^1\cdot 13^1} c_8^2 + \tfrac{31^1\cdot 151^1\cdot 3617^1}{3^4\cdot 5^1\cdot 11^1\cdot 13^1} c_{16}\Big]\\
L_9 =& \tfrac{1}{3^7\cdot 5^3\cdot 7^1\cdot 19^1}\Big[-\tfrac{2^1\cdot 43867^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} c_2^9 + \tfrac{41^1\cdot 53^1\cdot 827^1}{3^1\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} c_2^7 c_4 - \tfrac{1337617^1}{5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} c_2^6 c_6 - \tfrac{541^1\cdot 761^1}{3^1\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1} c_2^5 c_4^2\\
&+ \tfrac{29^1\cdot 79^1\cdot 3467^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} c_2^5 c_8 + \tfrac{3931^1\cdot 17981^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} c_2^4 c_4 c_6 - \tfrac{61^1\cdot 491^1\cdot 1013^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} c_2^4 c_{10} + \tfrac{2^1\cdot 31^1\cdot 475957^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} c_2^3 c_4^3\\
&- \tfrac{2^2\cdot 1697^1\cdot 25951^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} c_2^3 c_4 c_8 - \tfrac{2^1\cdot 70003^1}{7^1\cdot 11^1\cdot 13^1\cdot 17^1} c_2^3 c_6^2 + \tfrac{29^1\cdot 53^1\cdot 181^1\cdot 433^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} c_2^3 c_{12} - \tfrac{2^1\cdot 93133^1}{3^2\cdot 5^1\cdot 13^1\cdot 17^1} c_2^2 c_4^2 c_6\\
&+ \tfrac{2^1\cdot 83892287^1}{3^1\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} c_2^2 c_4 c_{10} + \tfrac{743^1\cdot 73597^1}{3^1\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} c_2^2 c_6 c_8 - \tfrac{887^1\cdot 210719^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 17^1} c_2^2 c_{14} - \tfrac{13^1\cdot 7297^1}{5^1\cdot 7^2\cdot 11^1\cdot 17^1} c_2 c_4^4\\
&+ \tfrac{2^1\cdot 8359009^1}{5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} c_2 c_4^2 c_8 + \tfrac{2^2\cdot 83^1\cdot 15667^1}{3^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} c_2 c_4 c_6^2 - \tfrac{47^1\cdot 1291^1\cdot 22709^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} c_2 c_4 c_{12} - \tfrac{641^1\cdot 121169^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} c_2 c_6 c_{10}\\
&- \tfrac{67^1\cdot 101^1\cdot 13883^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} c_2 c_8^2 + \tfrac{43^1\cdot 237398563^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} c_2 c_{16} + \tfrac{2^4\cdot 31^1\cdot 139^1\cdot 293^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} c_4^3 c_6 - \tfrac{2^2\cdot 31^1\cdot 314173^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} c_4^2 c_{10}\\
&- \tfrac{2^2\cdot 89^1\cdot 11071^1}{3^1\cdot 5^1\cdot 7^2\cdot 13^1\cdot 17^1} c_4 c_6 c_8 + \tfrac{2^1\cdot 23^1\cdot 1933^1\cdot 6857^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} c_4 c_{14} - \tfrac{2^3\cdot 89^1\cdot 17929^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} c_6^3 + \tfrac{2^1\cdot 367^1\cdot 1150249^1}{3^2\cdot 5^1\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} c_6 c_{12}\\
&+ \tfrac{2^2\cdot 13569497^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} c_8 c_{10} - \tfrac{2^1\cdot 43867^1\cdot 131071^1}{3^2\cdot 7^2\cdot 11^1\cdot 13^1\cdot 17^1} c_{18}\Big]
\end{aligned}
$$

We now consider the higher structures in the complex case as well.

Expansion 36. *L-genus for complex String manifolds ($c_1 = c_2 = 0$) in terms of Chern classes.*

L-genus with complex String structure

$$
\begin{aligned}
L_0 &= 1\\
L_1 &= 0\\
L_2 &= \tfrac{1}{3^2 \cdot 5^1}\Big[+ 7^1 c_4 \Big]\\
L_3 &= \tfrac{1}{3^3 \cdot 5^1 \cdot 7^1}\Big[- 2^1 \cdot 31^1 c_6 \Big]\\
L_4 &= \tfrac{1}{3^3 \cdot 5^2 \cdot 7^1}\Big[- \tfrac{19^1}{3^1} c_4^2 + 127^1 c_8 \Big]\\
L_5 &= \tfrac{1}{3^4 \cdot 5^1 \cdot 11^1}\Big[+ \tfrac{2^4}{5^1} c_4 c_6 - \tfrac{2^1 \cdot 73^1}{3^1} c_{10} \Big]\\
L_6 &= \tfrac{1}{3^5 \cdot 5^3 \cdot 7^2 \cdot 11^1 \cdot 13^1}\Big[+ 2^1 \cdot 1453^1 c_4^3 - \tfrac{159287^1}{3^1} c_4 c_8 - \tfrac{167^1 \cdot 241^1}{3^1} c_6^2 + \tfrac{2^1 \cdot 23^1 \cdot 89^1 \cdot 691^1}{3^1} c_{12} \Big]\\
L_7 &= \tfrac{1}{3^6 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1}\Big[- \tfrac{2^3 \cdot 2087^1}{3^1 \cdot 5^2 \cdot 7^1} c_4^2 c_6 + \tfrac{2^1 \cdot 23^2}{3^1} c_4 c_{10} + \tfrac{2^1 \cdot 97^1 \cdot 107^1}{5^2 \cdot 7^1} c_6 c_8 - \tfrac{2^2 \cdot 8191^1}{5^1} c_{14} \Big]\\
L_8 &= \tfrac{1}{3^7 \cdot 5^3 \cdot 7^1 \cdot 17^1}\Big[- \tfrac{13687^1}{3^1 \cdot 7^1 \cdot 11^1 \cdot 13^1} c_4^4 + \tfrac{2^1 \cdot 101^1 \cdot 6491^1}{3^1 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1} c_4^2 c_8 + \tfrac{2^3 \cdot 311^1}{5^1 \cdot 11^1} c_4 c_6^2 - \tfrac{191^1 \cdot 14143^1}{3^1 \cdot 5^1 \cdot 11^1 \cdot 13^1} c_4 c_{12} - \tfrac{29^1 \cdot 31^1 \cdot 79^1}{3^1 \cdot 5^1 \cdot 13^1} c_6 c_{10}\\
&\qquad - \tfrac{167^1 \cdot 2663^1}{5^1 \cdot 7^1 \cdot 11^1 \cdot 13^1} c_8^2 + \tfrac{31^1 \cdot 151^1 \cdot 3617^1}{5^1 \cdot 11^1 \cdot 13^1} c_{16} \Big]\\
L_9 &= \tfrac{1}{3^8 \cdot 5^3 \cdot 7^3 \cdot 13^1 \cdot 17^1 \cdot 19^1}\Big[+ \tfrac{2^4 \cdot 31^1 \cdot 139^1 \cdot 293^1}{3^1 \cdot 5^1 \cdot 11^1} c_4^3 c_6 - \tfrac{2^2 \cdot 31^1 \cdot 314173^1}{3^1 \cdot 11^1} c_4^2 c_{10} - \tfrac{2^2 \cdot 89^1 \cdot 11071^1}{5^1} c_4 c_6 c_8 + \tfrac{2^1 \cdot 23^1 \cdot 1933^1 \cdot 6857^1}{3^1 \cdot 11^1} c_4 c_{14}\\
&\qquad - \tfrac{2^3 \cdot 89^1 \cdot 17929^1}{3^1 \cdot 5^1 \cdot 11^1} c_6^3 + \tfrac{2^1 \cdot 367^1 \cdot 1150249^1}{3^1 \cdot 5^1 \cdot 11^1} c_6 c_{12} + \tfrac{2^2 \cdot 13569497^1}{3^1 \cdot 11^1} c_8 c_{10} - \tfrac{2^1 \cdot 43867^1 \cdot 131071^1}{3^1 \cdot 11^1} c_{18} \Big]
\end{aligned}
$$

4.4. **The Todd genus.** Recall from the definition of the Todd genus (see Section 4) that its characteristic power series is $\frac{x_j}{1-e^{-x_j}}$. As with the Â-genus and L-genus, the characteristic power series were plugged into the corresponding function, in order to produce the expansion below. However, unlike the two genera considered earlier, the Todd genus is expressed in terms of the Chern classes.

Expansion 37. *Todd genus in terms of Chern classes.*

Todd genus in terms of Chern classes

$$
\begin{aligned}
T_0 =& 1\\
T_1 =& \tfrac{1}{2^1}\big[c_1\big]\\
T_2 =& \tfrac{1}{2^2.3^1}\big[c_1^2 + c_2\big]\\
T_3 =& \tfrac{1}{2^3.3^1}\big[c_1c_2\big]\\
T_4 =& \tfrac{1}{2^2.3^1.5^1}\Big[- \tfrac{1^1}{2^2.3^1}c_1^4 + \tfrac{1^1}{3^1}c_1^2c_2 + \tfrac{1^1}{2^2.3^1}c_1c_3 + \tfrac{1^1}{2^2}c_2^2 - \tfrac{1^1}{2^2.3^1}c_4\Big]\\
T_5 =& \tfrac{1}{2^5.3^1.5^1}\Big[- \tfrac{1^1}{3^1}c_1^3c_2 + \tfrac{1^1}{3^1}c_1^2c_3 + c_1c_2^2 - \tfrac{1^1}{3^1}c_1c_4\Big]\\
T_6 =& \tfrac{1}{2^4.3^1.7^1}\Big[+ \tfrac{1^1}{2^1.3^2.5^1}c_1^6 - \tfrac{1^1}{3^1.5^1}c_1^4c_2 + \tfrac{1^1}{2^2.3^2}c_1^3c_3 + \tfrac{11^1}{2^2.3^2.5^1}c_1^2c_2^2 - \tfrac{1^1}{2^2.3^2}c_1^2c_4 + \tfrac{11^1}{2^2.3^2.5^1}c_1c_2c_3\\
&- \tfrac{1^1}{2^1.3^2.5^1}c_1c_5 + \tfrac{1^1}{2^1.3^2}c_2^3 - \tfrac{1^1}{2^2.5^1}c_2c_4 - \tfrac{1^1}{2^2.3^2.5^1}c_3^2 + \tfrac{1^1}{2^1.3^2.5^1}c_6\Big]\\
T_7 =& \tfrac{1}{2^6.3^1.7^1}\Big[+ \tfrac{1^1}{3^2.5^1}c_1^5c_2 - \tfrac{1^1}{3^2.5^1}c_1^4c_3 - \tfrac{1^1}{3^2}c_1^3c_2^2 + \tfrac{1^1}{3^2.5^1}c_1^3c_4 + \tfrac{11^1}{2^1.3^2.5^1}c_1^2c_2c_3 - \tfrac{1^1}{3^2.5^1}c_1^2c_5 + \tfrac{1^1}{3^2}c_1c_2^3\\
&- \tfrac{1^1}{2^1.5^1}c_1c_2c_4 - \tfrac{1^1}{2^1.3^2.5^1}c_1c_3^2 + \tfrac{1^1}{3^2.5^1}c_1c_6\Big]\\
T_8 =& \tfrac{1}{2^4.3^3}\Big[- \tfrac{1^1}{2^4.5^2.7^1}c_1^8 + \tfrac{1^1}{2^1.5^2.7^1}c_1^6c_2 - \tfrac{1^1}{2^3.3^1.5^2}c_1^5c_3 - \tfrac{1^1}{2^3.3^1.7^1}c_1^4c_2^2 + \tfrac{1^1}{2^3.3^1.5^2}c_1^4c_4 + \tfrac{13^1}{2^3.3^1.5^2.7^1}c_1^3c_2c_3\\
&- \tfrac{1^1}{2^4.3^1.5^2}c_1^3c_5 + \tfrac{1^1}{2^1.3^1.5^2.7^1}c_1^2c_2^3 - \tfrac{19^1}{2^4.3^1.5^2.7^1}c_1^2c_2c_4 + \tfrac{1^1}{2^4.5^2.7^1}c_1^2c_3^2 + \tfrac{1^1}{2^4.3^1.5^2}c_1^2c_6 + \tfrac{1^1}{2^3.3^1.7^1}c_1c_2^2c_3\\
&- \tfrac{1^1}{3^1.5^2.7^1}c_1c_2c_5 - \tfrac{13^1}{2^4.3^1.5^2.7^1}c_1c_3c_4 + \tfrac{1^1}{2^4.5^2.7^1}c_1c_7 + \tfrac{1^1}{2^4.5^2}c_2^4 - \tfrac{17^1}{2^3.3^1.5^2.7^1}c_2^2c_4 - \tfrac{1^1}{2^1.3^1.5^2.7^1}c_2c_3^2\\
&+ \tfrac{13^1}{2^4.3^1.5^2.7^1}c_2c_6 + \tfrac{1^1}{2^4.5^2.7^1}c_3c_5 + \tfrac{1^1}{2^4.3^1.5^1.7^1}c_4^2 - \tfrac{1^1}{2^4.5^2.7^1}c_8\Big]\\
T_9 =& \tfrac{1}{2^5.3^3}\Big[- \tfrac{1^1}{2^4.5^2.7^1}c_1^7c_2 + \tfrac{1^1}{2^4.5^2.7^1}c_1^6c_3 + \tfrac{1^1}{2^4.5^2}c_1^5c_2^2 - \tfrac{1^1}{2^4.5^2.7^1}c_1^5c_4 - \tfrac{29^1}{2^4.3^1.5^2.7^1}c_1^4c_2c_3 + \tfrac{1^1}{2^4.5^2.7^1}c_1^4c_5\\
&- \tfrac{1^1}{2^3.5^2}c_1^3c_2^3 + \tfrac{13^1}{2^3.3^1.5^2.7^1}c_1^3c_2c_4 + \tfrac{1^1}{2^1.3^1.5^2.7^1}c_1^3c_3^2 - \tfrac{1^1}{2^4.5^2.7^1}c_1^3c_6 + \tfrac{1^1}{2^3.3^1.7^1}c_1^2c_2^2c_3 - \tfrac{1^1}{3^1.5^2.7^1}c_1^2c_2c_5\\
&- \tfrac{13^1}{2^4.3^1.5^2.7^1}c_1^2c_3c_4 + \tfrac{1^1}{2^4.5^2.7^1}c_1^2c_7 + \tfrac{1^1}{2^4.5^2}c_1c_2^4 - \tfrac{17^1}{2^3.3^1.5^2.7^1}c_1c_2^2c_4 - \tfrac{1^1}{2^1.3^1.5^2.7^1}c_1c_2c_3^2\\
&+ \tfrac{13^1}{2^4.3^1.5^2.7^1}c_1c_2c_6 + \tfrac{1^1}{2^4.5^2.7^1}c_1c_3c_5 + \tfrac{1^1}{2^4.3^1.5^1.7^1}c_1c_4^2 - \tfrac{1^1}{2^4.5^2.7^1}c_1c_8\Big]
\end{aligned}
$$

4.4.1. *Todd genus with simplifications.* Here we present expansions of the Todd genus with vanishing one or several characteristic classes.

In particular, complex spaces with vanishing first Chern class are very prominent in algebraic geometry, complex differential geometry and mathematical physics (see [15], [16], and [11]). Observe the significance of c_1 characteristic class for the Todd genus - by setting $c_1 = 0$ all odd Todd genus expansions vanished as well at least up to the degree presented.

Expansion 38. *Todd genus for Calabi-Yau manifolds ($c_1 = 0$).*

Todd genus for Calabi-Yau spaces

$$
\begin{aligned}
T_2 =& \tfrac{1}{2^2.3^1}\big[c_2\big]\\
T_4 =& \tfrac{1}{2^4.3^1.5^1}\Big[c_2^2 - \tfrac{1^1}{3^1}c_4\Big]\\
T_6 =& \tfrac{1}{2^5.3^1.7^1}\Big[+ \tfrac{1^1}{3^2}c_2^3 - \tfrac{1^1}{2^1.5^1}c_2c_4 - \tfrac{1^1}{2^1.3^2.5^1}c_3^2 + \tfrac{1^1}{3^2.5^1}c_6\Big]\\
T_8 =& \tfrac{1}{2^5.3^3.5^1}\Big[+ \tfrac{1^1}{2^3.5^1}c_2^4 - \tfrac{17^1}{2^2.3^1.5^1.7^1}c_2^2c_4 - \tfrac{1^1}{3^1.5^1.7^1}c_2c_3^2 + \tfrac{13^1}{2^3.3^1.5^1.7^1}c_2c_6 + \tfrac{1^1}{2^3.5^1.7^1}c_3c_5 + \tfrac{1^1}{2^3.3^1.7^1}c_4^2 - \tfrac{1^1}{2^3.5^1.7^1}c_8\Big]
\end{aligned}
$$

Sometimes a complex manifold might have vanishing higher Chern classes without the lower ones necessarily being zero. These can be viewed as complex analogues of the p_1-structures encountered in the case of Â-genus and L-genus. These can be done similarly and we will not list them.

The next expansions present Todd genus simplifications with more than one vanishing Chern class. Observe how drastically the following expansions simplify from their initial form in *Expansion 37* or even from the above expansions involving vanishing of a single Chern class.

Expansion 39. *Todd genus for complex String manifolds ($c_1 = c_2 = 0$).*

Todd genus with complex String stucture

$$\begin{aligned}
T_4 &= \tfrac{1}{2^4.3^2.5^1}\left[-1^1 c_4\right]\\
T_6 &= \tfrac{1}{2^5.3^3.5^1.7^1}\left[-\tfrac{1^1}{2^1}c_3^2 + c_6\right]\\
T_8 &= \tfrac{1}{2^8.3^3.5^1.7^1}\left[+\tfrac{1^1}{5^1}c_3c_5 + \tfrac{1^1}{3^1}c_4^2 - \tfrac{1^1}{5^1}c_8\right]\\
T_{10} &= \tfrac{1}{2^5.3^4.5^1.11^1}\left[+\tfrac{1^1}{2^5.5^1}c_3^2c_4 - \tfrac{1^1}{2^4.3^1.7^1}c_3c_7 - \tfrac{1^1}{3^1.5^1.7^1}c_4c_6 - \tfrac{1^1}{2^5.3^1.7^1}c_5^2 + \tfrac{1^1}{2^4.3^1.7^1}c_{10}\right]
\end{aligned}$$

Expansion 40. *Todd genus with complex Fivebrane structure ($c_1 = c_2 = c_3 = c_4 = 0$).*

Todd genus complex Fivebrane structure

$$\begin{aligned}
T_6 &= \tfrac{1}{2^5.3^3.5^1.7^1}\left[c_6\right]\\
T_8 &= \tfrac{1}{2^8.3^3.5^2.7^1}\left[-1^1 c_8\right]\\
T_{10} &= \tfrac{1}{2^9.3^5.5^1.7^1.11^1}\left[-\tfrac{1^1}{2^1}c_5^2 + c_{10}\right]
\end{aligned}$$

Expansion 41. *Todd-genus simplification - vanishing of the odd Chern classes ($c_{2i+1} = 0$).*

Todd genus with even Chern classes

$$\begin{aligned}
T_0 &= 1\\
T_2 &= \tfrac{1}{2^2.3^1}\left[c_2\right]\\
T_4 &= \tfrac{1}{2^4.3^1.5^1}\left[c_2^2 - \tfrac{1^1}{3^1}c_4\right]\\
T_6 &= \tfrac{1}{2^5.3^1.7^1}\left[+\tfrac{1^1}{3^2}c_2^3 - \tfrac{1^1}{2^1.5^1}c_2c_4 + \tfrac{1^1}{3^2.5^1}c_6\right]\\
T_8 &= \tfrac{1}{2^7.3^3.5^1}\left[+\tfrac{1^1}{2^1.5^1}c_2^4 - \tfrac{17^1}{3^1.5^1.7^1}c_2^2c_4 + \tfrac{13^1}{2^1.3^1.5^1.7^1}c_2c_6 + \tfrac{1^1}{2^1.3^1.7^1}c_4^2 - \tfrac{1^1}{2^1.5^1.7^1}c_8\right]\\
T_{10} &= \tfrac{1}{2^5.3^3.5^1.7^1.11^1}\left[+\tfrac{1^1}{2^4}c_2^5 - \tfrac{73^1}{2^5.3^1.5^1}c_2^3c_4 + \tfrac{109^1}{2^5.3^2.5^1}c_2^2c_6 + \tfrac{29^1}{2^5.3^1.5^1}c_2c_4^2 - \tfrac{43^1}{2^5.3^2.5^1}c_2c_8 - \tfrac{1^1}{3^2.5^1}c_4c_6 - \tfrac{1^1}{2^5.3^2}c_5^2 + \tfrac{1^1}{2^4.3^2}c_{10}\right]
\end{aligned}$$

4.4.2. *Todd genus of realification.* The Todd genus may be written in terms of the Pontrjagin classes as well, due to the relation between Chern and Pontrjagin classes shown in *Equation* (3.2). The expansion below straightforwardly implements that relation by first imposing vanishing all odd degrees of the Todd genus and then converting the Chern classes to Pontrjagin classes in even degrees.

Expansion 42. *Todd genus in terms of Pontrjagin classes of realification.*

Todd genus in terms of Pontrjagin classes

$$\begin{aligned}
T_0 &= 1\\
T_2 &= \tfrac{1}{2^2.3^1}\left[-1^1 p_1\right]\\
T_4 &= \tfrac{1}{2^4.3^1.5^1}\left[p_1^2 - \tfrac{1^1}{3^1}p_2\right]\\
T_6 &= \tfrac{1}{2^5.3^1.7^1}\left[-\tfrac{1^1}{3^2}p_1^3 + \tfrac{1^1}{2^1.5^1}p_1p_2 - \tfrac{1^1}{3^2.5^1}p_3\right]\\
T_8 &= \tfrac{1}{2^7.3^3.5^1}\left[+\tfrac{1^1}{2^1.5^1}p_1^4 - \tfrac{17^1}{3^1.5^1.7^1}p_1^2p_2 + \tfrac{13^1}{2^1.3^1.5^1.7^1}p_1p_3 + \tfrac{1^1}{2^1.3^1.7^1}p_2^2 - \tfrac{1^1}{2^1.5^1.7^1}p_4\right]\\
T_{10} &= \tfrac{1}{2^5.3^3.5^1.7^1.11^1}\left[-\tfrac{1^1}{2^4}p_1^5 + \tfrac{73^1}{2^5.3^1.5^1}p_1^3p_2 - \tfrac{109^1}{2^5.3^2.5^1}p_1^2p_3 - \tfrac{29^1}{2^5.3^1.5^1}p_1p_2^2 + \tfrac{43^1}{2^5.3^2.5^1}p_1p_4 + \tfrac{1^1}{3^2.5^1}p_2p_3 - \tfrac{1^1}{2^4.3^2}p_5\right]
\end{aligned}$$

Expansion 43. *String or p_1-structure - Todd genus in terms of Pontrjagin classes with $p_1 = 0$.*

Todd genus with String or p_1-structure

$$
\begin{aligned}
T_0 =& 1\\
T_2 =& 0\\
T_4 =& \tfrac{1}{2^4.3^2.5^1}\big[-1^1 p_2\big]\\
T_6 =& \tfrac{1}{2^5.3^3.5^1.7^1}\big[-1^1 p_3\big]\\
T_8 =& \tfrac{1}{2^8.3^3.5^1.7^1}\Big[+\tfrac{1^1}{3^1}p_2^2 - \tfrac{1^1}{5^1}p_4\Big]\\
T_{10} =& \tfrac{1}{2^5.3^5.5^1.7^1.11^1}\Big[+\tfrac{1^1}{5^1}p_2p_3 - \tfrac{1^1}{2^4}p_5\Big]
\end{aligned}
$$

Expansion 44. *Todd genus in terms of Pontrjagin classes with Fivebrane structure ($p_1 = 0 = p_2$).*

Todd genus with Fiverbane structure

$$
\begin{aligned}
T_0 =& 1\\
T_2 =& 0\\
T_4 =& 0\\
T_6 =& -\tfrac{1}{2^5.3^3.5.7}p_3\\
T_8 =& -\tfrac{1}{2^8.3^3.5^2.7}p_4\\
T_{10} =& -\tfrac{1}{2^9.3^5.5\cdot 7\cdot 11}p_5\\
T_{12} =& \tfrac{1}{2^{12}.3^6.5^3.7^2.11\cdot 13}[+3^1\cdot 7^1\cdot 101^1 p_3^2 - 2^1\cdot 691^1 p_6]
\end{aligned}
$$

Other higher cases can be done similarly, as the expressions for the Todd genus are generally much simpler than those of other genera.

5. Relations among genera

Having computed the genera expansions for various manifolds, we can now relate the genera with each other for the corresponding structures, since all of them can be expressed in both the Pontrjagin and Chern classes. Some such relations can also be predicted theoretically; see Proposition 4 for an illustration of one case.

Relations among the genera are important, as they provide information about which manifolds are comparable and in what ways are their structures are similar. In fact, relations among the genera lead to nontrivial effects in geometry, topology, and physics. In particular, in mathematical approaches to quantum field theory and string theory they lead to what is called *cancellation of anomalies*, thereby rendering the theories properly defined (see [2], [28], and [29]). The relations presented here are intended as ready-made results and references for people working on the relevant topics.

It is explicitly indicated in each case, which expansions are being related. The method of computing was substitution - we expressed one genus in terms of its characteristic classes degree by degree (e.g. from *Expansions 29* we have $p_2 = \frac{45}{7}L_2, \quad p_3 = \frac{945}{62}L_3$, ...), then inserted these expressions into the other genus for the corresponding structure (e.g., L-genus for string manifolds corresponds to the Â-genus for String manifolds). The complexity of the resulting equation was the main reason for curtailing the number of equations at a certain degree in each case.

5.1. **Â and L genera.** Since both the Â-genus and L-genus are expressed in terms of the Pontrjagin classes, then the relations can be computed without any further conversions. In the first case below, we relate Â-genus and L-genus of String manifolds (see *Expansion 20* and *29*). Observe how the $\hat{A}_2$ and L_2 as well as $\hat{A}_3$ and L_3 have a linear relationship.

Expansion 45. *Relation between Â-genus and L-genus for String manifolds ($p_1 = 0$).*

Relation between Â-genus and L-genus for String manifolds

$A_2 = \frac{1}{2^5 \cdot 7^1}\big[-1L_2\big]$

$A_3 = \frac{1}{2^7 \cdot 31^1}\big[-1L_3\big]$

$A_4 = \frac{1}{2^{11} \cdot 7^2 \cdot 127^1}\big[+3^1 \cdot 5^1 \cdot 17^1 L_2^2 - 2^2 \cdot 7^2 L_4\big]$

$A_5 = \frac{1}{2^{12} \cdot 7^1 \cdot 31^1 \cdot 73^1}\big[+3^3 \cdot 5^1 L_2 L_3 - 2^1 \cdot 31^1 L_5\big]$

$A_6 = \frac{1}{2^{16} \cdot 7^3 \cdot 23^1 \cdot 31^2 \cdot 89^1 \cdot 127^1}\Big[-3^2 \cdot 5^2 \cdot 11^1 \cdot 31^2 \cdot 163^1 L_2^3 + 2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 17^1 \cdot 31^2 L_2 L_4 + 2^1 \cdot 7^3 \cdot 127^1\big(3^4 \cdot 7^2 L_3^2 - 2^2 \cdot 31^2 L_6\big)\Big]$

Next we relate *Expansions 21* and *30*. Again notice the linear relationship in the first two equations.

Expansion 46. *Relation between Â-genus and L-genus of p_2-structure manifolds ($p_2 = 0$).*

Relation between Â-genus and L-genus with p_2-structure

$A_1 = \frac{1}{2^3}\big[-1L_1\big]$

$A_2 = \frac{1}{2^7}\big[-7L_2\big]$

$A_3 = \frac{1}{2^{10} \cdot 31^1}\big[-3^3 L_1^3 - 2^3 L_3\big]$

$A_4 = \frac{1}{2^{15} \cdot 3^1 \cdot 5^1 \cdot 31^1 \cdot 127^1}\big[+37^1 \cdot 41^1 L_1^4 + 2^5 \cdot 3^3 \cdot 5^1 \cdot 7^1 L_1 L_3 - 2^6 \cdot 3^1 \cdot 5^1 \cdot 31^1 L_4\big]$

$A_5 = \frac{1}{2^{18} \cdot 5^1 \cdot 7^1 \cdot 31^1 \cdot 73^1 \cdot 127^1}\big[-3^3 \cdot 17^1 \cdot 26863^1 L_1^5 - 2^4 \cdot 3^2 \cdot 5^2 \cdot 17^1 \cdot 251^1 L_1^2 L_3 + 2^6 \cdot 3^2 \cdot 5^2 \cdot 17^1 \cdot 31^1 L_1 L_4 - 2^7 \cdot 5^1 \cdot 31^1 \cdot 127^1 L_5\big]$

The equations below were written based on *Expansions 22* and *31*.

Expansion 47. *Relation between Â-genus and L-genus of p_3-structure manifolds ($p_3 = 0$).*

Relation between Â-genus and L-genus with p_3-structure

$A_1 = \frac{1}{2^3}\big[-1L_1\big]$

$A_2 = \frac{1}{2^7 \cdot 7^1}\big[+3^2 L_1^2 - 2^2 L_2\big]$

$A_3 = \frac{1}{2^{10} \cdot 13^1}\big[-3^2 L_1^3 - 2^2 \cdot 11^1 L_3\big]$

$A_4 = \frac{1}{2^{15} \cdot 5^1 \cdot 7^2 \cdot 127^1}\Big[+3^2 \cdot 13^1 \cdot 389^1 L_1^4 - 2^3 \cdot 3^2 \cdot 5^2 \cdot 59^1 L_1^2 L_2 + 2^4 \cdot 5^1 \cdot \big(3^2 \cdot 5^2 L_2^2 - 2^2 \cdot 7^2 L_4\big)\Big]$

For the next result we used *Expansions 23* and *32*. Observe the liner relationship between in degrees 3 through 5.

Expansion 48. *Relation between Â-genus and L-genus of Fivebrane manifolds ($p_1 = p_2 = 0$).*

Relation between Â-genus and L-genus with Fivebrane structure

$A_3 = \frac{1}{2^7 \cdot 31^1}\big[-1L_3\big]$

$A_4 = \frac{1}{2^9 \cdot 127^1}\big[-1L_4\big]$

$A_5 = \frac{1}{2^{11} \cdot 7^1 \cdot 73^1}\big[-1L_5\big]$

$A_6 = \frac{1}{2^{15} \cdot 3^1 \cdot 5^1 \cdot 23^1 \cdot 31^2 \cdot 89^1}\big[+3^5 \cdot 5^1 \cdot 7^2 L_3^2 - 2^2 \cdot 17^1 \cdot 31^2 L_6\big]$

$A_7 = \frac{1}{2^{16} \cdot 31^1 \cdot 127^1 \cdot 8191^1}\big[+3^3 \cdot 5^1 \cdot 7^1 \cdot 17^1 L_3 L_4 - 2^1 \cdot 31^1 \cdot 127^1 L_7\big]$

In the next result we relate *Expansions 24* and *33*. It is indeed an interesting observation that from degree 4 to 7 the genera are related linearly. It appears that there is a clear trade-off between vanishing of the characteristic classes and the simplicity of the resulting relations among the two genera.

Expansion 49. *Relation between Â-genus and L-genus of Ninebrane manifolds ($p_1 = p_2 = p_3 = 0$).*

Relation between Â-genus and L-genus with Ninebrane structure

$$A_4 = \frac{1}{2^9 \cdot 127^1}\big[-1L_4\big]$$

$$A_5 = \frac{1}{2^{11} \cdot 7^1 \cdot 73^1}\big[-1L_5\big]$$

$$A_6 = \frac{1}{2^{13} \cdot 23^1 \cdot 89^1}\big[-1L_6\big]$$

$$A_7 = \frac{1}{2^{15} \cdot 8191^1}\big[-1L_7\big]$$

$$A_8 = \frac{1}{2^{19} \cdot 7^1 \cdot 31^1 \cdot 127^2 \cdot 151^1}\big[+3^2 \cdot 5^2 \cdot 17^2 L_4^2 + 2^2 \cdot 127^2 L_8\big]$$

$$A_9 = \frac{1}{2^{20} \cdot 7^1 \cdot 73^1 \cdot 127^1 \cdot 131071^1}\big[+3^2 \cdot 5^1 \cdot 11^1 \cdot 17^1 \cdot 31^1 L_4 L_5 - 2^1 \cdot 7^1 \cdot 73^1 \cdot 127^1 L_9\big]$$

The result below was computed from *Expansions 25* and *34*.

Expansion 50. *Relation between Â-genus and L-genus of manifolds with vanishing odd Pontrjagin classes ($p_{2i+1} = 0$).*

In this case the odd classes on both sides are trivial and we are left with only even degree classes, i.e. those of dimension multiple of 8.

Relation between Â-genus and L-genus with even Pontrjagin classes

$$A_2 = \frac{1}{2^5 \cdot 7^1}\big[-1L_2\big]$$

$$A_4 = \frac{1}{2^{11} \cdot 7^2 \cdot 127^1}\big[+3^2 \cdot 5^2 L_2^2 - 2^2 \cdot 7^2 L_4\big]$$

$$A_6 = \frac{1}{2^{16} \cdot 7^3 \cdot 23^1 \cdot 89^1 \cdot 127^1}\big[-3^2 \cdot 5^2 \cdot 11^1 \cdot 163^1 L_2^3 + 2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 17^1 L_2 L_4 - 2^3 \cdot 7^3 \cdot 127^1 L_6\big]$$

5.2. **Todd and L genera.** In order to compare either L-genus or Â-genus to Todd genus one needs to ensure that both have either complex bundles or real bundles. Therefore, we use the Todd genus of complexification in terms of Pontrjagin classes (see Expansion 42) and relate it to the standard L-genus and Â-genus expansions (see *Expansion 28* and *19*, respectively).

Expansion 51. *L-genus in terms of Todd genus of complexification.*

L-genus in terms of Todd genus of complexification (or realification)

$$T_1 = \frac{1}{2^2}\big[-1L_1\big]$$

$$T_2 = \frac{1}{2^4 \cdot 7^1}\big[+2^2 L_1^2 - 1^1 L_2\big]$$

$$T_3 = \frac{1}{2^6 \cdot 7^1 \cdot 31^1}\big[-2^1 \cdot 3^3 L_1^3 + 2^1 \cdot 19^1 L_1 L_2 - 7^1 L_3\big]$$

$$T_4 = \frac{1}{2^8 \cdot 7^2 \cdot 31^1 \cdot 127^1}\big[+2^1 \cdot 3^3 \cdot 331^1 L_1^4 - 2^1 \cdot 3^4 \cdot 5^1 \cdot 29^1 L_1^2 L_2 + 2^1 \cdot 7^2 \cdot 79^1 L_1 L_3 + 31^1(2^3 \cdot 11^1 L_2^2 - 7^2 L_4)\big]$$

5.3. **Todd and Â genera.** For completeness let us restate that the equations below were obtained from *Expansions 42* and *19*. We illustrate a theoretical relationship as follows.

Proposition 4. *For any oriented real vector bundle E we have the relation $\mathrm{Td}(E \otimes \mathbb{C}) = \hat{A}(E)^2$.*

Proof. Start with the series

$$\mathrm{Td}(E) = \prod_{i=0}^{n} \frac{x_i}{1 - e^{-x_i}} \ .$$

Now, with $\bar{E}$ the complex conjugate of E, the splitting principle gives

$$\begin{aligned} E \otimes \mathbb{C} &\cong E \oplus \bar{E} \\ &\cong (L_1 \oplus \ldots \oplus L_n) \oplus (\bar{L}_1 \oplus \ldots \oplus \bar{L}_n) \end{aligned}$$

such that $c_i(\bar{L}) = -c_i(L)$ and $c_i(\bar{E}) = (-1)^j c_j(E)$. Note that we have the exchange $x_i \leftrightarrow -x_i$ for $E \leftrightarrow \bar{E}$. Therefore,

$$\begin{aligned} \mathrm{Td}(E \oplus \mathbb{C}) &= \prod_{i=1}^{n} \frac{x_i}{1-e^{-x_i}} \cdot \frac{-x_i}{1-e^{x_i}} \\ &= \prod_{i=1}^{n} \frac{x_i}{1-e^{-x_i}} \cdot \frac{x_i}{e^{x_i}-1} \\ &= \prod_{i=1}^{n} \frac{x_i^2}{e^{x_i}+e^{-x_i}-1-e^{x_i}e^{x_i}} \\ &= \prod_{i=1}^{n} \left[\frac{x_i}{e^{x_i/2}-e^{-x_i/2}}\right]^2 \\ &= \prod_{i=1}^{n} \left[\frac{x_{i/2}}{\frac{1}{2}(e^{x_i/2}-e^{-x_i/2})}\right]^2 \\ &= \prod_{i=1}^{n} \left[\frac{x_{i/2}}{\sinh(x_{i/2})}\right]^2 \\ &= \left[\hat{A}(E)\right]^2 . \end{aligned}$$

□

Expansion 52. *Â-genus in terms of Todd genus of complexification.*

Observation 1. *Note the relations in term of components*

$$\begin{aligned} T_{2n} &= A_n^2 + 2(A_1 A_{2n-1} + A_2 A_{2n-2} + \ldots + A_{2n} A_0) , \\ T_{2n+1} &= 2(A_{2n+1} A_0 + A_{2n} A_1 + A_{2n-1} A_2 + \ldots + A_n A_{n+1}) . \end{aligned}$$

Considering how complex the expressions for the Â-genus and the Todd genus of complexification are, particularly in higher degrees, it is perhaps surprising how the two genera can be related through simple formulas with a clear pattern. In fact, the pattern evident in the formulas above is an expanded form of the relation between Todd and Â- genus demonstrated theoretically above.

Â-genus in terms of Todd genus of complexification (or realification)

$T_1 = \frac{1}{1^1}\left[2A_1\right]$

$T_2 = \frac{1}{1^1}\left[A_1^2 + 2^1 A_2\right]$

$T_3 = 2^1\left[A_1 A_2 + A_3\right]$

$T_4 = \frac{1}{1^1}\left[A_2^2 + 2^1(A_1 A_3 + A_4)\right]$

$T_5 = 2^1\left[A_2 A_3 + A_1 A_4 + A_5\right]$

$T_6 = A_3^2 + 2^1\left[A_2 A_4 + A_1 A_5 + A_6\right]$

References

[1] M. Ando, M. J. Hopkins, and N. P. Strickland, *Elliptic spectra, the Witten genus and the theorem of the cube*, Invent. Math. **146** (2001), no. 3, 595-687.
[2] R. Bertlmann, *Anomalies in Quantum Field Theory*, Clarendon Press, UK, 1996.
[3] A. Borel, *Topology of Lie groups and characteristic classes*, Bulletin of the American Mathematical Society, 61, 297-432, 1955.
[4] A. Borel, F. Hirzebruch, *Characteristic classes and homogeneous spaces, I*, Amer. J. Math. 80, 458-538, 1958.
[5] A. Borel, F. Hirzebruch, *Characteristic classes and homogeneous spaces, II*, Amer. J. Math. 81, 315-382, 1959.
[6] A. Borel, F. Hirzebruch, *Characteristic classes and homogeneous spaces, III*, Amer. J. Math. 82, 491-504, 1960.

[7] R. Bott, L. W. Tu, *Differential Forms in Algebraic Topology*, New York: Springer-Verlag, 1982.
[8] S.-S. Chern, *Complex manifolds without potential theory*, Springer-Verlag, Berlin, 1995.
[9] J. Gallier, S. S. Shatz, *Complex Algebraic Geometry*, University of Pennsylvania, 2007, retrieved from ftp://ftp.cis.upenn.edu/pub/cis610/public_html/calg5.pdf
[10] P. Gilkey, R. Ivanova, S. Nikcevic, *Characteristic Classes*, Encyclopedia of Mathematical Physics (Elsevier Academic) eds. J.-P. Francoise, G.L. Naber and Tsou S. T., volume 1, page 488-495, Oxford: Elsevier, 2006.
[11] M. Gross, D. Huybrechts, D. Joyce, *Calabi-Yau manifolds and related geometries*, Universitext, Berlin, New York: Springer-Verlag, 2003.
[12] F. Hirzebruch, *Topological Methods in Algebraic Geometry*, Springer-Verlag Berlin Heidelberg, 1978.
[13] F. Hirzebruch and M. Kreck, *On the concept of genus in topology and complex analysis*, Notices Amer. Math. Soc. **56** (2009), no. 6, 713-719.
[14] M. Hopkins, *The $\hat{A}$ class*, Harvard University, 2015, retrieved from http://www.math.harvard.edu/archive/272b_spring_05/handouts/A-roof/A-roof.pdf
[15] T. Hübsch, *Calabi-Yau Manifolds: a Bestiary for Physicists*, Singapore, New York: World Scientific, 1994.
[16] D. Husemoller, *Fibre Bundles*, McGraw-Hill, 1966.
[17] O. Iena, *On Symbolic Computations with Chern Classes: Remarks on the Library CHERN.LIB for SINGULAR*, Mathematics Subject Classification, 2010, retrieved from http://orbilu.uni.lu/bitstream/10993/22395/1/ChernLib.pdf
[18] O. Iena, *On Different Approaches to Compute the Chern Classes of a tensor Product of Two Vector Bundles*, 2016, http://orbilu.uni.lu/bitstream/10993/27418/1/ChernProd.pdf
[19] A. Lascoux, *Classes de Chern d'un produit tensoriel*, C. R. Acad. Sci. Paris Ser, A-B, 286(8):A385-A387, 1978.
[20] Q. Lu, S. S.-T. Yau, A. Libgober, *Singularities and Complex Geometry*, American Mathematical Society / International Press, 1997.
[21] G. Luke and A. Mishchenko, *Vector bundles and their applications*, Kluwer Academic Publishers, Dordrecht, 1998.
[22] L. Manivel, *Chern Classes of Tensor Products*, Int. J. Math., 27, 1650079, 2016, arXiv:1012.0014v1.
[23] I. G. Macdonald, *Symmetric functions and Hall polynomials*, reprint of the 1998 2nd edition, Oxford University Press, reprint of the 1998 2nd edition edition, 2015.
[24] I. Madsen and J. Tornehave, *From calculus to cohomology: de Rham cohomology and characteristic classes*, Cambridge University Press, 1997.
[25] C. McTague, *Computing Hirzebruch L-Polynomials*, Blog: Carl McTague, Jan 2014, retrieved from https://www.mctague.org/carl/blog/2014/01/05/computing-L-polynomials/
[26] J. Milnor and J.D. Stasheff, *Characteristic classes*, Annals of Math. Studies, Princeton University Press, Princeton, 1974.
[27] S. Morita, *Geometry of characteristic classes*, Amer. Math. Soc., Providence, RI, 2001.
[28] M. Nakahara, *Geometry, Topology and Physics*, Institute of Physics Publishing, Bristol and Philadelphia, 2003.
[29] H. Sati, *Ninebrane structures*, Int. J. Geom. Methods Mod. Phys. 12 1550041, 2015, arXiv:1405.7686v2.
[30] H. Sati, U. Schreiber, J. Stasheff, *Fivebrane structures*, Rev. Math. Phys. 21:1197-1240, 2009, arXiv:0805.0564v3.
[31] H. Sati, U. Schreiber, J. Stasheff, *Twisted differential String and Fivebrane structures*, Commun. Math. Phys. 315, 169-213, 2012, arXiv:0910.4001v2.
[32] Z. Teitler, *An Informal Introduction to Computing with Chern Classes in Algebraic Geometry*, Boise State University, March 31, 2014.
[33] Wikipedia `https://en.wikipedia.org/wiki/Genus_of_a_multiplicative_sequence`
[34] Y. Zhang, *A brief introduction to characteristic classes from the differential viewpoint*, Cornell University, April 24, 2011.