Value at Risk Analysis

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Some key risk characteristics

- Risk is any uncertainty that affects a system in an unknown fashion and brings great fluctuation in value and outcome.
- Reminder: Risk is the outcome of uncertainty; fluctuations can be measured, in a probabilistic sense
- Risk has a time horizon
- Risk measurement has to be set against a benchmark

Statistics of risk

- Risk analysis is using the “typical” statistical measures
  - Mean
  - Variance
  - Skewness
  - Kurtosis - Used for catastrophic, extreme tail events

Common risk measures

- Probability of Occurrence
  - Probability of failure of a project, probability of default, migration probabilities, transition matrices

- Standard Deviation, Variance and Coefficient of Variation
  - Two-sided measures
  - Sufficient only under normality

Common risk measures

- Semi-standard deviation (downside risk)
  - $\hat{s}_{semi} = \sqrt{T^{-1} \sum_{t=1}^{T} \min(X_t - \bar{X}, 0)^2}$
- Volatility
  - Used mostly in finance and real-options
  - Std. deviation of an asset's logarithmic returns over time
  - $\hat{s}_{volatility} = \sqrt{T^{-1} \sum_{t=1}^{T} \ln\left(\frac{X_t}{X_{t-1}}\right)^2}$

Common risk measures

- Value at Risk - VaR
  - The amount of capital reserves at risk given a particular holding period at a particular probability of loss
    (e.g. 1-year 99.9% VaR)
- Expected Shortfall
  - The expected capital reserve given a particular holding period in the worst q% of the cases
- Unexpected Loss, Worst Case, etc.

Calculating VaR and ES

History of VaR

- Developed in early 1990's by J.P. Morgan
- The "*4:15pm*" *report*
- J.P. Morgan launched *RiskMetrics*® (1994)
- VaR has been widely used since that time.
- Currently, researchers are looking into more advanced "VaR-like" measures.

Definition

- The VaR calculation is aimed at making a statement of the following form:
  - We are *99*% certain that we will not lose more than *$10,000* dollars in the next *3* days
    - $10,000 is the *3*-day Value-At-Risk at a *99*% confidence level
- VaR is the maximum amount at risk to be lost
  - …over a given period (e.g. one year)
  - …at a particular level of confidence (e.g. 99%)

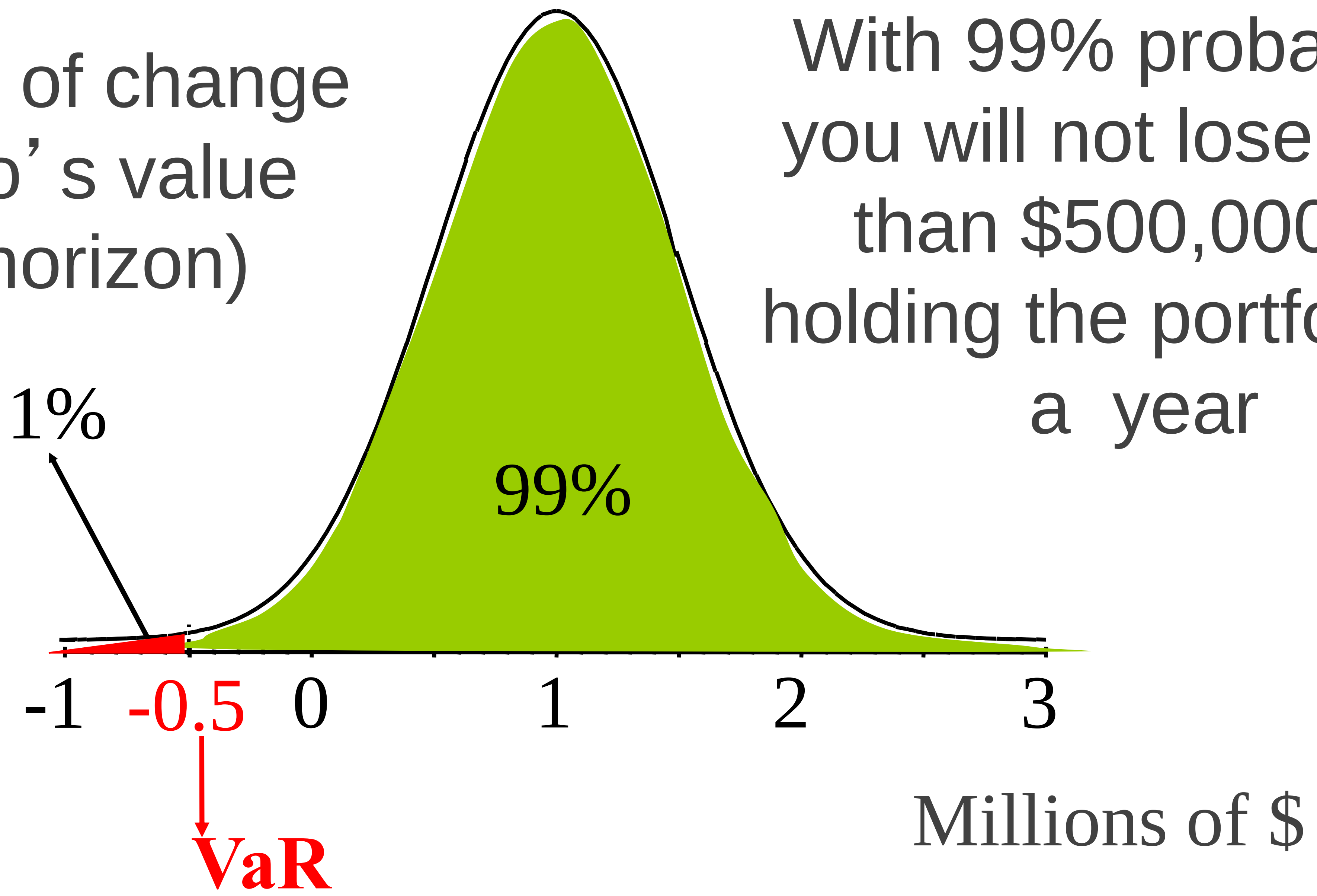
Definition

- VaR is associated with a percentile (quartile) of a distribution
- It focuses on the tail of a distribution

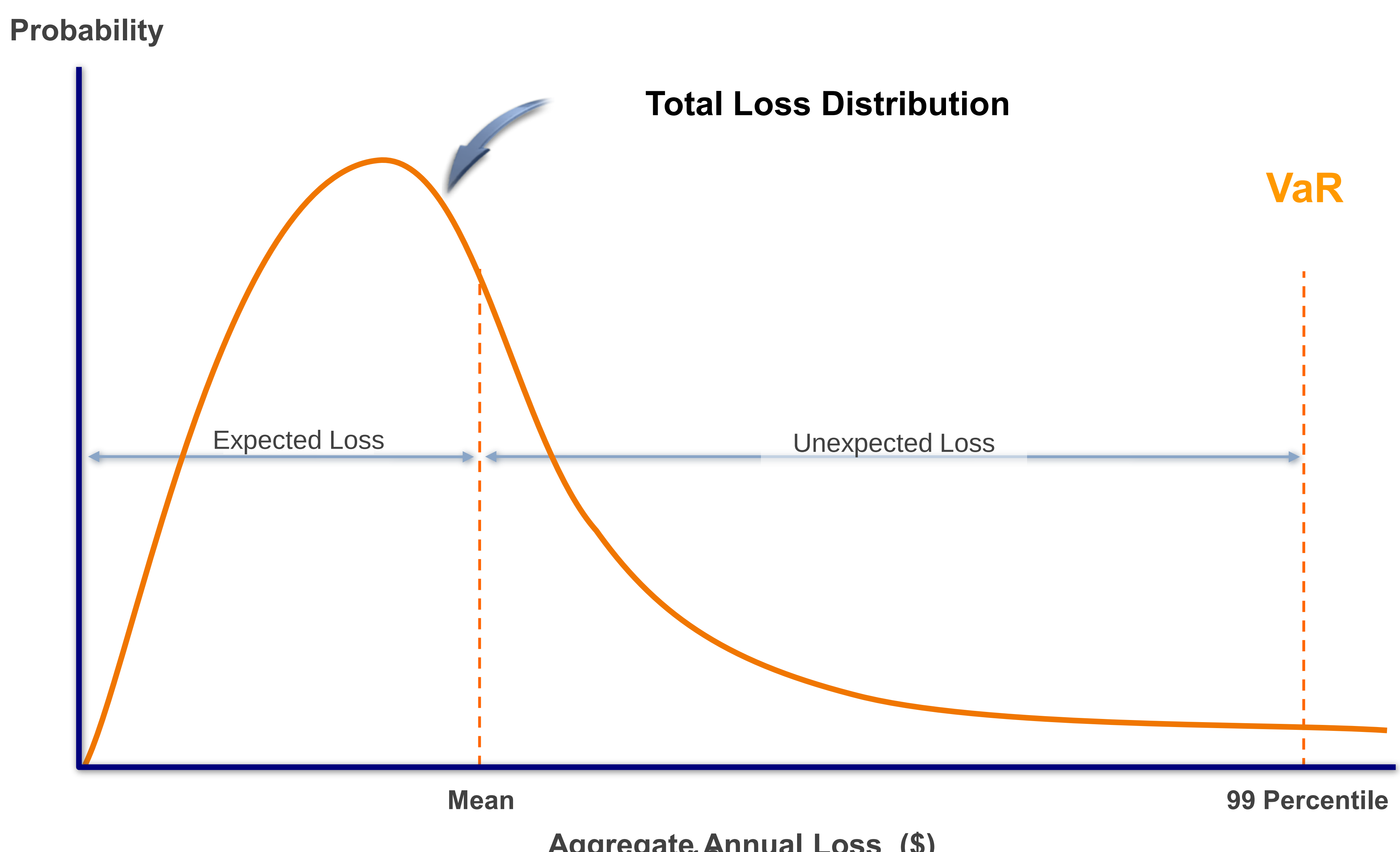
Visualizing VaR

Distribution of change in portfolio's value (1 year horizon)

With 99% probability, you will not lose more than $500,000 by holding the portfolio for a year



Visualizing VaR



VaR Estimation: Main steps

- Identify the variable of interest (Asset value, Portfolio value, Credit losses, Insurance claims, etc.)
- Identify the key risk factors that impact the variable of interest (e.g. assets prices, interest rates, duration, volatility, default probabilities etc.)
- Perform perturbations in the risk factors to calculate the impact in the variable of interest
- KEY QUESTION: How do we perturb/update the risk factors?

VaR Estimation: Main steps

- Three main approaches
  - Delta – Normal or Variance-Covariance Approach
  - Historical Simulation
  - Monte Carlo Simulation

Delta – Normal Approach

- Suppose that the value, V, of an asset is a function of a normally distributed risk factor, RF.
- V(RF) is a non-linear function
- How can we calculate VaR, by taking advantage of the normality assumption?

Delta – Normal Approach

- Value the instrument at the initial value of the risk factor:
  $$V(RF_0)$$
- Calculate Δ, the first derivative of V w.r.t. RF, evaluated at the initial value
  - Modified duration for a fixed-income position
  - Delta for a derivative

Delta – Normal Approach

- This is a linear relationship!
- The worst loss for V is attained for an extreme value of RF
- RF is normal: Use the standard deviation of RF and an α level to calculate VaR of the instrument

Delta – Normal Approach

Taylor Series expansion:

$$dV = \frac{\partial V}{\partial RF} \cdot dRF + \frac{1}{2} \cdot \frac{\partial^2 V}{\partial RF^2} \cdot dRF^2 + \cdots$$

Take only the first term into account, evaluated at RF_0 :

$$dV = \left.\frac{\partial V}{\partial RF}\right|_{RF_0} \cdot dRF \Rightarrow$$

$$dV = D_0 \cdot dRF$$

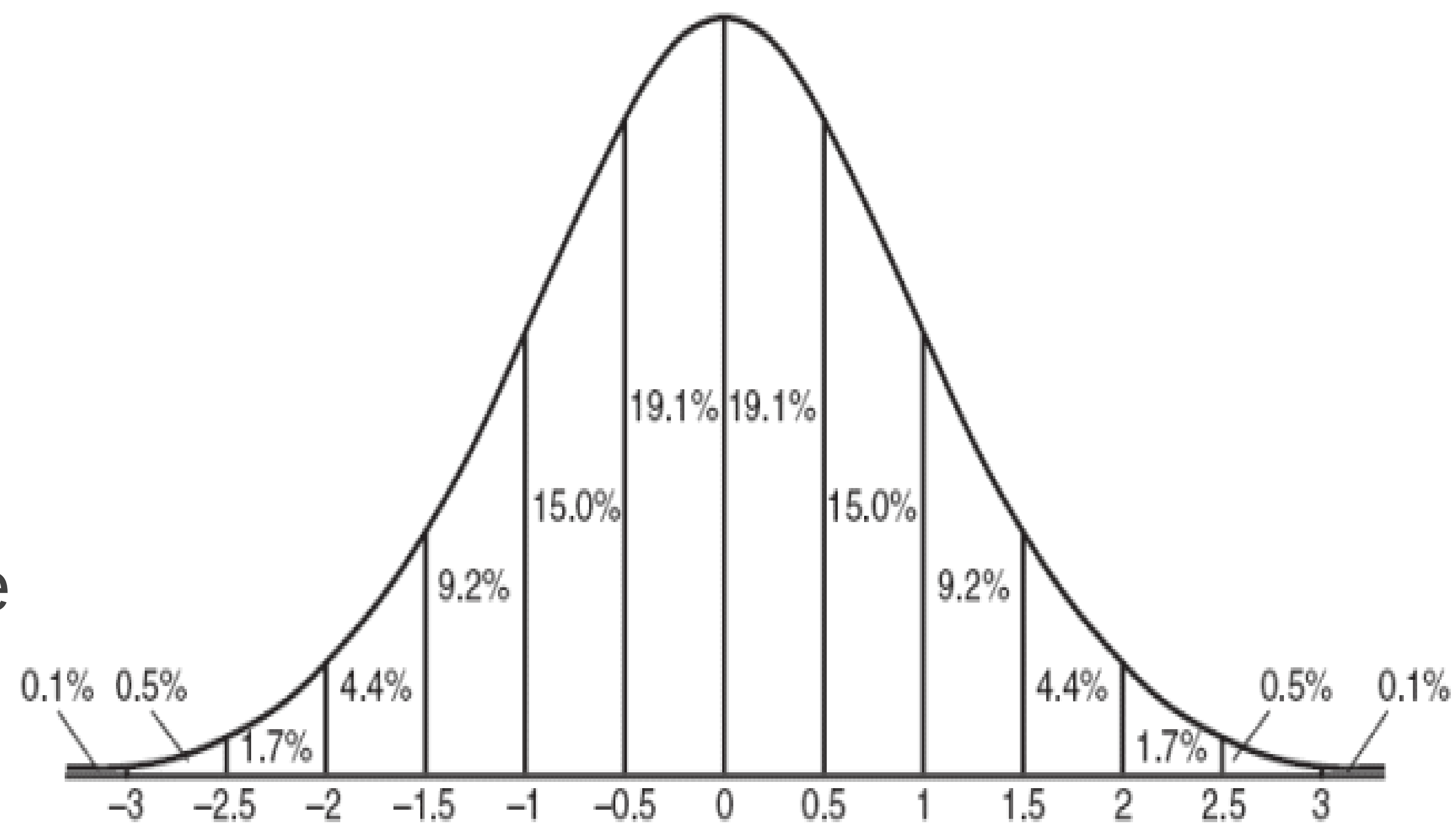
Delta – Normal Approach

- Suppose that the variable of interest is a portfolio consisting of *N* units in a certain stock, S. The price of the stock at time *t* is denoted by S_t.
- Value of portfolio: N x S(t)
- Change in portfolio value = N x $\Delta(S_t)$
- Assuming the price is a random walk: $S_t = S_{t-1} + e_t$, $e_t \sim N(0,\sigma)$
- $\Delta(S_t) = e_t$: NO APPROXIMATION NEEDED IN THIS CASE!!!

Variance-Covariance Approach

- Suppose the variable of interest (e.g. the daily change of a portfolio) is normally distributed.

- In order to calculate any percentile, all we need is the variance (σ^2) of the variable of interest



Variance-Covariance Approach

- “Popular” percentiles used in VaR (left tail)
  - 0.1% (i.e. 99.9% conf. level): -3.09xσ
  - 0.5% (i.e. 99.5% conf. level): -2.58xσ
  - 1.0% (i.e. 99% conf. level): -2.33xσ
  - 5.0% (i.e. 97.5% conf. level): -1.64xσ

Variance-Covariance Approach

- In the case of a portfolio with more than one position, we need
  - The variance of each position
  - The covariance among the positions

Variance Covariance Approach: Single position portfolio

- $100,000 invested in Apple today.
- Daily standard deviation of Apple’s returns: 2.46%.
- Average daily return: 0%
- Assume normally distributed returns.
- What is the daily VaR of your position, at a 99% confidence level?

Variance Covariance Approach: Single position portfolio

- The percentile of the return is -2.33 standard deviations away from the mean (recall mean is 0)

- Thus, the 99% VaR of the position is given by

$$VaR = \$100{,}000 \times (-2.33) \times 2.46\% = -\$5{,}731.8$$

With 99% probability, you expect not to lose more than $5,731.8 by holding the Apple stock for one day.

Variance Covariance Approach: Two positions portfolio

- Your portfolio consists of two positions:
  - $300,000 invested as follows: $200,000 in MSFT and $100,000 in Apple
  - σ_{MSFT} = 1.5% per day
  - σ_{APPLE} = 2.5% per day
  - Correlation of returns = 0.316 = 31.6%
  - Assume normally distributed, zero-mean, daily returns
  - What is the 99% VaR of the portfolio?

Variance Covariance Approach: Example

- We need to find the variance of the portfolio's return.
  - Portfolio's return = (MSFT return)*2/3 + (Apple return)*1/3
  - The variance of the portfolio's return is given by:

$$\sigma^2_{PORT} = \left(\tfrac{2}{3}\right)^2 \cdot \sigma^2_{MSFT} + \left(\tfrac{1}{3}\right)^2 \cdot \sigma^2_{APPLE} + 2 \cdot \left(\tfrac{2}{3}\right) \cdot \left(\tfrac{1}{3}\right) \cdot Cov(APPLE, MSFT) \Rightarrow$$

$$\sigma^2_{PORT} = \left(\tfrac{4}{9}\right) \cdot (1.5\%)^2 + \left(\tfrac{1}{9}\right) \cdot (2.5\%)^2 + \left(\tfrac{4}{9}\right) \cdot 0.316 \cdot 1.5\% \cdot 2.5\% \Rightarrow$$

$$\sigma^2_{PORT} = 0.000222 \Rightarrow$$

$$\sigma_{PORT} = 0.0149 = 1.49\%$$

The 99% VaR is given by

$$VaR = \$300{,}000 \times (-2.33) \times 1.49\% = -\$10{,}415.1$$

Variance Covariance Approach

- Under the assumption of normality, we can get the following relation between 1-day and n-day VaR:
  - $VaR_{N\text{-days}} = \text{sqrt}(N) \times VaR_{1\text{-Day}}$
- In general, the relation between *a* and *b* periods VaR is:
  - $VaR_a = \text{sqrt}(a/b) \times VaR_b$

Historical Simulation

- Non-parametric (distribution free) methodology
- Based solely on historical data
- Main idea: If history suggests that Apple’s daily returns were below -4% only 1% of the times, what you think the VaR at a 99% confidence level should be?

Historical Simulation
One position portfolio

- $100,000 invested in Apple today

- You have 500 observations on Apple's daily returns. You want to compute the daily VaR of your portfolio, at the 99% confidence level.

  - The 99% VaR will be a loss value that will not be exceeded 99% of the time; alternatively, this loss will be exceeded only 1% of the time

    - The 1% of 500 days is 5. We should find a loss observation in our dataset that is exceeded only 5 times.

Historical Simulation (Example)

- Using the 500 observations on daily returns, calculate the portfolios value (\$100,000xR_{APPLE})

- Sort the 500 observations, from worst to best (biggest loss to the biggest profit).

- The 99% VaR will be the 6th observation of your sorted dataset.

Historical Simulation
Two position portfolio

- \$200,000 invested in MSFT today , \$100,000 invested in Apple
- You have 500 daily observations on both returns
- Calculate the portfolio's value using each one of the historical daily returns, i.e. compute: $\$200{,}000 \text{x} R_{MSFT} + \$100{,}000 \text{x} R_{APPLE}$
- Sort the 500 portfolio values from worst to best (biggest loss to biggest profit).
- The 99% VaR will be the 6^{th} observation in your sorted dataset.
- Using 500 observations from 02/19/2006 to 02/19/2008, we get:
- $VaR_{99\%}$ = -\$10,156.71

Historical Simulation: *Key Assumptions*

- The past will repeat itself
- The historical period covered is long enough to get a good representation of “tail” events

Monte Carlo Simulation

- Estimate VaR through the simulation results of statistical/mathematical models.

- Main principle

  - Simulate the value of the portfolio using some statistical/financial model that explains the behavior of the random variables of interest.

Monte Carlo Simulation

- If we have “enough” simulations, …
  - Simulated Distribution of portfolio’s value→
    “True”, unknown, distribution of portfolio’s value
- Use the empirical distribution to find the VaR at any point you wish
  - Can you guess how this step is performed?

Monte Carlo Simulation

- MC is not easy to use, but is able to handle
  - Non-normal models
  - Nonlinear models
  - Multidimensional problems

MC
One position portfolio

- You have a portfolio with 2500 Apple stocks. The current stock price is $124.63. How can we use MC to estimate the 1-day ahead 99% VaR of the portfolio's value?

- What is the variable of interest?

  - Portfolio's Value

- How is the variable of interest computed?

  - Portfolio's Value = $2500 \text{x}(P_{\text{APPLE, 1 DAY AHEAD}})$

MC: Example
One position portfolio

- The "key" is Apple's price, 1 day ahead. How does the price of Apple evolves from one day to the next?

- Use the random walk model

$$p_{t+1} = p_t + \sigma \times Z_{t+1}$$

  - $p_{t+1} = ln(Price_{APPLE,t+1})$
  - Z_{t+1}: ~ i.i.d. Std. Normal
  - σ: *Daily Volatility* (standard deviation of returns)

MC: Example
One position portfolio

- We now have a model for p_{t+1}.

- If we manage to simulate p_{t+1}, we can get Apple's price, since
  $p_{t+1} = \ln(Price_{APPLE,t+1}) \Rightarrow$
  $Price_{APPLE,t+1} = \exp(p_{t+1})$

- Thus, the question becomes:

  - How can I simulate p_{t+1}?

MC: Example
One position portfolio

- Note that $\Delta p_{t+1} = p_{t+1} - p_t = \sigma Z_{t+1}$
  - $\Delta p_{t+1} \sim N(0, \sigma^2)$: We can use any statistical package to simulate it!
  - Once we have simulated $\Delta p_{t,}$ how can I get p_{t+1}?
    - $p_{t+1} = \Delta p_{t+1} + p_t$

MC: Example
One position portfolio

- Let's use 10,000 draws. In each draw:
  - Draw a value from $N(0,\sigma^2)$; treat this as a realization of Δp_{t+1}
  - Use $p_{t+1} = p_t + \Delta p_{t+1}$ to get an estimate of p_{t+1}
  - Get an estimate of Apple's price tomorrow, using $Price_{APPLE,\ t+1} = \exp(p_{t+1})$
  - Get the Portfolio's Value = $2{,}500 \times Price_{t+1}$

MC Example
One position portfolio

- Following the previous steps for each one of the 10,000 draws will give us 10,000 simulated portfolio-change values.

- These values create the empirical distribution of the portfolio's change in value

- Use the empirical distribution to get VaR

  - For example, to get VaR at 99%, sort the observations, from the worst to the best. The VaR will be the 101st observation. Why?

MC Example
Two positions portfolio

- 2500 stocks of Apple ($124.63 each) and 1700 stocks of MSFT ($28.42 each)
- The math are getting trickier…
- The model is now

$$\text{In matrix form: } \mathbf{P_t} = \mathbf{P_{t-1}} + \Phi_{\mathbf{t}},\ \text{i.e.}$$

$$\begin{bmatrix} p_{MSFT,t} \\ p_{APPLE,t} \end{bmatrix} = \begin{bmatrix} p_{MSFT,t-1} \\ p_{APPLE,t-1} \end{bmatrix} + \begin{bmatrix} \Phi_{1,t} \\ \Phi_{2,t} \end{bmatrix}$$

- Φ_t is distributed normally with mean **μ** and variance **Σ**

MC Example: Two Positions Portfolio

- The logic is the same as in previous example.
- We want to simulate, simultaneously, $\Delta p_{MSFT,t+1}$ and $\Delta p_{APPLE,t+1}$
- Let's use 10,000 draws. In each draw:
  - Draw a value from a *bivariate* normal distribution with mean 0 and variance-covariance matrix $\mathbf{\Sigma}$; this will give us one draw for $\mathbf{\Phi_{t+1}}$ or in other words, one draw for the pair $\Delta p_{MSFT,t+1}$ and $\Delta p_{APPLE,t+1}$.
    - Note: Use of the "Cholesky Decomposition of Σ".

MC Example
Two Positions Portfolio

- Once you obtain the 10,000 simulated values for $\Delta p_{MSFT,t+1}$ and $\Delta p_{APPLE,t+1}$, use the same steps as before to get the $Price_{APPLE,\ t+1}$ and $Price_{MSFT,\ t+1}$

- Then,
  Portfolio's value = $2500x(Price_{APPLE,\ t+1})$ $+1700x(Price_{MSFT,\ t+1})$

- This will give us 10,000 values on the portfolio's change, i.e. the empirical distribution of the portfolio's change

- Use the empirical distribution to get VaR

  - To get VaR at 99%, sort the observations, from the worst to the best. The VaR will be the 101st observation. Doing this in SAS, using daily returns from 02/19/2006 to 02/19/2008 gives a 99% VaR value of -$11,077.

  - With a probability of 99%, your will not lose more than $11,077, if you hold the portfolio for one more day.

Note on MC Assumptions

- Due to the normality of Δp_t, you could have also used the variance approach to calculated the VaR (how?)
- However, this example illustrates the MC principal and can be used in many complicated models. For example, can you use the variance-covariance approach if Z_t is a “mixture” of normals? I.e. if 20% of the times $\mathbf{\Phi_t}$ is normal with covariance matrix Σ_1 and 50% of the times it is normal with covariance matrix Σ_2 and 30% of the times is normal with covariance Σ_3?

MC
Key Assumptions

- The model used is an accurate representation of the reality
- The number of draws is enough to capture the tail behavior

Comparing the three approaches

	Covariance	Historical	Monte Carlo
Attractions	Intuitive	Very intuitive and easy to explain	Extremely powerful and flexible
	Easy formula for VaR	Non-parametric (Distribution free)	Handles non-linearity, non-normality, etc.
	Ideal for linear and normally distributed factors	Very easy to implement	Ideal for complex and exotic positions
Limitations	Normality assumption	Problems obtaining data	Hard to explain
	Mot suited for non-linear models	Complete dependence on *particular* dataset	Computer-time intensive
	Covariance matrix might not be well-behaved	Length of estimation	Requires considerable human and financial investment

Suggestion

- For simple linear, normal models → Variance approach
- For advanced models → MC approach

Drawbacks of VaR

- VaR ignores the distribution of a portfolio's return beyond its VaR
- The 99.9% VaR for an investment in stock A is $100K. The 99.9% VaR for an investment in stock B is also $100K.
  Are you indifferent between the two?
- Stock A: The loss can be up to $250K
- Stock B: The loss can be up to $950K
- <u>VaR ignores the magnitude of the worst returns</u>

Drawbacks of VaR

- Under non-normality, VaR may not capture diversification

  VaR fails to satisfy the 'subadditivity property',

  Risk (A + B) ≤ Risk(A) + Risk(B)

  That is, the VaR of a portfolio with two securities may be larger than the sum of the VaRs of the securities in the portfolio.

An Alternative to VaR: CVaR

- The Conditional Value at Risk (CVaR) or Expected Shortfall (ES) is a measure that doesn’t have the two drawbacks of VaR diescussed before.

Definition of CVaR

- Given a confidence level and a time horizon, a portfolio's CVaR is the *expected loss* one suffers given that a ‘bad’ event occurs.

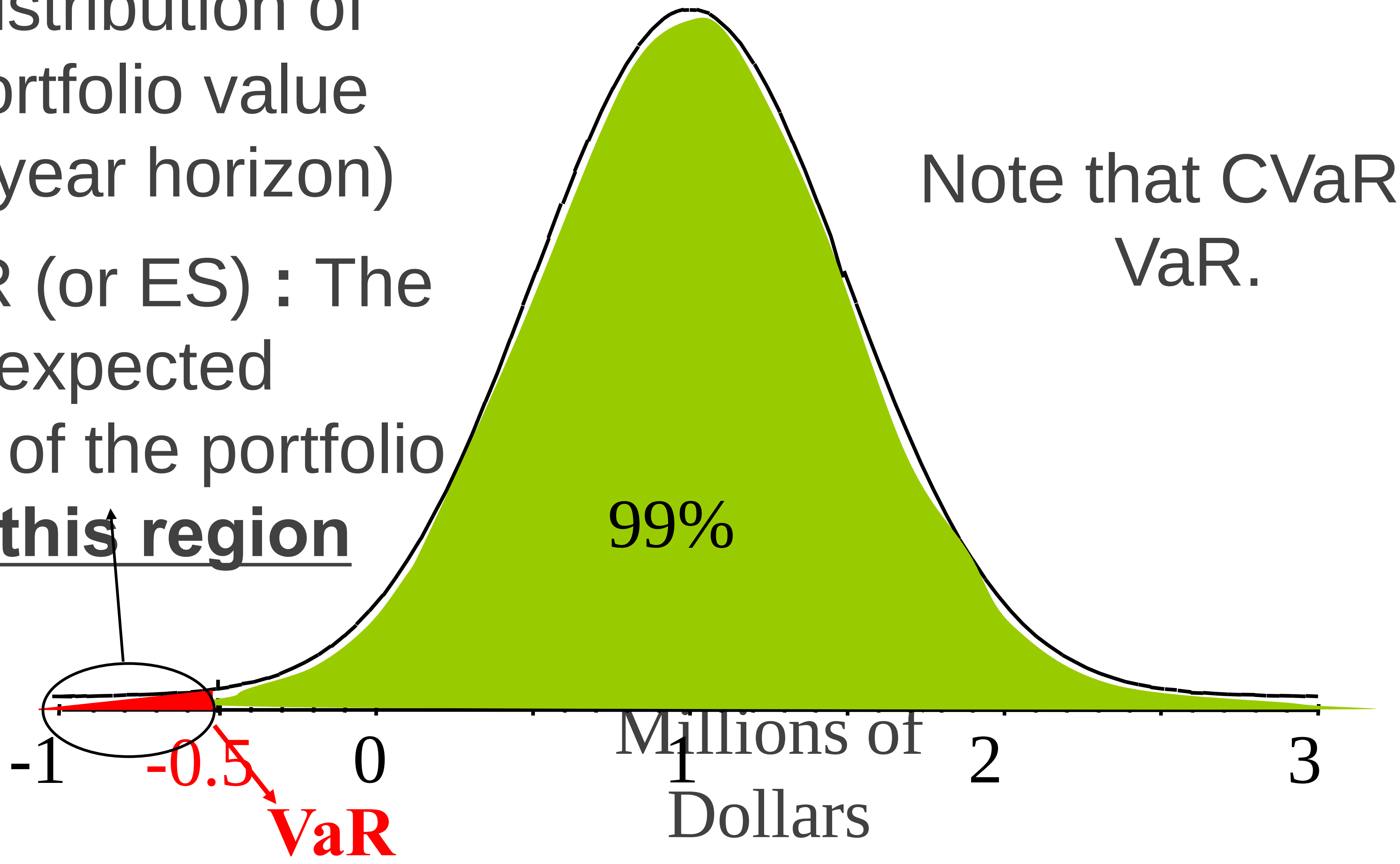
- In other words, the CVaR is a conditional expectation. If my loss exceeds the VaR level, what I should expect it to be equal to?

Visualizing CVaR

Distribution of portfolio value (1 year horizon)

CVaR (or ES) : The expected value of the portfolio value of the portfolio **in this region**

Note that CVaR > VaR.



CVaR Estimation
Variance-Covariance Approach

- In the case of the variance-covariance approach, the CVaR can be calculated as follows:

$$CVaR = \frac{\exp(-\frac{q_a^2}{2})}{a\sqrt{2\pi}} \times \sigma$$

where α is the percentile we are working on (e.g. 1%), q_α is the tail 100α percentile of a standard normal distribution (e.g. -2.33) and σ is the standard deviation.

CVaR Estimation
Historical Approach

- Suppose you have 1000 observations for the daily return on Apple.
  - Recall that 1% times 1000 is 50.
- In order to find Apple's CVaR at the 99% confidence level, you need to:
  - Sort the data from worst to best
  - Recall that the 51st worst value is the VaR
  - The CVaR is simply the average of the first 50 values in you sorted dataset (i.e the average of the values that are worst than VaR)

CVaR Estimation
MC Approach

- Follow the steps described earlier to create the 10,000 simulated, sorted, portfolio's values.

- Take the average value of the first 100 observations (i.e. the average of all values that are worst than the 99% VaR)

  - This is the 99% CVaR

Recent developments

- CVaR has one main drawback…
  - What you think is CVaR’s estimation error? Bigger or smaller than VaR’s?
- This problem can be overcome using Extreme Value Theory

Questions-Comments


THANK
YOU