NON-STATIONARY MODELS

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INTRODUCTION TO NON-STATIONARITY

Objectives

- Discuss signs of non-stationarity, such as trending and seasonality.
- Discuss possible solutions to different forms of non-stationarity.

Stationarity

- A stationary time series has a constant mean and variance.
- A time series with *long-term* trend or seasonal components cannot be stationary because the mean of the series depends on the time that the value is observed.
  - Population gradually increasing over time.
  - Average sales in December are always higher than average sales in March.

Accounting for Non-stationarity

- Non-stationarity is typically handled with the same approach as most analytical problems – make it look stationary, then solve it!
- How do we typically "account for" non-stationarity?
  - Trend
    - Linear Regression → Deterministic Trend
    - Differencing → Stochastic Trend
  - Seasonality?
    - Linear Regression → Residuals are Stationary
    - Differencing → Stochastic Trend on Seasons

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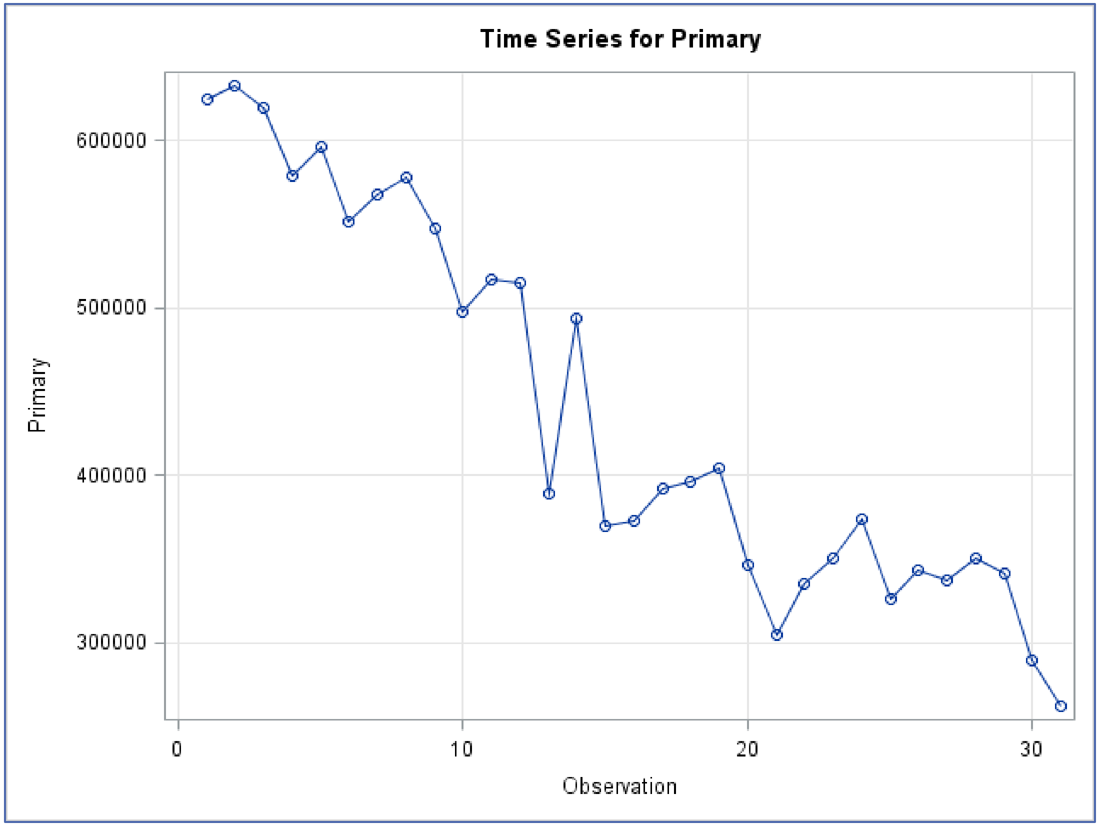
DETERMINISTIC TREND

Objectives

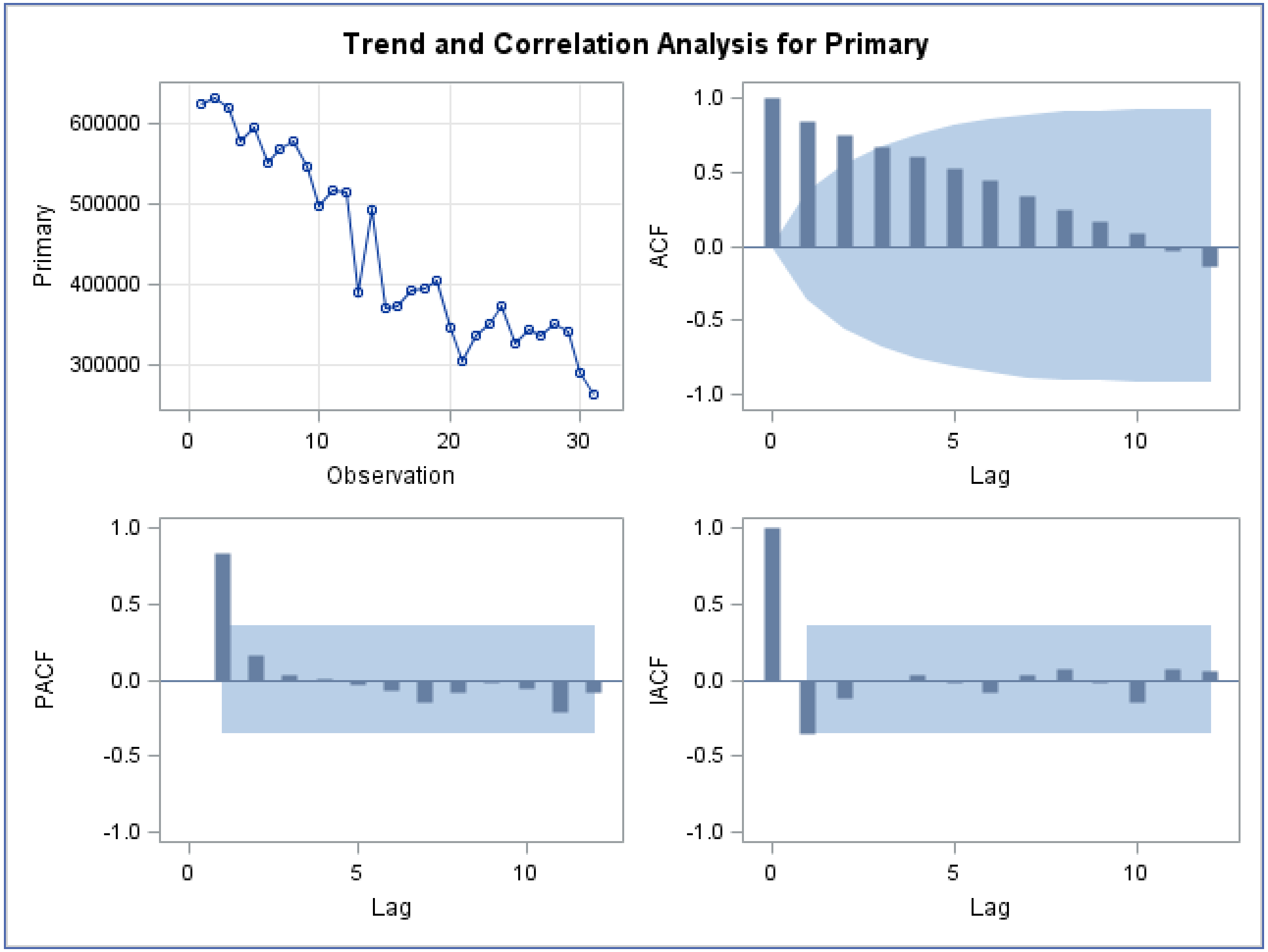
- Discuss the differences between deterministic and stochastic trends.
- Detail the deterministic trend model.
- Discuss the topic of applying time series to residuals from a trend model.

Annual Lead Production

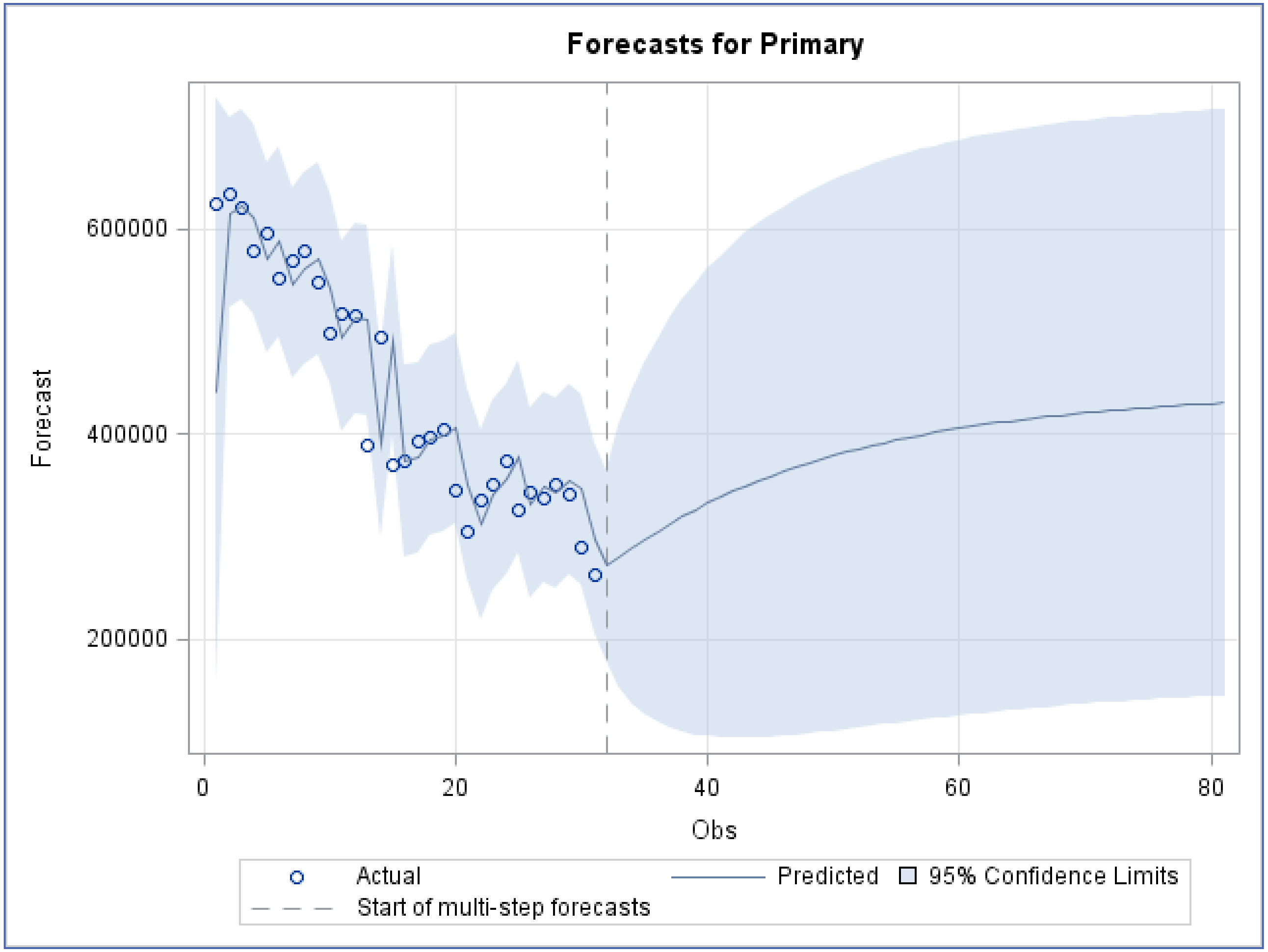
- A series of data may be trending over time.
- Trending series are not stationary because they do not hover around a mean.



Annual Lead Production



Annual Lead Production



Two Types of Trend

- Deterministic:
  - Mathematical Function of Time – linear, quadratic, logarithmic, exponential, etc.
  - Mathematical Function of Other Variables – regression with time series residuals.
- Stochastic:
  - Future time values depend on past values plus error.
  - A common stochastic trend model is a random walk with drift.

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Linear Trend Model

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$$Y_t = \beta_0 + \beta_1 t + Z_t$$

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Intercept

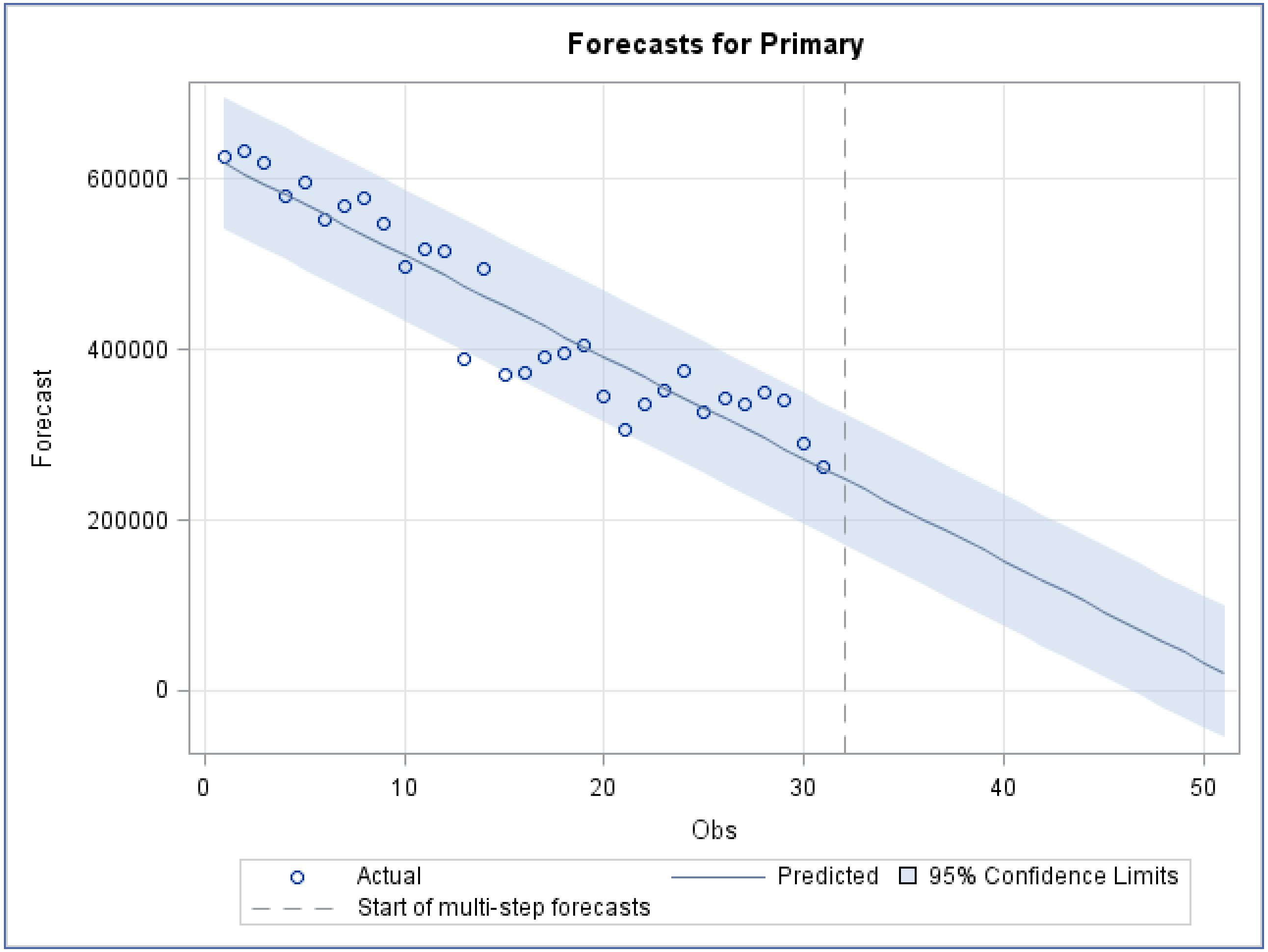
Residuals

Slope

Linear Trend

```
proc arima data=Time.Leadyear plot=all;
     identify var=Primary nlag=12 crosscorr=Time;
     estimate Input=Time;
run;
quit;
```

Linear Trend Model – Forecast



Linear Trend Model – Residuals

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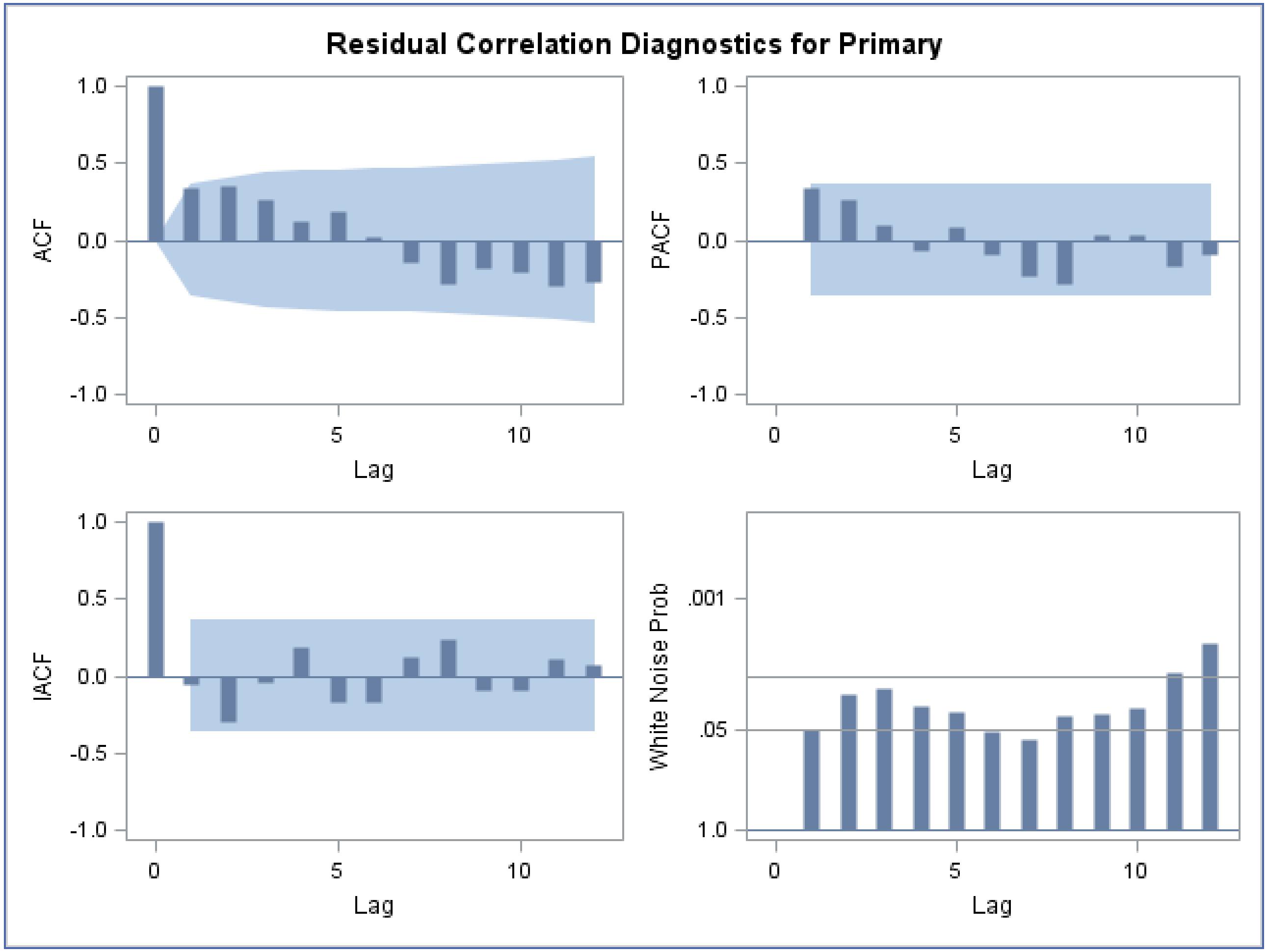
$$Y_t = \beta_0 + \beta_1 t + Z_t$$

Residuals – what is left after accounting for the trend!

Linear Trend Model – Residuals



Linear Trend Model – Residuals

- The linear trend model is rather straight-forward:

$$Y_t = \beta_0 + \beta_1 t + Z_t$$

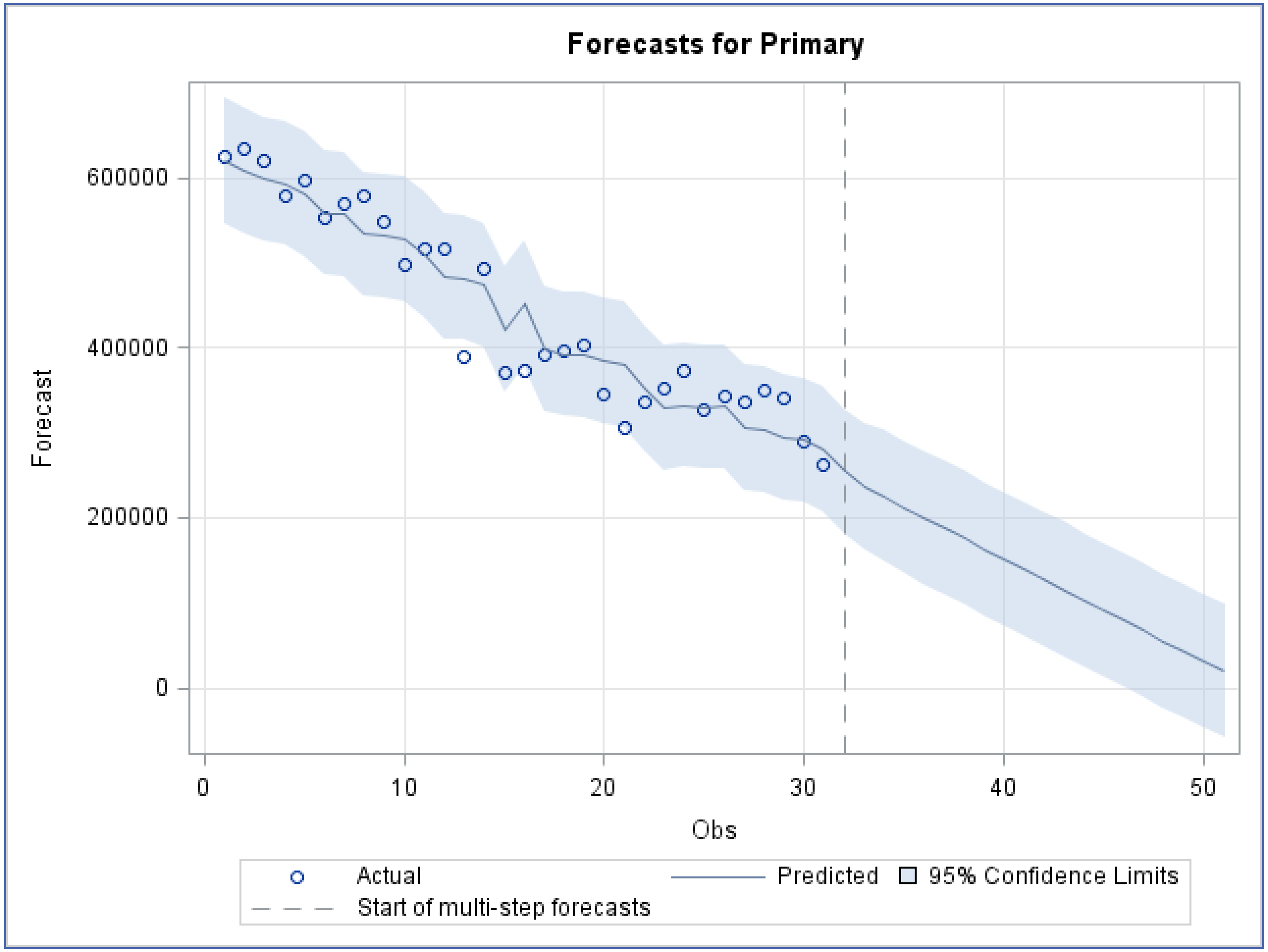
- The residuals can have a time series pattern to them.
- For example, the residuals may have an AR(1) pattern:

$$Z_t = \phi_1 Z_{t-1} + e_t$$

Linear Trend + Residual Pattern

```
proc arima data=Time.Leadyear plot=all;
     identify var=Primary nlag=12 crosscorr=Time;
     estimate Input=Time p=2;
run;
quit;
```

Linear Trend + Residual Pattern



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- In these models, the process reverts to the **linear trend** NOT THE MEAN!

Common Trend Models

- We are not limited to only having a linear trend:
  - Quadratic Trend:

$$Y_t = \beta_0 + \beta_1 t + \beta_2 t^2 + Z_t$$

  - Logarithmic Trend:

$$Y_t = \beta_0 + \beta_1 \log(t) + Z_t$$

  - Exponential Trend:

$$Y_t = \exp(\beta_0 + \beta_1 t) + Z_t \rightarrow \log(Y_t) = \beta_0 + \beta_1 t$$

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Linear Regression Model

- The linear regression model with time series residuals is also rather straight-forward:

$$Y_t = \beta_0 + \beta_1 X_1 + \cdots + \beta_k X_k + Z_t$$

Linear Regression Model

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$$Y_t = \beta_0 + \underbrace{\beta_1 X_1 + \cdots + \beta_k X_k}_{\text{Deterministic Portion}} + Z_t$$

Intercept

Deterministic Portion

Residuals

Linear Regression Model

- The linear regression model with time series residuals is also rather straight-forward:

$$Y_t = \beta_0 + \beta_1 X_1 + \cdots + \beta_k X_k + Z_t$$

Residuals – what is left after accounting for the trend!

Linear Regression Model – Residuals

- The linear regression model with time series residuals is also rather straight-forward:

$$Y_t = \beta_0 + \beta_1 X_1 + \cdots + \beta_k X_k + Z_t$$

- The residuals can have a time series pattern to them.
- For example, the residuals may have an AR(1) pattern:

$$Z_t = \phi_1 Z_{t-1} + e_t$$

- In these models, the process reverts to the **linear regression model** NOT THE MEAN!

How to Model?

- There are 2 different ways to model time series residuals in a regression model:
  1. PROC ARIMA only
     - Use the CROSSCORR option with a list of inputs.

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- There are 2 different ways to model time series residuals in a regression model:
  1. PROC ARIMA only
     - Use the CROSSCORR option with a list of inputs.
  2. Combination of PROC GLM and PROC ARIMA
     - Run a regression model in PROC GLM (could use PROC REG, but GLM has CLASS statement).
     - Output the residuals into another data set.
     - Model the residuals in PROC ARIMA.
     - Combine the forecasts from both.

STOCHASTIC TREND

Objectives

- Discuss the stochastic trend model.
- Review random walk models.
- Introduce the topic of differencing and the ARIMA model.

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Two Types of Trend

- Deterministic:
  - Mathematical Function of Time – linear, quadratic, logarithmic, exponential, etc.
- Stochastic:
  - Future time values depend on past values plus error.
  - A common stochastic trend model is a random walk with drift:

$$Y_t = \omega + Y_{t-1} + e_t$$

Random Walk with Drift

- The random walk with drift model has the same properties of a trending series:

$$Y_t = \omega + Y_{t-1} + e_t$$

$$\begin{aligned} Y_t &= \omega + (\omega + Y_{t-2} + e_{t-1}) + e_t \\ &= 2\omega + Y_{t-2} + e_{t-1} + e_t \end{aligned}$$

$$Y_t = 3\omega + Y_{t-3} + e_{t-2} + e_{t-1} + e_t$$

$$\vdots$$

$$Y_t = \omega t + Y_0 + \sum_{i=1}^{t} e_i$$

Random Walk with Drift

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$$Y_t = \omega + Y_{t-1} + e_t$$

$$Y_t = \omega + (\omega + Y_{t-2} + e_{t-1}) + e_t$$
$$= 2\omega + Y_{t-2} + e_{t-1} + e_t$$

$$Y_t = 3\omega + Y_{t-3} + e_{t-2} + e_{t-1} + e_t$$

$$\vdots$$

$$Y_t = \omega t + Y_0 + \sum_{i=1}^{t} e_i$$

Stochastic Trend: Differencing

- Random Walk with Drift:

$$Y_t = \omega + Y_{t-1} + e_t$$

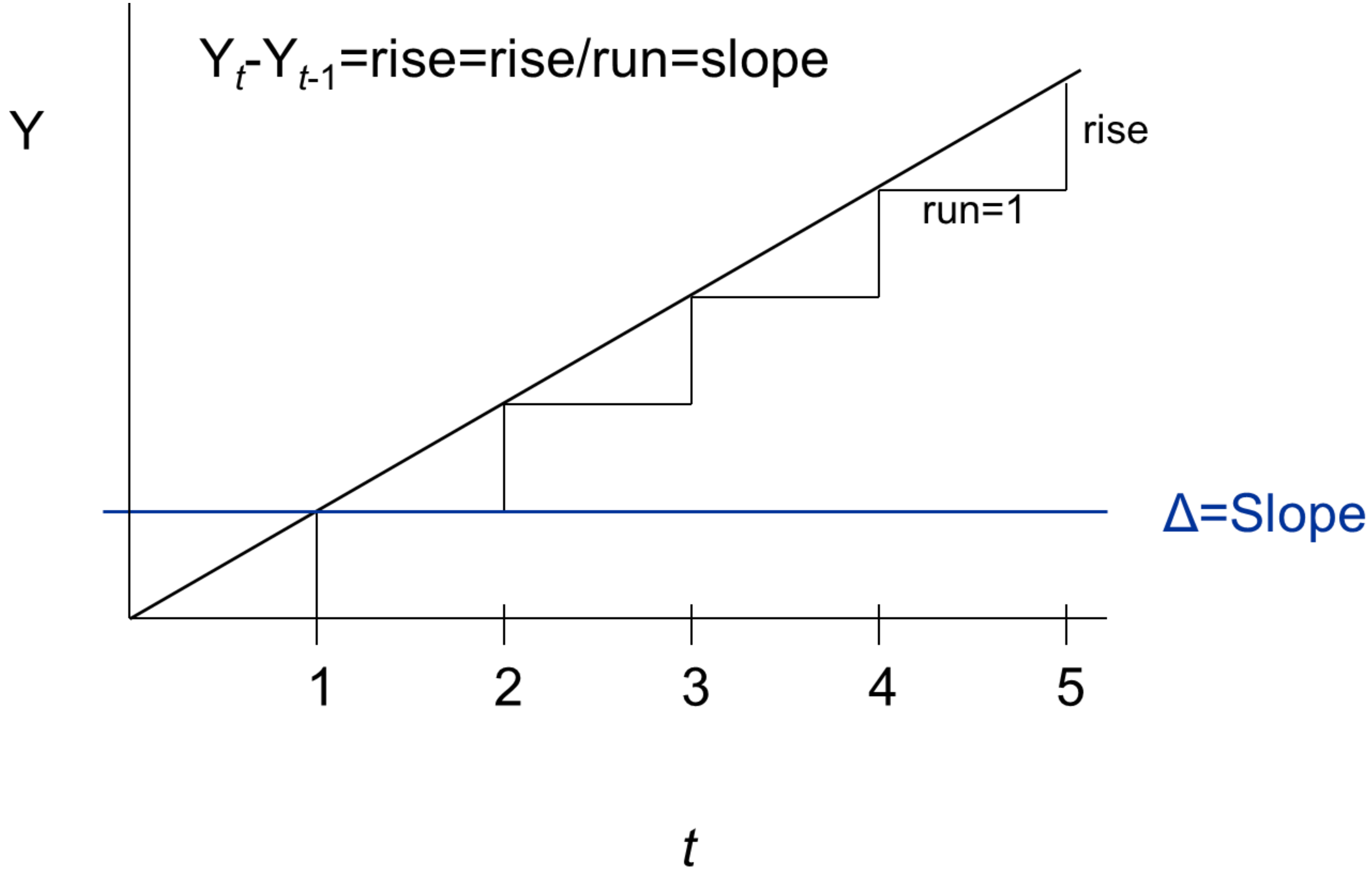
Stochastic Trend: Differencing

- Random Walk with Drift:

$$Y_t = \omega + Y_{t-1} + e_t$$

$$Y_t - Y_{t-1} = \omega + e_t$$

Stochastic Trend: Differencing



Stochastic Trend: Differencing

- Random Walk with Drift:

$$Y_t = \omega + Y_{t-1} + e_t$$

- General Model with Stochastic Trend:

$$Y_t - Y_{t-1} = Z_t$$

Patterns may exist in the differences!

ARIMA Models

- Models were differences are being modeled instead of the original series are called **autoregressive integrated moving average models** – ARIMA models.
- ARIMA(p, d, q) models have an autoregressive order of *p,* a moving average of order *q*, and a difference of order *d*.
- For example, an ARIMA(1,1,1) model is using an ARMA(1,1) model to model the first differences:

$$Y_t - Y_{t-1} = Z_t$$

$$Z_t = \phi Z_{t-1} + e_t - \theta e_{t-1}$$

Stochastic Trend: Differencing

```
proc arima data=Time.Leadyear plot=all;
     identify var=Primary(1) nlag=12;
run;
quit;
```

Stochastic Seasonal Functions

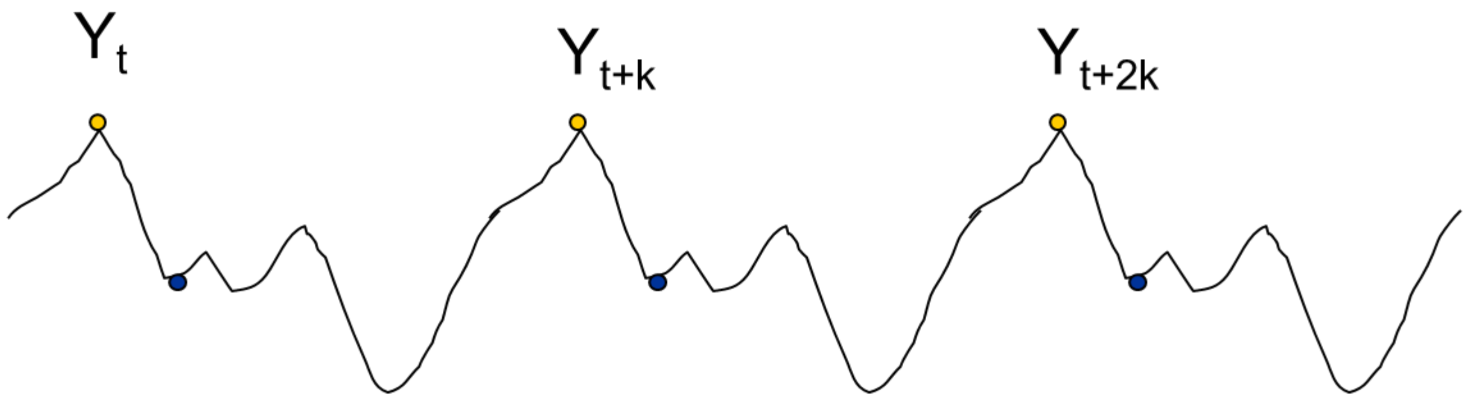
- For seasonal data with period *S*, express the current value as a function that includes the value *S* time units in the past.

$$Y_t = Y_{t-S} + \cdots$$

$$Y_t - Y_{t-S} = Z_t \longrightarrow \text{Difference of order } S$$

- Examples:
  - Monthly → January is a function of last January
  - Daily → Sunday is a function of last Sunday

Seasonal Differencing



Stochastic Trend: Seasonal Differencing

```
proc arima data=Time.USAirlines plot=all;
     identify var=Passengers nlag=40;
     identify var=Passengers(12) nlag=40;
     identify var=Passengers(1 12) nlag=40;
run;
quit;
```

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UNIT ROOT TESTING

Objectives

- Introduce the Dickey-Fuller and Augmented Dickey-Fuller test for unit roots.
- Discuss the implications of over-differencing.

The Dickey-Fuller Unit Root Test

- This test provides a statistical test for first differencing.
- The null hypothesis is that first differencing is required (non-stationary data).
- The alternative hypothesis has 3 forms:
  1. Zero Mean
  2. Single Mean
  3. Trend

The Dickey-Fuller Test – Zero Mean

- Model:

$$Y_t = \phi Y_{t-1} + e_t$$

- Null Hypothesis:

$$H_0: \phi = 1$$

- Alternative Hypothesis:

$$H_a: |\phi| < 1$$

The Dickey-Fuller Test – Zero Mean

- Model:

$$Y_t = \phi Y_{t-1} + e_t$$

- Null Hypothesis:

$$H_0: \phi = 1 \longleftarrow \text{Non-stationary!}$$

- Alternative Hypothesis:

$$H_a: |\phi| < 1 \longleftarrow \text{Stationary!}$$

The Dickey-Fuller Test – Single Mean

- Model:

$$Y_t - \mu = \phi(Y_{t-1} - \mu) + e_t$$

- Null Hypothesis:

$$H_0 : \phi = 1$$

- Alternative Hypothesis:

$$H_a : |\phi| < 1$$

The Dickey-Fuller Test – Single Mean

- Model:

$$Y_t - \mu = \phi(Y_{t-1} - \mu) + e_t$$

- Null Hypothesis:

$$H_0\colon \phi = 1$$ ← Non-stationary!

- Alternative Hypothesis:

$$H_a\colon |\phi| < 1$$ ← Stationary!

The Dickey-Fuller Test – Trend

- Model:

$$Y_t - \beta_0 - \beta_1 t = \phi(Y_{t-1} - \beta_0 - \beta_1(t-1)) + e_t$$

- Null Hypothesis:

$$H_0: \phi = 1$$

- Alternative Hypothesis:

$$H_a: |\phi| < 1$$

The Dickey-Fuller Test – Trend

- Model:

$$Y_t - \beta_0 - \beta_1 t = \phi(Y_{t-1} - \beta_0 - \beta_1(t-1)) + e_t$$

- Null Hypothesis:

$$H_0: \phi = 1 \longleftarrow \text{Non-stationary!}$$

- Alternative Hypothesis:

$$H_a: |\phi| < 1 \longleftarrow \text{Stationary!}$$

Augmented Dickey-Fuller (ADF) Test

- Unit roots are not limited to only AR(1) models that are random walks.
- Unit roots can exist for any AR(p) model.
- Higher order models are tested with the ADF tests.
- Lag 0 tests are equivalent to what we have previously seen.
- Lag 1 tests consider an AR(2) model.
- Lag 2 tests consider an AR(3) model and so on.

Augmented Dickey-Fuller (ADF) Test

- Characteristic polynomial of an AR(p) model:

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p$$

- Null Hypothesis:

$$H_0\text{: polynomial has root} = 1$$

- Alternative Hypothesis:

$$H_a\text{: polynomial is for stationary process}$$

Augmented Dickey-Fuller (ADF) Test

- The Rho test is the regression coefficient-based test statistic.
  - Superior power properties for lag 1 tests.
- The Tau test is the studentized test.
  - Superior power properties for all lags but 1.
- The F test is the regression F test for the full model and the null hypothesis restricted reduced model, except that the distribution is not the usual F distribution used in ordinary regression.
  - Poorest power properties – seldom recommended.

Augmented Dickey-Fuller Testing

```
proc arima data=Time.Ebay9899 plot=all;
     identify var=DailyHigh nlag=10 stationarity=(adf=2);
     identify var=DailyHigh(1) nlag=10 stationarity=(adf=2);
run;
quit;
```

Seasonal ADF Test

- The Augmented Dickey-Fuller test can be extended to check for seasonal lags as well.
- The tests will be differenced on specified seasonal lengths instead of single differences.
- Tests only able to be checked for seasons up to length 12.

Seasonal ADF Testing

```
proc arima data=Time.USA_TX_NOAA plot=all;
    identify var=Temperature nlag=60
             stationarity=(adf=2 dlag=12);
    identify var=Temperature(12) stationarity=(adf=2);
run;
quit;
```

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