Two-player
Zero-Sum
Games

Game 1

$$\begin{array}{c|cc|c} & 1 & 2 & \\ \hline 1 & 3 & -1 & p_1 \\ 2 & -2 & 1 & p_2 \\ \hline & q_1 & q_2 & \end{array}$$

Pure strategy = a single row/col
Mixed strategy = a distribution over pure strats

1. Row player announces mixed strat $p = (p_1, p_2)$
2. Col player responds w/ mixed strat $q = (q_1, q_2)$

Payoff of col 1: $3p_1 - 2p_2$
Payoff of col 2: $-p_1 + p_2$

Col player's best strat = $\min \{ 3p_1 - 2p_2 , -p_1 + p_2 \}$

Row " " " = $\underset{\text{mixed strat } q}{\text{maximize}} \{ \min \{ 3p_1 - 2p_2 , -p_1 + p_2 \} \}$

Fact: Can calculate $\underset{\text{mixed strat } p}{\text{maximize}} \left\{ \min \left\{ 3 \cdot p_1 - 2 \cdot p_1 ,\ -1 \cdot p_1 + 1 \cdot p_2 \right\} \right\}$ with LP.

Pf:

$$
\begin{aligned}
\text{maximize} \quad & z \\
\text{subject to} \quad & z \le 3p_1 - 2p_2 \\
& z \le -p_1 + p_2 \\
& p_1 + p_2 = 1 \\
& p_1 \geq 0,\ p_2 \geq 0.
\end{aligned}
\qquad = LP_1
$$

Note: $z = \min\{3p_1 - 2p_2, -p_1 + p_2\}$. $\square$

Game 2

Same as Game 1, except col player goes 1st
and row player goes 2nd

	1	2	
1	3	-1	p_1
2	-2	1	p_2
	q_1	q_2	

Payoff of row 1: $3 \cdot q_1 - 1 \cdot q_2$

Payoff of row 2: $-2 \cdot q_1 + 1 \cdot q_2$

Row player's best strat = $\max\{3q_1 - q_2, -2q_1 + q_2\}$

Col " " " = $\underset{\text{mixed strat } q}{\text{minimize}} \{ \max \{ 3q_1 - q_2, -2q_1 + q_2 \} \}$

LP_2 = minimize z,

subject to $3q_1 - q_2 \leq z$

$-2q_1 + q_2 \leq z$

$q_1 + q_2 = 1, \quad q_1 \geq 0, q_2 \geq 0.$

Game 1

1. Row player first
2. Col player second

	1	2
1	3	-1
2	-2	1

Game 2

1. Col player first
2. Row player second

$$\max_{p} \left\{ \min_{q} \left\{ \text{Score}(p,q) \right\} \right\} \leq \min_{q} \left\{ \max_{p} \left\{ \text{Score}(p,q) \right\} \right\}$$

$$\begin{array}{ccc} \| & & \| \\ LP_1 & \xleftrightarrow{\text{(dual)}} & LP_2 \end{array}$$

$\therefore$ Strong duality $\Rightarrow$ $LP_1 = LP_2 = \text{Value(Game)}$ (Definition)

(Min-Max Theorem)

$\Rightarrow$ order of play doesn't change value

$\Rightarrow$ $\exists$ optimal strat for ROW, irrespective of COL