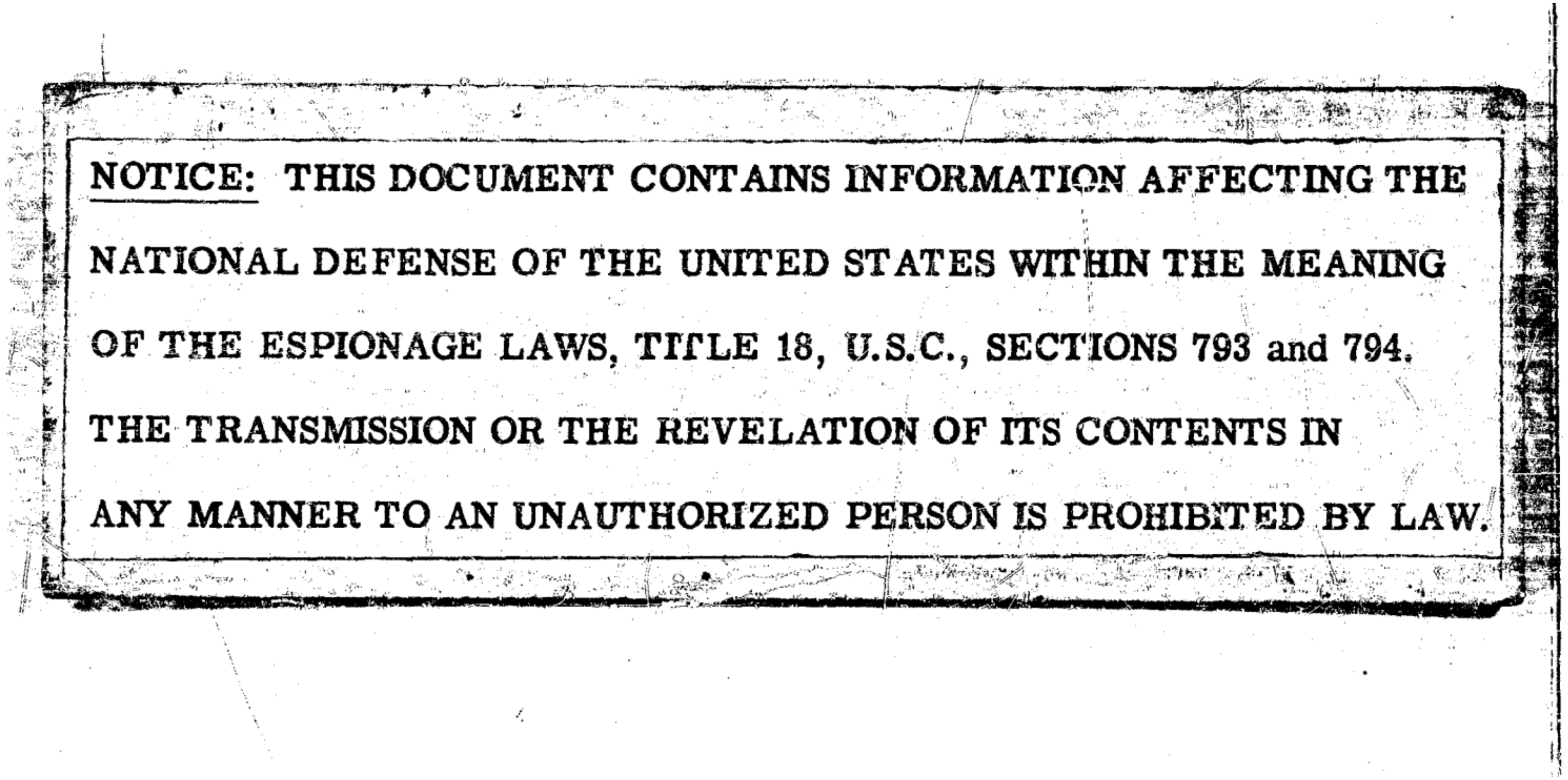
# Lecture 16

## Maximum flow



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# Admin corner

- Professor Haghtalab is gone forever

- My OHes: **Tuesdays 3-4pm**
- I'm going to do an experiment: **slides** or **no slides**?

## Midterm:

- Midterm 1 regrade requests will close **Monday 10/23**

## Homeworks:

- Homework 8 has been released and will be due next **Wednesday 10/25**

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U. S. AIR FORCE

# PROJECT RAND

## RESEARCH MEMORANDUM

# FUNDAMENTALS OF A METHOD FOR EVALUATING RAIL NET CAPACITIES (U)

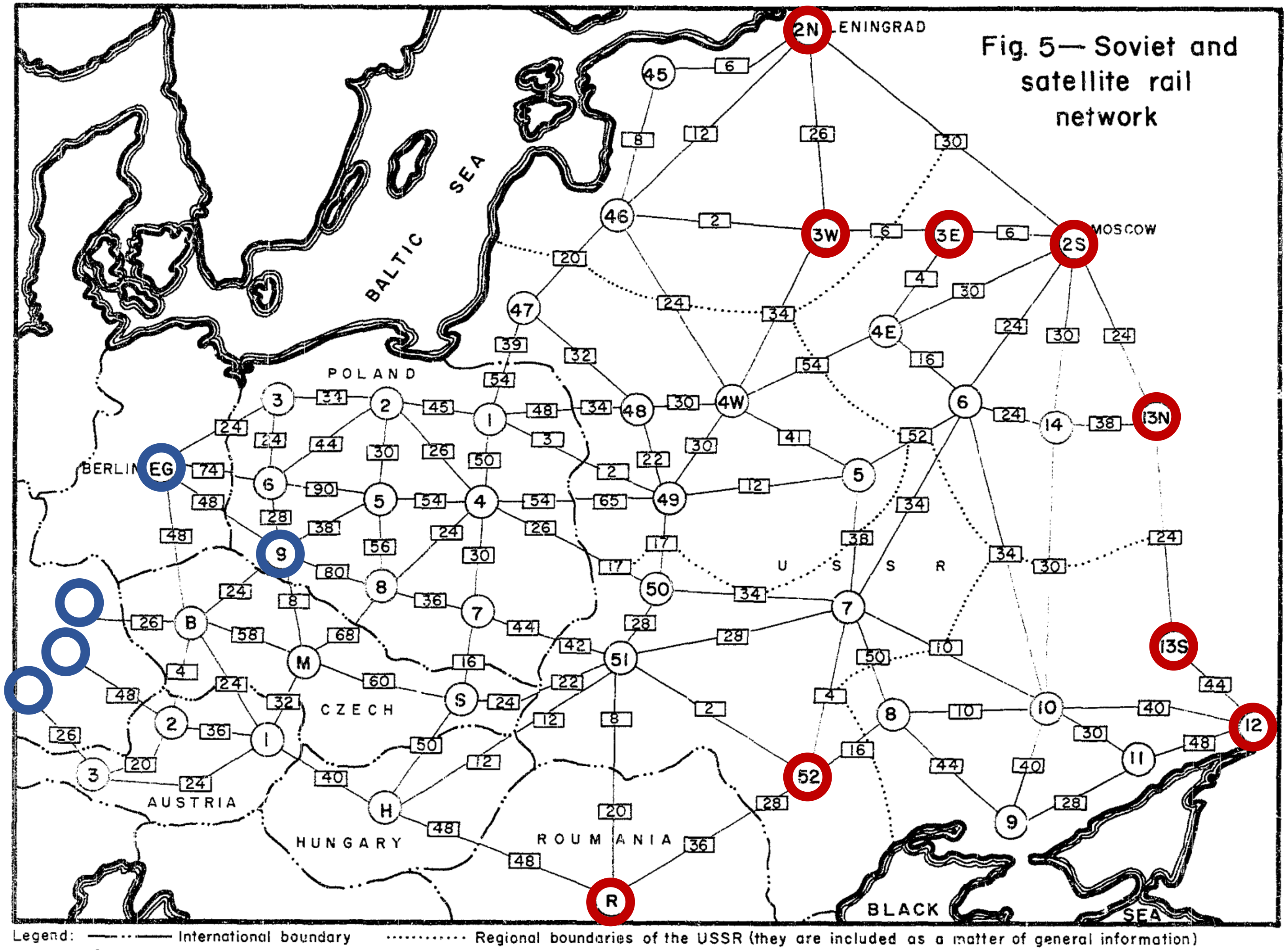
T. E. Harris
F. S. Ross

RM-1573

October 24, 1955

Copy No. 37

SECRET

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30

= a capacity of 30,000 tons

: source

: destination

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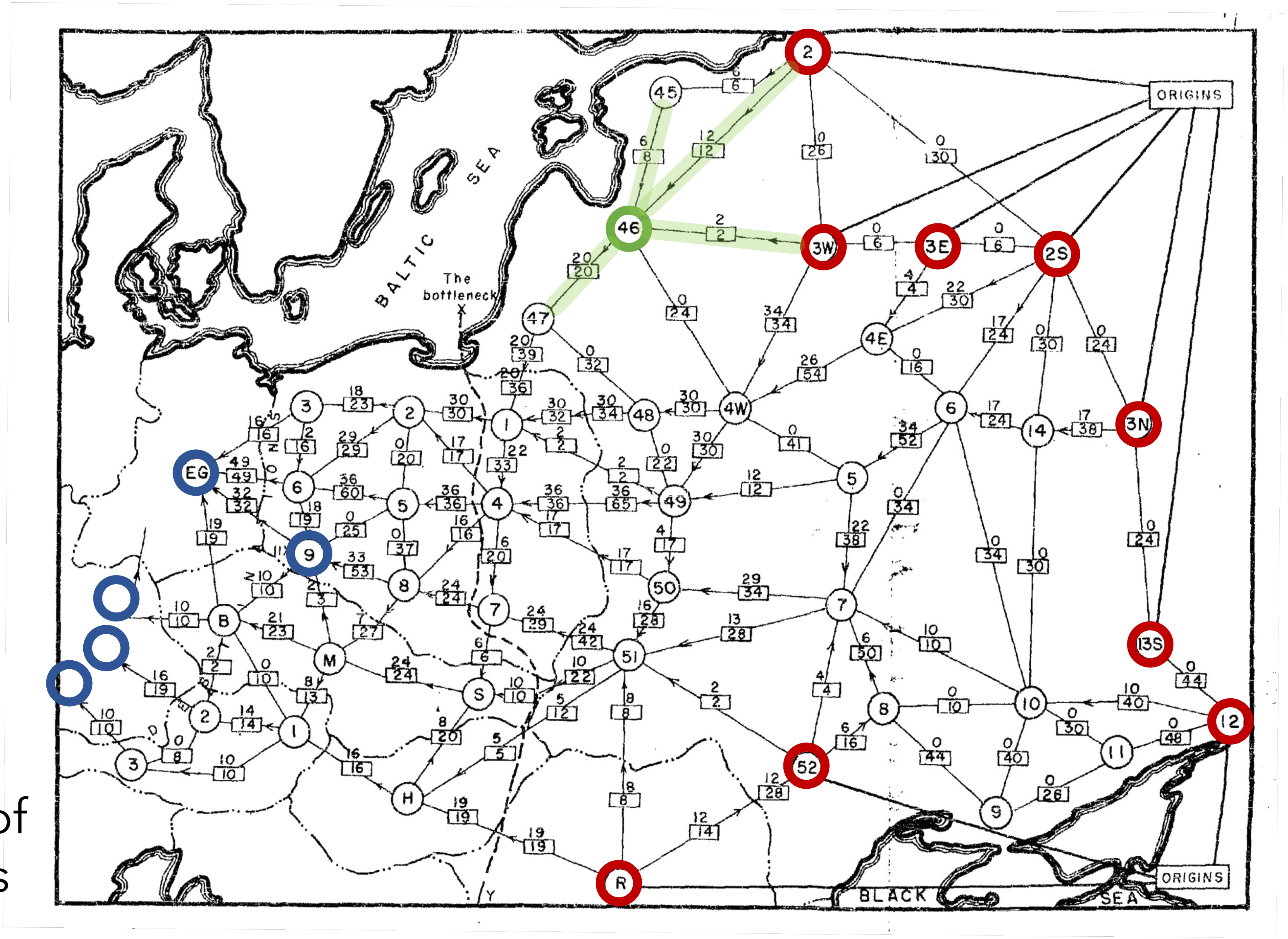


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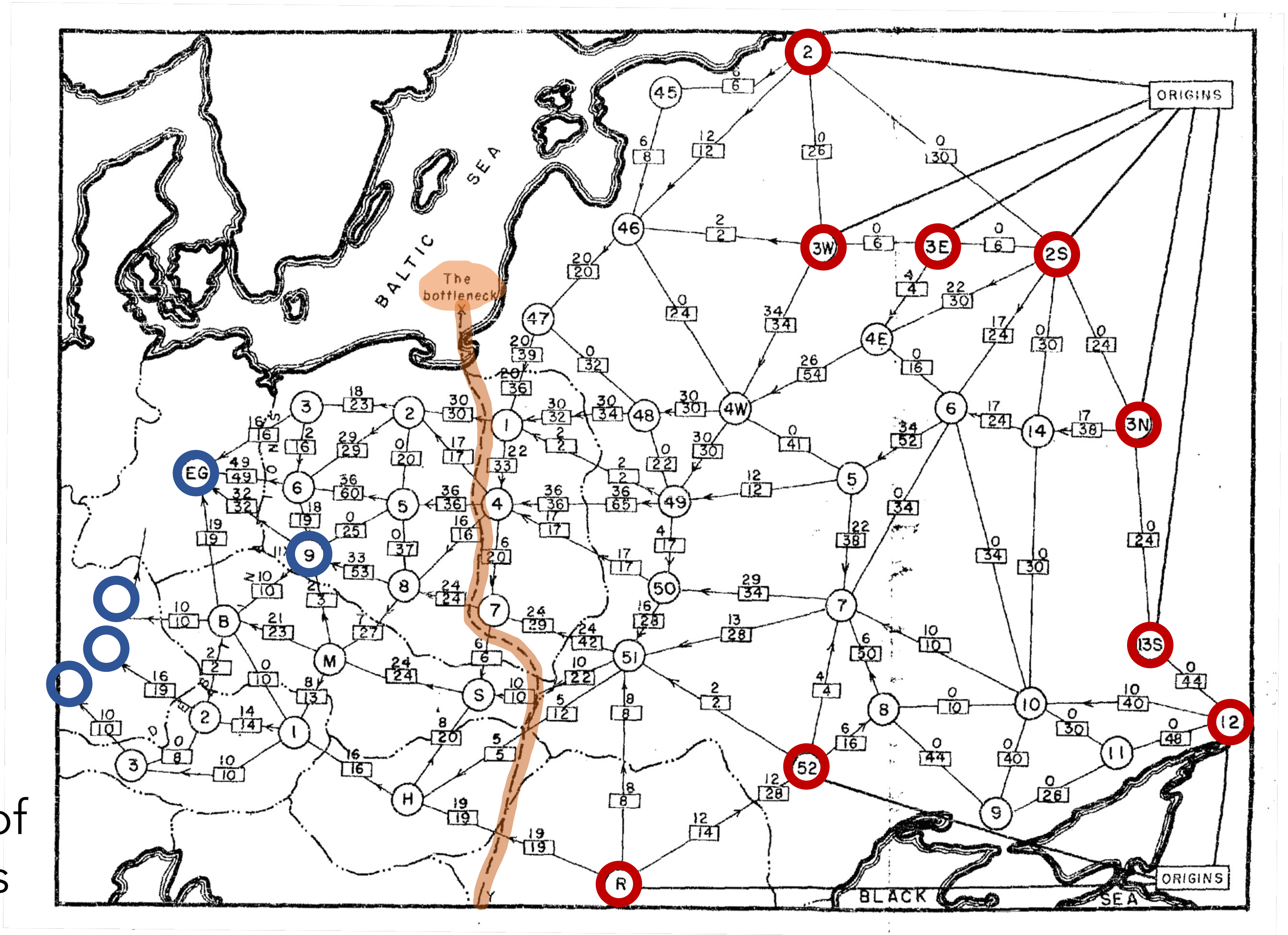


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30

= a capacity of 30,000 tons

: source

: destination

Total East → West capacity: **163,000 tons**

Optimal due to **the bottleneck**

Harris and Ross solved this problem
using a **greedy** algorithm they called "flooding"

But flooding would sometimes output **incorrect** solutions

So they approached their colleagues Ford and Fulkerson,
who devised an alg known as the **Ford-Fulkerson algorithm**

**MAXIMAL FLOW THROUGH A NETWORK**

L. R. FORD, JR. AND D. R. FULKERSON

**Introduction.** The problem discussed in this paper was formulated by T. Harris as follows:

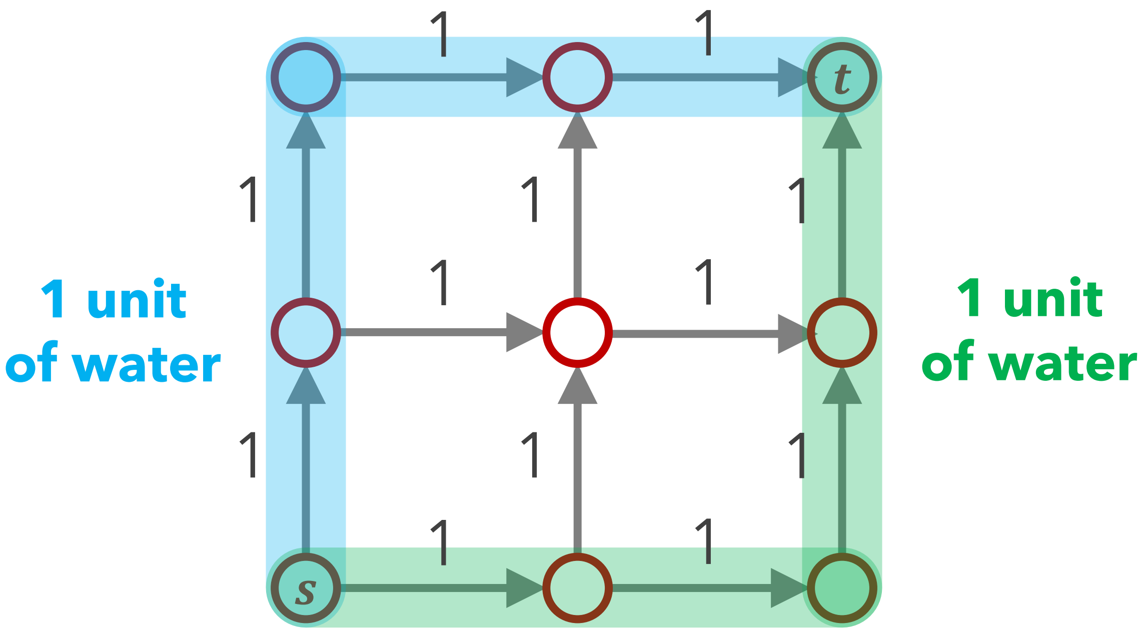
We will see this algorithm today.

**See also:** "On the history of the transportation and maximum flow problems"
by Alexander Schrijver

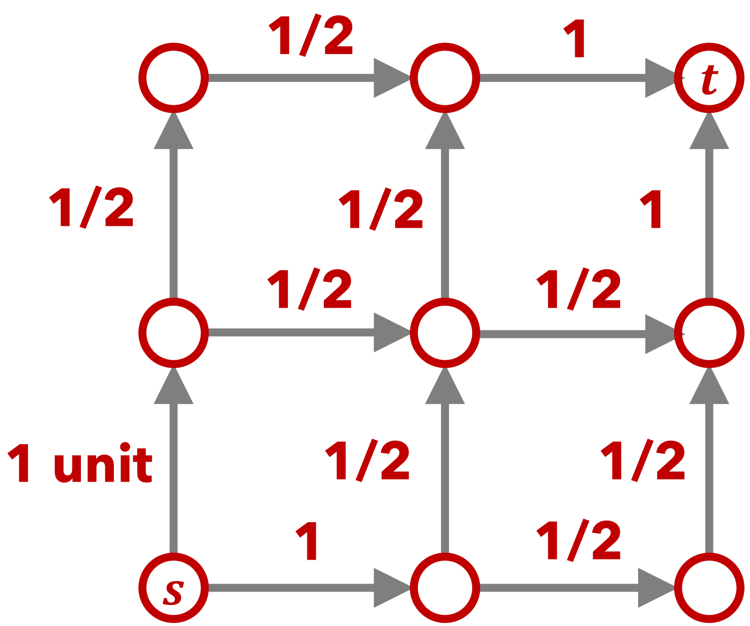
# Maximum flow

**Input:**
1. Directed graph $G = (V, E)$
2. One "source vertex" $s \in V$
3. One "sink vertex" $t \in V$
4. For each edge $e \in E$, a "capacity" $c_e \in \mathbb{Z}^+$ (or $\mathbb{R}^+$)

**Goal:** Route the maximum amount of water from $s$ to $t$




2 units of flow (water)



# Maximum flow

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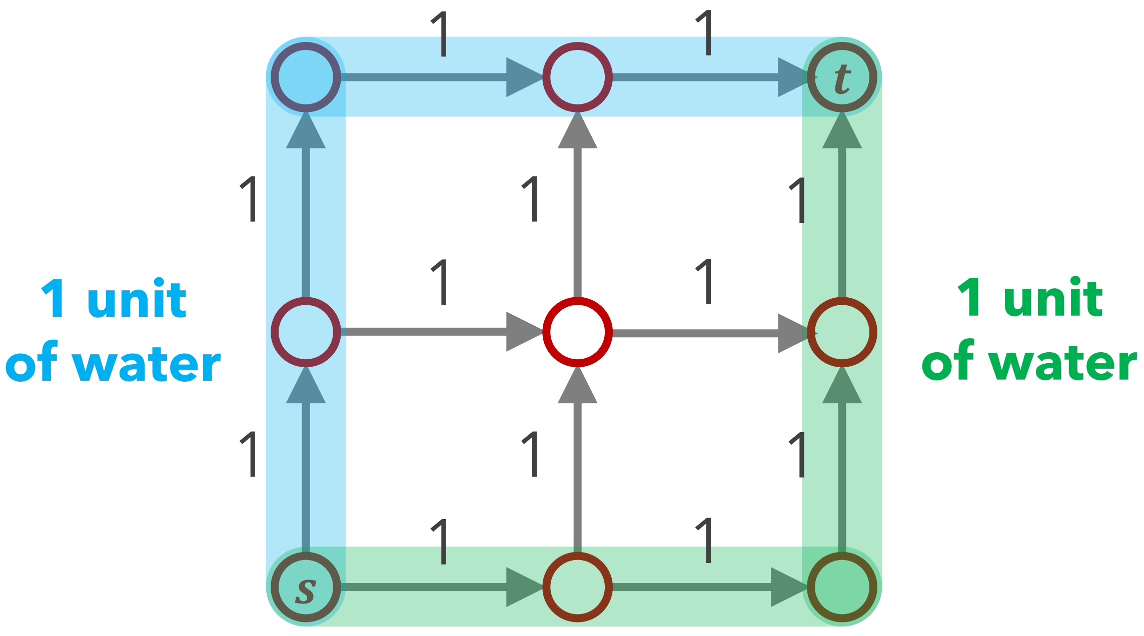
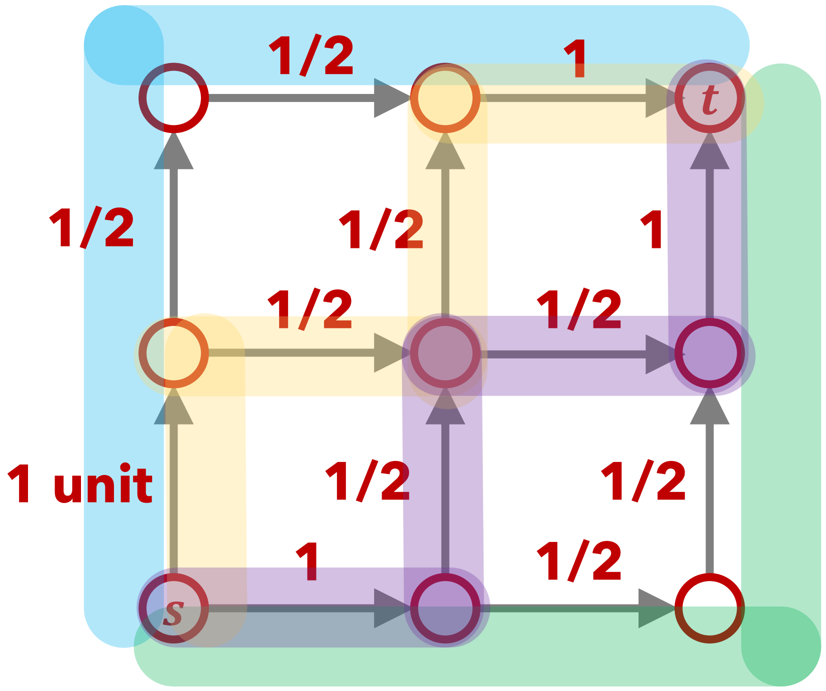



2 units of flow (water)



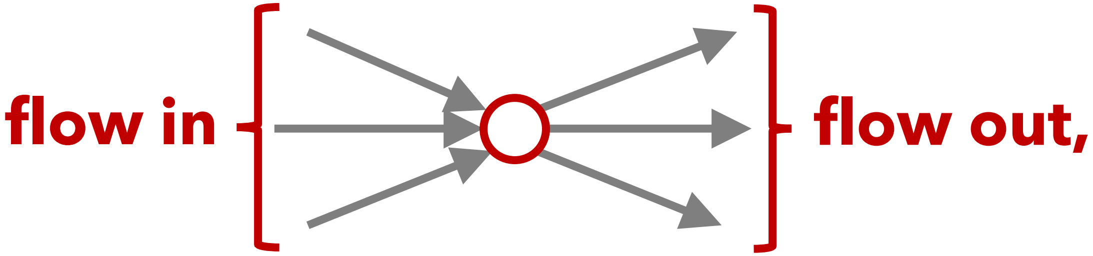

2 units of flow

**Def:** A **flow** assigns a number $f_e$ to each directed edge $e \in E$ such that

**(Nonnegativity)** $f_e \geq 0$

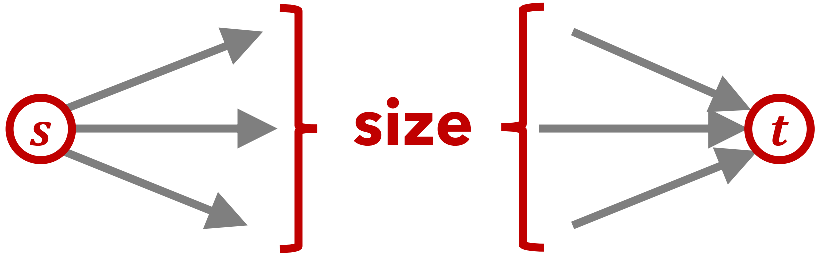
**(Capacity)** $f_e \leq c_e$

**(Flow in = flow out)** for each vertex $v \neq s, t$,




$$\sum_{u \to v} f_{u,v} = \sum_{v \to w} f_{v,w}$$

**Def:** The **size** of a flow $f$ is the total quantity sent from $s$ to $t$.




$$\textbf{size}(f) = \sum_{s \to v} f_{s,v} = \sum_{v \to t} f_{v,t}$$

**Maximum flow**

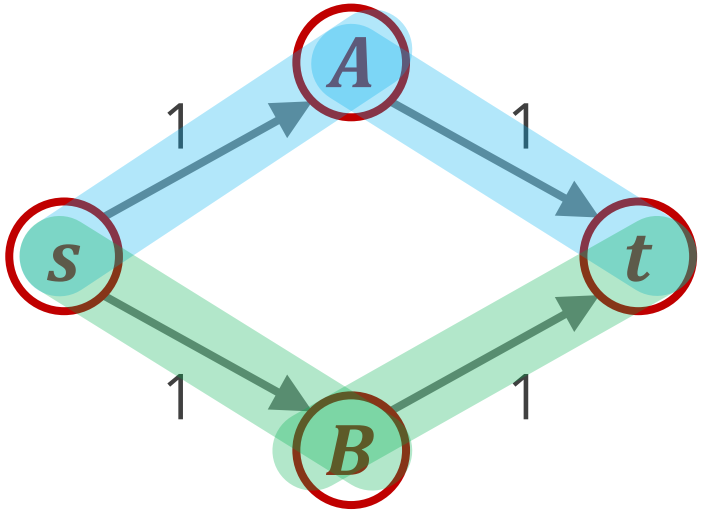
maximize $\textbf{size}(f)$

s.t. $\{f_e\}$ is a flow

**= a linear program!**

# **Max flow algorithm:** first try (Harris and Ross' "flooding" method)

1. Find a path $P$ from $s$ to $t$ which is not yet saturated
2. Send more flow along $P$
3. Repeat

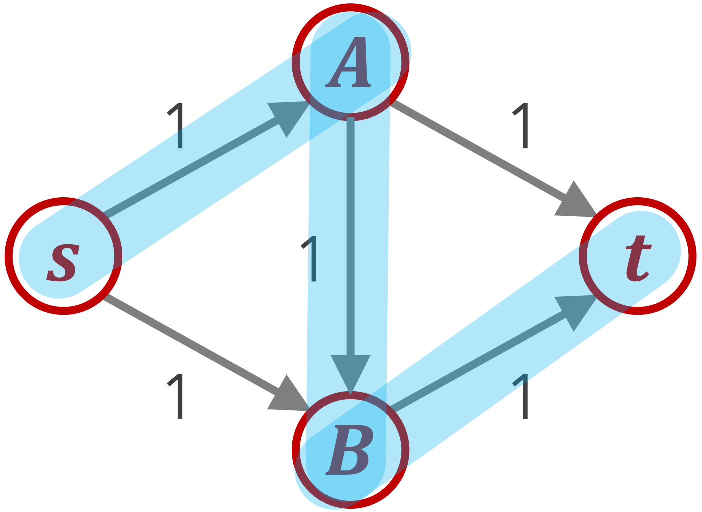



$s \to A \to t$: **1 unit**

$s \to B \to t$: **1 unit**

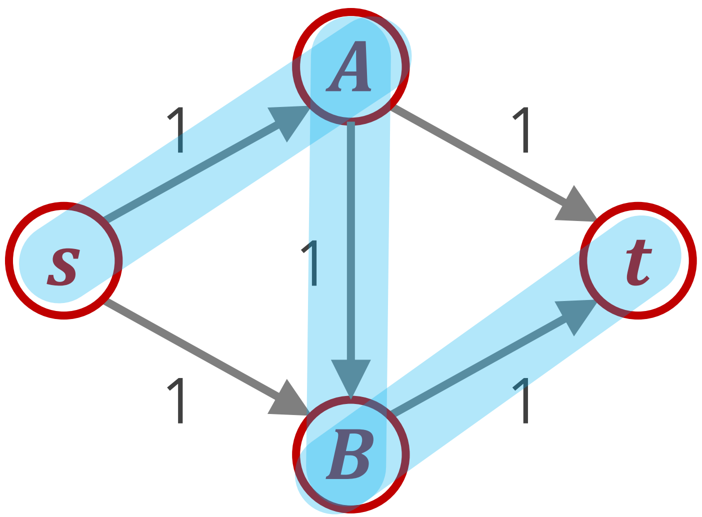
---

total: **2 units**




$s \to A \to B \to t$: **1 unit**

**Alg fails!**

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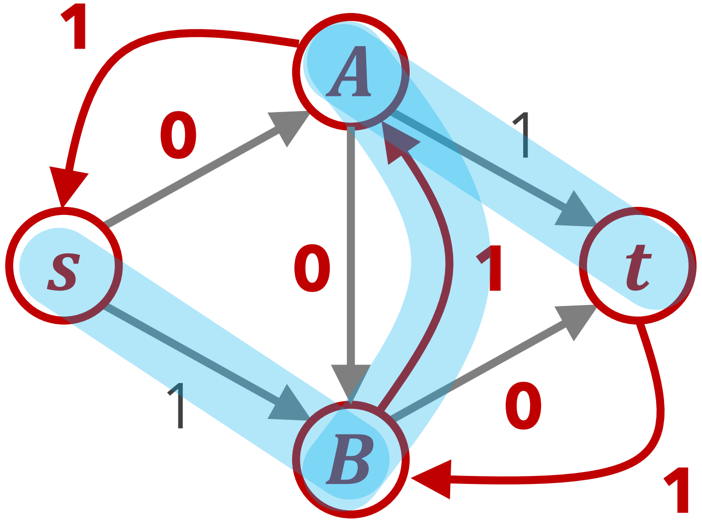
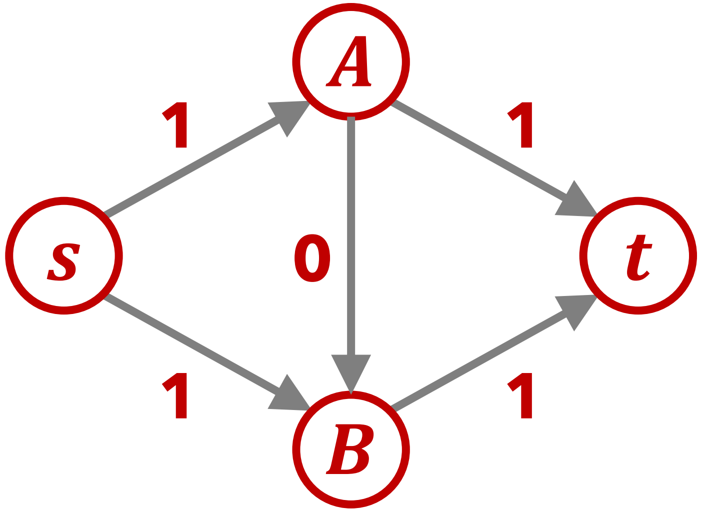



**"residual graph"**

$s \to B \to A \to t$: **1 unit**

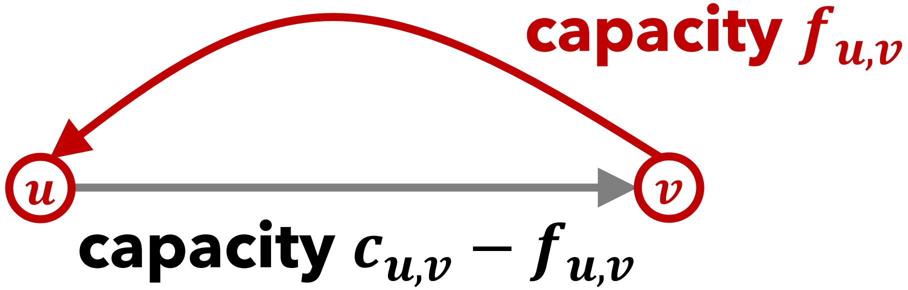



**2 units total!**

**Def:** Given a graph $G$ and a flow $f$ on $G$, the residual graph $G_f$ is as follows.

For all edges $(u, v)$:

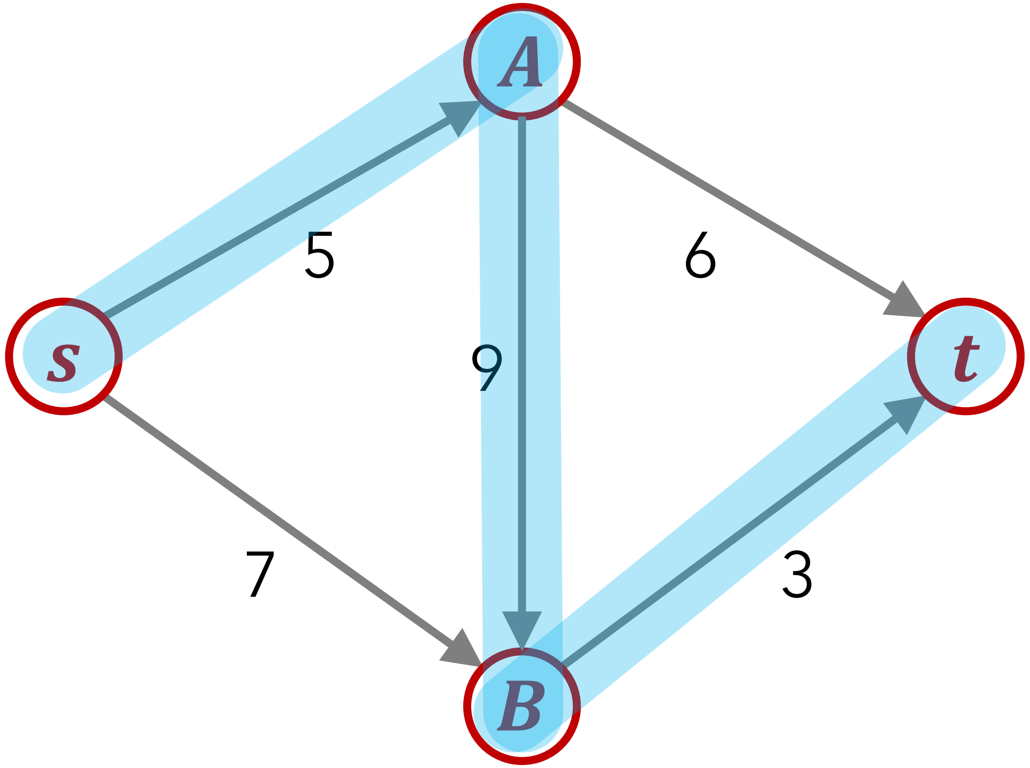

(in original graph)




(in residual graph)

# Ford-Fulkerson algorithm

1. Find a path ***P*** from ***s*** to ***t*** **in the residual graph** which is not yet saturated
2. Send more flow along ***P*** = an **augmenting path**
3. Repeat




$s \to A \to B \to t$: **3 units**

# Ford-Fulkerson algorithm

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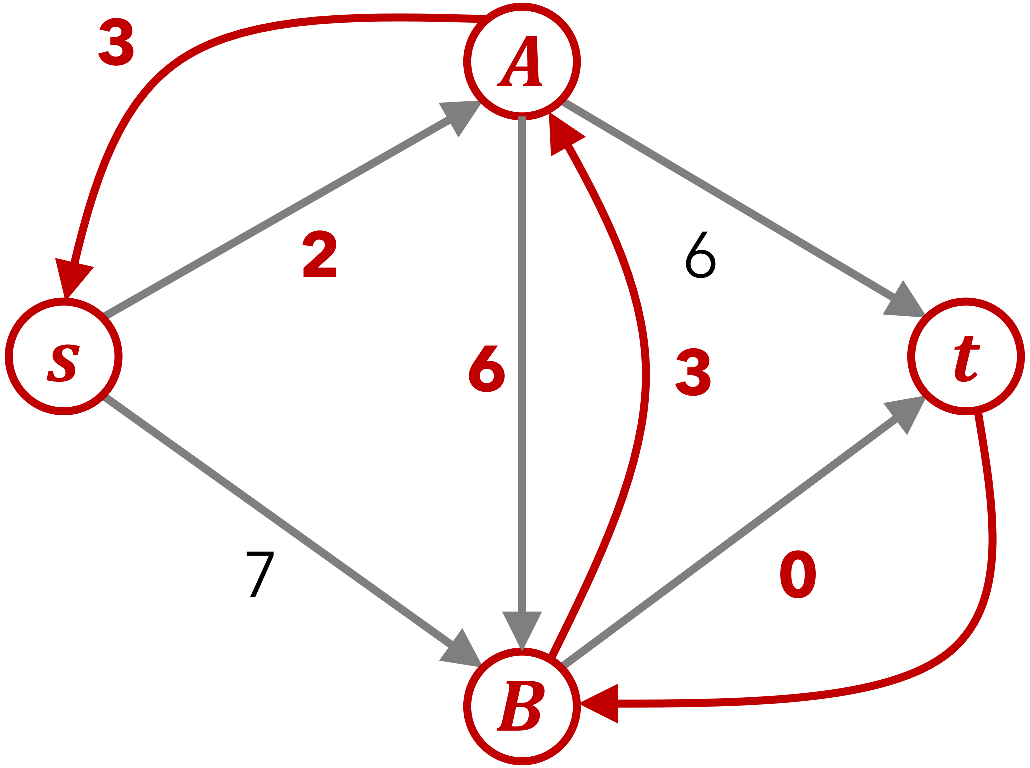


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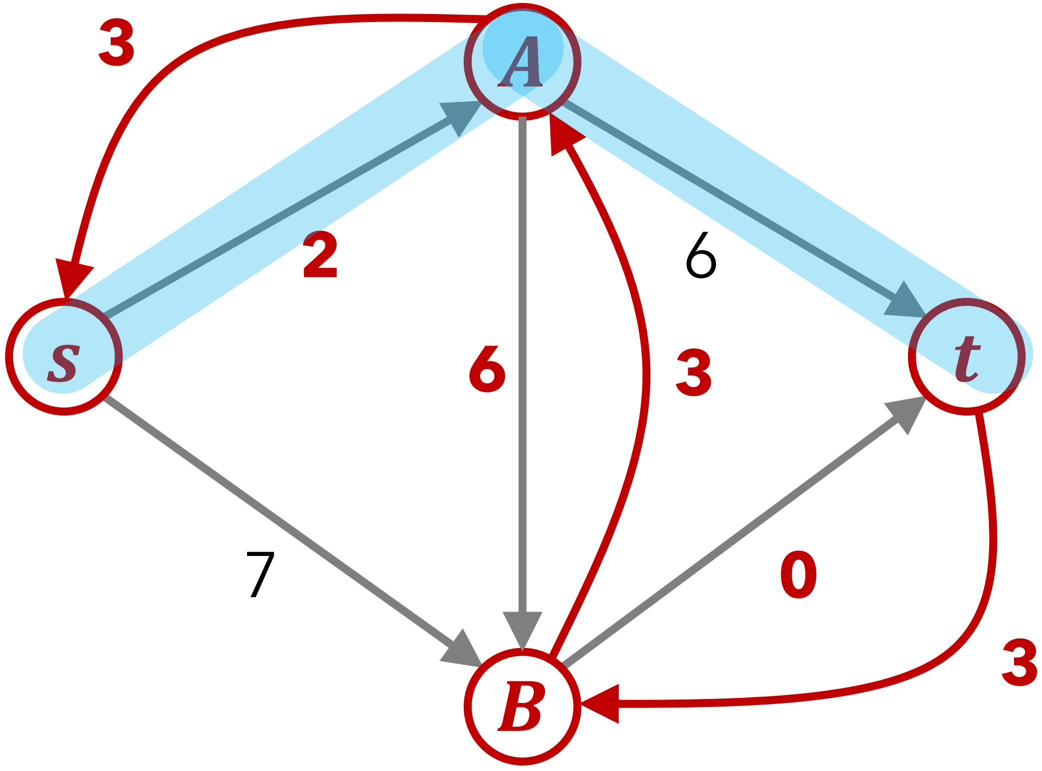


$s \to A \to B \to t$: **3 units**

$s \to A \to t$: **2 units**

# Ford-Fulkerson algorithm

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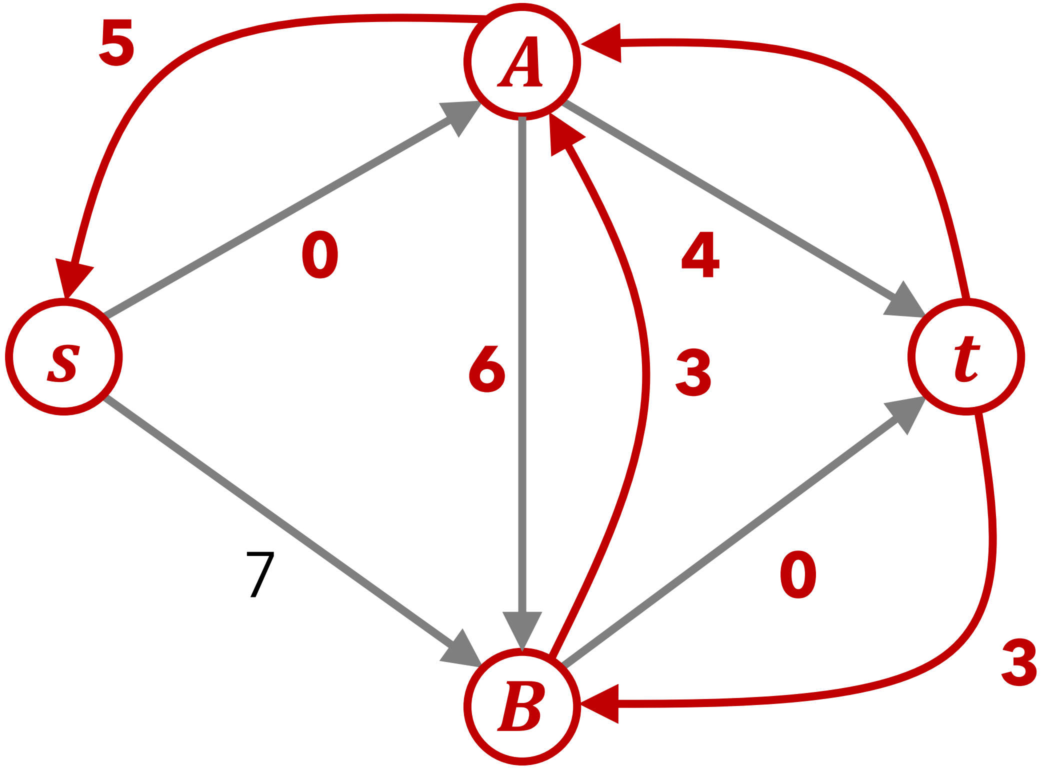


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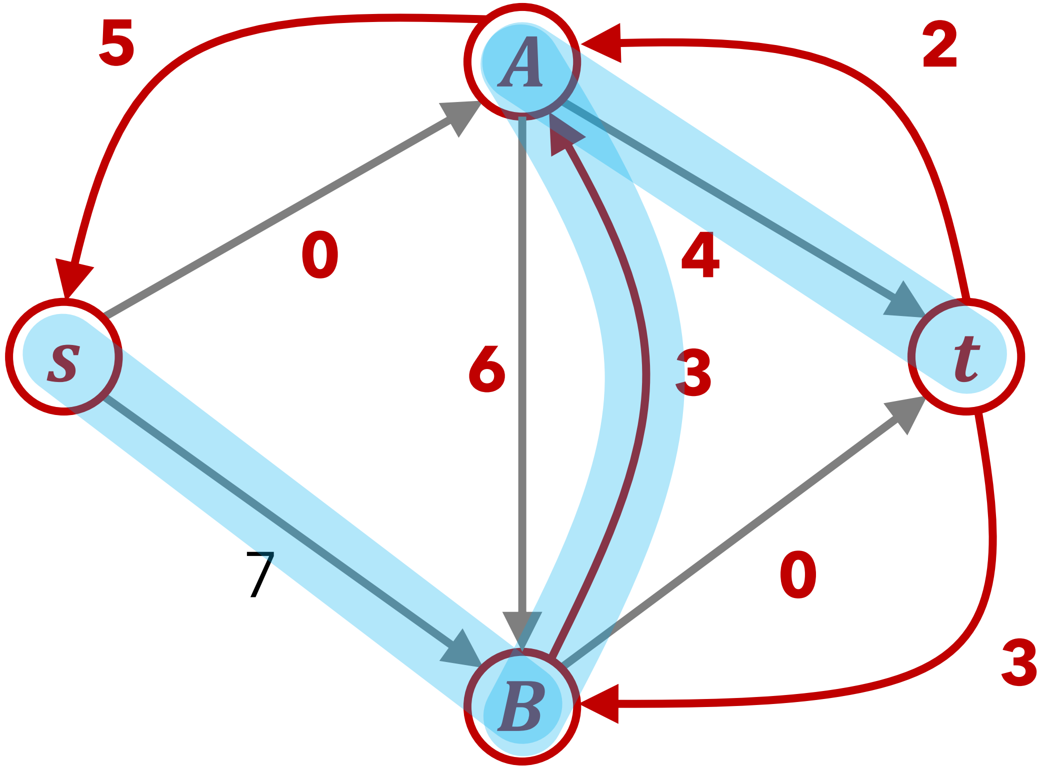


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$s \to A \to t$: **2 units**

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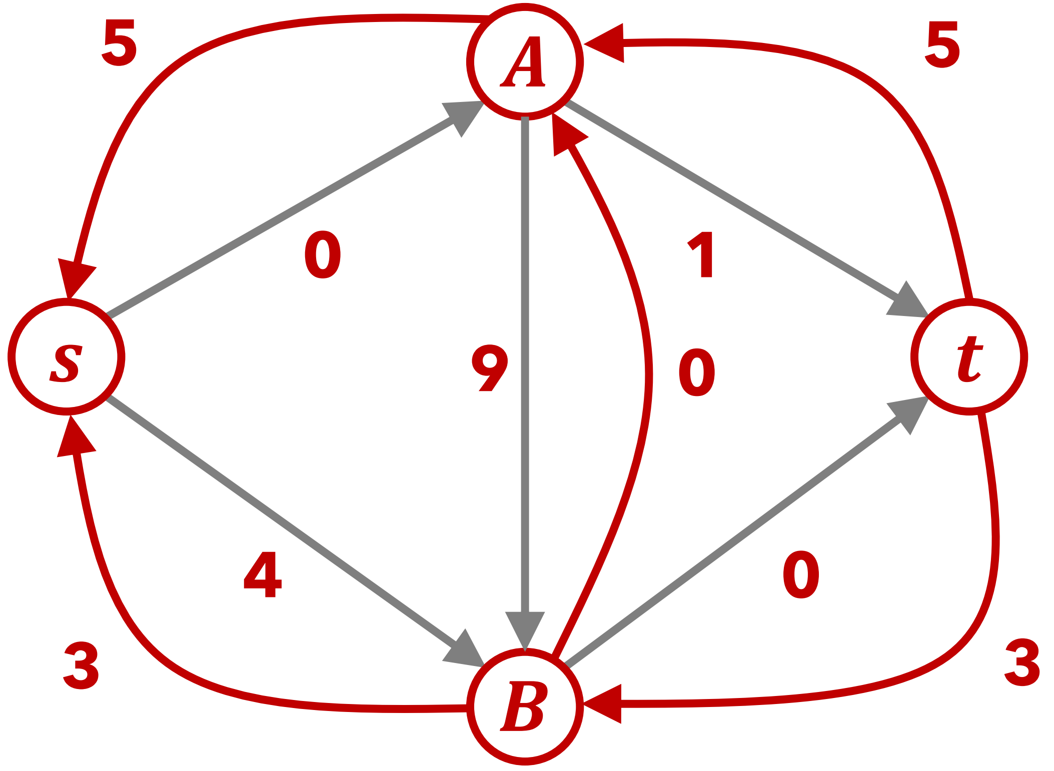


$s \to A \to B \to t$: **3 units**

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$s \to B \to A \to t$: **3 units**

total: **8 units**

# Ford-Fulkerson algorithm

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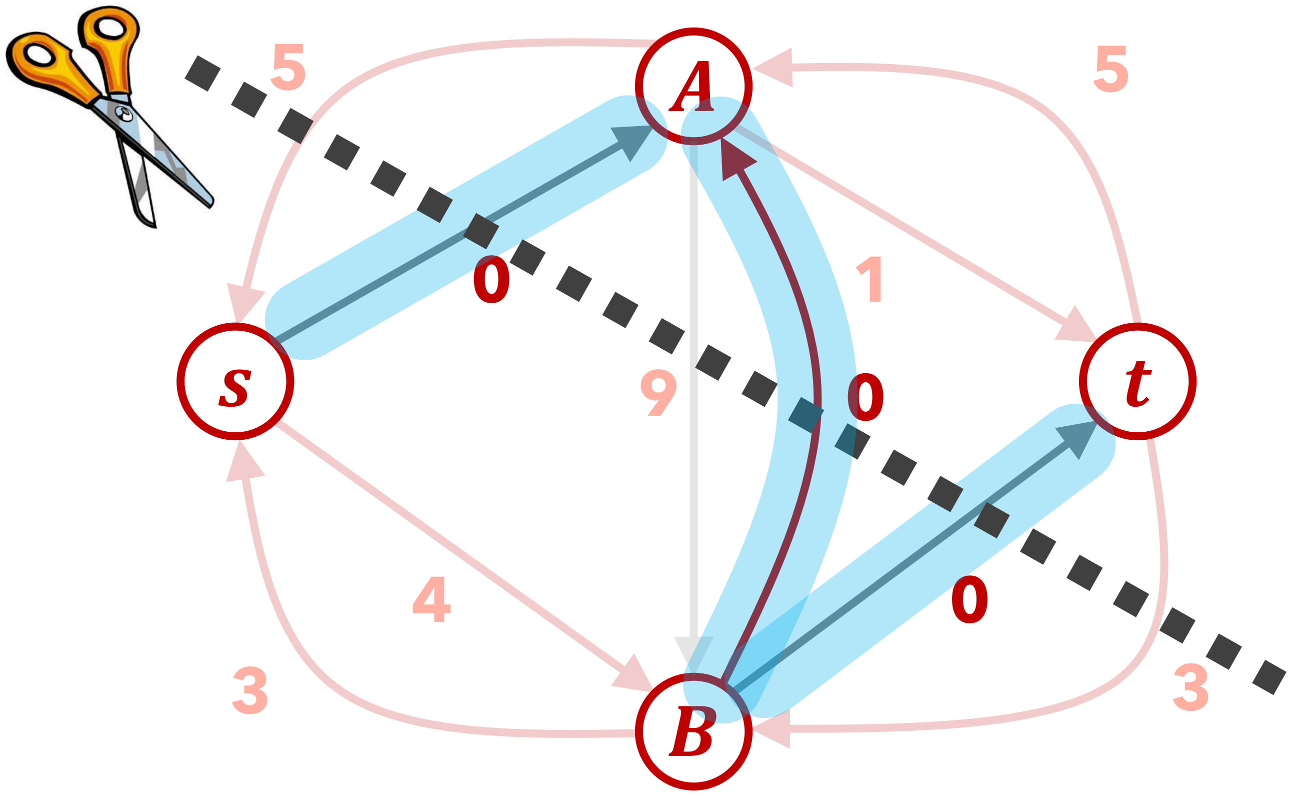


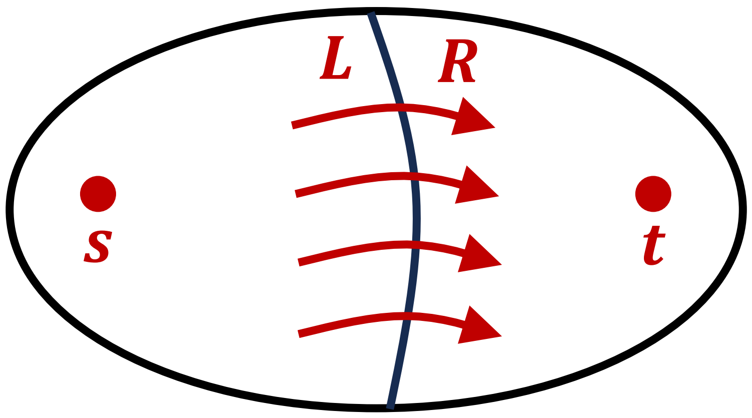

$s \to A \to B \to t$: **3 units**

$s \to A \to t$: **2 units**

$s \to B \to A \to t$: **3 units**

total: **8 units**

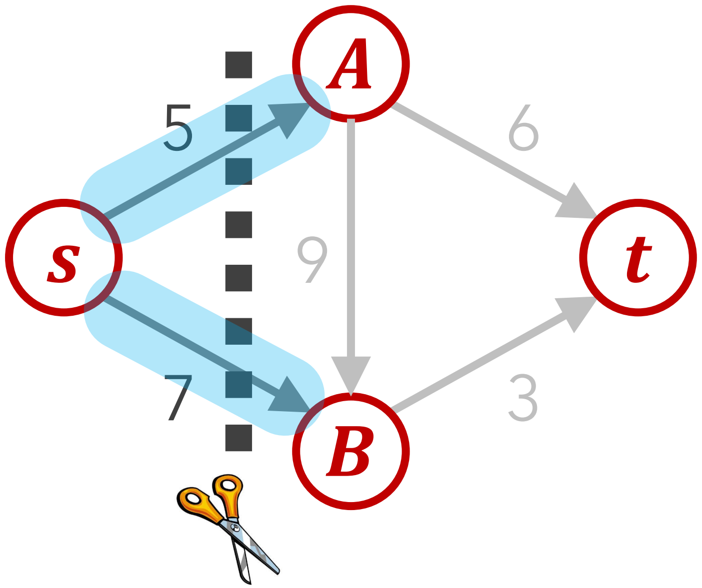
**Def:** An ***s*-*t* cut** is a partition $V = L \cup R$ of the vertices such that $s \in L$ and $t \in R$




**Def:** The **capacity** of the **cut** is $\text{capacity}(L, R) = \sum_{\substack{u \to v \\ u \in L, v \in R}} c_{u,v}$

**Thm:** For any flow $f$ and any cut $(L, R)$,

$$\text{size}(f) \le \text{capacity}(L, R).$$



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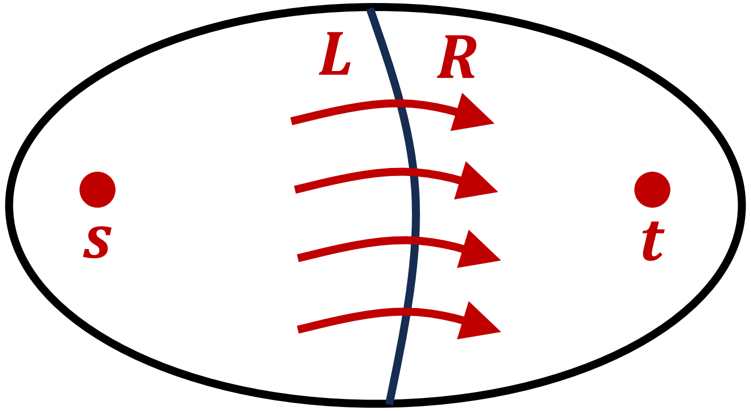


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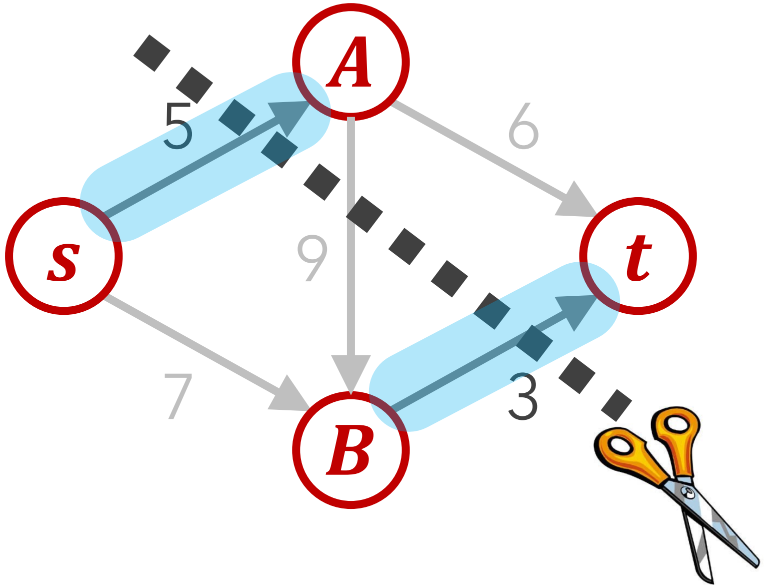
**Thm:** For any flow $f$ and any cut $(L, R)$,

$$\text{size}(f) \leq \text{capacity}(L, R).$$

**Def:** The **Min-cut** is the cut with **minimum** capacity.

**Aka:** Max-flow ≤ Min-cut

(**Harris and Ross' "bottleneck"**)



**Thm:** Max-flow = Min-cut

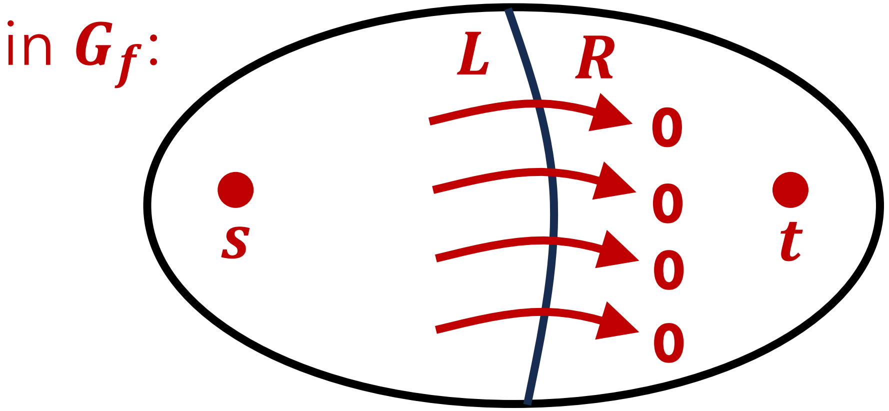
**Pf:** Only need to show "≥"

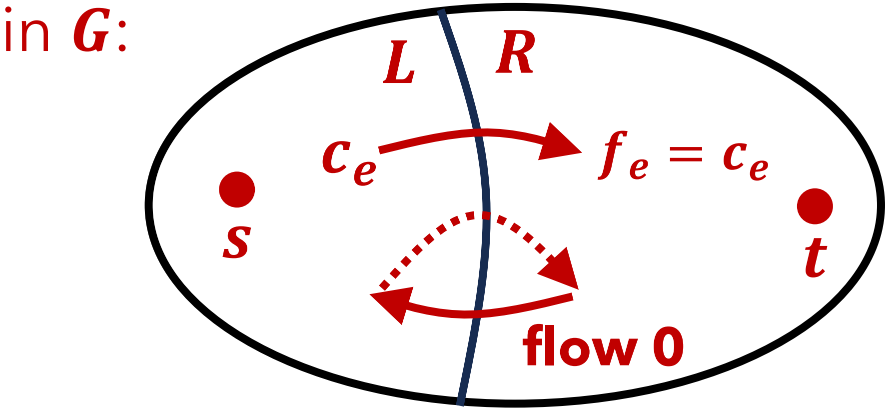
Run Ford-Fulkerson on $G$. Let $f$ be the flow it outputs.

Then no $s \to t$ in residual graph $G_f$.

Set $L$ = vertices reachable from $s$ in $G_f$.

$R$ = everything else.




Max-flow ≥ $\text{size}(f) = \text{capacity}(L, R)$ ≥ Min-cut. □

**Thm:** Ford-Fulkerson outputs a **maximum flow**.

**Runtime** $\approx$ # of augmenting paths $\leq \boldsymbol{U}$, where $\boldsymbol{U}$ = Max-flow

(× the time to find the paths) (in graphs of integer weights)

$\leq \mathbf{O}(m+n) \cdot \boldsymbol{U}$

Is this a good runtime?

Suppose each capacity $\boldsymbol{c_e}$ was $\leq \boldsymbol{C}$.

Then $\boldsymbol{U} \leq \boldsymbol{m} \cdot \boldsymbol{C}$.

Each $\boldsymbol{c_e}$ is a $\mathbf{log_2}(\boldsymbol{C})$–bit integer.

So $\boldsymbol{U}$ can be **exponential** in the input length!

This is a **pseudo-polynomial** algorithm

(it is polynomial in the **numerical value** of the input)

Recall: Knapsack

**Runtime** $\approx$ # of augmenting paths $\leq \boldsymbol{U}$, where $\boldsymbol{U}$ = Max-flow

(× the time to find the paths) (in graphs of integer weights)

$\leq \mathbf{O}(m+n)\cdot\boldsymbol{U}$

**Surprise:** If all the capacities are integral, then the Max-flow is integral.

(all the capacities/flows are integers)

**Other algorithms:**

Dinitz 1970/Edmonds-Karp 1972: Always pick the shortest augmenting path

Runs in time $\mathbf{O}(n\,m^2)$!

⋮

(many, many more)

⋮

Chen-Kyng-Liu-Peng-Gutenberg-Sachdeva 2022: $\mathbf{O}(m^{1+o(1)}\cdot\log(\boldsymbol{U}))$

(only 112 pages!)