Two-player Zero-Sum Games

Zero-sum games

Input: A payoff matrix M

$$\left.\begin{array}{l}\text{Row player: picks row } r \\ \text{Col player: picks col } c\end{array}\right\} \rightarrow \left\{\begin{array}{l}\text{Win } +A[r,c] \\ \text{Win } -A[r,c]\end{array}\right.$$

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

Types of strategies:

"Pure strategy": a single row/column

e.g. row player always picks rock (beaten by paper col)

"Mixed strategy": probability distribution over pure strategies

e.g. $\Pr[\text{Rock}] = 1/3$, $\Pr[\text{Paper}] = 1/3$, $\Pr[\text{Scissors}] = 1/3$

Note: average score $= 0$ (no matter what col player does)

Game 1

Turn Order:

1. Row player announces mixed strat $P = (p_1, p_2)$
2. Col player responds w/ mixed strat $q = (q_1, q_2)$

$p_1 = \Pr[\text{Row } 1]$

$p_2 = \Pr[\text{Row } 2]$

	1	2
1	3	−1
2	−2	1
	q_1	q_2

Def: Row player's average score is $\text{Score}(p,q) = 3\cdot p_1 q_1 - 1\cdot p_1 q_2 - 2\cdot p_2 q_1 + 1\cdot p_1 q_2$

Col player's best strat: $\underset{\text{mixed strat } q}{\text{minimize}} \{\text{Score}(p,q)\}$

(should pick best col) $= \underset{\text{pure strat } q}{\min} \{\text{Score}(p,q)\} = \min\{\underbrace{3\cdot p_1 - 2\cdot p_2}_{\text{col 1 score}}, \underbrace{-1\cdot p_1 + 1\cdot p_2}_{\text{col 2 score}}\}$

Row player's best strat: $\underset{\text{mixed strat } p}{\text{maximize}} \{\min\{3\cdot p_1 - 2\cdot p_1, -1\cdot p_1 + 1\cdot p_2\}\}$

Fact: Can calculate $\underset{\text{mixed strat } p}{\text{maximize}} \left\{ \min \left\{ 3 \cdot p_1 - 2 \cdot p_1 ,\ -1 \cdot p_1 + 1 \cdot p_2 \right\} \right\}$ with LP.

Pf:

$$
\begin{aligned}
\text{maximize} \quad & z \\
\text{subject to} \quad & z \leq 3p_1 - 2p_2 \\
& z \leq -p_1 + p_2 \\
& p_1 + p_2 = 1 \\
& p_1 \geq 0,\ p_2 \geq 0.
\end{aligned}
\qquad = LP_1
$$

Note: $z = \min\{3p_1 - 2p_2, -p_1 + p_2\}$.

$\square$

Game 2

Same as Game 1, except col player goes 1st
and row player goes 2nd

$$\begin{array}{c|cc|c} & 1 & 2 & \\ \hline 1 & 3 & -1 & p_1 \\ 2 & -2 & 1 & p_2 \\ \hline & q_1 & q_2 & \end{array}$$

Payoff of row 1: $3 \cdot q_1 - 1 \cdot q_2$

Payoff of row 2: $-2 \cdot q_1 + 1 \cdot q_2$

Row player's best strat $= \max\{3q_1 - q_2, -2q_1 + q_2\}$

Col " " " $= \underset{\text{mixed strat } q}{\text{minimize}} \{ \max\{3q_1 - q_2, -2q_1 + q_2\}\}$

$LP_2 =$ minimize Z,

subject to $3q_1 - q_2 \le Z$

$-2q_1 + q_2 \le Z$

$q_1 + q_2 = 1, \quad q_1 \geq 0, q_2 \geq 0.$

Game 1

1. Row player first
2. Col player second

	1	2
1	3	-1
2	-2	1

Game 2

1. Col player first
2. Row player second

$$\max_{p}\left\{\min_{q}\left\{\text{Score}(p,q)\right\}\right\} \leq \min_{q}\left\{\max_{p}\left\{\text{Score}(p,q)\right\}\right\}$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$LP_1 \xleftrightarrow{\text{(dual)}} LP_2$$

$\therefore$ Strong duality $\Rightarrow$ $LP_1 = LP_2 = \text{Value(Game)}$ (Definition)

(Min-Max Theorem)

$\Rightarrow$ order of play doesn't change value

$\Rightarrow$ $\exists$ optimal strat for ROW, irrespective of COL