Note: Your TA probably will not cover all the problems. This is totally fine, the discussion worksheets are not designed to be finished in an hour. They are deliberately made long so they can serve as a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

1 Counting inversions

This problem arises in the analysis of *rankings*. Consider comparing two rankings. One way is to label the elements (books, movies, etc.) from 1 to k according to one of the rankings, then order these labels according to the other ranking, and see how many pairs are "out of order".

We are given a sequence of k distinct numbers $n_1, \cdots, n_k$. We say that two indices $i < j$ form an inversion if $n_i > n_j$, that is if the two elements n_i and n_j are "out of order". Provide a divide and conquer algorithm to determine the number of inversions in the sequence $n_1, \cdots, n_k$ in time $\mathcal{O}(k \log k)$. Only an algorithm description and runtime analysis is needed.

Hint: Modify merge sort to count during merging.

2 Maximum Subarray Sum

Given an array A of n integers, the maximum subarray sum is the largest sum of any contiguous subarray of A (including the empty subarray). In other words, the maximum subarray sum is:

$$\max_{i \le j} \sum_{k=i}^{j} A[k]$$

For example, the maximum subarray sum of $[-2, 1, -3, 4, -1, 2, 1, -5, 4]$ is 6, the sum of the contiguous subarray $[4, -1, 2, 1]$.

Design an $O(n \log n)$-time algorithm that finds the maximum subarray sum.

Give a 3-part solution.

Hint 1: Split the array into two equally-sized pieces. What are the possibilities for the subarray, and how does this apply if we want to use divide and conquer?

3 Pareto Optimality

Given a set of points $P = \{(x_1, y_1), (x_2, y_2) \dots (x_n, y_n)\}$, a point $(x_i, y_i) \in P$ is Pareto-optimal if there does not exist any $j \neq i$ such that such that $x_j > x_i$ and $y_j > y_i$. In other words, there is no point in P above and to the right of (x_i, y_i). Design a $O(n \log n)$-time divide-and-conquer algorithm that given P, outputs all Pareto-optimal points in P. Only an algorithm description and runtime analysis is needed.

(*Hint: Split the array by x-coordinate. Show that all points returned by one of the two recursive calls is Pareto-optimal, and that you can get rid of all non-Pareto-optimal points in the other recursive call in linear time.*)

4 Complex numbers review

A *complex number* is a number that can be written in the rectangular form $a + bi$ (i is the imaginary unit, with $i^2 = -1$). The following famous equation (*Euler's formula*) relates the polar form of complex numbers to the rectangular form:

$$re^{i\theta} = r(\cos\theta + i\sin\theta)$$

In polar form, $r \geq 0$ represents the distance of the complex number from 0, and θ represents its angle. Note that since $\sin(\theta) = \sin(\theta + 2\pi), \cos(\theta) = \cos(\theta + 2\pi)$, we have $re^{i\theta} = re^{i(\theta+2\pi)}$ for any r, θ.

The n-th *roots of unity* are the n complex numbers satisfying $\omega^n = 1$. They are given by

$$\omega_k = e^{2\pi ik/n}, \qquad k = 0, 1, 2, \ldots, n-1$$

(a) Let $x = e^{2\pi i 3/10}, y = e^{2\pi i 5/10}$ which are two 10-th roots of unity. Compute the product $x \cdot y$. Is this an n-th root of unity for some n? Is it a 10-th root of unity?

What happens if $x = e^{2\pi i 6/10}, y = e^{2\pi i 7/10}$?

For all your answers, simplify if possible.

(b) Show that for any n-th root of unity $\omega \neq 1$, $\sum_{k=0}^{n-1} \omega^k = 0$, when $n > 1$.

Hint: Use the formula for the sum of a geometric series $\sum_{k=0}^{n} \alpha^k = \frac{\alpha^{n+1}-1}{\alpha-1}$. It works for complex numbers too!

(c) (i) Find all ω such that $\omega^2 = -1$.

(ii) Find all ω such that $\omega^4 = -1$.

5 (OPTIONAL) Monotone matrices

A m-by-n matrix A is *monotone* if $n \geq m$, each row of A has no duplicate entries, and it has the following property: if the minimum of row i is located at column j_i, then $j_1 < j_2 < j_3 \dots j_m$. For example, the following matrix is monotone (the minimum of each row is bolded):

$$\begin{bmatrix} \mathbf{1} & 3 & 4 & 6 & 5 & 2 \\ 7 & 3 & \mathbf{2} & 5 & 6 & 4 \\ 7 & 9 & 6 & 3 & 10 & \mathbf{0} \end{bmatrix}$$

Give an efficient (i.e., better than $O(mn)$-time) algorithm that finds the minimum in each row of an m-by-n monotone matrix A.

Give a 3-part solution. You do not need to write a formal recurrence relation in your runtime analysis; an informal summary of the runtime analysis such as "proof by picture" is fine.